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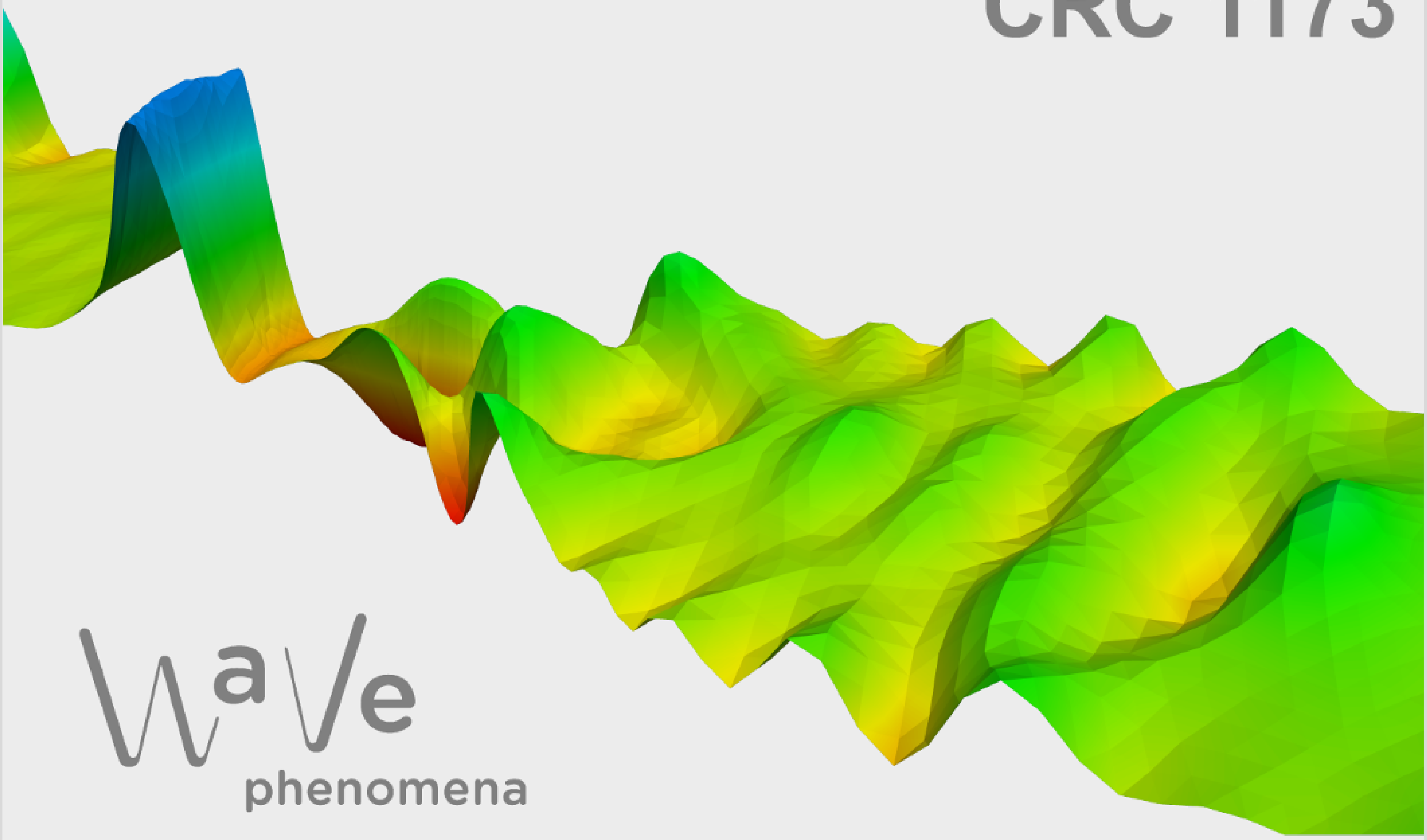

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SEISMIC IMAGING WITH GENERALIZED RADON TRANSFORMS: STABILITY OF THE BOLKER CONDITION

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Dedicated to Professor Victor Guillemin on the occasion of his 85th birthday

ABSTRACT. Generalized Radon transforms are Fourier integral operators which are used, for instance, as imaging models in geophysical exploration. They appear naturally when linearizing about a known background compression wave speed. In this work we first consider a linearly increasing background velocity in two spatial dimensions. We verify the Bolker condition for the zero-offset scanning geometry and provide meaningful arguments for it to hold even if the common offset is positive. Based on this result we suggest an imaging operator for which we calculate the top order symbol in the zero-offset case to study how it maps singularities. Second, to support the usage of background models obtained from linear regression we present a stability result for the Bolker condition under perturbations of the background velocity and of the offset.

1. INTRODUCTION

Generalized Radon transforms serve, for instance, as linear models in seismic imaging in the acoustic regime. To this end the nonlinear inverse problem of recovering the wave speed from reflected wave fields is linearized about a known background velocity: We start from the acoustic wave equation

$$\frac{1}{\nu_{\mathrm{p}}^2}\partial_t^2 u - \Delta_{\mathbf{x}} u = \delta(\mathbf{x}-\mathbf{x}_s)\delta(t), \quad u|_{t=0} = \partial_t u|_{t=0} = 0, \tag{1.1}$$

where $\nu_{\mathrm{p}} = \nu_{\mathrm{p}}(\mathbf{x})$ is the velocity (sound speed) and $\mathbf{x}_s$ denotes the position of the source. So, the pressure wave $u = u(t;\mathbf{x},\mathbf{x}_s)$, $\mathbf{x}\in\mathbb{R}^d$, $d\in\{2,3\}$, at time $t\geq 0$ is initiated solely by the source at time $t=0$. The corresponding nonlinear inverse problem entails the recovery of ν_{p} from measurements of $u(\cdot;\mathbf{x}_r,\mathbf{x}_s)$ over a time interval at several pairs $(\mathbf{x}_s,\mathbf{x}_r)$ of source and receiver positions.

For the linearization we make the ansatz

$$\frac{1}{\nu_{\mathrm{p}}^2(\mathbf{x})} = \frac{1+n(\mathbf{x})}{v^2(\mathbf{x})}$$

with an a priori known background velocity $v = v(\mathbf{x})$ which satisfies the geometric optics assumption, i.e., points on the surface are connected to points in the subsurface by unique

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characteristic rays. Now, n is the object we seek. It is a dimensionless quantity which records the high frequency content of ν_{p}.

Using the principles of wave propagation in geometric optics one derives the following linear integral equation for n,

$$Fn(t;\mathbf{x_r},\mathbf{x_s}) = \int_0^t (t-s)^{d-2}(\widetilde{u}-u)(s;\mathbf{x_r},\mathbf{x_s})\mathrm{d}s, \tag{1.2}$$

where the operator F is given by

$$Fw(t;\mathbf{x_r},\mathbf{x_s}) = \int \frac{w(\mathbf{x})}{v^2(\mathbf{x})} A(\mathbf{x},\mathbf{x_s})A(\mathbf{x},\mathbf{x_r})\delta\big(t-\tau(\mathbf{x},\mathbf{x_s})-\tau(\mathbf{x},\mathbf{x_r})\big)\mathrm{d}\mathbf{x} \tag{1.3}$$

with the amplitude A and the travel time τ from the progressing wave expansion of the reference solution $\widetilde{u}$ which solves (1.1), however, with ν_{p} replaced by v. So, the right hand side of (1.2) is available from the measurements and the computed reference solution. Further, τ and A can be computed as well, the former from the eikonal equation

$$|\nabla_{\mathbf{x}}\tau(\cdot,\mathbf{x_s})| = \frac{1}{v}, \quad \tau(\mathbf{x_s},\mathbf{x_s}) = 0, \tag{1.4}$$

and the latter from the transport equation

$$\operatorname{div}(A^2\nabla_{\mathbf{x}}\tau) = 0. \tag{1.5}$$

The operator F is a generalized Radon transform as $Fn(t;\mathbf{x_r},\mathbf{x_s})$ is an integral mean over the reflection isochrone connecting points of equal travel time t to source and to receiver. We recall the representation of F as Fourier integral operator (for a definition see next section). Assuming that pairs of source and receiver points are parametrized by a variable s we have that

$$Fw(\mathbf{s},t) = \frac{1}{2\pi}\int \frac{w(\mathbf{x})}{v^2(\mathbf{x})} A(\mathbf{x},\mathbf{x_s}(\mathbf{s}))A(\mathbf{x},\mathbf{x_r}(\mathbf{s}))\mathrm{e}^{\imath\omega(t-\varphi(\mathbf{s},\mathbf{x}))}\,\mathrm{d}\mathbf{x}\mathrm{d}\omega \tag{1.6}$$

with

$$\varphi(\mathbf{s},\mathbf{x}) := \tau(\mathbf{x},\mathbf{x_s}(\mathbf{s})) + \tau(\mathbf{x},\mathbf{x_r}(\mathbf{s})).$$

Hence, $\{\mathbf{x} : t = \varphi(\mathbf{s},\mathbf{x})\}$ is the reflection isochrone at time t with respect to $\mathbf{x_s}(\mathbf{s})$ and $\mathbf{x_r}(\mathbf{s})$. For all the details we refer to, e.g., [22, Sec. 6] or [12]. See also [2, Appendix E] and [5, 7].

As there is no inversion formula known for F one defines imaging operators mimicking well-known reconstruction formulas of filtered backprojection type from X-ray computerized tomography, see, e.g., [17]. For instance, given the data y (right hand side of (1.2)), the output of Kirchhoff migration, the traditional inversion procedure in geophysics, can be written as $F^\dagger Ky$ where K is a convolution filter and $F^\dagger$ denotes a dual transform (generalized backprojection). The corresponding imaging operator $F^\dagger KF$ is a kind of low pass filter superimposed with a smoothing operator, see [1]. Hence, prominent features of n are in fact visible in $F^\dagger KFn$.

In a series of papers [11, 12, 13] we have demonstrated the potential of imaging operators of the type $KF^*\psi F$ from an analytical as well as a numerical point of view. Here, F^* is a backprojection operator (i.e., the formal, possibly weighted, L^2 adjoint of F), K is a suitable pseudodifferential operator and ψ is a smooth cutoff function. Under a technical assumption (the Bolker condition (2.3)) these imaging operators are pseudodifferential operators and we have computed their top order symbols to understand how they map singularities. In case of a constant background velocity v and if source and receiver positions are offset by a constant vector (common offset data acquisition geometry), we have thus been able to construct explicit K's such that $KF^*\psi Fn$ enhances features (discontinuities) of n relatively independent of location and offset.

In the present work we extend our results to the linear background velocity model in two spatial dimensions. This velocity model approximates well seismic wave propagation in Tertiary basins [21, Lesson 37] and is, for that reason, also derived by linear regression from well log measurements for other geological formations, see, e.g., [4]. Moreover, sound velocity in the oceans can be calculated by an empirical formula which depends on temperature, salinity and depth [16]. For depths below 1000m, salinity and temperature can be considered constant and the formula is then essentially linear in depth.

First, for the common zero offset scanning geometry, we verify the Bolker condition, compute and study the top order symbol of $KF^*\psi F$ which reveals a fundamentally different mapping property compared to the constant background velocity model: singularities of n with a vertical tangent are visible in $KF^*\psi Fn$ (for an adequate choice of ψ). If the offset is positive we provide overwhelming numerical combined with analytical evidence for the Bolker condition to hold. Second, to strengthen the usage of linear models obtained by regression we explore how stable the properties of $KF^*\psi F$ are under perturbations of the velocity model. This will be done in a rather general framework which even covers stability of the Bolker condition under a perturbation of the offset.

The layout of the paper is as follows. In the following section we compile background material on Fourier integral operators and microlocal analysis on which our accomplishments are based. The experienced reader can skip it. Section 3 is then devoted to the study of the linear velocity model where we first validate the Bolker condition for zero offset. We succeed here because we find an explicit parameterization of the reflection isochrones. Unfortunately, in the positive offset case, we only have an implicit parameterization which prevents a complete rigorous proof. We are nevertheless able to show that the Bolker condition cannot hold near the surface. In the last part of Section 3 we study the top order symbol of $\Lambda = KF^\dagger\psi F$ for zero offset and where $K = \Delta$ is the Laplacian. We characterize visible and invisible singularities and find how the top order symbol depends asymptotically on increasing depth. The latter result leads to the definition of K's counteracting the depth dependence. A first numerical experiment illustrates these findings.

The Bolker condition actually is a condition on the phase function of a Fourier integral operator. The phase function of our operator (1.6) depends on the travel time. Therefore, Section 4 prepares our stability results by providing a stability analysis for the travel time as a solution of the eikonal equation (1.4) under perturbation of the wave speed. To this end we study the corresponding characteristic system (ray system). Finally, we show in Section 5 that if the phase functions of two Fourier integral operators are sufficiently close and one of them satisfies the Bolker condition, so does the other. This is then applied twice to our seismic situation: once for a small offset and once for a perturbation in the wave speed, using the insight from Section 4.

In three appendices we have outsourced technical calculations which would otherwise make Section 3 overly technical.

2. Basic material

In this section we not only collect essential standard textbook material on Fourier integral and pseudodifferential operators but we also introduce our notation. See, e.g., [15, 18, 23, 24] for proofs and further details.

If X is an open subset of $\mathbb{R}^d$ and $f\colon X \to \mathbb{R}$, then we define $\nabla_{\mathbf{x}} f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots \frac{\partial f}{\partial x_d}\right)$. We let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}_0^d$, we use the standard notation for the differential operator D^α by $D^\alpha f = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} f$.

2.1. **Fourier Integral Operators.** For positive integers d_X and d_Y let $X \subset \mathbb{R}^{d_X}$ and $Y \subset \mathbb{R}^{d_Y}$ be open subsets. Let N be a positive integer.

Definition 2.1 (Symbol)**.** *A function $p \in C^\infty(Y \times X \times \mathbb{R}^N \setminus \{\mathbf{0}\})$ is a* symbol of order $m \in \mathbb{Z}$ *if for every compact set $\mathrm{K} \subset Y \times X$ and all multi-indices $\alpha \in \mathbb{N}_0^N$, $\beta \in \mathbb{N}_0^{d_X}$, and $\gamma \in \mathbb{N}_0^{d_Y}$ there exists a positive constant $C = C(\mathrm{K}, \alpha, \beta, \gamma)$ such that*

$$|D_\xi^\alpha D_{\mathbf{x}}^\beta D_{\mathbf{y}}^\gamma p(\mathbf{y}, \mathbf{x}, \xi)| \le C(1 + |\xi|)^{m - |\alpha|}$$

holds for all $(\mathbf{y}, \mathbf{x}) \in \mathrm{K}$ and all ξ with $|\xi| \ge 1$ and if p is locally integrable on $K \times S^{N-1}$. The set of all symbols of order m is denoted by $S^m(Y \times X \times \mathbb{R}^N)$.

The symbol p of order m is elliptic *if for each compact subset K of $Y \times X$ there are positive constants c and M such that*

$$|p(\mathbf{y}, \mathbf{x}, \xi)| \ge c\,(1 + |\xi|)^m \tag{2.1}$$

for all $(\mathbf{y}, \mathbf{x}) \in \mathrm{K}$ and all ξ with $|\xi| \ge M$.

Let $(\mathbf{y}_0, \mathbf{x}_0, \xi_0) \in Y \times X \times (\mathbb{R}^N \setminus \{\mathbf{0}\})$. Then, the symbol p is microlocally elliptic near $(\mathbf{x}_0, \xi_0)$ *if there are an open neighborhood U of $\mathbf{x}_0$, a conic open neighborhood V of ξ_0, and positive constants C and M such that (2.1) holds for all $\mathbf{x} \in U$ and $\xi \in V$ with $|\xi| \ge M$.*

Definition 2.2 (Phase function)**.** *A real-valued function $\Phi \in C^\infty(Y \times X \times \mathbb{R}^N \setminus \{\mathbf{0}\})$ with arguments $(\mathbf{y}, \mathbf{x}, \xi)$ is called a* phase function *if it is positively homogeneous of degree 1 in ξ and $(\nabla_{\mathbf{y}} \Phi, \nabla_\xi \Phi)$ as well as $(\nabla_{\mathbf{x}} \Phi, \nabla_\xi \Phi)$ do not vanish on $Y \times X \times \mathbb{R}^N \setminus \{\mathbf{0}\}$.*

The phase function is nondegenerate *if the set $\{\nabla_{(\mathbf{y}, \mathbf{x}, \xi)} \partial_{\xi_j} \Phi : j = 1, \ldots, N\}$ is linearly independent on the manifold*

$$\Sigma_\Phi = \{(\mathbf{y}, \mathbf{x}, \xi) \in Y \times X \times \mathbb{R}^N \setminus \{\mathbf{0}\} : \nabla_\xi \Phi(\mathbf{y}, \mathbf{x}, \xi) = 0\}.$$

Definition 2.3 (Fourier integral operator)**.** *Given a symbol $p \in S^m(Y \times X \times \mathbb{R}^N \setminus \{\mathbf{0}\})$ and a nondegenerate phase function $\Phi \in C^\infty(Y \times X \times \mathbb{R}^N \setminus \{\mathbf{0}\})$ we define the* Fourier integral operator *(FIO) F applied to $u \in C_0^\infty(X)$ by*

$$Fu(\mathbf{y}) = \int_{\mathbb{R}^N} \int_X p(\mathbf{y}, \mathbf{x}, \xi) u(\mathbf{x}) \mathrm{e}^{\imath \Phi(\mathbf{y}, \mathbf{x}, \xi)} \,\mathrm{d}\mathbf{x}\,\mathrm{d}\xi$$

where the integral exists as an oscillatory integral which represents a distribution in general, see [15, Chap. I]. *The operator F maps $C_0^\infty(X)$ continuously to $C^\infty(Y)$ and can be extended as a continuous map from $\mathcal{E}'(X)$ to $\mathcal{D}'(Y)$.*

To simplify notation, and because the sets we consider are all subsets of Euclidean space, we will identify cotangent bundles with subsets of Euclidean space; if Ω is an open subset of $\mathbb{R}^d$, we identify $T^*(\Omega)$ with $\Omega \times \mathbb{R}^d$.

With the FIO F we associate the set

$$\mathcal{C} = \big\{\big(\mathbf{y}, \nabla_{\mathbf{y}} \Phi(\mathbf{y}, \mathbf{x}, \xi); \mathbf{x}, -\nabla_{\mathbf{x}} \Phi(\mathbf{y}, \mathbf{x}, \xi)\big) : (\mathbf{y}, \mathbf{x}, \xi) \in \Sigma_\Phi\big\} \subset T^*(Y) \times T^*(X)$$

which is called the *canonical relation* of F.

The canonical relation encodes how the FIO propagates singularities. To describe this more precisely, we define singularities of a distribution as the elements of the distribution's wave front set.

Definition 2.4. *Let $\Omega \subseteq \mathbb{R}^d$ be open and let $u \in \mathcal{D}'(\Omega)$ be a distribution.*
a) u *is* microlocally C^∞ *at* $(\mathbf{x}_0, \xi_0) \in T^*(\Omega)$ *if for some* $\phi \in C_0^\infty(\Omega)$ *with* $\phi(\mathbf{x}_0) \neq 0$ *and some conic neighborhood* V *of* ξ_0 *in* $\mathbb{R}^d \backslash \{0\}$*, the Fourier transform* $\widehat{\phi u}$ *is rapidly decaying on* V*, that is, for every* $M \in \mathbb{N}$ *exists a constant* $C = C(M) > 0$ *such that*

$$|\widehat{\phi u}(\xi)| \leq C(1 + |\xi|)^{-M} \quad \textit{for all } \xi \in V.$$

b) *The* wave front set $\mathrm{WF}(u)$ *of* u *is given by*

$$\mathrm{WF}(u) = \{(\mathbf{x}, \xi) \in T^*(\Omega) : u \textit{ is not microlocally } C^\infty \textit{ at } (\mathbf{x}, \xi)\}.$$

For any $u \in \mathcal{E}'(X)$ we have

$$\mathrm{WF}(Fu) \subset \Pi_{\mathrm{L}} \Pi_{\mathrm{R}}^{-1} \mathrm{WF}(u) = \{(\mathbf{y}, \eta) \in T^*(Y) : \exists (\mathbf{x}, \xi) \in \mathrm{WF}(u) \colon (\mathbf{y}, \eta; \mathbf{x}, \xi) \in \mathcal{C}\} \tag{2.2}$$

which is the statement of the Hörmander-Sato lemma. Above we used the two canonical projections $\Pi_{\mathrm{L}} \colon \mathcal{C} \to T^*(Y)$ and $\Pi_{\mathrm{R}} \colon \mathcal{C} \to T^*(X)$ onto the left and right components of $\mathcal{C}$, respectively. The *Bolker condition* is satisfied if the left projection

$$\Pi_{\mathrm{L}} \colon \mathcal{C} \to T^*(Y) \backslash \{\mathbf{0}\} \ \text{ is an injective immersion.} \tag{2.3}$$

Assume that F^*F, the composition of F with its formal L^2-adjoint operator F^*, is well defined. Then, under (2.3), F^*F is a pseudodifferential operator, see [14]. Pseudodifferential operators are introduced in the next subsection: they are FIOs with favorable qualities for imaging.

2.2. **Pseudodifferential Operators.** Pseudodifferential operators are FIOs where $X = Y$, $d_X = N$, and $\Phi(\mathbf{y}, \mathbf{x}, \xi) = (\mathbf{y} - \mathbf{x}) \cdot \xi$ is the nondegenerate phase function.

In the applications we consider in the next sections, the symbols of the pseudodifferential operators depend only on the two variables $\mathbf{x}$ and ξ. All concepts and results of the previous subsection carry over. Since $X \subset \mathbb{R}^N$ we write $S^m(X)$ instead of $S^m(X \times \mathbb{R}^N)$. Hence, for $p \in S^m(X)$, the linear map $P \colon \mathcal{E}'(X) \to \mathcal{D}'(X)$,

$$Pu(\mathbf{y}) = \int_{\mathbb{R}^N} \int_X p(\mathbf{x}, \xi) u(\mathbf{x}) \mathrm{e}^{\imath(\mathbf{y}-\mathbf{x})\cdot\xi} \, \mathrm{d}\mathbf{x} \, \mathrm{d}\xi, \tag{2.4}$$

is a pseudodifferential operator (ΨDO) of order m. Here, p is called the *full symbol* of the operator P. The *principal symbol* $\sigma(P)$ *of* P is the equivalence class of p in the quotient space $S^m(X)/S^{m-1}(X)$.

Since ΨDOs are FIOs with specific phase functions, one might expect the symbol p in (2.4) to be a function of $(\mathbf{x}, \mathbf{y}, \xi)$ as in Definition 2.3. However, ΨDOs with symbol $p(\mathbf{x}, \xi)$ generate the same class of operators modulo smoothing operators[1] as those with symbol $p(\mathbf{x}, \mathbf{y}, \xi)$ [18, Theorem 4.5, p. 188].

The ΨDO P is *elliptic* (respectively: *microlocally elliptic*) if its symbol is elliptic (respectively: microlocally elliptic).

Let P be a ΨDO of order m. When we write $\sigma(P)$ as a function, we understand this as the equivalence class of the function modulo $S^{m-1}(X)$. We will introduce some more technical terminology in Section 5.

ΨDOs do not create singularities: The Hörmander-Sato inclusion (2.2) for a ΨDO P reads

$$\mathrm{WF}(Pu) \subset \mathrm{WF}(u) \quad \text{for any } u \in \mathcal{E}'(X)$$

and is known as *pseudo local property* of ΨDOs. In case P is elliptic we even have equality:

$$\mathrm{WF}(Pu) = \mathrm{WF}(u) \quad \text{for any } u \in \mathcal{E}'(X).$$

[1] Smoothing operators map $\mathcal{E}'$ into C^∞.

A finer analysis of how ΨDOs affect singularities allows a microlocalization with respect to the Sobolev space H^r, $r \in \mathbb{R}$. A distribution $u \in \mathcal{D}'(X)$ is *microlocally* H^r at $(\mathbf{x}_0, \xi_0) \in T^*(X)$ if there are a neighborhood $U \subset X$ of $\mathbf{x}_0$ and a conic neighborhood $V \subset \mathbb{R}^d \setminus \{\mathbf{0}\}$ of ξ_0 such that

$$\int_V |\widehat{\phi u}(\xi)|^2 (1 + |\xi|^2)^r \, \mathrm{d}\xi < \infty \qquad \text{for all } \phi \in C_0^\infty(U).$$

Now, we define the H^r*-wave front set of* u by

$$\mathrm{WF}^r(u) = \{(\mathbf{x}, \xi) \in T^*(X) : u \text{ is not microlocally } H^r \text{ at } (\mathbf{x}, \xi)\},$$

see [18]. Note that H^r-wave front sets are indeed a refinement of wave front sets: $\mathrm{WF}(u) = \mathrm{cl}\big(\cup_{r \in \mathbb{R}} \mathrm{WF}^r(u)\big)$.

Theorem 2.5. *Let P be a* ΨDO *of order m. If P is microlocally elliptic at $(\mathbf{x}_0, \xi_0) \in T^*(X)$, we have*

$$(\mathbf{x}_0, \xi_0) \in \mathrm{WF}^r(u) \textit{ if and only if } (\mathbf{x}_0, \xi_0) \in \mathrm{WF}^{r-m}(Pu)$$

for $u \in \mathcal{E}'(\Omega)$ and $r \in \mathbb{R}$.

The proof of the theorem above is given by the same argument as in [3, Proposition A.6] which is based on [15, Theorem 4.3.2].

3. Linear velocity model

In this section we restrict ourselves to two spatial dimensions, i.e., $d = 2$, and consider the background wave speed

$$v(\mathbf{x}) = b + a x_2, \; x_2 > 0, \tag{3.1}$$

where a and b are positive constants (the positive direction of the x_2-axis points downwards to the interior of the earth). Finally, we position sources and receivers according to the common offset data acquisition geometry on the line $x_2 = 0$ with common offset $\alpha \geq 0$. Thus, source and receiver positions are determined by a real parameter s via

$$\mathbf{x_s}(s) = (s - \alpha, 0)^\top, \qquad \mathbf{x_r}(s) = (s + \alpha, 0)^\top.$$

Under those assumptions F from (1.3) can be represented as the FIO

$$Fw(s,t) = \int_{\mathbb{R}} \int_X \frac{1}{2\pi} \, \Theta(s, \mathbf{x}) w(\mathbf{x}) \mathrm{e}^{\imath \omega (t - \varphi(s,\mathbf{x}))} \mathrm{d}\mathbf{x} \mathrm{d}\omega, \tag{3.2}$$

compare (1.6). For defining the preimage and image spaces of F we set

$$X = \{\mathbf{x} \in \mathbb{R}^2 : x_2 > x_{\min}\} \quad \text{and} \quad Y = S \times \,]t_{\min}, \infty[\tag{3.3}$$

where

$$x_{\min} := \frac{b}{a} \left(\sqrt{1 + \frac{a^2 \alpha^2}{b^2}} - 1 \right), \quad t_{\min} := \frac{2}{a} \operatorname{asinh}\left(\frac{a\,\alpha}{b}\right), \tag{3.4}$$

and $S \subset \mathbb{R}$ being the bounded open set which contains the parameters of the source/receiver pairs used for data recording. Note that $x_{\min}$ and $t_{\min}$ are both zero in the zero-offset case, $\alpha = 0$. The lower bounds $x_{\min}$ and $t_{\min}$ in the definitions of X and Y, respectively, are needed to ensure the Bolker condition (2.3) for $F\colon \mathcal{E}'(X) \to \mathcal{D}'(Y)$. If X contains points with $x_2 < x_{\min}$ the Bolker condition is violated, as we will show.

Further,

$$\Theta(s, \mathbf{x}) := A(\mathbf{x}, \mathbf{x_s}(s)) A(\mathbf{x}, \mathbf{x_r}(s)) / v^2(\mathbf{x}) \tag{3.5}$$

is a symbol in $S^0(Y \times X \times \mathbb{R})$. An explicit representation of A is given in Appendix A, see (A.2) and (A.3). Moreover, the travel time from point $\mathbf{x}$ to source and receiver is also explicitly known to be

$$\begin{aligned}\varphi(s,\mathbf{x}) &:= \tau(\mathbf{x},\mathbf{x_s}(s)) + \tau(\mathbf{x},\mathbf{x_r}(s)) \\ &= \frac{1}{a}\operatorname{acosh}\left(1 + \frac{a^2}{2b}\frac{(x_1+\alpha-s)^2+x_2^2}{b+ax_2}\right) \\ &\qquad\qquad + \frac{1}{a}\operatorname{acosh}\left(1 + \frac{a^2}{2b}\frac{(x_1-\alpha-s)^2+x_2^2}{b+ax_2}\right),\end{aligned}$$

see [21, Lesson 41].

In the notation of Section 2.1 we have $N = 1$, $d_X = d_Y = 2$, and the nondegenerate phase function $\Phi(\mathbf{y},\mathbf{x},\xi) = \omega(t-\varphi(s,\mathbf{x}))$ where $\mathbf{y} = (s,t)$ and $\xi = \omega$. Note that $\nabla_{\mathbf{x}}\varphi$ is never zero for $\alpha \geq 0$ and $x_2 > x_{\min}$. This is easy to see for $\alpha = 0$ and we refer to Remark 3.2 below for $\alpha > 0$. Hence, the canonical relation of F is

$$\mathcal{C} = \left\{(s,\varphi(s,\mathbf{x}), -\omega\partial_s\varphi, \omega \mathrm{d}t; \mathbf{x}, \omega\nabla_{\mathbf{x}}\varphi) : s\in\mathbb{R},\ \mathbf{x}\in X,\ \omega\neq 0\right\} \subset T^*(Y)\times T^*(X) \tag{3.6}$$

and note that

$$\mathbb{R}\times X\times\mathbb{R}\backslash\{0\} \ni (s,\omega,\mathbf{x}) \mapsto (s,\varphi(s,\mathbf{x}), -\omega\partial_s\varphi, \omega\mathrm{d}t; \mathbf{x}, \omega\nabla_{\mathbf{x}}\varphi) \tag{3.7}$$

define smooth global coordinates on $\mathcal{C}$.

To prove the necessary injectivity we need to recover $(\mathbf{x}, \omega\nabla_{\mathbf{x}}\varphi) \in T^*(X)$ uniquely from any given $(s,\varphi(s,\mathbf{x}), -\omega\partial_s\varphi, \omega\mathrm{d}t) \in T^*(Y)$. Since s and ω are immediately known from the projection, the goal is to find $\mathbf{x}\in X$ using that s, $t = \varphi(s,\mathbf{x})$, and $\partial_s\varphi(s,\mathbf{x})$ are known.

In the following two subsections we will investigate the Bolker condition, first we will verify it for $\alpha = 0$ and then provide overwhelming evidence for it to hold even for $\alpha > 0$.

3.1. **Bolker condition for the zero offset case.**

First we explore the zero offset situation ($\alpha = 0$ yielding $x_{\min} = t_{\min} = 0$) where source and receiver locations coincide: $\mathbf{x_s}(s) = \mathbf{x_r}(s) = (s,0)^\top$. We then have

$$\varphi(s,\mathbf{x}) = \frac{2}{a}\operatorname{acosh}\left(1+\frac{a^2}{2b}\frac{(x_1-s)^2+x_2^2}{b+ax_2}\right) \tag{3.8}$$

with partial derivative

$$\partial_s\varphi = -\frac{2a}{b(b+ax_2)}\frac{x_1-s}{\sqrt{H}\sqrt{H+2}} \tag{3.9}$$

where

$$H = \frac{a^2}{2b}\frac{(x_1-s)^2+x_2^2}{b+ax_2}. \tag{3.10}$$

As $(s,\omega,\mathbf{x})$ parametrize $\mathcal{C}$, see (3.7), we obtain

$$\Pi_{\mathrm{L}}(s,\omega,\mathbf{x}) = \left(s,\ \frac{2}{a}\operatorname{acosh}(1+H),\ \frac{\omega\,2a}{b(b+ax_2)}\frac{x_1-s}{\sqrt{H}\sqrt{H+2}},\ \omega\right).$$

To show injectivity let $t > 0$ be given. We introduce new (polar-)coordinates

$$x_1 = s + r\cos\vartheta, \quad x_2 = c + r\sin\vartheta,$$

with $\vartheta \in [-\pi/2, 3\pi/2[$,

$$c = \frac{b}{a}\left(\cosh\frac{at}{2} - 1\right) > 0, \quad \text{and} \quad r = \sqrt{c^2 + \frac{2b}{a}c} = \frac{b}{a}\sinh\frac{at}{2}. \tag{3.11}$$

Observe that $x_2 > 0$ iff $\vartheta \in \mathcal{I}(c) :=]-\delta(c), \pi + \delta(c)[$ with $\delta(c) = \arcsin(c/r)$. In the new coordinates we have

$$\mathcal{L}_{s,t} = \{\mathbf{x}(\vartheta) : \vartheta \in \mathcal{I}(c)\}$$

as the expression H is independent of ϑ:

$$H = \frac{ac}{b} = \cosh\frac{at}{2} - 1 \tag{3.12}$$

yielding

$$\sqrt{H}\sqrt{H+2} = \sinh\frac{at}{2} = \frac{a}{b}\,r. \tag{3.13}$$

It remains to determine $\vartheta \in \mathcal{I}(c)$ smoothly from knowing

$$d = \frac{2a}{b(b + ax_2)}\,\frac{x_1 - s}{\sqrt{H}\sqrt{H+2}} = \frac{2\cos\vartheta}{b + ac + ar\sin\vartheta} =: f(\vartheta).$$

We show that f is strictly decreasing in $\mathcal{I}(c)$ by studying its derivative

$$f'(\vartheta) = \frac{-2\big((ac+b)\sin\vartheta + ar\big)}{(b + ac + ar\sin\vartheta)^2}.$$

Since $\sin\vartheta > -c/r$ for $\vartheta \in \mathcal{I}(c)$ we obtain

$$-(ac+b)\sin\vartheta - ar < (ac+b)\frac{c}{r} - ar < 0.$$

Hence, f' is negative and f is strictly decreasing. Therefore Π_{L} is injective.

Remark 3.1. *The level sets $\mathcal{L}_{s,t}$ are circles with centers and radii depending on s, t, and a, b:*

$$\left(s,\ \frac{b}{a}\left(\cosh\frac{at}{2} - 1\right)\right), \quad \frac{b}{a}\sinh\frac{at}{2}.$$

In the limit $t \to \infty$ the "north pole" of the circle converges to $(s, -b/a)$.

To show that Π_{L} is an immersion we compute the determinant of the Jacobian $D\Pi_{\mathrm{L}}$. We rearrange the components of Π_{L} and use the identity $\partial_s\varphi = -\partial_{x_1}\varphi$:

$$\Pi_{\mathrm{L}}(s, \omega, \mathbf{x}) = \big(s,\ \omega,\ \varphi,\ \omega\partial_{x_1}\varphi\big).$$

Thus,

$$D\Pi_{\mathrm{L}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \partial_{x_1}\varphi \\ 0 & 0 & \partial_{x_1}\varphi & \partial_{x_2}\varphi \\ 0 & 0 & \omega\partial_{x_1}^2\varphi & \omega\partial_{x_2}\partial_{x_1}\varphi \end{pmatrix}$$

and

$$\det D\Pi_{\mathrm{L}} = \omega\big(\partial_{x_1}\varphi\,\partial_{x_2}\partial_{x_1}\varphi - \partial_{x_2}\varphi\,\partial_{x_1}^2\varphi\big). \tag{3.14}$$

Using the *Symbolic Math Toolbox* of MATLAB (R2017b) we find that

$$\det D\Pi_{\mathrm{L}} = -8\omega\,\frac{a(x_1 - s)^2 + x_2(ax_2 + 2b)}{(b + ax_2)\big((x_1 - s)^2 + x_2^2\big)\big(a^2(x_1 - s)^2 + (2b + ax_2)^2\big)}.$$

The determinant does not vanish since $\omega \neq 0$ and x_2, a, and b are positive, that is, Π_{L} is an injective immersion. Thus, the Bolker condition (2.3) holds.

3.2. Bolker condition for the positive offset case.

Let $\alpha > 0$ be the offset. In a first step towards the proof of Bolker we derive a parametrization of the isochrone

$$\mathcal{L}_{s,t} = \Big\{\mathbf{x} \in \mathbb{R}^2 : \varphi(s,\mathbf{x}) = t\Big\} \cap X.$$

W.l.o.g. set $s = 0$. The idea to obtain a parametrization of $\mathcal{L}_{0,t}$ is to intersect two isochrones of the previous setting where source and receiver are located at the same position (zero offset). To be precise: we intersect the zero offset isochrone about $(-\alpha, 0)$ for travel time $\vartheta \in [0,t]$ with the zero offset isochrone about $(\alpha, 0)$ for travel time $t - \vartheta$. All intersection points are in $\mathcal{L}_{0,t}$ and by letting ϑ vary in $[0,t]$ we get finally all of $\mathcal{L}_{0,t}$.

The resulting system of nonlinear equations is

$$(x_1 + \alpha)^2 + (x_2 - c_1)^2 = r_1^2 = c_1^2 + 2bc_1/a, \tag{3.15}$$

$$(x_1 - \alpha)^2 + (x_2 - c_2)^2 = r_2^2 = c_2^2 + 2bc_2/a, \tag{3.16}$$

where

$$c_1 = c_1(\vartheta) = b(\cosh a\vartheta - 1)/a \quad \text{and} \quad c_2 = c_2(\vartheta) = b(\cosh a(t-\vartheta) - 1)/a,$$

compare (3.11). Subtracting (3.16) from (3.15) and assuming $\vartheta \neq t/2$, i.e. $c_1 \neq c_2$, lead to

$$x_2 = -b/a - 2\alpha\, x_1/(c_2 - c_1). \tag{3.17}$$

This expression for x_2 plugged into (3.16) yields the quadratic equation

$$\Big(1 + \frac{4\alpha^2}{(c_1 - c_2)^2}\Big)x_1^2 + 2\alpha\Big(\frac{2\alpha(b/a + c_2)}{c_2 - c_1} - 1\Big)x_1 + \frac{b^2}{a^2} + \alpha^2 = 0$$

having the two solutions

$$x_1^{\pm} = x_1^{\pm}(\vartheta) = (c_1 - c_2)\,\mathrm{dx}^{\pm}$$

with

$$\mathrm{dx}^{\pm} := \frac{\alpha(2b + a(c_1 + c_2)) \pm \sqrt{\triangle}}{a\big((c_1 - c_2)^2 + 4\alpha^2\big)}$$

where

$$\triangle = 4a^2\alpha^2(c_1c_2 - \alpha^2) + 4ab\alpha^2(c_1 + c_2) - b^2(c_1 - c_2)^2.$$

If $\triangle < 0$ then there exist no intersection points. We have that $\triangle \geq 0$ if and only if $t \geq t_{\min}$ and $\vartheta \in [\vartheta_{\min}, \vartheta_{\max}]$ where

$$\vartheta_{\min\,/\,\max} = \frac{t}{2} \mp \frac{t_{\min}}{2},$$

see first paragraph of Appendix B. In view of (3.17) we finally obtain

$$x_2^{\pm} = x_2^{\pm}(\vartheta) = -\frac{b}{a} + 2\alpha\,\mathrm{dx}^{\pm}$$

which also holds in case $c_1 = c_2$.

By construction the pairs $(x_1^{\pm}, x_2^{\pm})$ solve (3.16) but (3.15) as well because we obtain the same pairs when we replace c_2 by c_1 and α by $-\alpha$.

Remark 3.2. *The isochrone $\mathcal{L}_{0,t}$, $t > t_{\min}$, is the set of intersection points of two circles. These circles intersect at two points ($\Delta > 0$) with normal directions that are not collinear. Hence, $\nabla_{\mathbf{x}}\varphi \neq 0$. For $t = t_{\min}$ the circles touch at one point ($\Delta = 0$). Thus, $\nabla_{\mathbf{x}}\varphi = 0$.*

In Remark 4.5 below the situation of a more general wave speed is discussed.

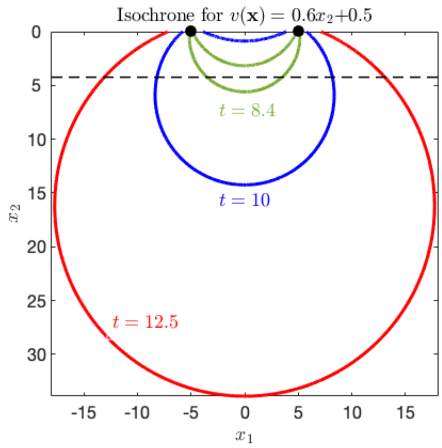

FIGURE 1. Isochrones $\mathcal{L}_{0,t}$ for $t \in \{8.4, 10, 12.5\}$ for wave speed $v(\mathbf{x}) = 0.5 + 0.6\,x_2$, $x_2 > 0$. Source and receiver positions are indicated by black dots. The offset is $\alpha = 5$. Here, $t_{\min} \approx 8.31$, $\underline{t} \approx 12.12$ and $x_{\min} \approx 4.24$ which is indicated by the dashed horizontal line.

From symmetry arguments we know two points explicitly on $\mathcal{L}_{0,t}$, namely $(0, p_\pm)$ where

$$p_\pm = \frac{b}{a}\Big(\cosh\frac{at}{2} - 1\Big) \pm \sqrt{\frac{b^2}{a^2}\sinh^2\frac{at}{2} - \alpha^2} \quad \text{for} \quad t > t_{\min}.$$

These points correspond to the parameter value $\vartheta = t/2$. In case

$$t < \underline{t} = \frac{4}{a}\,\text{asinh}\Big(\frac{a}{b}\frac{\alpha}{2}\Big)$$

these two points are positive and because of the symmetry of $\mathcal{L}_{0,t}$ with respect to the line $x_1 = 0$ the curve $\mathcal{L}_{0,t}$ has a horizontal tangent at $(0, p_\pm)$. As a consequence the equation $\omega\partial_s\varphi(0,\cdot) = 0$, $\omega \neq 0$, has the two solutions $(0, p_\pm)$ and Π_{L} would fail to be injective if X contained points with depth-coordinates x_2 less than

$$x_{\min} = \frac{b}{a}\Big(\cosh\frac{a\,t_{\min}}{2} - 1\Big). \tag{3.18}$$

Observe that $x_{\min}$ is the common limit of p_+ and p_- as $t \searrow t_{\min}$.

Further, for $t \geq t_{\min}$ and $\vartheta \in [\vartheta_{\min}, \vartheta_{\max}]$,

$$2\alpha\,\frac{a}{b}\,\mathrm{dx}^- = \frac{2\alpha\big(\alpha(2b + a(c_1 + c_2)) - \sqrt{\triangle}\big)}{b\big((c_1 - c_2)^2 + 4\alpha^2\big)} < \cosh\frac{a\,t_{\min}}{2}, \tag{3.19}$$

see Appendix B. Thus,

$$x_2^- = -\frac{b}{a} + 2\alpha\,\mathrm{dx}^- = -\frac{b}{a} + \frac{b}{a}\frac{2\alpha\big(\alpha(2b + a(c_1 + c_2)) - \sqrt{\triangle}\big)}{b\big((c_1 - c_2)^2 + 4\alpha^2\big)} < x_{\min} \quad \text{for } t > t_{\min}. \tag{3.20}$$

We conclude that

$$\mathcal{L}_{0,t} = \big\{\mathbf{x}_t^+(\vartheta) := (x_1^+(\vartheta), x_2^+(\vartheta)) : \vartheta \in [\vartheta_{\min}, \vartheta_{\max}]\big\} \cap X \quad \text{for } t > t_{\min},[2]$$

see Figure 1 for an illustration[3]. Recalling the geometric definition of $\mathbf{x}_t^+(\vartheta)$ as intersections of circles, see (3.15) and (3.16), it is obvious that there is a proper subinterval $[\vartheta_{\min,2}, \vartheta_{\max,2}]$ of

[2]We use the subscript t in $\mathbf{x}_t^+$ to emphasize the dependence on t.

[3]Under http://www.math.kit.edu/ianm3/~rieder/media/plot_isochrones.m we provide a MATLAB-function to plot isochrones for different a, b, α and t.

$[\vartheta_{\min}, \vartheta_{\max}]$ such that

$$\mathcal{L}_{0,t} = \left\{\mathbf{x}_t^+(\vartheta) : \vartheta \in [\vartheta_{\min,2}, \vartheta_{\max,2}]\right\} \quad \text{for } t > t_{\min}. \tag{3.21}$$

Analytic expressions for $\vartheta_{\min,2} = \vartheta_{\min,2}(t)$ and $\vartheta_{\max,2} = \vartheta_{\max,2}(t)$ are hard, if not even impossible, to find. One has to solve $x_2^+(\cdot) = x_{\min}$ with $x_{\min}$ from (3.4).

To finish the proof for the injectivity of Π_{L}, to determine the value of ϑ for each preimage (which will determine the one preimage with $\mathbf{x}$ in $\mathcal{L}_{0,t}$), we have to solve the following task: Given $d \in \{\partial_s\varphi(0, \mathbf{x}_t^+(\theta)) : \theta \in [\vartheta_{\min,2}, \vartheta_{\max,2}]\}$ determine smoothly a unique ϑ such that $d = \partial_s\varphi(0, \mathbf{x}_t^+(\vartheta))$.

In view of (3.10), (3.12), and (3.13) we obtain

$$\partial_s\varphi(0, \mathbf{x}_t^+(\vartheta)) = \frac{2}{b + a\, x_2^+(\vartheta)} \left(\frac{x_1^+(\vartheta) + \alpha}{r_1} + \frac{x_1^+(\vartheta) - \alpha}{r_2}\right). \tag{3.22}$$

This is an odd function in ϑ with respect to $t/2$. Numerous numerical experiments confirm $\partial_s\varphi(0, \mathbf{x}_t^+(\cdot))$ to be strictly increasing in $[\vartheta_{\min,2}, \vartheta_{\max,2}]$, however, an analytic proof is still missing. But consult Appendix C for analytic arguments in case t is sufficiently large. Further, see Figure 2 for some plots of $\partial_s\varphi(0, \mathbf{x}_t^+(\cdot))$. The numerical values used for a, b, α, and t are noted on top of the plots.[4]

Conclusion 3.3. *We have overwhelming numerical and some analytical evidence that the left projection* Π_{L} *for* $\alpha > 0$ *is injective.*

To show that Π_{L} is an immersion we recall the zero offset situation. The representation (3.14) for $\det D\Pi_{\mathrm{L}}$ holds true also for non-zero offset. Since the explicit expression of $\det D\Pi_{\mathrm{L}}$ computed by the *Symbolic Math Toolbox* of MATLAB is complicated and involved, we take a different route.

Define the mapping

$$P\colon X \to \mathbb{R}^2, \quad \mathbf{x} \mapsto \big(\varphi(0, \mathbf{x}), \partial_{x_1}\varphi(0, \mathbf{x})\big)^\top,$$

and observe that $\det D\Pi_{\mathrm{L}} = \omega \det DP$. Further, we have previously shown that X is the disjoint union of level sets: $X = \bigcup_{t > t_{\min}} \mathcal{L}_{0,t} \cap X$, see (3.21). In other words, the mapping

$$\mathbf{X}\colon \left\{(t, \vartheta) : t > t_{\min}, \vartheta \in [\vartheta_{\min,2}(t), \vartheta_{\max,2}(t)]\right\} \to X, \quad (t, \vartheta) \mapsto \mathbf{x}_t^+(\vartheta),$$

is one-to-one and onto. For $Q := P \circ \mathbf{X}$ we find that

$$Q(t, \vartheta) = \big(t, \partial_{x_1}\varphi(0, \mathbf{x}_t^+(\vartheta))\big)^\top = \big(t, -\partial_s\varphi(0, \mathbf{x}_t^+(\vartheta))\big)^\top.$$

Hence,

$$DQ(t, \vartheta) = \begin{pmatrix} 1 & 0 \\ -\partial_t\partial_s\varphi(0, \mathbf{x}_t^+(\vartheta)) & -\partial_\vartheta\partial_s\varphi(0, \mathbf{x}_t^+(\vartheta)) \end{pmatrix}$$

yielding

$$\det DQ(t, \vartheta) = -\partial_\vartheta\partial_s\varphi(0, \mathbf{x}_t^+(\vartheta)).$$

Above we gave numerical evidence that $\partial_s\varphi(0, \mathbf{x}_t^+(\cdot))$ is strictly increasing on the interval $[\vartheta_{\min,2}(t), \vartheta_{\max,2}(t)]$ for all $t \geq t_{\min}$, see Figure 2 right column. Thus,

$$\det DQ(t, \vartheta) < 0.$$

[4]For the reader's own experiments, the MATLAB-function used to plot the graphs of $\partial_s\varphi(0, \mathbf{x}_t^+(\cdot))$ shown in Figure 2 can be downloaded following this link:
`http://www.math.kit.edu/ianm3/~rieder/media/plot_partial_s_phi3.m`.

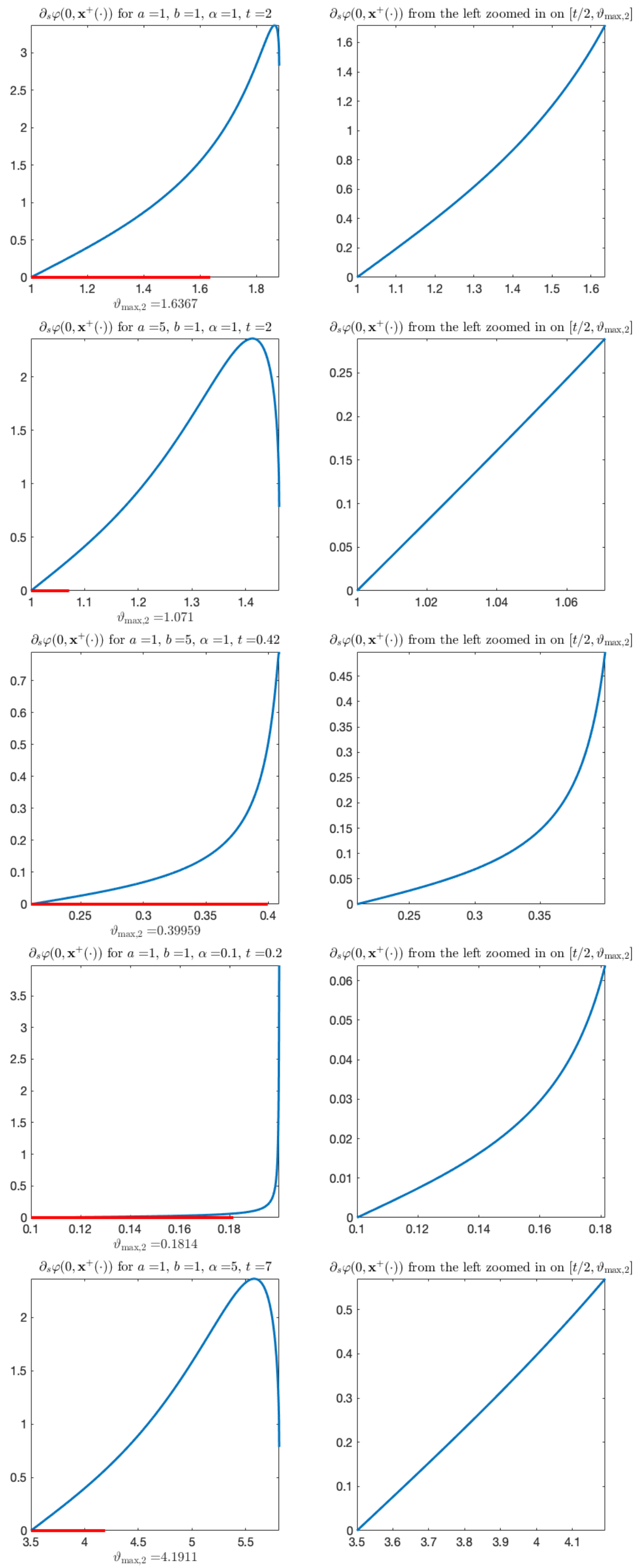

FIGURE 2. The function $\partial_s\varphi(0, \mathbf{x}_t^+(\cdot))$ in different scenarios. Left: over $[t/2, \vartheta_{\max}]$, the red line indicates $[t/2, \vartheta_{\max,2}]$. Right: over $[t/2, \vartheta_{\max,2}]$ where it is always strictly increasing. The value for $\vartheta_{\max,2}$ has been determined by solving equation (C.2) numerically.

Since,

$$0 > \det DQ(t,\vartheta) = (\det DP)(\mathbf{X}(t,\vartheta))\ \det D\mathbf{X}(t,\vartheta)$$

we must have $(\det DP)(\mathbf{X}(t,\vartheta)) \neq 0$ for all $t > t_{\min}$ and all $\vartheta \in [\vartheta_{\min,2}(t), \vartheta_{\max,2}(t)]$. Thus, $\det DP$ and, hence, $\det D\Pi_{\mathrm{L}}$ cannot vanish on X.

Conclusion 3.4. *We have overwhelming numerical and some analytical evidence that the Bolker condition* (2.3) *is satisfied for the FIO* $F\colon \mathcal{E}'(X) \to \mathcal{D}'(Y)$ *as defined by* (3.2) *and* (3.3) *for* $\alpha > 0$.

3.3. An analysis of the top order symbol for the zero offset case.

In this section, we calculate the top order symbol of our imaging operator

$$\Lambda := \Delta F^{\dagger}\psi F \tag{3.23}$$

for offset $\alpha = 0$ where $\psi\colon Y \to [0,\infty)$ is a smooth compactly supported cutoff function and Δ is the (negative) Laplacian with symbol $|\cdot|^2$. Further, $F^{\dagger}$ is a *generalized backprojection* operator:

$$F^{\dagger}u(\mathbf{x}) = \iint\limits_{Y} W(s,\mathbf{x})u(s,t)\delta(t-\varphi(s,\mathbf{x}))\mathrm{d}t\,\mathrm{d}s = \int_{\mathbb{R}} W(s,\mathbf{x})u(s,\varphi(s,\mathbf{x}))\mathrm{d}s \tag{3.24}$$

with a smooth positive weight W. The formal L^2-adjoint F^* has weight $W = \Theta$ and the generalized backprojection used by Beylkin [1] has weight $W = 1/\Theta$. In view of (3.5), (A.2), and (A.3), Θ is a smooth positive function. We include the smooth cutoff function $\psi\colon Y \to [0,\infty)$ because $F\colon \mathcal{E}'(X) \to \mathcal{D}'(Y)$ but $F^{\dagger}\colon \mathcal{E}'(Y) \to \mathcal{D}'(X)$, so they cannot be composed directly.

To calculate this symbol, we first analyze the preimages of $\Pi_{\mathrm{R}}\colon \mathcal{C} \to T^*(X)$. This will allow us to calculate the symbol of the imaging operator at $(\mathbf{x},\xi) \in T^*(X)\backslash\{\mathbf{0}\}$ by multiplying the symbols of ψF and of $\Delta F^{\dagger}$ at each preimage and then adding the results. The natural projection is

$$\Pi_{\mathrm{R}}\colon \mathcal{C} \to T^*(X)\backslash\{\mathbf{0}\}, \qquad (s,\omega,\mathbf{x}) \mapsto (\mathbf{x}, \omega\nabla_{\mathbf{x}}\varphi) \tag{3.25}$$

where we are using coordinates of (3.6) on $\mathcal{C}$.

We show that $(\mathbf{x},\xi) \in T^*(X)\backslash\{\mathbf{0}\}$, has exactly two preimages in $\mathcal{C}$ under Π_{R} unless $\xi_1 = 0$. To this end we need to find (s,ω) from $\omega\nabla_{\mathbf{x}}\varphi = \xi$, i.e., from

$$\omega\partial_{x_1}\varphi = \xi_1 \quad \text{and} \quad \omega\partial_{x_2}\varphi = \xi_2. \tag{3.26}$$

First, assume $\xi_1 = 0$. Using (3.9) and that $\partial_s\varphi = -\partial_{x_1}\varphi$, one sees there is only one solution, $s = x_1$. Using (3.26) one sees that $\omega = \xi_2/\partial_{x_2}\varphi(x_1,\mathbf{x})$ (note that $\partial_{x_2}\varphi(x_1,\mathbf{x}) \neq 0$, since $\xi_2 \neq 0$ and $\omega \neq 0$). Therefore, there is only one preimage in this case: $\Pi_{\mathrm{R}}^{-1}(\mathbf{x},\xi) = \{(x_1, \xi_2/\partial_{x_2}\varphi(x_1,\mathbf{x}), \mathbf{x})\}$

Now, assume $\xi_1 \neq 0$ and let $q := \xi_2/\xi_1$. Using (3.26) yields that

$$\frac{\partial_{x_2}\varphi}{\partial_{x_1}\varphi} = \frac{\xi_2}{\xi_1} = q.$$

Since $\partial_{x_1}\varphi = -\partial_s\varphi$, see (3.9), and

$$\partial_{x_2}\varphi = \frac{2}{b(b+ax_2)}\,\frac{ax_2/b - H}{\sqrt{H}\sqrt{H+2}}$$

where H is given by (3.12), we obtain

$$q = \frac{x_2(2b + ax_2) - a\ell^2}{2\ell(b + ax_2)} \quad \text{with } \ell := x_1 - s.$$

Completing the square we find the two solutions s_+ and s_- for ℓ where

$$s_\pm = s_\pm(\mathbf{x}, q) := x_1 - \ell_{1,2} = x_1 + \frac{q(b + ax_2)}{a} \pm \sqrt{\frac{x_2}{a}(2b + ax_2) + \left(\frac{q(b + ax_2)}{a}\right)^2}. \tag{3.27}$$

Finally, using the coordinates (3.7) on $\mathcal{C}$, we have the preimage of $(\mathbf{x}, \xi)$:

$$\Pi_{\mathrm{R}}^{-1}(\mathbf{x}, \xi) = \begin{cases} \big\{\big(x_1, \xi_2/\partial_{x_2}\varphi(x_1, \mathbf{x}), \mathbf{x}\big)\big\} & : \ \xi_1 = 0,\ \xi_2 \neq 0, \\ \big\{\big(s, \xi_1/\partial_{x_1}\varphi(s, \mathbf{x}), \mathbf{x}\big) : \ s \in \{s_+, s_-\}\big\} & : \ \xi_1 \neq 0. \end{cases} \tag{3.28}$$

Since φ from (3.8) satisfies the Bolker condition, Λ is a ΨDO of order 1 ([12, Theorem 3.3]). Further, our representation of the top order symbol $\sigma(\Lambda)$ given in [12, Theorem 3.7] for a constant v is valid also for any Radon transform (3.2) defined by a function φ for which the Bolker condition holds. Thus,

$$\sigma(\Lambda)(\mathbf{x}, \xi) = 2\pi\, |\xi|^2 \sum_{(s,\omega,\mathbf{x}) \in \Pi_{\mathrm{R}}^{-1}(\mathbf{x},\xi)} \frac{\psi(s, \varphi(s, \mathbf{x}))W(s, \mathbf{x})\Theta(s, \mathbf{x})}{|\omega B(s, \mathbf{x})|} \tag{3.29}$$

where

$$B(s, \mathbf{x}) = \det \begin{pmatrix} \nabla_{\mathbf{x}}\varphi(s, \mathbf{x}) \\ \partial_s \nabla_{\mathbf{x}}\varphi(s, \mathbf{x}) \end{pmatrix}$$

is the Beylkin determinant (which does not vanish). This calculation is done in generality in [19, pp. 337-338], and one argues microlocally around each preimage then takes the sum over the finite number of preimages of $(\mathbf{x}, \xi)$ under Π_{R}.

To analyze the symbol of the imaging operator near $\xi_1 = 0$, we note the following limits, which follow from (3.27):

$$\begin{aligned} s_+(\mathbf{x}, q) &\xrightarrow{q \to \infty} \infty, & s_-(\mathbf{x}, q) &\xrightarrow{q \to \infty} x_1, \\ s_+(\mathbf{x}, q) &\xrightarrow{q \to -\infty} x_1, & s_-(\mathbf{x}, q) &\xrightarrow{q \to -\infty} -\infty. \end{aligned} \tag{3.30}$$

We emphasize that the sum on the right of (3.29) is smooth even at $(\mathbf{x}, (0, \xi_2))$ because, by (3.30), one of the two values $s_\pm(\mathbf{x}, \xi_2/\xi_1)$ for $(\mathbf{x}, \xi)$, $\xi_1 \neq 0$, grows without bound as $\xi_1 \to 0$ (the other value converges to x_1 by (3.30)). Hence, the cutoff function ψ becomes zero as the one value of $s_\pm$ becomes unbounded. Put differently, the sum in (3.29) transitions continuously from two terms to one term as $\xi_1 \to 0$ because one of the values of s in the sum becomes unbounded and the other converges to the preimage for $\xi_1 = 0$.

Next we explore properties of Λ inspecting its top order symbol: In view of Theorem 2.5 we want to know where is it microlocally elliptic? Further, how does it behave asymptotically as $x_2 \to \infty$?

To this end we first consider $1/|\omega B(s, \mathbf{x})|$. Using $\xi^\top = \omega \nabla_{\mathbf{x}}\varphi(s, \mathbf{x})$ for $(s, \omega, \mathbf{x}) \in \Pi_{\mathrm{R}}^{-1}(\mathbf{x}, \xi)$ and $\partial_s \varphi = -\partial_{x_1}\varphi$ we find that

$$\begin{aligned} |\omega B(s, \mathbf{x})| &= \left| \det \begin{pmatrix} \omega\nabla_{\mathbf{x}}\varphi(s, \mathbf{x}) \\ \partial_s \nabla_{\mathbf{x}}\varphi(s, \mathbf{x}) \end{pmatrix} \right| = \left| \det \begin{pmatrix} \xi^\top \\ \partial_{x_1} \nabla_{\mathbf{x}}\varphi(s, \mathbf{x}) \end{pmatrix} \right| \\ &= \left| \xi_1\, \partial^2_{x_1,x_2}\varphi(s, \mathbf{x}) - \xi_2\, \partial^2_{x_1}\varphi(s, \mathbf{x}) \right|. \end{aligned}$$

Further,

$$\partial^2_{x_1,x_2}\varphi(s,\mathbf{x}) = \frac{a^3}{b^3}\,\frac{(s-x_1)\big(a(s-x_1)^2 + x_2(ax_2+b)\big)}{(ax_2+b)^2(H+2)^{3/2}H^{3/2}}$$

and

$$\partial^2_{x_1}\varphi(s,\mathbf{x}) = \frac{1}{2}\,\frac{a^3}{b^3}\,\frac{x_2^2(ax_2+2b)^2 - a^2(s-x_1)^4}{(ax_2+b)^3(H+2)^{3/2}H^{3/2}}$$

with H from (3.10).

In case $\xi_1 = 0$ and $\xi_2 \neq 0$ we have $s = x_1$ leading to

$$\frac{1}{|\omega B(s,\mathbf{x})|} = \frac{1}{|\xi_2|\,|\partial^2_{x_1}\varphi(x_1,\mathbf{x})|}.$$

The situation is a bit more involved in the general situation of $\xi_1 \neq 0$. Setting

$$S_\pm := s_\pm(\mathbf{x},q) - x_1 = \frac{q(b+ax_2)}{a} \pm \sqrt{\frac{x_2}{a}(2b+ax_2) + \left(\frac{q(b+ax_2)}{a}\right)^2} \tag{3.31}$$

we have

$$\frac{1}{|\omega B(s_\pm,\mathbf{x})|} = N_\pm(\mathbf{x},\xi)$$

where

$$N_\pm(\mathbf{x},\xi) := \frac{b^3}{a^3}\,\frac{(H+2)^{3/2}H^{3/2}(ax_2+b)^3}{\left|\frac{1}{2}\left(x_2^2(ax_2+2b)^2 - a^2S_\pm^4\right)\xi_2 - (ax_2+b)S_\pm\big(aS_\pm^2 + x_2(ax_2+b)\big)\xi_1\right|}$$

using the abbreviation H from (3.10).

The following result characterizes visible and invisible singularities with respect to Λ.

Proposition 3.5. *Let* $(\mathbf{y},\eta) \in T^*(X)$ *and define*

$$C(\mathbf{y}) := C_+(\mathbf{y}) \cup C_-(\mathbf{y}) \cup \big\{\xi \in \mathbb{R}^2 : \xi_1 = 0,\ \psi\big(y_1,\varphi(y_1,\mathbf{y})\big) > 0\big\}$$

where

$$C_\pm(\mathbf{y}) = \big\{\xi \in \mathbb{R}^2 : \xi_1 \neq 0,\ \psi\big(s_\pm(\mathbf{y},\xi_2/\xi_1),\varphi(s_\pm(\mathbf{y},\xi_2/\xi_1)),\mathbf{y}\big) > 0\big\}.$$

a) (visible singularity) *If* $\eta \in C(\mathbf{y})$ *then* Λ *is microlocally elliptic of order* 1 *at* $(\mathbf{y},\eta)$ *which yields*

$$(\mathbf{y},\eta) \in \mathrm{WF}^r(u) \quad\Longleftrightarrow\quad (\mathbf{y},\eta) \in \mathrm{WF}^{r-1}(\Lambda u)$$

for any $u \in \mathcal{E}'(X)$ *and any* $r \in \mathbb{R}$.

b) (invisible singularity) *If* $\eta \notin C(\mathbf{y})$ *then* Λu *is microlocally* $\mathcal{C}^\infty$ *at* $(\mathbf{y},\eta)$ *for any* $u \in \mathcal{E}'(X)$.

Proof. a) According to Theorem 2.5 we only need to validate the statement about the microlocal ellipticity of Λ.

First, let $\eta_1 > 0$. Define $\overline{m} := \eta_2/\eta_1$ and the cone

$$V_\epsilon = \big\{(\lambda, m\lambda)^\top : \lambda \geq 0, m \in [\overline{m}-\epsilon, \overline{m}+\epsilon]\big\}$$

where $\epsilon > 0$. Obviously, V_ϵ is a conic neighborhood of η and

$$\forall\,\xi \in V_\epsilon\backslash\{\mathbf{0}\}:\quad \overline{m}-\epsilon \leq \frac{\xi_2}{\xi_1} \leq \overline{m}+\epsilon. \tag{3.32}$$

Let $B_\rho \subset X$ be a closed ball centered at $\mathbf{y}$ with a sufficiently small radius $\rho > 0$. Since $\eta \in C(\mathbf{y})$ and $\eta_1 \neq 0$ we have $\eta \in C_+(\mathbf{y})$ or $\eta \in C_-(\mathbf{y})$, say, $\eta \in C_+(\mathbf{y})$. Using

$$\frac{1}{|\omega B(s_+,\mathbf{x})|} = \frac{N_+\big(\mathbf{x},(1,\xi_2/\xi_1)\big)}{|\xi_1|}$$

we obtain

$$\sigma(\Lambda)(\mathbf{x},\xi) \geq |\xi|^2 \, \frac{\Sigma(\mathbf{x},\xi)}{|\xi_1|}$$

with numerator

$$\Sigma(\mathbf{x},\xi) = \psi\big(s_+(\mathbf{x},\xi_2/\xi_1), \varphi(s_+(\mathbf{x},\xi_2/\xi_1),\mathbf{x})\big) W\big(s_+(\mathbf{x},\xi_2/\xi_1),\mathbf{x}\big) \\ \times \Theta\big(s_+(\mathbf{x},\xi_2/\xi_1),\mathbf{x}\big) N_+\big(\mathbf{x},(1,\xi_2/\xi_1)\big).$$

In view of (3.32) and by continuity we may decrease ϵ and ρ such that Σ attains a positive minimum in $B_\rho \times V_\epsilon\backslash\{\mathbf{0}\}$:

$$c_{\epsilon,\rho} := \min\big\{\Sigma(\mathbf{x},\xi) : \mathbf{x}\in B_\rho,\ \xi \in V_\epsilon\backslash\{\mathbf{0}\}\big\} > 0.$$

Hence,

$$\forall\, \xi \in V_\epsilon\backslash\{\mathbf{0}\},\ \forall\, \mathbf{x}\in B_\rho : \quad \sigma(\Lambda)(\mathbf{x},\xi) \geq c_{\epsilon,\rho}\,\frac{|\xi|}{|\xi_1|}\,|\xi| \geq c_{\epsilon,\rho}\,|\xi|.$$

The case $\eta_1 < 0$ can be handled similarly.

Finally, we consider $\eta \in C(\mathbf{y})$ with $\eta_1 = 0$. Assume $\eta_2 > 0$. Here, we choose $\epsilon > 0$ and

$$V_\epsilon = \big\{(m\lambda,\lambda)^\top : \lambda \geq 0, m \in [-\epsilon,\epsilon]\big\}$$

as conic neighborhood of η with the property that

$$\forall\, \xi \in V_\epsilon\backslash\{\mathbf{0}\} : \quad \left|\frac{\xi_1}{\xi_2}\right| \leq \epsilon. \tag{3.33}$$

Let $\xi \in V_\epsilon\backslash\{\mathbf{0}\}$. Note that $|\xi_2/\xi_1| \geq 1/\epsilon$ (where $1/0 := \infty$). Consider for the time being $\xi_1 \geq 0$. Then, for any $\delta > 0$ we can find $\epsilon = \epsilon(\delta) > 0$ and $\rho = \rho(\delta) > 0$ such that

$$\forall\, \xi \in V_\epsilon\backslash\{\mathbf{0}\},\ \forall\, \mathbf{x}\in B_\rho : \ s_-(\mathbf{x},\xi_2/\xi_1) \in [x_1-\delta, x_1+\delta] \quad \text{and} \quad s_+(\mathbf{x},\xi_2/\xi_1) \geq 1/\delta \tag{3.34}$$

where B_ρ is as above (in case of $\xi_1 \leq 0$: $s_+(\mathbf{x},\xi_2/\xi_1) \in [x_1-\delta, x_1+\delta]$ and $s_-(\mathbf{x},\xi_2/\xi_1) \leq -1/\delta$). Thus, for δ sufficiently small

$$\psi\big(s_-(\mathbf{x},\xi_2/\xi_1), \varphi(s_-(\mathbf{x},\xi_2/\xi_1),\mathbf{x})\big) > 0 \quad \text{and} \quad \psi\big(s_+(\mathbf{x},\xi_2/\xi_1), \varphi(s_+(\mathbf{x},\xi_2/\xi_1),\mathbf{x})\big) = 0$$

for any $\xi \in V_\epsilon\backslash\{\mathbf{0}\}$, $\xi_1 \geq 0$, and $\mathbf{x}\in B_\rho$. Now, $\sigma(\Lambda)(\mathbf{x},\xi)$ consists of one term only (namely the one with $s = s_-$). We write

$$\frac{1}{|\omega B(s_-,\mathbf{x})|} = \frac{N_-\big(\mathbf{x},(\xi_1/\xi_2,1)\big)}{|\xi_2|}$$

to get

$$\sigma(\Lambda)(\mathbf{x},\xi) \geq |\xi|^2 \, \frac{\Sigma(\mathbf{x},\xi)}{|\xi_2|}$$

with

$$\Sigma(\mathbf{x},\xi) = \psi\big(s_-(\mathbf{x},\xi_2/\xi_1), \varphi(s_-(\mathbf{x},\xi_2/\xi_1)),\mathbf{x})\big) W\big(s_-(\mathbf{x},\xi_2/\xi_1),\mathbf{x}\big) \\ \times \Theta\big(s_-(\mathbf{x},\xi_2/\xi_1),\mathbf{x}\big) N_-\big(\mathbf{x},(\xi_1/\xi_2,1)\big).$$

By continuity, (3.33), and (3.34) we may decrease δ such that

$$c_{\epsilon,\rho} := \min\big\{\Sigma(\mathbf{x},\xi) : \mathbf{x}\in B_\rho,\ \xi \in V_\epsilon\backslash\{\mathbf{0}\}\big\} > 0.$$

Similar arguments in case $\xi_1 \leq 0$ let us conclude with

$$\forall\, \xi \in V_\epsilon\backslash\{\mathbf{0}\},\ \forall\, \mathbf{x}\in B_\rho : \quad \sigma(\Lambda)(\mathbf{x},\xi) \geq c_{\epsilon,\rho}\,\frac{|\xi|}{|\xi_2|}\,|\xi| \geq c_{\epsilon,\rho}\,|\xi|.$$

The case $\eta_2 < 0$ can be treated analogously.

The proof of part b) follows the lines of [20, Rem. 3.3]. □

The result of the above proposition differs fundamentally from a similar result in the situation of a constant sound speed $v(\cdot) = b$ where singularities $(\mathbf{y}, \eta)$ of n with $\eta_2 = 0$ are not visible in Λn (whatever the choice of ψ and S is), see [10]. The increasing sound speed (3.1), however, allows to recover those singularities, in principle.

Now we investigate how the top order symbol $\sigma(\Lambda)(\mathbf{x}, \xi)$ behaves as depth increases, that is, as $x_2 \to \infty$ while x_1 and ξ are kept fixed. In case $\xi_1 = 0$ and $\xi_2 \neq 0$ we have $s = x_1$ leading to

$$\frac{1}{|\omega B(s, \mathbf{x})|} = \frac{1}{\left|\xi_2\, \partial_{x_1}^2 \varphi(s, \mathbf{x})\right|} \asymp \frac{a^4}{4} \frac{x_2^2}{|\xi_2|} \quad \text{as } x_2 \to \infty \tag{3.35}$$

where $\asymp$ indicates that the terms are asymptotically equal. The situation is a bit more involved in the general situation of $\xi_1 \neq 0$. From (3.31) we obtain

$$S_\pm \asymp \widetilde{q}_\pm\, x_2 \quad \text{as } x_2 \to \infty$$

with $\widetilde{q}_\pm := q \pm \sqrt{1 + q^2}$ and $q = \xi_2/\xi_1$. Using

$$(H+2)^{3/2} H^{3/2} \asymp \frac{a^3}{8b^3} \left(1 + \widetilde{q}_\pm^2\right)^3 x_2^3 \quad \text{as } x_2 \to \infty$$

we arrive at

$$\frac{1}{|\omega B(s_\pm, \mathbf{x})|} \asymp \frac{a}{8} \frac{(1+\widetilde{q}_\pm^2)^2}{|(1-\widetilde{q}_\pm^2)q/2 - \widetilde{q}_\pm|} \frac{x_2^2}{|\xi_1|} \quad \text{as } x_2 \to \infty. \tag{3.36}$$

Next, we investigate the asymptotics of $\Theta(s, \mathbf{x})$ as $x_2 \to \infty$. Now let $(s, \omega, \mathbf{x}) \in \Pi_{\mathrm{R}}^{-1}(\mathbf{x}, \xi)$, $\xi_1 \neq 0$. By, (3.5), (A.2), (A.3), and (3.31),

$$\Theta(s_\pm, \mathbf{x}) \asymp \frac{\mathrm{C}_A^2}{1 + \widetilde{q}_\pm^2} \frac{2b}{a^2} \frac{1}{x_2^3} \quad \text{as } x_2 \to \infty. \tag{3.37}$$

In case $\xi_1 = 0$ we have $s = x_1$ and

$$\Theta(s, \mathbf{x}) \asymp \frac{2b^2}{a^2} \frac{\mathrm{C}_A^2}{x_2^3} \quad \text{as } x_2 \to \infty. \tag{3.38}$$

We summarize our results in the following proposition. For its compact formulation we introduce new notation:

$$\Psi_\pm(\mathbf{x}, q) := \psi\big(s_\pm(\mathbf{x}, q), \varphi(s_\pm(\mathbf{x}, q), \mathbf{x})\big).$$

Proposition 3.6. *Let $(\mathbf{x}, \xi) \in T^*(X)$. Set $q = \xi_2/\xi_1$ for $\xi_1 \neq 0$, $\widetilde{q}_\pm = q \pm \sqrt{1+q^2}$, and $\Xi_\pm = |(1 - \widetilde{q}_\pm^2)q - 2\widetilde{q}_\pm|$.*

If $W = \Theta$ in (3.24) (i.e. $F^\dagger = F^*$) then

$$\sigma(\Lambda)(\mathbf{x}, \xi) \asymp \begin{cases} 2\pi \dfrac{b^2}{a^3} \dfrac{\mathrm{C}_A^4}{x_2^4} \left(\dfrac{\Psi_+(\mathbf{x}, q)}{\Xi_+} + \dfrac{\Psi_-(\mathbf{x}, q)}{\Xi_-} \right) \dfrac{|\xi|^2}{|\xi_1|} & : \ \xi_1 \neq 0, \\[2ex] 2\pi\, b^4 \dfrac{\mathrm{C}_A^4}{x_2^4}\, \psi\big(x_1, \varphi(x_1, \mathbf{x})\big) \dfrac{|\xi|^2}{|\xi_2|} & : \ \xi_1 = 0, \end{cases}$$

as $x_2 \to \infty$.

If $W = 1/\Theta$ in (3.24) then

$$\sigma(\Lambda)(\mathbf{x},\xi) \asymp \begin{cases} \frac{\pi}{4}\, a\, x_2^2 \left(\Psi_+(\mathbf{x},q)\frac{(1+\widetilde{q}_+^2)^2}{\Xi_+} + \Psi_-(\mathbf{x},q)\frac{(1+\widetilde{q}_-^2)^2}{\Xi_-}\right)\frac{|\xi|^2}{|\xi_1|} & : \ \xi_1 \neq 0, \\ \frac{\pi}{2}\, a^4 x_2^2\, \psi\big(x_1, \varphi(x_1,\mathbf{x})\big)\frac{|\xi|^2}{|\xi_2|} & : \ \xi_1 = 0, \end{cases}$$

as $x_2 \to \infty$.

Proof. We only need to combine (3.29) with (3.35), (3.36), (3.37), and (3.38). □

The above proposition clearly reveals that the top order symbols for both weights depend on x_2. Hence, jumps in n having the same height but being located at different depths should be reconstructed with different jump height in Λn. While the weight $W = \Theta$ diminishes, the weight $W = 1/\Theta$ magnifies jumps. These shortcomings can be overcome by a slight modification of Λ. Indeed, let M_2 be the operator which multiplies a distribution in $\mathcal{D}'(X)$ by the monomial x_2. Then, the imaging operators

$$\Lambda_{\text{mod},1} = \Delta \mathrm{M}_2^4 F^* \psi F \ \text{ for } W = \Theta \quad \text{and} \quad \Lambda_{\text{mod},2} = \Delta \mathrm{M}_2^{-2} F^\dagger \psi F \ \text{ for } W = 1/\Theta$$

are still ΨDOs of order 1 with top order symbols which are asymptotically independent of the depth variable. Hence, jumps in n should be reconstructed relatively independently of their depths (provided the jumps of n are visible in $\Lambda_{\text{mod},i} n$, compare Proposition 3.5).

Remark 3.7. *We expect statements analogous to Propositions 3.5 and 3.6 to hold even for $\alpha > 0$ because the geometry of the isochrones (3.21) that determine the visible singularities are similar to those spheres which are the isochrones for $\alpha = 0$, compare Remark 3.1 and see Figure 1.*

We finish this section with a numerical example where the underlying background sound speed is $v(\mathbf{x}) = 0.5 + 0.1x_2$ and the used common offset is $\alpha = 5$. Thus, the characteristic values are

$$t_{\min} \approx 17.63 \quad \text{and} \quad x_{\min} \approx 2.07,$$

that is, the Bolker condition is satisfied in the sense of Conclusion 3.4 for $X = \{\mathbf{x} \in \mathbb{R}^2 : x_2 > x_{\min}\}$ and $Y = S \times (t_{\min}, \infty)$ but it is violated off these sets. We use the phantom n shown in the left of Figure 3 together with some isochrones to travel times close to $t_{\min}$. The isochrone for $t = t_{\min}$ is the geodesic connecting source with receiver.

The numerical approach of [11] has been adapted to non-constant background velocity and yields the numerical approximations to $\Lambda n = \Delta F^* \psi g$ and $\Lambda_{\text{mod},1} n = \Delta \mathrm{M}_2^2 F^* \psi g$ presented in the bottom of Figure 3 from discrete data $g = Fn$. The reconstructions exhibit some cutoff-artifacts but the parts of the singular support of n with horizontal normal directions are visible as predicted by Proposition 3.5. Moreover, while the ellipticity of Λ deteriorates with depth, the ellipticity of $\Lambda_{\text{mod},1}$ is asymptotically independent of it.

The illustration and numerical approximations in Figure 3 were kindly provided by Kevin Ganster[5]. The underlying algorithm and further examples will be published elsewhere.

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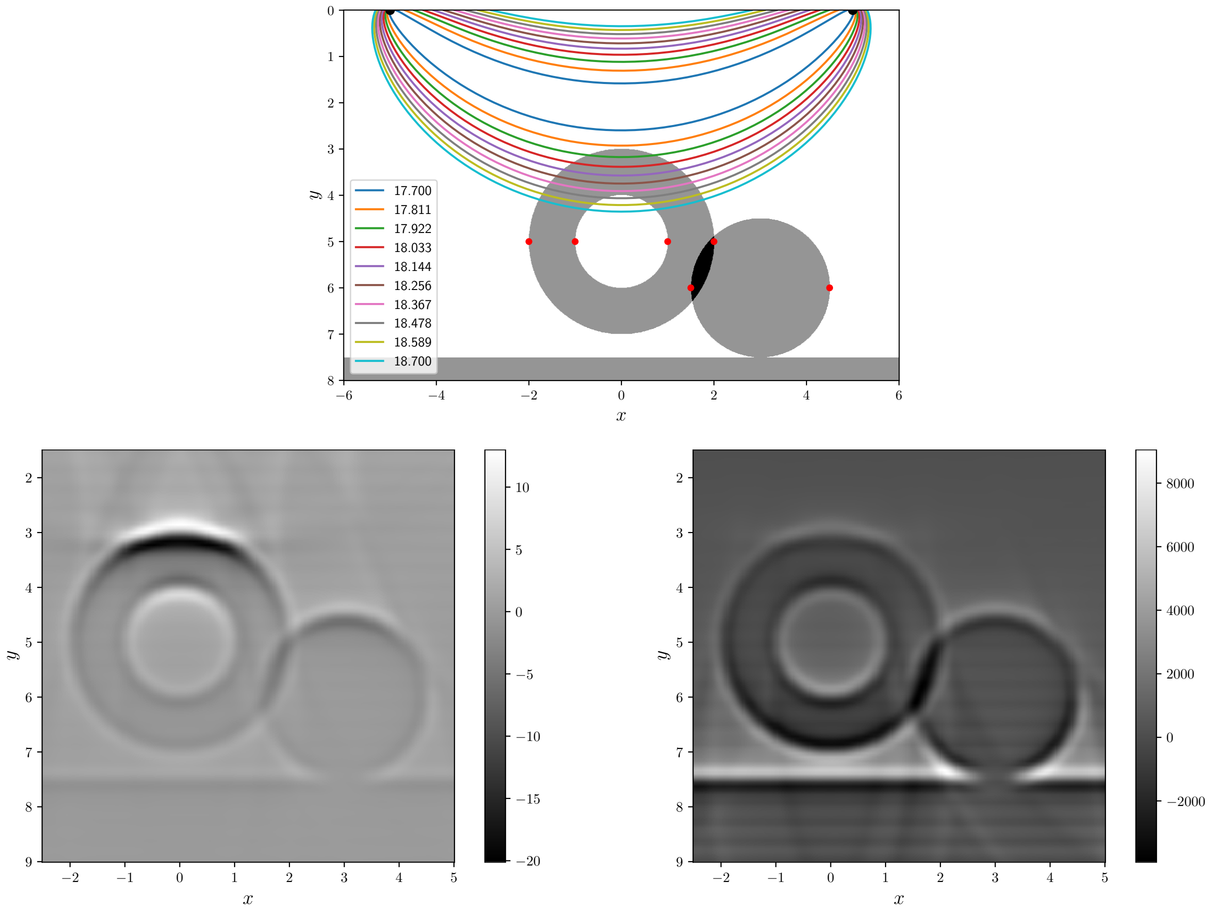

FIGURE 3. Top: Illustration of phantom n. It consists of a superposition of indicator functions of circular disks and a half-space. The colors white, grey, and black represent the numerical values 0, 1, and 2 respectively. Moreover, the colored curves are isochrones $\mathcal{L}_{0,t}$ for ten selected travel times as specified in the legend. The black dots mark source and receiver positions.
Bottom: Numerical approximations of $\Lambda n = \Delta F^* \psi g$ (left) and $\Lambda_{\mathrm{mod},1} n = \Delta \mathrm{M}_2^4 F^* \psi g$ (right) computed from $g(s,t) = Fn(s,t)$ for discrete values $s \in [-10, 10]$ and $t \in [17.628, 47.628]$. Both reconstructions show those parts of the singular support of n with horizontal normal directions (indicated by red dots in the top image).

4. STABILITY OF THE TRAVEL TIME WITH RESPECT TO SOUND SPEED

In this section we study the dependence of the phase function Φ in the FIO representation (1.6) of the operator F on the background sound speed $v = v(\mathbf{x})$. We recall that the phase function is given by

$$\Phi(\mathbf{s}, t, \mathbf{x}, \omega) = \omega(t - \varphi(\mathbf{s}, \mathbf{x})),$$

where

$$\varphi(\mathbf{s}, \mathbf{x}) = \tau(\mathbf{x}, \mathbf{x_s}(\mathbf{s})) + \tau(\mathbf{x}, \mathbf{x_r}(\mathbf{s}))$$

and $\tau(\mathbf{x}, \mathbf{x_s})$ is the solution to the eikonal equation (1.4). Hence we first consider this equation in the following section.

4.1. **Solving the eikonal equation.** We denote by $\mathbb{R}^d_+ = \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_d > 0\}$ the subsurface (here $d = 2$ or $d = 3$). In $\mathbb{R}^d_+$ we consider the eikonal equation

$$|\nabla\tau(\mathbf{x})|^2 = \frac{1}{v(\mathbf{x})^2} \tag{4.1}$$

for the travel time τ of rays starting from a fixed point $\mathbf{x_0} \in \partial\mathbb{R}^d_+$ on the surface. Here $v \in C^\infty(\mathrm{cl}(\mathbb{R}^d_+))$ denotes the smooth and positive sound speed in the subsurface, and we write $C^\infty(\mathrm{cl}(\mathbb{R}_+))$ for the set of all functions $g\colon\ \mathrm{cl}(\mathbb{R}^d_+) \to \mathbb{R}$ that are C^∞ in $\mathbb{R}^d_+$ and, together with all derivatives, have continuous extensions to $\mathrm{cl}(\mathbb{R}^d_+)$.

We also need that the solution τ is a smooth function of the initial condition $\mathbf{x}_0 \in \partial\mathbb{R}^d_+$ in order to get a smooth phase function, but we will suppress this dependence for the moment.

We study (4.1) as a special case of the equation

$$H(\mathbf{x}, u, \nabla u) = 0$$

for a real-valued function $u = u(\mathbf{x})$ on a subset of $\mathbb{R}^d$ where the Hamiltonian is given by

$$H(\mathbf{x}, u, \mathbf{p}) = \frac{1}{2}\Big(\mathbf{p}\cdot\mathbf{p} - \frac{1}{v(\mathbf{x})^2}\Big) \quad \text{for } \mathbf{p} \in \mathbb{R}^d.$$

Note that $H(\mathbf{x}, u, \mathbf{p}) = H(\mathbf{x}, \mathbf{p})$ does not depend on u in our case.

According to [9, Chapter 10], solutions are thus obtained via solving the characteristic system

$$\begin{aligned} \dot{\mathbf{x}} &= \partial_{\mathbf{p}} H(\mathbf{x}, \mathbf{p}) = \mathbf{p}, \\ \dot{\mathbf{p}} &= -\partial_{\mathbf{x}} H(\mathbf{x}, \mathbf{p}) = -\tfrac{\nabla v(\mathbf{x})}{v(\mathbf{x})^3}, \\ \dot{u} &= \mathbf{p}\cdot\partial_{\mathbf{p}} H(\mathbf{x}, \mathbf{p}) - H(\mathbf{x}, \mathbf{p}), = \tfrac{1}{2}\Big(|\mathbf{p}|^2 + \tfrac{1}{v(\mathbf{x})^2}\Big). \end{aligned} \tag{4.2}$$

We see that u is obtained by simple integration and thus we can concentrate on the $(\mathbf{x}, \mathbf{p})$-subsystem.

The initial conditions corresponding to those in (1.4) are

$$\mathbf{x}(0, \xi) = \mathbf{x}_0, \qquad \mathbf{p}(0, \xi) = \frac{1}{v(\mathbf{x}_0)}\xi, \qquad u(0, \xi) = 0, \tag{4.3}$$

where $\xi \in S^{d-1}_+ = \{\xi \in \mathbb{R}^d : |\xi| = 1, \xi_d > 0\}$ as we are considering the subsurface.

Remark 4.1. *If there is an $\varepsilon_0 > 0$ such that $v(\mathbf{x}) = v_0 > 0$ for $\mathbf{x} \in \mathrm{cl}(\mathbb{R}^d_+)$ with $x_d \in [0, \varepsilon_0]$, then if a ray starts from $\mathbf{x_0} \in \partial\mathbb{R}^d_+$ in a direction $\xi \in S^{d-1}_+$, we have an explicit formula for $t \in [0, \varepsilon_0 v_0]$, namely*

$$\mathbf{x}(t, \xi) = \mathbf{x_0} + \frac{t}{v_0}\xi, \qquad \mathbf{p}(t, \xi) = \frac{1}{v_0}\xi, \qquad u(t, \xi) = \frac{t}{v_0^2},$$

which, for $\mathbf{x} \in \mathbb{R}^d_+$ with $|\mathbf{x} - \mathbf{x_0}| \le \varepsilon_0$, leads via $v_0(\mathbf{x} - \mathbf{x_0}) = t\xi$ to $t = v_0|\mathbf{x} - \mathbf{x_0}|$ and the well-known $\tau(\mathbf{x}) = u(\mathbf{x}) = \frac{|\mathbf{x}-\mathbf{x_0}|}{v_0}$.

We want to solve the system (4.2) for $t \ge 0$ with initial conditions (4.3). Assuming

$$v(\mathbf{x}) \text{ is bounded away from } 0 \text{ and } \nabla v \text{ is bounded,} \tag{4.4}$$

the system (4.2) with initial conditions (4.3) has a unique global solution: local existence and uniqueness hold by Picard-Lindelöf, and since the right hand side of the $(\mathbf{x}, \mathbf{p})$-subsystem has linear growth in $\mathbf{p}$ the solution exists globally (this is an application of the Gronwall lemma). Since v is defined on $\mathrm{cl}(\mathbb{R}^d_+)$, "globally" means here that, for fixed $\xi \in S^{d-1}_+$, the maximal

t-interval is either $[0, \infty)$ and we set $T_{\max}(\xi) := \infty$ or it is a compact interval $[0, T_{\max}(\xi)]$ with $\mathbf{x}(T_{\max}(\xi), \xi) \in \partial\mathbb{R}^d_+$, which means that the ray resurfaces.

In order to obtain a solution τ of the eikonal equation (4.1) we parametrize S^{d-1}_+ by ξ' in the open unit ball B_{d-1} in $\mathbb{R}^{d-1}$ via $\xi = \left(\xi', \sqrt{1-|\xi'|^2}\right)$. We let $T_{\max}(\xi') := T_{\max}(\xi)$ for $\xi' \in B_{d-1}$ and denote by

$$Q_{\max} := \{(t, \xi') \in (0, \infty) \times B_{d-1} : t \in (0, T_{\max}(\xi'))\}$$

the (open) parameter set for the family of maximal solutions of (4.2) with (4.3). We introduce the map

$$\Psi_{\mathbf{x}_0} \colon Q_{\max} \to \mathbb{R}^d_+, \qquad (t, \xi') \mapsto \mathbf{x}(t, \xi),$$

where the subscript $\mathbf{x}_0$ refers to the point $\mathbf{x}_0 \in \partial\mathbb{R}^d_+$ we fixed at the beginning. (Of course, also the solutions $\mathbf{x}$, u, $\mathbf{p}$, and the maximal existence time $T_{\max}(\xi')$ depend on $\mathbf{x}_0$ but we skip this dependence in notation.)

If the sound speed is constant, $v = v_0$, then $Q_{\max} = (0, \infty) \times B_{d-1}$ and $\Psi_{\mathbf{x}_0} \colon (0, \infty) \times B_{d-1} \to \mathbb{R}^d_+$ is a diffeomorphism (see Remark 4.1 above with $\varepsilon_0 \to \infty$). For our linear velocity model only the ray with $\xi' = 0$ does not resurface and we have $T_{\max}(\xi') \to 0$ for $|\xi'| \to 1$ (see Appendix A), but also here the map $\Psi_{\mathbf{x}_0}$ above is a diffeomorphism. Observe in both cases that the diffeomorphism $\Psi^{-1}_{\mathbf{x}_0}$ "degenerates" as we approach the initial point $\mathbf{x}_0$ from the subsurface.

In the general case, rays may intersect in the subsurface, and this is something we want to exclude. So we let

$$Q_{\mathrm{uniq}} := \{(t, \xi') \in Q_{\max} : (\Psi_{\mathbf{x}_0})^{-1}(\Psi_{\mathbf{x}_0}(t, \xi')) = (t, \xi')\}.$$

This means that $\Psi_{\mathbf{x}_0}(Q_{\mathrm{uniq}})$ is the set of points in the subsurface that are hit (exactly once) by a unique ray from $\mathbf{x}_0$. In the two examples of a constant $v = v_0$ or a linearly growing v we clearly have $Q_{\mathrm{uniq}} = Q_{\max}$.

In the following we thus consider $\Psi_{\mathbf{x}_0} \colon Q_{\mathrm{uniq}} \to \mathbb{R}^d_+$. We might only have that a suitable restriction of this map $\Psi_{\mathbf{x}_0}$ is a diffeomorphism. Hence we assume for the sound speed v:

(4.5) there exists a continuous $T \colon B_{d-1} \to (0, \infty)$ such that the restriction of $\Psi_{\mathbf{x}_0}$ to $Q := \{(t, \xi') : \xi' \in B_{d-1}, t \in (0, T(\xi'))\}$ is a diffeomorphism onto $\Psi_{\mathbf{x}_0}(Q)$.

Since $Q \subset Q_{\mathrm{uniq}}$, in particular each point $\mathbf{x}$ in the subset $\Psi_{\mathbf{x}_0}(Q)$ of the subsurface is hit exactly once by a unique ray emanating from the fixed point $\mathbf{x_0}$ on the surface. Under (4.5) the function

$$\tau(\mathbf{x}) := \tau(\mathbf{x}, \mathbf{x}_0) := u(\Psi^{-1}_{\mathbf{x}_0}(\mathbf{x})), \qquad \mathbf{x} \in \Psi_{\mathbf{x}_0}(Q),$$

is the desired solution to the eikonal equation (4.1), existing on $\Psi_{\mathbf{x}_0}(Q)$. As explained above, in our linear velocity model, we can take $Q = Q_{\max}$ and have $\Psi_{\mathbf{x}_0}(Q_{\max}) = \mathbb{R}^d_+$. Varying the source point $\mathbf{x}_0$ of the rays in an open set $L \subset \mathrm{bd}(\mathbb{R}^d_+)$ we assume moreover

(4.6) v is such that the map $\Psi_{\mathbf{x}_0} \colon Q \to \mathbb{R}^d_+$ depends smoothly on $\mathbf{x}_0 \in L$.

Here we assume for simplicity that the same Q can be used for all $\mathbf{x}_0 \in L$. In particular, this means that the Q in (4.5) has to be a subset of Q_{uniq} for every $\mathbf{x}_0 \in L$. Obviously, this is no restriction if v only depends on depth, i.e., only depends on x_d. The assumption (4.6) is clearly satisfied for the linear velocity model where $(\mathbf{x}, \mathbf{x}_0) \mapsto \tau(\mathbf{x}, \mathbf{x}_0)$ is smooth on $\mathbb{R}^d_+ \times \partial\mathbb{R}^d_+$ (see Appendix A).

4.2. **Perturbation of the sound speed.** Let v_0 be a given sound speed satisfying (4.4), (4.5), and (4.6) in place of v. We now assume that $v_1 \in C^\infty(\mathrm{cl}(\mathbb{R}^d_+))$ is another sound speed satisfying (4.4) such that

$$\text{the support of } v_1 - v_0 \text{ is contained in } \{\mathbf{x} \in \mathbb{R}^d_+ : x_d \geq \varepsilon_0\} \text{ for some } \varepsilon_0 > 0. \tag{4.7}$$

For $j = 0, 1$, we denote by $\mathbf{x}^j, \mathbf{p}_j, u_j$ the solution to the characteristic system (4.2) with sound speed v replaced by v_j and with the same initial values (4.3), which then induces a function $T^j_{\max}\colon B_{d-1} \to (0, \infty]$, parameter sets $Q^j_{\max}$ and Q^j_{uniq} and a map $\Psi^j_{\mathbf{x}_0}\colon Q^j_{\text{uniq}} \to \mathbb{R}^d_+$ as before. We denote by $\tau_0(\cdot)$ the solution to (4.1) with sound speed v_0 for a fixed initial value $\mathbf{x}_0 \in \partial\mathbb{R}^d_+$ (which exists by assumption (4.5) for v_0). By a perturbation argument, we shall obtain a solution $\tau_1(\cdot)$ to (4.1) with sound speed v_1. The assumption $\varepsilon_0 > 0$ guarantees that $\Psi^0_{\mathbf{x}_0}$ and $\Psi^1_{\mathbf{x}_0}$ coincide for small values of t. For the perturbation argument we thus can stay away from $\mathbf{x}_0$ where the diffeomorphism $\Psi^0_{\mathbf{x}_0}$ degenerates. More precisely, we shall consider compact subsets $K \subset Q$ of the form

$$K = \{(t, \xi') : \xi' \in K_0, t \in [a(\xi'), b(\xi')]\} \tag{4.8}$$

where $K_0 \subset B_{d-1}$ is the compact closure of a smooth domain and $a, b\colon K_0 \to (0, \infty)$ are smooth and satisfy, for any $\xi' \in K_0$,

$$0 < a(\xi') < b(\xi') < T(\xi') \quad \textit{and} \quad \{\mathbf{x}(t, \xi') : t \in (0, a(\xi')]\} \subset \{\mathbf{x} \in \mathbb{R}^d_+ : x_d \in (0, \varepsilon_0)\}.$$

The following is the main part of the perturbation result.

Proposition 4.2. *Let $v_0 \in C^\infty(\mathrm{cl}(\mathbb{R}^d_+))$ be a sound speed satisfying (4.4), (4.5), and (4.6) in place of v. Let $K \subset Q$ be compact and of the form (4.8) above. Let $\delta > 0$ and let $K_1 \subset \mathbb{R}^d_+$ be the compact closure of an open neighborhood of $\Psi^0_{\mathbf{x}_0}(K)$. If $v_1 \in C^\infty(\mathrm{cl}(\mathbb{R}^d_+))$ satisfies (4.4) and (4.7) and is sufficiently close to v_0 in C^2-norm on K_1 then $K \subset Q^1_{\max}$ and $\Psi^1_{\mathbf{x}_0}\colon Q^1_{\max} \to \mathbb{R}^d_+$, $(t, \xi') \mapsto \mathbf{x}^1(t, \xi')$, gives rise to a diffeomorphism of an open set $U \supset K$ with $\mathrm{cl}(U) \subset Q$ and $\Psi^0_{\mathbf{x}_0}(K) \subset \Psi^1_{\mathbf{x}_0}(U)$ and the solutions τ_0 and $\tau_1(\mathbf{x}) := u_1((\Psi^1_{\mathbf{x}_0})^{-1}(\mathbf{x}))$ satisfy*

$$|\tau_0(\mathbf{x}) - \tau_1(\mathbf{x})| \leq \delta, \tag{4.9}$$

for all $\mathbf{x} \in \Psi^0_{\mathbf{x}_0}(K)$.

If $C \subset \partial\mathbb{R}^d_+$ is compact, then K_2 is the compact closure of an open neighborhood of $\bigcup_{\mathbf{x}_0 \in C} \Psi^0_{\mathbf{x}_0}(K)$, and if v_0 and v_1 are sufficiently close in C^2-norm on K_2 then, for $\mathbf{x}_0 \in C$, $\tau_1(\cdot, \mathbf{x}_0)$ exists on $\Psi^0_{\mathbf{x}_0}(K)$ and we have

$$|\tau_0(\mathbf{x}, \mathbf{x}_0) - \tau_1(\mathbf{x}, \mathbf{x}_0)| \leq \delta$$

for all $\mathbf{x} \in \Psi^0_{\mathbf{x}_0}(K)$ and $\mathbf{x}_0 \in C$.

Proof. Clearly we obtain the solutions $\mathbf{x}^1, \mathbf{p}_1, u_1$ by letting, for $\xi' \in K_0$ and $t \in [0, a(\xi')]$,

$$\mathbf{x}^1(t, \xi) = \mathbf{x}^0(t, \xi), \quad \mathbf{p}_1(t, \xi) = \mathbf{p}_0(t, \xi), \quad u_1(t, \xi) = u_0(t, \xi),$$

and then solving (4.2) with initial conditions

$$\mathbf{x}^1(a(\xi'), \xi) = \mathbf{x}^0(a(\xi'), \xi), \quad \mathbf{p}_1(a(\xi'), \xi) = \mathbf{p}_0(a(\xi'), \xi), \quad u_1(a(\xi'), \xi) = u_0(a(\xi'), \xi),$$

which are non-degenerate. Here we have used assumption (4.7).

If $v_1 - v_0$ is sufficiently small in C^1-norm on K_1 then the solutions $\mathbf{x}^0(t, \xi')$ and $\mathbf{x}^1(t, \xi')$, as well as $u_0(t, \xi')$ and $u_1(t, \xi')$ can be made arbitrarily close in sup-norm on K, since solutions to ODE systems depend continuously on the right hand side and on parameters.

Here we need also that derivatives of $\Psi^0_{\mathbf{x}_0}$ and $\Psi^1_{\mathbf{x}_0}$ with respect to t and the parameter ξ' are close to each other. This follows by the same arguments, as $\partial_t \Psi^0_{\mathbf{x}_0}$ and $\partial_{\xi'} \Psi^0_{\mathbf{x}_0}$ are given as solutions to ODE systems involving derivatives of the right hand side of (4.2). We thus need that v_1 is close to v_0 in C^2-norm on K_1.

On the compact set K we have $\inf|\det D\Psi^0_{\mathbf{x}_0}(t,\xi')| > 0$, and if $D\Psi^0_{\mathbf{x}_0}(t,\xi')$ and $D\Psi^1_{\mathbf{x}_0}(t,\xi')$ are sufficiently close on K we infer that $D\Psi^1_{\mathbf{x}_0}(t,\xi') \in \mathbb{R}^{d\times d}$ is regular for each $(t,\xi') \in K$. Hence $\Psi^1_{\mathbf{x}_0}$ is locally an isomorphism on an open superset U of K and we may assume $\mathrm{cl}(U) \subset Q$. Thus it only remains to show that $\Psi^1_{\mathbf{x}_0}$ is injective on K.

By smoothness of a and b on the compact set K_0, the set K has a Lipschitz boundary and there exists $c > 0$ such that, for any two points $(t_1,\xi_1'), (t_2,\xi_2') \in K$, we find a C^1-curve $\gamma\colon [0,1] \to K$ connecting these two points with length $L \le c|(t_1,\xi_1') - (t_2,\xi_2')|$. Then we have

$$\begin{aligned}\Psi^1_{\mathbf{x}_0}(t_2,\xi_2') - \Psi^1_{\mathbf{x}_0}(t_1,\xi_1') &= \int_0^1 (D\Psi^1_{\mathbf{x}_0})\big(\gamma(r)\big)\dot\gamma(r)\,\mathrm{d}r \\ &= \Psi^0_{\mathbf{x}_0}(t_2,\xi_2') - \Psi^0_{\mathbf{x}_0}(t_1,\xi_1') + \int_0^1 \big(D\Psi^1_{\mathbf{x}_0} - D\Psi^0_{\mathbf{x}_0}\big)\big(\gamma(r)\big)\dot\gamma(r)\,\mathrm{d}r.\end{aligned}$$

By compactness of K we obtain a constant $\eta_0 > 0$ such that

$$|\Psi^0_{\mathbf{x}_0}(t_2,\xi_2') - \Psi^0_{\mathbf{x}_0}(t_1,\xi_1')| \ge \eta_0|(t_2,\xi_2') - (t_1,\xi_1')| \quad \text{for all } (t_1,\xi_1'), (t_2,\xi_2') \in K.$$

Since

$$\Big|\int_0^1 \big(D\Psi^1_{\mathbf{x}_0} - D\Psi^0_{\mathbf{x}_0}\big)\big(\gamma(r)\big)\dot\gamma(r)\,\mathrm{d}r\Big| \le \sup_{(t,\xi')\in K}\big|\big(D\Psi^1_{\mathbf{x}_0} - D\Psi^0_{\mathbf{x}_0}\big)(t,\xi')\big|\,L\,|(t_2,\xi_2') - (t_1,\xi_1')|,$$

we thus obtain, if $D\Psi^1_{\mathbf{x}_0}$ and $D\Psi^0_{\mathbf{x}_0}$ are sufficiently close on K,

$$|\Psi^1_{\mathbf{x}_0}(t_2,\xi_2') - \Psi^1_{\mathbf{x}_0}(t_1,\xi_1')| \ge \frac{\eta_0}{2}|(t_2,\xi_2') - (t_1,\xi_1')| \quad \text{for all } (t_1,\xi_1'), (t_2,\xi_2') \in K.$$

In particular, $\Psi^1_{\mathbf{x}_0}$ is injective on K, and we find U as desired.

We may run the same arguments with a superset $K' \supset K$ of the form (4.8) satisfying $K \subset \mathrm{int}(K')$ and $\Psi^0_{\mathbf{x}_0}(K') \subset \mathrm{int}(K_1)$. The boundary points of $\Psi^0_{\mathbf{x}_0}(K')$ and $\Psi^1_{\mathbf{x}_0}(K')$ are close if v_0 and v_1 are close in C^2-norm on K_1. Hence we can arrange for $\Psi^0_{\mathbf{x}_0}(K) \subset \Psi^0_{\mathbf{x}_0}(K') \cap \Psi^1_{\mathbf{x}_0}(K')$.

Letting $\tau_1(\mathbf{x}) := u_1((\Psi^1_{\mathbf{x}_0})^{-1}(\mathbf{x}))$ for $\mathbf{x} \in \Psi^1_{\mathbf{x}_0}(K')$ we have, for $\mathbf{x} \in \Psi_{\mathbf{x}_0}(K') \cap \Psi^1_{\mathbf{x}_0}(K')$,

$$\begin{aligned}|\tau_0(\mathbf{x}) - \tau_1(\mathbf{x})| &\le |u_1((\Psi^1_{\mathbf{x}_0})^{-1}(\mathbf{x})) - u_0((\Psi^1_{\mathbf{x}_0})^{-1}(\mathbf{x}))| + |u_0((\Psi^1_{\mathbf{x}_0})^{-1}(\mathbf{x})) - u_0((\Psi^0_{\mathbf{x}_0})^{-1}(\mathbf{x}))| \\ &\le \sup_{(t,\xi')\in K'}|(u_1 - u_0)(t,\xi')| + \sup_{(t,\xi')\in K'}|\nabla u_0(t,\xi')|\,|(\Psi^1_{\mathbf{x}_0})^{-1}(\mathbf{x}) - (\Psi^0_{\mathbf{x}_0})^{-1}(\mathbf{x})|.\end{aligned}$$

We know that $u_1 - u_0$ can be made small in sup-norm on K', and from the arguments above it is clear that $(\Psi^1_{\mathbf{x}_0})^{-1}$ and $(\Psi^0_{\mathbf{x}_0})^{-1}$ are as close as we wish on $\Psi^0_{\mathbf{x}_0}(K)$ if v_0 and v_1 are sufficiently close in C^2-norm on K_1.

Using (4.6) and another compactness argument we prove the last assertion. □

For application of Theorem 5.2 below to our situation, we need closeness of the corresponding phase functions φ_0 and φ_1 in C^3-norm, see (5.14) and the definition of P_j, (5.4). We recall that, e.g., φ_0 is given by

$$\varphi_0(\mathbf{s},\mathbf{x}) = \tau_0(\mathbf{x},\mathbf{x_s}(\mathbf{s})) + \tau_0(\mathbf{x},\mathbf{x_r}(\mathbf{s}))$$

where $\mathbf{x_s},\mathbf{x_r}\colon S' \to \partial\mathbb{R}^d_+$ are smooth parameterizations of the source/receiver pairs over an open set $S' \subset \mathbb{R}^{d-1}$. In the situation of Proposition 4.2, i.e. if v_0 satisfies (4.4), (4.5), and (4.6), and

$$\mathbf{x_s}(S') \cup \mathbf{x_r}(S') \subset L, \tag{4.10}$$

we have that φ_0 is defined at least on the set

$$U_0 := \{(\mathbf{s},\mathbf{x}) \in S' \times \mathbb{R}^d_+ : \mathbf{x} \in \Psi^0_{\mathbf{x_s}(\mathbf{s})}(Q) \cap \Psi^0_{\mathbf{x_r}(\mathbf{s})}(Q)\}, \tag{4.11}$$

this set is open, and $\varphi_0 \colon U_0 \to \mathbb{R}$ is smooth.

The following notion seems natural in the given situation.

Definition 4.3. For a subset $K \subset U_0$ we define the *ray closure* $r(K) \subset \mathrm{cl}(\mathbb{R}^d_+)$ (with respect to v_0) to be the union of all trajectories of the parts of rays connecting $\mathbf{x_s}(\mathbf{s})$ or $\mathbf{x_r}(\mathbf{s})$ and $\mathbf{x}$ where $(\mathbf{s},\mathbf{x}) \in K$.

The motivation for this definition is that, for any $(\mathbf{s},\mathbf{x}) \in K$, the travel time from $\mathbf{x_s}(\mathbf{s})$ or $\mathbf{x_r}(\mathbf{s})$ to the point $\mathbf{x}$ is affected by the values of the velocity on the trajectory of the ray that hits $\mathbf{x}$ before it hits. Observe that the ray closure of a compact set K is a compact subset of $\mathrm{cl}(\mathbb{R}^d_+)$.

Theorem 4.4. *Let v_0 satisfy (4.4), (4.5), and (4.6). Let $S' \subset \mathbb{R}^{d-1}$ be open and assume that $\mathbf{x_s}, \mathbf{x_r} \colon S' \to \partial\mathbb{R}^d_+$ are smooth and satisfy (4.10). Let $v_1 \in C^\infty(\mathrm{cl}(\mathbb{R}^d_+))$ satisfy (4.4) and (4.7). Let $K \subset U_0$ be compact and $\delta_0 > 0$. If $v_1 - v_0$ is sufficiently small in C^5-norm on the compact closure $M \subset \mathrm{cl}(\mathbb{R}^d_+)$ of an open neighborhood of the ray closure of K then*

$$\|\varphi_1 - \varphi_0\|_{C^3(K)} \le \delta_0.$$

Moreover, if $\nabla_{\mathbf{x}}\varphi_0 \neq 0$ on K and $\|v_1 - v_0\|_{C^5(M)}$ is sufficiently small, then we have in addition that $\nabla_{\mathbf{x}}\varphi_1 \neq 0$ on K.

Proof. Applying Proposition 4.2 we obtain φ_1. We have to look at derivatives of φ_0 and φ_1 with respect to $\mathbf{x}$ and the parameter s, i.e. at derivatives of solutions to the eikonal equation (4.1) with respect to $\mathbf{x}$ and initial values $\mathbf{x}_0$. Hence we need derivatives of solutions u_0 (and u_1) to (4.2) with respect to t, ξ' and $\mathbf{x}_0$ as well as derivatives of the diffeomorphisms $(\Psi^0)^{-1}$ and $(\Psi^1)^{-1}$. The latter boils down to derivatives of Ψ^0 and Ψ^1. All these derivatives are solutions to ODE systems involving derivatives of the right hand side in (4.2). The first claim now follows by the same arguments as in the proof of Proposition 4.2.

For the proof of the second claim we put

$$\delta_1 := \inf\left\{\max_{j=1,\dots,d} |\partial_{x_j}\varphi_0(\mathbf{s},\mathbf{x})| : (\mathbf{s},\mathbf{x}) \in K\right\}.$$

By compactness of K and assumption we have $\delta_1 > 0$. Now we choose $0 < \tilde{\delta}_0 < \min\{\delta_0, \delta_1\}$ and apply the first claim with $\tilde{\delta}_0$ in place of δ_0. Then we have, for $(\mathbf{s},\mathbf{x}) \in K$,

$$\max_{j=1,\dots,d} |\partial_{x_j}\varphi_1(\mathbf{s},\mathbf{x})| \ge \max_{j=1,\dots,d} |\partial_{x_j}\varphi_0(\mathbf{s},\mathbf{x})| - \tilde{\delta}_0 \ge \delta_1 - \tilde{\delta}_0 > 0,$$

which proves the claim. □

Recall that φ_0 is just a part of the phase function

$$\Phi_0(\mathbf{s}, t, \mathbf{x}, \omega) = \omega\,(t - \varphi_0(\mathbf{s},\mathbf{x}))\,,$$

and in order to satisfy Definition 2.2 we need φ_0 defined on a set $S' \times X_0$, where $X_0 \subset \mathbb{R}^d_+$ is open, and we need that $-\nabla_{\mathbf{x}}\Phi_0 = \omega\nabla_{\mathbf{x}}\varphi_0(\mathbf{s},\mathbf{x})$ does not vanish on $S' \times X_0$ where $\omega \neq 0$, i.e. we need $\nabla_{\mathbf{x}}\varphi_0 \neq 0$ on $S' \times X_0$. In a first step we let

$$\Xi_0 := \bigcap_{s \in S'} \Psi^0_{\mathbf{x_s}(s)}(Q) \cap \Psi^0_{\mathbf{x_r}(s)}(Q). \tag{4.12}$$

Observe that $S' \times \Xi_0 \subset U_0$. Then we set

$$X_0 := \mathrm{int}\left(\{\mathbf{x} \in \Xi_0 : \nabla_{\mathbf{x}}\varphi_0(\mathbf{s},\mathbf{x}) \neq 0 \text{ for all } \mathbf{s} \in S'\}\right). \tag{4.13}$$

Finally, for our applications below, the second claim in Theorem 4.4, applied to a compact subset $K \subset S' \times X_0$ makes sure that in the situation of Theorem 4.4 also the perturbed function φ_1 gives rise to a phase function

$$\Phi_1(\mathbf{s}, t, \mathbf{x}, \omega) = \omega\left(t - \varphi_1(\mathbf{s}, \mathbf{x})\right),$$

in the sense of Definition 2.2.

Remark 4.5. *The condition $\nabla_{\mathbf{x}}\varphi_0(\mathbf{s}, \mathbf{x}) \neq 0$ means $\nabla_{\mathbf{x}}\tau_0(\mathbf{x_s}(\mathbf{s}), \mathbf{x}) \neq -\nabla_{\mathbf{x}}\tau_0(\mathbf{x_r}(\mathbf{s}), \mathbf{x})$. By construction via the characteristic system (4.2), $\nabla_{\mathbf{x}}\tau_0(\mathbf{x_s}(\mathbf{s}), \mathbf{x}) = -\nabla_{\mathbf{x}}\tau_0(\mathbf{x_r}(\mathbf{s}), \mathbf{x})$ means that the ray emanating from $\mathbf{x_s}(\mathbf{s})$ and the ray emanating from $\mathbf{x_r}(\mathbf{s})$ meet smoothly at $\mathbf{x}$ (cp. [22,* p. 35]*) or, in other words, it means that the prolongation of the ray from $\mathbf{x_s}(\mathbf{s})$ to $\mathbf{x}$ eventually hits $\mathbf{x_r}(\mathbf{s})$. So we have to exclude points of Ξ_0 lying on trajectories of rays that directly connect source $\mathbf{x_s}(\mathbf{s})$ and receiver $\mathbf{x_r}(\mathbf{s})$. We mention here that the argument* [22, p. 35]*, where no points have to be excluded, relies on the assumption that the velocity v_0 is constant close to the surface (and on uniqueness of connecting rays).*

In our linear velocity model, where all (but one) of the rays starting from a fixed point $\mathbf{x}_0$ resurface, and in the common offset geometry with parameter $\alpha > 0$ source $\mathbf{x_s}(\mathbf{s})$ and receiver $\mathbf{x_r}(\mathbf{s})$ have a fixed distance 2α. Rays connecting $\mathbf{x_s}(\mathbf{s})$ and $\mathbf{x_r}(\mathbf{s})$ only reach a certain maximal depth. In other words, points in X_0 need to have a certain minimal depth, given by $x_{\min}$ in (3.4)*.*

5. Stability of the imaging operator with respect to phase function

The main result of this section, Theorem 5.2, asserts that if the phase functions of FIOs F_0 and F_1 are close enough in a precise way and if F_0 satisfies the Bolker condition, then F_1 also satisfies the Bolker condition. In section 5.3, we will apply this to the seismic operator with small offset and to operators with travel time close to ones for which the forward operator satisfies the Bolker condition.

5.1. **The setup.** First we provide some notation. Let U and V be subsets of $\mathbb{R}^d$. If $\mathrm{cl}(U)$ is compact, then we say U is *precompact*. If U is precompact and $\mathrm{cl}(U) \subset V$, then we write $U \Subset V$.

If M is a matrix in $\mathbb{R}^{m\times n}$, we define *the sup norm* of M, $\|M\|$, to be the maximum of the absolute values of the entries of M.

If G is a differentiable map with domain B and $\mathrm{cl}(A) \subset B$, then we will say G is an immersion (respectively, injective) on A if G is an immersion (respectively, injective) on some open neighborhood of $\mathrm{cl}(A)$. Let X be an open subset of $\mathbb{R}^d$ and $\ell \in \mathbb{N}$. Finally, let G be a C^ℓ function from X to either $\mathbb{R}^n$ or $\mathbb{R}^{m\times n}$ for some m, n. If $A \Subset X$, then for $m \in \mathbb{N}$, we let $\|G\|_{C^m(A)}$ denote the maximum of the sup norm of the component functions of G and their derivatives up to order m on A.

Let X be an open subset of $\mathbb{R}^2_+$, and let S' be an open subset of $\mathbb{R}$. For $j = 0, 1$, assume that F_j is an FIO from $\mathcal{E}'(X)$ to $\mathcal{D}'(S' \times (0, \infty))$ with phase function

$$\Phi_j(s, t, \mathbf{x}, \omega) = \omega(t - \varphi_j(s, \mathbf{x})), \tag{5.1}$$

where $\varphi_j \colon S' \times X \to (0, \infty)$ is smooth.

In this case, F_j is a FIO given by

$$F_j n(s, t) = \int \exp\left(\imath\, \Phi_j(s, t, \mathbf{x}, \omega)\right) \Theta_j(s, t, \mathbf{x}) n(\mathbf{x})\, \mathrm{d}\mathbf{x}\, \mathrm{d}\omega \tag{5.2}$$

where Θ_j is a symbol satisfying Definition 2.1.

The canonical relation for F_j is

$$\mathcal{C}_j = \{(s,t,-\omega\partial_s\varphi_j,\omega\,\mathrm{d}t;\mathbf{x},\omega\nabla_{\mathbf{x}}\varphi_j) : (s,\mathbf{x}) \in S'\times X, \omega\in\mathbb{R}\backslash\{0\}, t=\varphi_j(s,\mathbf{x})\},$$

and it can be given global coordinates

$$S'\times X\times\mathbb{R}\backslash\{0\} \ni (s,\mathbf{x},\omega) \mapsto (s,\varphi_j(s,\mathbf{x}),-\omega\partial_s\varphi_j(s,\mathbf{x}),\omega\,\mathrm{d}t;\mathbf{x},\omega\nabla_{\mathbf{x}}\varphi_j(s,\mathbf{x})) \in \mathcal{C}_j. \tag{5.3}$$

For $j=0,1$, we will let Π_{L}^j be the left projection from $\mathcal{C}_j$ to $T^*(S'\times(0,\infty))$ and Π_{R}^j be the corresponding right projection. Let A be a subset of $S'\times X$. Then the projection Π_{L}^j is injective (or an immersion) on $A\times\mathbb{R}\backslash\{0\}$ (using coordinates (5.3)) if and only if

$$P_j(s,\mathbf{x}) = (s,\varphi_j(s,\mathbf{x}),\partial_s\varphi_j(s,\mathbf{x})) \tag{5.4}$$

has the same property for $(s,\mathbf{x})\in A$. We introduce the function P_j to simplify the calculations since ω is given by the $\mathrm{d}t$ coordinate of Π_{L} in (5.3).

Note that if $A \Subset S'\times X$, then

$$\|\varphi_1-\varphi_0\|_{C^3(A)} \geq \|P_1-P_0\|_{C^2(A)}\,. \tag{5.5}$$

Our next proposition is a key to the proof of Theorem 5.2 below.

Proposition 5.1. *Let $S\Subset S'$ and $\Omega\Subset X$ both be open and let P_j be as given in (5.4) where F_j and φ_j are as given in (5.1) and (5.2) for $j=0,1$. Assume F_0 satisfies the Bolker condition. Then, there is a $\delta_1>0$ such that if*

$$\|\varphi_1-\varphi_0\|_{C^3(S'\times X)} < \delta_1, \tag{5.6}$$

then,

(a) P_1 is an immersion above $\mathrm{cl}(S\times\Omega)$, so $\mathcal{C}_1$ is a local canonical graph, and
(b) There is an $\varepsilon_1>0$ depending on δ_1 (and φ_0, S, S', Ω and X) such that for all $(s,\mathbf{x})\in\mathrm{cl}(S\times\Omega)$, the function $P_1\colon B_{\varepsilon_1}(s,\mathbf{x})\to\mathbb{R}^3$ is injective.

Proof. We need to take some estimates on a superset of $S\times\Omega$ with compact closure in $S'\times X$, so let $\tilde S$ and $\tilde\Omega$ be open sets such that $S\Subset\tilde S\Subset S'$ and $\Omega\Subset\tilde\Omega\Subset X$.

First, we prove part (a). Because F_0 satisfies the Bolker condition and $\mathrm{cl}(\tilde S\times\tilde\Omega)$ is compact, there is an $m>0$ such that the derivative matrix satisfies

$$|\det(DP_0(s,\mathbf{x}))| \geq m \quad \text{for all} \quad (s,\mathbf{x})\in\mathrm{cl}(\tilde S\times\tilde\Omega). \tag{5.7}$$

The determinant function is continuous on the space of 3×3 matrices in sup norm, so there is a $\delta_1>0$ such that

$$\begin{gathered}\text{if}\quad \|\varphi_1-\varphi_0\|_{C^3(\mathrm{cl}(\tilde S\times\tilde\Omega))} < 2\delta_1,\ \text{then}\\ \forall(s,\mathbf{x})\in\mathrm{cl}(\tilde S\times\tilde\Omega),\ |\det(DP_1(s,\mathbf{x}))-\det(DP_0(s,\mathbf{x}))| < m/2.\end{gathered} \tag{5.8}$$

Therefore, $|\det(DP_1(s,\mathbf{x}))|\geq m/2$ for all $(s,\mathbf{x})\in\mathrm{cl}(\tilde S\times\tilde\Omega)$, so P_1 is an immersion on a neighborhood of $\mathrm{cl}(\tilde S\times\tilde\Omega)$ and Π_{L}^1 is an immersion above $\tilde S\times\tilde\Omega$. Then, Π_{R}^1 must be an immersion above this set by [15, Proposition 4.1.4] because $\mathcal{C}_1$ is a Lagrangian manifold. Therefore, $\mathcal{C}_1$ is a local canonical graph above $\tilde S\times\tilde\Omega\times\mathbb{R}\backslash\{0\}$.

Without loss of generality, we will assume $\delta_1\leq 1$.

Now, we prove part (b). For $A\in\mathbb{R}^{3\times3}$, let $\|A\|_{\mathrm{Op}}$ denote the operator norm of the map $\mathbb{R}^3\ni\mathbf{y}\mapsto A\mathbf{y}$. Then, $A\mapsto\|A\|_{\mathrm{Op}}$ is a continuous map in the sup norm on $\mathbb{R}^{3\times3}$. This is true because all norms are equivalent on finite dimensional normed linear spaces, due to the Heine Borel Theorem.

Since the map $A \mapsto A^{-1}$ is continuous on $\mathrm{Gl}(3)$, we see that

$$A \mapsto \left\|A^{-1}\right\|_{\mathrm{Op}} \quad \text{is continuous on } \mathrm{Gl}(3). \tag{5.9}$$

We now prove that there is a $d > 0$ such that $\left\|DP_1^{-1}(s,\mathbf{x})\right\|_{\mathrm{Op}} \leq d$ for all $(s,\mathbf{x}) \in \mathrm{cl}(\tilde{S} \times \tilde{\Omega})$ and all φ_1 satisfying (5.6).

Let $C_0 = DP_0(\mathrm{cl}(\tilde{S} \times \tilde{\Omega}))$. As the derivative DP_0 is a continuous function, C_0 is a compact subset of $\mathrm{Gl}(3)$. Now, let C be the union of all closed balls of radius $\sqrt{3}\delta_1$ in $\mathbb{R}^{3\times 3}$ centered at points of C_0. Since C_0 is a compact subset of $\mathrm{Gl}(3)$, we may assume $C \subset \mathrm{Gl}(3)$ (by making δ_1 smaller if needed).

By (5.5), for each $(s,\mathbf{x}) \in \mathrm{cl}(\tilde{S} \times \tilde{\Omega})$, $|DP_1(s,\mathbf{x}) - DP_0(s,\mathbf{x})| \leq \sqrt{3}\delta_1$. Therefore, $DP_1(\mathrm{cl}(\tilde{S} \times \tilde{\Omega})) \subset C$. By (5.9), there is a maximum $d > 0$ on the compact set C to the continuous function $A \mapsto \|A^{-1}\|_{\mathrm{Op}}$. Therefore, for all $(s,\mathbf{x}) \in \mathrm{cl}(\tilde{S} \times \tilde{\Omega})$, $\left\|DP_1^{-1}(s,\mathbf{x})\right\|_{\mathrm{Op}} \leq d$.

Let $c = 1/d$, then

$$\forall (s,\mathbf{x}) \in \mathrm{cl}(\tilde{S} \times \tilde{\Omega}), \forall \mathbf{y} \in \mathbb{R}^3,\ \|(DP_1(s,\mathbf{x}))\mathbf{y}\| \geq c\,\|\mathbf{y}\|. \tag{5.10}$$

For the rest of the proof let $p^k_{ij}(s,\mathbf{x})$ denote the ij entry of the 3×3 matrix $DP^k_j(s,\mathbf{x})$ for $k = 0, 1$.

We claim there is an $L > 0$ depending only $\tilde{S} \times \tilde{\Omega}$ and φ_0 such that all first and second derivatives of φ_1 have Lipschitz norm bounded by L on any convex subset of $\mathrm{cl}(\tilde{S} \times \tilde{\Omega})$. First, by compactness of $\mathrm{cl}(\tilde{S} \times \tilde{\Omega})$, there is an $L' > 0$ such that all second and third derivatives of φ_0 are bounded above in sup norm by L'. Then, since (5.5) holds and $\delta_1 \leq 1$, all second and third derivatives of φ_1 are bounded above in absolute value by $L = L' + 1$. Then, a straightforward Mean Value Theorem argument shows that all first and second derivatives of φ_1 have Lipschitz norms bounded above by L on any convex subset of $\mathrm{cl}(\tilde{S} \times \tilde{\Omega})$.

This implies that

$$\left|p^1_{ij}(s,\mathbf{x}) - p^1_{ij}(t,\mathbf{y})\right| \leq L\,|(s,\mathbf{x}) - (t,\mathbf{y})| \tag{5.11}$$

for all $(s,\mathbf{x})$ and $(t,\mathbf{y})$ in any convex subset of $\mathrm{cl}(\tilde{S} \times \tilde{\Omega})$.

Let

$$\varepsilon_1 = \frac{c}{6L}. \tag{5.12}$$

Perhaps by making ε_1 smaller, we can assume that the open ball $B_{\varepsilon_1}(s,\mathbf{x}) \subset \tilde{S} \times \tilde{\Omega}$ for all $(s,\mathbf{x}) \in \mathrm{cl}(S \times \Omega)$. By (5.11), if $(s,\mathbf{x}) \in \mathrm{cl}(S \times \Omega)$ and $(t,\mathbf{y})$ is in the convex set $B_{\varepsilon_1}(s,\mathbf{x}) \subset \tilde{S} \times \tilde{\Omega}$, then

$$|(s,\mathbf{x}) - (t,\mathbf{y})| < \varepsilon_1 \implies \left|p^1_{ij}(s,\mathbf{x}) - p^1_{ij}(t,\mathbf{y})\right| < \frac{c}{6}. \tag{5.13}$$

This is exactly inequality (16.14) in the proof of Theorem 16.9 in [6] where the c in (5.10) is exactly the constant in inequality (16.13) in [6]. The rest of our proof follows word for word the proof of Theorem 16.9 in [6]. The conclusion of that theorem is that P_1 is injective on $B_{\varepsilon_1}(s,\mathbf{x})$. Since $(s,\mathbf{x}) \in \mathrm{cl}(S \times \Omega)$ is arbitrary, ε_1 is independent of $(s,\mathbf{x})$, and for every $(s,\mathbf{x}) \in \mathrm{cl}(S \times \Omega)$, P_1 is injective on $B_{\varepsilon_1}(s,\mathbf{x})$. ☐

5.2. **Nearby travel times and the Bolker condition.** We now state the main theorem of Section 5.

Theorem 5.2. *Let F_0 and F_1 be FIO from domain $\mathcal{E}'(X)$ to $\mathcal{D}'(S' \times \mathbb{R}_+)$ given by* (5.2). *We assume $F_0 \colon \mathcal{E}'(X) \to \mathcal{D}'(S' \times \mathbb{R}_+)$ satisfies the Bolker condition.*

Let $\Omega \Subset X$ and $S \Subset S'$ both be open. Then, there is a $\delta_0 > 0$ such that if

$$\|\varphi_1 - \varphi_0\|_{C^3(S' \times X)} < \delta_0 \tag{5.14}$$

then $F_1 \colon \mathcal{E}'(\Omega) \to \mathcal{D}'(S \times \mathbb{R}_+)$ satisfies the Bolker condition.

Proof. Our assumptions allow us to use Proposition 5.1 for $S \times \Omega \subset S' \times X$. Let δ_1 be as in (5.8) and let $\varepsilon_1 > 0$ be as in part (b) of Proposition 5.1 for $S \times \Omega$. Then, there is a $\delta_0 \in]0, \delta_1]$ such that

$$\begin{gathered} \text{for all } s \in S \text{ and } \mathbf{x} \text{ and } \mathbf{y} \text{ in } \Omega, \\ |P_0(s, \mathbf{x}) - P_0(s, \mathbf{y})| < 2\delta_0 \implies |(s, \mathbf{x}) - (s, \mathbf{y})| < \varepsilon_1. \end{gathered} \tag{5.15}$$

This is an immediate consequence of the fact that P_0^{-1} is uniformly continuous from the compact set $P_0\left(\mathrm{cl}(S \times \Omega)\right)$ to $\mathrm{cl}(S \times \Omega)$ because P_0 is a smooth injection on $S' \times X$ and therefore P_0^{-1} is also a smooth injection.

Now assume $\|\varphi_1 - \varphi_0\|_{C^3(S' \times X)} < \delta_0$. Let $(s, \mathbf{x})$ and $(t, \mathbf{y})$ be in Ω and assume $P_1(s, \mathbf{x}) = P_1(t, \mathbf{y})$. By the definition of P_1, $s = t$. Now, using (5.14) and the triangle inequality, one sees that $|P_0(s, \mathbf{x}) - P_0(s, \mathbf{y})| < 2\delta_0$. By (5.15), $|\mathbf{x} - \mathbf{y}| < \varepsilon_1$, so $(s, \mathbf{y}) \in B_{\varepsilon_1}(s, \mathbf{x})$. Since P_1 is injective on this ball by Proposition 5.1, $\mathbf{x} = \mathbf{y}$. Therefore, P_1 is injective on $S \times \Omega$ and $F_1 \colon \mathcal{E}'(\Omega) \to \mathcal{D}'(S \times \mathbb{R}_+)$ satisfies the Bolker condition. □

Somewhat related estimates have been used to prove injectivity of Radon transforms with measures that are close to real analytic measures using the injectivity of Radon transforms with those real analytic measures [8, Section 5].

5.3. **Applications of our stability results.** In this section, we apply the results of the previous sections to show that the Bolker condition holds for a broader range of operators than just those for the linear velocity model, as long as they are close to operators satisfying Bolker.

5.3.1. *Small Offset.* In Section 3, we showed that the seismic operator with linear wave speed in dimension two satisfies the Bolker condition for zero offset, $\alpha = 0$. We now show that the operator with sufficiently small offset $\alpha > 0$ also satisfies this condition. However, our theorem is more general, and we will prove it for any travel time for which the zero-offset operator satisfies the Bolker condition. The proof rests on the fact that φ_α for small offset, α, is close to φ_0 (which satisfies the Bolker condition), and this allows the use of Theorem 5.2.

Assume that the travel time τ is smooth from $\mathbb{R}^2_+ \times \partial(\mathbb{R}^2_+)$ to $(0, \infty)$. For $\alpha \geq 0$ define

$$\begin{gathered} \Phi_\alpha(s, t, \mathbf{x}, \omega) = \omega(t - \varphi_\alpha(s, \mathbf{x})), \\ \text{where} \\ \varphi_\alpha(s, \mathbf{x}) = \tau(\mathbf{x}, (s - \alpha, 0)) + \tau(\mathbf{x}, (s + \alpha, 0)) \end{gathered} \tag{5.16}$$

and let

$$F_\alpha n(s, t) = \int \exp\left(\imath\, \Phi_\alpha(s, t, \mathbf{x}, \omega)\right) \Theta_\alpha(s, t, \mathbf{x}) n(\mathbf{x})\, \mathrm{d}\mathbf{x}\, \mathrm{d}\omega \tag{5.17}$$

where Θ_α is a symbol according to Definition 2.1.

Theorem 5.3. *Let X be an open subset of $\mathbb{R}^2_+$. Using the notation of (5.16) (5.17), assume $F_0 \colon \mathcal{E}'(X) \to \mathcal{D}'(\mathbb{R} \times (0, \infty))$ is an FIO satisfying the Bolker condition. Let $\Omega \Subset X$ be open and let $S \Subset \mathbb{R}$ be open.*

Then, there is an $\alpha_0 > 0$ that depends on $\tau, S,\ X,$ and Ω, such that $F_\alpha \colon \mathcal{E}'(\Omega) \to \mathcal{D}'(S \times (0, \infty)$ satisfies the Bolker condition for all $\alpha \in [0, \alpha_0]$.

Remark 5.4. *Our theorem is valid in a somewhat more general setting. Assume that* $t_0 \geq 0$*,* $\overline{\alpha} > 0$*, and* $F_{\overline{\alpha}}\colon \mathcal{E}'(X) \to \mathcal{D}'(S' \times (t_0, \infty))$ *is an FIO satisfying the Bolker condition (assuming the function* $A(\mathbf{x}, \mathbf{x}_s)$ *in (1.6) is smooth). Then, our proof below shows there is a* $\delta > 0$ *depending on* $\varphi_{\overline{\alpha}}$*,* S'*,* X*,* S*, and* Ω *such that* $F_\alpha\colon \mathcal{E}'(\Omega) \to \mathcal{D}'(S \times (t_0, \infty))$ *also satisfies Bolker for* $|\alpha - \overline{\alpha}| < \delta$*. This is true because our proof rests on compactness and uniform continuity arguments that can be used to show* φ_α *is sufficiently close to* $\varphi_{\overline{\alpha}}$ *if* α *is sufficiently close to* $\overline{\alpha}$*.*

Theorem 5.3 can be applied to linear, increasing wave speed, (3.1) as we now discuss. We will write the constant $x_{\min}$ *in (3.4) as* $x_{\min} = x_{\min}(\alpha)$ *since its dependence on* α *is important.*

To apply Theorem 5.3, we first recall that F_0 *satisfies the Bolker condition as shown in Section 3.1. Then, we choose an* $\alpha_2 > 0$ *and choose* $x_0 > x_{\min}(\alpha_2)$*. We let*

$$X = \left\{(x_1, x_2) \,\middle|\, x_2 > x_0\right\}.$$

Therefore, isochrones intersect X *only for* $t > t_{\min}(\alpha_2)$*. This allows us to use the arguments in Section 3, including Remark 3.2, to assert* F_α *is an FIO for* $\alpha \in [0, \alpha_2]$*. Next, we apply Theorem 5.3 to conclude, for some* $\alpha_0 \in (0, \alpha_2]$*, that*

(5.18) $F_\alpha\colon \mathcal{E}'(\Omega) \to \mathcal{D}'(S \times (0, \infty))$ *satisfies the Bolker condition for all* $\alpha \in [0, \alpha_0]$*.*

The statement (5.18) includes no specific condition on $\mathbf{x}$*, but there is an implicit condition since* $\Omega \Subset X$*, so* x_2 *is bounded away from* x_0 *for all points in* Ω*. By the discussion in Section 3.2 for linear wave speed, the Bolker condition holds only if* $x_2 > x_{\min}(\alpha_0)$ *for all* $\mathbf{x} \in \Omega$*. Once* Ω *is chosen, this gives an implicit restriction on* α_0*, namely* $x_{\min}(\alpha_0) < x_2$ *for all* $\mathbf{x} \in \Omega$*.*

Proof of Theorem 5.3. As mentioned at the beginning of this section, all we need to show is that φ_α is sufficiently close to φ_0 for α sufficiently close to zero.

We first show that F_α is an FIO for sufficiently small α. When F_α is an FIO, its canonical relation is

$$\mathcal{C}_\alpha = \left\{(s, t, -\omega\partial_s\varphi_\alpha, \omega\,\mathrm{d}t; \mathbf{x}, \omega\nabla_{\mathbf{x}}\varphi_\alpha) : (s, \mathbf{x}) \in S \times \Omega, \omega \in \mathbb{R}\backslash\{0\}, t = \varphi_\alpha(s, \mathbf{x})\right\},$$

and $\mathcal{C}_\alpha$ can be given coordinates

$$(s, \mathbf{x}, \omega) \mapsto (s, \varphi_\alpha(s, \mathbf{x}), -\omega\partial_s\varphi_\alpha, \omega\,\mathrm{d}t; \mathbf{x}, \omega\nabla_{\mathbf{x}}\varphi_\alpha)\,. \tag{5.19}$$

Let $\alpha_2 > 0$. Let X be an open, precompact, convex set containing $\mathrm{cl}(\Omega)$ and let S' be an open, precompact, convex set containing $\mathrm{cl}(S + [-\alpha_2, \alpha_2])$. Then, $S' \times X$ is an open, precompact, convex set in $\mathbb{R}^3$.

Note that the symbol of F_0, $\frac{A(\mathbf{x}, \mathbf{x}_s)^2}{v^2(\mathbf{x})}$ (see (1.6)) is smooth by assumption. This means that the function $A(\mathbf{x}, \mathbf{x}_s)$ must be smooth for $s \in \mathbb{R}$ and $\mathbf{x} \in X$, and all α. Therefore, the symbol of F_α is smooth.

To show that Φ_α is a nondegenerate phase function on $S' \times (0, \infty) \times X \times \mathbb{R}\backslash\{0\}$ for small α, we note that $\frac{\partial}{\partial t}\Phi_\alpha = \omega$ is nonzero for all α. Therefore, we only need to check the $\mathbf{x}$ derivative. As F_0 is assumed to be an FIO satisfying Definition 2.2, $-\nabla_{\mathbf{x}}\Phi_0 = \omega\nabla_{\mathbf{x}}\varphi_0(s, \mathbf{x})$ is nowhere zero on $\mathbb{R} \times \mathbb{R}^2_+$.

Note that the differentiable map

$$[-\alpha_2, \alpha_2] \times (\mathrm{cl}(S') + [-\alpha_2, \alpha_2]) \times \mathrm{cl}(X) \ni (\alpha, s, \mathbf{x}) \mapsto \nabla_{\mathbf{x}}\tau(\mathbf{x}, (s - \alpha, \mathbf{x})$$

is uniformly continuous because this domain is compact. Therefore, $(\alpha, s, \mathbf{x}) \mapsto \nabla_{\mathbf{x}}\varphi_\alpha(s, \mathbf{x})$ is uniformly continuous on $[0, \alpha_2] \times \mathrm{cl}(S' \times X)$. Now, using uniform continuity and that $\nabla_{\mathbf{x}}\varphi_0$ is bounded away from zero on the compact set $\mathrm{cl}(S' \times X)$, there is an $\alpha_1 \in (0, \alpha_2]$ such that $\nabla_{\mathbf{x}}\varphi_\alpha(s, \mathbf{x})$ is bounded away from zero for all $(\alpha, s, \mathbf{x}) \in [0, \alpha_1] \times S' \times X$. This shows for

$\alpha \in [0, \alpha_1]$ that Φ_α is a nondegenerate phase function according to Definition 2.2, and F_α is an FIO from $\mathcal{E}'(X)$ to $\mathcal{D}'(S' \times (0, \infty))$ satisfying Definition 2.3.

Let δ_0 be as in Theorem 5.2 for this φ_0, S, S', Ω, and X. We now show that for some $\alpha_0 \in (0, \alpha_1]$,

$$\forall \alpha \in [0, \alpha_0],\ \|\varphi_\alpha - \varphi_0\|_{C^3(\mathrm{cl}(S' \times X))} < \delta_0. \tag{5.20}$$

This follows immediately since the function $(\alpha, s, \mathbf{x}) \mapsto \varphi_\alpha(s, \mathbf{x})$ and its derivatives up to order 3 in $(s, \mathbf{x})$ are uniformly continuous on the compact set $[0, \alpha_1] \times \mathrm{cl}(S' \times X)$, therefore there is an $\alpha_0 \in (0, \alpha_1]$ such that

$$\left|\partial^\beta \varphi_\alpha(s, \mathbf{x}) - \partial^\beta \varphi_0(s, \mathbf{x})\right| < \delta_0, \quad \text{for } \alpha \in [0, \alpha_0] \text{ and } (s, \mathbf{x}) \in \mathrm{cl}(S \times \Omega)$$

for all partial derivatives in $(s, \mathbf{x})$ up to order three, i.e., for $|\beta| \leq 3$. Taking the sup over all these derivatives shows that (5.20) holds for all $\alpha \in [0, \alpha_0]$.

By Theorem 5.2, this implies that $F_\alpha \colon \mathcal{E}'(\Omega) \to \mathcal{D}'(S \times (0, \infty)$ satisfies the Bolker condition for all $\alpha \in [0, \alpha_0]$. ☐

5.3.2. *Seismic operators with close traveltimes.* In this section we show that, if the velocities for two seismic experiments are close and the associated seismic operator for one satisfies the Bolker condition, then the other seismic operator does, too, as long as the operators agree near the surface, $\partial\mathbb{R}^d$. We use the results of Section 4 to relate the velocities to the traveltime and then use Theorem 5.2.

Theorem 5.5. *Let $v_0 \in C^\infty(\mathrm{cl}(\mathbb{R}^d_+))$ satisfy (4.4), (4.5), and (4.6). Let $S' \subset \mathbb{R}^{d-1}$ be open and assume that $\mathbf{x_s}, \mathbf{x_r} \colon S' \to \partial\mathbb{R}^d_+$ are smooth and satisfy (4.10). Let Ξ_0 be given by (4.12), assume that X_0 defined in (4.13) is not empty, and that v_0 induces an FIO $F_0 \colon \mathcal{E}'(X_0) \to \mathcal{D}'(S' \times (0, \infty))$ satisfying the Bolker condition.*

Let $v_1 \in C^\infty(\mathrm{cl}(\mathbb{R}^d_+))$ satisfy (4.4) and (4.7). Let $S \Subset S'$ and $\Omega \Subset X_0$ be open.

If $v_1 - v_0$ is sufficiently small in C^5-norm on the compact closure M of an open neighborhood $\tilde{M}$ of the ray closure (Definition 4.3) of $\mathrm{cl}(S \times \Omega)$ and if the amplitude function A_0 is smooth on $(\tilde{M} \cap \mathbb{R}^d_+) \times L$, where L is from (4.6) and (4.10), then v_1 induces an FIO $F_1 \colon \mathcal{E}'(\Omega) \to \mathcal{D}'(S \times (0, \infty))$ that satisfies the Bolker condition.

Proof. We choose $\tilde{S}'$ and X such that $S \Subset \tilde{S}' \Subset S'$, $\Omega \subset X \Subset X_0$, and M is still the closure of an open neighborhood of the ray closure of $\mathrm{cl}(\tilde{S}' \times X)$. We shall apply Theorem 5.2 to $F_j \colon \mathcal{E}'(X) \to \mathcal{D}'(\tilde{S}' \times (0, \infty))$, $j = 0, 1$. Clearly, $F_0 \colon \mathcal{E}'(X) \to \mathcal{D}'(\tilde{S}' \times (0, \infty))$ is an FIO that satisfies the Bolker condition.

We have to check that v_1 induces an FIO $F_1 \colon \mathcal{E}'(X) \to \mathcal{D}'(\tilde{S}' \times (0, \infty))$ and we have to make $\varphi_1 - \varphi_0$ small in C^3-norm on $\mathrm{cl}(\tilde{S}' \times X)$, which is a compact subset of $S' \times X_0$. To this end we use Theorem 4.4, which asserts the required smallness of $\varphi_1 - \varphi_0$, if $v_1 - v_0$ is sufficiently small in C^5-norm on M, which is the compact closure of an open neighborhood of the ray closure of $\mathrm{cl}(\tilde{S}' \times X)$.

Moreover, by Theorem 4.4 we also have that v_1 induces a phase function Φ_1 on $\mathrm{cl}(\tilde{S}' \times X)$. For the amplitude function $A_1(\mathbf{x}, \mathbf{x}_0)$ we solve the transport equation (1.5) (with A_1 and τ_1 in place of A and τ, respectively) as in [2, eq. (E3.9)], see also (A.1). Observe that the ray Jacobian appearing there is just the determinant of $D\Psi^j_{\mathbf{x}_0}(t, \xi')$ in our situation. As initial values we can take those of $A_0(\mathbf{x}, \mathbf{x}_0)$ for $\mathbf{x}$ with $x_d \in (0, \varepsilon_0)$ since $v_1(\mathbf{x}) = v_0(\mathbf{x})$ for such $\mathbf{x}$. Hence we conclude that v_1 gives rise to an FIO $F_1 \colon \mathcal{E}'(X) \to \mathcal{D}'(\tilde{S}' \times (0, \infty))$. Now application of Theorem 5.2 finishes the proof. ☐

APPENDIX A. THE AMPLITUDE FUNCTION

To find an explicit representation of the function $\Theta = \Theta(s, \mathbf{x})$, see (3.5), we will use (1.5) to get hold of A along seismic rays which are the characteristic curves of the eikonal equation (1.4), see, e.g., [2, Appendix E2]. We rely on the ray system (4.2) ($t \geq 0$ is the running parameter)

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{p}, \quad \mathbf{r}(0) = \mathbf{x_s}; \qquad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\frac{\nabla v(\mathbf{r})}{v^3(\mathbf{r})}, \quad \mathbf{p}(0) = \frac{\boldsymbol{\xi}}{v(\mathbf{r}(0))},$$

with a unit vector $\boldsymbol{\xi} = (\xi_1, \xi_2)^\top \in \mathbb{R}^2$ (we denote the rays here by $\mathbf{r}$ rather than by $\mathbf{x}$ as in (4.2) to comply with the notation of (3.5)). Note that $\mathbf{p}(t) = \nabla\tau(\mathbf{r}(t))$. Before we proceed with solving the system we establish the connection of the rays to the amplitude: via the divergence theorem follows from (1.5) that

$$A^2(\mathbf{r}(t), \mathbf{r}(0)) = \frac{\mathrm{C}_A^2}{|\det \mathrm{J}\mathbf{r}(t)|} \tag{A.1}$$

where $\mathrm{C}_A > 0$ is a suitable constant and $\mathrm{J}\mathbf{r}$ is the Jacobian of $\mathbf{r}$ with respect to t and a variable which parameterizes $\boldsymbol{\xi}$, the initial directions of the rays, see [2, eq. (E3.9)].

For the time being let $\mathbf{r}(0) = \mathbf{0}$ yielding $\mathbf{p}(0) = \boldsymbol{\xi}/b$. Then, the ray system has the following explicit solution

$$r_1(t) = p_1(0)t = \frac{\xi_1}{b}\, t, \quad r_2(t) = \frac{b}{a}\left(\sqrt{1 - \frac{a^2}{b^4}\xi_1^2\, t^2 + 2\frac{a}{b^2}\xi_2\, t} - 1\right),$$

which can be verified by plugging in. We are only interested in down-going rays, so that $\xi_2 > 0$. Further, $r_2 \geq 0$ for $t \in [0, t_{\max}]$ where $t_{\max} = \frac{2b^2}{a}\frac{\xi_2}{\xi_1^2}$, $\xi_1 \neq 0$ (these rays re-surface at $t_{\max}$). In case $\xi_1 \neq 0$, the orbits of the rays are arcs on the circles with centers $z = \left(\frac{b}{a}\frac{\xi_2}{\xi_1}, -\frac{b}{a}\right)$ and radii $R = |z|$. This fact was already reported in [21].

As $\xi_2 = \sqrt{1 - \xi_1^2}$ we use ξ_1 as additional parameter for the rays, that is, $\mathbf{r} = \mathbf{r}(t, \xi_1)$. Hence,

$$|\det \mathrm{J}\mathbf{r}| = |\partial_t r_1 \partial_{\xi_1} r_2 - \partial_t r_2 \partial_{\xi_1} r_1| = \frac{t}{\xi_2\sqrt{b^4 - a^2\xi_1^2\, t^2 + 2\, ab^2\xi_2\, t}}.$$

For given $\mathbf{x} \in X$, $x_1 \neq 0$, we now find the unique ray connecting $\mathbf{0}$ with $\mathbf{x}$. To this end we need to determine the corresponding ξ_1 where we use the feature that rays follow circular arcs with centers z as explained above. For symmetry reasons we may assume that $x_1 > 0$. Thus,

$$\left(x_2 + \frac{b}{a}\right)^2 + (x_1 - z_1)^2 = \frac{b^2}{a^2} + z_1^2$$

yielding

$$z_1 = \frac{1}{2x_1}\left(\left(x_2 + \frac{b}{a}\right)^2 + x_1^2 - \frac{b^2}{a^2}\right).$$

Since $z_1 = \frac{b}{a}\frac{\sqrt{1-\xi_1^2}}{\xi_1}$ we get

$$\begin{aligned}\xi_1 = \xi_1(\mathbf{x}) &= \frac{1}{\sqrt{1 + \frac{a^2}{4b^2\, x_1^2}\left((x_2 + \frac{b}{a})^2 + x_1^2 - \frac{b^2}{a^2}\right)^2}} \\ &= \frac{x_1}{|\mathbf{x}|\sqrt{1 + \frac{a}{b}x_2 + \frac{a^2}{4b^2}|\mathbf{x}|^2}}.\end{aligned}$$

Further,

$$\mathbf{r}(t(\mathbf{x}), \xi_1(\mathbf{x})) = \mathbf{x} \quad \text{for } t(\mathbf{x}) = \frac{b\,x_1}{\xi_1(\mathbf{x})} = b\,|\mathbf{x}|\sqrt{1 + \frac{a}{b}x_2 + \frac{a^2}{4b^2}|\mathbf{x}|^2}.$$

We conclude that

$$A^2(\mathbf{x}, \mathbf{0}) = \mathrm{C}_A^2\, \frac{\xi_2(\mathbf{x})\sqrt{b^4 - a^2\xi_1^2(\mathbf{x})\,t^2(\mathbf{x}) + 2\,ab^2\xi_2(\mathbf{x})\,t(\mathbf{x})}}{t(\mathbf{x})}. \tag{A.2}$$

For arbitrary source position we get

$$A(\mathbf{x}, \mathbf{x_s}(s)) = A(\mathbf{x} - \mathbf{x_s}(s), \mathbf{0}). \tag{A.3}$$

Remark A.1. *We do have that* $\lim_{a\searrow 0} A(\mathbf{x}, \mathbf{0}) = \mathrm{C}_A\sqrt{b}\sqrt{x_2}/|\mathbf{x}|$. *This limit is an amplitude belonging to the wave speed* $v(\mathbf{x}) = b$ *with travel time* $\tau(\mathbf{x}, \mathbf{0}) = |\mathbf{x}|/b$. *Indeed,* $\sqrt{x_2}/|\mathbf{x}|$ *solves* (1.5) *for that* τ. *Another solution is* $1/\sqrt{|\mathbf{x}|}$. *Observe that the quotient of both amplitudes is bounded from above and from below by positive constants on each cone given by* $|x_1| \le cx_2$ *where* $c > 0$. *In particular, both amplitudes have the same asymptotic behavior as* $x_2 \to \infty$ *while* x_1 *remains bounded.*

APPENDIX B. PROOF OF (3.19)

We set $\varepsilon = \vartheta - t/2 \in [-t/2, t/2]$ and express c_1 as well as c_2 as functions of ε. Then,

$$c_1 + c_2 = 2b(\cosh(at/2)\cosh(a\varepsilon) - 1)/a, \quad c_1c_2 = b^2(\cosh(a\varepsilon) - \cosh(at/2))^2/a^2.$$

Further,

$$(c_1 - c_2)^2 = R^2\sinh^2(a\varepsilon) \quad \text{with } R = 2b\sinh(at/2)/a.$$

It follows that

$$\Delta = b^2(R^2 - 4\alpha^2)\Big(1 + \frac{a^2\alpha^2}{b^2} - \cosh^2(a\varepsilon)\Big).$$

Hence, $\Delta \ge 0$ if and only if $t \ge t_{\min}$ and $\varepsilon \in [-\varepsilon_*, \varepsilon_*]$ where $\cosh^2(a\varepsilon_*) = 1 + a^2\alpha^2/b^2$, that is, $\varepsilon_* = t_{\min}/2$.

In this notation, (3.19) is equivalent to

$$\frac{2b\alpha\cosh(at/2)\cosh(a\varepsilon) - \sqrt{\Delta}}{4\alpha^2 + R^2\sinh^2(a\varepsilon)} < \frac{b}{2\alpha}\cosh(a\varepsilon_*) \quad \text{for } \varepsilon \in [-\varepsilon_*, \varepsilon_*]. \tag{B.1}$$

As the left hand side is even in ε we restrict our attention to $[0, \varepsilon_*]$. We observe (B.1) to hold for $\varepsilon = 0$ since

$$\begin{aligned} &2\alpha\cosh(at/2) - \sqrt{R^2 - 4\alpha^2}\,a\alpha/b < 2\alpha\cosh(a\varepsilon_*) \\ &\qquad \iff b(\cosh(at/2) - \cosh(a\varepsilon_*)) < a\sqrt{R^2/4 - \alpha^2} \end{aligned}$$

which is true for $t > t_{\min}$. Further, (B.1) is also true for $\varepsilon = \varepsilon_*$ and $t > t_{\min}$ according to

$$\begin{aligned} &4\alpha^2\cosh(at/2)\cosh(a\varepsilon_*) < 4\alpha^2 + 4\alpha^2\big(\cosh^2(at/2) - \cosh^2(a\varepsilon_*)\big) \\ &\qquad \iff \cosh(at/2)\cosh(a\varepsilon_*) < 1 + \cosh^2(at/2) - \cosh^2(a\varepsilon_*). \end{aligned}$$

To validate the general case we rewrite (B.1) equivalently into

$$R^2\cosh(a\varepsilon_*) \le f(\varepsilon)$$

with

$$f(\varepsilon) := R^2\cosh(a\varepsilon_*)\cosh^2(a\varepsilon) - 4\alpha^2\big(\cosh(at/2) - \cosh(a\varepsilon_*)\big)\cosh(a\varepsilon) + 2\alpha\sqrt{\Delta}/b.$$

We have just established that

$$f(0) \geq R^2 \cosh(a\varepsilon_*) \quad \text{and} \quad f(\varepsilon_*) \geq R^2 \cosh(a\varepsilon_*).$$

To finish the proof of (B.1) we consider the derivative of f:

$$f'(\varepsilon) = 2a \sinh(a\varepsilon)\big(g(\varepsilon) + h(\varepsilon)\big)$$

where

$$g(\varepsilon) = R^2 \cosh(a\varepsilon_*) \cosh(a\varepsilon) - 2\alpha^2\big(\cosh(at/2) - \cosh(a\varepsilon_*)\big)$$

and

$$h(\varepsilon) = -\frac{\alpha\sqrt{R^2 - 4\alpha^2}\cosh(a\varepsilon)}{\sqrt{\cosh^2(a\varepsilon_*) - \cosh^2(a\varepsilon)}}.$$

The function g is positive and strictly increasing whereas h is negative and strictly decreasing to $-\infty$ in $[0, \varepsilon_*[$. Further, there is at most one $\widetilde{\varepsilon} \in]0, \varepsilon_*[$ such that $g(\widetilde{\varepsilon}) = -h(\widetilde{\varepsilon})$.

We distinguish three cases.

(1) $g(0) + h(0) > 0$: Then, $g + h > 0$ in a neighborhood of $\varepsilon = 0$ and $\widetilde{\varepsilon}$ exists, that is, $g + h \geq 0$ on $[0, \widetilde{\varepsilon}]$. Hence, $f'|_{[0,\widetilde{\varepsilon}]} \geq 0$ on $[0, \widetilde{\varepsilon}]$ and $f|_{[0,\widetilde{\varepsilon}]} \geq R^2 \cosh(a\varepsilon_*)$ since $f(0) \geq R^2 \cosh(a\varepsilon_*)$. However, $f|_{[\widetilde{\varepsilon},\varepsilon_*]} \geq R^2 \cosh(a\varepsilon_*)$ as well since $g+h$ is negative on $[\widetilde{\varepsilon}, \varepsilon_*[$ implying $f'|_{[\widetilde{\varepsilon},\varepsilon_*[} \leq 0$ which yields the stated estimate by $f(\varepsilon_*) \geq R^2 \cosh(a\varepsilon_*)$.

(2) $g(0) + h(0) < 0$: Then, $g + h < 0$ in a neighborhood of $\varepsilon = 0$ which readily implies that $g + h \leq 0$ on all of $[0, \varepsilon_*]$ because $g + h$ has one zero at most and has to approach $-\infty$. Hence, $f'|_{[0,\varepsilon_*[} \leq 0$ and $f|_{[0,\varepsilon_*]} \leq R^2 \cosh(a\varepsilon_*)$ by $f(\varepsilon_*) \geq R^2 \cosh(a\varepsilon_*)$.

(3) $g(0) + h(0) = 0$: Then, either $g + h > 0$ or $g + h < 0$ in a neighborhood of $\varepsilon = 0$ and we can proceed as in (1) or (2), respectively.

Appendix C. On the injectivity of Π_{L} for positive offset

Here we prove injectivity of the function $\partial_s\varphi(0, \mathbf{x}_t^+(\cdot))$ over $[\vartheta_{\min,2}, \vartheta_{\max,2}]$, see (3.22), if t is sufficiently large.

Using the notation of Appendix B we get, for $\varepsilon \in [-\varepsilon_*, \varepsilon_*]$,

$$\begin{aligned} f(\varepsilon) := \partial_s\varphi\big(0, \mathbf{x}_t^+(t/2 + \varepsilon)\big) &= \frac{4\sinh(a\varepsilon)}{(R^2 - T^2)\sinh^2(a\varepsilon) + R^2} \\ &\quad \times \left(\frac{R^2}{a\alpha}\cosh(a\varepsilon) - \frac{T(4\alpha^2 + R^2\sinh^2(a\varepsilon))}{a\alpha T\cosh(a\varepsilon) + b\sqrt{R^2 - 4\alpha^2}\sqrt{\cosh^2(a\varepsilon_*) - \cosh^2(a\varepsilon)}}\right) \\ &= \underbrace{\frac{4\sinh(a\varepsilon)/a/\alpha}{(1 - T^2/R^2)\sinh^2(a\varepsilon) + 1}}_{=:\, g(\varepsilon)} \\ &\quad \times \underbrace{\frac{a\alpha T(R^2 - 4\alpha^2) + \cosh(a\varepsilon) bR^2\sqrt{R^2 - 4\alpha^2}\sqrt{\cosh^2(a\varepsilon_*) - \cosh^2(a\varepsilon)}}{a\alpha TR^2\cosh(a\varepsilon) + bR^2\sqrt{R^2 - 4\alpha^2}\sqrt{\cosh^2(a\varepsilon_*) - \cosh^2(a\varepsilon)}}}_{=:\, h(\varepsilon)} \end{aligned}$$

where $T = 2b\cosh(at/2)/a$. Since $T/R \geq 1$ the function g is strictly increasing in $[0, \varepsilon_*]$ with $g(\varepsilon_*) = 4/((1 - T^2/R^2)a^2\alpha^2/b + b)$. Further, the function h is strictly decreasing in $[0, \varepsilon_*]$ from $h(0) = 1 - 4\alpha^2 T/(T + \sqrt{R^2 - 4\alpha^2})/R^2$ to $h(\varepsilon_*) = (R^2 - 4\alpha^2)/R^2/\sqrt{1 + a^2\alpha^2/b^2}$.

Further, $h'(0) = 0$ and $\lim_{\varepsilon\to\varepsilon_*} h'(\varepsilon) = -\infty$. Hence, f strictly increases to its maximal value attained at an $\varepsilon_{\max} \in]0, \varepsilon_*[$ and then decreases strictly on $[\varepsilon_{\max}, \varepsilon_*]$.

To validate the required injectivity we need to show that $\varepsilon_{*,2} := \vartheta_{\max,2} - t/2$ is less or equal to $\varepsilon_{\max}$. However, explicit values for both, $\varepsilon_{\max}$ and $\varepsilon_{*,2}$, are hard to find. We can, nevertheless, guarantee that $\varepsilon_{*,2} \leq \varepsilon_{\max}$ for large t. Indeed, by $\lim_{t\to\infty} T/R = 1$ and $\lim_{t\to\infty} R = \infty$ we find f to converge uniformly on $[0, \varepsilon_*]$ to

$$f_\infty(\varepsilon) := \frac{4\sinh(a\varepsilon)}{a\alpha}\,\frac{a\alpha + \cosh(a\varepsilon) b\sqrt{\cosh^2(a\varepsilon_*) - \cosh^2(a\varepsilon)}}{a\alpha\cosh(a\varepsilon) + b\sqrt{\cosh^2(a\varepsilon_*) - \cosh^2(a\varepsilon)}}.$$

This limit function attains its maximum, say, at $\varepsilon_{\max}^\infty$ and $\varepsilon_{\max} \approx \varepsilon_{\max}^\infty$ for large t. Since

$$\varepsilon_{*,2} \to 0 \text{ for } t \to \infty \tag{C.1}$$

and $\varepsilon_{\max}^\infty$ only depends on a, b, and α, we have the claimed injectivity for large t.

We close this section with a proof of (C.1). Recall that $\varepsilon_{*,2}$ is the positive solution of $x_2^+(t/2+\varepsilon) = x_{\min}$ with $x_{\min}$ from (3.18). In view of (3.19) and (3.20) (where the minus sign in front of the square root has to be replaced by a plus sign) this equation reads

$$\sqrt{1 + \frac{a^2\alpha^2}{b^2}} = \underbrace{2\alpha\,\frac{a\alpha T\cosh(a\varepsilon)/b + \sqrt{R^2 - 4\alpha^2}\sqrt{\cosh^2(a\varepsilon_*) - \cosh^2(a\varepsilon)}}{R^2\sinh^2(a\varepsilon) + 4\alpha^2}}_{=:\,D(\varepsilon)}. \tag{C.2}$$

For D we have $D(0) = \cosh(at/2) + \sqrt{\sinh^2(at/2) - a^2\alpha^2/b^2} \to \infty$ as $t \to \infty$ but $D(\varepsilon) \to 0$ as $t \to \infty$ for any $\varepsilon \neq 0$. Hence, (C.1) holds true.

References

[1] G. Beylkin, *Imaging of discontinuities in the inverse scattering problem by inversion of a causal generalized Radon transform*, J. Math. Phys., 26 (1985), pp. 99–108, http://dx.doi.org/10.1063/1.526755.

[2] N. Bleistein, J. K. Cohen, and J. W. Stockwell, Jr., *Mathematics of multidimensional seismic imaging, migration, and inversion*, vol. 13 of Interdisciplinary Applied Mathematics, Springer-Verlag, New York, 2001, http://dx.doi.org/10.1007/978-1-4613-0001-4. Geophysics and Planetary Sciences.

[3] L. Borg, J. Frikel, J. S. Jørgensen, and E. T. Quinto, *Analyzing reconstruction artifacts from arbitrary incomplete X-ray CT data*, SIAM J. Imaging Sci., 11 (2018), pp. 2786–2814, https://doi.org/10.1137/18M1166833.

[4] T. Brocher, *A regional view of urban sedimentary basins in northern California based on oil industry compressional-wave velocity and density logs*, Bull. Seismol. Soc. Am., 95 (2005), p. 2093–2114, http://dx.doi.org/10.1785/0120050025.

[5] R. Courant and D. Hilbert, *Methods of mathematical physics. Vol. II*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1989. Partial differential equations, Reprint of the 1962 original, A Wiley-Interscience Publication.

[6] P. M. Fitzpatrick, *Advanced Calculus, Second Edition*, American Mathematical Society, Providence, RI, USA, 2009.

[7] F. G. Friedlander, *Sound pulses*, Cambridge University Press, New York, 1958.

[8] B. Frigyik, P. Stefanov, and G. Uhlmann, *The X-ray transform for a generic family of curves and weights*, J. Geom. Anal., 18 (2008), pp. 81–97, https://doi.org/10.1007/s12220-007-9007-6.

[9] M. GIAQUINTA AND S. HILDEBRANDT, *Calculus of variations.* II, vol. 311 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 1996. The Hamiltonian formalism.
[10] C. GRATHWOHL, *Seismic imaging with the elliptic Radon transform in 3D: analytical and numerical aspects*, PhD thesis, Karlsruhe Institute of Technology, 2020, https://doi.org/10.5445/IR/1000105093.
[11] C. GRATHWOHL, P. KUNSTMANN, E. T. QUINTO, AND A. RIEDER, *Approximate inverse for the common offset acquisition geometry in 2D seismic imaging*, Inverse Problems, 34 (2018), p. 014002, https://doi.org/10.1088/1361-6420/aa9900.
[12] C. GRATHWOHL, P. KUNSTMANN, E. T. QUINTO, AND A. RIEDER, *Microlocal analysis of imaging operators for effective common offset seismic reconstruction*, Inverse Problems, 34 (2018), pp. 114001, 24, https://doi.org/10.1088/1361-6420/aadc2a.
[13] C. GRATHWOHL, P. KUNSTMANN, E. T. QUINTO, AND A. RIEDER, *Imaging with the elliptic Radon transform in three dimensions from an analytical and numerical perspective*, SIAM J. Imaging Sci., 13 (2020), pp. 2250–2280, https://doi.org/10.1137/20M1332657.
[14] V. GUILLEMIN AND S. STERNBERG, *Geometric asymptotics*, American Mathematical Society, Providence, R.I., 1977. Mathematical Surveys, No. 14.
[15] L. HÖRMANDER, *Fourier integral operators. I*, Acta Math., 127 (1971), pp. 79–183, http://link.springer.com/article/10.1007/BF02392052.
[16] K. V. MACKENZIE, *Nine-term equation for sound speed in the oceans*, The Journal of the Acoustical Society of America, 70 (1981), pp. 807–812, https://doi.org/10.1121/1.386920.
[17] F. NATTERER AND F. WÜBBELING, *Mathematical methods in image reconstruction*, SIAM Monographs on Mathematical Modeling and Computation, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001, https://doi.org/10.1137/1.9780898718324.
[18] B. E. PETERSEN, *Introduction to the Fourier transform and pseudo-differential operators*, vol. 19 of Monographs and Studies in Mathematics, Pitman (Advanced Publishing Program), Boston, MA, 1983.
[19] E. T. QUINTO, *The dependence of the generalized Radon transform on defining measures*, Trans. Amer. Math. Soc., 257 (1980), pp. 331–346, https://doi.org/10.2307/1998299.
[20] E. T. QUINTO, A. RIEDER, AND T. SCHUSTER, *Local inversion of the sonar transform regularized by the approximate inverse*, Inverse Problems, 27 (2011), pp. 035006, 18, http://dx.doi.org/10.1088/0266-5611/27/3/035006.
[21] M. M. SLOTNICK, *Lessons in Seismic Computing*, Soc. of Expl. Geophys., Tulsa, Oklahoma, 1959, https://doi.org/10.1190/1.9781560802563. Edited by R. A. Geyer.
[22] W. W. SYMES, *Mathematics of reflection seismology*, tech. report, The Rice Inversion Project, Rice University, Houston, TX, USA, 1998, http://www.trip.caam.rice.edu/downloads/preamble.pdf.
[23] F. TRÈVES, *Introduction to pseudodifferential and Fourier integral operators. Vol. 1*, Plenum Press, New York-London, 1980. Pseudodifferential operators, The University Series in Mathematics.
[24] F. TRÈVES, *Introduction to pseudodifferential and Fourier integral operators. Vol. 2*, Plenum Press, New York-London, 1980. Fourier integral operators, The University Series in Mathematics.

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