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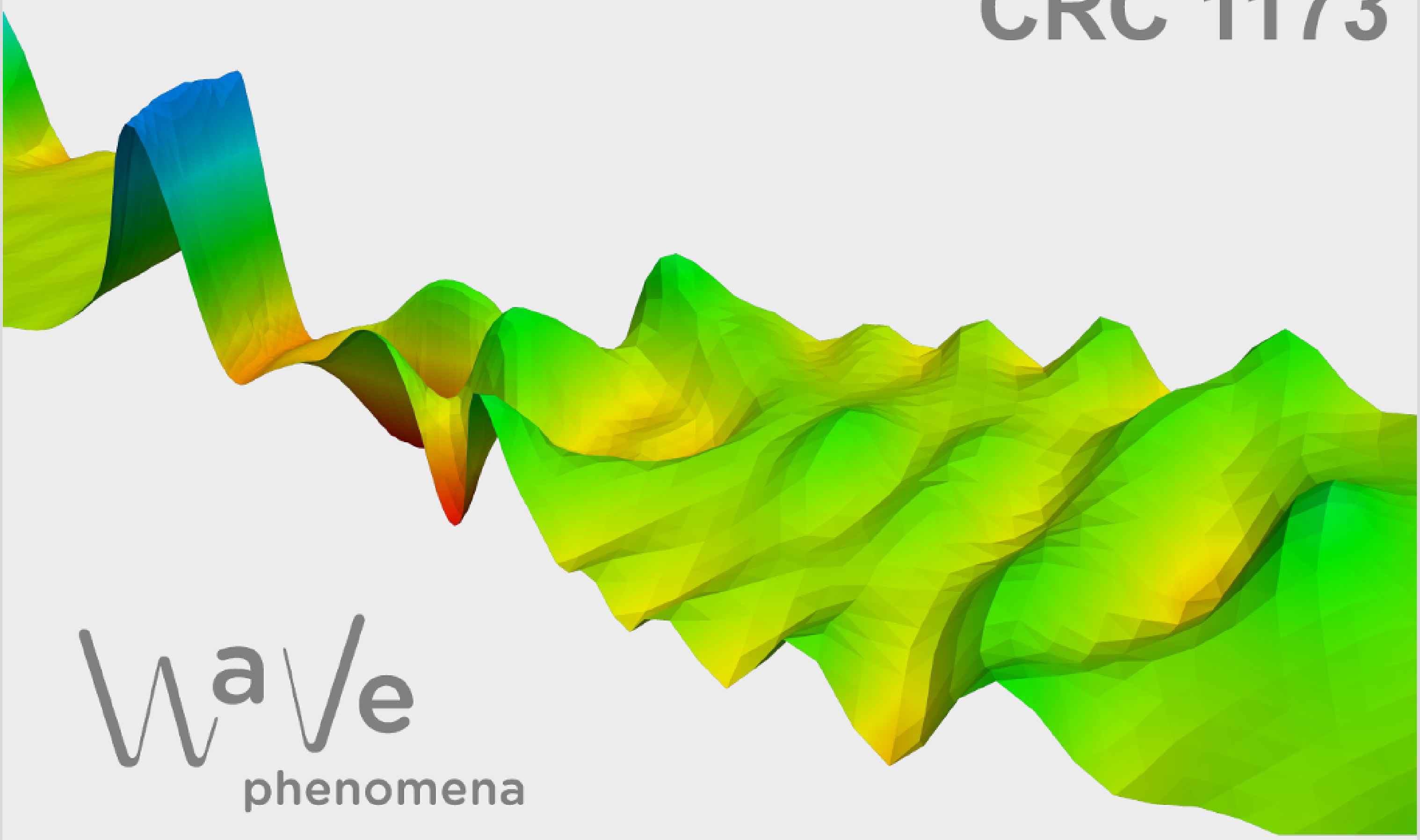

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ON GAGLIARDO-NIRENBERG INEQUALITIES WITH VANISHING SYMBOLS

RAINER MANDEL[1]

Abstract. We prove interpolation inequalities of Gagliardo-Nirenberg type involving Fourier symbols that vanish on hypersurfaces in $\mathbb{R}^d$.

1. Introduction

In a recent paper by Fernández, Jeanjean, Mariş and the author the following inequality of Gagliardo-Nirenberg type was proved

$$\|u\|_q \lesssim \|(|D|^s - 1)u\|_2^{1-\kappa} \|u\|_2^{\kappa} \qquad (u \in \mathcal{S}(\mathbb{R}^d)). \tag{1}$$

Here, $(|D|^s - 1)u = \mathcal{F}^{-1}((|\xi|^s - 1)\hat{u}(\xi))$, the symbol $\lesssim$ stands for $\leq C$ for some positive number C independent of u and the parameters $s > 0, \kappa \geq \frac{1}{2}, 2 < q < \infty, d \geq 2$ are supposed to satisfy $\frac{d}{s}(\frac{1}{2} - \frac{1}{q}) \leq 1 - \kappa \leq \frac{d+1}{2}(\frac{1}{2} - \frac{1}{q})$, see [14, Theorem 2.6]. In this paper we investigate such inequalities in greater generality both by extending the analysis to a larger class of exponents but also allowing for more general Fourier symbols. We expect applications in the context of normalized solutions of elliptic PDEs and orbital stability [2, 9, 24] or long-time behaviour [26] of time-dependent PDEs just as the classical Gagliardo-Nirenberg Inequality [23]. In [14] and [21] applications of (1) to variational existence results and symmetry breaking phenomena for biharmonic nonlinear Schrödinger equations are given. Further interesting research directions regard optimal constants as well as the existence and qualitative properties of maximizers in such inequalities as in [3, 12, 20, 26, 27]. We refer to [6, 7, 11] for interpolation inequalities in different spaces like Lorentz spaces, Besov spaces, BMO or weighted Lebesgue spaces.

We shall be concerned with inequalities of the form

$$\|u\|_q \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa} \|P_2(D)u\|_{r_2}^{\kappa} \tag{2}$$

where $q, r_1, r_2 \in (1, \infty), \kappa \in (0, 1)$ and P_1, P_2 are smooth Fourier symbols that may vanish on some smooth compact hypersurface $S \subset \mathbb{R}^d$ with at least $k \in \{0, \dots, d-1\}$ non-vanishing principal curvatures in each point. We will assume that P_i vanishes of order $\alpha_i \leq 1$ on S and behaves like $|\cdot|^{s_i}$ at infinity, see Assumption $(A1), (A2)$ below for a precise statement. This covers (1) as a special case where $(\alpha_1, \alpha_2, s_1, s_2) = (1, 0, s, 0)$ and S is the unit sphere in $\mathbb{R}^d$, so $k = d - 1$. As an application of our results for (2) we obtain the following generalization of [14, Theorem 2.3] to general exponents $(r_1, r_2) \neq (2, 2)$.

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Theorem 1. *Assume* $d = 1, \kappa \in (0,1), s > 0$. *Then*

$$\|u\|_q \lesssim \|(|D|^s - 1)u\|_{r_1}^{1-\kappa} \|u\|_{r_2}^{\kappa} \qquad (u \in \mathcal{S}(\mathbb{R}))$$

holds provided that $q, r_1, r_2 \in (1,\infty)$ *satisfy*

$$1 - \kappa \le \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} \le (1-\kappa)s.$$

In the higher-dimensional case we restrict ourselves to $r_1 = r_2 = r \in (1,2]$ and $q \in [2,\infty)$ to avoid heavy notation. Our generalization of [14, Theorem 2.6] reads as follows.

Theorem 2. *Assume* $d \in \mathbb{N}, d \ge 2, \kappa \in (0,1), s > 0$. *Then*

$$\|u\|_q \lesssim \|(|D|^s - 1)u\|_{r}^{1-\kappa} \|u\|_{r}^{\kappa} \qquad (u \in \mathcal{S}(\mathbb{R}^d))$$

holds provided that the exponents $r, q \in (1,\infty)$ *satisfy*

$$\frac{2(1-\kappa)}{d+1} \le \frac{1}{r} - \frac{1}{q} \le \frac{(1-\kappa)s}{d} \quad \text{and} \quad \frac{1}{2} \ge \frac{1}{r}, \frac{1}{q'} \ge \frac{d+1-2\kappa}{2d}.$$

Both results arise as special cases of Theorem 3 and Theorem 4 where the inequalities (2) are proved for suitable exponents $q, r_1, r_2 \in (1,\infty)$ and symbols $P_1, P_2 \in C^\infty(\mathbb{R}^d)$ that satisfy the following conditions:

- $(A1)$ There is a compact hypersurface $S = \{\xi \in \mathbb{R}^d : F(\xi) = 0\}$ with $F \in C^\infty(\mathbb{R}^d)$, $|\nabla F| \neq 0$ on S and at least k nonvanishing principal curvatures in each point, $\{P_i(\xi) = 0\} \subset S$ and $P_i(\xi) = a_{i+}(\xi)F(\xi)_+^{\alpha_i} + a_{i-}(\xi)F(\xi)_-^{\alpha_i}$ for smooth functions a_{i+}, a_{i-} near S where $\alpha_i \le 1$.
- (A2) For any open neighbourhood of S there is $C > 0$ such that $|\partial^\alpha(1/P_i(\xi))| \le C(1 + |\xi|)^{-s_i - |\alpha|}$ holds for $|\alpha| \le \lceil \frac{d}{2} \rceil + 1$ outside of this neighbourhood where $s_1, s_2 \in \mathbb{R}$.

In the case $d = 1$ assumption $(A1)$ is supposed to mean $S = \{\xi \in \mathbb{R} : F(\xi) = 0\} = \{\xi_1, \ldots, \xi_m\}$ with $F \in C^\infty(\mathbb{R})$, $F' \neq 0$ on S, $\{P_i(\xi) = 0\} \subset S$ and $P_i(\xi) = a_{i+}(\xi)F(\xi)_+^{\alpha_i} + a_{i-}(\xi)F(\xi)_-^{\alpha_i}$ for smooth functions a_{i+}, a_{i-} near S. Here, $F(\xi)_+ = \max\{F(\xi), 0\}$ and $F(\xi)_- = -\min\{F(\xi), 0\}$. The probably most relevant examples are given by $P_i(\xi) = |F(\xi)|^{\alpha_i}$ or $P_i(\xi) = |F(\xi)|^{\alpha_i - 1}F(\xi)$.

Our approach makes use of estimates from [10, 22] that, roughly speaking, can be used to find estimates of the form $\|u\|_q \lesssim \|P_1(D)u\|_r$. We will comment on such inequalities in Remark 2. It turns out that it is not sufficient to interpolate these estimates naively (using the Riesz-Thorin Theorem, say) to get satisfactory results which at least reproduce the above-mentioned results from [14]. For this reason we split up the corresponding operators dyadically, both for frequencies close to S and at infinity. A combination of the resulting estimates will allow to conclude. We stress that the proof from [14] does not carry over from the $L^2(\mathbb{R}^d)$-setting since Plancherel's Theorem does not have a counterpart in $L^r(\mathbb{R}^d)$ with $r \neq 2$.

2. Preliminaries

In the following we decompose a given Schwartz function $u \in \mathcal{S}(\mathbb{R}^d)$ in frequency space. We start by separating the frequencies close to the critical surface from the others by defining

$$u_1 := \mathcal{F}^{-1}(\tau\hat{u}),\ u_2 := \mathcal{F}^{-1}((1-\tau)\hat{u}) \quad \text{where } \tau \in C_0^\infty(\mathbb{R}^d),\ \tau = 1 \text{ near } S. \tag{3}$$

This function τ is considered as fixed from now on. It is not surprising that our analysis related to u_1 only involves the parameters α_1, α_2 that measure how $P_i(\xi)$ vanishes as $\operatorname{dist}(\xi, S) \to 0$. Accordingly, the parameters s_1, s_2 only play a role in our estimates involving u_2. For both u_1 and u_2 we introduce a dyadic decomposition into infinitely many annular regions in order to prove our estimates via Bourgain's summation argument [5]. We will need the following abstract version of this result from [8, p.604].

Lemma 1. *Let $\beta_1, \beta_2 \in \mathbb{R}, \theta \in (0,1)$, let (X_1, X_2) and (Y_1, Y_2) be real interpolation pairs of Banach spaces. For $j \in \mathbb{N}$ let $\mathcal{T}_j$ be linear operators satisfying*

$$\|\mathcal{T}_j f\|_{Y_1} \leq M_1 2^{\beta_1 j}\|f\|_{X_1}, \qquad \|\mathcal{T}_j f\|_{Y_2} \leq M_2 2^{\beta_2 j}\|f\|_{X_2}.$$

Then we have

$$\|\sum_{j\in\mathbb{N}} \mathcal{T}_j f\|_{(Y_1,Y_2)_{\theta,\infty}} \leq C M_1^{1-\theta} M_2^{\theta} \|f\|_{(X_1,X_2)_{\theta,1}} \tag{4}$$

provided that $(1-\theta)\beta_1 + \theta\beta_2 = 0$ with $\beta_1, \beta_2 \neq 0$. In the case $(1-\theta)\beta_1 + \theta\beta_2 < 0$ we have for all $r \in [1,\infty]$

$$\|\sum_{j\in\mathbb{N}} \mathcal{T}_j f\|_{(Y_1,Y_2)_{\theta,r}} \leq C M_1^{1-\theta} M_2^{\theta} \|f\|_{(X_1,X_2)_{\theta,r}}. \tag{5}$$

The whole point of this result is (4); the estimate (5) is a rather trivial consequence of the summability (over $\mathbb{N}$) of the interpolated bounds

$$\|\mathcal{T}_j f\|_{(Y_1,Y_2)_{\theta,r}} \lesssim 2^{j(\beta_1(1-\theta)+\beta_2\theta)} \|f\|_{(X_1,X_2)_{\theta,r}} \quad \text{for all } r \in [1,\infty].$$

Here, $(Y_1, Y_2)_{\theta,r}, (X_1, X_2)_{\theta,r}$ denote real interpolation spaces [4]. In our context, this Lemma will be applied to the Banach spaces $Y_i := L^{q_i}(\mathbb{R}^d)$ where the exponents q_1, q_2 are supposed to satisfy $\frac{1}{q} = \frac{1-\kappa}{q_1} + \frac{\kappa}{q_2}$. The spaces X_i are defined as the completion of the Schwartz functions $\mathcal{S}(\mathbb{R}^d)$ with respect to $\|u\|_{X_i} := \|P_i(D)u\|_{r_i}$, which we will abbreviate by $X_i := P_i(D)^{-1}L^{r_i}(\mathbb{R}^d)$. Note that $P_i(D)^{-1}$ is meaningful in the sense of distributions whenever the parameters α_1, α_2 from (A1) are smaller than 1 because then the singularity of the Fourier symbol is integrable. Moreover, for any given Schwartz function $\|u\|_{X_i} = 0$ holds if and only if $u = 0$. Hence, $\|\cdot\|_{X_i}$ is a well-defined norm on the set of Schwartz functions. Instead of defining the spaces X_i in the case $\max\{\alpha_1, \alpha_2\} = 1$, we will treat this case simply by passing to the limit $\max\{\alpha_1, \alpha_2\} \nearrow 1$ in our final estimate (2). Here, we will use that our bounds depend locally uniformly on $\alpha_1, \alpha_2 \in (-\infty, 1]$.

The link to Gagliardo-Nirenberg-type inequalities is provided by the general interpolation property [4, Theorem 3.1.2], namely

$$\|f\|_{(X_1,X_2)_{\kappa,r}} \leq \|f\|_{X_1}^{1-\kappa}\|f\|_{X_2}^{\kappa} \qquad (0 < \kappa < 1, 1 \leq r \leq \infty).$$

In fact, choosing X_1, X_2 as above we obtain for $u \in \mathcal{S}(\mathbb{R}^d)$

$$\|u\|_{(X_1,X_2)_{\kappa,r}} \leq \|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa} \qquad (0<\kappa<1, 1\leq r\leq\infty). \tag{6}$$

3. Dyadic decompositions and related estimates

We first provide the dyadic decomposition in Fourier space related to frequencies away from the critical surface. To this end we use a dyadic partition of unity, i.e., we choose

$$\eta \in C_0^\infty(\mathbb{R}), \quad \operatorname{supp}(\eta) \subset [-2,-\frac{1}{2}]\cup[\frac{1}{2},2], \quad \sum_{j\in\mathbb{Z}} \eta(2^j\cdot) = 1 \text{ almost everywhere on } \mathbb{R} \tag{7}$$

(see [4, Lemma 6.1.7]) and define for some fixed $\xi_0 \in S$

$$\begin{aligned} T_j f &:= \mathcal{F}^{-1}\left(\eta(2^j|\xi-\xi_0|)\hat{f}\right) = K_j * f \\ \text{where} \quad K_j(x) &:= \mathcal{F}^{-1}\left(\eta(2^j|\xi-\xi_0|)\right)(x) = 2^{-jd}\mathcal{F}^{-1}\left(\eta(|\cdot|)\right)(2^{-j}x)e^{ix\cdot\xi_0}. \end{aligned} \tag{8}$$

The only reason for introducing $\xi_0 \in S$ is that for any such ξ_0 we have $T_j u_2 = 0$ for $j \geq j_0$ where $j_0 \in \mathbb{Z}$ only depends on ξ_0 and τ. This is because $\hat{u}_2(\xi) = (1-\tau(\xi))\hat{u}(\xi)$ does not contain frequencies close to S. As a consequence, only the bounds for $j \searrow -\infty$ will be of importance.

Lemma 2. *Assume $d \in \mathbb{N}$ and let $\eta \in C_0^\infty(\mathbb{R})$. Then we have for $j \in \mathbb{Z}$*

$$\|T_j\|_{p\to q} \lesssim 2^{-jd(\frac{1}{p}-\frac{1}{q})} \qquad \textit{for } 1\leq p\leq q\leq\infty.$$

Proof. For all $r \in [1,\infty]$ we have $\|K_j\|_r = 2^{-jd}\|\mathcal{F}^{-1}(\eta(|\cdot|))(2^{-j}\cdot)\|_r \lesssim 2^{-j\frac{d}{r'}}$. Hence, for any given p,q such that $1\leq p\leq q\leq\infty$ we get for $\frac{1}{r} := 1+\frac{1}{q}-\frac{1}{p}$ from Young's Convolution Inequality

$$\|T_j f\|_q \lesssim \|K_j\|_r\|f\|_p \lesssim 2^{-j\frac{d}{r'}}\|f\|_p \lesssim 2^{-jd(\frac{1}{p}-\frac{1}{q})}\|f\|_p.$$

□

The above Lemma will be used to analyze the validity of Gagliardo-Nirenberg inequalities in the large frequency regime. To analyze the frequencies close to the critical surface S we consider operators of the form

$$\begin{aligned} \tilde{T}_j f &:= \mathcal{F}^{-1}\left(\eta\left(2^j(\xi_d-\psi(\xi'))\right)\chi(\xi')\hat{f}(\xi)\right) = \tilde{K}_j * f \\ \text{where} \quad \tilde{K}_j &:= \mathcal{F}^{-1}\left(\eta\left(2^j(\xi_d-\psi(\xi'))\right)\chi(\xi')\right). \end{aligned} \tag{9}$$

Here we used the notation $\xi = (\xi',\xi_d) \in \mathbb{R}^{d-1}\times\mathbb{R} \simeq \mathbb{R}^d$. In the degenerate case $d=1$ we interpret $\eta(2^j(\xi_d-\psi(\xi')))\chi(\xi')$ as $\eta(2^j(\xi-c))$ for some constant $c\in\mathbb{R}$. In the case $d\geq 2$ the functions $\eta \in C_0^\infty(\mathbb{R}^d)$ and χ,ψ are required to satisfy

$$\begin{aligned} &\psi \in C^\infty(\mathbb{R}^{d-1}),\ \chi \in C_0^\infty(\mathbb{R}^{d-1}) \text{ and at least } k \in \{0,\ldots,d-1\} \\ &\text{eigenvalues of the Hessian } D^2\psi \text{ are non-zero on } \operatorname{supp}(\chi). \end{aligned} \tag{10}$$

The reason is that S may locally be written as the graph of some function ψ with these properties. Our analysis of the mapping properties of $\tilde{T}_j$ follows [22, Section 4]. Contrary to the situation for T_j, only the bounds for $j \nearrow +\infty$ will be of importance. We give a separate treatment in the case $d = 1$ because it is much simpler. As above, Young's Convolution Inequality gives the following.

Lemma 3. *Assume $d = 1$ and $\eta \in C_0^\infty(\mathbb{R})$. Then we have*

$$\|\tilde{T}_j\|_{p\to q} \lesssim 2^{-j(\frac{1}{p}-\frac{1}{q})} \qquad \text{for } 1 \le p \le q \le \infty,\ j \in \mathbb{Z}.$$

The results in the higher-dimensional case are much more complicated and depend on the number $k \in \{0, \ldots, d-1\}$ of non-vanishing principal curvatures of S. We first analyze the kernel function $\tilde{K}_j$, which is entirely analogous to [22, Lemma 4.4].

Proposition 1. *Assume $d \in \mathbb{N}, d \ge 2$, let χ, ψ, k be as in* (10) *and $\eta \in C_0^\infty(\mathbb{R})$. Then the kernel function $\tilde{K}_j$ satisfies*

$$\|\tilde{K}_j\|_r \lesssim 2^{-j(\frac{2d-k}{2}-\frac{2d-k-1}{r})} \text{ if } 1 \le r \le 2, \qquad \|\tilde{K}_j\|_\infty \lesssim 2^{-j}. \tag{11}$$

Proof. The bound $\|\tilde{K}_j\|_2 \lesssim 2^{-j/2}$ follows from Plancherel's identity and (9). To prove (11) it thus suffices to show $\|\tilde{K}_j\|_1 \lesssim 2^{-j(\frac{k+2}{2}-d)}$ as well as $\|\tilde{K}_j\|_\infty \lesssim 2^{-j}$, which in turn follows from the following pointwise bounds

$$\begin{aligned} |\tilde{K}_j(x)| &\lesssim_N 2^{-j}(1+|x'|)^{-N} \quad \text{if } |x'| \ge c|x_d|, \\ |\tilde{K}_j(x)| &\lesssim 2^{-j}(1+|x_d|)^{-\frac{k}{2}} \quad \text{if } |x'| \le c|x_d|. \end{aligned}$$

To prove those we adapt the proof from [22]. We have

$$\tilde{K}_j(x) = c_d\, 2^{-j}(\mathcal{F}^{-1}\eta)(2^{-j}x_d) \int_{\mathbb{R}^{d-1}} e^{i(x'\cdot\xi'+x_d\psi(\xi'))}\chi(\xi')\, d\xi'$$

for some dimensional constant $c_d > 0$. We choose $c > 0$ sufficiently large such that the smooth phase function $\Phi(\xi') = x' \cdot \xi' + x_d\psi(\xi')$ satisfies $|\nabla\Phi(\xi')| \ge c^{-1}|x'|$ for all $\xi' \in \mathbb{R}^{d-1}$ in the case $|x'| \ge c|x_d|$. In view of $\chi \in C_0^\infty(\mathbb{R}^{d-1})$ the method of non-stationary phase gives

$$\begin{aligned} |\tilde{K}_j(x)| &\lesssim_N 2^{-j}|(\mathcal{F}^{-1}\eta)(2^{-j}x_d)|(1+|x'|)^{-N} \\ &\lesssim_{N,M} 2^{-j}(1+2^{-j}|x_d|)^{-M}(1+|x'|)^{-N} \quad \text{for } |x'| \ge c|x_d|. \end{aligned}$$

In the second estimate we used that $\mathcal{F}^{-1}\eta$ is a Schwartz function. The theory of oscillatory integrals gives (see [25, p.361])

$$|\tilde{K}_j(x)| \lesssim_M 2^{-j}(1+2^{-j}|x_d|)^{-M}(1+|x_d|)^{-\frac{k}{2}} \qquad \text{for } |x'| \le c|x_d|.$$

□

Next we use Proposition 1 to find reasonable upper bounds for the operator norms of $\tilde{T}_j$ as maps from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ where $1 \le p \le q \le \infty$. The latter condition is mandatory since $\tilde{T}_j$ is a translation-invariant operator covered by Hörmander's result from [17, Theorem 1.1]. The bounds have a simple expression except for the points belonging to the following set

$$\mathcal{E} := \Big\{(p,q) \in (1,\infty)^2 : \frac{1}{p} = \frac{k+2}{2(k+1)},\ \frac{1}{q} \le \frac{k^2}{2(k+1)(k+2)} \quad \text{or}$$

$$\frac{1}{q} = \frac{k}{2(k+1)}, \ \frac{1}{p} \geq \frac{k^2+6k+4}{2(k+1)(k+2)} \Big\}.$$

The necessity of removing these points will be commented on later in Remark 2. Our findings are visualized in the Riesz diagram from Figure 3.

Lemma 4. *Assume $d \in \mathbb{N}, d \geq 2$ and let χ, ψ, k are as in (10) and $\eta \in C_0^\infty(\mathbb{R})$. Then*

$$\|\tilde{T}_j\|_{p\to q} \lesssim 2^{-jA(p,q)} \qquad \text{for } 1 \leq p \leq q \leq \infty,\ j \in \mathbb{Z}, j \geq j_0$$

holds for $1 \leq p \leq q \leq \infty$ with $(p,q) \notin \mathcal{E}$ where $A(p,q) := \min\{A_0, A_1, A_2, A_2', A_3, A_3', A_4, A_4'\}$ is given by $A_i = A_i(p,q), A_i' = A_i(q',p')$ and

$$A_0 = 1, \qquad A_1 = \frac{k+2}{2}\left(\frac{1}{p} - \frac{1}{q}\right), \qquad A_2 = -\frac{k}{2} + \frac{k+1}{p},$$

$$A_3 = \frac{2d-k}{2} - \frac{2d-k-1}{q}, \qquad A_4 = \frac{k+2}{2}\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{2d-k-2}{2} - \frac{2d-k-2}{q}.$$

For $(p,q) \in \mathcal{E}$ we have $\|\tilde{T}_j\|_{p\to q} \lesssim_\varepsilon 2^{-j(A(p,q)+\varepsilon)}$ for any $\varepsilon > 0$.

Proof. We first analyze the range $0 \leq \frac{1}{q} \leq \frac{1}{2} \leq \frac{1}{p} \leq 1$ in the nondegenerate case $k \geq 1$. Plancherel's Theorem gives

$$\|\tilde{T}_j f\|_2 = \|\eta\left(2^j(\xi_d - \psi(\xi'))\right)\chi(\xi')\hat{f}\|_2 \lesssim \|\hat{f}\|_2 = \|f\|_2$$

due to $\eta, \chi \in L^\infty(\mathbb{R}^d)$. The Stein-Tomas Theorem for surfaces with k non-vanishing principal curvatures [25, p.365] yields as in [22, Lemma 4.3]

$$\|\tilde{T}_j f\|_q \lesssim 2^{-\frac{j}{2}}\|f\|_2, \quad \|\tilde{T}_j f\|_2 \lesssim 2^{-\frac{j}{2}}\|f\|_{q'} \qquad \text{if } \frac{1}{q} \leq \frac{k}{2(k+2)}.$$

The Fourier Restriction-Extension operator $f \mapsto \mathcal{F}^{-1}(\hat{f}\, d\sigma_M)$ for compact pieces M of hypersurfaces with k non-vanishing principal curvatures has the mapping properties from [22, Corollary 5.1], so it is bounded for $(\frac{1}{p}, \frac{1}{q})$ belonging to the pentagonal region

$$\frac{1}{p} > \frac{k+2}{2(k+1)}, \quad \frac{1}{q} < \frac{k}{2(k+1)}, \quad \frac{1}{p} - \frac{1}{q} \geq \frac{2}{k+2}.$$

So for these exponents and $M_t := \{\xi \in \mathrm{supp}(\chi) : \xi_d - \psi(\xi') = t\}$ with induced surface measure $d\sigma_{M_t} = (1 + |\nabla\psi(\xi')|^2)^{1/2}\, d\xi'$ we have for $\hat{g}(\xi) := \chi(\xi')\hat{f}(\xi)(1 + |\nabla\psi(\xi')|^2)^{-1/2}$

$$\|\tilde{T}_j f\|_q \lesssim \int_{\mathbb{R}} |\eta(2^j t)| \|\mathcal{F}^{-1}(\hat{g}\, d\sigma_{M_t})\|_q\, dt \lesssim \int_{\mathbb{R}} |\eta(2^j t)| \|g\|_p\, dt \lesssim 2^{-j}\|f\|_p.$$

Moreover, restricted weak-type bounds from $L^{p,1}(\mathbb{R}^d)$ to $L^{q,\infty}(\mathbb{R}^d)$ even hold for all (p,q) belonging to the closure of the above-mentioned pentagon, which implies $\|\tilde{T}_j f\|_{q,\infty} \lesssim 2^{-j}\|f\|_{p,1}$ in the same manner. Interpolating all these bounds shows that $\|\tilde{T}_j\|_{p\to q} \lesssim 2^{-j\min\{A_0,A_1,A_2,A_2'\}}$ holds except for the red points in Figure 3, which finishes the analysis in the case $1 \leq p \leq 2 \leq q \leq \infty$ and $k \geq 1$.

For $2 \leq p \leq q \leq \infty$ or $1 \leq p \leq q \leq 2$ we use Proposition 1. We have

$$\|\tilde{T}_j\|_{1\to 1} + \|\tilde{T}_j\|_{\infty\to\infty} \lesssim \|\tilde{K}_j\|_1 \lesssim 2^{-j(\frac{k+2}{2}-d)}$$

Interpolating this with the above bounds gives $\|\tilde{T}_j\|_{p\to q} \lesssim 2^{-j\min\{A_3,A_3',A_4,A_4'\}}$ and hence the claim for $k \geq 1$.

Finally, in the degenerate case $k = 0$ we interpolate the estimates

$$\|\tilde{T}_j\|_{\infty\to\infty} + \|\tilde{T}_j\|_{1\to 1} \lesssim \|\tilde{K}_j\|_1 \lesssim 2^{-j(1-d)}, \qquad \|\tilde{T}_j\|_{1\to\infty} \lesssim \|\tilde{K}_j\|_\infty \lesssim 2^{-j},$$
$$\|\tilde{T}_j\|_{2\to 2} \lesssim 1, \qquad \|\tilde{T}_j\|_{1\to 2} + \|\tilde{T}_j\|_{2\to\infty} \lesssim \|\tilde{K}_j\|_2 \lesssim 2^{-j/2}$$

to obtain

$$\|\tilde{T}_j\|_{p\to q} \lesssim 2^{-j\min\{A_1,A_4,A_4'\}} = 2^{-j\min\{A_0,A_1,A_2,A_3,A_3',A_4,A_4'\}}.$$

This finishes the proof. □

The optimality of our constants is not clear in general. In the range $1 \leq p \leq q < 2$ or $2 < p \leq q \leq \infty$ we expect that improvements are possible. It would be interesting to see whether recent results and techniques for oscillatory integral operators by Guth, Hickman, Iliopolou [15] and Kwon, Lee [19] (Proposition 2.4, Proposition 2.5) can be adapted. Any advance in this direction provides a larger range of exponents q, r_1, r_2 for which our Gagliardo-Nirenberg inequalities hold.

4. Gagliardo-Nirenberg Inequalities

We start with the frequencies away from the critical surface S, set $\overline{s} := (1-\kappa)s_1 + \kappa s_2$. Our aim is to prove that the general Gagliardo-Nirenberg inequality (2) holds in this frequency regime whenever the parameters belong to the following set:

$$\mathcal{B} := \Big\{(q,r_1,r_2,\kappa) \in (1,\infty)^3 \times (0,1) : \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} \geq \frac{1}{q} \geq \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{\overline{s}}{d} \text{ with } \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} = \frac{1}{q} \text{ only if } (1-\kappa)s_1 + \kappa s_2 > 0\Big\}. \tag{12}$$

Proposition 2. *Assume $d \in \mathbb{N}$, $\kappa \in (0,1)$ and (A2) for $s_1, s_2 \in \mathbb{R}$. Then*

$$\|u_2\|_q \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa} \|P_2(D)u\|_{r_2}^{\kappa} \qquad (u \in \mathcal{S}(\mathbb{R}^d))$$

holds provided that $(q, r_1, r_2, \kappa) \in \mathcal{B}$.

Proof. Define $\mathcal{T}_j u := T_j(u_2)$ where T_j and $u_2 = \mathcal{F}^{-1}((1-\tau(\xi))\hat{u})$ were defined in (8),(3), respectively. Since we have $\tau = 1$ on an open neighbourhood of ξ_0, there is $j_0 \in \mathbb{Z}$ such that $\sum_{j=-\infty}^{j_0} \eta(2^j|\xi - \xi_0|) = 1$ on $\mathrm{supp}(1-\tau)$, see (7). This implies

$$u_2 = \mathcal{F}^{-1}\left(\sum_{j=-\infty}^{j_0} \eta(2^j|\xi-\xi_0|)(1-\tau(\xi))\hat{u}(\xi)\right) = \sum_{j=-\infty}^{j_0} \mathcal{T}_j u \tag{13}$$

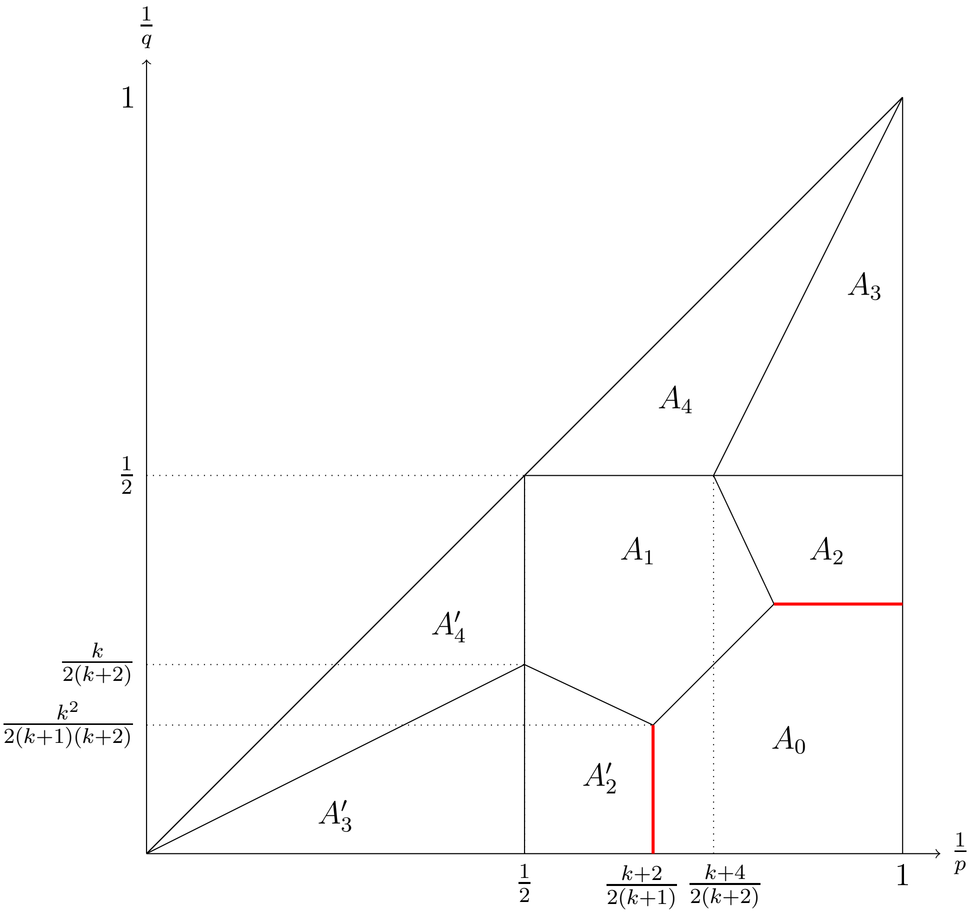

FIGURE 1. Riesz diagram with the bounds for the mapping constant of $\tilde{T}_j$ from Lemma 4. The exceptional points are coloured in red.

in the sense of distributions. To determine the mapping properties of $\mathcal{T}_j$ with the aid of Lemma 2 set $\eta_i(z) := \eta(z)|z|^{-s_i}$ for $z \in \mathbb{R}$ where s_i is taken from Assumption (A_2). Then $\eta \in C_0^\infty(\mathbb{R}), 0 \notin \operatorname{supp}(\eta)$ implies $\eta_i \in C_0^\infty(\mathbb{R})$ for $i = 1, 2$ and we have for $i = 1, 2$ and $j \in \mathbb{Z}$

$$\begin{aligned}
\mathcal{T}_j u &= \mathcal{F}^{-1}\left(\eta(2^j|\xi - \xi_0|)\hat{u}_2(\xi)\right) \\
&= \mathcal{F}^{-1}\left(\eta_i(2^j|\xi - \xi_0|)\,(2^j|\xi - \xi_0|)^{s_i}\,\hat{u}_2(\xi)\right) \\
&= 2^{js_i}\mathcal{F}^{-1}\left(\eta_i(2^j|\xi - \xi_0|)m_i(\xi)P_i(\xi)\hat{u}(\xi)\right)
\end{aligned}$$

where

$$m_i(\xi) := \frac{(1 - \tau(\xi))|\xi - \xi_0|^{s_i}}{P_i(\xi)}.$$

The Mikhlin Multiplier Theorem [4, Theorem 6.1.6] and assumption (A), notably $|\partial^\alpha(1/P_i)(\xi)| \lesssim (1 + |\xi|)^{-s_i - |\alpha|}$ for $0 \leq |\alpha| \leq \lfloor \frac{d}{2} \rfloor + 1$ and $\xi \in \operatorname{supp}(1 - \tau)$ (by choice of τ), imply that m_i is an $L^{r_i}(\mathbb{R}^d) - L^{r_i}(\mathbb{R}^d)$-multiplier. Here we used $r_i \in (1, \infty)$. This implies for $r_i \leq q_i \leq \infty$, exploiting Lemma 2,

$$\begin{aligned}\|\mathcal{T}_j u\|_{q_i} &\lesssim 2^{js_i}\|\mathcal{F}^{-1}(\eta_i(2^j|\xi-\xi_0|)m_i(\xi)\widehat{P_i(D)u}(\xi))\|_{q_i}\\ &\lesssim 2^{j(s_i-d(\frac{1}{r_i}-\frac{1}{q_i}))}\|\mathcal{F}^{-1}(m_i(\xi)\widehat{P_i(D)u}(\xi))\|_{r_i}\\ &\lesssim 2^{j(s_i-d(\frac{1}{r_i}-\frac{1}{q_i}))}\|P_i(D)u\|_{r_i}\\ &= 2^{j(s_i-d(\frac{1}{r_i}-\frac{1}{q_i}))}\|u\|_{X_i}.\end{aligned}$$

We use Bourgain's summation argument to conclude. Eq. (5) from Lemma 1 yields the bound $\|u_2\|_q \lesssim \|u\|_{(X_1,X_2)_{\kappa,q}}$ provided that

$$(1-\kappa)\left(s_1-d\left(\frac{1}{r_1}-\frac{1}{q_1}\right)\right)+\kappa\left(s_2-d\left(\frac{1}{r_2}-\frac{1}{q_2}\right)\right)>0,\quad \frac{1}{q}=\frac{1-\kappa}{q_1}+\frac{\kappa}{q_2},\quad r_i\le q_i\le\infty.$$

Such q_1,q_2 may be chosen if and only if

$$\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}\ge\frac{1}{q}>(1-\kappa)\left(\frac{1}{r_1}-\frac{s_1}{d}\right)+\kappa\left(\frac{1}{r_2}-\frac{s_2}{d}\right). \tag{14}$$

Since the above bound yields the desired inequality via

$$\|u_2\|_q \overset{(13)}{=} \|\sum_{j=-\infty}^{j_0}\mathcal{T}_j u\|_q \overset{(5)}{\lesssim} \|u\|_{(X_1,X_2)_{\kappa,q}} \overset{(6)}{\lesssim} \|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa},$$

the claim is proved for exponents as in (14).

Restricted weak-type estimates are obtained with the aid of (4). As above we get $\|u_2\|_{q,\infty}\lesssim \|u\|_{(X_1,X_2)_{\kappa,1}}$ provided that

$$\begin{aligned}&(1-\kappa)\left(s_1-d\left(\frac{1}{r_1}-\frac{1}{q_1}\right)\right)+\kappa\left(s_2-d\left(\frac{1}{r_2}-\frac{1}{q_2}\right)\right)=0,\\ &\frac{1}{q}=\frac{1-\kappa}{q_1}+\frac{\kappa}{q_2},\quad r_i\le q_i\le\infty,\quad q_1\neq q_2,\quad s_i-d\left(\frac{1}{r_i}-\frac{1}{q_i}\right)\neq 0.\end{aligned}$$

Such q_1,q_2 can be chosen if and only if there is $q_2\in[1,\infty]$ such that

$$\begin{aligned}&\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}\ge\frac{1}{q}=(1-\kappa)\left(\frac{1}{r_1}-\frac{s_1}{d}\right)+\kappa\left(\frac{1}{r_2}-\frac{s_2}{d}\right),\\ &\frac{1}{q}-\frac{1-\kappa}{r_1}\le\frac{\kappa}{q_2}\le\frac{\kappa}{r_2},\quad q_2\neq q,\quad \frac{1}{q}-(1-\kappa)\left(\frac{1}{r_1}-\frac{s_1}{d}\right)\neq\frac{\kappa}{q_2}\neq\kappa\left(\frac{1}{r_2}-\frac{s_2}{d}\right).\end{aligned}$$

In particular, the weak bounds hold in the case

$$\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}>\frac{1}{q}=(1-\kappa)\left(\frac{1}{r_1}-\frac{s_1}{d}\right)+\kappa\left(\frac{1}{r_2}-\frac{s_2}{d}\right). \tag{15}$$

In order to turn the weak-type estimates into strong estimates we use once more real interpolation. Choose

$$\frac{1}{\tilde{q}}=\frac{1}{q}+\varepsilon,\ \frac{1}{q^*}=\frac{1}{q}-\varepsilon,\quad \tilde{\kappa}=\kappa+\delta,\ \kappa^*=\kappa-\delta$$

for $\varepsilon := \delta(\frac{1}{r_2} - \frac{s_2}{d} - \frac{1}{r_1} + \frac{s_1}{d})$ and $\delta > 0$ sufficiently small. Note that $q \neq 1$ implies $|\varepsilon| > 0$. So (15) holds for $(\tilde{q}, \tilde{\kappa})$, (q^*, κ^*) and we have $\frac{1}{q} = \frac{1}{2}(\frac{1}{\tilde{q}} + \frac{1}{q^*})$, $\frac{1}{2}(\tilde{\kappa} + \kappa^*) = \kappa$. So the reiteration property [4, Theorem 3.5.3] gives

$$\begin{aligned}\|u_2\|_q &\lesssim \|u_2\|_{(L^{\tilde{q},\infty}, L^{q^*,\infty})_{\frac{1}{2},q}} \\ &\lesssim \|u\|_{((X_1,X_2)_{\tilde{\kappa},1}, (X_1,X_2)_{\kappa^*,1})_{\frac{1}{2},q}} \\ &\lesssim \|u\|_{(X_1,X_2)_{\kappa,q}} \\ &\overset{(6)}{\lesssim} \|P_1(D)u\|_{r_1}^{1-\kappa} \|P_2(D)u\|_{r_2}^{\kappa}.\end{aligned}$$

□

Remark 1.

(a) *The above interpolation procedure only partially applies in the case* $\frac{1}{q} = \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2}, (1-\kappa)s_1 + \kappa s_2 = 0$. *Our computations show that it works out under the additional assumptions* $s_1 = s_2 = 0$ *or* $q = r_1 = r_2$, *which does not seem to be optimal in general. We believe that an explicit characterization of the interpolation space* $(X_1, X_2)_{\kappa,1}$ *would be useful to get less restrictive sufficient conditions in this endpoint case. Formally, one might expect that this space resembles* $P_1(D)^{\kappa-1}P_2(D)^{-\kappa}L^{r,1}(\mathbb{R}^d)$ *with* $\frac{1}{r} = \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2}$. *Analogous formulas for weighted Lorentz spaces may, however, fail, see Corollary 6.2 and Counterexample 6.3 in [1]. For that reason we believe a characterization of* $\|u\|_{(X_1,X_2)_{\kappa,1}}$ *to be nontrivial.*

(b) *The classical Gagliardo-Nirenberg inequality* $\|\nabla^j v\|_q \lesssim \|\nabla^m v\|_{r_1}^{1-\kappa} \|v\|_{r_2}^{\kappa}$ *from [23] holds for* $j, m \in \mathbb{N}$ *provided that* $\frac{1}{q} - \frac{j}{d} = (1-\kappa)(\frac{1}{r_1} - \frac{m}{d}) + \frac{\kappa}{r_2}$ *and* $\frac{j}{m} \leq 1 - \kappa < 1$. *The above result shows that the large frequency part of this estimate holds provided that* $\frac{1}{q} - \frac{j}{d} \geq (1-\kappa)(\frac{1}{r_1} - \frac{m}{d}) + \frac{\kappa}{r_2}$ *and* $\frac{j}{m} < 1 - \kappa < 1$. *To see this it suffices to replace* v *by* $\nabla^{-j}u$ *and evaluate* (12) *for* $s_1 = m - j, s_2 = -j$. *As mentioned above, the endpoint case* $1 - \kappa = \frac{j}{m}$ *is unfortunately not reproduced.*

Next we establish the interpolation inequality for frequencies close to the critical surface S. Here, the assumption $(A1)$ will be needed and the estimates depend on the parameters α_1, α_2. The basic strategy is the same as in the previous Proposition, but T_j and Lemma 2 need to be replaced by $\tilde{T}_j$ and Lemma 4. An explicit characterization of the admissible exponents is possible in general, but we prefer to avoid the heavy computations. So we describe the set of parameters in an abstract way following the same interpolation scheme as above with $s_i - d(\frac{1}{r_i} - \frac{1}{q_i})$ replaced by $\alpha_i - A_\varepsilon(r_i, q_i)$ where

$$A_\varepsilon(r,q) := \begin{cases} \frac{1}{r} - \frac{1}{q} & \text{, if } d = 1, \\ A(r,q) + \varepsilon \cdot \mathbb{1}_{(p,q)\in\mathcal{E}} & \text{, if } d \geq 2. \end{cases} \qquad (\varepsilon > 0)$$

Accordingly, we obtain our bounds in a completely analogous manner. Since the summation index will range from some $j = j_0$ to $+\infty$ instead of $j = j_0$ to $-\infty$, the crucial inequalities will be opposite to those before. For notational simplicity we introduce $\overline{\alpha} := (1-\kappa)\alpha_1 + \kappa\alpha_2$.

Using (5) we shall obtain strong bounds for parameters in

$$\begin{aligned}\mathcal{A}_1 := \{(q,r_1,r_2,\kappa) \in (1,\infty)^3 \times (0,1) :\ &\text{There are } \varepsilon>0,\ q_1\in[r_1,\infty],\ q_2\in[r_2,\infty],\ \text{such that}\\ &\frac{1}{q}=\frac{1-\kappa}{q_1}+\frac{\kappa}{q_2} \text{ and } (1-\kappa)A_\varepsilon(r_1,q_1)+\kappa A_\varepsilon(r_2,q_2)>\overline{\alpha}\}.\end{aligned}$$

Using instead (4) with $Y_1=Y_2=L^q(\mathbb{R}^d)$ we obtain strong bounds for

$$\begin{aligned}\mathcal{A}_2 := \{(q,r_1,r_2,\kappa) \in (1,\infty)^3 \times (0,1) :\ &q\geq \max\{r_1,r_2\} \text{ and there is } \varepsilon>0 \text{ such that}\\ &(1-\kappa)A_\varepsilon(r_1,q)+\kappa A_\varepsilon(r_2,q)=\overline{\alpha}\}.\end{aligned}$$

Using (4) for general $q_1\neq q_2$ we obtain restricted weak-type bounds for

$$\begin{aligned}\mathcal{A}_3^w := \{(q,r_1,r_2,\kappa) \in (1,\infty)^3 \times (0,1) :\ &\text{There are } \varepsilon>0,\ q_1\in[r_1,\infty],\ q_2\in[r_2,\infty] \text{ such that}\\ &\frac{1}{q}=\frac{1-\kappa}{q_1}+\frac{\kappa}{q_2},\ \alpha_i\neq A_\varepsilon(r_i,q_i) \text{ and } (1-\kappa)A_\varepsilon(r_1,q)+\kappa A_\varepsilon(r_2,q)=\overline{\alpha}\}.\end{aligned}$$

Finally, interpolating all endpoint estimates with each other yields

$$\begin{aligned}\mathcal{A}_3 := \{(q,r_1,r_2,\kappa) \in (1,\infty)^3 \times (0,1) :\ &\text{There are } \tilde{q}\neq q^*,\tilde{\kappa},\kappa^* \text{ and } \theta\in(0,1) \text{ with}\\ &(\tilde{q},r_1,r_2,\tilde{\kappa}),(q^*,r_1,r_2,\kappa^*)\in\mathcal{A}_3^w\cup\mathcal{A}_2 \text{ and } \frac{1}{q}=\frac{1-\theta}{\tilde{q}}+\frac{\theta}{q^*},\ \kappa=(1-\theta)\tilde{\kappa}+\theta\kappa^*\}.\end{aligned}$$

We thus conclude, just as before, that the Gagliardo-Nirenberg inequality holds in this frequency regime for parameters from $\mathcal{A}:=\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{A}_3$.

Proposition 3. *Assume $d\in\mathbb{N},\kappa\in(0,1)$ and (A1) for $\alpha_1,\alpha_2\leq 1$. Then*

$$\|u_1\|_q \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa} \qquad (u\in\mathcal{S}(\mathbb{R}^d))$$

holds provided that $(q,r_1,r_2,\kappa)\in\mathcal{A}$.

Proof. We concentrate on the case $d\geq 2$ and $\alpha_1,\alpha_2<1$. Recall $u_1=\mathcal{F}^{-1}(\tau(\xi)\hat{u})$ where τ was chosen in (3). According to Assumption (A) there are $\tau_1,\ldots,\tau_L\in C_0^\infty(\mathbb{R}^d)$ such that $\tau_1+\ldots+\tau_L=\tau$ holds and the critical surface S is locally given as the graph of some function ψ_l as in (10). More precisely, $S\cap\operatorname{supp}(\tau_l)=\{\xi\in\operatorname{supp}(\tau_l):\tilde{\xi}_d=\psi_l(\tilde{\xi}')$ where $\tilde{\xi}=\Pi_l\xi\}$. Here, Π_l denotes some permutation of coordinates in $\mathbb{R}^d$. Since P_i vanishes of order α_i near the surface in the sense of Assumption (A1), we may write

$$\begin{aligned}&P_i(\xi)^{-1}\tau_l(\xi)=\left[\tau_{li+}(\xi)(\tilde{\xi}_d-\eta_l(\tilde{\xi}'))_+^{-\alpha_i}+\tau_{li-}(\xi)(\tilde{\xi}_d-\eta_l(\tilde{\xi}'))_-^{-\alpha_i}\right]\chi_l(\tilde{\xi}')\\ &\text{with}\quad \tau_{li+},\tau_{li-}\in C_0^\infty(\mathbb{R}^d),\ \chi_l\in C_0^\infty(\mathbb{R}^{d-1}),\ \tilde{\xi}:=\Pi_l\xi.\end{aligned} \tag{16}$$

In view of this we define $\mathcal{T}$, its local versions $\mathcal{T}^l$ and the dyadic operators $\mathcal{T}_j^l$ via

$$\begin{aligned}\mathcal{T}u &:= u_1=\mathcal{F}^{-1}(\tau(\xi)\hat{u}(\xi)), \qquad \mathcal{T}^l u:=\mathcal{F}^{-1}(\tau_l(\xi)\hat{u}(\xi)),\\ \mathcal{T}_j^l u &:= \mathcal{F}^{-1}\left(\tau_l(\xi)\hat{u}(\xi)\,\eta(2^j(\tilde{\xi}_d-\psi_l(\tilde{\xi}')))\chi_l(\tilde{\xi}')\right) \quad (\tilde{\xi}=\Pi_l\xi).\end{aligned}$$

So there is $j_0 \in \mathbb{Z}$ such that

$$\mathcal{T} = \sum_{l=1}^{L} \mathcal{T}^l = \sum_{l=1}^{L} \sum_{j=j_0}^{\infty} \mathcal{T}_j^l. \tag{17}$$

As in the previous lemma we introduce $\eta_i(z) := \eta(z)(z_+^{-\alpha_i} + z_-^{-\alpha_i})$ so that $\eta_i \in C_0^\infty(\mathbb{R})$ due to $\eta \in C_0^\infty(\mathbb{R}), 0 \notin \mathrm{supp}(\eta)$. Then Lemma 4 yields

$$\begin{aligned}
\|\mathcal{T}_j^l u\|_{q_i} &= \|\mathcal{F}^{-1}\left(\eta(2^j(\tilde{\xi}_d - \psi_l(\tilde{\xi}')))\chi_l(\tilde{\xi}')\,\tau_l(\xi)\hat{u}(\xi)\right)\|_{q_i} \\
&= \|\mathcal{F}^{-1}\left(\eta(2^j(\tilde{\xi}_d - \psi_l(\tilde{\xi}')))\chi_l(\tilde{\xi}')\,P_i(\xi)^{-1}\tau_l(\xi)\widehat{P_i(D)u}(\xi)\right)\|_{q_i} \\
&\overset{(16)}{=} 2^{j\alpha_i}\|\mathcal{F}^{-1}\left(\eta_i(2^j(\tilde{\xi}_d - \psi_l(\tilde{\xi}')))\chi_l(\tilde{\xi}')(\tau_{li+}(\xi) + \tau_{li-}(\xi))\widehat{P_i(D)u}(\xi)\right)\|_{q_i} \\
&\lesssim 2^{j(\alpha_i - A(r_i,q_i))}\|\mathcal{F}^{-1}((\tau_{li+}(\xi) + \tau_{li-}(\xi))\widehat{P_i(D)u}(\xi))\|_{r_i} \\
&\lesssim 2^{j(\alpha_i - A(r_i,q_i))}\|P_i(D)u\|_{r_i} \\
&= 2^{j(\alpha_i - A(r_i,q_i))}\|u\|_{X_i}.
\end{aligned}$$

Note at this point that these estimates are uniform with respect to α_i (and j, of course). In fact, these parameters only enter the definition of η_i in a way that the bounds from Lemma 4 persist as $\max\{\alpha_1, \alpha_2\} \to 1$. Here, it is crucial that the support of η does not contain zero. For parameters in $\mathcal{A}_1$ we thus obtain

$$\|u_1\|_q \leq \sum_{l=1}^{L} \|\sum_{j=j_0}^{\infty} \mathcal{T}_j^l u\|_q \lesssim \|u\|_{(X_1,X_2)_{\kappa,1}} \overset{(6)}{\lesssim} \|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa}$$

and similarly the claim follows for parameters in $\mathcal{A}$ using the same interpolation scheme as in Proposition 2.

In the case $d = 1$ the analysis is essentially the same because (16) still holds with $\chi_l \equiv 1$ and $\tilde{\xi}_d - \eta_l(\tilde{\xi}') = \xi - \xi_l$ for $S = \{\xi_1, \ldots, \xi_l\}$, see Assumption $(A1)$ and the explanations following it. Replacing Lemma 4 by Lemma 3 the result follows along the same lines as above given our definition for A_ε in the case $d = 1$.

As anticipated, the case $\max\{\alpha_1, \alpha_2\} = 1$ is obtained by passing to the limit $\max\{\alpha_1, \alpha_2\} \nearrow 1$ in the inequality $\|u_1\|_q \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa}$. For instance, to prove the bound for $\alpha_1 = 1, \alpha_2 < 1$ we apply these bounds to the functions $u^\varepsilon := P_1(D)^\varepsilon u$ and obtain $\|u_1^\varepsilon\|_q \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa}$ uniformly with respect to $\varepsilon \searrow 0$. So $u_1^\varepsilon \rightharpoonup u_1$ gives the claim. The analogous argument works for $\alpha_1 < 1, \alpha_2 = 1$ and finally for $\alpha_1 = \alpha_2 = 1$. □

4.1. **The one-dimensional case and Proof of Theorem 1.** We first discuss the one-dimensional case where it is possible to give a precise statement in the general framework. We concentrate on parameters α_i, s_i such that

$$\overline{\alpha} := (1-\kappa)\alpha_1 + \kappa\alpha_2 > 0, \qquad \overline{s} := (1-\kappa)s_1 + \kappa s_2 > 0$$

in order to provide results that we believe to be optimal.

Theorem 3. *Assume $d = 1, \kappa \in (0,1)$ and that $(A1), (A2)$ hold for $s_1, s_2 \in \mathbb{R}, \alpha_1, \alpha_2 \leq 1$ such that $\overline{\alpha}, \overline{s} > 0$. Then*

$$\|u\|_q \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa} \|P_2(D)u\|_{r_2}^{\kappa} \qquad (u \in \mathcal{S}(\mathbb{R}))$$

holds provided that $q, r_1, r_2 \in (1,\infty)$ satisfy

$$\overline{\alpha} \leq \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} \leq \overline{s}.$$

Proof. We derive this result from the estimates that we established in the previous section. In view of $u = u_1 + u_2$, see (3), it suffices to combine Proposition 2 and Proposition 3. The estimate related to frequencies away from the critical surface is valid provided that (12) holds, i.e.,

$$(1-\kappa)\left(\frac{1}{r_1} - \frac{s_1}{d}\right) + \kappa\left(\frac{1}{r_2} - \frac{s_2}{d}\right) \leq \frac{1}{q} \leq \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2}.$$

This is satisfied under our assumptions. So it is sufficient to show that the estimate from Proposition 3 holds in the case

$$\frac{1}{q} \leq \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \overline{\alpha}.$$

Indeed, one subsequently verifies (using $\overline{\alpha} > 0$) that (q, r_1, r_2, κ) belongs to $\mathcal{A}_1$ iff $\frac{1}{q} < \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \overline{\alpha}$, to $\mathcal{A}_3^w = \mathcal{A}_3$ iff $\frac{1}{q} = \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} + \overline{\alpha}$, and finally to $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$ iff $\frac{1}{q} \leq \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \overline{\alpha}$. This proves the claim. □

Proof of Theorem 1: This follows from Theorem 3 for the symbols $P_1(D) = |D|^s - 1, P_2(D) = I$ that satisfy the hypotheses of the Theorem for $(\alpha_1, \alpha_2, s_1, s_2) = (1, 0, s, 0)$. □

4.2. **The higher-dimensional case and Proof of Theorem 2.** Given the results of the previous section, explicit criteria require for a characterization of $\mathcal{A}$. In the general case, this appears to be rather cumbersome to do analytically (no problem though assuming computer assistance). To simplify the discussion we concetrate on the special case $r_1 = r_2 = r \in (1, 2]$ and $q \in [2, \infty)$.

Theorem 4. *Assume $d \in \mathbb{N}, d \geq 2, \kappa \in (0,1)$ and that $(A1), (A2)$ holds for $s_1, s_2 \in \mathbb{R}, \alpha_1, \alpha_2 \leq 1$ such that $\overline{\alpha}, \overline{s} > 0$. Then*

$$\|u\|_q \lesssim \|P_1(D)u\|_r^{1-\kappa} \|P_2(D)u\|_r^{\kappa} \qquad (u \in \mathcal{S}(\mathbb{R}^d))$$

holds provided that the exponents $r, q \in (1, \infty)$ satisfy

$$\frac{2\overline{\alpha}}{k+2} \leq \frac{1}{r} - \frac{1}{q} \leq \frac{\overline{s}}{d} \qquad \text{and} \qquad \min\left\{\frac{1}{r}, \frac{1}{q'}\right\} \geq \max\left\{\frac{1}{2}, \frac{k+2\overline{\alpha}}{2(k+1)}\right\}.$$

with $\min\{\frac{1}{r}, \frac{1}{q'}\} = \frac{k+2}{2(k+1)}, \min\{\frac{1}{q}, \frac{1}{r'}\} \leq \frac{k^2}{2(k+1)(k+2)}$ only if $\overline{\alpha} < 1$.

Proof. We determine a subset of $\mathcal{A}$ via the ansatz $q_1 = q_2 := q$. We find $(q, r, r, \kappa) \in \mathcal{A}_2 \cup \mathcal{A}_3$ if $A_\varepsilon(r, q) \geq \overline{\alpha}$ for some $\varepsilon > 0$. In the range $1 < r \leq 2 \leq q < \infty$ we have

$$A_\varepsilon(r,q) = A(r,q) + \varepsilon \mathbb{1}_{(p,q)\in\mathcal{E}}, \quad A(r,q) = \min\left\{1, \frac{k+2}{2}\left(\frac{1}{r} - \frac{1}{q}\right), \frac{k+2}{2} - \frac{k+1}{q}, -\frac{k}{2} + \frac{k+1}{r}\right\},$$

see Figure 3. Then $A_\varepsilon(r, q) \geq \overline{\alpha}$ for small enough $\varepsilon > 0$ is equivalent to

$$1 \geq \overline{\alpha}, \quad \frac{1}{r} - \frac{1}{q} \geq \frac{2\overline{\alpha}}{k+2}, \quad \min\left\{\frac{1}{r}, \frac{1}{q'}\right\} \geq \max\left\{\frac{1}{2}, \frac{k+2\overline{\alpha}}{2(k+1)}\right\}, \quad (r,q) \in \mathcal{E} \Rightarrow \overline{\alpha} < 1.$$

Since these conditions hold under our assumptions, the interpolation inequality holds for the frequencies close to the critical surface S thanks to Proposition 2. On the other hand, Proposition 3 yields the inequality for the remaining frequencies since our assumptions imply $\frac{1}{r} - \frac{1}{q} \leq \frac{\overline{s}}{d}$. This proves the claim. □

Proof of Theorem 2: This follows from Theorem 4 for the symbols $P_1(D) = |D|^s - 1, P_2(D) = I$. The hypotheses of the Theorem hold for $(\alpha_1, \alpha_2, s_1, s_2, k) = (1, 0, s, 0, d-1)$ because S is the unit sphere with $d - 1$ non-vanishing principal curvatures. □

Remark 2. *Our estimates are uniform with respect to $\kappa \in (0, 1)$, see (4),(5). So they persist in the limit $\kappa \searrow 0$ or $\kappa \nearrow 1$. In particular we obtain $\|u\|_q \lesssim \|P_1(D)u\|_r$ provided that $d \geq 2$ and*

$$\begin{aligned} &\frac{2\alpha_1}{k+2} \leq \frac{1}{r} - \frac{1}{q} \leq \frac{s_1}{d} \quad \textit{and} \quad \min\left\{\frac{1}{r}, \frac{1}{q'}\right\} \geq \max\left\{\frac{1}{2}, \frac{k+2\alpha_1}{2(k+1)}\right\} \quad \textit{where} \\ &\min\left\{\frac{1}{r}, \frac{1}{q'}\right\} = \frac{k+2}{2(k+1)}, \ \min\left\{\frac{1}{q}, \frac{1}{r'}\right\} \leq \frac{k^2}{2(k+1)(k+2)} \Rightarrow \alpha_1 < 1. \end{aligned} \tag{18}$$

In the case $\alpha_1 = 1$ this gives

$$\frac{2}{k+2} \leq \frac{1}{r} - \frac{1}{q} \leq \frac{s_1}{d} \quad \textit{and} \quad \min\left\{\frac{1}{r}, \frac{1}{q'}\right\} > \frac{k+2}{2(k+1)},$$

which generalizes results by Kenig, Ruiz, Sogge [18, Theorem 2.3] and Gutiérrez [16, Theorem 6] for the Helmholtz operator $-\Delta - 1 = |D|^2 - 1$ where $(k, s_1) = (d-1, 2)$ and $d \geq 3$. For $d = 2$ see [13]. It also shows that the bounds $A_\varepsilon(p, q)$ from Lemma 4 cannot be replaced by $A(p, q)$. Indeed, otherwise the above argument would imply the above inequality to hold for $\min\{\frac{1}{r}, \frac{1}{q'}\} = \frac{k+2}{2(k+1)}$, which is known to be false in general, see [22, Section 4.3].

5. An extension

In [14] it was shown that a "local" version of Gagliardo-Nirenberg inequalities is of interest, too. Here one looks for a larger set of exponents where (2) holds under the additional hypothesis $\|P_1(D)u\|_{r_1} \leq R\|P_2(D)u\|_{r_2}$ where $R > 0$ is fixed, see Corollary 2.10 in that paper. A simple consequence of our estimates above is the following.

Corollary 1. *Assume $d \in \mathbb{N}, \kappa \in (0, 1)$ and (A1), (A2) for $s_1, s_2 \in \mathbb{R}, \alpha_1, \alpha_2 \leq 1$. Then the inequality*

$$\|u\|_q \lesssim (R^{\kappa-\kappa_1} + R^{\kappa-\kappa_2})\|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa}$$

holds for all $u \in \mathcal{S}(\mathbb{R}^d)$ *satisfying* $\|P_1(D)u\|_{r_1} \leq R\|P_2(D)u\|_{r_2}$ *provided that* $\kappa_1, \kappa_2 \in [0, \kappa]$ *and* $(q, r_1, r_2, \kappa_1) \in \mathcal{B}, (q, r_1, r_2, \kappa_2) \in \mathcal{A}$.

Proof. Choose κ_1, κ_2 as required. Then Proposition 2 gives

$$\begin{aligned}\|u_1\|_q &\lesssim \|P_1(D)u\|_{r_1}^{1-\kappa_1}\|P_2(D)u\|_{r_2}^{\kappa_1}\\ &= (\|P_1(D)u\|_{r_1}\|P_2(D)u\|_{r_2}^{-1})^{\kappa-\kappa_1} \cdot \|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa}\\ &\lesssim R^{\kappa-\kappa_1}\|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa}.\end{aligned}$$

Similarly, Proposition 3 implies

$$\|u_2\|_q \lesssim R^{\kappa-\kappa_2}\|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa}.$$

Summing up these inequalities gives the claim. □

In the context of our particular example $P_1(D) = |D|^s - 1, s > 0$ and $P_2(D) = I$ this gives the following generalization of [14, Corollary 2.10].

Corollary 2. *Assume* $d \in \mathbb{N}, \kappa \in (0,1), s > 0$. *Then*

$$\|u\|_q \lesssim (R^\kappa + 1)\|(|D|^s - 1)u\|_r^{1-\kappa}\|u\|_r^\kappa$$

holds for all $u \in \mathcal{S}(\mathbb{R}^d)$ *satisfying* $\|(|D|^s - 1)u\|_r \leq R\|u\|_r$ *provided that*

$$d = 1, \quad 1 < r, q < \infty, \quad 1 - \kappa \leq \frac{1}{r} - \frac{1}{q} \leq s \qquad \text{or}$$

$$d \geq 2, \quad 1 < r \leq 2 \leq q < \infty, \quad \frac{2(1-\kappa)}{k+2} \leq \frac{1}{r} - \frac{1}{q} \leq \frac{s}{d} \quad \text{and}$$

$$\min\left\{\frac{1}{r}, \frac{1}{q'}\right\} \geq \max\left\{\frac{1}{2}, \frac{k+2-2\kappa}{2(k+1)}\right\}.$$

Proof. This corresponds to the special case $(\alpha_1, \alpha_2, s_1, s_2, k, r_1, r_2, \kappa_1, \kappa_2) = (1, 0, s, 0, d - 1, r, r, 0, \kappa)$ in Corollary 1. □

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