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Analysis of the 2021 Bundestag elections. 3/4. Tackling the Bundestag growth

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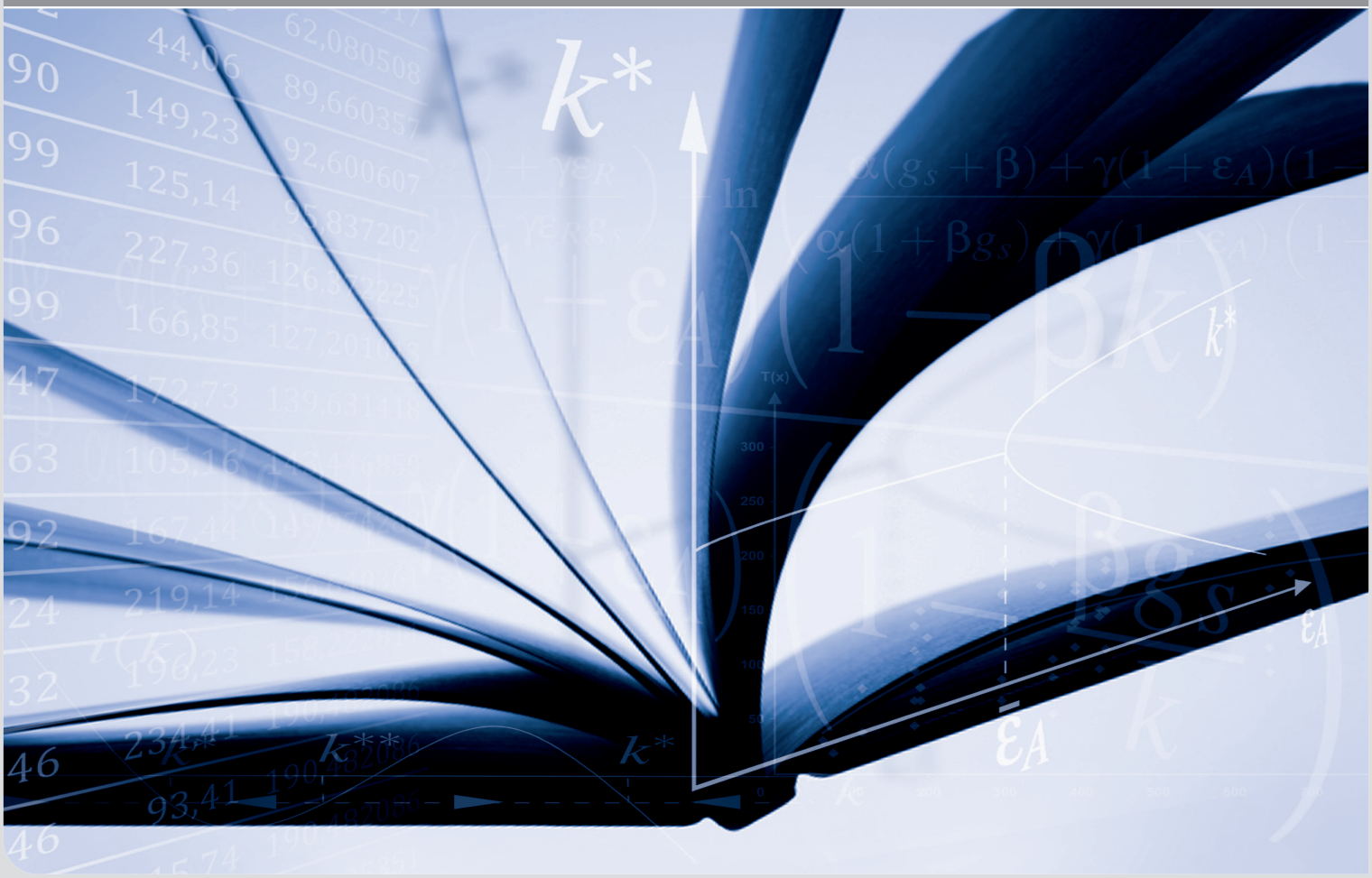

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Abstract

This is the third of four papers devoted to the 2021 German federal elections continuing our analysis of the 2009, 2013 and 2017 Bundestag elections. Currently, only China has a parliament larger than the German Bundestag, which still grows due to the increasing number of overhang mandates. The unfettered growth of the Bundestag — caused by allotting too many direct mandates to parties that received too few second votes — can be prevented by relaxing the principle of 'one man—one vote' and introducing *adjustable vote weights* of Bundestag members. Such a practice could make numerous adjustment (leveling) seats unnecessary and the basic 598 Bundestag seats sufficient under most circumstances. For this purpose, the members of the overrepresented parties (because they receive too many direct mandates) should have vote power $= 1$ and the members of other parties should have adjustment vote weights > 1. We explain the adjustment vote weights using the example of the 2021 Bundestag.

The second point discussed is the incomplete compliance of the Sainte-Laguë/Schepers method, which dates back to 1832 and is used to apportion the Bundestag, with the mathematical standards of the 21st century. This method results in apportionments that are often not the best ones found by discrete optimization.

Keywords: Representative democracy, elections, theory of voting, proportional representation.

JEL Classification: D71

Contents

1 Introduction

This is the third of four papers devoted to the 2021 German federal elections continuing our analysis of the Bundestag elections in 2009, 2013 and 2017 by the methods of the mathematical theory of democracy [Tangian 2014, 2017, 2020] . We discuss the Bundestag growth and propose a solution to constrain it.

The German two-vote electoral system embodies two major historical concepts of political representation coined during the American and French Revolutions. The *descriptive concept* (leading to proportional representation) — that is, the parliament portrays the society in miniature[1] — is implemented in the first vote *(Erststimme)*, with which local candidates are elected within constituencies and delegated to the federal parliament (Bundestag). These *direct mandate* holders from 299 German constituencies fill 299 Bundestag seats.

The *agent concept* (leading to majoritarianism) — that is, the parliament is a committee of political experts who make majority decisions as the people's trustees and not simply as their fellow countrymen[2] — is embodied in the second vote *(Zweitstimme)* for a party. The second vote serves two purposes: (1) to qualify parties receiving at least 5% of the second votes nationwide for seats in the Bundestag, and (2) to apportion the number of seats allotted to each qualifying party in accordance with the second votes. The second vote also guarantees a certain minimum number of seats for party representatives from each German state (*Land*), which together with direct mandates can result in so-called overhang mandates that exceed the state quotas of the parties. To provide seats for all 'obligatory' mandate holders in proportion to the second votes, another 299 Bundestag seats are allocated, which are *adjustment or leveling seats*. If the required proportion between party factions is unattainable within the regular 299 + 299 = 598 seats, some extra adjustment seats are added.

Currently, only China has a parliament larger than the German Bundestag, which continues to grow due to the increasing number of overhang mandates: the 2005, 2009, 2013, 2017 and 2021 Bundestags required 16, 24, 33, 111 and 138 adjustment seats, respectively, and the 2021 Bundestag has as many as 736 members. Such growth makes the Bundestag more expensive for taxpayers: its annual budget is already approaching a billion Euros [Finthammer 2018]. In 2016, Norbert Lammert, then president of the Bundestag, proposed to restrict it to 630 members by allocating mandates according to quotas for each of the German states, which should be proportional to their population [Roßner 2016]. This idea found no approval among the German parties, neither large nor small [Finthammer 2018]. Only in October 2019, after predictions that the next Bundestag could exceed 800 seats, did some 100 German experts in constitutional law write an open letter suggesting to constrain its size by reducing the number of effective constituencies, and the Bundestag vice-president, Thomas Oppermann, called for such a reform without delay [Spiegel online 2019, Zeit online 2019].

These and other proposals require a profound change in the existing election system. But a mathematical solution to the problem does not require such changes and is much simpler. The unfettered growth of the Bundestag — caused by allotting too many direct mandates to parties that received too few second votes — can be prevented by replacing 'the ideal of one man, one vote' [Balinski and Young 1982] with *adjustment vote weights* for Bundestag members — the idea that remounts to power indices in the game theory and their political applications [Shapley and Shubik 1954, Mazurkiewicz and Mercik 2005, Varela and Prado-Dominguez 2012, Holler and Nurmi 2013]. Adjustment vote weights could make numerous adjustment seats unnecessary and the basic 598 Bundestag seats sufficient under most circum-

[1]The descriptive concept was defended in America by John Adams (1735–1826), one of the key Founding Fathers, the first Vice President and the Second President of the United States from 1797–1801. In France, the same viewpoint was shared by Honore Gabriel Riqueti, comte de Mirabeau (1749–1791), a statesman, a moderate revolutionary and promoter of a British-like constitutional monarchy [Manin 1997, p. 111].

[2]The agent concept was promoted by American Federalists, particularly by Alexander Hamilton (1755?–1804), one of key Founding Father of the United States and James Madison (1751–1836), the fourth President of the USA from 1809–1817. In France, the concept of political representative as professional was developed by Emmanuel Joseph Sieyès (1748–1836), clergyman and political writer [Manin 1997, pp. 2–3, 129–131].

stances. For this purpose, the members of the most overrepresented party (because it has too many direct mandates) should have vote weight $= 1$ and the members of other parties should have adjustment vote weights > 1. Thereby, one can reduce the Bundestag on the one hand, and, on the other hand, refine the balance of party powers in the Bundestag, bringing it to the exact ratio of votes cast for the parties, as opposed to the current approximate ratio.

Relaxations of the rule 'one man—one vote' are not that uncommon. For instance, the chairperson of a committee with an even number of members may be given 1.5 votes to avoid a tie. In joint-stock companies, the vote power of each shareholder is proportional to his/her percentage of shares [Edelman et al. 2014], etc. Besides, the range of adjustment vote weights in the Bundestag is expected to be quite moderate. For the 2021 Bundestag reduced to 630 seats recommended by Norbert Lammert, the adjustment vote weights would vary within 29% only. Moreover, adjustment vote weights do not in the least contradict the established apportionment practices, since they already include adjustments — adjustment seats and rounding (= adjusting) fractional numbers of seats obtained in computations to integer numbers of seats. As for implementation, the fraction-valued voting is easy to implement in voting consoles with electronic deputy cards.

The second point discussed in this paper is the incomplete compliance of the Sainte-Laguë/Schepers method used to apportion the Bundestag with the mathematical standards of the 21st century. In fact, it was originally invented in 1832 by Daniel Webster [Webster/Sainte-Laguë method 2021] and reinvented in 1910 by Sainte-Lagfuë as a modification of the very similar method by [D'Hondt 1882]. The d'Hondt method was used to apportion the Bundestag until 2009 when it was replaced by the Sainte-Laguë method in Schepers' computational version. As heuristic methods of the 19th century, both d'Hondt and Sainte-Laguë methods result in apportionments that are often not the best ones found by discrete optimization. In particular, the 2021 official Bundestag apportionment with three tolerated non-adjusted overhangs is not optimal at all. This apportionment was done in two steps — at first an apportionment with no overhangs was found and then three overhangs were added, whereas it should have been done in one run under certain optimality criteria subject to constraints.

In Section 2, 'Official apportionment of the Bundestag', the so called 'divisor procedure' — a particular form of the Sainte-Laguë/Schepers method — is explained and analyzed.

In Section 3, 'Non-optimality of the Sainte-Laguë/Schepers method of apportionment', two alternative apportionments of the 2021 Bundestag are computed that are better fitted to the party quotas derived from the second votes.

In Section 4, 'Adjustment vote weights', it is shown how vote weighting can compensate deviations of faction sizes from quotas.

In Section 5, 'Medium-sized Bundestag', three hypothetical reductions of the 2021 Bundestag are considered as examples.

In Section 6 'D'Hondt and Sainte-Laguë/Schepers methods versus optimization', the performance of the d'Hondt and Sainte-Laguë/Schepers methods is compared with that of two optimization models.

In Section 7, 'Conclusions', the main findings are recapitulated and put into context.

Section 8, 'Annex: D'Hondt and Saint-Laguë apportionment methods', explains two main methods for allocating parliamentary seats.

2 Official apportionment of the Bundestag

We focus on the 2021 Bundestag apportionment, taking for granted the parties' minimum entitlement to seats determined by certain rules we do not consider. The allocation of seats to eligible parties, together with finding the appropriate Bundestag's size, is illustrated in Tables 1 and 2, which show screenshots of Tables 6.1.4 and 6.2.3 from the official report [Bundeswahlleiter 2021, pp. 421 and 451] with the English translation of their headings added above.

Table 1: Oberverteilung: Erhöhung der Gesamtzahl der Sitze für die Parteien (= Top distribution: Increasing the total number of seats for the parties)

Party	Minimum seat entitlement (Maximum of sum of quota of seats and sum of minimum number of seats	Threatening overhang	Second votes	Divisor	Seats after increase		Remaining overhang (Column 7)	Total seats (Column 8)	Seats according to quotas of seats (Column 9)	Increase by … seats (Difference between Columns 8 and 9)
					unrounded	rounded				
	1	2	3	4	5	6	7	8	9	10
Partei	**Mindest-sitzanspruch** (Maximum aus Summe Sitzkontin-gente und Summe Mindest-sitzzahlen)	**Drohender Überhang**	**Zweit-stimmen**	**Divisor**	**Sitze nach Erhöhung**		**Verblei-bender Überhang** (Spalte 7)	**Sitze insgesamt** (Spalte 8)	**Sitze nach Sitzkontin-genten** (Spalte 9)	**Erhöhung um … Sitze** (Differenz aus Spalten 8 und 9)
					ungerundet	**gerundet**				
CDU	122	12	8.775.471		151,567	**152**	–	152	122	30
SPD	170	10	11.955.434		206,491	**206**	–	206	170	36
AfD	69	1	4.803.902		82,971	**83**	–	83	69	14
FDP	76	–	5.319.952	57.898	91,884	**92**	–	92	76	16
DIE LINKE	32	–	2.270.906		39,222	**39**	–	39	32	7
GRÜNE	94	–	6.852.206		118,349	**118**	–	118	94	24
CSU	45	11	2.402.827		41,501	**42**	**3**	45	34	11
SSW	1	–	55.578		0,959	**1**	–	1	1	–
Insgesamt	**609**	**34**	**42.436.276**			**733**	**3**	**736**	**598**	**138**

Source: [Bundeswahlleiter 2021, Screenshot of Table 6.1.4 in p. 421]

Table 2: Ermittlung der Divisorspanne und des endgültigen Divisors für ”6.1.4 Erhöhung der Gesamtzahl der Sitze fr die Parteien” (= Finding the divisor range and the final divisor for ”6.1.4 Increasing the total number of seats for the parties)”

Party	Second votes	Minimum entitlement of parties to seats									
		without overhangs		with overhangs							
		−0.5	Party divisor	−0.5	Party divisor	−1.5	Party divisor	−2.5	Party divisor	−3.5	Party divisor
	1	2	3	4	5	6	7	8	9	10	11
Partei	**Zweitstimmen**	**Mindestsitzanspruch der Parteien**									
		ohne Überhang		**mit Überhang**							
		-0,5	**Parteien-Divisor**	**-0,5**	**Parteien-Divisor**	**-1,5**	**Parteien-Divisor**	**-2,5**	**Parteien-Divisor**	**-3,5**	**Parteien-Divisor**
CDU	8.775.471	121,5	72.226,099	121,5	72.226,099	120,5	72.825,485	119,5	73.434,904	118,5	74.054,608
SPD	11.955.434	169,5	70.533,534	169,5	70.533,534	168,5	70.952,131	167,5	71.375,725	166,5	71.804,408
AfD	4.803.902	68,5	70.129,956	68,5	70.129,956	67,5	71.168,919		–		–
FDP	5.319.952	75,5	70.462,940		–		–		–		–
DIE LINKE	2.270.906	31,5	72.092,254		–		–		–		–
GRÜNE	6.852.206	93,5	73.285,626		–		–		–		–
CSU	2.402.827	33,5	71.726,179	44,5	53.996,112	43,5	55.237,402	42,5	56.537,106	41,5	57.899,446
SSW	55.578	0,5	111.156		–		–		–		–
Insgesamt	**42.436.276**										

Source: [Bundeswahlleiter 2021, Screenshot of Table 6.2.3 in p. 451]

In Table 1, Column 1 displays the minimum number of Bundestag seats, to which a party is entitled according to the German electoral rules. The bottom row, 'Total', summarizes the columns; thus, the 2021 Bundestag must have at least 609 seats. Column 2 warns of threatening overhang mandates, which emerge at the level of German states but are mostly compensated at the federal level. Column 3 contains the number of valid 'second votes' cast for the party. Column 4, 'Divisor', indicates the number of votes needed by a party to get one Bundestag seat. The divisor must adjust the rounded quotient of the party votes divided by the divisor to the quota of the party in the Bundestag with the accuracy of ± 0.5 seat (Columns 5 and 6 give the exact and rounded quotients, respectively). For example,

$$\begin{aligned} \text{Minimum number of seats for the FDP} &= \frac{5\,319\,952}{57\,898} \\ &= 91.884 \\ &\approx 92 \text{ seats.} \end{aligned}$$

Moreover, to reduce the total number of Bundestag seats, three overhang mandates are tolerated and for this reason are excluded from consideration (in Column 6, CSU has 42 seats instead of 45 in Column 1). Column 8 displays the final allocation of Bundestag seats to the parties, and Columns 9–10 show how this apportionment relates to the basic 598 Bundestag seats.

The divisor in Column 4 is found by the Sainte-Laguë/Schepers method (see Section 8.2 at the end of the paper) reformulated in terms of 'divisor procedure'. Its application to the outcomes of the 2021 elections is traced in Table 2. Column 1 displays the number of votes cast for the parties. Because of final rounding, the party's minimum number of seats for intermediate computations is reduced by 0.5, as in Column 2. Column 3 reveals that the 11 'threatening overhangs' of CSU are in fact real because the CSU with no overhangs should have 33.5 seats instead of 44.5. The divisor procedure begins in Columns 4–5 where only the parties with threatening overhangs are retained; cf. with Column 2 of Table 1. In Columns 6, 8 and 9, the party factions with still threatening overhangs are successively reduced by one seat (since AfD has one threatening overhang, its faction is reduced only once and then it is no longer considered) and their individual divisors are recalculated in Columns 7, 9 and 11. Since only three overhangs are tolerated, the procedure stops at Columns 10–11 and the smallest divisor, in this case for the CSU, is selected, rounded and used in Column 4 of Table 1.

This procedure can be significantly simplified. The first task is finding the appropriate Bundestag size, which we explain using Table 3. Columns 1–2 show the conversion of the number of votes cast for the party into the party quota in the Bundestag. For example,

$$\begin{aligned} \text{CSU quota} &= \frac{\text{Number of votes for the CSU}}{\text{Total number of votes for the parties eligible for Bundestag seats}} \\ &= \frac{2\,402\,827}{42\,436\,276} \\ &\approx 0.0566 \quad (= 5.66\%). \end{aligned}$$

In Section 'No overhangs', Column 3 shows the minimum number of seats the party must get to within the accuracy of 0.5. It is Column 1 of Table 1 reordered and reduced by 0.5. Columns 4–5 show the fraction- and integer-valued Bundestag size that enables the party quota accommodate all the seats the party is entitled to. For example,

$$\begin{aligned} \text{Size of the Bundestag with the CSU quota that accommodates 44.5 seats} &= \frac{44.5}{0.0566} \\ &= 785.9 \\ &\approx 786 \text{ seats} \end{aligned}$$

The largest Bundestag size is in the CSU's row (framed), indicating that the CSU faction is most underprivileged, so the first tolerated must be the overhang of the CSU. Having removed one CSU's overhang

Table 3: Minimum Bundestag with successively eliminated overhangs

	Elections result		No overhangs			−1 overhang			−2 overhangs			−3 overhangs		
	Votes received	Quota of the party in the Bundestag, in %	Minimum seat entitlement	Size of the Bundestag with sufficiently large party quota	Minimum integer Bundestag size	Minimum seat entitlement	Size of the Bundestag with sufficiently large party quota	Minimum integer Bundestag size	Minimum seat entitlement	Size of the Bundestag with sufficiently large party quota	Minimum integer Bundestag size	Minimum seat entitlement	Size of the Bundestag with sufficiently large party quota	Minimum integer Bundestag size
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
SPD	11955434	→ 28.17	169.5	601.6	602	169.5	601.6	602	169.5	601.6	602	169.5	601.6	602
CDU	8775471	→ 20.68	121.5	587.5	588	121.5	587.5	588	121.5	587.5	588	121.5	587.5	588
GRÜNE	6852206	→ 16.15	93.5	579.1	580	93.5	579.1	580	93.5	579.1	580	93.5	579.1	580
FDP	5319952	→ 12.54	75.5	602.2	603	75.5	602.2	603	75.5	602.2	603	75.5	602.2	603
AfD	4803902	→ 11.32	68.5	605.1	606	68.5	605.1	606	68.5	605.1	606	68.5	605.1	606
CSU	2402827	→ 5.66	44.5	785.9	786	43.5	768.3	769	42.5	750.6	751	41.5	732.9	733
LINKE	2270906	→ 5.35	31.5	588.6	589	31.5	588.6	589	31.5	588.6	589	31.5	588.6	589
SSW	55578	→ 0.13	0.5	381.8	382	0.5	381.8	382	0.5	381.8	382	0.5	381.8	382
Sum/ Max	42436276	→100.00	605.0	785.9	786	604.0	768.3	769	603.0	750.6	751	602.0	732.9	733

from consideration, the same analysis is repeated in Section '−1 overhang' where the number of CSU mandates is reduced to 43.5 (Column 6). The most underprivileged is again the CSU, and the next tolerated is the overhang of the CSU. (If the faction of some other party implied the largest Bundestag, we would reduce it instead of the CSU faction.) Proceeding in this way, we find that all the three tolerated overhangs are that of the CSU, implying the Bundestag size to be equal to 733 seats (the framed bottom-right element of the table), as in Column 6 of Table 1.

It remains to allocate 733 seats to the eligible parties. For this purpose, we multiply 733 seats by party quotas, obtain fraction-valued faction sizes, round them and test whether their total is equal to 733. In our case, it is larger, and the party quotas in Column 2 are proportionally reduced by multiplying them by a common factor $1-\varepsilon$, while keeping control over the total of the resulting rounded numbers of seats. For instance, multiplying all the quotas by factor 0.99995, we obtain the sum of 733 seats with the same Bundestag apportionment as in Column 6 of Table 1. (If the total of integer-valued faction sizes were smaller than 733 then the quotas should be proportionally increased by multiplying them by a factor $1+\varepsilon$, while keeping control over the total of the resulting rounded numbers of seats.) After the Bundestag seats without three overhangs have been distributed, the final apportionment is obtained by adding three 'tolerated' overhangs to the CSU faction.

3 Non-optimality of the Sainte-Laguë/Schepers method of apportionment

Let us first analyze the apportionment of the Bundestag with 733 seats and no overhangs (CSU with 42 seats) from Column 6 of Table 1. Its detailed characteristics are displayed in the upper section of Table 4. Column 1, 'Party quota', replicates Column 2 of Table 3. Columns 2–4 show the composition of the party factions consisting of the minimum number of seats the party is entitled to and adjustment (leveling) seats. Column 5 expresses the number of party seats in percentage of the Bundestag seats. For example,

$$\text{Size of the SPD faction} = \frac{206 \text{ seats}}{733 \text{ seats}} \times 100\% \approx 28.104\% \text{ of the Bundestag seats .}$$

Table 4: Apportionments of the Bundestag with 733 seats and no overhangs

Official apportionment by the **Sainte-Laguë/Schepers** method

	Party quota in Bundestag, in %	Minimum seat entitlement	Adjustment seats	Seats	Seats, in % of the Bundestag seats	Absolute deviation from the quota, in % of all seats	Absolute deviation from the quota, in number of seats	Relative deviation from the quota, in % of the quota	Adjustment vote weight	Party's votes (Seats × Vote weight)	Party's share of power, in %
	1	2	3	4	5	6	7	8	9	10	11
SPD	28.173	170	+ 36	= 206	→ 28.104	−0.069	−0.506	−0.245	1.044	215.111 →	28.173
CDU	20.679	122	+ 30	= 152	→ 20.737	0.058	0.422	0.278	1.039	157.895 →	20.679
GRÜNE	16.147	94	+ 24	= 118	→ 16.098	−0.049	−0.358	−0.302	1.045	123.290 →	16.147
FDP	12.536	76	+ 16	= 92	→ 12.551	0.015	0.109	0.118	1.040	95.720 →	12.536
AfD	11.320	69	+ 14	= 83	→ 11.323	0.003	0.022	0.027	1.041	86.435 →	11.320
CSU	5.662	42	+ 0	= 42	→ 5.730	0.068	0.496	1.195	1.029	43.233 →	5.662
LINKE	5.351	32	+ 7	= 39	→ 5.321	−0.031	−0.225	−0.574	1.048	40.860 →	5.351
SSW	0.131	1	+ 0	= 1	→ 0.136	0.005	0.040	4.167	1.000	1.000 →	0.131
Sum / Range	100.000	606	+ 127	= 733	→100.000	0.137	1.002	4.741	0.048	763.544 →	100.000

Apportionment with minimum **absolute** deviations from party quotas

	Party quota in Bundestag, in %	Minimum seat entitlement	Adjustment seats	Seats	Seats, in % of the Bundestag seats	Absolute deviation from the quota, in % of all seats	Absolute deviation from the quota, in number of seats	Relative deviation from the quota, in % of the quota	Adjustment vote weight	Party's votes (Seats × Vote weight)	Party's share of power, in %
	1	12	13	14	15	16	17	18	19	20	21
SPD	28.173	170	+ 36	= 206	→ 28.104	−0.069	−0.506	−0.245	1.044	215.111 →	28.173
CDU	20.679	122	+ 30	= 152	→ 20.737	0.058	0.422	0.278	1.039	157.895 →	20.679
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LINKE	5.351	32	+ 7	= 39	→ 5.321	−0.031	−0.225	−0.574	1.048	40.860 →	5.351
SSW	0.131	1	+ 0	= 1	→ 0.136	0.005	0.040	4.167	1.000	1.000 →	0.131
Sum / Range	100.000	606	+ 127	= 733	→100.000	0.137	1.002	4.741	0.048	763.544 →	100.000

Apportionment with minimum **relative** deviations from party quotas

	Party quota in Bundestag, in %	Minimum seat entitlement	Adjustment seats	Seats	Seats, in % of the Bundestag seats	Absolute deviation from the quota, in % of all seats	Absolute deviation from the quota, in number of seats	Relative deviation from the quota, in % of the quota	Adjustment vote weight	Party's votes (Seats × Vote weight)	Party's share of power, in %
	1	22	23	24	25	26	27	28	29	30	31
SPD	28.173	170	+ 36	= 206	→ 28.104	−0.069	−0.506	−0.245	1.044	215.111 →	28.173
CDU	20.679	122	+ 29	= 151	→ 20.600	−0.079	−0.578	−0.382	1.046	157.895 →	20.679
GRÜNE	16.147	94	+ 24	= 118	→ 16.098	−0.049	−0.358	−0.302	1.045	123.290 →	16.147
FDP	12.536	76	+ 16	= 92	→ 12.551	0.015	0.109	0.118	1.040	95.720 →	12.536
AfD	11.320	69	+ 14	= 83	→ 11.323	0.003	0.022	0.027	1.041	86.435 →	11.320
CSU	5.662	42	+ 0	= 42	→ 5.730	0.068	0.496	1.195	1.029	43.233 →	5.662
LINKE	5.351	32	+ 8	= 40	→ 5.457	0.106	0.775	1.975	1.021	40.860 →	5.351
SSW	0.131	1	+ 0	= 1	→ 0.136	0.005	0.040	4.167	1.000	1.000 →	0.131
Sum / Range	100.000	606	+ 127	= 733	→100.000	0.185	1.353	4.549	0.046	763.544 →	100.000

This figure differs from the quota of 28.173% by −0.069% (Column 6). Being converted into the number of seats it gives (see Column 7)

$$\text{Absolute deviation of the SPD faction from the quota} = -0.00069 \times 733 \text{ seats} = -0.506 \text{ seat}\,.$$

The deviations from quotas in Columns 6–7 are expressed in *absolute* units that are the same for all the parties — percentages of Bundestag seats or seats. To reflect the fact that the importance of one seat (or 1% of the seats) is perceived differently by small and large factions, Column 8 shows *relative* deviations from the quotas measured in the percentage of the quota size. For example,

$$\text{Relative deviation of the SPD faction from the quota} = \frac{-0.506 \text{ seat}}{206 \text{ seats}} \times 100\% \approx -0.245\%\,.$$

The maximum deviations from the quotas, both positive and negative, are framed, and their max-min ranges are shown below and are framed as well. Column 7 confirms that, indeed, the actual party factions fit to the quotas to within 0.5 seat (with a minor inaccuracy for the SPD) — as required by the Bundestag apportionment rules.

The upper section of Table 5 analyzes what happens to the Bundestag with 733 seats when three overhangs of the CSU are added. Due to such a Bundestag enlargement, the CSU exceeds its quota by 3.326 seats (Column 7). The SPD, on the contrary, is 1.351 seat short of its quota. The relative deviations from quotas displayed in Column 8 are more significant, ranging up to 8.961%.

Now one can ask: Is that apportionment of the Bundestag with 736 seats optimal? The middle and lower sections of Table 5 call this in question. The middle and lower sections describe its really optimal apportionments with minimum *absolute* and minimum *relative* deviations of factions from the quotas, respectively; cf. Columns 6 and 16 and Columns 8 and 28. These apportionments differ from the official one in passing one seat from CDU either to SPD or LINKE.

Here, we apply double *lexicographic optimization*, that is, we first find all apportionments that are optimal under one criterion (for example, those with minimum absolute deviations from quotas), and, if there is a choice, select the one that is best with respect to the other criterion (with minimum relative deviations from quotas). As one can see, the apportionments in the upper and lower sections of Table 5 have the same range of absolute deviations from quotas (0.635% at the bottom of Columns 6 and 26) but the apportionment in the lower section has a smaller range of relative deviations (8.961% in Column 8 versus 8.769% in Column 28).

The non-optimality of the Sainte-Laguë method is caused by its heuristic nature. It just finds one apportionment that satisfied certain conditions (fits factions to quotas with the accuracy of 0.5 seat) neglecting the fact that there can be several such apportionments of different quality. The optimization approach, on the contrary, not only respects all legal constraints but also minimizes the deviations of factions from quotas, making the allocation of seats more fair.

Rigorously speaking, adding overhangs even to an optimal apportionment does not guarantee that the resulting apportionment will be optimal. Indeed, the official Sainte-Laguë allocation of 733 seats with no overhangs is optimal (coincidentally but anyway): the upper two sections of Table 4 coincide, i.e., this apportionment minimizes the absolute deviations from the quotas; cf. Columns 6 and 16 of Table 4. However, the Bundestag with three additional overhangs is not optimal with respect to the same criterion; cf. Columns 6 and 16 of Table 5.

We conclude that the official Sainte-Laguë/Schepers method to allocate Bundestag seats is not only cumbersome and requiring numerous adjustment seats but also not optimal — meaning that it is not most fair. In this case, the modern optimization approach helps to make the right choice.

4 Adjustment vote weights

The major constraint of apportionment is the principle ‘one man—one vote’, requiring significant assembly enlargements to accurately fit factions to quotas. This principle is however not universal. As

Table 5: Apportionments of the Bundestag with 736 seats

Official apportionment by the **Sainte-Laguë/Schepers** method

	Party quota in Bundestag, in %	Minimum seat entitlement	Adjustment seats	Seats	Seats, in % of the Bundestag seats	Absolute deviation from the quota, in % of all seats	Absolute deviation from the quota, in number of seats	Relative deviation from the quota, in % of the quota	Adjustment vote weight	Party's votes (Seats × Vote weight)	Party's share of power, in %
	1	2	3	4	5	6	7	8	9	10	11
SPD	28.173	170	+ 36	= 206	→ 27.989	−0.184	−1.351	−0.651	1.087	223.901	→ 28.173
CDU	20.679	122	+ 30	= 152	→ 20.652	−0.027	−0.199	−0.131	1.081	164.346	→ 20.679
GRÜNE	16.147	94	+ 24	= 118	→ 16.033	−0.114	−0.842	−0.709	1.088	128.328	→ 16.147
FDP	12.536	76	+ 16	= 92	→ 12.500	−0.036	−0.267	−0.290	1.083	99.632	→ 12.536
AfD	11.320	69	+ 14	= 83	→ 11.277	−0.043	−0.317	−0.381	1.084	89.967	→ 11.320
CSU	5.662	45	+ 0	= 45	→ 6.114	0.452	3.326	7.982	1.000	45.000	→ 5.662
LINKE	5.351	32	+ 7	= 39	→ 5.299	−0.052	−0.386	−0.980	1.090	42.529	→ 5.351
SSW	0.131	1	+ 0	= 1	→ 0.136	0.005	0.036	3.742	1.041	1.041	→ 0.131
Sum / Range	100.000	609	+ 127	= 736	→ 100.000	0.635	4.677	8.961	0.090	794.744	→ 100.000

Apportionment with minimum **absolute** deviations from party quotas

	Party quota in Bundestag, in %	Minimum seat entitlement	Adjustment seats	Seats	Seats, in % of the Bundestag seats	Absolute deviation from the quota, in % of all seats	Absolute deviation from the quota, in number of seats	Relative deviation from the quota, in % of the quota	Adjustment vote weight	Party's votes (Seats × Vote weight)	Party's share of power, in %
	1	12	13	14	15	16	17	18	19	20	21
SPD	28.173	170	+ 37	= 207	→ 28.125	−0.048	−0.351	−0.169	1.082	223.901	→ 28.173
CDU	20.679	122	+ 29	= 151	→ 20.516	−0.163	−1.199	−0.788	1.088	164.346	→ 20.679
GRÜNE	16.147	94	+ 24	= 118	→ 16.033	−0.114	−0.842	−0.709	1.088	128.328	→ 16.147
FDP	12.536	76	+ 16	= 92	→ 12.500	−0.036	−0.267	−0.290	1.083	99.632	→ 12.536
AfD	11.320	69	+ 14	= 83	→ 11.277	−0.043	−0.317	−0.381	1.084	89.967	→ 11.320
CSU	5.662	45	+ 0	= 45	→ 6.114	0.452	3.326	7.982	1.000	45.000	→ 5.662
LINKE	5.351	32	+ 7	= 39	→ 5.299	−0.052	−0.386	−0.980	1.090	42.529	→ 5.351
SSW	0.131	1	+ 0	= 1	→ 0.136	0.005	0.036	3.742	1.041	1.041	→ 0.131
Sum / Range	100.000	609	+ 127	= 736	→ 100.000	0.615	4.525	8.961	0.090	794.744	→ 100.000

Apportionment with minimum **relative** deviations from party quotas

	Party quota in Bundestag, in %	Minimum seat entitlement	Adjustment seats	Seats	Seats, in % of the Bundestag seats	Absolute deviation from the quota, in % of all seats	Absolute deviation from the quota, in number of seats	Relative deviation from the quota, in % of the quota	Adjustment vote weight	Party's votes (Seats × Vote weight)	Party's share of power, in %
	1	22	23	24	25	26	27	28	29	30	31
SPD	28.173	170	+ 36	= 206	→ 27.989	−0.184	−1.351	−0.651	1.087	223.901	→ 28.173
CDU	20.679	122	+ 29	= 151	→ 20.516	−0.163	−1.199	−0.788	1.088	164.346	→ 20.679
GRÜNE	16.147	94	+ 24	= 118	→ 16.033	−0.114	−0.842	−0.709	1.088	128.328	→ 16.147
FDP	12.536	76	+ 16	= 92	→ 12.500	−0.036	−0.267	−0.290	1.083	99.632	→ 12.536
AfD	11.320	69	+ 14	= 83	→ 11.277	−0.043	−0.317	−0.381	1.084	89.967	→ 11.320
CSU	5.662	45	+ 0	= 45	→ 6.114	0.452	3.326	7.982	1.000	45.000	→ 5.662
LINKE	5.351	32	+ 8	= 40	→ 5.435	0.083	0.614	1.559	1.063	42.529	→ 5.351
SSW	0.131	1	+ 0	= 1	→ 0.136	0.005	0.036	3.742	1.041	1.041	→ 0.131
Sum / Range	100.000	609	+ 127	= 736	→ 100.000	0.635	4.677	8.769	0.088	794.744	→ 100.000

mentioned in the introduction, the vote weights of shareholders are not equal but proportional to the number of shares they have. Similarly, members of parliament can have adjustable vote weights. Since the methodology of apportionment is already includes different forms of adjustment — leveling seats and rounding numbers of seats — adjustable vote weights do not in the least contradict the established practice.

To trace the implementation of this suggestion, let us consider the official Bundestag apportionment from the upper section of Table 5. Dividing party quotas by faction sizes in % of the Bundestag seats (see Columns 1 and 6), we obtain

$$\begin{array}{rcccl} \text{Adjustment vote weight of the SPD deputy} & = & \frac{28.17\%}{27.99\%} & = & 1.0064 \\ \text{Adjustment vote weight of the CDU deputy} & = & \frac{20.68\%}{20.65\%} & = & 1.0015 \\ \ldots & & & & \\ \text{Adjustment vote weight of the CSU deputy} & = & \frac{5.66\%}{6.11\%} & = & 0.9264 \\ \ldots & & & & \\ \text{Adjustment vote weight of the SSW deputy} & = & \frac{0.13\%}{0.14\%} & = & 0.9286 \end{array}$$

For easier comparisons of vote weights, we divide all of them by their minimum, in our case the vote weight of CSU deputy. Thereby, the minimum weight = 1 (of the CSU deputy) with all other weights being > 1; see Column 9 of Table 5.

The total of the votes of each party faction is the number of its seats (Column 4) multiplied by the adjustment vote weight (Column 9) resulting in the figure in Column 10. Reducing these faction votes to the total of 100% we obtain the faction's share of power (Column 11) that is equal to the party quota (Column 1).

It should be noted that the the minimum and maximum adjustment vote weights are inherent in the factions with the maximum (positive) and minimum (negative) deviation from the quota, respectively. This is illustrated by the correspondence of minima and maxima in Columns 8–9, 18–19 and 28–29 of the apportionment tables. This relationship is reversible, implying the following statement:

While minimizing the range of relative deviations from quotas, the range of adjustment vote weights is minimized as well, and vice versa.

In other words, the optimization problem of minimizing *relative* deviations of factions from quotas has a dual formulation in terms of minimizing the range of adjustment vote weights. Since a greater range of adjustment vote weights means a higher inequality of Bundestag members, the problem of minimizing *relative* deviations from quotas can be interpreted as *equalizing individual powers* of Bundestag members. Taking this into account and not to confuse minimizing the range of *absolute* or *relative* deviations from quotas, the latter problem will be referred to as minimizing the range of adjustment vote weights.

Thus, for a given Bundestag size, the apportionment with minimum *absolute* deviations of factions from quotas is optimal regarding the faction sizes. If the adjustment vote weights are used, the apportionment with minimum *relative* deviations of factions from quotas is optimal regarding the deputy powers. In this paper, the corresponding apportionments are displayed in the middle and lower sections of the apportionment tables, which all have the same design. The apportionment made by the officially adopted Sainte-Laguë/Schepers method is always at the top.

5 Medium-sized Bundestag

The adjustment vote weights remove all formal problems posed by apportionment. Regardless of the faction accuracy, they bring the faction powers in line with the party quotas. In other words, the Bundestag size can be reduced, even at the price of violating the faction ratio, without any change of the balance of faction powers.

The critical question is however to which degree the Bundestag members can differ in their vote weight, i.e., to which degree the inequality of Bundestag members is acceptable. It should probably be restricted

Table 6: Apportionment of the Bundestag with 609 seats (the minimum the parties are entitled to)

		The **only possible** apportionment									
	Party quota in Bundestag, in %	Minimum seat entitlement	Adjustment seats	Seats	Seats, in % of the Bundestag seats	Absolute deviation from the quota, in % of all seats	Absolute deviation from the quota, in number of seats	Relative deviation from the quota, in % of the quota	Adjustment vote weight	Party's votes (Seats × Vote weight)	Party's share of power, in %
	1	2	3	4	5	6	7	8	9	10	11
SPD	28.173	170	+0	= 170	→ 27.915	−0.258	−1.572	−0.916	1.317	223.901	→ 28.173
CDU	20.679	122	+0	= 122	→ 20.033	−0.646	−3.936	−3.126	1.347	164.346	→ 20.679
GRÜNE	16.147	94	+0	= 94	→ 15.435	−0.712	−4.336	−4.409	1.365	128.328	→ 16.147
FDP	12.536	76	+0	= 76	→ 12.479	−0.057	−0.346	−0.454	1.311	99.632	→ 12.536
AfD	11.320	69	+0	= 69	→ 11.330	0.010	0.060	0.086	1.304	89.967	→ 11.320
CSU	5.662	45	+0	= 45	→ 7.389	1.727	10.517	30.500	1.000	45.000	→ 5.662
LINKE	5.351	32	+0	= 32	→ 5.255	−0.097	−0.590	−1.809	1.329	42.529	→ 5.351
SSW	0.131	1	+0	= 1	→ 0.164	0.033	0.202	25.377	1.041	1.041	→ 0.131
Sum / Range	100.000	609	+0	= 609	→100.000	2.439	14.853	34.909	0.365	794.744	→100.000

— not to make certain Bundestag members too privileged or underprivileged. Since this ethical question is beyond our study, we can only provide examples of medium-sized Bundestag with estimations of adjustment vote weights.

The first example is the Bundestag with 609 seats the parties are minimum entitled to; see Column 2 of Table 6. Since no adjustment seats are assumed — Column 3 includes but 0s — there are no alternative apportionments. For the only possible one, the maximum deviation of factions from quotas is within 1.727% of the Bundestag seats. If adjustment vote weights are used then they differ from 1 to 1.365 (Column 9), i.e. the degree of inequality of Bundestag members is within 36.5%.

The second example is the Bundestag with 630 seats — the restriction proposed by the 12th President of the Bundestag (2005–2017) Norbert Lammert [Roßner 2016]. Table 7 shows three apportionments obtained by the Sainte-Laguë/Schepers method and two optimization models. In all three apportionments, the maximum deviation from quotas is under 1.481% (framed in Columns 6, 16 and 26). If adjustment vote weight are used then the optimal apportionment in the lower section of the table results in their range from 1 to 1.287 (Column 29), i.e. the degree of inequality of Bundestag members is within 28.7%.

The third example is presented in Table 8 — the Bundestag with the basic 598 seats, which is even fewer than the minimum the parties are entitled to. Nevertheless, we consider this case assuming that the parties' minimum entitlement to seats is reduced to their direct mandates only, plus one seat for the SSW, the party of ethnical minority treated in a special way. Table 8 displays three apportionments obtained, as previously, by the Sainte-Laguë/Schepers method and two optimization models. In all three apportionments, the maximum deviation from quotas is under 2.44% (Columns 6, 16 and 26). If adjustment vote weights are used then the optimal apportionment in the lower section of the table results in their range from 1 to 1.365 (Column 29), i.e. the degree of inequality of Bundestag members is within 36.5%.

To conclude, the correct balance of party powers in the Bundestag with 609, 630 or 598 seats could be attained if adjusted vote weights were used. Whether the degree of inequality of Bundestag members of about 30% is acceptable or not is an open question.

6 D'Hondt and Sainte-Laguë/Schepers methods versus optimization

As mentioned at the end of Section 3, the Sainte-Laguë/Schepers method of apportionment is not optimal. It finds one of several apportionments sufficiently accurately fitted to the party quotas but not necessarily the best one. Now we compare the Sainte-Laguë/Schepers method and the d'Hondt method that has been used to apportion the Bundestag until 2009 with two discrete optimization models that find the

Table 7: Apportionments of the Bundestag with 630 seats (proposed by Norbert Lammert)

		Apportionment by the **Sainte-Laguë/Schepers** method									
	Party quota in Bundestag, in %	Minimum seat entitlement	Adjustment seats	Seats	Seats, in % of the Bundestag seats	Absolute deviation from the quota, in % of all seats	Absolute deviation from the quota, in number of seats	Relative deviation from the quota, in % of the quota	Adjustment vote weight	Party's votes (Seats × Vote weight)	Party's share of power, in %
	1	2	3	4	5	6	7	8	9	10	11
SPD	28.173	170	+ 5	= 175	→ 27.778	−0.395	−2.488	−1.402	1.279	223.901	→ 28.173
CDU	20.679	122	+ 6	= 128	→ 20.317	−0.362	−2.279	−1.749	1.284	164.346	→ 20.679
GRÜNE	16.147	94	+ 6	= 100	→ 15.873	−0.274	−1.726	−1.697	1.283	128.328	→ 16.147
FDP	12.536	76	+ 2	= 78	→ 12.381	−0.155	−0.979	−1.239	1.277	99.632	→ 12.536
AfD	11.320	69	+ 1	= 70	→ 11.111	−0.209	−1.318	−1.848	1.285	89.967	→ 11.320
CSU	5.662	45	+ 0	= 45	→ 7.143	1.481	9.328	26.150	1.000	45.000	→ 5.662
LINKE	5.351	32	+ 1	= 33	→ 5.238	−0.113	−0.713	−2.116	1.289	42.529	→ 5.351
SSW	0.131	1	+ 0	= 1	→ 0.159	0.028	0.175	21.198	1.041	1.041	→ 0.131
Sum / Range	100.000	609	+ 21	= 630	→ 100.000	1.876	11.816	28.266	0.289	794.744	→ 100.000

		Apportionment with minimum **absolute** deviations from party quotas									
	Party quota in Bundestag, in %	Minimum seat entitlement	Adjustment seats	Seats	Seats, in % of the Bundestag seats	Absolute deviation from the quota, in % of all seats	Absolute deviation from the quota, in number of seats	Relative deviation from the quota, in % of the quota	Adjustment vote weight	Party's votes (Seats × Vote weight)	Party's share of power, in %
	1	12	13	14	15	16	17	18	19	20	21
SPD	28.173	170	+ 6	= 176	→ 27.937	−0.236	−1.488	−0.838	1.272	223.901	→ 28.173
CDU	20.679	122	+ 7	= 129	→ 20.476	−0.203	−1.279	−0.982	1.274	164.346	→ 20.679
GRÜNE	16.147	94	+ 6	= 100	→ 15.873	−0.274	−1.726	−1.697	1.283	128.328	→ 16.147
FDP	12.536	76	+ 1	= 77	→ 12.222	−0.314	−1.979	−2.506	1.294	99.632	→ 12.536
AfD	11.320	69	+ 1	= 70	→ 11.111	−0.209	−1.318	−1.848	1.285	89.967	→ 11.320
CSU	5.662	45	+ 0	= 45	→ 7.143	1.481	9.328	26.150	1.000	45.000	→ 5.662
LINKE	5.351	32	+ 0	= 32	→ 5.079	−0.272	−1.713	−5.082	1.329	42.529	→ 5.351
SSW	0.131	1	+ 0	= 1	→ 0.159	0.028	0.175	21.198	1.041	1.041	→ 0.131
Sum / Range	100.000	609	+ 21	= 630	→ 100.000	1.795	11.307	31.232	0.329	794.744	→ 100.000

		Apportionment with minimum **relative** deviations from party quotas									
	Party quota in Bundestag, in %	Minimum seat entitlement	Adjustment seats	Seats	Seats, in % of the Bundestag seats	Absolute deviation from the quota, in % of all seats	Absolute deviation from the quota, in number of seats	Relative deviation from the quota, in % of the quota	Adjustment vote weight	Party's votes (Seats × Vote weight)	Party's share of power, in %
	1	22	23	24	25	26	27	28	29	30	31
SPD	28.173	170	+ 4	= 174	→ 27.619	−0.554	−3.488	−1.965	1.287	223.901	→ 28.173
CDU	20.679	122	+ 6	= 128	→ 20.317	−0.362	−2.279	−1.749	1.284	164.346	→ 20.679
GRÜNE	16.147	94	+ 6	= 100	→ 15.873	−0.274	−1.726	−1.697	1.283	128.328	→ 16.147
FDP	12.536	76	+ 2	= 78	→ 12.381	−0.155	−0.979	−1.239	1.277	99.632	→ 12.536
AfD	11.320	69	+ 1	= 70	→ 11.111	−0.209	−1.318	−1.848	1.285	89.967	→ 11.320
CSU	5.662	45	+ 0	= 45	→ 7.143	1.481	9.328	26.150	1.000	45.000	→ 5.662
LINKE	5.351	32	+ 2	= 34	→ 5.397	0.045	0.287	0.850	1.251	42.529	→ 5.351
SSW	0.131	1	+ 0	= 1	→ 0.159	0.028	0.175	21.198	1.041	1.041	→ 0.131
Sum / Range	100.000	609	+ 21	= 630	→ 100.000	2.034	12.816	28.115	0.287	794.744	→ 100.000

Table 8: Apportionments of the Bundestag with 598 seats (for minimum party entitlement reduced to direct mandates)

		Apportionment by the **Sainte-Laguë/Schepers** method									
	Party quota in Bundestag, in %	Minimum seat entitlement	Adjustment seats	Seats	Seats, in % of the Bundestag seats	Absolute deviation from the quota, in % of all seats	Absolute deviation from the quota, in number of seats	Relative deviation from the quota, in % of the quota	Adjustment vote weight	Party's votes (Seats × Vote weight)	Party's share of power, in %
	1	2	3	4	5	6	7	8	9	10	11
SPD	28.173	121	+ 44	= 165	→ 27.592	−0.581	−3.473	−2.061	1.357	223.901	→ 28.173
CDU	20.679	98	+ 23	= 121	→ 20.234	−0.445	−2.661	−2.152	1.358	164.346	→ 20.679
GRÜNE	16.147	16	+ 79	= 95	→ 15.886	−0.261	−1.559	−1.615	1.351	128.328	→ 16.147
FDP	12.536	0	+ 74	= 74	→ 12.375	−0.162	−0.967	−1.290	1.346	99.632	→ 12.536
AfD	11.320	16	+ 50	= 66	→ 11.037	−0.283	−1.695	−2.504	1.363	89.967	→ 11.320
CSU	5.662	45	+ 0	= 45	→ 7.525	1.863	11.140	32.900	1.000	45.000	→ 5.662
LINKE	5.351	3	+ 28	= 31	→ 5.184	−0.167	−1.001	−3.128	1.372	42.529	→ 5.351
SSW	0.131	1	+ 0	= 1	→ 0.167	0.036	0.217	27.683	1.041	1.041	→ 0.131
Sum / Range	100.000	300	+ 298	= 598	→ 100.000	2.444	14.613	36.028	0.372	794.744	→ 100.000

		Apportionment with minimum **absolute** deviations from party quotas									
	Party quota in Bundestag, in %	Minimum seat entitlement	Adjustment seats	Seats	Seats, in % of the Bundestag seats	Absolute deviation from the quota, in % of all seats	Absolute deviation from the quota, in number of seats	Relative deviation from the quota, in % of the quota	Adjustment vote weight	Party's votes (Seats × Vote weight)	Party's share of power, in %
	1	12	13	14	15	16	17	18	19	20	21
SPD	28.173	121	+ 45	= 166	→ 27.759	−0.413	−2.473	−1.468	1.349	223.901	→ 28.173
CDU	20.679	98	+ 24	= 122	→ 20.401	−0.278	−1.661	−1.344	1.347	164.346	→ 20.679
GRÜNE	16.147	16	+ 79	= 95	→ 15.886	−0.261	−1.559	−1.615	1.351	128.328	→ 16.147
FDP	12.536	0	+ 73	= 73	→ 12.207	−0.329	−1.967	−2.624	1.365	99.632	→ 12.536
AfD	11.320	16	+ 50	= 66	→ 11.037	−0.283	−1.695	−2.504	1.363	89.967	→ 11.320
CSU	5.662	45	+ 0	= 45	→ 7.525	1.863	11.140	32.900	1.000	45.000	→ 5.662
LINKE	5.351	3	+ 27	= 30	→ 5.017	−0.335	−2.001	−6.253	1.418	42.529	→ 5.351
SSW	0.131	1	+ 0	= 1	→ 0.167	0.036	0.217	27.683	1.041	1.041	→ 0.131
Sum / Range	100.000	300	+ 298	= 598	→ 100.000	2.276	13.613	39.153	0.418	794.744	→ 100.000

		Apportionment with minimum **relative** deviations from party quotas									
	Party quota in Bundestag, in %	Minimum seat entitlement	Adjustment seats	Seats	Seats, in % of the Bundestag seats	Absolute deviation from the quota, in % of all seats	Absolute deviation from the quota, in number of seats	Relative deviation from the quota, in % of the quota	Adjustment vote weight	Party's votes (Seats × Vote weight)	Party's share of power, in %
	1	22	23	24	25	26	27	28	29	30	31
SPD	28.173	121	+ 44	= 165	→ 27.592	−0.581	−3.473	−2.061	1.357	223.901	→ 28.173
CDU	20.679	98	+ 23	= 121	→ 20.234	−0.445	−2.661	−2.152	1.358	164.346	→ 20.679
GRÜNE	16.147	16	+ 79	= 95	→ 15.886	−0.261	−1.559	−1.615	1.351	128.328	→ 16.147
FDP	12.536	0	+ 73	= 73	→ 12.207	−0.329	−1.967	−2.624	1.365	99.632	→ 12.536
AfD	11.320	16	+ 50	= 66	→ 11.037	−0.283	−1.695	−2.504	1.363	89.967	→ 11.320
CSU	5.662	45	+ 0	= 45	→ 7.525	1.863	11.140	32.900	1.000	45.000	→ 5.662
LINKE	5.351	3	+ 29	= 32	→ 5.351	0.000	−0.001	−0.003	1.329	42.529	→ 5.351
SSW	0.131	1	+ 0	= 1	→ 0.167	0.036	0.217	27.683	1.041	1.041	→ 0.131
Sum / Range	100.000	300	+ 298	= 598	→ 100.000	2.444	14.613	35.524	0.365	794.744	→ 100.000

Table 9: Pearson correlations of the curves in Figure 1.

	Relative optimization upper	d'Hondt upper	Sainte-Laguë upper	Absolute optimization upper	Relative optimization lower	d'Hondt lower	Sainte-Laguë lower	Absolute optimization lower
Relative optimization upper	1.000	0.993	0.993	0.996	0.996	0.996	0.994	0.989
d'Hondt upper	0.993	1.000	0.994	0.996	0.997	0.995	0.994	0.989
Sainte-Laguë upper	0.993	0.994	1.000	0.997	0.997	0.997	0.992	0.992
Absolute optimization upper	0.996	0.996	0.997	1.000	0.999	0.998	0.996	0.993
Relative optimization lower	0.996	0.997	0.997	0.999	1.000	0.999	0.996	0.993
d'Hondt lower	0.996	0.995	0.997	0.998	0.999	1.000	0.995	0.993
Sainte-Laguë lower	0.994	0.994	0.992	0.996	0.996	0.995	1.000	0.990
Absolute optimization lower	0.989	0.989	0.992	0.993	0.993	0.993	0.990	1.000

apportionments, which are in a sense 'more fair'. The 'more fair' means that the deviation of party faction from the quota should not be negligible for one party but large for an other; on the contrary, the range of the fitting errors should be minimized, reducing the inequality between privileged and underprivileged parties. Since the fitting-error-minimization can be done in terms of either *absolute* or *relative* deviations from the quotas, we consider both options.

The curves of Figure 1 trace the performance of the four models. The four upper curves plot the range of faction fitting errors, as if plotting the bottom (framed) elements of Columns 6, 16 and 26 of the apportionment tables (similar tables can be imagined for the d'Hondt method). These curves characterize the accuracy of faction sizes in *absolute* units — percentages of the Bundestag seats (even for the model that minimizes the range of *relative* deviations from quotas, or, equivalently, the range of adjustment vote weights).

The four lower curves characterize the inequality of Bundestag members if the adjustment vote weights were used. They plot the range of adjustment vote weights — as if the bottom (framed) elements of Columns 9, 19 and 29. In other words, the upper curves evaluate the four methods from the viewpoint of the first optimization criterion (minimizing the range of faction fitting errors in absolute terms), whereas the lower curves evaluate the four methods from the viewpoint of the second optimization criterion (equalizing the adjustment vote weights = equalizing the individual powers of Bundestag members).

Since we apply lexicographic optimization, the apportionments with minimum absolute deviations from quotas that are traced by black curves have the largest relative deviations from quotas (among the apportionments obtained by the four methods) and vice versa. Correspondingly, the black curve is minimum among the upper curves that plot absolute deviations and the black curve is maximum among the lower curves that plot relative deviations. Conversely, the blue curve that characterizes the apportionments with minimum relative deviations from quotas is minimum among the lower curves that plot relative deviations, and the blue curve is maximum among the upper curves that plot absolute deviations.

The curves that characterize the performance of the Sainte-Laguë/Schepers and d'Hondt methods run between the curves that characterize that of the optimization models. The Sainte-Laguë/Schepers curves run close to the curves of the model that optimizes the relative deviations of factions from quotas, and the d'Hondt curves run between the Sainte-Laguë/Schepers curves and the ones that optimizes the absolute deviations. As the Bundestag size grows, the four curves of each group approach each other and coincide more frequently, however, they never cross, respecting the vertical order that is opposite in the upper and lower groups of curves. All of these are well seen in the detailed Figures 2–3.

We conclude that the Sainte-Laguë/Schepers and d'Hondt apportionments are in a sense intermediate between two optimal ones, sometimes coinciding with one of them. Table 9 demonstrates that all the curves in Figure 1 are highly correlated, meaning that the Sainte-Laguë/Schepers method, the d'Hondt method and two optimization models are consistent on a large scale. On a small scale, as show Figures 2–3, there is no clear regularity in the way the Sainte-Laguë/Schepers and d'Hondt curves approach

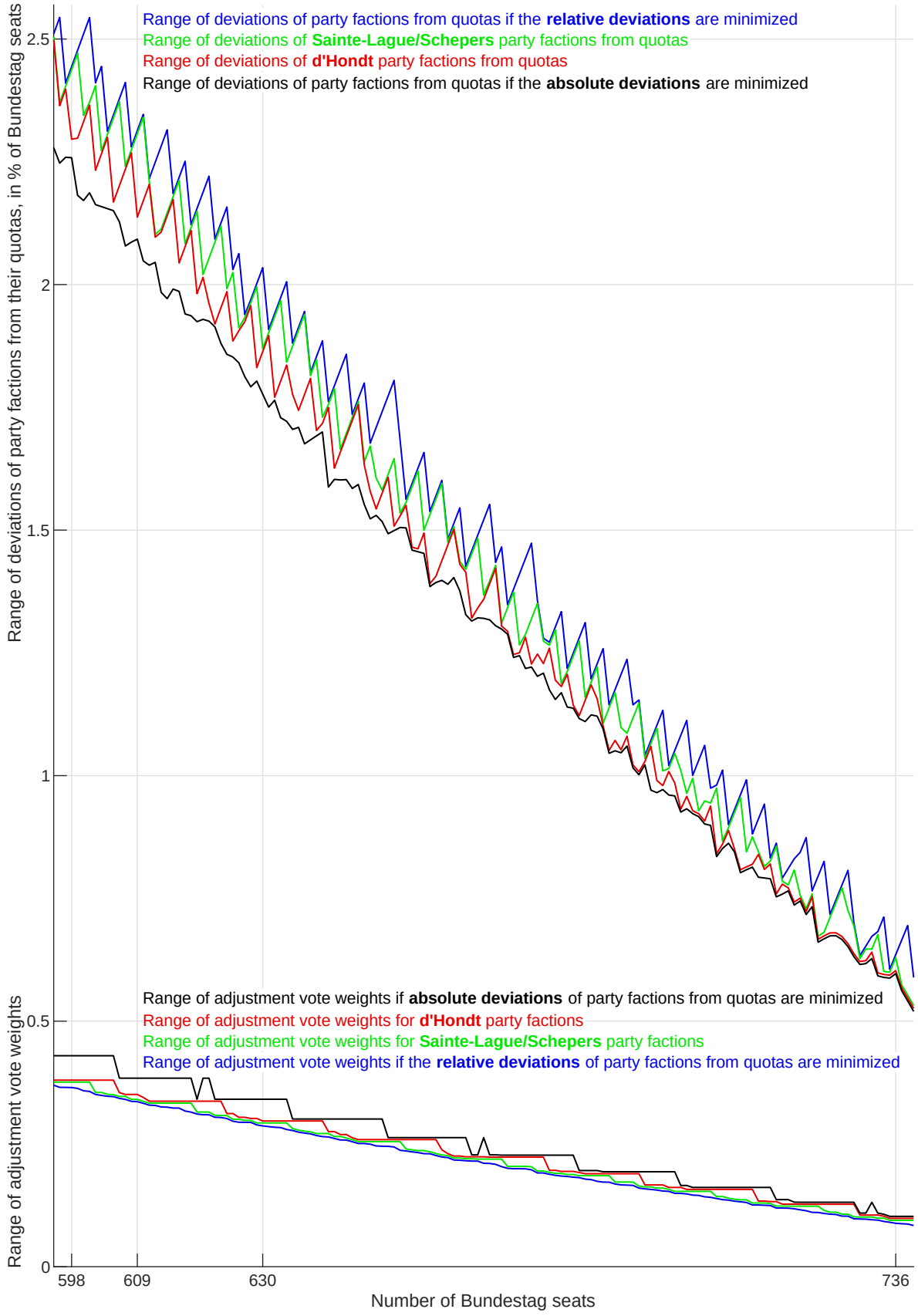

Figure 1: Above: Range of deviations of party factions from their quotas. Below: Range of adjustment vote weights.

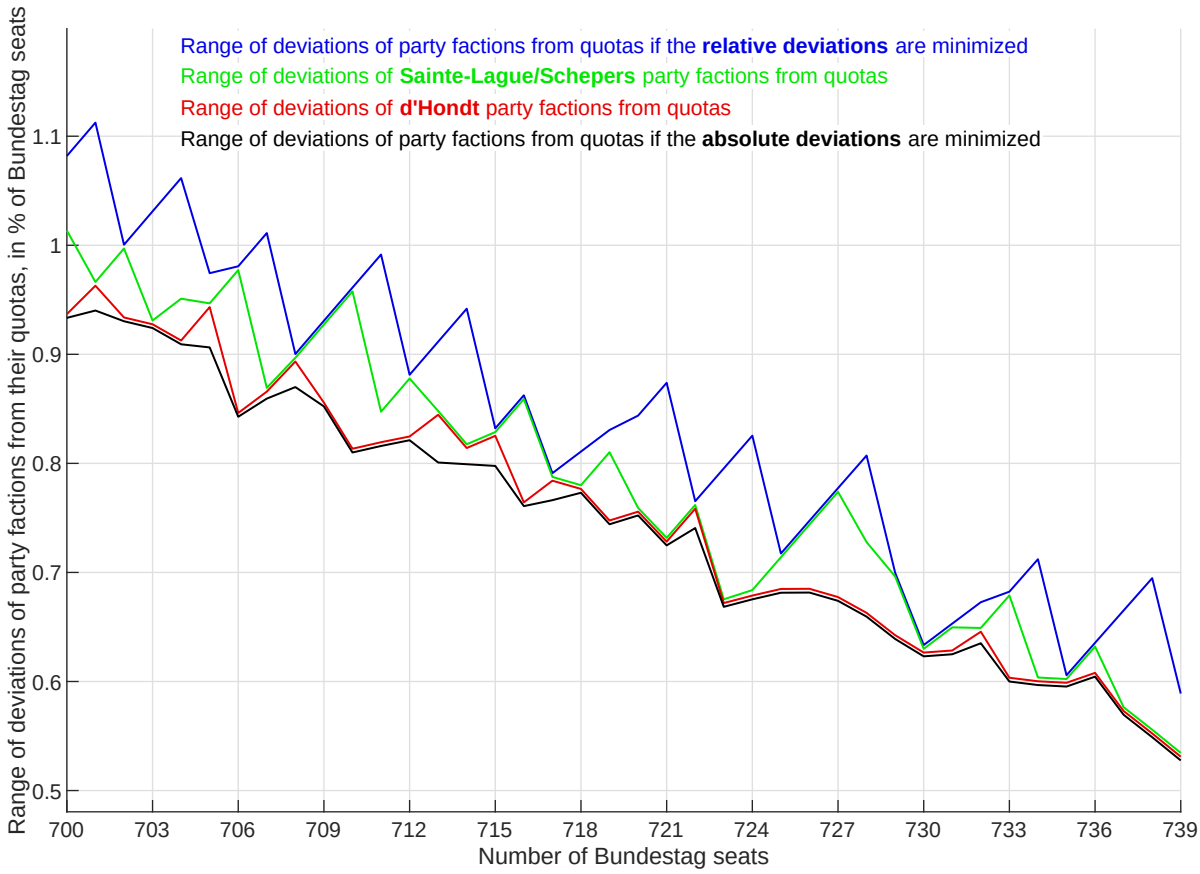

Figure 2: Detail of Figure 1: the upper curves for large Bundestag sizes.

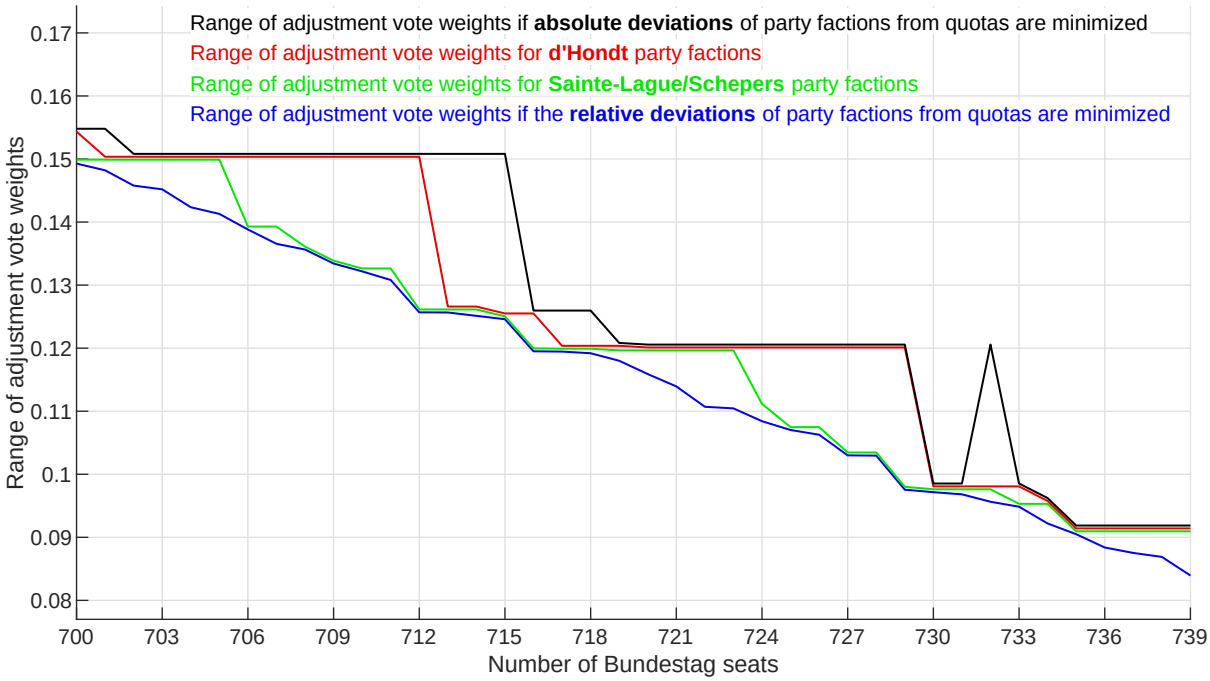

Figure 3: Detail of Figure 1: the lower curves for large Bundestag sizes.

or join the curves of the optimization models. In this respect, the Saite-Laguë/Schepers and d'Hondt methods are somewhat short of optimization consistency.

7 Conclusions

The Sainte-Laguë/Schepers method of apportionment is not as perfect as it is commonly believed. It finds one of solutions but not necessarily the best one. It is shown that better (or more fair) apportionments can be found using discrete optimization techniques.

The second point is the vicious practice of obtaining an apportionment with non-adjusted overhangs in two steps: first find an apportionment with no overhangs and then add a few ones. This task should be done in one run using optimality criteria subject to constraints.

Oversized assemblies can be avoided by introducing adjustment vote weights that extend the adjustment practices used in apportionment anyway. This device can reduce the assembly size on the one hand, and, on the other hand, refine the balance of party powers with respect to the votes cast for the parties.

8 Annex: D'Hondt and Saint-Laguë apportionment methods

8.1 D'Hondt method

In 1882, Belgian lawyer and mathematician Victor d'Hondt (1841–1902) published his method for allocating parliamentary seats to parties in proportion to the votes they received at elections [D'Hondt 1882]. The eponymous method is used in many countries and, in particular, was applied in the German Bundestag until 1985 [D'Hondt-Verfahren 2021]. The D'Hondt method has numerous mathematical advantages but is also known for slightly favoring large parties over small ones [Balinski and Young 1979, Lijphart 2003, Pukelsheim 2007, D'Hondt-Verfahren 2021].

The idea of the method is as follows. The party with the most electoral votes 'purchases' its first parliamentary seat by 'spending' a half of the total votes it received in the election. At each successive step, the currently 'richest' party acquires a seat. For its first seat, the party 'pays' an amount that leaves it with only 1/2 of its original number of votes; then for its next seat it pays an amount that leaves it with only 1/3 of its original number of votes, then 1/4, and so on. At every round, however, the next seat goes up to the 'highest bidder' — the party with the most votes to spend — until the seats are exhausted. In this way, the biggest winners can acquire several seats before a minor party ever gets to make its first 'purchase'.

Thus, to allocate the next available seat, the algorithm finds the party i with the *largest remainder of votes*:

$$\text{while} \quad \sum_{i=1}^{n} s_i < S \quad \text{find} \;\; i : \max_{i=1,\ldots,n} \underbrace{\left(\frac{V_i}{s_i + 1} \right)}_{\substack{i\text{th party's} \\ \text{remainder} \\ \text{of votes}}} \quad \Rightarrow \quad s_i = s_i + 1 \,, \tag{1}$$

where

$i = 1, \ldots, n$ are labels of n parties eligible for parliamentary seats,

s_i is the number of seats that have already been allocated to the i party (initially $s_i = 0$),

S is the total number of parliamentary seats to be allocated, and

V_i is the total number of electoral votes that party i received at the election.

8.2 Webster/Sainte-Laguë/Schepers method

The Webster/Sainte-Laguë/Schepers method strives to complete the same task as the D'Hondt method and is very similar to it. It is named after the American statesman Daniel Webster (1782–1852), who proposed it in 1832 for proportional allocation of seats in the United States congressional apportionment [Balinski and Young 1982], and the French mathematician André Sainte-Laguë (1882–1950), who independently rediscovered it and studied its properties [Sainte-Laguë 1910]. Together with the D'Hondt method, it is widely used worldwide, sometimes interchangeably.

In 1980, the German physicist and electoral expert Hans Schepers (1928–2021), having studied the D'Hondt method used by the German Bundestag, discovered that it disadvantaged smaller parties and suggested an improved version equivalent to the Sainte-Laguë method [Pukelsheim 2002]. At first it was adopted only for certain Bundestag commissions, but since 2009 it has been used to allocate seats both in the German Bundestag and the European Parliament [Sainte-Laguë-Verfahren 2021].

The idea of the method is the same as that of d'Hondt, but the progression of 'payments' for the seats is different. The party with most electoral votes 'purchases' its first parliamentary seat by 'spending' 2/3 of its votes. At each successive step, the seat goes to the currently 'richest' party, who 'pays' at first an amount that leaves it with only 1/3 of its original votes, then an amount that leaves it with only 1/5 of its original votes, then 1/7, etc. The procedure continues as long as there are still seats to be apportioned. As one can see, the biggest winners 'spend' their votes much faster than under the D'Hondt method, thereby giving way to smaller parties.

Correspondingly, the allocation algorithm is slightly modified. In its loop (1), the divisor $s_i + 1$ is replaced by $2s_i + 1$ as follows:

$$\text{while} \quad \sum_{i=1}^{n} s_i < S \quad \text{find } i : \max_{i=1,\ldots,n} \underbrace{\left(\frac{V_i}{2s_i + 1} \right)}_{\substack{i\text{th party's} \\ \text{remainder} \\ \text{of votes}}} \quad \Rightarrow \quad s_i = s_i + 1 \ .$$

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