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A scattering problem for a local perturbation of an open periodic waveguide

Andreas Kirsch

Department of Mathematics, Karlsruhe Institute of Technology (KIT), Karlsruhe, 76131, Germany

Correspondence
Andreas Kirsch, Department of Mathematics, Karlsruhe Institute of Technology (KIT), Karlsruhe 76131, Germany.
Email: andreas.kirsch@kit.edu

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In this paper, we consider the propagation of waves in an open waveguide in $\mathbb{R}^2$ where the index of refraction is a local perturbation of a function which is periodic along the axis of the waveguide (which we choose to be the x_1 axis) and equal to one for $|x_2| > h_0$ for some $h_0 > 0$. Motivated by the limiting absorption principle (proven in an earlier paper by the author), we formulate a radiation condition which allows the existence of propagating modes and prove uniqueness, existence, and stability of a solution under the assumption that no bound states exist. In the second part, we determine the order of decay of the radiating part of the solution in the direction of the layer and in the direction orthogonal to it. Finally, we show that it satisfies the classical Sommerfeld radiation condition and allows the definition of a far field pattern.

KEYWORDS

periodic refractive index, radiation condition, scattering, waveguide

MSC CLASSIFICATION

35J05; 35B27; 78A50

1 | INTRODUCTION

Let $k > 0$ be the wavenumber which is fixed throughout the paper and $n \in L^\infty(\mathbb{R}^2)$ the real valued index of refraction which is assumed to be 2π-periodic with respect to x_1 and equals to 1 for $|x_2| > h_0$ for some $h_0 > 0$. Furthermore, let $q \in L^\infty(\mathbb{R}^2)$ and $f \in L^2(\mathbb{R}^2)$ have compact support in $Q := (0, 2\pi) \times (-h_0, h_0)$. It is the aim to solve

$$\Delta u + k^2(n+q)u = -f \text{ in } \mathbb{R}^2 \tag{1}$$

subject to a suitable radiating condition stated below.

The solution of (1) is understood in the variational sense; that is,

$$\int_{\mathbb{R}^2} \left[\nabla u \cdot \nabla \psi - k^2(n+q)u\psi\right] dx = \int_Q f\,\psi\,dx \tag{2}$$

for all $\psi \in H^1(\mathbb{R}^2)$ with compact support. By standard regularity theorems, it is known that for $|x_2| > h_0$, the solution u is a classical solution of the Helmholtz equation and thus analytic.

As mentioned above, a further condition is needed to assure uniqueness (see Definition 2.5 below). In contrast to the closed waveguide, that is, where $\mathbb{R}^2$ is replaced by $\mathbb{R} \times (a_-, a_+)$ and where boundary conditions for $x_2 = a_\pm$ are added, not only a radiation condition in the direction of periodicity, that is, x_1, is needed but also one in direction of x_2. The radiation condition should be in accordance with the limiting absorption principle; that is, the solution u should be the limit (as $\varepsilon > 0$ tends to zero) of the solutions $u_\varepsilon \in H^1(\mathbb{R}^2)$ corresponding to wave numbers $k + i\varepsilon$ instead of k.

Candidates are the Sommerfeld radiation condition (see, e.g., Colton & Kress[1] for bounded scatterers in free space or Brandsmeier et al[2] for periodic open waveguides) or the "upward propagating radiation condition" which is popular for scattering problems by rough surfaces (see, e.g., Chandler-Wilde & Zhang[3]). However, one of the basic differences between the scattering by bounded (penetrable or impenetrable) obstacles in free space and (unbounded) layers is the existence of guided (or propagating) modes in the latter case which don't exist for the scattering by bounded obstacles in free space. Therefore, Sommerfeld's radiation condition is too restrictive, while the upward propagating radiation condition is not sufficient for uniqueness, that is, not restrictive enough. The special case of layered media, that is, where n is constant with respect to x_1, is well studied in the literature; see, e.g., previous studies[4–9] for different types of radiation conditions based on spectral representations of the scattered field (or the radiating part of the scattered field) with respect to the (point oder continuous) spectrum of the transverse contribution of the Helmholtz operator. In any case, this leads to a decomposition of the scattered field into a radiating part and a guided part. The radiating part decays in all directions, while the components of the guided part do not decay with respect to x_1. Since they decay exponentially with respect to x_2, they are also called surface waves.

In Kirsch[10,11,13] and Kirsch and Lechleiter,[12] we introduced a new kind of radiation condition which has been derived rigorously from the limiting absorption principle for unperturbed (i.e., $q = 0$) problems. For closed waveguides, this radiation condition is equivalent to the condition based on the dispersion curves (see, e.g. Fliss & Joly[14] and also Remark 2.4 below). In Sections 3 and 5, we investigate uniqueness, existence, and continuous dependence on f of Equation 1 complemented by this radiation condition. This seems to be new for this kind of problems. For the proof of uniqueness in Section 3, we were inspired by Furuya.[15] We had, however, to modify his proof considerably because of the full-space waveguide instead of the half-space waveguide considered in Furuya.[15] The Floquet-Bloch transform is a basic tool in the analysis of periodic problems and replaces the role of the Fourier transform for layered media. It transforms the problem in $\mathbb{R}^2$ into a class of quasi-periodic (with respect to x_1) problems in $Q^\infty := (0, 2\pi) \times \mathbb{R}$. Section 4 is devoted to the analysis of quasi-periodic problems, in particular smoothness with respect to the Floquet-parameter. The results obtained in this section (Theorems 4.1–4.3) are not surprising, and one can skip this section if one is only interested in the main arguments.

In Section 6, we will investigate the asymptotic behavior of the radiating part of the solution in the direction of the waveguide and orthogonal to it. While for closed waveguides, the radiating part decays exponentially along the waveguide, we will show that the radiating part for open waveguides behaves only as $\mathcal{O}(|x_1|^{-3/2})$ in the direction of the waveguide and as $\mathcal{O}(|x_2|^{-1/2})$ orthogonal to it. We will show Sommerfeld's radiation condition for the radiating part and introduce its far field pattern. These results seem to be new as well.

2 | THE OPEN WAVEGUIDE RADIATION CONDITION AND FIRST CONSEQUENCES

As mentioned above, the field will have a decomposition into a propagating and a radiating part. The loss of exponential decay of the radiating part is a consequence of the existence of cut-off values while the propagative wave numbers determine the behavior of the guided part along the waveguide. These quantities are defined as follows.

Definition 2.1. $\alpha \in [-1/2, 1/2]$ is called a cut-off value if there exists $\ell \in \mathbb{Z}$ such that $|\alpha + \ell| = k$.

$\alpha \in [-1/2, 1/2]$ is called a propagative wave number if there exists a non-trivial $u \in H^1_{\alpha,loc}(\mathbb{R}^2) := \{u \in H^1_{loc}(\mathbb{R}^2) : u(\cdot, x_2) \text{is}\, \alpha - \text{quasi-periodic}\}$ such that

$$\Delta u + k^2 n u = 0 \text{ in } \mathbb{R}^2, \tag{3a}$$

and u satisfies the Rayleigh expansion

$$u(x) = \sum_{\ell \in \mathbb{Z}} u_\ell^\pm e^{i(\ell+\alpha)x_1} e^{i\sqrt{k^2-(\ell+\alpha)^2}(\pm x_2 - h_0)} \text{ for } \pm x_2 > h_0 \tag{3b}$$

for some $u_\ell^\pm \in \mathbb{C}$ where the convergence is uniform for $|x_2| \geq h_0 + \varepsilon$ for every $\varepsilon > 0$. Here, and in all of the paper, we choose the square root function to be holomorphic in the cutted plane $\mathbb{C} \setminus (i\mathbb{R}_{\leq 0})$. In particular, $\sqrt{t} = i\sqrt{|t|}$ for $t \in \mathbb{R}_{<0}$. We recall that a function $u(\cdot, x_2)$ is α-quasi-periodic if $u(x_1 + 2\pi, x_2) = e^{2\pi\alpha i} u(x_1, x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$. The functions u are called guided (or propagating or Floquet) modes.

Throughout this paper, we make the following assumption.

Assumption 2.2. Let $|\ell + \alpha| \neq k$ for every propagative wave number $\alpha \in [-1/2, 1/2]$ and every $\ell \in \mathbb{Z}$; that is, no cut-off value is a propagative wave number.

Under Assumption 2.2, it can be shown (see, e.g., Kirsch & Lechleiter[12]) that at most, a finite number of propagative wave numbers exists in the interval $[-1/2, 1/2]$. Furthermore, if α is a propagative wave number with mode u, then $-\alpha$ is a propagative wave number with mode $\overline{u}$. Therefore, we can number the propagative wave numbers in $[-1/2, 1/2]$ such they are given by $\{\hat{\alpha}_j \, : \, j \in J\}$, where $J \subset \mathbb{Z}$ is finite and symmetric with respect to 0 and $\hat{\alpha}_{-j} = -\hat{\alpha}_j$ for $j \in J$. Furthermore, it is known that (under Assumption 2.2) every mode u is evanescent, that is, exponentially decaying as $|x_2|$ tends to infinity, that is, satisfying $|u(x)| \leq c e^{-\delta |x_2|}$ for $|x_2| \geq h_0$ and some $c, \delta > 0$ which are independent of x. The corresponding space

$$X_j \, := \left\{ u \in H^1_{\hat{\alpha}_j, loc}(\mathbb{R}^2) \, : \, u \text{ satisfies (3a) and (3b) for } \alpha = \hat{\alpha}_j \right\} \tag{4}$$

of modes is finite dimensional with some dimension $m_j > 0$. We note that the elements of X_j are in $H^2(Q^\infty)$ and even analytic for $|x_2| > h_0$. They decay exponentially as $|x_2|$ tends to infinity. On X_j, we define the sesqui-linear form $E \, : \, X_j \times X_j \to \mathbb{C}$ by

$$E(\phi, \psi) \, := -2i \int_{Q^\infty} \frac{\partial \phi}{\partial x_1} \overline{\psi} \, dx, \ \ \phi, \psi \in X_j \, , \tag{5}$$

where $Q^\infty \, := (0, 2\pi) \times \mathbb{R}$. We note that E is Hermitian and make the assumption that E is non-degenerated on every X_j; that is,

Assumption 2.3. For every $j \in J$ and $\psi \in X_j$, the linear form $E(\cdot, \psi) \, : \, X_j \to \mathbb{C}$ is non-trivial on X_j; that is, there exists $\phi \in X_j$ with $E(\phi, \psi) \neq 0$.

Remark 2.4. This condition is equivalent to the requirement that the group velocity does not vanish. Indeed, assume that for all α, there exists eigenvalues $\mu_\nu(\alpha) \in \mathbb{R}$ and corresponding eigenfunctions $u_\nu(\alpha) \in H^1_\alpha(Q^\infty)$ that satisfy $\Delta u_\nu(\alpha) + \mu_\nu(\alpha) n u_\nu(\alpha) = 0$ in Q^∞. Then, $\hat{\alpha}$ is a propagative wave number if $\mu_\nu(\hat{\alpha}) = k^2$ for some ν. We transform u_ν to its periodic form by setting $\tilde{u}_\nu(x) \, := e^{-i\alpha x_1} u_\nu(x)$. Then, $\tilde{u}_\nu(\alpha)$ is 2π-periodic with respect to x_1 and satisfies $\Delta \tilde{u}_\nu(\alpha) + 2i\alpha \, \partial \tilde{u}_\nu(\alpha)/\partial x_1 + (\mu_\nu(\alpha) n - \alpha^2) \tilde{u}_\nu(\alpha) = 0$ in Q^∞. Assuming that $\tilde{u}_\nu(\alpha)$ is differentiable with respect to α, we differentiate this equation and set $\alpha = \hat{\alpha}$. This yields

$$\Delta \tilde{u}_\nu'(\hat{\alpha}) + 2i\hat{\alpha} \frac{\partial \tilde{u}_\nu'(\hat{\alpha})}{\partial x_1} + (k^2 n - \hat{\alpha}^2) \tilde{u}_\nu'(\hat{\alpha}) = -2i \frac{\partial \tilde{u}_\nu(\hat{\alpha})}{\partial x_1} + [2\hat{\alpha} - \mu_\nu'(\hat{\alpha}) n] \, \tilde{u}_\nu(\hat{\alpha})$$

in Q^∞. We multiply this equation by $\overline{\tilde{u}_\nu(\hat{\alpha})}$, integrate over Q^∞, and use Green's second theorem. This yields

$$2i \int_{Q^\infty} \overline{\tilde{u}_\nu(\hat{\alpha})} \left[\frac{\partial \tilde{u}_\nu(\hat{\alpha})}{\partial x_1} + i\hat{\alpha} \tilde{u}_\nu(\hat{\alpha}) \right] dx + \mu_\nu'(\hat{\alpha}) \int_{Q^\infty} n \, |\tilde{u}_\nu(\hat{\alpha})|^2 \, dx = 0.$$

Formulated with u_ν instead of $\tilde{u}_\nu$, this reads as

$$2i \int_{Q^\infty} \overline{u_\nu(\hat{\alpha})} \frac{\partial u_\nu(\hat{\alpha})}{\partial x_1} dx + \mu_\nu'(\hat{\alpha}) \int_{Q^\infty} n \, |u_\nu(\hat{\alpha})|^2 \, dx = 0.$$

Therefore, the condition of Assumption 2.3 (for $m_j = 1$) is equivalent to $\mu_\nu'(\hat{\alpha}) \neq 0$.

The Hermitian sesqui-linear form E defines the cones $\{\psi \in X_j \, : \, E(\psi, \psi) \gtrless 0\}$ of propagating waves traveling to the right and left, respectively. We construct a basis of X_j with elements in these cones by taking any inner product $(\cdot, \cdot)_{X_j}$ and consider the following eigenvalue problem in X_j for every fixed $j \in J$. Determine $\lambda_{\ell,j} \in \mathbb{R}$ and non-trivial $\hat{\phi}_{\ell,j} \in X_j$ with

$$E(\hat{\phi}_{\ell,j}, \psi) = -2i \int_{Q^\infty} \frac{\partial \hat{\phi}_{\ell,j}}{\partial x_1} \overline{\psi} \, dx = \lambda_{\ell,j} \left(\hat{\phi}_{\ell,j}, \psi \right)_{X_j} \quad \text{for all } \ \psi \in X_j \tag{6}$$

and $\ell = 1, \dots, m_j$. We normalize the basis such that $\left(\hat{\phi}_{\ell,j}, \hat{\phi}_{\ell',j}\right)_{X_j} = \delta_{\ell,\ell'}$ for $\ell, \ell' = 1, \dots, m_j$. Then, $\lambda_{\ell,j} = E(\hat{\phi}_{\ell,j}, \hat{\phi}_{\ell,j})$, and Assumption 2.3 is equivalent to $\lambda_{\ell,j} \neq 0$ for all $\ell = 1, \dots, m_j$ and $j \in J$.

Now, we are able to formulate the radiation condition. In all of the paper, we make Assumptions 2.2 and 2.3 without mentioning this always. To simplify the notation, we define the space $H^1_*(\mathbb{R}^2)$ by

$$H^1_*(\mathbb{R}^2) := \left\{u \in H^1_{loc}(\mathbb{R}^2) : u \in H^1(W_R) \text{ for all } R > 0\right\}, \tag{7}$$

where $W_R = \mathbb{R} \times (-R, R)$.

Definition 2.5. Let $\psi_+, \psi_- \in C^\infty(\mathbb{R})$ be any functions with $\psi_\pm(x_1) = 1$ for $\pm x_1 \geq \sigma_0$ (for some $\sigma_0 > 2\pi + 1$) and $\psi_\pm(x_1) = 0$ for $\pm x_1 \leq \sigma_0 - 1$.

A solution $u \in H^1_{loc}(\mathbb{R}^2)$ of (1), that is,

$$\Delta u + k^2(n+q)u = -f \text{ in } \mathbb{R}^2, \tag{8}$$

satisfies the open waveguide radiation condition with respect to an inner product $(\cdot,\cdot)_{X_j}$ in X_j if u has a decomposition into $u = u_{rad} + u_{prop}$ with a radiating part $u_{rad} \in H^1_*(\mathbb{R}^2)$ and a propagating part u_{prop} which satisfy the following conditions.

(a) The propagating part u_{prop} has the form

$$u_{prop}(x) = \sum_{j\in J}\left[\psi_+(x_1)\sum_{\ell:\lambda_{\ell,j}>0} a_{\ell,j}\,\hat{\phi}_{\ell,j}(x) + \psi_-(x_1)\sum_{\ell:\lambda_{\ell,j}<0} a_{\ell,j}\,\hat{\phi}_{\ell,j}(x)\right] \tag{9}$$

for $x \in \mathbb{R}^2$ and some $a_{\ell,j} \in \mathbb{C}$. Here, for every $j \in J$, the scalars $\lambda_{\ell,j} \in \mathbb{R}$ and $\hat{\phi}_{\ell,j} \in \hat{X}_j$ for $\ell = 1, \dots, m_j$ are given by the eigenvalues and corresponding eigenfunctions, respectively, of the self adjoint eigenvalue problem (6).

(b) $u_{rad} \in H^1_*(\mathbb{R}^2)$ satisfies the generalized angular spectrum radiation condition

$$\int_{-\infty}^{\infty}\left|(\text{sign } x_2)\frac{\partial(\mathcal{F}u_{rad})(\omega,x_2)}{\partial x_2} - i\sqrt{k^2-\omega^2}\,(\mathcal{F}u_{rad})(\omega,x_2)\right|^2 d\omega \to 0,\ |x_2| \to \infty, \tag{10}$$

where the Fourier transform is defined as

$$(\mathcal{F}\phi)(\omega) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(s)e^{-is\omega}\,ds,\ \omega \in \mathbb{R},$$

considered as an unitary operator from $L^2(\mathbb{R})$ onto itself.

This radiation condition has a natural extension to the scattering by an infinitely long penetrable cylinder with periodic (with respect to the axis of the cylinder) refractive index; see Kirsch.[10,11] In this case, the one-dimensional Fourier transform in the angular spectrum radiation condition (10) has to be replaced by the cylindrical Fourier transform.

It has been shown in Kirsch and Lechleiter[12] for the case of a half plane problem that this open waveguide radiation condition for the inner product $(\phi,\psi)_{X_j} = 2k\int_{Q^\infty} n\phi\overline{\psi}\,dx$ is a consequence of the limiting absorption principle. A second motivation is the following result on the direction of the energy flow which will play a central role in the proof of uniqueness.

Lemma 2.6. *Let u_{prop} be given by (9). With $I_r := \{r\} \times \mathbb{R}$ and $r + Q^\infty := (r, r+2\pi) \times \mathbb{R}$ for $|r| \geq \sigma_0$, we have*

$$4\pi\, Im\int_{I_r}\overline{u_{prop}}\,\frac{\partial u_{prop}}{\partial x_1}\,ds = 2\,Im\int_{r+Q^\infty}\overline{u_{prop}}\,\frac{\partial u_{prop}}{\partial x_1}\,dx = \begin{cases}\sum\limits_{j\in J}\sum\limits_{\lambda_{\ell,j}>0}\lambda_{\ell,j}\,|a_{\ell,j}|^2,\ r > \sigma_0,\\ \sum\limits_{j\in J}\sum\limits_{\lambda_{\ell,j}<0}\lambda_{\ell,j}\,|a_{\ell,j}|^2,\ r < -\sigma_0.\end{cases}$$

Proof. We only consider $r > \sigma_0$. Then, $u_{prop}(x) = \sum_{j\in J}\sum_{\lambda_{\ell,j}>0} a_{\ell,j}\,\hat{\phi}_{\ell,j}(x)$ for $x_1 > \sigma_0$. First, we fix $j\in J$ and define $u_j^+(x) := \sum_{\lambda_{\ell,j}>0} a_{\ell,j}\,\hat{\phi}_{\ell,j}$. Since we fix j in the first part, we drop the index j and write u^+ for u_j^+. Furthermore, we define $v(x) := (x_1-r)u^+(x)$. Then, $\frac{\partial v}{\partial x_1} = u^+ + (x_1-r)\frac{\partial u^+}{\partial x_1}$ and $\Delta v + k^2 n v = 2\frac{\partial u^+}{\partial x_1}$. Therefore, with $r+Q^\infty := (r, r+2\pi)\times\mathbb{R}\subset\mathbb{R}^2$,

$$\begin{aligned}
2\int_{Q^\infty}\overline{u^+}\,\frac{\partial u^+}{\partial x_1}\,dx &= 2\int_{r+Q^\infty}\overline{u^+}\,\frac{\partial u^+}{\partial x_1}\,dx = \int_{r+Q^\infty}\overline{u^+}\left(\Delta v + k^2 n v\right)dx \\
&= \int_{r+Q^\infty} v\left(\Delta\overline{u^+} + k^2 n\overline{u^+}\right)dx + \int_{r+\partial Q^\infty}\left(\overline{u^+}\,\frac{\partial v}{\partial \nu} - v\,\frac{\partial\overline{u^+}}{\partial\nu}\right)ds \\
&= -\int_{I_r}|u^+|^2 ds + \int_{I_{r+2\pi}}\left[\overline{u^+}\left(u^+ + 2\pi\,\frac{\partial u^+}{\partial x_1}\right) - 2\pi u^+\,\frac{\partial\overline{u^+}}{\partial x_1}\right]ds \\
&= 2\pi\int_{I_r}\left(\overline{u^+}\,\frac{\partial u^+}{\partial x_1} - u^+\,\frac{\partial\overline{u^+}}{\partial x_1}\right)ds = 4\pi i\,\mathrm{Im}\int_{I_r}\overline{u^+}\,\frac{\partial u^+}{\partial x_1}\,ds,
\end{aligned}$$

which shows the first equality for fixed $j\in J$. Furthermore, with $L_j^+ := \{\ell : \lambda_{\ell,j} > 0\}$,

$$4\pi\ \mathrm{Im}\int_{I_r}\overline{u_j^+}\,\frac{\partial u_j^+}{\partial x_1}\,ds = E(u_j^+, u_j^+) = \sum_{\ell,\ell'\in L_j^+}\overline{a_{\ell,j}}\,a_{\ell',j}\,E\left(\hat{\phi}_{\ell,j}, \hat{\phi}_{\ell',j}\right) = \sum_{\ell\in L_j^+}\lambda_{\ell,j}\,|a_{\ell,j}|^2$$

by the orthonormalization of $\hat{\phi}_{\ell,j}$ where we indicated the dependence on j. In the second part, we take $j, j'\in J$, apply Green's theorem in $r+Q^\infty$, and use the quasi-periodicities of u_j^+ and $u_{j'}^+$.

$$\begin{aligned}
0 &= \int_{r+\partial Q^\infty}\left(\overline{u_j^+}\,\frac{\partial u_{j'}^+}{\partial\nu} - u_{j'}^+\,\frac{\partial\overline{u_j^+}}{\partial\nu}\right)ds \\
&= -\int_{I_r}\left(\overline{u_j^+}\,\frac{\partial u_{j'}^+}{\partial x_1} - u_{j'}^+\,\frac{\partial\overline{u_j^+}}{\partial x_1}\right)ds + \int_{I_{r+2\pi}}\left(\overline{u_j^+}\,\frac{\partial u_{j'}^+}{\partial x_1} - u_{j'}^+\,\frac{\partial\overline{u_j^+}}{\partial x_1}\right)ds \\
&= \left(e^{i(\hat{\alpha}_{j'}-\hat{\alpha}_j)2\pi} - 1\right)\int_{I_r}\left(\overline{u_j^+}\,\frac{\partial u_{j'}^+}{\partial x_1} - u_{j'}^+\,\frac{\partial\overline{u_j^+}}{\partial x_1}\right)ds.
\end{aligned}$$

Therefore, the last integral vanishes for $j \neq j'$. Thus, we have

$$\begin{aligned}
&4\pi i\,\mathrm{Im}\int_{I_r}\overline{u_{prop}}\,\frac{\partial u_{prop}}{\partial x_1}\,ds \\
&= 2\pi\int_{I_r}\left[\overline{u_{prop}}\,\frac{\partial u_{prop}}{\partial x_1} - u_{prop}\,\frac{\partial\overline{u_{prop}}}{\partial x_1}\right]ds = 2\pi\sum_{j\in J}\int_{I_r}\left[\overline{u_j^+}\,\frac{\partial u_j^+}{\partial x_1} - u_j^+\,\frac{\partial\overline{u_j^+}}{\partial x_1}\right]ds \\
&= 4\pi i\sum_{j\in J}\mathrm{Im}\int_{I_r}\overline{u_j^+}\,\frac{\partial u_j^+}{\partial x_1}\,ds = i\sum_{j\in J}\sum_{\ell\in L_j^+}\lambda_{\ell,j}\,|a_{\ell,j}|^2.
\end{aligned}$$

□

As the next step, we prove a first result on the asymptotic behavior of u_{rad} which will be needed in the proof of uniqueness.

Because $q\psi_\pm$ vanishes identically by our choice of $\psi_\pm$, we observe that the radiating part u_{rad} satisfies

$$\Delta u_{rad} + k^2(n+q)\,u_{rad} = -f - \sum_{j\in J}\sum_{\ell=1}^{m_j} a_{\ell,j}\varphi_{\ell,j} \text{ in } \mathbb{R}^2, \tag{11a}$$

where

$$\varphi_{\ell,j}(x) = \begin{cases} 2\psi_+'(x_1)\,\frac{\partial\hat\phi_{\ell,j}(x)}{\partial x_1} + \psi_+''(x_1)\,\hat\phi_{\ell,j}(x) \text{ if } \lambda_{\ell,j} > 0, \\ 2\psi_-'(x_1)\,\frac{\partial\hat\phi_{\ell,j}(x)}{\partial x_1} + \psi_-''(x_1)\,\hat\phi_{\ell,j}(x) \text{ if } \lambda_{\ell,j} < 0. \end{cases} \tag{11b}$$

We note that f has compact support in Q and $\varphi_{\ell,j}$ vanish for $|x_1| \geq \sigma_0$ and are evanescent; that is, there exist $\hat c, \delta > 0$ with $|\varphi_{\ell,j}(x)| \leq \hat c \exp(-\delta|x_2|)$ for all $x \in \mathbb{R}^2$.

Using the result of Lemma 7.1 of the appendix, we are able to show the following asymptotic behavior * for u_{rad}. We set

$$\varphi := \sum_{j\in J}\sum_{\ell=1}^{m_j} a_{\ell,j}\varphi_{\ell,j} \text{ and } \phi := u_{rad}|_\Gamma \in H^{1/2}(\Gamma)$$

for abbreviation where $\Gamma := \Gamma_{h_0} \cup \Gamma_{-h_0}$ with $\Gamma_{\pm h_0} := \mathbb{R} \times \{\pm h_0\}$ and note that $u_{rad} \in H^1_*(\mathbb{R}^2)$ satisfies

$$\Delta u_{rad} + k^2 u_{rad} = -\varphi \text{ for } |x_2| > h_0,\ u_{rad} = \phi \text{ on } \Gamma, \tag{12}$$

and the generalized angular spectrum radiation condition (10).

Lemma 2.7. *Let Assumptions 2.2 and 2.3 hold, and let $u \in H^1_{loc}(\mathbb{R}^2)$ be a solution of (8) satisfying the radiation condition of Definition 2.5. Then, the radiating part u_{rad} has the form*

$$u_{rad}(x) = \int_{-\sigma_0}^{\sigma_0}\int_{h_0}^{\infty} \varphi(y)\,G^+(x,y)\,dy_2\,dy_1 + \frac{i}{2}\int_{\Gamma_{h_0}} \phi(y)\,\frac{\partial}{\partial y_2}H_0^{(1)}(k|x-y|)\,ds(y) \tag{13}$$

for $x_2 > h_0$, where the Green's function $G^+(x,y)$ is defined as $G^+(x,y) := \frac{i}{4}\left[H_0^{(1)}(k|x-y|) - H_0^{(1)}(k|x-y^*|)\right]$ for $x, y \in \mathbb{R}^2$ with $x_2, y_2 > h_0$ and $x \neq y$. Here, $y^* := (y_1, 2h_0 - y_2)^\top$ is the reflected point at the line $\Gamma_{h_0} := \mathbb{R} \times \{h_0\}$, and $H_0^{(1)}$ denotes the Hankel function of the first kind and order zero. An analogous representation holds for $x_2 < -h_0$.

u_{rad} satisfies a stronger form of the radiation condition (10), namely,

$$\left|(\text{sign } x_2)\,\frac{\partial(\mathcal{F}u_{rad})(\omega,x_2)}{\partial x_2} - i\sqrt{k^2-\omega^2}\,(\mathcal{F}u_{rad})(\omega,x_2)\right| \leq \frac{c}{\delta + \sqrt{|\omega^2-k^2|}}\,e^{-\delta|x_2|} \tag{14}$$

for almost all $\omega \in \mathbb{R}$ and $|x_2| > h_0$, where $c > 0$ is independent of ω and x. Here, $\delta > 0$ is again a constant such that $|\varphi_{\ell,j}(x)| \leq \hat c \exp(-\delta|x_2|)$ for all x.

Furthermore, there exists $c > 0$ with

$$|u_{rad}(x)| + |\nabla u_{rad}(x)| \leq c(1+|x_2|)\,\rho(x_1) \tag{15}$$

for all $x \in \mathbb{R}^2$ with $|x_2| \geq h_0 + 1$, where $\rho \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is given by

$$\rho(x_1) := \sum_{\sigma\in\{+,-\}}\int_{\mathbb{R}} \frac{|u_{rad}(y_1,\sigma h_0)|}{(1+|x_1-y_1|)^{3/2}}\,dy_1 + \frac{1}{1+|x_1|^{3/2}},\ x_1 \in \mathbb{R}. \tag{16}$$

Proof. First, we note that $\rho \in L^2(\mathbb{R})$ because the first term can be expressed as the convolution of the L^2-function $|u_{rad}(\cdot, \pm h_0)|$ and the L^1-function $y_1 \mapsto (1+|y_1|)^{-3/2}$. It is also bounded by the inequality of Cauchy-Schwarz.

We restrict ourselves to the upper half plane $\mathbb{R}^2_{h_0} := \{x \in \mathbb{R}^2 : x_2 > h_0\}$. In Lemma 7.1 of the appendix uniqueness of (12), (10) has been shown and that the volume potential in (13) satisfies (12) for $\phi = 0$ and the

* We will sharpen this result in Section 6.

estimates (14) and (15). It remains to study the line integral in (13) which we denote by $v(x)$. (Again, we consider only the upper half plane) The function $v(\cdot,x_2)$ is a convolution of the L^2–function $\phi(\cdot,h_0)$ and the function $\chi(y_1)=\frac{i}{2}\frac{\partial}{\partial y_2}H_0^{(1)}(k\sqrt{y_1^2+(x_2-y_2)^2})|_{y_2=h_0}$ (for fixed $x_2>h_0$). It is $\chi\in W^{1,1}(\mathbb{R})$ by the asymptotic behavior of the Hankel functions (see Zhang & Chandler-Wilde[16]). Indeed, for all $a>0$, there exists $c=c(a)>0$ with

$$\left|\frac{\partial}{\partial y_2}H_0^{(1)}(k|x-y|)\right|+\left|\nabla_x\frac{\partial}{\partial y_2}H_0^{(1)}(k|x-y|)\right|\le c\,\frac{|x_2|+|y_2|+1}{|x-y|^{3/2}} \tag{17}$$

for all $x,y\in\mathbb{R}^2$ with $|x-y|\ge a$. Taking the Fourier transform with respect to x_1, we get first $(\mathcal{F}v)(\cdot,x_2)=\sqrt{2\pi}(\mathcal{F}\phi)(\mathcal{F}\chi)$ by our normalization of the Fourier transform and thus, using $(\mathcal{F}\chi)(\omega)=\frac{1}{\sqrt{2\pi}}e^{i\sqrt{k^2-\omega^2}(x_2-h_0)}$ (differentiate the formulas 3. and 4. in Gradshteyn & Ryshik,[17] section 6.677, with respect to z), $(\mathcal{F}v)(\omega,x_2)=(\mathcal{F}\phi)(\omega,h_0)e^{i\sqrt{k^2-\omega^2}(x_2-h_0)}$ for $x_2>h_0$ which satisfies the radiation condition (14) trivially. Furthermore, from Parseval's identity, we get

$$\int_{h_0}^{H}\int_{-\infty}^{\infty}\left[|v(x)|^2+|\nabla v(x)|^2\right]dx_1\,dx_2=\int_{h_0}^{H}\int_{-\infty}^{\infty}(1+\omega^2+|k^2-\omega^2|)\,|(\mathcal{F}v)(\omega,x_2)|^2d\omega\,dx_2$$

$$=\int_{h_0}^{H}\int_{-\infty}^{\infty}(1+\omega^2+|k^2-\omega^2|)\,|(\mathcal{F}\phi)(\omega,h_0)|^2e^{-2\,\mathrm{Im}\sqrt{k^2-\omega^2}(x_2-h_0)}d\omega\,dx_2$$

$$\le 2(1+k^2)\int_{h_0}^{H}\int_{|\omega|<k}(1+\omega^2)\,|(\mathcal{F}\phi)(\omega,h_0)|^2d\omega\,dx_2$$

$$+\,2(1+k^2)\int_{|\omega|>k}(1+\omega^2)\,|(\mathcal{F}\phi)(\omega,h_0)|^2\int_{h_0}^{H}e^{-2\sqrt{\omega^2-k^2}(x_2-h_0)}dx_2\,d\omega$$

$$\le c_H\int_{-\infty}^{\infty}\sqrt{1+\omega^2}\,|(\mathcal{F}\phi)(\omega,h_0)|^2d\omega=c_H\|\phi\|^2_{H^{1/2}(\Gamma_{h_0})}.$$

This shows that $v\in H^1(W_H^+)$ for all $H>h_0$ where $W_H^+:=\mathbb{R}\times(h_0,H)$.

Finally, using (17), $v(x)$ is estimated by

$$|v(x)|\le c(x_2+h_0+1)\int_{-\infty}^{\infty}\frac{|u_{rad}(y_1,h_0)|}{[(x_1-y_1)^2+1]^{3/4}}\,dy_1$$

for $x_2>h_0+1$ which proves the desired estimate (15). □

3 | UNIQUENESS

In this section, we follow the proof of uniqueness given by Furuya[15] for the half-plane case. We have to modify his approach, however, because the free space Green's function; that is, the fundamental solution $\frac{i}{4}H_0^{(1)}(k|x-y|)$ does not decay as fast as the Green's function $G^+(x,y)$ for the half-plane as $|x_1|$ tends to infinity. Therefore, we can't use his integral representations.

We begin with the following technical result.

Lemma 3.1. *Let Assumptions 2.2 and 2.3 hold, and let $u\in H^1_{loc}(\mathbb{R})$ be a solution of (1) satisfying the open waveguide radiation condition of Definition 2.5. Analogously to $\rho(x_1)$ of (16) (see Lemma 2.7), we define*

$$\rho_N(x_1):=\sum_{\sigma\in\{+,-\}}\int_{-N}^{N}\frac{|u_{rad}(y_1,\sigma h_0)|}{(1+|x_1-y_1|)^{3/2}}\,dy_1+\frac{1}{1+|x_1|^{3/2}},\ x_1\in\mathbb{R},N\in\mathbb{N}. \tag{18}$$

Then, there exists $c > 0$ and a sequence (N_m) in $\mathbb{N}$ converging to infinity such that

$$\int\limits_{|x_1|>N_m} \rho_{N_m}(x_1)^2\,dx_1 \;\le\; \frac{c}{\sqrt{N_m}}\,, \quad \int\limits_{|x_1|<N_m} |\rho(x_1)-\rho_{N_m}(x_1)|^2\,dx_1 \;\le\; \frac{c}{\sqrt{N_m}},$$

and

$$\int\limits_{N_m<|x_1|<N_m+1} \rho(x_1)^2\,dx_1 \;\le\; \frac{c}{\sqrt{N_m}}$$

for all $m \in \mathbb{N}$.

Proof. We define the sets $J_N := (-N-\sqrt{N}, -N+\sqrt{N}) \cup (N-\sqrt{N}, N+\sqrt{N})$. As in Chandler-Wilde,[18] we first note that for every $m \in \mathbb{N}$, there exists $N_m \ge m$ with $\|u_{rad}(\cdot,h_0)\|_{L^2(J_{N_m})} + \|u_{rad}(\cdot,-h_0)\|_{L^2(J_{N_m})} \le \frac{1}{N_m^{1/4}}$. Indeed, otherwise, there exists $m \in \mathbb{N}$ such that $\|u_{rad}(\cdot,h_0)\|_{L^2(J_N)} + \|u_{rad}(\cdot,-h_0)\|_{L^2(J_N)} \ge \frac{1}{N^{1/4}}$ for all $N \ge m$. Since $J_{N^2} \cap J_{M^2} = \emptyset$ for $N \ne M$, we would have

$$\sum_{\sigma\in\{-1,+1\}} \int\limits_{|x_1|>m^2-m} |u_{rad}(x_1,\sigma h_0)|^2\,dx_1 \ge \sum_{\sigma\in\{-1,+1\}} \sum_{N=m}^{\infty} \int\limits_{J_{N^2}} |u_{rad}(x_1,\sigma h_0)|^2\,dx_1$$

$$\ge \sum_{N=m}^{\infty} \frac{1}{N} \;=\; \infty,$$

a contradiction to $u_{rad}(\cdot,\pm h_0) \in L^2(\mathbb{R})$.

We set $N_m^- := N_m - \sqrt{N_m}$ for abbreviation and estimate for $|x_1| > N_m$:

$$\int\limits_{|y_1|<N_m} \frac{|u_{rad}(y_1,\sigma h_0)|}{(1+|x_1-y_1|)^{3/2}}\,dy_1$$

$$= \int\limits_{|y_1|<N_m^-} \frac{|u_{rad}(y_1,\sigma h_0)|}{(1+|x_1-y_1|)^{3/2}}\,dy_1 + \int\limits_{N_m^-<|y_1|<N_m} \frac{|u_{rad}(y_1,\sigma h_0)|}{(1+|x_1-y_1|)^{3/2}}\,dy_1$$

$$\le \|u_{rad}\|_{L^2(\mathbb{R})} \sqrt{\int\limits_{|y_1|<N_m^-} \frac{dy_1}{(1+|x_1|-|y_1|)^3}} + \|u_{rad}\|_{L^2(J_{N_m})} \sqrt{\int\limits_{N_m^-<|y_1|<N_m} \frac{dy_1}{(1+|x_1|-|y_1|)^3}}$$

$$\le \frac{c}{1+|x_1|-N_m^-} + \frac{1}{N_m^{1/4}}\,\frac{c}{1+|x_1|-N_m}$$

and thus

$$\int\limits_{|x_1|>N_m} \rho_{N_m}(x_1)^2\,dx_1 \le \frac{8}{(1+N_m)^2} + c\int\limits_{|x_1|>N_m} \frac{dx_1}{(1+|x_1|-N_m^-)^2}$$

$$+\frac{c}{\sqrt{N_m}} \int\limits_{|x_1|>N_m} \frac{dx_1}{(1+|x_1|-N_m)^2}$$

$$\le \frac{8}{(1+N_m)^2} + \frac{c}{1+\sqrt{N_m}} + \frac{c}{\sqrt{N_m}}.$$

Analogously, with $N_m^+ := N_m + \sqrt{N_m}$, we estimate for $|x_1| < N_m$:

$$
\begin{aligned}
\rho(x_1) - \rho_{N_m}(x_1) &= \int\limits_{|y_1|>N_m} \frac{|u_{rad}(y_1,\sigma h_0)|}{(1+|x_1-y_1|)^{3/2}}\,dy_1 \\
&= \int\limits_{|y_1|>N_m^+} \frac{|u_{rad}(y_1,\sigma h_0)|}{(1+|x_1-y_1|)^{3/2}}\,dy_1 + \int\limits_{N_m<|y_1|<N_m^+} \frac{|u_{rad}(y_1,\sigma h_0)|}{(1+|x_1-y_1|)^{3/2}}\,dy_1 \\
&\leq \|u_{rad}\|_{L^2(\mathbb{R})} \sqrt{\int\limits_{|y_1|>N_m^+} \frac{dy_1}{(1+|y_1|-|x_1|)^3}} + \|u_{rad}\|_{L^2(J_{N_m})} \sqrt{\int\limits_{N_m<|y_1|<N_m^+} \frac{dy_1}{(1+|y_1|-|x_1|)^3}} \\
&\leq \frac{c}{1+N_m^+-|x_1|} + \frac{1}{N_m^{1/4}}\,\frac{c}{1+N_m-|x_1|}
\end{aligned}
$$

and thus $\int_{|x_1|<N_m} |\rho(x_1) - \rho_{N_m}(x_1)|^2\,dx_1 \leq c/\sqrt{N_m}$ as before. Finally, for $N_m < |x_1| < N_m+1$, we estimate

$$
\begin{aligned}
\rho(x_1) &= \int\limits_{J_{N_m}} \frac{|u_{rad}(y_1,\sigma h_0)|}{(1+|x_1-y_1|)^{3/2}}\,dy_1 \\
&\quad + \int\limits_{|y_1|<N_m^-} \frac{|u_{rad}(y_1,\sigma h_0)|}{(1+|x_1-y_1|)^{3/2}}\,dy_1 + \int\limits_{|y_1|>N_m^+} \frac{|u_{rad}(y_1,\sigma h_0)|}{(1+|x_1-y_1|)^{3/2}}\,dy_1 \\
&\leq c\|u_{rad}\|_{L^2(J_{N_m})} + \|u_{rad}\|_{L^2(\mathbb{R})} \sqrt{\int\limits_{|y_1|<N_m^-} \frac{dy_1}{(1+|x_1|-|y_1|)^3}} \\
&\quad + \|u_{rad}\|_{L^2(\mathbb{R})} \sqrt{\int\limits_{|y_1|>N_m^+} \frac{dy_1}{(1+|y_1|-|x_1|)^3}} \\
&\leq \frac{c}{N_m^{1/4}} + \frac{c}{1+|x_1|-N_m^-} + \frac{c}{1+N_m^+-|x_1|} \leq \frac{c'}{N_m^{1/4}}.
\end{aligned}
$$

Integration with respect to x_1 yields the last assertion. □

After these preparations, we turn to the proof of uniqueness. From the following theorem, $f = 0$ implies that already the propagating part u_{prop} has to vanish.

Theorem 3.2. *Let again $f \in L^2(\mathbb{R}^2)$ and $q \in L^\infty(\mathbb{R}^2)$ have support in $Q := (0,2\pi) \times (-h_0,h_0)$ and $u \in H^1_{loc}(\mathbb{R}^2)$ a solution of $\Delta u + k^2(n+q)u = -f$ in $\mathbb{R}^2$ satisfying the open waveguide radiation condition of Definition 2.5. Then,*

$$
Im \int\limits_Q f\,\overline{u}\,dx \leq -\frac{1}{4\pi} \sum_{j\in J} \sum_{\ell:\lambda_{\ell,j}>0} \lambda_{\ell,j}\,|a_{\ell,j}|^2 + \frac{1}{4\pi} \sum_{j\in J} \sum_{:\ell:\lambda_{\ell,j}<0} \lambda_{\ell,j}\,|a_{\ell,j}|^2 \leq 0. \tag{19}
$$

Proof. Choose $\psi_N \in C^\infty(\mathbb{R})$ with $\psi_N(x_1) = 1$ for $|x_1| \leq N$ and $\psi_N(x_1) = 0$ for $|x_1| \geq N+1$. For $N > \sigma_0 + 1$ and $H > h_0 + 1$, we define the regions $D_{N,H} := (-N,N) \times (-H,H)$ and $W_{N,H}^- := (-N-1,-N) \times (-H,H)$ and $W_{N,H}^+ := (N,N+1) \times (-H,H)$ and the vertical and horizontal segments $I_{\pm N,H} := \{\pm N\} \times (-H,H)$ and $\Gamma_{N,\pm H} :=$

$(-N,N)\times\{\pm H\}$. We apply Green's theorem in $D_{N+1,H}$ to $v(x) := \psi_N(x_1)\,u(x)$ (note that $Q\subset D_{N,H}$):

$$\begin{aligned}
&\sum_{\sigma\in\{+,-\}}\sigma\int\limits_{\Gamma_{N+1,\sigma H}}\psi_N^2\,\overline{u}\,\frac{\partial u}{\partial x_2}\,ds\\
&=\sum_{\sigma\in\{+,-\}}\sigma\int\limits_{\Gamma_{N+1,\sigma H}}\overline{v}\,\frac{\partial v}{\partial x_2}\,ds=\int\limits_{D_{N+1,H}}\left[|\nabla v|^2+\overline{v}\,\Delta v\right]dx\\
&=\int\limits_{D_{N,H}}\left[|\nabla u|^2+\overline{u}\,\Delta u\right]dx+\int\limits_{W^+_{N,H}}\left[|\nabla v|^2+\overline{v}\,\Delta v\right]dx+\int\limits_{W^-_{N,H}}\left[|\nabla v|^2+\overline{v}\,\Delta v\right]dx;
\end{aligned}$$

that is, with $\Delta u=-k^2(n+q)u-f$,

$$\begin{aligned}
\operatorname{Im}\int\limits_{Q}f\,\overline{u}\,dx=\ &\operatorname{Im}\int\limits_{W^+_{N,H}}\left[|\nabla v|^2+\overline{v}\,\Delta v\right]dx+\operatorname{Im}\int\limits_{W^-_{N,H}}\left[|\nabla v|^2+\overline{v}\,\Delta v\right]dx\\
&-\sum_{\sigma\in\{+,-\}}\sigma\operatorname{Im}\int\limits_{\Gamma_{N+1,\sigma H}}\psi_N^2\,\overline{u}\,\frac{\partial u}{\partial x_2}\,ds.
\end{aligned}\tag{20}$$

We note that $\Delta v=-\psi_N k^2(n+q)u+2\psi_N'\frac{\partial u}{\partial x_1}+\psi_N''u$ and $\nabla v=\psi_N\nabla u+u\psi_N'e^{(1)}$ in $W^\pm_{N,H}$ where $e^{(1)} := (1,0)^\top$. The decomposition $u=u_{rad}+u_{prop}$ yields four terms in each of the three integrals on the right-hand side of (20).

(a) First, we look at the first two integrals on the right-hand side of (20). We define $v^{(1)}(x) := \psi_N(x_1)u_{rad}(x)$ and $v^{(2)}(x) := \psi_N(x_1)u_{prop}(x)$ and estimate the terms

$$a^\pm_{N,H}(j,\ell) := \int\limits_{W^\pm_{N,H}}\left[\nabla\overline{v^{(j)}}\cdot\nabla v^{(\ell)}+\overline{v^{(j)}}\,\Delta v^{(\ell)}\right]dx$$

for $j,\ell\in\{1,2\}$. Then, with (15),

$$\begin{aligned}
|a^+_{N,H}(1,1)|&\le c\,\|u_{rad}\|^2_{H^1(W^+_{N,h_0+1})}+c\,\|u_{rad}\|^2_{H^1(W^+_{N,H}\setminus W^+_{N,h_0+1})}\\
&\le c\|u_{rad}\|^2_{H^1(W^+_{N,h_0+1})}+c\int\limits_{N}^{N+1}\int\limits_{h_0+1<|x_2|<H}x_2^2\,\rho(x_1)^2\,dx_2\,dx_1\\
&\le c\,\gamma_{N,H}\qquad\text{with}
\end{aligned}\tag{21}$$

$$\gamma_{N,H} := \|u_{rad}\|^2_{H^1(Q_N)}+H^3\int\limits_{N<|x_1|<N+1}\rho(x_1)^2\,dx_1\tag{22}$$

and $Q_N := W^+_{N,h_0+1}\cup W^-_{N,h_0+1}=\{x\in\mathbb{R}^2 : N<|x_1|<N+1,\ |x_2|<h_0+1\}$. Analogously, since $\|u_{prop}\|_{H^1(W^+_{N,H})}$ and $\|\nabla u_{prop}\|_{H^1(W^+_{N,H})}$ are bounded with respect to N and H,

$$|a^+_{N,H}(1,2)|+|a^+_{N,H}(2,1)|\le c\left[\|u_{rad}\|^2_{H^1(W^+_{N,h_0+1})}+\|u_{rad}\|^2_{H^1(W^+_{N,H}\setminus W^+_{N,h_0+1})}\right]^{1/2}\le c\sqrt{\gamma_{N,H}}.$$

For $a^+_{N,H}(2,2)$, we apply Green's theorem:

$$a^+_{N,H}(2,2) = -\int\limits_{I_{N,H}} \overline{u_{prop}}\,\frac{\partial u_{prop}}{\partial x_1}\,ds + \sum_{\sigma\in\{+,-\}} \sigma \int\limits_{\substack{N<x_1<N+1\\ x_2=\sigma H}} \psi_N^2\,\overline{u_{prop}}\,\frac{\partial u_{prop}}{\partial x_2}\,ds$$
$$= -\int\limits_{I_N} \overline{u_{prop}}\,\frac{\partial u_{prop}}{\partial x_1}\,ds + \beta^+_{N,H}$$

with $I_N := \{N\} \times \mathbb{R}$ and

$$|\beta^+_{N,H}| \le \sum_{\sigma\in\{+,-\}} \left| \int\limits_{\substack{N<x_1<N+1\\ x_2=\sigma H}} \psi_N^2\,\overline{u_{prop}}\,\frac{\partial u_{prop}}{\partial x_2}\,ds \right| + \left| \int\limits_{I_N\setminus I_{N,H}} \overline{u_{prop}}\,\frac{\partial u_{prop}}{\partial x_1}\,ds \right| \le c\,e^{-2\delta H}.$$

The same estimates hold for $a^-_{N,H}(j,\ell)$, that is, the integrals over $W^-_{N,H}$. Therefore, using Lemma 2.6, we have shown that

$$\begin{aligned}
&\mathrm{Im}\int\limits_{W^+_{N,H}} \left[|\nabla v|^2 + \overline{v}\,\Delta v\right] dx + \mathrm{Im}\int\limits_{W^-_{N,H}} \left[|\nabla v|^2 + \overline{v}\,\Delta v\right] dx \\
&\le -\mathrm{Im}\int\limits_{I_N} \overline{u_{prop}}\,\frac{\partial u_{prop}}{\partial x_1}\,ds + \mathrm{Im}\int\limits_{I_{-N}} \overline{u_{prop}}\,\frac{\partial u_{prop}}{\partial x_1}\,ds + c\,e^{-2\delta H} + c\left[\gamma_{N,H} + \sqrt{\gamma_{N,H}}\right] \qquad (23)\\
&\le -\frac{1}{4\pi}\sum_{j\in J}\sum_{\lambda_{\ell,j}>0} \lambda_{\ell,j}\,|a_{\ell,j}|^2 + \frac{1}{4\pi}\sum_{j\in J}\sum_{\lambda_{\ell,j}<0} \lambda_{\ell,j}\,|a_{\ell,j}|^2 + c\,e^{-2\delta H} + c\left[\gamma_{N,H} + \sqrt{\gamma_{N,H}}\right].
\end{aligned}$$

(b) Now, we look at the third integral on the right-hand side of (20) and decompose again u into $u = u_{rad} + u_{prop}$. Using Cauchy-Schwarz and (15), we estimate for $\sigma \in \{-1, 1\}$

$$\begin{aligned}
&\int\limits_{\Gamma_{N+1,\sigma H}} \psi_N^2 \left| \overline{u_{rad}}\,\frac{\partial u_{prop}}{\partial x_2} + \overline{u_{prop}}\,\frac{\partial u_{rad}}{\partial x_2} + \overline{u_{prop}}\,\frac{\partial u_{prop}}{\partial x_2} \right| ds \\
&\le c\,\|u_{rad}\|_{L^2(\Gamma_{N+1,\sigma H})} \left\| \frac{\partial u_{prop}}{\partial x_2} \right\|_{L^2(\Gamma_{N+1,\sigma H})} + c\,\|u_{prop}\|_{L^2(\Gamma_{N+1,\sigma H})} \left\| \frac{\partial u_{rad}}{\partial x_2} \right\|_{L^2(\Gamma_{N+1,\sigma H})} \\
&+ c\,\|u_{prop}\|_{L^2(\Gamma_{N+1,\sigma H})} \left\| \frac{\partial u_{prop}}{\partial x_2} \right\|_{L^2(\Gamma_{N+1,\sigma H})} \le c\left[H\,\|\rho\|_{L^2(\mathfrak{R})}\sqrt{N} + N\right] e^{-\delta H}.
\end{aligned}$$

Finally, we consider $\int_{\Gamma_{N+1,\pm H}} \psi_N^2\,\overline{u_{rad}}\,\frac{\partial u_{rad}}{\partial x_2}\,ds$. We approximate u_{rad} by functions $u^{N,H}_{rad}$ which satisfy the homogeneous Helmholtz equation for $|x_2| > H$. To do this, we restrict ourselves to the region $x_2 > h_0$ and set $u^{N,H}_{rad} := u^+_N + w^+_H$ for $x_2 > h_0$ where u^+_N is the unique radiating solution of $\Delta u^+_N + k^2 u^+_N = 0$ for $x_2 > h_0$ and $u^+_N(x_1, h_0) = u_{rad}(x_1, h_0)$ for $|x_1| < N$ and $u^+_N(x_1, h_0) = 0$ for $|x_1| > N$, while the function w^+_H is defined as the unique radiating solution of

$$\Delta w^+_H + k^2 w^+_H = \begin{cases} -\sum_{j\in J}\sum_{\ell=1}^{m_j} a_{\ell,j}\,\varphi_{\ell,j} & \text{for } h_0 < x_2 < H, \\ 0 & \text{for } x_2 > H, \end{cases}$$

and $w_H^+ = 0$ for $x_2 = h_0$. Then, u_N^+ and w_H^+ are given by (compare with (13))

$$u_N^+(x) = \frac{i}{2}\int_{-N}^{N} u_{rad}(y_1, h_0)\,\frac{\partial}{\partial y_2} H_0^{(1)}(k\sqrt{(x_1-y_1)^2+(x_2-h_0)^2})\,dy_1\,,\ x_2 > h_0,$$

$$w_H^+(x) = \sum_{j\in J}\sum_{\ell=1}^{m_j} a_{\ell,j}\int_{h_0}^{H}\int_{-\sigma_0}^{\sigma_0} G^+(x,y)\,\varphi_{\ell,j}(y)\,dy_1 dy_2\,,\ x_2 > h_0,$$

and it is easy to show by modifying the proof of Lemma 2.7 that

$$\begin{aligned} \left|u_{rad}^{N,H}(x)\right| + \left|\nabla u_{rad}^{N,H}(x)\right| &\le c x_2\,\rho_N(x_1), \\ \left|u_{rad}(x) - u_{rad}^{N,H}(x)\right| + \left|\nabla\left(u_{rad}(x) - u_{rad}^{N,H}(x)\right)\right| &\le c x_2\,[\rho(x_1) - \rho_N(x_1)] + \frac{c x_2}{|x|^{3/2}}\,e^{-\delta H}, \end{aligned}$$

for all $x \in \mathbb{R}^2$ with $x_2 \ge h_0 + 1$, where $\rho, \rho_N \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ are given by (16) and (18), respectively. The functions $u_{rad}^{N,H}$ for $x_2 < -h_0$ are defined analogously. With $\Gamma_{\infty,\pm H} := \mathbb{R} \times \{\pm H\}$, we decompose

$$\begin{aligned} &\int_{\Gamma_{N+1,\pm H}} \psi_N^2\,\overline{u_{rad}}\,\frac{\partial u_{rad}}{\partial x_2}\,ds \\ &= \int_{\Gamma_{\infty,\pm H}} \overline{u_{rad}^{N,H}}\,\frac{\partial u_{rad}^{N,H}}{\partial x_2}\,ds + \int_{\Gamma_{N+1,\pm H}} \psi_N^2\left[\overline{u_{rad}}\,\frac{\partial u_{rad}}{\partial x_2} - \overline{u_{rad}^{N,H}}\,\frac{\partial u_{rad}^{N,H}}{\partial x_2}\right] ds \\ &\quad - \int_{\Gamma_{\infty,\pm H}\setminus\Gamma_{N+1,\pm H}} \overline{u_{rad}^{N,H}}\,\frac{\partial u_{rad}^{N,H}}{\partial x_2}\,ds + \int_{\Gamma_{N+1,\pm H}\setminus\Gamma_{N,\pm H}} (\psi_N^2 - 1)\,\overline{u_{rad}^{N,H}}\,\frac{\partial u_{rad}^{N,H}}{\partial x_2}\,ds \\ &= \int_{\Gamma_{\infty,\pm H}} \overline{u_{rad}^{N,H}}\,\frac{\partial u_{rad}^{N,H}}{\partial x_2}\,ds + \eta_{N,\pm H}, \end{aligned}$$

where

$$\begin{aligned} |\eta_{N,\pm H}| &\le c\,\|u_{rad} - u_{rad}^{N,H}\|_{L^2(\Gamma_{N+1,\pm H})}\left\|\frac{\partial u_{rad}}{\partial x_2}\right\|_{L^2(\Gamma_{N+1,\pm H})} \\ &\quad + c\,\|u_{rad}^{N,H}\|_{L^2(\Gamma_{N+1,\pm H})}\left\|\frac{\partial u_{rad}}{\partial x_2} - \frac{\partial u_{rad}^{N,H}}{\partial x_2}\right\|_{L^2(\Gamma_{N+1,\pm H})} \\ &\quad + c\,\|u_{rad}^{N,H}\|_{L^2(\Gamma_{\infty,\pm H}\setminus\Gamma_{N,\pm H})}\left\|\frac{\partial u_{rad}^{N,H}}{\partial x_2}\right\|_{L^2(\Gamma_{\infty,\pm H}\setminus\Gamma_{N,\pm H})} \\ &\le\ cH^2\,\|\rho\|_{L^2(\mathbb{R})}\sqrt{\int_{|x_1|<N} |\rho(x_1) - \rho_N(x_1)|^2\,dx_1} + cH^2\int_{|x_1|>N} \rho_N(x_1)^2\,dx_1. \end{aligned} \tag{24}$$

Now, we show that the imaginary part of $\sigma\int_{\Gamma_{\infty,\sigma H}} \overline{u_{rad}^{N,H}}\,\frac{\partial u_{rad}^{N,H}}{\partial x_2}\,ds$ is non-negative. Indeed, we take the Fourier transform $\hat{u}_{N,H}(\omega, x_2) = (\mathcal{F}u_{rad}^{N,H})(\omega, x_2)$ for $\sigma x_2 > H$. Then,

$$\int_{\Gamma_{\infty,\sigma H}} \overline{u_{rad}^{N,H}}\,\frac{\partial u_{rad}^{N,H}}{\partial x_2}\,ds = \int_{-\infty}^{\infty} \overline{\hat{u}_{N,H}(\omega, \sigma H)}\,\hat{u}'_{N,H}(\omega, \sigma H)\,d\omega. \tag{25}$$

Furthermore, $\hat{u}_{N,H}''(\omega,x_2)+(k^2-\omega^2)\,\hat{u}_{N,H}=0$ for $|x_2|>H$ and satisfies the radiation condition (14). Therefore, $\hat{u}_{N,H}$ has the form

$$\hat{u}_{N,H}(\omega,x_2)=\begin{cases}\hat{u}_{N,H}(\omega,H)\,e^{i\sqrt{k^2-\omega^2}(x_2-H)} & \text{for } x_2>H,\\ \hat{u}_{N,H}(\omega,-H)\,e^{i\sqrt{k^2-\omega^2}(-x_2-H)} & \text{for } x_2<-H,\end{cases}$$

and thus, $\sigma\overline{\hat{u}_{N,H}(\omega,\sigma H)}\,\hat{u}_{N,H}'(\omega,\sigma H)=i\,|\hat{u}_{N,H}(\omega,\sigma H)|^2\sqrt{k^2-\omega^2}$, and its imaginary part is therefore non-negative.

At this point, we set $N:=N_m$, where (N_m) is the sequence from Lemma 3.1. Then, from (22) and (24) in combination with the estimates of Lemma 3.1, we conclude that $\gamma_{N_m,H}\le c\,\|u_{rad}\|^2_{H^1(Q_{N_m})}+c\,\frac{H^3}{\sqrt{N_m}}$ and $|\eta_{N_m,\pm H}|\le c\,\frac{H^2}{N_m^{1/4}}$. We choose $H=H_m$ such that the reminders converge to zero, for example, $H_m:=N_m^{1/10}$. Then,

$$\sum_{\sigma\in\{-1,+1\}}\sigma\limsup_{m\to\infty}\left[\operatorname{Im}\int\limits_{\Gamma_{N_m+1,\sigma H_m}}\psi_{N_m}\,\overline{u_{rad}}\,\frac{\partial u_{rad}}{\partial x_2}\,ds\right]\ge 0,$$

and, from (23),

$$\liminf_{m\to\infty}\left[\operatorname{Im}\int\limits_{W^+_{N_m,H_m}}\left[|\nabla v|^2+\overline{v}\,\Delta v\right]dx+\operatorname{Im}\int\limits_{W^-_{N_m,H_m}}\left[|\nabla v|^2+\overline{v}\,\Delta v\right]dx\right]$$
$$\le-\frac{1}{4\pi}\sum_{j\in J}\sum_{\lambda_{\ell,j}>0}\lambda_{\ell,j}\,|a_{\ell,j}|^2+\frac{1}{4\pi}\sum_{j\in J}\sum_{\lambda_{\ell,j}<0}\lambda_{\ell,j}\,|a_{\ell,j}|^2.$$

Estimate (19) follows now from (20). □

We are now able to prove (partial) uniqueness.

Theorem 3.3. *Let Assumptions 2.2 and 2.3 hold, and let $u\in H^1_{loc}(\mathbb{R}^2)$ solve the problem (1) for $f=0$ and the open waveguide radiation condition of Definition 2.5. Then, u is a bound state; that is, $u\in H^1(\mathbb{R}^2)$. In other words, k^2 is in the point spectrum of $-\frac{1}{n+q}\Delta$. In the unperturbed case $q=0$, there are no bound states; that is, $u=0$ follows.*

Proof. From (19) of the previous theorem, we conclude that the coefficients $a_{\ell,j}$ vanish. Therefore, $u=u_{rad}\in H^1(W_H)$ for all $H>h_0$ where again $W_H:=\mathbb{R}\times(-H,H)$. We now show that $u=u_{rad}$ is a bound state under a smoothness assumption on its Fourier transform. The latter property is shown in Corollary 4.5 below.

From Green's theorem applied in W_H, we conclude (compare with (20)) that

$$\operatorname{Im}\sum_{\sigma\in\{+,-\}}\sigma\int\limits_{-\infty}^{\infty}\overline{u(x_1,\sigma H)}\,\frac{\partial u(x_1,\sigma H)}{\partial x_2}\,dx_1=\operatorname{Im}\int\limits_{W_H}\left[|\nabla u|^2-k^2(n+q)\,|u|^2\right]dx=0.$$

Transforming this equation to the Fourier space, we observe just as in (25) that $(\mathcal{F}u)(\omega,\pm H)$ vanishes for all $|\omega|<k$. For $|\omega|>k$, we conclude again that

$$(\mathcal{F}u)(\omega,x_2)=(\mathcal{F}u)(\omega,\pm H)\,e^{-\sqrt{\omega^2-k^2}(\pm x_2-H)}\quad\text{for }\pm x_2>H,\tag{26}$$

and thus, for $|\omega|>k$,

$$\int\limits_H^{\infty}|(\mathcal{F}u)(\omega,x_2)|^2\,dx_2=|(\mathcal{F}u)(\omega,H)|^2\int\limits_H^{\infty}e^{-2\sqrt{\omega^2-k^2}(x_2-H)}\,dx_2=\frac{|(\mathcal{F}u)(\omega,H)|^2}{2\sqrt{\omega^2-k^2}}.$$

The integrand vanishes for $|\omega|<k$. The analogous formula holds for the integral $\int_{-\infty}^{-H}|(\mathcal{F}u)(\omega,x_2)|^2\,dx_2$. Now, we use the fact that $(\mathcal{F}u)(\cdot,\pm H)$ is continuous in a neighborhood of $\omega=\pm k$ which we will prove in Corollary 4.5 below (set $g:=k^2qu$ in this corollary). Therefore, the integral is integrable with respect to $\omega\in\mathbb{R}$ and, by Parseval's theorem, $u\in H^1(\mathbb{R}^2)$. This implies that u is a bound state.

In the case $q = 0$, we recall that u satisfies the differential equation $\Delta u + k^2 nu = 0$ in $\mathbb{R}^2$ and, because of (26), the generalized angular spectral radiation condition (10). Theorem 4.1 below implies that almost all $\alpha \in (-1/2, 1/2)$ are propagative wave numbers which contradicts the fact that there exist only finitely many of them. □

Remark 3.4. In the case of general q, we call this a partial uniqueness result in contrast to the complete uniqueness result where in addition the absence of bound states has to be shown. For general q, such a complete uniqueness result is not known to the author. For the unperturbed case $q = 0$, however, we have shown above the absence of bound states under Assumptions 2.2. However, this assumption is not needed as proven in Hoang and Radosz.[19]

4 | THE FLOQUET-BLOCH TRANSFORM AND QUASI-PERIODIC PROBLEMS

In this section, we collect properties of the Floquet-Bloch transform and quasi-periodic scattering problems. These results are essential for proving existence of a solution and the asymptotics of the radiating part u_{rad}. As a standard reference for the Floquet-Bloch transform, we recommend Kuchment's monograph.[20] For $\phi \in C_0^\infty(\mathbb{R})$, the Floquet-Bloch transform F is defined by

$$(F\phi)(x_1, \alpha) := \sum_{\ell \in \mathbb{Z}} \phi(x_1 + 2\pi\ell) e^{-i2\pi\ell\alpha}, \ x_1, \alpha \in \mathbb{R}.$$

Then, $(F\phi)(\cdot, \alpha)$ is α−quasi-periodic, and $(F\phi)(x_1, \cdot)$ is periodic with period 1. Therefore, we can restrict ourselves to $x_1 \in [0, 2\pi]$ and $\alpha \in [-1/2, 1/2]$. Setting $R := (0, 2\pi) \times (-1/2, 1/2)$ for abbreviation, F has an extension to an unitary operator from $L^2(\mathbb{R})$ into $L^2(R)$; that is,

$$\int_{-\infty}^{\infty} v(x_1)\overline{\psi(x_1)}\, dx_1 = \int_{-1/2}^{1/2}\int_0^{2\pi} (Fv)(x_1, \alpha)\overline{(F\psi)(x_1, \alpha)}\, dx_1\, d\alpha, \ v, \psi \in L^2(\mathbb{R}). \tag{27}$$

The inverse transform is given by

$$\phi(x_1) = \int_{-1/2}^{1/2} (F\phi)(x_1, \alpha)\, d\alpha, \ x_1 \in \mathbb{R}, \tag{28}$$

where $(F\phi)(\cdot, \alpha)$ has to be extended α-quasi-periodically into $\mathbb{R}$. We note the following connection between the Fourier transform $\mathcal{F}\phi$ of $\phi \in L^2(\mathbb{R})$ and the Fourier coefficients $\tilde{\phi}_\ell(\alpha)$ of the α-quasi-periodic function $(F\phi)(\cdot, \alpha)$:

$$(\mathcal{F}\phi)(\ell + \alpha) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} (F\phi)(x_1, \alpha) e^{-i(\ell+\alpha)x_1}\, dx_1 = \tilde{\phi}_\ell(\alpha), \ \ell \in \mathbb{Z}, \tag{29}$$

which is easily seen by decomposing $\mathbb{R}$ in the definition of the Fourier transform into $\mathbb{R} = \bigcup_{\ell \in \mathbb{Z}} (2\pi\ell, 2\pi(\ell + 1))$.

With a slight abuse of notation, we use the symbol of F also for functions u of two variables. Therefore, let

$$(Fu)(x_1, x_2, \alpha) := \sum_{\ell \in \mathbb{Z}} u(x_1 + 2\pi\ell, x_2) e^{-i2\pi\ell\alpha} \tag{30}$$

for $x \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$ denote the Floquet-Bloch transform of $u(\cdot, x_2)$ with respect to x_1. Then, it is well known (see, e.g., Lechleiter[21]) that F maps $H^s(W_H)$ onto

$$L^2\left((-1/2, 1/2), H^s_\alpha(Q^H)\right) := \left\{ v \in L^2 : \begin{array}{l} v(\cdot, \alpha) \in H^s_\alpha(Q^H) \text{for almost all}\, \alpha \,\text{and} \\ \alpha \mapsto \|v(\cdot, \alpha)\|_{H^s(Q^H)} \text{is in} L^2(-1/2, 1/2) \end{array} \right\}$$

for all $s \in \mathbb{R}$. Here, $W_H := \mathbb{R} \times (-H, H)$ and $Q^H := (0, 2\pi) \times (-H, H)$, and $H^s_\alpha(Q^H)$ denotes the subspace of $H^s(Q^H)$ consisting of α-quasi-periodic functions. It can be characterized by the decay of the Fourier coefficients; that is, $\psi \in H^s_\alpha(\mathbb{R}^2)$ if, and only if, $\int_{-\infty}^{\infty} \sum_{\ell \in \mathbb{Z}} (1 + \ell^2 + \omega^2)^s |\hat{\psi}_\ell(\omega)|^2 d\omega < \infty$ where $\hat{\psi}_\ell(\omega)$ are the Fourier coefficients of the Fourier transform $\hat{\psi}(x_1, \omega)$ with respect to x_2 which is itself α−quasi-periodic with respect to x_1.

From (11a), we note that u_{rad} satisfies $\Delta u_{rad} + k^2 n u_{rad} = -g$ in $\mathbb{R}^2$ where $g = f + k^2 q u_{rad} + \sum_{j\in J}\sum_{\ell=1}^{m_j} a_{\ell,j}\varphi_{\ell,j}$ and thus $\Delta(Fu_{rad})(\cdot,\alpha) + k^2 n(Fu_{rad})(\cdot,\alpha) = -(Fg)(\cdot,\alpha)$ in $Q^\infty := (0,2\pi)\times\mathbb{R}$. The right-hand side $(Fg)(\cdot,\alpha)$ is not compactly supported with respect to x_2. Nevertheless, we can rewrite the problem as a variational equation in a bounded domain by well-known techniques using the Dirichlet-Neumann operator. This is done in the following theorem where we write u for u_{rad}. In Theorem 4.2, we will prove existence and in Theorem 4.3 smoothness of the solution with respect to the parameter α.

First, analogously to $Q^H := (0,2\pi)\times(-H,H)$ define the regions $Q_+^H := (0,2\pi)\times(H,\infty)$ and $Q_-^H := (0,2\pi)\times(-\infty,-H)$, recall $W_H := \mathbb{R}\times(-H,H)$ and let $\Gamma^H := ((0,2\pi)\times\{H\})\cup((0,2\pi)\times\{-H\})$ denote the horizontal part of ∂Q^H.

Theorem 4.1. *Let $g\in L^2(\mathbb{R}^2)$ with $g(x)=0$ for $|x_1|>\sigma_0$ and $|g(x)|\le \hat{c}e^{-\delta|x_2|}$ in $\mathbb{R}^2$ for some $\sigma_0,\hat{c},\delta>0$.*

(a) For every $\alpha\in[-1/2,1/2]$, there exists a unique α-quasi-periodic solution $w_\alpha^\pm \in H^1_{\alpha,loc}(Q_\pm^H)$ of $\Delta w_\alpha^\pm + k^2 w_\alpha^\pm = -(Fg)(\cdot,\alpha)$ in $Q_\pm^H$, $w_\alpha^\pm = 0$ for $x_2=\pm H$, which satisfies the generalized Rayleigh condition

$$\sum_{\ell\in\mathbb{Z}}\left|(\mathrm{sign}\,x_2)\frac{dw_{\alpha,\ell}^\pm(x_2)}{dx_2} - i\sqrt{k^2-(\ell+\alpha)^2}\,w_{\alpha,\ell}^\pm(x_2)\right|^2 \to 0,\ x_2\to\pm\infty,$$

where $w_{\alpha,\ell}^\pm(x_2) = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} w_\alpha^\pm(x)e^{-i(\ell+\alpha)x_1}dx_1$ are the Fourier coefficients of $w_\alpha^\pm(\cdot,x_2)$.

(b) Let $u\in H^1_{loc}(\mathbb{R}^2)$ with $u\in H^1(W_H)$ for every $H>h_0$ satisfy $\Delta u + k^2 nu = -g$ in $\mathbb{R}^2$ and the generalized angular spectrum radiation condition (10). Then, for almost all $\alpha\in(-1/2,1/2)$, the transform $\tilde{u}_\alpha := (Fu)(\cdot,\alpha)\in H^1_{\alpha,loc}(Q^\infty)$ satisfies

$$\Delta\tilde{u}_\alpha + k^2 n\tilde{u}_\alpha = -(Fg)(\cdot,\alpha)\ \text{in}\ Q^\infty \tag{31a}$$

in the variational sense and the generalized Rayleigh condition; that is,

$$\sum_{\ell\in\mathbb{Z}}\left|(\mathrm{sign}\,x_2)\,\tilde{u}'_{\alpha,\ell}(x_2) - i\sqrt{k^2-(\ell+\alpha)^2}\,\tilde{u}_{\alpha,\ell}(x_2)\right|^2 \to 0,\ |x_2|\to\infty. \tag{31b}$$

(c) For fixed $\alpha\in[-1/2,1/2]$, the problem (31a)-(31b) is equivalent to the variational equation

$$\int_{Q^H}[\nabla\tilde{u}_\alpha\cdot\nabla\overline{\psi} - k^2 n\,\tilde{u}_\alpha\overline{\psi}]\,dx - \int_{\Gamma^H}(\Lambda_\alpha\tilde{u}_\alpha)\overline{\psi}\,ds = \int_{Q^H}(Fg)(\cdot,\alpha)\overline{\psi}\,dx + \int_{\Gamma^H}\frac{\partial w_\alpha}{\partial\nu}\overline{\psi}\,ds \tag{32}$$

for all $\psi\in H^1_\alpha(Q^H)$ where $\Lambda_\alpha : H^{1/2}_\alpha(\Gamma^H)\to H^{-1/2}_\alpha(\Gamma^H)$ is the α-quasi-periodic Dirichlet-to-Neumann operator given by

$$(\Lambda_\alpha\phi)(x_1,\pm H) := \frac{i}{\sqrt{2\pi}}\sum_{\ell\in\mathbb{Z}}\sqrt{k^2-(\ell+\alpha)^2}\,\phi_\ell(\pm H)e^{i(\ell+\alpha)x_1},\ x_1\in(0,2\pi), \tag{33}$$

for $\phi\in H^{1/2}_\alpha(\Gamma^H)$, and where $\partial w_\alpha/\partial\nu := \pm\partial w_\alpha^\pm/\partial x_2$ for $x_2=\pm H$ with the solutions $w_\alpha^\pm$ from part (a).

(d) For fixed $\alpha\in[-1/2,1/2]$, the variational equation (32) can be written as

$$(I-K_\alpha)\tilde{u}_\alpha = r_\alpha\ \text{in}\ H^1_\alpha(Q^H), \tag{34}$$

where $r_\alpha\in H^1_\alpha(Q^H)$ and K_α is a compact linear operator from $H^1_\alpha(Q^H)$ into itself. The operator $I-K_\alpha$ is invertible if, and only if, α is not a propagative wave number.

Proof.

(a) We show that $w_\alpha^\pm$ is given by $w_\alpha^\pm(x) = \sum_{\ell\in\mathbb{Z}} w_{\alpha,\ell}^\pm(x_2)\, e^{i(\ell+\alpha)x_1}$ for $x \in Q_\pm^H$ where the Fourier coefficients are given by

$$w_{\alpha,\ell}^\pm(x_2) := \frac{i}{2\sqrt{k^2-(\ell+\alpha)^2}} \int_H^\infty (Fg)_\ell(\pm y_2,\alpha) \left[e^{i\sqrt{k^2-(\ell+\alpha)^2}|x_2\mp y_2|} - e^{i\sqrt{k^2-(\ell+\alpha)^2}(\pm(x_2+y_2)-2H)} \right] dy_2 ,\ \pm x_2 > H ,\ \ell \in \mathbb{Z}. \tag{35}$$

Indeed, to show the generalized Rayleigh condition for w_α^+, we split the integral from H to x_2 and from x_2 to ∞ and compute

$$\frac{d}{dx_2} w_{\alpha,\ell}^+(x_2) - i\sqrt{k^2-(\ell+\alpha)^2}\, w_{\alpha,\ell}^+(x_2) = -\frac{1}{2}\int_{x_2}^\infty (Fg)_\ell(y_2,\alpha)\, e^{i\sqrt{k^2-(\ell+\alpha)^2}(y_2-x_2)}\, dy_2 .$$

For $|\ell+\alpha| > k$, we use the Cauchy-Schwarz inequality and estimate

$$\begin{aligned}
&\left| \frac{d}{dx_2} w_{\alpha,\ell}^+(x_2) - i\sqrt{k^2-(\ell+\alpha)^2}\, w_{\alpha,\ell}^+(x_2) \right|^2 \\
&\le \frac{1}{4}\int_{x_2}^\infty |(Fg)_\ell(y_2,\alpha)|^2\, dy_2 \int_{x_2}^\infty e^{-2\sqrt{(\ell+\alpha)^2-k^2}(y_2-x_2)}\, dy_2 \\
&= \frac{1}{8\sqrt{(\ell+\alpha)^2-k^2}} \int_{x_2}^\infty |(Fg)_\ell(y_2,\alpha)|^2\, dy_2
\end{aligned}$$

and thus

$$\begin{aligned}
&\sum_{|\ell+\alpha|>k} \left| \frac{d}{dx_2} w_{\alpha,\ell}^+(x_2) - i\sqrt{k^2-(\ell+\alpha)^2}\, w_{\alpha,\ell}^+(x_2) \right|^2 \\
&\le c \sum_{\ell\in\mathbb{Z}} \int_{x_2}^\infty |(Fg)_\ell(y_2,\alpha)|^2\, dy_2 = c \int_0^{2\pi}\int_{x_2}^\infty |(Fg)(y_1,y_2,\alpha)|^2\, dy_2\, dy_1 ,
\end{aligned}$$

and this tends to zero as x_2 tends to infinity. For $|\ell+\alpha| \le k$, we estimate

$$\begin{aligned}
&\left| \frac{d}{dx_2} w_{\alpha,\ell}^+(x_2) - i\sqrt{k^2-(\ell+\alpha)^2}\, w_{\alpha,\ell}^+(x_2) \right| \\
&\le \frac{1}{2}\int_{x_2}^\infty |(Fg)_\ell(y_2,\alpha)|\ dy_2 \le \frac{1}{2\sqrt{2\pi}} \int_{x_2}^\infty\int_0^{2\pi} |(Fg)(y_1,y_2,\alpha)|\ dy_1\, dy_2 ,
\end{aligned}$$

and this tends to zero as x_2 tends to infinity because $(Fg)(\cdot,\alpha) \in L^1(Q_+^H)$. In the same way, it is shown that

$$\sum_{\ell\in\mathbb{Z}} \left[\left| \frac{dw_{\alpha,\ell}^\pm(x_2)}{dx_2} \right| + \sqrt{1+\ell^2}\, |w_{\alpha,\ell}^\pm(x_2)| \right] \le c \left[\|(Fg)(\cdot,\alpha)\|_{L^2(Q_\pm^H)} + \|(Fg)(\cdot,\alpha)\|_{L^1(Q_\pm^H)} \right]$$

for $\pm x_2 > H$ and thus $w_\alpha^\pm \in H^1_{\alpha,loc}(Q_\pm^H)$ with

$$\|w_\alpha^\pm\|_{H^1(Q_\pm^H \setminus Q_\pm^{H'})} \le c_{H,H'} \left[\|(Fg)(\cdot,\alpha)\|_{L^2(Q_\pm^H)} + \|(Fg)(\cdot,\alpha)\|_{L^1(Q_\pm^H)} \right] \tag{36}$$

for $H' > H$. We omit the proofs of uniqueness and the fact that $w_\alpha^\pm$ satisfies the differential equation.

(b) The variational form of $\Delta u + k^2 n u = -g$ is given by

$$\int_{\mathbb{R}^2} [\nabla u \cdot \nabla \overline{\psi} - k^2 n u \overline{\psi}]\, dx = \int_Q g \overline{\psi}\, dx$$

for all $\psi \in H^1(\mathbb{R}^2)$ which vanish for $|x_2| > H$ for some $H > h_0$. From (27) and the fact that F commutes with differentiation yields the equivalent form

$$\int_{Q^\infty} \int_{-1/2}^{1/2} [\nabla_x (Fu)(x,\alpha) \cdot \nabla_x \overline{\psi(x,\alpha)} - k^2 n(x) (Fu)(x,\alpha) \overline{\psi(x,\alpha)}]\, d\alpha\, dx$$
$$= \int_Q \int_{-1/2}^{1/2} (Fg)(x,\alpha) \overline{\psi(x,\alpha)}\, d\alpha\, dx$$

for all $\psi \in L^2\left((-1/2,1/2), H^1_\alpha(Q^\infty)\right)$ which vanish for $|x_2| > H$ for some $H > h_0$. For any $\psi_1 \in L^2(-1/2,1/2)$ and any $\psi_2 \in H^1_{per}(Q^\infty) := \left\{\psi \in H^1(Q^\infty) : \psi(\cdot, x_2) \text{is}\, 2\pi - \text{periodic}\right\}$ which vanishes for $|x_2| > H$ for some $H > h_0$, we set $\psi(x,\alpha) := e^{i\alpha x_1} \psi_1(\alpha) \psi_2(x)$. Substituting this $\psi \in L^2\left((-1/2,1/2), H^1_\alpha(Q^\infty)\right)$ into the variational equation and the fact that $\int_{-1/2}^{1/2} \chi(\alpha) \overline{\psi_1(\alpha)} d\alpha = 0$ for all $\psi_1 \in L^2(-1/2,1/2)$ implies that χ vanishes almost everywhere yields

$$\int_{Q^\infty} [\nabla_x (Fu)(x,\alpha) \cdot \nabla_x \overline{(e^{i\alpha x_1} \psi_2(x))} - k^2 n(x) (Fu)(x,\alpha) \overline{(e^{i\alpha x_1} \psi_2(x))}]\, dx$$
$$= \int_Q (Fg)(x,\alpha) \overline{(e^{i\alpha x_1} \psi_2(x))}\, dx$$

for almost all α and thus

$$\int_{Q^\infty} [\nabla_x (Fu)(x,\alpha) \cdot \nabla \overline{\psi(x)} - k^2 n(x) (Fu)(x,\alpha) \overline{\psi(x)}]\, dx = \int_Q (Fg)(x,\alpha) \overline{\psi(x)}\, dx$$

for all $\psi \in H^1_\alpha(Q^\infty)$ which vanish for $|x_2| > H$ for some $H > h_0$. This is the variational form of (31a). We leave it to the reader to show that the application of (29) to (10) implies the generalized Rayleigh condition (31b).

(c) We define v_α by $v_\alpha := \tilde{u}_\alpha$ in Q^H and $v_\alpha := \tilde{u}_\alpha + w_\alpha^\pm$ in $Q^H_\pm$. Then, v_α solves (in a variational form) $\Delta v_\alpha + k^2 v_\alpha = 0$ in $Q^\infty \setminus \overline{Q^H}$ and $\Delta v_\alpha + k^2 n v_\alpha = -(Fg)(\cdot,\alpha)$ in Q^H and $v_\alpha|_- = v_\alpha|_+$ on Γ^H and $\partial v_\alpha/\partial \nu|_- = \partial v_\alpha/\partial \nu|_+ - \partial w_\alpha/\partial \nu$ on Γ^H. The reduction of this problem to the variational equation (32) is standard and omitted.

(d) Using

$$\int_{\Gamma^H} (\Lambda_\alpha \tilde{u}) \overline{\psi}\, ds = i \sum_{\sigma \in \{+,-\}} \sum_{\ell \in \mathbb{Z}} \sqrt{k^2 - (\ell + \alpha)^2}\, \tilde{u}_\ell(\sigma h_0) \overline{\psi_\ell(\sigma h_0)},$$

we write (32) in the form

$$\int_Q [\nabla \tilde{u}_\alpha \cdot \nabla \overline{\psi} + \tilde{u}_\alpha \overline{\psi}]\, dx + \sum_{\sigma \in \{+,-\}} \sum_{|\ell| \geq k+1} |\ell|\, \tilde{u}_{\alpha,\ell}(\sigma h_0) \overline{\psi_\ell(\sigma h_0)}$$
$$- \int_Q (k^2 n + 1) \tilde{u}_\alpha \overline{\psi}]\, dx - i \sum_{\sigma \in \{+,-\}} \sum_{|\ell| < k+1} \sqrt{k^2 - (\ell + \alpha)^2}\, \tilde{u}_{\alpha,\ell}(\sigma h_0) \overline{\psi_\ell(\sigma h_0)}$$
$$- \sum_{\sigma \in \{+,-\}} \sum_{|\ell| \geq k+1} \left[|\ell| - \sqrt{(\ell + \alpha)^2 - k^2}\right] \tilde{u}_{\alpha,\ell}(\sigma h_0) \overline{\psi_\ell(\sigma h_0)}$$
$$= \int_Q (Fg)(\cdot,\alpha) \overline{\psi}\, dx + \int_{\Gamma^H} \frac{\partial w_\alpha}{\partial \nu} \overline{\psi}\, ds \quad \text{for all} \;\; \psi \in H^1_\alpha(Q).$$

Since the two terms in the first line describe a coercive sesqui-linear form, we can write these two terms as $(A\tilde{u}_\alpha, \psi)_{H^1(Q)}$ for some isomorphism A from $H^1_\alpha(Q)$ onto itself (theorem of Lax-Milgram). By the representation theorem of Riesz and the compact embedding of $H^1(Q^H)$ in $L^2(Q^H)$, the remaining parts can be written as $(B_\alpha\tilde{u}_\alpha, \psi)_{H^1(Q)}$ for some compact operator B_α. (Note that $|\ell| - \sqrt{(\ell+\alpha)^2 - k^2}$ is bounded!) Application of the theorem of Riesz to the right-hand side yields an equation of the type $(A - B_\alpha)\tilde{u}_\alpha = R_\alpha$ which has the form (34) with $K_\alpha = A^{-1}B_\alpha$ and $r_\alpha = A^{-1}R_\alpha$. □

The following result answers the question of existence of solutions of quasi-periodic problems of the form (31a)-(31b).

Theorem 4.2. *Let Assumptions 2.2 and 2.3 hold and let $g_\alpha \in L^2(Q^\infty)$ such that there exist $\hat{c}, \delta > 0$ with $|g_\alpha(x)| + |\partial g_\alpha(x)/\partial\alpha| \le \hat{c}e^{-\delta|x_2|}$ for almost all $x \in Q^\infty$ and all $\alpha \in [-1/2, 1/2]$. Furthermore, for any propagative wave number $\hat{\alpha}_j \in [-1/2, 1/2]$, let the orthogonality condition*

$$\int_{Q^\infty} g_{\hat{\alpha}_j}(x)\overline{\hat{\phi}(x)}\,dx = 0 \tag{37}$$

hold for all modes $\hat{\phi} \in X_j$ corresponding to the propagative wave number $\hat{\alpha}_j$.

Then, for every $\alpha \in [-1/2, 1/2]$, there exists a α-quasi-periodic solution $v_\alpha \in H^1_{\alpha,loc}(Q^\infty)$ of the equation

$$\Delta v_\alpha + k^2 n v_\alpha = -g_\alpha \ \text{in} \ Q^\infty \tag{38}$$

satisfying the generalized Rayleigh radiation condition (31b).

Proof. From parts (c) and (d) of the previous theorem, we know that (38) is equivalent to the variational equation (32) (with g_α replacing $(Fg)(\cdot,\alpha)$) and

$$L_\alpha v_\alpha = R_\alpha \ \text{in} \ H^1_\alpha(Q^H), \tag{39}$$

where $r_\alpha \in H^1_\alpha(Q^H)$ and the linear and bounded operator L_α from $H^1_\alpha(Q^H)$ into itself are defined as

$$(L_\alpha v, \psi)_{H^1(Q^H)} = \int_Q [\nabla v \cdot \nabla\overline{\psi} - k^2 n v\overline{\psi}]\,dx - \int_{\Gamma^H} (\Lambda_\alpha v)\overline{\psi}\,ds,$$

$$(R_\alpha, \psi)_{H^1(Q^H)} = \int_Q g_\alpha\overline{\psi}\,dx + \int_{\Gamma^H} \frac{\partial w_\alpha}{\partial\nu}\overline{\psi}\,ds$$

for all $v, \psi \in H^1_\alpha(Q^H)$. Then, L_α is Fredholm with index zero and α is a propagative wave number if, and only if, L_α fails to be invertible. For propagative wave numbers α, this form (39) allows the application of Fredholm's theorem; that is, $L_\alpha v_\alpha = R_\alpha$ is solvable if, and only if, R_α is orthogonal to the null space of the adjoint L_α^* of L_α. This is indeed the case for this particular form of the right-hand side and follows directly from the following properties of the operators L_α and the right-hand side R_α. Let $\alpha = \hat{\alpha}$ be a propagative wave number.

(i) The null spaces $\mathcal{N}(L_{\hat{\alpha}})$ and $\mathcal{N}(L_{\hat{\alpha}}^*)$ of $L_{\hat{\alpha}}$ and $L_{\hat{\alpha}}^*$, respectively, coincide and are given by the restrictions to Q^H of the space of corresponding modes.
(ii) The Riesz number of $L_{\hat{\alpha}}$ is one; that is, the geometric and algebraic multiplicities of the eigenvalue zero coincide.
(iii) For every mode $\hat{\phi}$ corresponding to $\hat{\alpha}$, we have

$$\left(R_{\hat{\alpha}}, \hat{\phi}\right)_{H^1(Q^H)} = \int_{Q^\infty} g_{\hat{\alpha}}(x)\overline{\hat{\phi}(x)}\,dx,$$

where $g_{\hat{\alpha}}$ is again the right hand side of (38).

Proof of (i): $L_\alpha^* \phi = 0$ is equivalent to $(L_\alpha \psi, \phi)_{H^1(Q^H)} = 0$ for all ψ; that is,

$$\int_{Q^H} [\nabla\psi \cdot \nabla\overline{\phi} - k^2 n \psi \overline{\phi}]\, dx - \int_{\Gamma^H} (\Lambda_\alpha \psi)\overline{\phi}\, ds = 0; \text{ that is,}$$

$$\int_{Q^H} [\nabla\psi \cdot \nabla\overline{\phi} - k^2 n \psi \overline{\phi}]\, dx - i \sum_{\sigma\in\{-1,+1\}} \sum_{\ell\in\mathbb{Z}} \sqrt{k^2 - (\ell+\alpha)^2}\, \psi_\ell(\sigma H)\overline{\phi_\ell(\sigma H)} = 0$$

for all $\psi \in H^1_\alpha(Q^H)$. If $\alpha = \hat{\alpha}$ this yields, by taking $\psi = \phi$ and the imaginary part, that $\phi_\ell(\pm H) = 0$ for $|\ell + \hat{\alpha}| < k$; that is, ϕ is evanescent and also $L_\alpha \phi = 0$.

Proof of (ii): Let ϕ with $L^2_{\hat{\alpha}} \phi = 0$. Then, $w := L_{\hat{\alpha}}\phi \in \mathcal{N}(L_{\hat{\alpha}}) = \mathcal{N}(L^*_{\hat{\alpha}})$, and thus, $\|w\|^2_{H^1(Q^H)} = (w, L_{\hat{\alpha}}\phi)_{H^1(Q^H)} = \left(L^*_{\hat{\alpha}} w, \phi\right)_{H^1(Q^H)} = 0$; that is, $w = 0$.

Proof of (iii): We compute (note that $w_{\hat{\alpha}}$ vanishes on Γ^H):

$$\begin{aligned}
\left(R_{\hat{\alpha}}, \hat{\phi}\right)_{H^1(Q^H)} &= \int_{Q^H} g_{\hat{\alpha}} \overline{\hat{\phi}}\, dx + \int_{\Gamma^H} \left[\frac{\partial w_{\hat{\alpha}}}{\partial \nu}\overline{\hat{\phi}} - \frac{\partial \overline{\hat{\phi}}}{\partial \nu} w_{\hat{\alpha}}\right] ds \\
&= \int_{Q^H} g_{\hat{\alpha}} \overline{\hat{\phi}}\, dx - \int_{Q^\infty\setminus Q^H} \left[\overline{\hat{\phi}}\,\Delta w_{\hat{\alpha}} - w_{\hat{\alpha}}\,\Delta\overline{\hat{\phi}}\right] dx \\
&= \int_{Q^H} g_{\hat{\alpha}} \overline{\hat{\phi}}\, dx - \int_{Q^\infty\setminus Q^H} \overline{\hat{\phi}}\left[\Delta w_{\hat{\alpha}} + k^2 w_{\hat{\alpha}}\right] dx = \int_{Q^\infty} g_{\hat{\alpha}} \overline{\hat{\phi}}\, dx.
\end{aligned}$$

This ends the proof of (i)–(iii) and, in particular, existence of a solution for every $\alpha \in [-1/2, 1/2]$ under the assumption (37). □

Smoothness with respect to α is shown in the following theorem.

Theorem 4.3. *Let all of the assumptions of the previous theorem hold and, in addition, let, for some open bounded neighborhood $U \subset \mathbb{C}$ of $[-1/2, 1/2]$, the mapping $\alpha \mapsto g_\alpha$ be holomorphic from U into $L^2(Q)$, thus analytic (see Colton & Kress[1]). Furthermore, let there exist $\hat{c}, \delta > 0$ with $|g_\alpha(x)| + |\partial g_\alpha(x)/\partial\alpha| \le \hat{c}e^{-\delta|x_2|}$ for almost all $x \in Q^\infty$ and all $\alpha \in U$. Furthermore, for any propagative wave number $\hat{\alpha}_j \in [-1/2, 1/2]$, let the orthogonality condition (37) hold. Then, we have the following:*

- *(a) The solution v_α of (38)-(31b) (which exists by the previous theorem) can be chosen such that the mapping $\alpha \mapsto v_\alpha$ is continuous as a mapping from $[-1/2, 1/2]$ into $H^1(Q^H)$ for every $H > h_0$. Again, $Q^H := (0, 2\pi) \times (-H, H)$.*
- *(b) Let $\hat{\alpha}$ be no cut-off value. Then, the mapping $\alpha \mapsto v_\alpha$ has an extension to an analytic mapping from a neighborhood $W \subset \mathbb{C}$ of $\hat{\alpha}$ to $H^1(Q^H)$ for every $H > h_0$.*
- *(c) In a neighborhood $(\hat{\alpha} - \delta, \hat{\alpha} + \delta) \subset \mathbb{R}$ of a cut-off value $\hat{\alpha} \in [-1/2, 1/2]$, the function v_α has the form*

$$v_\alpha = v^{(1)}_\alpha + \sqrt{\hat{\alpha} - \alpha}\, v^{(2)}_\alpha + \sqrt{\alpha - \hat{\alpha}}\, v^{(3)}_\alpha + |\alpha - \hat{\alpha}|\, v^{(4)}_\alpha \tag{40}$$

with analytic functions $\alpha \mapsto v^{(j)}_\alpha$, $j = 1, 2, 3, 4$, from a neighborhood $W \subset \mathbb{C}$ of $\hat{\alpha}$ into $H^1(Q^H)$.

Proof. We transform the equation $L_\alpha v_\alpha = R_\alpha$ into the 2π-periodic form and define the operator $J_\alpha : H^1_{per}(Q^H) \to H^1_\alpha(Q^H)$ by $(J_\alpha v)(x) := e^{i\alpha x_1} v(x)$ and set $\tilde{L}_\alpha := J_\alpha^{-1} L_\alpha J_\alpha : H^1_{per}(Q^H) \to H^1_{per}(Q^H)$ and $\tilde{r}_\alpha := J_\alpha^{-1} R_\alpha \in H^1_{per}(Q^H)$ where $H^1_{per}(Q^H)$ denotes again the space of periodic (with respect to x_1) functions. Then, $L_\alpha v_\alpha = R_\alpha$ is equivalent to $\tilde{L}_\alpha \tilde{v}_\alpha = \tilde{r}_\alpha$

where $\tilde{L}_\alpha$ and $\tilde{r}_\alpha$ are given by the forms

$$\begin{aligned}\left(\tilde{L}_\alpha v,\psi\right)_{H^1(Q^H)} = &\int\limits_{Q^H}\left[\nabla v\cdot\nabla\overline{\psi} - 2i\alpha\frac{\partial v}{\partial x_1}\overline{\psi} - (k^2 n-\alpha^2)v\overline{\psi}\right]dx \\ &- i\sum_{\sigma\in\{-1,+1\}}\sum_{\ell\in\mathbb{Z}}\sqrt{k^2-(\ell+\alpha)^2}\,v_\ell(\sigma H)\overline{\psi_\ell(\sigma H)},\end{aligned} \tag{41a}$$

$$\begin{aligned}(\tilde{r}_\alpha,\psi)_{H^1(Q^H)} &= \int\limits_{Q^H} e^{-i\alpha x_1}g_\alpha(x)\overline{\psi(x)}\,dx + \int\limits_{\Gamma^H} e^{-i\alpha x_1}\frac{\partial w_\alpha(x)}{\partial \nu}\overline{\psi(x)}\,ds \\ &= \int\limits_{Q^H} e^{-i\alpha x_1}g_\alpha(x)\overline{\psi(x)}\,dx \\ &\quad + \sum_{\sigma\in\{-1,+1\}}\sum_{\ell\in\mathbb{Z}}\int\limits_H^\infty g_{\alpha,\ell}(\sigma y_2)e^{i\sqrt{k^2-(\ell+\alpha)^2}(y_2-H)}dy_2\,\overline{\psi_\ell(\sigma H)}\end{aligned} \tag{41b}$$

for $v,\psi\in H^1_{per}(Q^H)$ where we used the form (35) of the Fourier coefficients of w_α. Then, $\tilde{L}_\alpha$ and $\tilde{r}_\alpha$ depend continuously on real $\alpha\in[-1/2,1/2]$ and, for any $\hat{\alpha}\in[-1/2,1/2]$ which is not a cut-off value, analytically on α in some neighborhood $W_1\subset U$ of $\hat{\alpha}$. We prove only the latter property. First, we note that the integral terms in the definitions of $\tilde{L}_\alpha$ and $\tilde{r}_\alpha$ are analytic with respect to α. Furthermore, there exist $c_+ > c_- > 0$ and a neighborhood $V\subset U$ of $[-1/2,1/2]$ with

$$c_+|\ell| \ge |\sqrt{k^2-(\ell+\alpha)^2}| \ge \operatorname{Im}\sqrt{k^2-(\ell+\alpha)^2} \ge c_-|\ell| \tag{42}$$

for all $\alpha\in V$ and $|\ell|\ge k+1$ and thus $\left|\frac{d}{d\alpha}\sqrt{k^2-(\ell+\alpha)^2}\right| = \frac{|\ell+\alpha|}{|\sqrt{k^2-(\ell+\alpha)^2}|} \le c$ for all $\alpha\in V$ and $|\ell|\ge k+1$. From this, we observe that the operator $\tilde{A}^+_\alpha$ corresponding to the series $\sum_{|\ell|\ge k+1}\sqrt{k^2-(\ell+\alpha)^2}\,v_\ell(\sigma H)\overline{\psi_\ell(\sigma H)}$ is analytic as a mapping from V into $\mathcal{L}(H^1_{per}(Q^H))$. The remaining part corresponding to the finite sum $\sum_{|\ell|<k+1}\sqrt{k^2-(\ell+\alpha)^2}\,v_\ell(\sigma H)\overline{\psi_\ell(\sigma H)}$ is obviously continuous for $\alpha\in V\cap[-1/2,1/2]$. If $\hat{\alpha}$ is not a cut-off value, then $k^2-(\ell+\alpha)^2\notin i\mathbb{R}_{\le 0}$ for α in some neighborhood $W_2\subset V$ of $\hat{\alpha}$ and thus the remaining parts—and thus also $\tilde{L}_\alpha$—depend analytically on α in W_2.

Next, we look at the right-hand side $\tilde{r}_\alpha$ and use similar arguments. With the product rule applied to $\frac{\partial}{\partial\alpha}\left[g_{\alpha,\ell}(\sigma y_2)e^{i\sqrt{k^2-(\ell+\alpha)^2}(y_2-H)}\right]$, we have to estimate the series

$$\sum_{\ell\in\mathbb{Z}}\int\limits_H^\infty\left[|\phi_{\alpha,\ell}(y_2)|\left|e^{i\sqrt{k^2-(\ell+\alpha)^2}(y_2-H)}\right|dy_2\,|\psi_\ell(\sigma H)|\right.$$

for $\phi_{\alpha,\ell}(y_2) := g_{\alpha,\ell}(\sigma y_2)$, $\phi_{\alpha,\ell}(y_2) := g_{\alpha,\ell}(\sigma y_2)(y_2-H)i\frac{d}{d\alpha}\sqrt{k^2-(\ell+\alpha)^2}$, and $\phi_{\alpha,\ell}(y_2) := \frac{\partial}{\partial\alpha}g_{\alpha,\ell}(\sigma y_2)$. We restrict ourselves to $\phi_{\alpha,\ell}(y_2) := g_{\alpha,\ell}(\sigma y_2)(y_2-H)i\frac{d}{d\alpha}\sqrt{k^2-(\ell+\alpha)^2}$. Using the estimate $\left|\frac{d}{d\alpha}\sqrt{k^2-(\ell+\alpha)^2}\right|\le c$ for $|\ell|\ge k+1$, it remains to estimate

$$\begin{aligned}&\left[\sum_{|\ell|\ge k+1}\int\limits_H^\infty\left[(y_2-H)|g_{\alpha,\ell}(\sigma y_2)|\left|e^{i\sqrt{k^2-(\ell+\alpha)^2}(y_2-H)}\right|dy_2\,|\psi_\ell(\sigma H)|\right]\right]^2 \\ &\le \sum_{|\ell|\ge k+1}\left|\int\limits_H^\infty(y_2-H)|g_{\alpha,\ell}(\sigma y_2)|\left|e^{i\sqrt{k^2-(\ell+\alpha)^2}(y_2-H)}\right|dy_2\right|^2\sum_{|\ell|\ge k+1}|\psi_\ell(\sigma H)|^2 \\ &\le \sum_{|\ell|\ge k+1}\int\limits_H^\infty|g_{\alpha,\ell}(\sigma y_2)|^2dy_2\int\limits_H^\infty(y_2-H)^2e^{-2c_-|\ell|(y_2-H)}dy_2\sum_{|\ell|\ge k+1}|\psi_\ell(\sigma H)|^2 \\ &\le c\|\psi\|^2_{H^1(Q^H)}\sum_{|\ell|\ge k+1}\int\limits_H^\infty|g_{\alpha,\ell}(\sigma y_2)|^2dy_2 \le c\|\psi\|^2_{H^1(Q^H)}\int\limits_{Q^H_\sigma}|g_\alpha(y)|^2dy,\end{aligned}$$

where again $Q_+^H := (0,2\pi) \times (H,\infty)$ and $Q_-^H := (0,2\pi) \times (-\infty,-H)$. For the finite series over $|\ell| < k+1$, we use that $|g_{\alpha,\ell}(\sigma y_2)| \le \hat{c} e^{-\delta y_2}$ and $|e^{i\sqrt{k^2-(\ell+\alpha)^2}(y_2-H)}| \le 1$ for real values of α which shows continuity of $\alpha \mapsto \tilde{r}_\alpha$. In the case that $\hat{\alpha}$ is not a cut-off value, we use for complex values α that there exists a neighborhood $W_3 \subset W_2$ of $\hat{\alpha}$ such that $\operatorname{Im}\sqrt{k^2-(\ell+\alpha)^2} \ge -\frac{\delta}{2}$ for all $|\ell| \le k+1$ and all $\alpha \in W$. Then,

$$\int_H^\infty (y_2-H)|g_{\alpha,\ell}(\sigma y_2)| \left| e^{i\sqrt{k^2-(\ell+\alpha)^2}(y_2-H)} \right| dy_2 \le c \int_H^\infty (y_2-H) e^{-\delta(y_2+H)/2} \, dy_2.$$

This shows the desired smoothness properties of $\tilde{A}_\alpha$ and $\tilde{r}_\alpha$.

Standard arguments on the perturbation of an invertible operator imply the continuous dependence of the solution $\tilde{v}_\alpha$ of $\tilde{L}_\alpha \tilde{v}_\alpha = \tilde{r}_\alpha$ on α in a neighborhood of $\hat{\alpha}$ provided $\hat{\alpha}$ is not a propagative wave number and analytic dependence provided $\hat{\alpha}$ is neither a propagative wave number nor a cut-off value. It remains to study the case where $\hat{\alpha}$ is a propagative wave number. Note that in this case, $\hat{\alpha}$ is not a cut-off value by assumption. Therefore, $\tilde{A}_\alpha$ and $\tilde{r}_\alpha$ depend analytically on α in a neighborhood of $\hat{\alpha}$. In this case, $\tilde{L}_{\hat{\alpha}}$ fails to be invertible, but (by the analytic Fredholm theory, see Colton & Kress[1]) $\tilde{L}_\alpha$ is invertible in a neighborhood of $\hat{\alpha}$.

Let P be the projection from $H^1_{per}(Q^H)$ into the null space $\mathcal{N} := \mathcal{N}(\tilde{L}_{\hat{\alpha}})$ along the direct decomposition $H^1_{per}(Q^H) = \mathcal{N} \oplus \mathcal{R}$ with range space $\mathcal{R} := \mathcal{R}(\tilde{L}_{\hat{\alpha}})$ (note that the Riesz number of $\tilde{L}_{\hat{\alpha}}$ is one) and set $Q := I - P$. Then, we project the equation $\tilde{L}_\alpha \tilde{v}_\alpha = \tilde{r}_\alpha$ onto the subspaces. With the ansatz $\tilde{v}_\alpha = v_\alpha^{\mathcal{N}} + v_\alpha^{\mathcal{R}} \in \mathcal{N} + \mathcal{R}$, we arrive at the equivalent equations

$$P\tilde{L}_\alpha(v_\alpha^{\mathcal{N}} + v_\alpha^{\mathcal{R}}) = P\tilde{r}_\alpha, \; Q\tilde{L}_\alpha(v_\alpha^{\mathcal{N}} + v_\alpha^{\mathcal{R}}) = Q\tilde{r}_\alpha.$$

Since $Q\tilde{L}_{\hat{\alpha}}|_{\mathcal{R}} = \tilde{L}_{\hat{\alpha}}|_{\mathcal{R}}$ is an isomorphism from $\mathcal{R}$ onto itself, the operators $B_\alpha := \left[Q\tilde{L}_\alpha|_{\mathcal{R}}\right]^{-1}$ exist for all α in a neighborhood $W \subset W_3$ of $\hat{\alpha}$ by a perturbation argument. Solving for $v_\alpha^{\mathcal{R}}$ from the second equation and substituting this into the first equation yields

$$P\tilde{L}_\alpha(I - B_\alpha Q\tilde{L}_\alpha)v_\alpha^{\mathcal{N}} = P\tilde{r}_\alpha - P\tilde{L}_\alpha B_\alpha Q\tilde{r}_\alpha \text{ in } \mathcal{N},$$

which we write as $C_\alpha v_\alpha^{\mathcal{N}} = s_\alpha$. We note that $C_{\hat{\alpha}} = 0$ and also $s_{\hat{\alpha}} = 0$. Therefore, $C_\alpha v_\alpha^{\mathcal{N}} = s_\alpha$ is equivalent to $\frac{1}{\alpha-\hat{\alpha}}[C_\alpha - C_{\hat{\alpha}}]v_\alpha^{\mathcal{N}} = \frac{1}{\alpha-\hat{\alpha}}[s_\alpha - s_{\hat{\alpha}}]$. Also, C_α and s_α are analytic in the neighborhood W of $\hat{\alpha}$ with derivatives C'_α and s'_α, respectively. We will show below that $C'_{\hat{\alpha}}$ is invertible in the finite dimensional space $\mathcal{N}$. Then, elementary arguments yield that $\alpha \mapsto v_\alpha^{\mathcal{N}}$ has an extension to an analytic function in all of W and $v_{\hat{\alpha}}^{\mathcal{N}}$ is the unique solution of $C'_{\hat{\alpha}} v_{\hat{\alpha}}^{\mathcal{N}} = s'_{\hat{\alpha}}$. This implies that also $\tilde{v}_\alpha$ depends analytically on α.

It remains to show that $C'_{\hat{\alpha}}$ is one to one. By the chain rule (note that $P\tilde{L}_{\hat{\alpha}} = \tilde{L}_{\hat{\alpha}}P = 0$), we compute $C'_{\hat{\alpha}}v = P\tilde{L}'_{\hat{\alpha}}v$ for $v \in \mathcal{N}$. Therefore, $C'_{\hat{\alpha}}v = 0$ is equivalent to

$$\int_{Q^H} \left[-2i\frac{\partial v}{\partial x_1}\overline{\psi} + 2\hat{\alpha} v \overline{\psi}\right] dx + \sum_{\sigma\in\{-1,+1\}} \sum_{|\ell+\alpha|>k} \frac{\ell+\alpha}{\sqrt{(\ell+\alpha)^2-k^2}} v_\ell(\sigma H)\overline{\psi_\ell(\sigma H)} = 0 \tag{43}$$

for all $\psi \in \mathcal{N}$. We extend v by

$$v(x) := \frac{1}{\sqrt{2\pi}} \sum_{|\ell+\alpha|>k} v_\ell(\pm H) e^{-\sqrt{(\ell+\alpha)^2-k^2}(|x_2|-H)} e^{i\ell x_1}, \pm x_2 > H,$$

and analogously ψ. Then, we observe that the second term on the left-hand side of (43) is just $\int_{Q^\infty\setminus Q^H} \left[-2i\frac{\partial v}{\partial x_1}\overline{\psi} + 2\hat{\alpha} v\overline{\psi}\right] dx$. Therefore, $C'_{\hat{\alpha}}v = 0$ is equivalent to

$$\int_{Q^\infty} \left[-2i\frac{\partial v}{\partial x_1}\overline{\psi} + 2\hat{\alpha} v\overline{\psi}\right] dx = 0$$

for all modes ψ corresponding to $\hat{\alpha}$. In terms of the quasi-periodic modes $\phi := J_{\hat{\alpha}} v$ and $\hat{\psi} := J_{\hat{\alpha}} \psi$, this is written as

$$\int_{Q^\infty} \frac{\partial \phi}{\partial x_1} \overline{\hat{\psi}}\, dx = 0$$

for all modes $\hat{\psi}$. Therefore, ϕ vanishes because $\hat{\alpha}$ is regular. This ends the proof of parts (a) and (b).

(c) We go back to the periodic equation $\tilde{L}_\alpha \tilde{v}_\alpha = \tilde{r}_\alpha$ where $\tilde{L}_\alpha$ and $\tilde{r}_\alpha$ are given by (41a)-(41b), respectively. The decomposition $k = \hat{\ell} + \kappa$ with $\hat{\ell} \in \mathbb{N} \cup \{0\}$ and $\kappa \in (-1/2, 1/2]$ shows that the propagative wave numbers in $[-1/2, 1/2]$ are given by $\hat{\alpha} = \kappa$ or $\hat{\alpha} = -\kappa$. We consider first the case $\hat{\alpha} = \kappa$ and assume first that $\kappa < 1/2$.

We look again at the second term in the definition (41a) of $\tilde{L}_\alpha$ which contains the square roots $\sqrt{k^2 - (\ell + \alpha)^2}$. We split the series into the series over $\ell \neq \hat{\ell}$ and the term with $\ell = \hat{\ell}$. This term defines the two-dimensional operator $E(\alpha)$ from $H^1_{per}(Q^H)$ into itself by

$$(E(\alpha)\phi, \psi)_{H^1(Q^H)} := i\sqrt{k + \hat{\ell} + \alpha}\,\left[\phi_{\hat{\ell}}(H)\overline{\psi_{\hat{\ell}}(H)} + \phi_{\hat{\ell}}(-H)\overline{\psi_{\hat{\ell}}(-H)}\right],\ \phi, \psi \in H^1_{per}(Q^H),$$

and the operator $\tilde{L}_\alpha$ has a decomposition in the form

$$\tilde{L}_\alpha = B(\alpha) - \sqrt{\kappa - \alpha}\, E(\alpha),$$

where E and B depend analytically on α in a neighborhood of $\alpha = \kappa$.

Now, we look at the right hand $\tilde{r}_\alpha$, given by (41b). We split the series again as above and decompose $e^{i\sqrt{k^2-(\hat{\ell}+\alpha)^2}(y_2-H)}$ into

$$\begin{aligned} e^{i\sqrt{k^2-(\hat{\ell}+\alpha)^2}(y_2-H)} &= \cos\left[\sqrt{k^2 - (\hat{\ell} + \alpha)^2}(y_2 - H)\right] \\ &\quad + i\sqrt{\kappa - \alpha}\sqrt{k + \hat{\ell} + \alpha}\,\frac{\sin\left[\sqrt{k^2 - (\hat{\ell} + \alpha)^2}(y_2 - H)\right]}{\sqrt{k^2 - (\hat{\ell} + \alpha)^2}} \\ &= a_1(y_2, \alpha) + \sqrt{\kappa - \alpha}\, a_2(y_2, \alpha) \end{aligned}$$

with analytic functions a_1, a_2 in a neighborhood of $\alpha = \kappa$ which satisfy

$$|a_j(y_2, \alpha)| \le c e^{-\operatorname{Im}\sqrt{k^2-(\hat{\ell}+\alpha)^2}(y_2-H)} \le c e^{\delta(y_2-H)/2}$$

for $j = 1, 2$ and $y_2 > H$ and α in a neighborhood $W_1 \subset U$ of κ. From this, we observe that $\int_H^\infty |g_{\alpha,\ell}(\sigma y_2)|\,|a_j(y_2, \alpha)|\, dy_2$ exist and $\tilde{r}_\alpha = \tilde{r}^{(1)}_\alpha + \sqrt{\kappa - \alpha}\,\tilde{r}^{(2)}_\alpha$ where $\tilde{r}^{(j)}_\alpha$ is analytic with respect to $\alpha \in W_1$ for $j = 1, 2$.

Therefore, $\tilde{L}_\alpha \tilde{v}_\alpha = \tilde{r}_\alpha$ is equivalent to

$$\left[B(\alpha) - \sqrt{\kappa - \alpha}\, E(\alpha)\right] \tilde{v}_\alpha = \tilde{r}^{(1)}_\alpha + \sqrt{\kappa - \alpha}\,\tilde{r}^{(2)}_\alpha. \tag{44}$$

Since the cut-off value $\hat{\alpha} = \kappa$ is not a propagative wave number by Assumption 2.2, we conclude that $\tilde{L}_\kappa = B(\kappa)$ is invertible and thus also $B(\alpha)$ in a neighborhood $W \subset W_1$ of κ. Since the operator on the left-hand side of (44) is a small perturbation of $B(\kappa) = \tilde{L}_\kappa$ the solution is given by the Neumann series as

$$\tilde{v}_\alpha = \sum_{m=0}^\infty [\sqrt{\kappa - \alpha}\, B(\alpha)^{-1} E(\alpha)]^m B(\alpha)^{-1} [\tilde{r}^{(1)}_\alpha + \sqrt{\kappa - \alpha}\,\tilde{r}^{(2)}_\alpha].$$

Therefore, sorting this series with respect to even and odd powers of $\sqrt{\kappa - \alpha} = \sqrt{\hat{\alpha} - \alpha}$, we conclude the form $\tilde{v}_\alpha = \tilde{v}^{(1)}_\alpha + \sqrt{\hat{\alpha} - \alpha}\,\tilde{v}^{(2)}_\alpha$ and $\tilde{v}^{(j)}_\alpha$ depend analytically on α in a neighborhood of $\alpha = \kappa = \hat{\alpha}$.

The case $\hat{\alpha} = -\kappa > -1/2$ is treated in the same way and leads to the singularity $\sqrt{\kappa + \alpha} = \sqrt{\alpha - \hat{\alpha}}$ in a neighborhood of $\hat{\alpha} = -\kappa$.

The cases $\kappa = 0$ or $\kappa = 1/2$ are more complicated. For example, if $\kappa = 0$, then $k = \hat{\ell} \in \mathbb{N}$, and one has to split the series in $\tilde{L}_\alpha$ into the series over $\ell \notin \{+\hat{\ell}, -\hat{\ell}\}$ and into the terms with $\pm\hat{\ell}$. This leads to the splittings

$$\tilde{L}_\alpha = B(\alpha) - \sqrt{-\alpha}\,E_+(\alpha) - \sqrt{\alpha}\,E_-(\alpha),\ \ \tilde{r}_\alpha = \tilde{r}_\alpha^{(1)} + \sqrt{-\alpha}\,\tilde{r}_\alpha^{(2)} + \sqrt{\alpha}\,\tilde{r}_\alpha^{(3)}.$$

In the Neumann series, also powers of $\sqrt{\alpha}\sqrt{-\alpha} = i|\alpha|$ appear which gives the forth term in (40). The case $\kappa = 1/2$ and $\hat{\alpha} = \pm 1/2$ is treated analogously. □

By the proof, we observe that all of the four terms in (40) appear only in the cases $\kappa = 0$ or $\kappa = 1/2$; that is, if $k \in \frac{1}{2}\mathbb{N}$.

Remark 4.4. During the proof, we have shown the existence of $\delta_H, c_H, c_H' > 0$ (independent of g_α) such that

$$\begin{aligned} \|v_\alpha\|_{H^1(Q^H)} &\le c_H \left[\sup_{\beta\in I} \|\tilde{r}_\beta\|_{H^1(Q^H)} + \sup_{\beta\in I} \|\partial\tilde{r}_\beta/\partial\beta\|_{H^1(Q^H)}\right] \\ &\le c_H' \left[\sup_{\beta\in I} \|g_\beta\|_{L^{(1,2)}(Q^\infty)} + \sup_{\beta\in I} \|\partial g_\beta/\partial\beta\|_{L^{(1,2)}(Q^\infty)}\right] \end{aligned} \tag{45a}$$

for all $\alpha \in I := \bigcup_{j\in J} [\hat{\alpha}_j - \delta_H, \hat{\alpha}_j + \delta_H] \subset \mathbb{R}$ and

$$\|v_\alpha\|_{H^1(Q^H)} \le c_H \|\tilde{r}_\alpha\|_{H^1(Q^H)} \le c_H' \|g_\alpha\|_{L^{(1,2)}(Q^\infty)} \tag{45b}$$

for all $\alpha \in [-1/2, 1/2] \setminus I$ where $\tilde{r}_\alpha$ is defined in (41b). For the second estimates, we use (41b) and (36). Here, $\|g\|_{L^{(1,2)}(Q^\infty)} = \|g\|_{L^1(Q^\infty)} + \|g\|_{L^2(Q^\infty)}$.

For the proof of Theorem 3.3, we needed the following implication of Theorem 4.3.

Corollary 4.5. *Let Assumptions 2.2 and 2.3 hold and let $u \in H^1_{loc}(\mathbb{R}^2)$ with $u \in H^1(W_H)$ for all $H > h_0$ satisfy $\Delta u + k^2 n u = -g$ in $\mathbb{R}^2$ where $g \in L^2(Q)$. Then, the Fourier transform $(\mathcal{F}u)(\cdot, x_2)$ of $u(\cdot, x_2)$ with respect to x_1 is continuous in a neighborhood of $\omega = \pm k$ for all $|x_2| > h_0$ and, even more, $(\mathcal{F}u)(\cdot, x_2) \in W^{1,1}(-R, R)$ for all $R > 0$.*

Proof. We decompose k again as $k = \hat{\ell} + \kappa$ with $\hat{\ell} \in \mathbb{N} \cup \{0\}$ and $\kappa \in (-1/2, 1/2]$. Then, $\pm\kappa$ are the cut-off values and, by (29),

$$(\mathcal{F}u)(\pm k, x_2) = (\mathcal{F}u)(\pm(\hat{\ell} + \kappa), x_2) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} (Fu)(x_1, x_2, \pm\kappa)\, e^{\mp i(\hat{\ell}+\kappa)x_1}\, dx_1,$$

where Fu denotes the Floquet-Bloch transform, defined in (30). Therefore, it suffices to prove continuity of $\alpha \mapsto (Fu)(\cdot, \alpha)$ in a neighborhood of $\pm\kappa$. By Theorem 4.1, Fu satisfies (38) with $g_\alpha = (Fg)(\cdot, \alpha)$. Furthermore, $\pm\kappa$ are no propagative wave numbers by Assumption 2.2. Application of Theorem 4.3 yields the desired continuity. Differentiation of the decomposition (40) yields that $\partial(Fu)(\cdot, \alpha)/\partial\alpha$ is integrable. □

5 | EXISTENCE

In this section, we will prove existence of a solution under the Assumptions 2.2 and 2.3 and, in the case that q does not vanish identically, under the additional assumption that no bound states exist. The main part deals with the unperturbed case $q = 0$ in which complete uniqueness has been shown in Theorem 3.3. The general case will follow by a compactness argument. Therefore, for given $f \in L^2(Q)$, we consider first the problem to determine $u \in H^1_{loc}(\mathbb{R}^2)$ which satisfies

$$\Delta u + k^2 n u = -f \text{ in } \mathbb{R}^2 \tag{46}$$

and the open waveguide radiation condition of Definition 2.5. We note that existence has been shown (for the half-plane problem or the case of scattering by an inhomogeneous cylinder in $\mathbb{R}^3$) in Kirsch[10] and Kirsch and Lechleiter[12] by the limiting absorption principle. In this section, we will give a direct proof; see also Kirsch.[11] With the propagative wave numbers $\hat{\alpha}_j$ for $j \in J$ and their modes $\hat{\phi}_{\ell,j}$, $\ell = 1, \ldots, m_j$, $j \in J$ (determined in (6)), we define the coefficients $a_{\ell,j} \in \mathbb{C}$ as

$$a_{\ell,j} := \frac{2\pi i}{|\lambda_{\ell,j}|} \int_Q f(x)\overline{\hat{\phi}_{\ell,j}(x)}\,dx, \quad \ell = 1, \ldots, m_j,\ j \in J. \tag{47}$$

Therefore, we have to solve Equation 11a for $q = 0$; that is,

$$\Delta u_{rad} + k^2 n u_{rad} = -g \text{ in } \mathbb{R}^2 \text{ with } g := f + \sum_{j\in J}\sum_{\ell=1}^{m_j} a_{\ell,j}\varphi_{\ell,j}, \tag{48}$$

where $\varphi_{\ell,j}$ are given by (11b). Furthermore, u_{rad} has to satisfy the generalized angular spectrum radiation condition (10). The plan is to take the Floquet-Bloch transform of this equation, show solvability for all $\alpha \in [-1/2, 1/2]$ (without exception) with Theorem 4.2 and continuity with respect to α with Theorem 4.3, and apply the inverse transform.

We note that the right-hand side g of (48) is in $L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ (and has even compact support with respect to x_1). Therefore, for every $\alpha \in [-1/2, 1/2]$, we try to solve the Floquet-Bloch transformed equation; that is, find $v_\alpha \in H^1_{\alpha,loc}(Q^\infty)$ with

$$\Delta v_\alpha + k^2 n v_\alpha = -(Fg)(\cdot,\alpha) \text{ in } Q^\infty = (0, 2\pi) \times \mathbb{R}, \tag{49}$$

satisfying the radiating condition (31b). Here, Fg denotes the Flochet-Bloch transform of g, defined in (30). The right-hand side Fg has no compact support but decays exponentially to zero as $|x_2|$ tends to infinity. Furthermore, Fg is analytic with respect to $\alpha \in \mathbb{C}$ (because the right hand side g of (48) vanishes for $|x_1| \geq \sigma_0$), and there exists $\hat{c}, \delta > 0$ such that $|(Fg)(x,\alpha)| + |\partial(Fg)(x,\alpha)/\partial\alpha| \leq \hat{c}e^{-\delta|x_2|}$ for almost all $x \in Q^\infty$ and all $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$. Therefore, to apply Theorem 4.2 of the previous section, we only have to show the orthogonality condition (37). This holds for the particular choice (47) of $a_{\ell,j}$ as we show now.

Lemma 5.1. *For every propagative wave number $\hat{\alpha}_j$ the right-hand side $g_{\hat{\alpha}_j} := (Fg)(\cdot,\hat{\alpha}_j)$ of (49) is orthogonal to the eigenspace X_j (see (4)) in $L^2(Q^\infty)$. Therefore, by Theorem 4.2, the problems (49) and (31b) are solvable for all $\alpha \in [-1/2, 1/2]$ without exception. Furthermore, by Theorem 4.3, for every $H > h_0$, the mapping $\alpha \mapsto v_\alpha$ is continuous from $[-1/2, 1/2]$ into $H^1(Q^H)$, and there exists $c_H > 0$ which is independent of f such that $\|v_\alpha\|_{H^1(Q^H)} \leq c_H \|f\|_{L^2(Q)}$ for all $\alpha \in [-1/2, 1/2]$.*

Proof. Recall the definition of g and thus $Fg = Ff + \sum_{j\in J}\sum_{\ell=1}^{m_j} a_{\ell,j} F\varphi_{\ell,j}$, where φ_{ℓ_j} are defined in (11b). Since $\hat{\phi}_{\ell,j}$ is $\hat{\alpha}_j$-quasi-periodic, it follows easily from the properties of the Floquet-Bloch transform that

$$(F\varphi_{\ell,j})(x,\alpha) = 2(F\psi'_\pm)(x_1, \alpha - \hat{\alpha}_j)\frac{\partial\hat{\phi}_{\ell,j}(x)}{\partial x_1} + (F\psi''_\pm)(x_1, \alpha - \hat{\alpha}_j)\,\hat{\phi}_{\ell,j}(x)$$

for ℓ with $\lambda_{\ell,j} \gtrless 0$. (Note that $\psi'_\pm \in L^2(\mathbb{R})$ in contrast to $\psi_\pm$ itself) Since $(F\psi'_\pm)(\cdot,\beta)$ is β-quasi-periodic, its Fourier series is given by

$$(F\psi'_\pm)(x_1,\beta) = \frac{1}{\sqrt{2\pi}}\sum_{\ell\in\mathbb{Z}} (\mathcal{F}\psi'_\pm)(\ell+\beta)\,e^{i(\ell+\beta)x_1},$$

where we used (29) for the relationship between the Fourier transform $\mathcal{F}\psi'_\pm$ and the Fourier coefficients of the Floquet-Bloch transform $(F\psi'_\pm)(\cdot,\beta)$.

With $(\mathcal{F}\psi'_\pm)(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi'_\pm(t)\,dt = \pm\frac{1}{\sqrt{2\pi}}$, we can write

$$(F\psi'_\pm)(x_1,\beta) = \begin{cases} \frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial x_1}\sum\limits_{\ell\in\mathbb{Z}} \frac{(\mathcal{F}\psi'_\pm)(\ell+\beta)}{i(\ell+\beta)}\,e^{i(\ell+\beta)x_1}, & \beta \notin \mathbb{Z}, \\ \pm\frac{1}{2\pi} + \frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial x_1}\sum\limits_{\ell\neq 0}\frac{(\mathcal{F}\psi'_\pm)(\ell)}{i\ell}\,e^{i\ell x_1}, & \beta \in \mathbb{Z}, \end{cases}$$

which we abbreviate as $(F\psi_\pm')(x_1,\beta) = \pm\frac{1}{2\pi}\delta_\beta + \frac{\partial}{\partial x_1}\rho_\pm(x_1,\beta)$ where $\delta_\beta := 0$ for $\beta \notin \mathbb{Z}$ and $\delta_\beta := 1$ for $\beta \in \mathbb{Z}$ and obvious meaning of $\rho_\pm$. This allows us to write

$$
\begin{aligned}
(F\varphi_{\ell,j})(x,\alpha) = \pm\frac{1}{\pi}\,\frac{\partial\hat{\phi}_{\ell,j}(x)}{\partial x_1}\,\delta_{\alpha-\hat{\alpha}_j}
&+ 2\frac{\partial}{\partial x_1}\rho_\pm(x_1,\alpha-\hat{\alpha}_j)\,\frac{\partial\hat{\phi}_{\ell,j}(x)}{\partial x_1} + \frac{\partial^2}{\partial x_1^2}\rho_\pm(x_1,\alpha-\hat{\alpha}_j)\,\hat{\phi}_{\ell,j}(x) \\
&= \pm\frac{1}{\pi}\,\frac{\partial\hat{\phi}_{\ell,j}(x)}{\partial x_1}\,\delta_{\alpha-\hat{\alpha}_j} + \Delta_x\tilde{v}^\pm_{\ell,j}(x,\alpha) + k^2 n(x)\tilde{v}^\pm_{\ell,j}(x,\alpha)
\end{aligned}
$$

for ℓ with $\lambda_{\ell,j} \gtrless 0$ where $\tilde{v}^\pm_{\ell,j}(x,\alpha) := \rho_\pm(x_1,\alpha-\hat{\alpha}_j)\hat{\phi}_{\ell,j}(x)$ is α-quasi-periodic.

Now, the proof of orthogonality is not difficult anymore. Let $\alpha = \hat{\alpha}_{j_0}$ for some $j_0 \in J$ and $\hat{\phi}_{\ell_0,j_0} \in X_{j_0}$. Then, $\int_{Q^\infty}\left[\Delta\tilde{v}^\pm_{\ell,j}(\cdot,\hat{\alpha}_{j_0}) + k^2 n\tilde{v}^\pm_{\ell,j}(\cdot,\hat{\alpha}_{j_0})\right]\overline{\hat{\phi}_{\ell_0,j_0}}\,dx$ vanishes by Green's second theorem and therefore

$$
\begin{aligned}
&\int_{Q^\infty}(Fg)(x,\hat{\alpha}_{j_0})\,\overline{\hat{\phi}_{\ell_0,j_0}(x)}\,dx \\
&= \int_{Q^\infty}(Ff)(x,\hat{\alpha}_{j_0})\,\overline{\hat{\phi}_{\ell_0,j_0}(x)}\,dx + \sum_{j\in J}\sum_{\ell=1}^{m_j}a_{\ell,j}\int_{Q^\infty}(F\varphi_{\ell,j})(x,\hat{\alpha}_{j_0})\,\overline{\hat{\phi}_{\ell_0,j_0}(x)}\,dx \\
&= \int_{Q^\infty}(Ff)(x,\hat{\alpha}_{j_0})\,\overline{\hat{\phi}_{\ell_0,j_0}(x)}\,dx + \sum_{j\in J}\sum_{\ell=1}^{m_j}a_{\ell,j}\,\mathrm{sign}\,\lambda_{\ell,j}\,\frac{1}{\pi}\,\delta_{\hat{\alpha}_{j_0}-\hat{\alpha}_j}\int_{Q^\infty}\frac{\partial\hat{\phi}_{\ell,j}(x)}{\partial x_1}\,\overline{\hat{\phi}_{\ell_0,j_0}(x)}\,dx \\
&= \int_{Q^\infty}(Ff)(x,\hat{\alpha}_{j_0})\,\overline{\hat{\phi}_{\ell_0,j_0}(x)}\,dx + \sum_{\ell=1}^{m_{j_0}}a_{\ell,j_0}\,\mathrm{sign}\,\lambda_{\ell,j_0}\,\frac{1}{\pi}\int_{Q^\infty}\frac{\partial\hat{\phi}_{\ell,j_0}(x)}{\partial x_1}\,\overline{\hat{\phi}_{\ell_0,j_0}(x)}\,dx \\
&= \int_{Q^\infty}(Ff)(x,\hat{\alpha}_{j_0})\,\overline{\hat{\phi}_{\ell_0,j_0}(x)}\,dx + a_{\ell_0,j_0}\,\mathrm{sign}\,\lambda_{\ell_0,j_0}\,\frac{1}{\pi}\int_{Q^\infty}\frac{\partial\hat{\phi}_{\ell_0,j_0}(x)}{\partial x_1}\,\overline{\hat{\phi}_{\ell_0,j_0}(x)}\,dx = 0
\end{aligned}
$$

by the properties of $\hat{\phi}_{\ell,j}$ from (6), the definition (47) of $a_{\ell,j}$, and the fact that $Ff = f$ because f has support in Q. Application of Theorem 4.2 yields existence. In (45a) and (45b) of Remark 4.4, the norm $\|v_\alpha\|_{H^1(Q^H)}$ is estimated by $\|g_\alpha\|_{L^1(Q^\infty)} + \|g_\alpha\|_{L^2(Q^\infty)}$ and its derivative with respect to α. We observe that $g_\alpha = (Fg)(\cdot,\alpha)$, defined in (48), depends linearly on $(Ff)(\cdot,\alpha) = f$ and $(F\varphi_{\ell,j})(\cdot,\alpha)$. Therefore,

$$
\|g_\alpha\|_{L^p(Q^\infty)} \le c\left[\|f\|_{L^p(Q)} + \sum_{j\in J}\sum_{\ell=1}^{m_j}|a_{\ell,j}|\right] \le c'\|f\|_{L^2(Q)}
$$

for all α where $p = 1$ or $p = 2$. The same estimate holds also for the derivative with respect to α. This proves boundedness of $f \mapsto v_\alpha$ from $L^2(Q)$ into $H^1(Q^H)$ uniformly with respect to $\alpha \in [-1/2, 1/2]$. □

Now, we are able to prove the main result of this section.

Theorem 5.2. *Let Assumptions 2.2 and 2.3 hold. Furthermore, in the case $q \neq 0$, we assume that no bound states exist; that is, there is no non-trivial solution $u \in H^1(\mathbb{R}^2)$ of $\Delta u + k^2(n+q)u = 0$ in $\mathbb{R}^2$. Then, there exists a unique solution $u \in H^1_{loc}(\mathbb{R}^2)$ of the source problem (8) satisfying the open waveguide radiation condition of Definition 2.5 for every $f \in L^2(Q)$. Furthermore, for every $H > h_0$, the mapping $f \mapsto u$ is bounded from $L^2(Q)$ into $H^1(W_H)$.*

Proof. For $q = 0$, the solution u is given by the inverse Floquet-Bloch transform

$$u(x) = \int_{-1/2}^{1/2} v_\alpha(x)\,d\alpha\,,\ x \in \mathbb{R}^2,$$

where v_α denotes the solution of (49) and (31b) which depends continuously on α and is extended as an α-quasi-periodic function into $\mathbb{R}^2$. By the uniform boundedness of $f \mapsto v_\alpha$ from $L^2(Q)$ into $H^1(Q^H)$, we conclude that $(x,\alpha) \mapsto v_\alpha(x)$ belongs to $L^2\left((-1/2,1/2), H^1_\alpha(Q^H)\right)$ and thus $u \in H^1(W_H)$ with $\|u\|_{H^1(W_H)} \le c_H \|f\|_{L^2(Q)}$ by the mapping property of the inverse Floquet-Bloch transform.

It remains to study the case of a general q. Let $S : L^2(Q) \to H^1(Q)$ be the linear and bounded operator which maps $f \in L^2(Q)$ into $u|_Q$ where u solves (8) for $q = 0$ and the radiation condition. For arbitrary q, the solution of (8) is equivalent to the fixpoint equation $u = S(f + k^2 qu)$ for $u \in L^2(Q)$. Since S is compact from $L^2(Q)$ into itself, uniqueness implies existence. □

6 | THE ASYMPTOTIC BEHAVIOR OF THE RADIATING PART

It is well known (see, e.g., Fliss & Joly[14]) that for closed waveguides the radiating part of the solution decays exponentially as $|x_1|$ tends to infinity. This follows also from the analog of Theorem 4.3. Indeed, in this case, no cut-off values exist, and Theorem 4.3 implies analyticity of $\alpha \mapsto (Fu_{rad})(\cdot\,,\alpha)$ in a neighborhood $W \subset \mathbb{C}$ of $[-1/2, 1/2]$. Then, we can modify the path $[-1/2, 1/2]$ of integration for the inverse transform

$$u_{rad}(x_1 + 2\pi\ell, x_2) = \int_{-1/2}^{1/2} (Fu_{rad})(x_1, x_2, \alpha)\, e^{i2\pi\ell\alpha}\, d\alpha\,,\ x \in Q^\infty\,,\ \ell \in \mathbb{Z},$$

depending on the sign of ℓ. We choose the path to be $\alpha = t + (\text{sign }\ell)\tau i$ for $t \in [-1/2, 1/2]$ where $\tau > 0$ is chosen such that Fu_{rad} is analytic in the strip $|\operatorname{Im}\alpha| \le \tau$. Then, it follows that $|u_{rad}(x_1 + 2\pi\ell, x_2)| \le c\, e^{-2\pi\tau|\ell|}$ for $|\ell| \ge 1$; that is, u_{rad} decays exponentially with respect to x_1. †

The situation is different in the case of an open waveguide because of the existence of cut-off values.

Theorem 6.1. *Let Assumptions 2.2 and 2.3 hold. For all $H > h_0$, there exists $c > 0$ such that $\|u_{rad}\|_{H^1(Q^H_\ell)} \le \frac{c}{|\ell|^{3/2}}$ for all $\ell \neq 0$. Here, $Q^H_\ell := (2\pi\ell, 2\pi(\ell+1)) \times (-H, H)$ for $\ell \in \mathbb{Z}$. In particular, $u_{rad} \in W^{1,1}(W_H)$ for all $H > h_0$ and $x \mapsto (1 + x_1^2)^{\rho/2} u_{rad}(x)$ is in $H^1(W_H)$ for all $\rho < 1$ and $H > h_0$ where again $W_H := \mathbb{R} \times (-H, H)$.*

Proof. Let again $k = \hat{\ell} + \kappa$ with $\hat{\ell} \in \mathbb{N} \cup \{0\}$ and $\kappa \in (-1/2, 1/2]$. For the different cases of κ, we define open sets I_1, I_2, and/or I_3 and corresponding functions $\psi_1, \psi_2, \psi_3 \in C^\infty(\mathbb{R})$ with $\operatorname{supp}\psi_j \subset I_j$ as follows.

Case I: If $|\kappa| < \frac{1}{2}$, we define $I := [-1/2, 1/2]$, $I_1 := (-1/2 - \varepsilon, 1/2 + \varepsilon) \setminus \{\pm\kappa\}$, $I_2 := (\kappa - \varepsilon, \kappa + \varepsilon)$, and $I_3 := (-\kappa - \varepsilon, -\kappa + \varepsilon)$ for some small $\varepsilon > 0$ (the latter only if $\kappa \neq 0$.) The functions ψ_j are chosen such that $\sum_j \psi_j(\alpha) = 1$ for all $\alpha \in I$ (partition of unity).

Case II: If $\kappa = 1/2$ we define $I := [0, 1]$, $I_1 := (-\varepsilon, 1+\varepsilon) \setminus \{1/2\}$, and $I_2 := (1/2 - \varepsilon, 1/2 + \varepsilon)$. The functions ψ_1, ψ_2 are chosen such that $\psi_1(\alpha) + \psi_2(\alpha) = 1$ for all $\alpha \in I$. In any case the inverse Floquet-Bloch transform is given by

$$u_{rad}(x_1 + 2\pi\ell, x_2) = \int_I (Fu_{rad})(x,\alpha)\, e^{i2\pi\ell\alpha} d\alpha = \sum_j \int_I \psi_j(\alpha)\, (Fu_{rad})(x,\alpha)\, e^{i2\pi\ell\alpha} d\alpha$$

for $x \in Q^H$ and $\ell \in \mathbb{Z}$. (Note that we can choose any interval of length one as domain of integration because of the periodicity of $(Fu_{rad})(x, \cdot)$) In the following, we restrict ourselves to the first case. The second case is treated as the case $\kappa = 0$.

† Actually, such an estimate holds only in the H^1-norm and not pointwise; see below.

The integrand of the term containing ψ_1 vanishes in a neighborhood of the cut-off values $\pm\kappa$ and is therefore smooth by Theorem 4.3, part (b). Furthermore, since $\psi_1 = 1$ in neighborhoods of $\pm 1/2$ and since $(Fu_{rad})(x,\cdot)$ is 1−periodic, partial integration (two times) yields

$$\left\| \int_{-1/2}^{1/2} \psi_1(\alpha)(Fu_{rad})(\cdot,\alpha)e^{i2\pi\ell\alpha}\,d\alpha \right\|_{H^1(Q^H)} \leq \frac{c}{\ell^2}.$$

Next, we consider the case containing ψ_j for $j \in \{2,3\}$, that is, by part (c) of Theorem 4.3 for $v_\alpha = (Fu_{rad})(\cdot,\alpha)$,

$$\begin{aligned}
&\int_{-1/2}^{1/2} \psi_j(\alpha)(Fu_{rad})(\cdot,\alpha)e^{i2\pi\ell\alpha}d\alpha \\
&= \int_{-1/2}^{1/2} \psi_j(\alpha)v_\alpha^{(1)}\,e^{i2\pi\ell\alpha}d\alpha + \int_{-1/2}^{1/2} \psi_j(\alpha)|\alpha-\hat{\alpha}|\,v_\alpha^{(4)}\,e^{i2\pi\ell\alpha}d\alpha \\
&\quad + \int_{-1/2}^{1/2} \psi_j(\alpha)\left[\sqrt{\hat{\alpha}-\alpha}\,v_\alpha^{(2)} + \sqrt{\alpha-\hat{\alpha}}\,v_\alpha^{(3)}\right] e^{i2\pi\ell\alpha}d\alpha,
\end{aligned}$$

where $\hat{\alpha} = \kappa$ or $\hat{\alpha} = -\kappa$ if $j = 2$ or $j = 3$, respectively. Two times partial integration of the first term gives $\mathcal{O}(1/\ell^2)$ (note that ψ_j vanishes near $\pm 1/2$). Also, the second term can be partially integrated twice and gives $\mathcal{O}(1/\ell^2)$. Partial integration of the third term yields

$$\begin{aligned}
&\int_{-1/2}^{1/2} \psi_j(\alpha)\left[\sqrt{\hat{\alpha}-\alpha}\,v_\alpha^{(2)} + \sqrt{\alpha-\hat{\alpha}}\,v_\alpha^{(3)}\right] e^{i2\pi\ell\alpha}d\alpha \\
&= -\frac{1}{i2\pi\ell}\int_{-1/2}^{1/2} \frac{\partial}{\partial\alpha}\left[\sqrt{\hat{\alpha}-\alpha}\,\psi_j(\alpha)v_\alpha^{(2)} + \sqrt{\alpha-\hat{\alpha}}\,\psi_j(\alpha)v_\alpha^{(3)}\right] e^{i2\pi\ell\alpha}d\alpha \\
&= \frac{1}{i4\pi\ell}\int_{\hat{\alpha}-\varepsilon}^{\hat{\alpha}+\varepsilon} \left(\frac{1}{\sqrt{\hat{\alpha}-\alpha}}\psi_j(\alpha)v_\alpha^{(2)} - \frac{1}{\sqrt{\alpha-\hat{\alpha}}}\psi_j(\alpha)v_\alpha^{(3)}\right) e^{i2\pi\ell\alpha}d\alpha \\
&\quad - \frac{1}{i2\pi\ell}\int_{\hat{\alpha}-\varepsilon}^{\hat{\alpha}+\varepsilon} \left(\sqrt{\hat{\alpha}-\alpha}\,\frac{\partial}{\partial\alpha}\left[\psi_j(\alpha)v_\alpha^{(2)}\right] + \sqrt{\alpha-\hat{\alpha}}\,\frac{\partial}{\partial\alpha}\left[\psi_j(\alpha)v_\alpha^{(3)}\right]\right) e^{i2\pi\ell\alpha}d\alpha.
\end{aligned}$$

The second term on the right-hand side is again of order $\mathcal{O}(1/\ell^2)$. For the first integral, we write

$$\begin{aligned}
&\int_{\hat{\alpha}-\varepsilon}^{\hat{\alpha}+\varepsilon} \left(\frac{1}{\sqrt{\hat{\alpha}-\alpha}}\psi_j(\alpha)v_\alpha^{(2)} - \frac{1}{\sqrt{\alpha-\hat{\alpha}}}\psi_j(\alpha)v_\alpha^{(3)}\right) e^{i2\pi\ell\alpha}d\alpha \\
&= v_{\hat{\alpha}}^{(2)}\int_{\hat{\alpha}-\varepsilon}^{\hat{\alpha}+\varepsilon} \frac{1}{\sqrt{\hat{\alpha}-\alpha}}e^{i2\pi\ell\alpha}d\alpha - v_{\hat{\alpha}}^{(3)}\int_{\hat{\alpha}-\varepsilon}^{\hat{\alpha}+\varepsilon} \frac{1}{\sqrt{\alpha-\hat{\alpha}}}e^{i2\pi\ell\alpha}d\alpha + \int_{\hat{\alpha}-\varepsilon}^{\hat{\alpha}+\varepsilon} \tilde{v}(\alpha)e^{i2\pi\ell\alpha}d\alpha
\end{aligned}$$

with

$$\tilde{v}(\alpha) := \frac{1}{\sqrt{\hat{\alpha}-\alpha}}\left[\psi_j(\alpha)v_\alpha^{(2)} - v_{\hat{\alpha}}^{(2)}\right] - \frac{1}{\sqrt{\alpha-\hat{\alpha}}}\left[\psi_j(\alpha)v_\alpha^{(3)} - v_{\hat{\alpha}}^{(3)}\right].$$

We show that $\tilde{v} \in W^{1,1}\left((-1/2,1/2), H^1(Q^H)\right)$. Indeed, for the first term, which we denote by $v_1(\alpha)$, we compute

$$\frac{\partial v_1(\alpha)}{\partial \alpha} = \frac{1}{2(\hat{\alpha}-\alpha)^{3/2}}\left[\psi_j(\alpha)v_\alpha^{(2)} - v_{\hat{\alpha}}^{(2)}\right] + \frac{1}{\sqrt{\hat{\alpha}-\alpha}}\frac{\partial}{\partial \alpha}\left[\psi_j(\alpha)v_\alpha^{(2)}\right].$$

We estimate (note that $\psi_j(\hat{\alpha}) = 1$)

$$\begin{aligned}
&\frac{1}{|\alpha-\hat{\alpha}|}\left\|\psi_j(\alpha)v_\alpha^{(2)} - v_{\hat{\alpha}}^{(2)}\right\|_{H^1(Q^H)} \\
&= \frac{1}{|\alpha-\hat{\alpha}|}\left\|\int_{\hat{\alpha}}^{\alpha}\frac{\partial}{\partial\beta}\left[\psi_j(\beta)v_\beta^{(2)}\right]d\beta\right\|_{H^1(Q^H)} \le \max_\beta\left\|\frac{\partial}{\partial\beta}\left[\psi_j(\beta)v_\beta^{(2)}\right]\right\|_{H^1(Q^H)}.
\end{aligned}$$

This shows that $\partial v_1/\partial\alpha$ satisfies an estimate of the form $\|\partial v_1(\alpha)/\partial\alpha\|_{H^1(Q^H)} \le c/\sqrt{|\alpha-\hat{\alpha}|}$. The second integral is estimated in the same way. Therefore, the integral $\int_{-1/2}^{1/2}\tilde{v}(\alpha)e^{i2\pi\ell\alpha}d\alpha$ is of order $\mathcal{O}(1/|\ell|)$ by partial integration. Finally, we compute

$$\begin{aligned}
&e^{-i2\pi\ell\hat{\alpha}}\sqrt{2\pi|\ell|}\int_{\hat{\alpha}-\varepsilon}^{\hat{\alpha}+\varepsilon}\frac{1}{\sqrt{\hat{\alpha}-\alpha}}e^{i2\pi\ell\alpha}d\alpha \\
&= \sqrt{2\pi|\ell|}\int_{-\varepsilon}^{\varepsilon}\frac{1}{\sqrt{\alpha}}e^{-i2\pi\ell\alpha}d\alpha \to \begin{cases}(1-i)\sqrt{2\pi}, & \ell\to\infty,\\ 0, & \ell\to-\infty,\end{cases}
\end{aligned}$$

by Lemma 7.3 of the appendix, and analogously

$$e^{-i2\pi\ell\hat{\alpha}}\sqrt{2\pi|\ell|}\int_{\hat{\alpha}-\varepsilon}^{\hat{\alpha}+\varepsilon}\frac{1}{\sqrt{\alpha-\hat{\alpha}}}e^{i2\pi\ell\alpha}d\alpha \to \begin{cases}0, & \ell\to\infty,\\ (1-i)\sqrt{2\pi}, & \ell\to-\infty.\end{cases}$$

Therefore, we conclude that

$$\lim_{\ell\to\pm\infty}\left[|\ell|^{3/2}e^{-i2\pi\ell\hat{\alpha}}\int_{-1/2}^{1/2}(Fu_{rad})(\cdot,\alpha)e^{i2\pi\ell\alpha}d\alpha\right] = -\frac{1+i}{4\pi}\begin{cases}\tilde{v}_{\hat{\alpha}}^{(2)}, & \ell\to\infty,\\ \tilde{v}_{\hat{\alpha}}^{(2)}, & \ell\to-\infty,\end{cases}$$

in $H^1(Q^H)$, and thus $\|u_{rad}\|_{H^1(Q_\ell^H)} \le \|\int_{-1/2}^{1/2}(Fu_{rad})(\cdot,\alpha)e^{i2\pi\ell\alpha}d\alpha\|_{H^1(Q^H)} \le c/|\ell|^{3/2}$.

To show that $u_{rad} \in W^{1,1}(W_H)$, we estimate

$$\begin{aligned}
\int_{W_H}[|u_{rad}| + |\nabla u_{rad}|]\,dx &= \sum_{\ell\in\mathbb{Z}}\int_{Q_\ell^H}[|u_{rad}| + |\nabla u_{rad}|]\,dx \\
&\le \sqrt{4\pi H}\sum_{\ell\in\mathbb{Z}}\left(\|u_{rad}\|_{L^2(Q_\ell^H)} + \|\nabla u_{rad}\|_{L^2(Q_\ell^H)}\right) \\
&\le c\sum_{\ell\in\mathbb{Z}}\|u_{rad}\|_{H^1(Q_\ell^H)} \le c'\left[\|u_{rad}\|_{H^1(Q^H)} + \sum_{|\ell|\ge1}\frac{1}{|\ell|^{3/2}}\right] < \infty.
\end{aligned}$$

Analogously, for $\rho \in [0,1)$,

$$\begin{aligned}\int\limits_{W_H} (1+x_1^2)^\rho\, |u_{rad}|^2\, dx &= \sum_{\ell\in\mathbb{Z}} \int\limits_{Q_\ell^H} (1+x_1^2)^\rho\, |u_{rad}|^2\, dx \\ &\le \sum_{\ell\in\mathbb{Z}} [1+(|\ell|+1)^2 4\pi^2]^\rho\, \|u_{rad}\|^2_{L^2(Q_\ell^H)} \\ &\le c\left[(1+4\pi^2)^\rho\, \|u_{rad}\|^2_{L^2(Q^H)} + \sum_{|\ell|\ge 1} [1+(|\ell|+1)^2 4\pi^2]^\rho |\ell|^{-3}\right] \\ &\le c'\left[1+\sum_{|\ell|\ge 1} |\ell|^{2\rho-3}\right] < \infty\end{aligned}$$

because $\rho < 1$. The proof for the derivative follows the same lines. □

We note that by the trace theorem $u_{rad}|_{\Gamma_{h_0}} \in L^2_\rho(\Gamma_{h_0})$ for all $\rho < 1$ where $\Gamma_{h_0} := \mathbb{R} \times \{h_0\}$ and

$$L^2_\rho(\Gamma_{h_0}) := \left\{ \phi \in L^2(\Gamma_{h_0}) : \int_{-\infty}^{\infty} (1+|t|^2)^\rho |\phi(t,h_0)|^2 dt < \infty \right\}, \tag{50}$$

equipped with its canonical norm $\|\cdot\|_{L^2_\rho(\Gamma_{h_0})}$.

After the investigation of the asymptotic behavior in x_1-direction, we turn to the study of the behavior in x_2-direction. We will prove the Sommerfeld radiation condition for u_{rad} in the upper and lower half planes $\{x \in \mathbb{R}^2 : x_2 > h_0 + \tau\}$ and $\{x \in \mathbb{R}^2 : x_2 < -h_0 - \tau\}$, respectively, for every $\tau > 0$. We note again that in $\mathbb{R}^2 \backslash W_{h_0} = \{x \in \mathbb{R}^2 : |x_2| > h_0\}$, the part u_{rad} satisfies the inhomogeneous Helmholtz equation

$$\Delta u_{rad} + k^2 u_{rad} = -\sum_{j\in J} \sum_{\ell=1}^{m_j} a_{\ell,j}\varphi_{\ell,j} \quad \text{for} \quad |x_2| > h_0,$$

where $\varphi_{\ell,j}$ are given by (11b) and the radiation condition (10).

Theorem 6.2. *Let Assumptions 2.2 and 2.3 hold. Furthermore, in the case $q \neq 0$, we assume that no bound states exist. Let $u \in H^1_{loc}(\mathbb{R}^2)$ be the unique solution of the source problem (8) satisfying the open waveguide radiation condition of Definition 2.5.*

(a) Then, u_{rad} satisfies the Sommerfeld radiation condition

$$\sup_{x\in\mathbb{R}^2,\, |x_2|\ge h_0+\tau} \sqrt{|x|}\,|u_{rad}(x)| + \sup_{x\in\mathbb{R}^2,\, |x_2|\ge h_0+\tau} \sqrt{|x|}\left|\frac{\partial u_{rad}(x)}{\partial r}\right| < \infty, \tag{51a}$$

for all $\tau > 0$, and

$$\sqrt{r}\sup_{x\in S_r^\tau}\left|\frac{\partial u_{rad}(x)}{\partial r} - ik\,u_{rad}(x)\right| \to 0,\ r\to\infty, \tag{51b}$$

for all $\tau > 0$ where $S_r^\tau := \{x \in \mathbb{R}^2 : |x_2| \ge h_0+\tau,\ |x| = r\}$.

(b) There exists a unique function $u^\infty \in C(S')$ with

$$\sup_{x\in S_r^\tau}\left|e^{-ikr}\sqrt{r}\,u_{rad}(x) - u^\infty(x/r)\right| \to 0,\ r\to\infty, \tag{52}$$

for all $\tau > 0$ where $S' = \{x \in \mathbb{R}^2 : |x| = 1, x_2 \neq 0\}$.

Proof. We restrict ourselves to the upper half plane $\{x \in \mathbb{R}^2 : x_2 > h_0\}$. Recall from (13) that $u_{rad}(x)$ is explicitly given as the sum of a volume potential $v_1(x)$ on $(-\sigma_0, \sigma_0)\times(h_0,\infty)$ and a double layer potential $v_2(x)$ on $\Gamma_{h_0} := \mathbb{R} \times \{h_0\}$.

We show the assertions separately for v_1 and v_2. Estimate (51a) for $v_1(x)$ follows directly from (A2) of Lemma 7.1 of the appendix for $h=h_0$. To show (51b), let $\varepsilon > 0$ be arbitrary. (A2) implies the existence of $h > h_0$ with

$$k\sqrt{|x|}\,|\tilde{v}_h(x)| + \sqrt{|x|}\left|\frac{\partial \tilde{v}_h(x)}{\partial r}\right| \le \frac{\varepsilon}{2} \quad \text{for all } \ x \in \mathbb{R}^2_{h_0},$$

where $\tilde{v}_h$ is defined in (A1) of Lemma 7.1 of the appendix. The function $v_1 - \tilde{v}_h$ is a volume potential on the compact rectangle $(-\sigma_0, \sigma_0)\times(h_0, h)$ and therefore satisfies the classical Sommerfeld radiation condition (51b); that is, there exists $R > 0$ with

$$\sqrt{|x|}\left|\frac{\partial[v_1(x) - \tilde{v}_h(x)]}{\partial r} - ik[v_1(x) - \tilde{v}_h(x)]\right| \le \frac{\varepsilon}{2} \quad \text{for all } \ x \in \mathbb{R}^2_{h_0},\ |x| \ge R.$$

The triangle inequality yields

$$\sqrt{|x|}\left|\frac{\partial v_1(x)}{\partial r} - ik v_1(x)\right| \le \varepsilon \quad \text{for all } \ x \in \mathbb{R}^2_{h_0},\ |x| \ge R,$$

which shows that v_1 satisfies (51b) even for $\tau = 0$.

Defining $v_1^\infty(\hat{x})$ by

$$v_1^\infty(\hat{x}) := \gamma \int_{-\sigma_0}^{\sigma_0}\int_{h_0}^{\infty} \varphi(y)\left[e^{-ik\hat{x}\cdot y} - e^{-ik\hat{x}\cdot y^*}\right] dy_2\, dy_1\,,\ \ |\hat{x}| = 1,$$

with $\gamma := \frac{e^{i\pi/4}}{\sqrt{8\pi k}}$, one shows estimate (52) in exactly the same way using the asymptotics

$$\frac{i}{4}H_0^{(1)}(k|x-y|) = \gamma\,\frac{e^{ik|x|}}{\sqrt{|x|}}\,e^{-iky\cdot\hat{x}} + \mathcal{O}\left(|x|^{-3/2}\right),\ \ |x| \to \infty,$$

uniformly with respect to $\hat{x} = x/|x|$ and y from compact sets (see Colton & Kress[1]), and the obvious estimate

$$\sup_{|\hat{x}|=1}\left|v_1^\infty(\hat{x}) - \gamma\int_{-\sigma_0}^{\sigma_0}\int_{h_0}^{h}\varphi(y)\left[e^{-ik\hat{x}\cdot y} - e^{-ik\hat{x}\cdot y^*}\right] dy_2\, dy_1\right| \le c\, e^{-\delta h}.$$

Now, we turn to the double layer potential $v_2(x)$. This function has been investigated in Hu et al.[22] We recall and simplify their arguments for the convenience of the reader. First, we recall the asymptotic behavior (17) of the Hankel function $H_0^{(1)}(k|x-y|)$ and their derivatives. Let $\phi \in L^2_\rho(\Gamma_{h_0})$ be any function for some $\rho < 1$ where $L^2_\rho(\Gamma_{h_0})$ is the weighted space from (50). We obtain for $x_2 \ge h_0 + \tau$:

$$\begin{aligned}
&\int\limits_{\substack{y\in\Gamma_{h_0}\\ |y_1|>1}} |\phi(y)|\left|\frac{\partial}{\partial y_2}H_0^{(1)}(k|x-y|)\right| ds(y)\\
&\le c(x_2+1)\int\limits_{|y_1|>1}|\phi(y_1,h_0)|\,\frac{1}{[(x_1-y_1)^2+(x_2-h_0)^2]^{3/4}}\,dy_1\\
&= c(x_2+1)\int\limits_{|y_1|>1}(1+y_1^2)^{\rho/2}|\phi(y_1,h_0)|\,\frac{1}{(1+y_1^2)^{\rho/2}[(x_1-y_1)^2+(x_2-h_0)^2]^{3/4}}\,dy_1\\
&\le c(x_2+1)\,\|\phi\|_{L^2_\rho(\Gamma_{h_0})}\sqrt{\int\limits_{|y_1|>1}\frac{1}{|y_1|^{2\rho}[(x_1-y_1)^2+(x_2-h_0)^2]^{3/2}}\,dy_1}.
\end{aligned}$$

Now, we apply Lemma 7.2 from the appendix with $q = 3/2$. Let first $|x_1| \le x_2 - h_0$. By the first estimate of Lemma 7.2, we have

$$\int\limits_{\substack{y \in \Gamma_{h_0} \\ |y_1| > 1}} |\phi(y)| \left| \frac{\partial}{\partial y_2} H_0^{(1)}(k|x-y|) \right| ds(y) \le c \|\phi\|_{L^2_\rho(\Gamma_{h_0})} \frac{1+x_2}{(x_2-h_0)^{3/2}} \le c' \|\phi\|_{L^2_\rho(\Gamma_{h_0})} \frac{1}{\sqrt{|x|}},$$

where we used $x_2 - h_0 \ge \frac{\tau}{h_0+\tau} x_2$ and thus $(x_2-h_0)^2 \ge \frac{1}{2}(x_1^2 + (x_2-h_0)^2) \ge \frac{1}{2} \frac{\tau^2}{(h_0+\tau)^2} |x|^2$ in the last estimate.

Second, let $|x_1| \ge x_2 - h_0$. Then, by the second estimate of Lemma 7.2 (note that $x_1 \ne 0$ because $x_2 \ge h_0 + \tau$),

$$\begin{aligned} &\int\limits_{\substack{y \in \Gamma_{h_0} \\ |y_1| > 1}} |\phi(y)| \left| \frac{\partial}{\partial y_2} H_0^{(1)}(k|x-y|) \right| ds(y) \\ &\le c \|\phi\|_{L^2_\rho(\mathbb{R})} (1+x_2) \sqrt{|x_1|^{-3} + |x_2-h_0|^{-2}|x_1|^{-2\rho}} \le c' \|\phi\|_{L^2_\rho(\mathbb{R})} \sqrt{|x|^{-1} + |x|^{-2\rho}}, \end{aligned}$$

because $x_1^2 \ge \frac{1}{2}(x_1^2 + (x_2-h_0)^2) \ge \frac{1}{2} \left(\frac{\tau^2}{(h_0+\tau)^2} \right. |x|^2$. Since $2\rho > 1$, we have shown the existence of a constant $\hat{c}$ which is independent of ϕ (but depends on $\tau > 0$) such that

$$\sqrt{|x|} \int\limits_{\substack{y \in \Gamma_{h_0} \\ |y_1| > 1}} |\phi(y)| \left| \frac{\partial}{\partial y_2} H_0^{(1)}(k|x-y|) \right| ds(y) \le \hat{c} \|\phi\|_{L^2_\rho(\Gamma_{h_0})} \tag{53}$$

for all $x \in \mathbb{R}^2$ with $x_2 \ge h_0 + \tau$. The same estimate holds also for the gradient $\nabla_x \frac{\partial}{\partial y_2} H_0^{(1)}(k|x-y|)$ by the same arguments.

Now, we specify the function ϕ. First, we set $\phi = u_{rad}$. The estimate (53) and the boundedness of $\sup_{x_2 \ge h_0+\tau} \sqrt{|x|} \int_{|y_1|<1} |u_{rad}(y)| \left| \frac{\partial}{\partial y_2} H_0^{(1)}(k|x-y|) \right| ds(y)$ for the double layer potential on the compact line segment $\{y \in \Gamma_{h_0} : |y_1| < 1\}$ implies the first estimate of (51a). The same argument holds also for the derivative.

Second, for any $a > 1$ we set $\phi_a(y) = u_{rad}(y)$ for $y \in \Gamma_{h_0}$, $|y_1| > a$, and $\phi_a(y) = 0$ for $y \in \Gamma_{h_0}$, $|y_1| < a$, and define v_a by

$$v_a(x) = \frac{i}{2} \int\limits_{\substack{y \in \Gamma_{h_0} \\ |y_1| < a}} u_{rad}(y) \frac{\partial}{\partial y_2} H_0^{(1)}(k|x-y|)\, ds(y), \ x \in \mathbb{R}^2_{h_0}.$$

Then, by (53),

$$\begin{aligned} \sqrt{|x|} \left| \frac{\partial v_2(x)}{\partial r} - ikv_2(x) \right| &\le \sqrt{|x|} \left| \frac{\partial v_a(x)}{\partial r} - ikv_a(x) \right| + \\ &+ \sqrt{|x|} \int\limits_{\substack{y \in \Gamma_{h_0} \\ |y_1| > 1}} |\phi_a(y)| \left[\left| \nabla_x \frac{\partial}{\partial y_2} H_0^{(1)}(k|x-y|) \right| + k \left| \frac{\partial}{\partial y_2} H_0^{(1)}(k|x-y|) \right| \right] ds(y) \\ &\le \sqrt{|x|} \left| \frac{\partial v_a(x)}{\partial r} - ikv_a(x) \right| + (1+k)\hat{c} \|\phi_a\|_{L^2_\rho(\Gamma_{h_0})} \end{aligned} \tag{54}$$

for all $x \in \mathbb{R}^2$ with $x_2 \ge h_0 + \tau$. Let now $\varepsilon > 0$ be arbitrary. We choose $a > 1$ such that

$$(1+k)\hat{c} \|\phi_a\|_{L^2_\rho(\Gamma_{h_0})} = (1+k)\hat{c} \sqrt{\int\limits_{|y_1|>a} |u_{rad}(y)|^2 (1+y_1^2)^\rho\, ds(y)} \le \frac{\varepsilon}{2}.$$

For this fixed a, we note that v_a is a double layer potential on a compact line segment. Therefore, v_a satisfies the classical Sommerfeld radiation condition, and we can find $R > 0$ such that

$$\sqrt{|x|}\left|\frac{\partial v_a(x)}{\partial r} - ik v_a(x)\right| \le \frac{\varepsilon}{2} \quad \text{for all } \ |x| \ge R.$$

By (54), this proves that v_2 satisfies Sommerfelds's radiation condition.

In the same way, ones proves (52) for v_2 with

$$v_2^\infty(\hat{x}) \ := \gamma \int\limits_{\Gamma_{h_0}} u_{rad}(y)\frac{\partial}{\partial y_2}e^{-ik\hat{x}\cdot y}\,ds(y), \ |\hat{x}| = 1$$

(see also[22]). We omit this part. □

Remark 6.3. Finally, we note that we can weaken the assumption with respect to the source f. Indeed, a careful inspection shows that we can take $f \in H^{-1}(\mathbb{R}^2)$ with support in K (as a distribution) where K is any compact subset of Q. For example, we can think of $f = \partial\chi/\partial x_j$ for some $\chi \in L^2(\mathbb{R}^2)$ with support in Q. We sketch the necessary modifications. In (2), the right-hand side has to be replaced by the dual form $\langle f, \overline{\psi}\rangle$. The Floquet-Bloch transform of f still coincides with f. In Theorems 4.1–4.3, the functions g and g_α have to be replaced by $f + \tilde{g}$ and $f + \tilde{g}_\alpha$, respectively, where $\tilde{g} \in L^2(\mathbb{R}^2)$ and $\tilde{g}_\alpha \in L^2(Q^\infty)$ decay exponentially with respect to x_2. The orthogonality condition (37) and the form (47) of $a_{\ell,j}$ have to be replaced by $\langle f, \overline{\hat{\phi}}\rangle + \int_{Q^\infty}\tilde{g}_{\hat{\alpha}_j}(x)\,\overline{\hat{\phi}(x)}\,dx = 0$ and $a_{\ell,j} = \frac{2\pi i}{|\lambda_{\ell,j}|}\langle f, \overline{\hat{\phi}_{\ell,j}}\rangle$, respectively. Then, Theorem 5.2 holds, and the mapping $f \mapsto u$ is bounded as a mapping from the closed subspace $\{f \in H^{-1}(\mathbb{R}^2) \,:\, \mathrm{supp} f \subset K\}$ into $H^1(W_H)$ for all $H > h_0$.

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CONFLICT OF INTEREST

This work does not have any conflict of interest.

ORCID

Andreas Kirsch https://orcid.org/0000-0003-3578-7504

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APPENDIX

In the first lemma, properties of the volume potential with certain non-compactly supported densities are shown. We set again $\mathbb{R}^2_{h_0} = \{x \in \mathbb{R}^2 : x_2 > h_0\}$ and $W_H^+ = \mathbb{R} \times (h_0, H)$.

Lemma 7.1. *Let $\varphi \in L^2(\mathbb{R}^2_{h_0})$ with $\varphi(x) = 0$ for $|x_1| > \sigma_0$ and $|\varphi(x)| \le \hat{c}\, e^{-\delta x_2}$ for $x_2 \ge h_0$ for some $\sigma_0, \hat{c}, \delta > 0$ (independent of x). Define*

$$v_h(x) = \int_{-\sigma_0}^{\sigma_0} \int_h^\infty \varphi(y)\, G^+(x,y)\, dy_2\, dy_1\,,\ x \in \mathbb{R}^2_{h_0}, \tag{A1}$$

for $h \ge h_0$, with the Green's function $G^+(x,y) := \frac{i}{4}\left[H_0^{(1)}(k|x-y|) - H_0^{(1)}(k|x-y^*|)\right]$ for $x, y \in \mathbb{R}^2_{h_0}$ with $x \ne y$. Here, $y^* := (y_1, 2h_0 - y_2)^\top$ is the reflected point at the line $x_2 = h_0$. Then, v_h and its gradient satisfy the estimate

$$|v_h(x)| + |\nabla v_h(x)| \le c\, \frac{1+x_2}{1+|x|^{3/2}}\,(1+h)\, e^{-\delta h/2}\,,\ x_2 > h_0, \tag{A2}$$

where c is independent of $x \in \mathbb{R}^2_{h_0}$ and $h \ge h_0$. In particular, $v_h \in H^1(W_H^+)$ for all $H > h_0$ and $h \ge h_0$. Furthermore, $v_h \in H^1_{loc}(\mathbb{R}^2_{h_0})$ is the unique solution of the boundary value problem

$$\Delta v_h + k^2 v_h = \begin{cases} 0 & \text{for } h_0 < x_2 < h, \\ -\varphi & \text{for } x_2 > h, \end{cases} \qquad v_h = 0 \ \text{ for } x_2 = h_0,$$

satisfying the generalized angular spectrum radiation condition (10).

Proof. First, we show (A2). We know from Chandler-Wilde and Ross[23] that for all $a > 0$, there exists $c > 0$ with

$$\left|G^+(x,y)\right| \le c\, \frac{(1+x_2)(1+y_2)}{|x-y|^{3/2}} \quad \text{for all } x, y \in \mathbb{R}^2_{h_0} \text{ with } |x-y| \ge a, \tag{A3a}$$

$$\left|\nabla_x G^+(x,y)\right| \le c\,\frac{(1+x_2)(1+y_2)}{|x-y|^{3/2}} \quad \text{for all } x,y\in\mathbb{R}^2_{h_0} \text{ with } |x-y|\ge a, \tag{A3b}$$

$$\left|G^+(x,y)\right| \le c\,|\ln|x-y|| \quad \text{for all } x,y\in\mathbb{R}^2_{h_0} \text{ with } 0<|x-y|\le a, \tag{A3c}$$

$$\left|\nabla_x G^+(x,y)\right| \le \frac{c}{|x-y|} \quad \text{for all } x,y\in\mathbb{R}^2_{h_0} \text{ with } 0<|x-y|\le a. \tag{A3d}$$

First, we consider $|x_1|\le 2\sigma_0$ and $x_2\le 2$ (if $h_0<2$, otherwise drop this case). In the definition of v_h, we split the region of integration with respect to y_2 into $\{y_2>h\colon |y_2-x_2|<1\}\cup\{y_2>h\colon |y_2-x_2|>1\}$ and use the estimates of G^+ in each of the regions. (Note that $|y_1|\le\sigma_0$) Therefore,

$$\begin{aligned}
|v_h(x)| \le{}& c\int\limits_{\substack{y_2>h\\|x_2-y_2|<1}}\int\limits_{-\sigma_0}^{\sigma_0} e^{-\delta y_2}\,|\ln|x-y||\,dy_1\,dy_2\\
&+c(1+x_2)\int\limits_{\substack{y_2>h\\|x_2-y_2|>1}}\int\limits_{-\sigma_0}^{\sigma_0} e^{-\delta y_2}\,\frac{1+y_2}{|x_2-y_2|^{3/2}}\,dy_1\,dy_2\\
\le{}& c e^{-\delta h}\int_{-1}^{1}\int_{-3\sigma_0}^{3\sigma_0}|\ln|z||\,dz_1\,dz_2+6\sigma_0 c\int_h^\infty e^{-\delta y_2}(1+y_2)\,dy_2 \le c'(1+h)e^{-\delta h}.
\end{aligned}$$

Let now $|x_1|\le 2\sigma_0$ and $x_2>2$. We split $\{y_2>h : |y_2-x_2|>1\}=\{y_2>h : 1<|y_2-x_2|<x_2/2\}\cup\{y_2>h : |y_2-x_2|>x_2/2\}$. Then,

$$\begin{aligned}
|v_h(x)| \le{}& c\int\limits_{\substack{y_2>h\\|x_2-y_2|<1}}\int\limits_{-\sigma_0}^{\sigma_0} e^{-\delta y_2}\,|\ln|x-y||\,dy_1\,dy_2\\
&+c(1+x_2)\int\limits_{\substack{y_2>h\\1<|x_2-y_2|<x_2/2}}\int\limits_{-\sigma_0}^{\sigma_0} e^{-\delta y_2}\,\frac{1+y_2}{|x_2-y_2|^{3/2}}\,dy_1\,dy_2\\
&+c(1+x_2)\int\limits_{\substack{y_2>h\\|x_2-y_2|>x_2/2}}\int\limits_{-\sigma_0}^{\sigma_0} e^{-\delta y_2}\,\frac{1+y_2}{|x_2-y_2|^{3/2}}\,dy_1\,dy_2\\
\le{}& c e^{-\delta h/2}e^{-\delta(x_2-1)/2}\int_{-1}^{1}\int_{-3\sigma_0}^{3\sigma_0}|\ln|z||\,dz_1\,dz_2\\
&+c(1+x_2)e^{-\delta x_2/4}\int_h^\infty(1+y_2)e^{-\delta y_2/2}dy_2+c\,\frac{1+x_2}{(x_2/2)^{3/2}}\int_h^\infty(1+y_2)e^{-\delta y_2}\,dy_2,
\end{aligned}$$

where we used the estimate $y_2=y_2/2+y_2/2\ge h/2+(x_2-1)/2$ in the first integral and $y_2\ge y_2/2+x_2/4$ in the second integral. Combining this with the estimate for $x_2\le 2$ implies

$$|v_h(x)| \le c\,\frac{1+x_2}{1+|x|^{3/2}}(1+h)e^{-\delta h/2} \tag{A4a}$$

for all $x_2 \geq h_0$ and $|x_1| \leq 2\sigma_0$ and $h \geq h_0$. Now, we consider $|x_1| > 2\sigma_0$. Then, $|y_1 - x_1| \geq |x_1| - \sigma_0 > |x_1|/2 \geq \sigma_0$, and thus,

$$|v_h(x)| \leq c(1+x_2)\int_h^\infty \int_{-\sigma_0}^{\sigma_0} e^{-\delta y_2} \frac{1+y_2}{|x-y|^{3/2}}\, dy_1\, dy_2.$$

We split the integral with respect to y_2 into $\{y_2 > h : |y_2 - x_2| > x_2/2\} \cup \{y_2 > h : |y_2 - x_2| < x_2/2\}$. For $|y_2 - x_2| > x_2/2$, we have $|x-y|^2 \geq \frac{1}{4}|x|^2$, and thus,

$$\int\limits_{\substack{y_2 > h \\ |x_2 - y_2| > x_2/2}} \int_{-\sigma_0}^{\sigma_0} e^{-\delta y_2} \frac{1+y_2}{|x-y|^{3/2}}\, dy_1\, dy_2 \leq \int\limits_{\substack{y_2 > h \\ |x_2 - y_2| > x_2/2}} (1+y_2) e^{-\delta y_2}\, dy_2 \frac{2\sigma_0}{(|x|/2)^{3/2}}$$

$$\leq \frac{c}{|x|^{3/2}} \int_h^\infty (1+y_2) e^{-\delta y_2}\, dy_2 \leq \frac{c'}{1+|x|^{3/2}} (1+h) e^{-\delta h}. \tag{A4b}$$

Finally, since $|y_2| \geq |x_2| - |y_2 - x_2| \geq x_2/2$ for $|y_2 - x_2| < x_2/2$, we have by estimating $y_2 \geq y_2/2 + x_2/4$

$$\int\limits_{\substack{y_2 > h \\ |x_2 - y_2| < x_2/2}} \int_{-\sigma_0}^{\sigma_0} e^{-\delta y_2} \frac{1+y_2}{|x-y|^{3/2}}\, dy_1\, dy_2 \leq e^{-\delta x_2/4} \int_h^\infty (1+y_2) e^{-\delta y_2/2}\, dy_2 \int_{-\sigma_0}^{\sigma_0} \frac{dy_1}{|x_1 - y_1|^{3/2}}$$

$$\leq c e^{-\delta x_2/4} \frac{1}{|x_1|^{3/2}} \int_h^\infty (1+y_2) e^{-\delta y_2/2}\, dy_2 \leq \frac{c'}{1+|x|^{3/2}} (1+h) e^{-\delta h/2} \tag{A4c}$$

(note that $|x_1| > 2\sigma_0$ and $x_2 \geq h_0$).

The proofs for the derivatives follow exactly the same lines. (Only the integral over $\ln|x-y|$ has to be replaced by the integral over $1/|x-y|$) Combining (A4a), (A4b), and (A4c) yields (A2).

From these estimates, it follows directly that $v_h \in H^1(W_H^+)$ for all $H > h_0$. By truncating the domain with respect to y_2 and using classical results on volume integrals on bounded domains, it is easily seen that v_h satisfies the differential equation.

To show the radiation condition (10), we take the Fourier transform with respect to x_1 and note that the integral with respect to y_1 is a convolution. By our normalization of the Fourier transform and the formulas 3. and 4. in Gradshteyn and Ryshik,[17] section 6.677, this yields

$$(\mathcal{F}v_h)(\omega, x_2) = \frac{i}{2\sqrt{k^2-\omega^2}} \int_h^\infty (\mathcal{F}\varphi)(\omega, y_2) \left[e^{i\sqrt{k^2-\omega^2}|x_2-y_2|} - e^{i\sqrt{k^2-\omega^2}(x_2+y_2-2h_0)} \right] dy_2,$$

and thus, for $x_2 > h$,

$$\frac{\partial(\mathcal{F}v_h)(\omega, x_2)}{\partial x_2} - i\sqrt{k^2-\omega^2}(\mathcal{F}v_h)(\omega, x_2) = \int_{x_2}^\infty (\mathcal{F}\varphi)(\omega, y_2) e^{i\sqrt{k^2-\omega^2}(y_2-x_2)}\, dy_2.$$

For $|\omega| < k$, we just estimate

$$\left| \frac{\partial(\mathcal{F}v_h)(\omega, x_2)}{\partial x_2} - i\sqrt{k^2-\omega^2}(\mathcal{F}v_h)(\omega, x_2) \right| \leq c \int_{x_2}^\infty e^{-\delta y_2}\, dy_2 = \frac{c}{\delta} e^{-\delta x_2}.$$

For $|\omega| > k$, we estimate

$$\left|\frac{\partial(\mathcal{F}v_h)(\omega,x_2)}{\partial x_2} - i\sqrt{k^2-\omega^2}\,(\mathcal{F}v_h)(\omega,x_2)\right|$$
$$\leq c\int_{x_2}^{\infty} e^{-\delta y_2-\sqrt{\omega^2-k^2}(y_2-x_2)}\,dy_2 \;=\; \frac{c}{\delta+\sqrt{\omega^2-k^2}}\,e^{-\delta x_2}.$$

Together, we have shown

$$\left|\frac{\partial(\mathcal{F}v_h)(\omega,x_2)}{\partial x_2} - i\sqrt{k^2-\omega^2}\,(\mathcal{F}v_h)(\omega,x_2)\right| \leq \frac{c}{\delta+\sqrt{|\omega^2-k^2|}}\,e^{-\delta x_2} \tag{A5}$$

for almost all $\omega \in \mathbb{R}$ and $x_2 > h$ where $c > 0$ is independent of ω and x. Squaring and integrating with respect to ω yields the generalized angular spectrum radiation condition (10).

Finally, we show uniqueness of the boundary value problem. Therefore, let $v \in H^1_{loc}(\mathbb{R}^2_{h_0})$ with $v \in H^1(W_H^+)$ for all $H > h_0$ be a solution for $\varphi = 0$ and $\phi = 0$. The Fourier transform $\hat{v}(\omega,x_2) := (\mathcal{F}v)(\omega,x_2)$ satisfies

$$\hat{v}''(\omega,x_2) + (k^2-\omega^2)\hat{v}(\omega,x_2) = 0\,,\ x_2 > h_0\,,\ \hat{v}(\omega,h_0) = 0,$$

for almost all ω and the radiation condition (10). The general solution of the differential equation and the initial condition is given by

$$\hat{v}(\omega,x_2) = a(\omega)\left[e^{i\sqrt{k^2-\omega^2}(x_2-h_0)} - e^{-i\sqrt{k^2-\omega^2}(x_2-h_0)}\right],\ x_2 > h_0,$$

for some $a(\omega) \in \mathbb{C}$, and thus, $\hat{v}'(\omega,x_2) - i\sqrt{k^2-\omega^2}\,\hat{v}(\omega,x_2) = 2a(\omega)i\sqrt{k^2-\omega^2}e^{-i\sqrt{k^2-\omega^2}(x_2-h_0)}$. Therefore,

$$\left|\hat{v}'(\omega,x_2) - i\sqrt{k^2-\omega^2}\,\hat{v}(\omega,x_2)\right|^2 = \begin{cases} 4|a(\omega)|^2\sqrt{k^2-\omega^2}, & |\omega| < k, \\ 4|a(\omega)|^2\sqrt{\omega^2-k^2}\,e^{\sqrt{\omega^2-k^2}(x_2-h_0)}, & |\omega| > k. \end{cases}$$

The radiation condition (10) implies $a(\omega) = 0$ for almost all ω; that is, v vanishes identically which proves uniqueness. □

We recall the following auxiliary result from Hu et al.[22]

Lemma 7.2. *For given $1/2 < \rho < 1$ and $q > 1/2$, define $I(x)$ by*

$$I(x) := \int_{|y_1|>1} \frac{1}{[(x_1-y_1)^2+x_2^2]^q\,|y_1|^{2\rho}}\,dy_1\,,\ x \in \mathbb{R}^2, x_2 \neq 0.$$

Then, there exists $c > 0$ with $I(x) \leq c|x_2|^{-2q}$ for all $x \in \mathbb{R}^2$ with $x_2 \neq 0$ and also $I(x) \leq c\left[|x_1|^{-2q} + |x_2|^{-2q+1}|x_1|^{-2\rho}\right]$ for $x \in \mathbb{R}^2$ with $x_2 \neq 0$ and $x_1 \neq 0$.

Proof.

(a) Obviously, for all $x \in \mathbb{R}^2$ with $x_2 \neq 0$, we have

$$I(x) \leq \int_{|y_1|>1} \frac{1}{|x_2|^{2q}\,|y_1|^{2\rho}}\,dy_1 = \frac{1}{|x_2|^{2q}}\int_{|y_1|>1}\frac{dy_1}{|y_1|^{2\rho}} = \frac{2}{2\rho-1}\,\frac{1}{|x_2|^{2q}}.$$

(b) We split the region of integration into y_1 with $|y_1-x_1| > |x_1|/2$ and $|y_1-x_1| < |x_1|/2$. For $x_1 \neq 0$, we have

$$\int_{\substack{|y_1|>1\\ |y_1-x_1|>|x_1|/2}} \frac{1}{[(y_1-x_1)^2+x_2^2]^q\,|y_1|^{2\rho}}\,dy_1 \leq \int_{|y_1|>1}\frac{1}{(|x_1|/2)^{2q}\,|y_1|^{2\rho}}\,dy_1 = \frac{2^{2q+1}}{2\rho-1}\,\frac{1}{|x_1|^{2q}}.$$

For y_1 with $|y_1 - x_1| < |x_1|/2$, we conclude that $|y_1| \geq |x_1|/2$; thus,

$$\begin{aligned}
&\int\limits_{\substack{|y_1|>1\\|y_1-x_1|<x_1/2}} \frac{1}{[(y_1-x_1)^2+x_2^2]^q\,|y_1|^{2\rho}}\,dy_1 \\
&\leq \int\limits_{|y_1|>1} \frac{1}{[(y_1-x_1)^2+x_2^2]^q}\,\frac{1}{(|x_1|/2)^{2\rho}}\,dy_1 \;=\; \frac{4^\rho}{|x_2|^{2q}|x_1|^{2\rho}} \int\limits_{|y_1|>1} \frac{1}{\left[\left(\frac{y_1-x_1}{|x_2|}\right)^2+1\right]^q}\,dy_1 \\
&\leq \frac{4^\rho\,|x_2|}{|x_2|^{2q}|x_1|^{2\rho}} \int\limits_{-\infty}^{\infty} \frac{1}{(t^2+1)^q}\,dt \;=\; \frac{c}{|x_2|^{2q-1}|x_1|^{2\rho}},
\end{aligned}$$

where we have used the substitution $t = \frac{y_1-x_1}{|x_2|}$. Therefore,

$$I(x) \;\leq\; c\left[|x_1|^{-2q} + |x_2|^{-2q+1}\,|x_1|^{-2\rho}\right],$$

which ends the proof. □

The following lemma is a simple consequence of the improper integrals $\int_0^\infty \frac{\cos t}{\sqrt{t}}\,dt = \int_0^\infty \frac{\sin t}{\sqrt{t}}\,dt = \sqrt{\frac{\pi}{2}}$.

Lemma 7.3. *For every $a > 0$ and $\sigma \in \{+1, -1\}$,*

$$\lim_{\sigma T\to\infty}\left[\sqrt{|T|}\int\limits_0^a \frac{1}{\sqrt{\alpha}}\,e^{-iT\alpha}\,d\alpha\right] = (1-i\sigma)\sqrt{\frac{\pi}{2}},$$

$$\lim_{\sigma T\to\infty}\left[\sqrt{|T|}\int\limits_{-a}^a \frac{1}{\sqrt{\alpha}}\,e^{-iT\alpha}\,d\alpha\right] = \begin{cases}(1-i)\sqrt{2\pi}, & \sigma = 1,\\ 0, & \sigma = -1.\end{cases}$$

Proof. Using the substitution $t = |T|\alpha = \sigma T\alpha$, the first formula follows from

$$\int\limits_0^a \frac{1}{\sqrt{\alpha}}\,e^{-iT\alpha}\,d\alpha \;=\; \frac{1}{\sqrt{|T|}}\int\limits_0^{a|T|}\frac{1}{\sqrt{t}}\,e^{-i\sigma t}\,dt \;=\; \frac{1}{\sqrt{|T|}}\int\limits_0^{a|T|}\frac{\cos t}{\sqrt{t}}\,dt - i\sigma\frac{1}{\sqrt{|T|}}\int\limits_0^{a|T|}\frac{\sin t}{\sqrt{t}}\,dt$$

and

$$\lim_{T\to\infty}\int\limits_0^T\frac{\cos t}{\sqrt{t}}\,dt \;=\; \lim_{T\to\infty}\int\limits_0^T\frac{\sin t}{\sqrt{t}}\,dt \;=\; \sqrt{\frac{\pi}{2}}.$$

For the second formula, we note that

$$\begin{aligned}
\int\limits_{-a}^a \frac{1}{\sqrt{\alpha}}\,e^{-iT\alpha}\,d\alpha &= \int\limits_0^a \frac{1}{\sqrt{\alpha}}e^{-iT\alpha}\,d\alpha + \frac{1}{i}\int\limits_0^a \frac{1}{\sqrt{\alpha}}e^{iT\alpha}\,d\alpha \\
&= (1-i)\mathrm{Re}\int\limits_0^a \frac{1}{\sqrt{\alpha}}e^{-iT\alpha}\,d\alpha - (1-i)\,\mathrm{Im}\int\limits_0^a \frac{1}{\sqrt{\alpha}}e^{-iT\alpha}\,d\alpha,
\end{aligned}$$

which yields the second assertion. □