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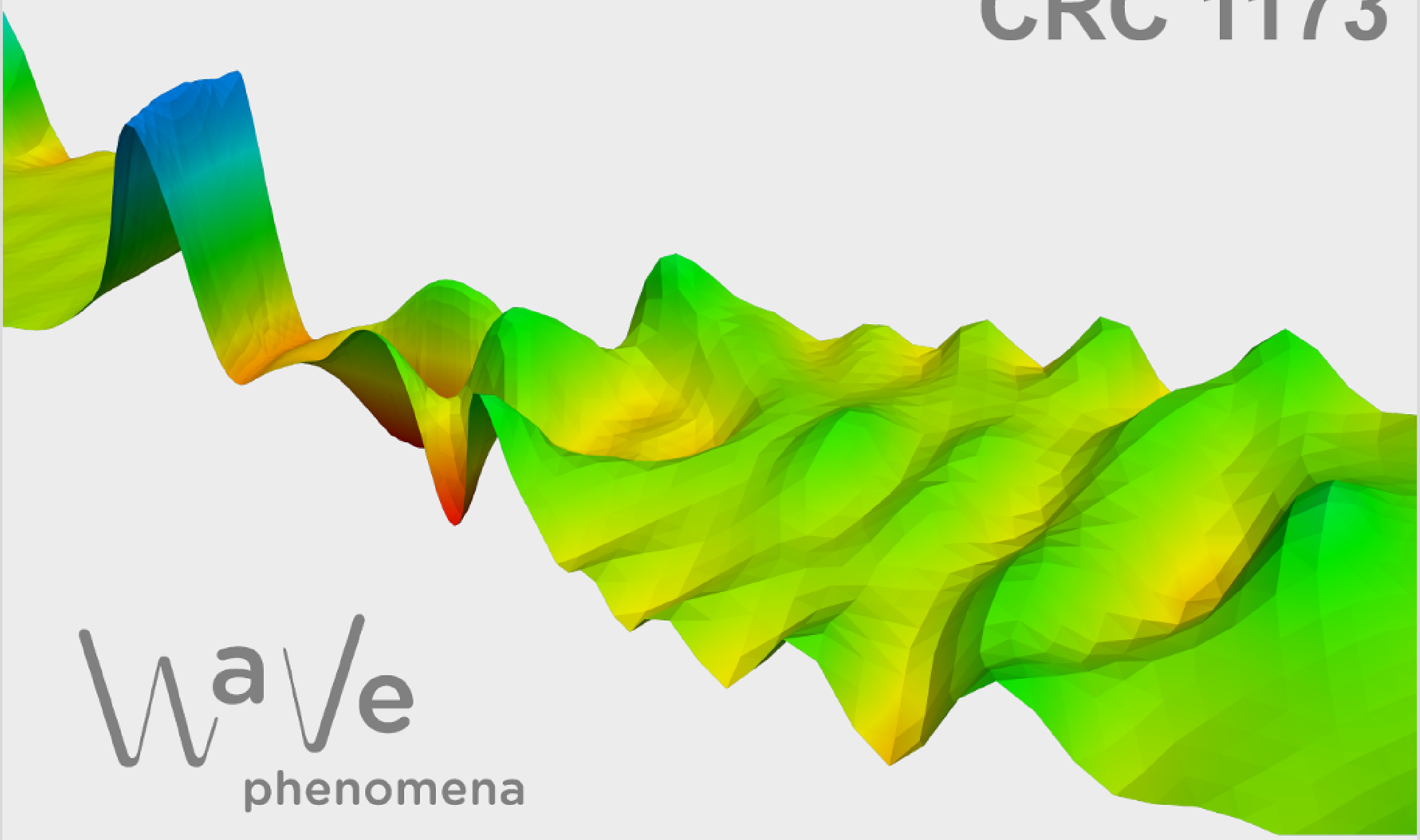

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INFINITE-ENERGY SOLUTIONS TO ENERGY-CRITICAL NONLINEAR SCHRÖDINGER EQUATIONS IN MODULATION SPACES

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ABSTRACT. We prove new well-posedness results for energy-critical nonlinear Schrödinger equations in modulation spaces, which are larger than the energy space. First, we remove the ε-derivative loss in L^p-smoothing estimates for the linear Schrödinger equation, if p is larger than the Tomas-Stein exponent. Next, we show local well-posedness results for nonlinear Schrödinger equations in modulation spaces containing the scaling critical L^2-based Sobolev space. The proof is carried out via bilinear refinements and adapted function spaces.

1. INTRODUCTION

In this paper we continue the study of modulation spaces as initial data for nonlinear Schrödinger equations in [27]. Modulation spaces in the present context are used to model initial data, which are decaying slower than functions in L^2-based Sobolev spaces. These spaces are natural because of their invariance under the linear Schrödinger evolution in contrast with the L^p-based Sobolev spaces for $p \neq 2$. Modulation spaces were introduced by Feichtinger [13]; see also subsequent joint works with Gröchenig [14, 15, 16]. The body of literature on modulation spaces is already huge, so we refer to [27, 10] and references therein for an overview with an emphasis on the use of modulation spaces in the context of dispersive equations.

In the work [27] L^p-smoothing estimates in modulation spaces were considered:

$$\|e^{it\Delta} u_0\|_{L^p([0,1],L^p(\mathbb{R}^d))} \lesssim \|u_0\|_{M^s_{p,2}(\mathbb{R}^d)}. \tag{1}$$

These turned out to be useful to prove well-posedness results for the cubic NLS

$$\begin{cases} i\partial_t u + \Delta u &= \pm |u|^2 u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \\ \quad u(0) &= u_0 \in M^s_{p,2}(\mathbb{R}^d). \end{cases} \tag{2}$$

The solution was placed in Strichartz spaces, in which the linear part was estimated by (1) and the nonlinear part was iterated with Strichartz estimates.

By frequency localization and rescaling arguments, the estimates (1) followed from ℓ^2-decoupling for the paraboloid due to Bourgain–Demeter [6]. Let $\mathcal{E}$ denote the Fourier extension operator for the paraboloid:

$$\mathcal{E}f(t,x) = \int_{\{\xi \in \mathbb{R}^d : |\xi| < 1\}} e^{i(x.\xi + t|\xi|^2)} f(\xi) d\xi.$$

Bourgain–Demeter proved the following estimates, which are sharp up to the ε-loss:

$$\|\mathcal{E}f\|_{L^p(B_{d+1}(0,R))} \lesssim_\varepsilon R^{s+\varepsilon} \big(\sum_\sigma \|\mathcal{E}f_\sigma\|^2_{L^p(w_{B_{d+1}(0,R)})} \big)^{1/2} \tag{3}$$

with $s = s(p,d)$ given by

$$s = \begin{cases} 0, & 2 \leq p \leq \frac{2(d+2)}{d}, \\ \frac{d}{4} - \frac{d+2}{2p}, & \frac{2(d+2)}{d} \leq p \leq \infty, \end{cases}$$

and f_σ denotes $f \cdot 1_{B(x_\sigma, R^{-1/2})}$ such that the family of $R^{-1/2}$-balls are finitely overlapping. In [27] was pointed out how the right-hand side is related to the modulation space norm of the initial value by rescaling and a kernel estimate. Thus, (3) indeed gives (1) with $s > s(p,d)$. It was also shown in [27] that $s \geq s(p,d)$ is necessary for (1) to hold true. In the present work, we remove the ε for $p > \frac{2(d+2)}{d}$:

Theorem 1.1. *Let $d \geq 1$ and $\frac{2(d+2)}{d} < p < \infty$. Then, we find (1) to hold for $s = s(p,d)$.*

For this purpose, we use an interpolation argument going back to Bourgain [2], in which an ε-derivative loss for Strichartz estimates on rational tori was removed. Moreover, it is known that (3) cannot hold true for $p = \frac{2(d+2)}{d}$ without R^ε-loss by a relation to Gauss sums (cf. [6, 2]). It is moreover conjectured that (3) is true with $s = 0$ for $p < \frac{2(d+2)}{d}$. For $d = 1$, $p = 4$ (3) follows with $\varepsilon = 0$ from a simple geometric argument (cf. [27, Theorem 1.1 (E)]).

Indeed, the present arguments follow closely Bourgain's treatment of the nonlinear Schrödinger equation on tori (cf. [2]; see also [24, 1]). As was already surmised in [27], the linear evolution on modulation spaces resembles the Fourier sums encountered in the periodic case. It will be interesting to explore further consequences of this transfer principle.

Following the proof of sharp smoothing estimates, we show bilinear refinements via Galilean invariance:

Proposition 1.2. *Let $d \geq 3$ and $N_1, N_2 \in 2^{\mathbb{N}_0}$ with $N_2 \lesssim N_1$. Then, we find the following estimate to hold:*

$$\|P_{N_1} e^{it\Delta} f_1 P_{N_2} e^{it\Delta} f_2\|_{L^2_{t,x}([0,1]\times\mathbb{R}^d)} \lesssim N_2^{\frac{d-2}{2}} \|P_{N_1} f_1\|_{M_{4,2}(\mathbb{R}^d)} \|P_{N_2} f_2\|_{M_{4,2}(\mathbb{R}^d)}. \tag{4}$$

Bilinear refinements go again back to Bourgain [2, 3].

Finally, we apply bilinear Strichartz estimates in modulation spaces to extend the local well-posedness theory of nonlinear Schrödinger equations. We consider the energy-critical nonlinear Schrödinger equation for $d \in \{3,4\}$:

$$\begin{cases} i\partial_t u + \Delta u &= \pm |u|^{\frac{4}{d-2}} u \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ \quad u(0) &= u_0 \in M^1_{4,2}(\mathbb{R}^d). \end{cases} \tag{5}$$

The equation (5) is energy critical because the scaling

$$u(t,x) \to \lambda^{\frac{d-2}{2}} u(\lambda^2 t, \lambda x)$$

leaves the energy invariant:

$$E[u] = \int_{\mathbb{R}^d} \frac{|\nabla u(t,x)|^2}{2} \pm \frac{d-2}{d+2} |u|^{\frac{d+2}{d-2}} dx.$$

The corresponding scaling critical Sobolev space is $\dot{H}^1(\mathbb{R}^d)$. For local well-posedness in $\dot{H}^1(\mathbb{R}^d)$ we refer to the survey by Bourgain [5]. Global well-posedness and scattering for the defocusing case is much harder and was proved for $d = 3$ by the I-team [12] and for $d = 4$ by Ryckman–Vişan [26]; see also references therein and

Bourgain's seminal contribution [4] in the radially symmetric case. Sharp conditions for global well-posedness and scattering of the focusing equation in the radial case were proved by Kenig–Merle [23]. By the embedding

$$\dot{H}^1(\mathbb{R}^d) \hookrightarrow M^1_{4,2}(\mathbb{R}^d),$$

the local well-posedness for initial data in the modulation space strengthens the local well-posedness result in $\dot{H}^1(\mathbb{R}^d)$. Previous results on infinite energy solutions to nonlinear Schrödinger equations are due to Braz e Silva *et al.* [7] with initial data in weak L^p-spaces. The results in [7] do not cover the energy critical equations though; see also [8]. Moreover, weak L^p-spaces are not invariant under the linear propagation in contrast with modulation spaces. The first resuls on infinite energy solutions are due to Cazenave–Weissler [9], who consider initial data with finite linear solution in a certain L^p-norm. The results in [9] do not cover the energy critical case. L^2-based Besov spaces were considered by Planchon [25].

In $d = 1$ I showed local well-posedness for any $s > 0$ by linear Strichartz estimates in [27]. This argument extends to $d = 2$ for $s > 0$, which is again the sharp analytic well-posedness up to endpoints. We remark how the arguments of [27] extend to L^2-critical equations for $d \in \{1, 2\}$, i.e., the quintic NLS on the real line or the cubic NLS in $\mathbb{R}^2$. Note that

$$L^2(\mathbb{R}^d) \sim M_{2,2}(\mathbb{R}^d) \hookrightarrow M^s_{p,2}(\mathbb{R}^d)$$

for $p \geq 2$ and $s \geq 0$. In this sense, the following well-posedness results are almost critical:

Theorem 1.3. *Let $s > 0$ and $T > 0$.*

(1) Then, the equation

$$\left\{ \begin{array}{ll} i\partial_t u + \Delta u &= \pm |u|^4 u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0) &= u_0 \in M^s_{6,2}(\mathbb{R}) + L^2(\mathbb{R}) \end{array} \right. \tag{6}$$

is locally well-posed in $X_T = C([0,T], L^2(\mathbb{R}) + M^s_{6,2}(\mathbb{R})) \cap L^6_t([0,T], L^6(\mathbb{R}))$ provided that $\|u_0\|_{M^s_{6,2}(\mathbb{R})+L^2(\mathbb{R})} \leq \varepsilon(T)$.

(2) The equation

$$\left\{ \begin{array}{ll} i\partial_t u + \Delta u &= \pm |u|^2 u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2, \\ u(0) &= u_0 \in M^s_{4,2}(\mathbb{R}^2) + L^2(\mathbb{R}^2) \end{array} \right. \tag{7}$$

is locally well-posed in $X_T = C([0,T], L^2(\mathbb{R}^2) + M^s_{4,2}(\mathbb{R}^2)) \cap L^4_t([0,T], L^4(\mathbb{R}^2))$ provided that $\|u_0\|_{M^s_{4,2}(\mathbb{R}^2)+L^2(\mathbb{R}^2)} \leq \varepsilon(T)$.

Note how above we choose the existence time in terms of the size of the initial data. It would be more practical to consider $T = T(u_0)$, which is not detailed for simplicity of presentation (see the proof of Theorem 1.4 below).

For $d \geq 3$ the derivative loss in the high frequencies of the L^4-Strichartz estimate has to be ameliorated via bilinear estimates. We show the following:

Theorem 1.4. *Let $d \in \{3, 4\}$. Then (5) is analytically locally well-posed in $X_T \hookrightarrow C([0,T], M^1_{4,2}(\mathbb{R}^d))$ in the critical sense: For any $u_0 \in M^1_{4,2}(\mathbb{R}^d)$ there is $T = T(u_0)$ such that there is a unique solution $u \in X_T$ to (5), and the data-to-solution mapping analytically depends on the initial value.*

The first local well-posedness results on energy critical nonlinear Schrödinger equations in the periodic setting are due to Herr–Tataru–Tzvetkov [19, 20]. In these works, improved bilinear or trilinear estimates were proved via orthogonality in time. This proof was simplified by Killip–Vişan [24], which is transferred to modulation spaces presently. Killip–Vişan pointed out how the estimates from Proposition 4.1 can be used to show the well-posedness result for the energy critical equation. A few remarks on global results in the periodic setting are in order: Herr–Tataru–Tzvetkov [19] proved global well-posedness for small initial data by energy conservation. Since the Sobolev embedding $H^1(\mathbb{T}^d) \hookrightarrow L^{\frac{d+2}{d-2}}(\mathbb{T}^d)$ is sharp, the straight-forward use of energy conservation requires smallness of the $H^1(\mathbb{T}^d)$ norm. Ionescu–Pausader [21] subsequently proved global well-posedness for large initial data in the defocusing case for $d = 3$. Nonetheless, the global results fundamentally build on energy conservation, which is not at disposal for initial data in $M^1_{4,2}(\mathbb{R}^d)$, since these possibly have infinite energy. Thus, global results, even in the defocusing case remain open for initial data in $M^1_{4,2}(\mathbb{R}^d)$. On the other hand, the classical blow-up arguments (cf. [23]) in the focusing case show that global solutions need not exist, if the energy is negative.

For further reading, we also refer to the very recent contribution by X. Chen and Holmer [11], in which *unconditional uniqueness* of solutions in $C([0,T], H^1(X^d))$ for energy critical Schrödinger equations is proved via a unified approach for $d \in \{3,4\}$ and $X \in \{\mathbb{T}, \mathbb{R}\}$.

Outline of the paper. In Section 2 we recall basic facts about modulation spaces, and we introduce the function spaces used in the proof of Theorem 1.4. In Section 3 we prove Theorem 1.1 by adapting Bourgain's interpolation argument for Strichartz estimates on the torus to modulation spaces. In Section 4 we show Proposition 4.1, by which we prove Theorem 1.4 in Section 5. Theorem 1.3 is proved in Section 5 with linear Strichartz estimates for comparison.

2. Preliminaries

2.1. **Modulation spaces.** The modulation spaces $M^s_{p,q}(\mathbb{R}^d)$ for $d \geq 1$, $s \in \mathbb{R}$, $p, q \in [1,\infty]$ are defined through an isometric decomposition in Fourier space. Let $(\sigma_k)_{k \in \mathbb{Z}^d}$ with $\sigma_k = \sigma(\cdot - k)$ and $\sigma \in C^\infty_c(B(0,1))$ denote a smooth partition of unity. We define

$$M^s_{p,q}(\mathbb{R}^d) = \{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{M^s_{p,q}(\mathbb{R}^d)} = \big\| \big(\langle k \rangle^s \|\sigma_k(D) f\|_{L^p(\mathbb{R}^d)} \big)_{k \in \mathbb{Z}^d} \big\|_{\ell^q} < \infty \}.$$

We write $M_{p,q}(\mathbb{R}^d) := M^0_{p,q}(\mathbb{R}^d)$ for brevity. We have the following embeddings in the standard Besov scale (cf. [27, Section 1]): By the embedding $\ell^{q_1} \hookrightarrow \ell^{q_2}$ for $q_1 \leq q_2$ and Bernstein's inequality, we have

$$M^s_{p,q_1}(\mathbb{R}^d) \hookrightarrow M^s_{p,q_2}(\mathbb{R}^d) \quad (q_1 \leq q_2),$$
$$M^s_{p_1,q}(\mathbb{R}^d) \hookrightarrow M^s_{p_2,q}(\mathbb{R}^d) \quad (p_1 \leq p_2).$$

By Plancherel's theorem, we have

$$M_{2,2}(\mathbb{R}^d) = L^2(\mathbb{R}^d). \tag{8}$$

Moreover, we have from kernel estimates with $p = 1$ and $p = \infty$ and interpolation with (8) the estimates

$$M_{p,p'} \hookrightarrow L^p \hookrightarrow M_{p,p} \quad (2 \leq p \leq \infty),$$
$$M_{p,p} \hookrightarrow L^p \hookrightarrow M_{p,p'} \quad (1 \leq p \leq 2).$$

Lastly, we note that

$$M^{s_1}_{p,q_1}(\mathbb{R}^d) \hookrightarrow M^{s_2}_{p,q_2}(\mathbb{R}^d)$$

provided that $s_1 - s_2 > d\big(\frac{1}{q_2} - \frac{1}{q_1}\big) > 0$ as a consequence of Hölder's inequality.

2.2. **Adapted function spaces.** We use U^p-/V^p-spaces taking values in modulation spaces as iteration spaces. U^p-/V^p-spaces based on L^2-based Sobolev spaces go back to unpublished notes of Tataru in the context of wave maps. For a careful introduction, we refer to the work by Hadac–Herr–Koch [17, 18]. However, there seems to be no literature on the case that the base space is not a Hilbert space. Hence, we choose to give the definition of U^p-/V^p-spaces and recall well-known aspects on the function spaces in the present context. The following presentation is very close to [17].

Let $\mathcal{Z}$ be the set of finite partitions $-\infty = t_0 < t_1 < \ldots < t_K = \infty$ and let $\mathcal{Z}_0$ be the set of finite partitions $-\infty < t_0 < t_1 < \ldots < t_K \leq \infty$. We consider U^p-/V^p-spaces taking values in modulation spaces $M_{p,q}(\mathbb{R}^d)$, but provided that we still have a suitable dual pairing, the following transpires to a more general case. Denote the value space in the following by E.

Definition 2.1. Let $1 < p < \infty$ and $\{t_k\}_{k=0}^K \in \mathcal{Z}$ and $\{\phi\}_{k=0}^{K-1} \subseteq E$ with $\sum_{k=0}^{K-1} \|\phi_k\|_E^p = 1$ and $\phi_0 = 0$. The function $a : \mathbb{R} \to E$ defined by
$a = \sum_{k=1}^{K} 1_{[t_{k-1},t_k)} \phi_{k-1}$ is said to be a U^p-atom. We define the atomic space

$$U^p(E) = \{u = \sum_{j=1}^{\infty} \lambda_j a_j \, : \, a_j : U^p - \text{atom}, \, (\lambda_j) \in \ell^1\}$$

with norm

$$\|u\|_{U^p} = \inf\{\sum_{j=1}^{\infty} |\lambda_j| \, : u = \sum_{j=1}^{\infty} \lambda_j a_j, \, (\lambda_j) \in \ell^1, \, a_j : U^p - \text{atom}\}.$$

Subspaces are considered as in [17, Proposition 2.2]. The spaces of p-variation were already considered by Wiener [28].

Definition 2.2. Let $1 \leq p < \infty$. $V^p(E)$ is defined as normed space of all functions $v : \mathbb{R} \to E$ such that $\lim_{t \to \pm\infty} v(t)$ exists, $v(\infty) := 0$ (this is purely conventional and does not necessarily coincide with the limit), and $v(-\infty) = \lim_{t\to-\infty} v(t)$. The norm is given by

$$\|v\|_{V^p} = \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \big(\sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_E^p\big)^{\frac{1}{p}}$$

is finite. Let V^p_- denote the closed subspace of V^p with $\lim_{t\to-\infty} v(t) = 0$.

Propositions 2.4, 2.5, and Corollary 2.6 from [17] carry over verbatim. Recall the duality $(M_{p,q}(\mathbb{R}^d))' \simeq M_{p',q'}(\mathbb{R}^d)$ for $1 < p, q < \infty$, which is established via the dual pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle : M_{p,q}(\mathbb{R}^d) \times M_{p',q'}(\mathbb{R}^d) &\to \mathbb{C} \\ (f,g) \qquad &\mapsto \int_{\mathbb{R}^d} f \overline{g} dx. \end{aligned}$$

We obtain an obvious variant of [17, Proposition 2.9] and have the following duality (cf. [17, Theorem 2.8]):

Theorem 2.3. *Let $1 < p < \infty$. We have*

$$(U^p(E))^* \simeq V^{p'}(E')$$

in the sense that

$$T : V^{p'}(E') \to (U^p(E))^*, \quad T(v) = B(\cdot, v)$$

is an isometric isomorphism.

We have an explicit description of B for sufficiently regular functions:

Proposition 2.4. *Let $1 < p < \infty$, $u \in V^1_-$ be absolutely continuous on compact intervals and $v \in V^{p'}(E')$. Then,*

$$B(u,v) = -\int_{-\infty}^{\infty} \langle u'(t), v(t) \rangle dt.$$

Later we rely on computing the U^p-norm with the aid of duality:

$$\|u\|_{U^p(E)} = \sup_{v \in V^{p'}(E') : \|v\|_{V^{p'}(E')} = 1} |B(u,v)|. \tag{9}$$

We remark that the spaces can as well be localized to an interval, in which case we write $U^p(I;E)$, $V^p(I;E)$. We furthermore define the space $DU^p(I;E)$:

$$DU^p(I;E) = \{ f = u' : u \in U^p(I;E) \}$$

with the derivative considered in the distributional sense and

$$\|f\|_{DU^p(I;E)} = \|u\|_{U^p(I;E)}.$$

By Theorem 2.3, we have $(DU^p(I;E))^* \simeq V^{p'}(I;E')$ with respect to a bilinear mapping, which for $f \in L^1(I) \hookrightarrow DU^p(I;E)$ is given by

$$\tilde{B}(f,v) = \int_a^b \langle f(t), v(t) \rangle dt.$$

We adapt U^p-/V^p-spaces to the linear Schrödinger propagation $e^{it\Delta}$ as usual:

$$\|u\|_{X^p_\Delta(I;E)} = \|e^{-it\Delta} u\|_{X^p(I;E)} \tag{10}$$

with $X \in \{U; V; DU\}$.

3. Sharp L^p-smoothing estimates

In this section we prove Theorem 1.1. Let $d \geq 1$ and $p^* = \frac{2(d+2)}{d}$ denote the Tomas–Stein exponent. In the following we show that for $p^* < p < \infty$ the estimate

$$\|e^{it\Delta} f\|_{L^p([0,1] \times \mathbb{R}^d)} \lesssim \|f\|_{M^s_{p,2}(\mathbb{R}^d)} \tag{11}$$

holds with $s = s(p,d) = \frac{d}{2} - \frac{d+2}{p}$. By the Littlewood-Paley square function estimate, we can reduce to the estimate

$$\|P_N e^{it\Delta} f\|_{L^p([0,1] \times \mathbb{R}^d)} \lesssim N^s \|P_N f\|_{M_{p,2}}. \tag{12}$$

Indeed, if (12) holds, then

$$\begin{aligned} \|e^{it\Delta} f\|_{L^p([0,1] \times \mathbb{R}^d)} &\sim \big\| \big(\sum_N |P_N e^{it\Delta} f|^2 \big)^{1/2} \big\|_{L^p([0,1] \times \mathbb{R}^d)} \\ &\leq \big(\sum_N \|P_N e^{it\Delta} f\|^2_{L^p([0,1] \times \mathbb{R}^d)} \big)^{1/2} \\ &\lesssim \big(\sum_N N^{2s} \|P_N f\|^2_{M_{p,2}} \big)^{1/2} \sim \|f\|_{M^s_{p,2}(\mathbb{R}^d)}. \end{aligned}$$

In the following we use an interpolation argument originally due to Bourgain [2] to deduce (11) for $s = s(p,d)$ from (11) for $s > s(p,d)$. For more recent presentations of Bourgain's interpolation argument and extensions, we refer to works by Killip–Vişan [24] and Barron [1]. The following presentation is close to Barron's proof [1] of sharp Strichartz estimates in the semi-periodic case. We opted to give an outline of the argument for the sake of self-containedness.

We turn to the proof of (12) for $p^* < p < \infty$. First, we normalize $\|f\|_{M_{p,2}(\mathbb{R}^d)} = 1$. Hence, we find by the Cauchy-Schwarz inequality

$$\|P_N f\|_{L^\infty(\mathbb{R}^d)} \lesssim N^{\frac{d}{2}} \|P_N f\|_{M_{\infty,2}(\mathbb{R}^d)} \lesssim N^{\frac{d}{2}} \|P_N f\|_{M_{p,2}(\mathbb{R}^d)}.$$

Let $F(t,x) = P_N e^{it\Delta} f(x)$. By the above, we can write

$$\|F\|^p_{L^p(\mathbb{R}^d \times [0,1])} = p \int_0^{CN^{\frac{d}{2}}} \mu^{p-1} |\{(t,x) : |F(t,x)| > \mu\}| d\mu.$$

Let $\delta > 0$ and choose $p^* < q < p$. Applying Hölder's inequality, the above display, and (12) in L^q for $s > s(q,d)$ gives

$$\int_0^{N^{\frac{d}{2}-\delta}} \mu^{p-1} |\{|F| > \mu\}| d\mu \lesssim_\varepsilon N^{q\left(\frac{d}{2} - \frac{d+2}{q}\right)} N^{(p-q)\left(\frac{d}{2}-\delta\right) + \varepsilon q}$$
$$\lesssim N^{p\left(\frac{d}{2} - \frac{d+2}{p}\right)}.$$

The ultimate estimate follows for $\varepsilon q \leq \delta(p-q)$.

Hence, in the following we focus on the estimate of the level sets, where $|F|$ takes large values:

$$A = \int_{N^{\frac{d}{2}-\delta}}^{CN^{\frac{d}{2}}} \mu^{p-1} |\{|F| > \mu\}| d\mu.$$

Let $\Omega = \{|F| > \mu\}$ for $\mu \geq N^{\frac{d}{2}-\delta}$ fixed, and set $\Omega_\omega = \{\Re(e^{i\omega} F) > \mu/2\}$. We choose $\omega \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ such that $|\Omega| \leq 4|\Omega_\omega|$ and estimate $|\Omega_\omega|$ instead. Note that

$$\mu^2 |\Omega_\omega|^2 \lesssim \langle 1_{\Omega_\omega}, K * 1_{\Omega_\omega} \rangle_{L^2_{x,t}}, \tag{13}$$

where K denotes the kernel

$$K(x,t) = \int_{\mathbb{R}^d} \psi(N^{-1}\xi) e^{i(x.\xi + t|\xi|^2)} d\xi, \quad \psi \in C^\infty_c(B(0,2) \backslash B(0,1/2)).$$

Like in Barron [1], we do not take advantage of the dispersive effects in $\mathbb{R}^d$ for $t \in [0,1]$, but discretize the kernel such that it resembles the exponential sum on the torus. By Galilean invariance, we find the following lemma.

Lemma 3.1 ([1, Lemma 4.2])**.** *If Q_0 is a cube of side length 1 centered at the origin, we have the pointwise estimate*

$$|K(x,t)| \lesssim \sup_{\alpha \in Q_0} \big| \sum_{\substack{k \in \mathbb{Z}^d, \\ |k| < cN}} \psi(N^{-1}(\alpha + k)) e^{i[(x+2t\alpha)\cdot k + t|k|^2]} \big|.$$

This is amenable to a pointwise estimate, originally used by Bourgain [2] for solutions to linear Schrödinger equations on the square torus. Let $q, a \in \mathbb{Z}$ with $1 \leq << N$ and $1 \leq a < q$ with $(a,q) = 1$, and set

$$S_{q,a} = \{t \in [0,1] : \big|t - \frac{a}{q}\big| \leq \frac{1}{qN}\}.$$

We have like in [1]:

$$|K(x,t)|1_{S_{q,a}}(t) \lesssim \frac{N^{\frac{d}{2}}}{q^{\frac{d}{2}}(1+N^d\big|t-\frac{a}{q}\big|^{\frac{d}{2}})}.$$

Next, like in [24], we consider the times where the kernel is large:

$$\mathcal{T} = \{t \in [0,1] : qN^2\big|t-\frac{a}{q}\big| \leq N^{2\rho} \text{ for some } q \leq N^{2\rho}, \text{ and } (a,q)=1\},$$

where $\rho > 0$ will be chosen later. We let $\tilde{K} = K1_{\mathcal{T}}(t)$ and observe from

$$|K-\tilde{K}| \lesssim N^{d-d\rho}$$

This yields that

$$|\langle 1_{\Omega_\omega}, (K-\tilde{K}) * 1_{\Omega_\omega}\rangle_{L^2_{x,t}}| \lesssim N^{d(1-\rho)}|\Omega_\omega|^2$$

and choosing $\mu \geq N^{\frac{n}{2}-\delta}$, we can absorb the contribution of $K-\tilde{K}$ into the left hand-side of (13). We remain with the contribution of $\tilde{K}$. For this, we use the following proposition (cf. [24, Section 2]):

Proposition 3.2 ([1, Proposition 4.3])**.** *Suppose $r > \frac{2(d+2)}{d}$. Then, we find the following estimate to hold:*

$$\|\tilde{K} * F\|_{L^r(\mathbb{R}^d\times[0,1])} \lesssim N^{2\left(\frac{d}{2}-\frac{d+2}{r}\right)}\|F\|_{L^{r'}}.$$

Applying Proposition 3.2 gives

$$\mu^2|\Omega_\omega|^2 \lesssim |\langle 1_{\Omega_\omega}, \tilde{K} * 1_{\Omega_\omega}\rangle| \lesssim |\Omega_\omega|^{\frac{2}{r'}} N^{2\left(\frac{d}{2}-\frac{d+2}{r}\right)}.$$

Consequently,

$$|\Omega| \leq 4|\Omega_\omega| \lesssim N^{\frac{r}{2}\left(d-\frac{2(d+2)}{r}\right)}\mu^{-r}$$

for any $r \in \left(\frac{2(d+2)}{d}, p\right)$. By the above, we can estimate the large level sets as

$$A \lesssim N^{\frac{r}{2}\left(d-\frac{2(d+2)}{r}\right)} \int_{N^{\frac{d}{2}-\delta}}^{CN^{\frac{d}{2}}} \mu^{p-r-1} d\mu \lesssim N^{p\left(\frac{d}{2}-\frac{d+2}{p}\right)}.$$

The proof of Theorem 1.1 is complete. □

4. Bilinear refinements

By Galilean invariance, we can show bilinear estimates with derivative loss only in the low frequency. In the context of Strichartz estimates on tori, we refer to [24, 2]. Starting point is the following linear Strichartz estimate:

$$\|e^{it\Delta}u_0\|_{L^4([0,1]\times\mathbb{R}^d)} \lesssim \|u_0\|_{M^s_{4,2}(\mathbb{R}^d)}. \tag{14}$$

Proposition 4.1. *Let $1 \leq K \ll N$ and suppose that (14) holds true. Then, we find the following estimate to hold:*

$$\|P_N e^{it\Delta}u_0 P_K e^{it\Delta}v_0\|_{L^2([0,1]\times\mathbb{R}^d)} \lesssim K^{2s}\|P_N u_0\|_{M_{4,2}}\|P_K v_0\|_{M_{4,2}}.$$

Proof. Let $(Q_{K'})_{K'}$ be a family of frequency projections to balls of size K in $\mathbb{R}^d$ whose supports are covering $B(0,2N)\backslash B(0,N/2)$ finitely overlapping. By almost orthogonality, we find

$$\begin{aligned}\|P_N e^{it\Delta} u_0 P_K e^{it\Delta} v_0\|^2_{L^2([0,1]\times\mathbb{R}^d)} &\lesssim \sum_{K'} \|P_N Q_{K'} e^{it\Delta} u_0 P_K e^{it\Delta} v_0\|^2_{L^2([0,1]\times\mathbb{R}^d)} \\ &= \sum_{K'} \|P_N Q_{K'} e^{it\Delta} u_0\|^2_{L^4([0,1]\times\mathbb{R}^d)} \|P_K e^{it\Delta} v_0\|^2_{L^4([0,1]\times\mathbb{R}^d)}.\end{aligned}$$

We apply (14) to the second factor and to the first factor after Galilean transform, which yields

$$\begin{aligned}&\lesssim K^{4s} \sum_{K'} \|Q_{K'} u_0\|^2_{M_{4,2}} \|P_K v_0\|^2_{M_{4,2}} \\ &\lesssim K^{4s} \|P_N u_0\|^2_{M_{4,2}} \|P_K v_0\|^2_{M_{4,2}}.\end{aligned}$$

The ultimate estimate follows by the finitely overlapping property and the definition of the modulation spaces. □

This yields Proposition 1.2 by Theorem 1.1. In the next step we use the transfer principle to derive an estimate for $V^2_\Delta M_{4,2}$-functions.

Proposition 4.2. *Let $K, N \in 2^{\mathbb{N}_0}$ and $1 \leq K \ll N$. Suppose that (14) holds. Then, we find the following estimate to hold:*

$$\|P_N u P_K v\|_{L^2_{t,x}([0,1]\times\mathbb{R}^d)} \lesssim K^{2s} \|P_N u\|_{V^2_\Delta M_{4,2}} \|P_K v\|_{V^2_\Delta M_{4,2}}. \tag{15}$$

Proof. By almost orthogonality, we can write

$$\|P_N u P_K v\|^2_{L^2} \lesssim \sum_{K'} \|Q_{K'} P_N u P_K v\|^2_{L^2_{t,x}([0,1]\times\mathbb{R}^d)}$$

with $(Q_{K'})_{K'}$ like above. We apply Hölder's inequality to find

$$\lesssim \sum_{K'} \|Q_{K'} P_N u\|^2_{L^4_{t,x}([0,1]\times\mathbb{R}^d)} \|P_K v\|^2_{L^4_{t,x}([0,1]\times\mathbb{R}^d)}.$$

We write $P_K v = \sum_m a_m g_m$ with g_m a $U^4_\Delta M_{4,2}$-atom:

$$g_m = \sum_j 1_{I_j^m} e^{it\Delta} f_j^m, \quad \sum_j \|f_j^m\|^4_{M_{4,2}} = 1.$$

Consequently,

$$\begin{aligned}\|P_K v\|_{L^4_{t,x}([0,1]\times\mathbb{R}^d)} &\leq \sum_m |a_m| \|P_K g_m\|_{L^4_{t,x}([0,1]\times\mathbb{R}^d)} \\ &\leq \sum_m |a_m| \big(\sum_j \|P_K e^{it\Delta} f_j^m\|^4_{L^4_{t,x}(I_j^m\times\mathbb{R}^d)} \big)^{\frac{1}{4}} \\ &\lesssim \sum_m |a_m| \big(\sum_j \|f_j^m\|^4_{M_{4,2}} \big)^{\frac{1}{4}} \\ &\lesssim K^s \sum_m |a_m| \lesssim K^s (1+\varepsilon) \|P_K v\|_{U^4_\Delta M_{4,2}}\end{aligned}$$

for any $\varepsilon > 0$ by choice of $(a_m) \in \ell^1$. Likewise, by an additional Galilean transform, we find

$$\|Q_{K'} P_N u\|_{L^4_{t,x}([0,1]\times\mathbb{R}^d)} \lesssim K^s \|Q_{K'} P_N u\|_{U^4_\Delta M_{4,2}}.$$

We use the embedding $V^2_\Delta \hookrightarrow U^4_\Delta$ and carry out the square sum over K' to find

$$\begin{aligned}
&\sum_{K'} \|Q_{K'} P_N u\|^2_{L^4_{t,x}([0,1]\times\mathbb{R}^d)} \|P_K v\|^2_{L^4_{t,x}([0,1]\times\mathbb{R}^d)} \\
&\lesssim K^{4s} \sum_{K'} \|Q_{K'} P_N u\|^2_{U^4_\Delta M_{4,2}} \|P_K v\|^2_{U^4_\Delta M_{4,2}} \\
&\lesssim K^{4s} \sum_{K'} \|Q_{K'} P_N u\|^2_{V^2_\Delta M_{4,2}} \|P_K v\|^2_{V^2_\Delta M_{4,2}} \\
&\lesssim K^{4s} \|P_N u\|^2_{V^2_\Delta M_{4,2}} \|P_K v\|^2_{V^2_\Delta M_{4,2}}.
\end{aligned}$$

The proof is complete. □

Hence, the estimate of Proposition 4.1 holds true with functions in $V^2_\Delta M_{4,2}$.

5. Local well-posedness in critical modulation spaces

This section is devoted to the proof of Theorems 1.3 and 1.4. We begin with the proof of Theorem 1.3, which is carried out via linear Strichartz estimates (cf. [22, Theorem 1.2]).

Proof of Theorem 1.3. We give the proof of (1) in detail. The key ingredients are still like in [27] smoothing and Strichartz estimates. Let $u_0 = f_1 + f_2$ with $f_1 \in M^s_{6,2}(\mathbb{R})$ and $f_2 \in L^2(\mathbb{R})$. Then, Theorem 1.1 yields

$$\|U(t) f_1\|_{L^6([0,T],L^6(\mathbb{R}))} \lesssim \langle T \rangle^{\frac{1}{6}} \|f_1\|_{M^s_{6,2}(\mathbb{R})}$$

and by Strichartz estimates we find

$$\|U(t) f_2\|_{L^6([0,T],L^6(\mathbb{R}))} \lesssim \|f_2\|_{L^2(\mathbb{R})}.$$

Furthermore, since $U(t)(L^2(\mathbb{R}) + M^s_{6,2}(\mathbb{R})) = L^2(\mathbb{R}) + M^s_{6,2}(\mathbb{R})$, we find

$$\|U(t) u_0\|_{L^\infty([0,T],L^2(\mathbb{R})+M^s_{6,2}(\mathbb{R}))} \lesssim \|u_0\|_{L^2(\mathbb{R})+M^s_{6,2}(\mathbb{R})}.$$

The nonlinear estimate is concluded by the inhomogeneous Strichartz estimates

$$\begin{aligned}
\Big\| \int_0^t e^{i(t-s)\Delta} (|u|^4 u)(s) ds \Big\|_{L^6([0,T],L^6(\mathbb{R}))} &\lesssim \||u|^4 u\|_{L^{6/5}([0,T],L^{6/5}(\mathbb{R}))} \\
&\lesssim \|u\|^5_{L^6([0,T],L^6(\mathbb{R}))}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Big\| \int_0^t e^{i(t-s)\Delta} (|u|^4 u)(s) ds \Big\|_{L^\infty([0,T],L^2(\mathbb{R}))} &\lesssim \||u|^4 u\|_{L^{6/5}([0,T],L^{6/5}(\mathbb{R}))} \\
&\lesssim \|u\|^5_{L^6([0,T],L^6(\mathbb{R}))}.
\end{aligned}$$

This finishes the proof of (1). The difference with the cubic NLS on $\mathbb{R}$ analyzed in [27] is that we cannot afford to apply Hölder's inequality in time. This gives the small data constraint. Regarding the claim (2), we note that in two dimensions, $p = q = 4$ are sharp Strichartz indices and by Theorem 1.1 we have the smoothing estimate

$$\|U(t) f\|_{L^4([0,T],L^4(\mathbb{R}^2))} \lesssim \langle T \rangle^{\frac{1}{4}} \|f\|_{M^s_{4,2}(\mathbb{R}^2)}$$

for $s > 0$. □

We turn to the proof of Theorem 1.4 in earnest. As iteration space, we consider $X^1 = \ell^2_N U^2_\Delta M^1_{4,s}$ (cf. Section 2). We have for the norm

$$\|u\|_{X^1} = \big(\sum_N N^2 \|P_N u\|^2_{U^2_\Delta M_{4,2}} \big)^{\frac{1}{2}}.$$

We let furthermore

$$\|v\|_{Y^s} = \big(\sum_N N^{2s} \|P_N u\|^2_{V^2_\Delta M_{4,2}} \big)^{\frac{1}{2}}$$

and have the embedding $X^s \hookrightarrow Y^s$.

With bilinear estimates in adapted function spaces like in [24] available, the arguments of the proof due to [24] apply to the local result. We have the following analog of [24, Proposition 4.1]:

Proposition 5.1. *Let $d \in \{3,4\}$ and $F(u) = \pm |u|^{\frac{4}{d-2}} u$. Then, for any $0 < T \leq 1$, we find the following estimates to hold:*

$$\| \int_0^t e^{i(t-s)\Delta} F(u(s)) ds \|_{X^1([0,T])} \lesssim \|u\|^{\frac{d+2}{d-2}}_{X^1([0,T])} \tag{16}$$

and

$$\begin{aligned} &\| \int_0^t e^{i(t-s)\Delta} [F(u+w)(s) - F(u(s))] ds \|_{X^1([0,T])} \\ &\lesssim \|w\|_{X^1([0,T])} \big(\|u\|_{X^1([0,T])} + \|w\|_{X^1([0,T])} \big)^{\frac{4}{d-2}}. \end{aligned} \tag{17}$$

The implicit constants do not depend on T.

Proof. We only have to prove (17) because (16) is a special case. By duality, it is enough to show

$$\begin{aligned} &\big| \int_0^T \int_{\mathbb{R}^d} [F(u+w)(t) - F(u)(t)] v(t,x) dx dt \big| \\ &\lesssim \|v\|_{Y^{-1}([0,T])} \|u\|_{X^1([0,T])} \big(\|u\|_{X^1([0,T])} + \|w\|_{X^1([0,T])} \big)^{\frac{4}{d-2}}. \end{aligned}$$

For the above display, it is enough to show

$$\begin{aligned} &\sum_{N_0 \geq 1} \sum_{N_1 \geq \ldots \geq N_{\frac{d+2}{d-2}} \geq 1} \big| \int_0^T \int_{\mathbb{R}^d} v_{N_0}(t,x) \prod_{j=1}^{\frac{d+2}{d-2}} u^{(j)}_{N_j})(t,x) dx dt \big| \\ &\lesssim \|v\|_{Y^{-1}} \prod_{j=1}^{\frac{d+2}{d-2}} \|u^{(j)}\|_{X^1([0,T])}. \end{aligned} \tag{18}$$

The proof of (18) follows from linear and bilinear Strichartz estimates combined with Bernstein's inequality. We shall only show the variant of the Killip–Vişan argument for $d = 3$ to avoid redundancy.

Case I: $d = 3$. By Littlewood–Paley theory, the two highest frequencies have to be comparable.

Case I.1: $N_0 \sim N_1 \geq \ldots \geq N_5$: We apply Proposition 4.1 to $v_{N_0} u^{(2)}_{N_2}$ and $u^{(1)}_{N_1} u^{(3)}_{N_3}$ and estimate the remaining factors in $L^\infty_{t,x}$. We write $\mathcal{N}_1 = \{(N_0, N_1, \ldots, N_5) :$

$N_0 \sim N_1 \geq \ldots \geq N_5\}$ for brevity. The estimates yield

$$\begin{aligned}
&\sum_{\mathcal{N}_1} \Big| \int_0^T \int_{\mathbb{R}^d} v_{N_0}(t,x) u^{(1)}_{N_1}(t,x) \ldots u^{(5)}_{N_5}(t,x) dx dt \Big| \\
&\lesssim \sum_{\mathcal{N}_1} \| v_{N_0} u^{(2)}_{N_2} \|_{L^2_{t,x}} \| u^{(1)}_{N_1} u^{(3)}_{N_3} \|_{L^2_{t,x}} \| u^{(4)}_{N_4} \|_{L^\infty_{t,x}} \| u^{(5)}_{N_5} \|_{L^\infty_{t,x}} \\
&\lesssim \sum_{\mathcal{N}_1} N_2^{\frac{1}{2}} N_3^{\frac{1}{2}} N_4^{\frac{3}{2}} N_5^{\frac{3}{2}} \| v_{N_0} \|_{V^2_\Delta M_{4,2}} \prod_{i=1}^5 \| u^{(i)}_{N_i} \|_{V^2_\Delta M_{4,2}} \\
&\lesssim \| v \|_{Y^{-1}} \prod_{i=1}^5 \| u^{(j)} \|_{Y^1}.
\end{aligned}$$

By the embedding $X^1 \hookrightarrow Y^1$ the proof of Case I.1 is complete.

Case I.2: $N_0 \lesssim N_1 \sim N_2 \geq N_3 \geq N_4 \geq N_5$. Denote the summation set with $\mathcal{N}_2$. We apply two bilinear estimates to $v_{N_0} u^{(1)}_{N_1}$ and $u^{(2)}_{N_2} u^{(3)}_{N_3}$ and $L^\infty_{t,x}$-estimates to the other factors to find

$$\begin{aligned}
&\sum_{\mathcal{N}_2} \Big| \int_0^T \int_{\mathbb{R}^d} v_{N_0}(t,x) u^{(1)}_{N_1}(t,x) \ldots u^{(5)}_{N_5}(t,x) dx dt \Big| \\
&\lesssim \sum_{\mathcal{N}_2} \| v_{N_0} u^{(1)}_{N_1} \|_{L^2_{t,x}} \| u^{(2)}_{N_2} u^{(3)}_{N_3} \|_{L^2_{t,x}} \| u^{(4)}_{N_4} \|_{L^\infty_{t,x}} \| u^{(5)}_{N_5} \|_{L^\infty_{t,x}} \\
&\lesssim \sum_{\mathcal{N}_2} N_0^{1/2} N_3^{\frac{1}{2}} N_4^{\frac{3}{2}} N_5^{\frac{3}{2}} \| v_{N_0} \|_{V^2_\Delta M_{4,2}} \prod_{i=1}^5 \| u^{(i)}_{N_i} \|_{V^2_\Delta M_{4,2}} \\
&\lesssim \sum_{\mathcal{N}_2} \frac{N_0^{\frac{3}{2}} N_4^{\frac{1}{2}} N_5^{\frac{1}{2}}}{N_1 N_2 N_3^{\frac{1}{2}}} \| v_{N_0} \|_{Y^{-1}} \prod_{i=1}^5 \| u^{(i)}_{N_i} \|_{Y^1} \\
&\lesssim \| v \|_{Y^{-1}} \prod_{i=1}^5 \| u^{(i)} \|_{Y^1}.
\end{aligned}$$

This finishes the proof of Case I. For the details of the proof of Case II for $d = 4$ we refer to [24]. □

We can complete the proof of Theorem 1.4 along the lines of [19, 24] with Proposition 5.1 at hand.

Proof of Theorem 1.4. For small initial data we can construct a solution on $[0,1]$ by showing that

$$\Phi(u)(t) := e^{it\Delta} u_0 \mp i \int_0^t e^{i(t-s)\Delta} F(u(s)) ds$$

is a contraction mapping within

$$B = \{ u \in X^1([0,1]) \cap C_t([0,1], M^1_{4,2}(\mathbb{R}^d)) \, : \, \| u \|_{X^1} \leq 2\eta \}$$

endowed with $d(u,v) := \| u - v \|_{X^1([0,1])}$. This is a consequence of Proposition 5.1 by observing that Φ maps B into itself by (16) and Φ is indeed contracting by (17). This proves Theorem 1.4 for small data.

For large initial data, we argue with a low frequency cutoff. Let $u_0 \in M^1_{4,2}(\mathbb{R}^d)$ with

$$\|u_0\|_{M^1_{4,2}(\mathbb{R}^d)} \leq A$$

for some $0 < A < \infty$. We consider

$$B = \{u \in X^1([0,T]) \cap C_t([0,T], M^1_{4,2}(\mathbb{R}^d)) \,:\, \|u\|_{X^1([0,T])} \leq 2A, \quad \|u_{>N}\|_{X^1([0,T])} \leq 2\delta\}$$

under the metric $d(u,v) := \|u-v\|_{X^1([0,T])}$.
First, we see that Φ indeed maps B to itself:

$$\begin{aligned}
\|\Phi(u)\|_{X^1} &\leq \|e^{it\Delta}u_0\|_{X^1} + \Big\| \int_0^t e^{i(t-s)\Delta} F(u_{\leq N}(s))ds \Big\|_{X^1} \\
&\quad + \Big\| \int_0^t e^{i(t-s)\Delta}[F(u)(s) - F(u_{\leq N})(s))]ds \Big\|_{X^1} \\
&\leq \|u_0\|_{M^1_{4,2}} + C\|F(u_{\leq N})\|_{L^1_t M^1_{4,2}} + C\|u_{\geq N}\|_{X^1}\|u\|_{X^1}^{\frac{4}{d-2}} \\
&\leq A + CT\|u_{\leq N}\|_{L^\infty_t M^1_{4,2}} \|u_{\leq N}\|^{\frac{4}{d-2}}_{L^\infty_t M^1_{\infty_1}} + C(2\delta)(2A)^{\frac{4}{d-2}} \\
&\leq A + CTN^{\frac{6}{d-2}}(2A)^{\frac{d+2}{d-2}} + C(2\delta)(2A)^{\frac{4}{d-2}} \leq 2A
\end{aligned}$$

provided δ is chosen small enough depending on A, and T is chosen small enough depending on A and N.

Next, we decompose $F(u) = F_1(u) + F_2(u)$, where

$$F_1(u) = O(u^2_{>N} u^{\frac{6-d}{d-2}}) \text{ and } F_2(u) = O(u_{\leq N}^{\frac{4}{d-2}} u).$$

We estimate with the Hölder-like inequality for modulation spaces (cf. [10, Theorem 4.3])

$$\begin{aligned}
&\|P_{>N}\Phi(u)\|_{X^1} \\
&\leq \|e^{it\Delta}P_{>N}u_0\|_{X^1} + \Big\| \int_0^t e^{i(t-s)\Delta} F_1(u(s))ds \Big\|_{X^1} \\
&\quad + \Big\| \int_0^t e^{i(t-s)\Delta} F_2(u(s))ds \Big\|_{X^1} \\
&\leq \|P_{>N}u_0\|_{M^1_{4,2}(\mathbb{R}^d)} + C\|u_{>N}\|^2_{X^1}\|u\|_{X^1}^{\frac{6-d}{d-2}} + C\|F_2(u)\|_{L^1_t M^1_{4,2}} \\
&\leq \delta + C(2\delta)(2A)^{\frac{6-d}{d-2}} + CT\|u\|_{L^\infty_t M^1_{4,2}} \|u_{\leq N}\|^{\frac{2d}{d-2}}_{L^\infty_t M^1_{\infty,1}} \\
&\leq \delta + C(2\delta)(2A)^{\frac{6-d}{d-2}} + CTN^{\frac{2d}{d-2}}(2A)^{\frac{d+2}{d-2}}.
\end{aligned}$$

We can bound the above by 2δ provided that δ is chosen small enough depending on A, and T is chosen small enough depending on A, δ, and N.
Next, we prove that Φ is a contraction. We decompose like above $F = F_1 + F_2$ and observe

$$F_1(u) - F_1(v) = O((u-v)(u_{>N} - v_{>N})(u^{\frac{6-d}{d-2}} + v^{\frac{6-d}{d-2}}))$$

and

$$F_2(u) - F_2(v) = O((u-v)(u_{\leq N} + v_{\leq N})^{\frac{4}{d-2}}) + O((u_{\leq N} - v_{\leq N})(u+v)(u_{\leq N} + v_{\leq N})^{\frac{6-d}{d-2}}).$$

By the above arguments for $u, v \in B$:

$$\begin{aligned}
&d(\Phi(u),\Phi(v))\\
&\lesssim \|u-v\|_{X^1}(\|u_{>N}\|_{X^1}+\|v_{>N}\|_{X^1})(\|u\|_{X^1}+\|v\|_{X^1})^{\frac{6-d}{d-2}}\\
&\quad+\|F_2(u)-F_2(v)\|_{L_t^1 M^1_{4,2}}\\
&\lesssim (4\delta)(4A)^{\frac{6-d}{d-2}} d(u,v)+T\|u-v\|_{L_t^\infty M^1_{4,2}}(\|u_{\leq N}\|_{L_t^\infty M^1_{\infty,1}}+\|v_{\leq N}\|_{L_t^\infty M^1_{\infty,1}})^{\frac{4}{d-2}}\\
&\quad+T(\|u\|_{L_t^\infty M^1_{4,2}}+\|v\|_{L_t^\infty M^1_{4,2}})\|u_{\leq N}-v_{\leq N}\|_{L_t^\infty M^1_{\infty,1}}\\
&\qquad\times\big(\|u_{\leq N}\|_{L_t^\infty M^1_{\infty,1}}+\|v_{\leq N}\|_{L_t^\infty M^1_{\infty,1}}\big)^{\frac{6-d}{d-2}}\\
&\lesssim [(4\delta)(4A)^{\frac{6-d}{d-2}}+TN^{\frac{4d}{d-2}}(4A)^{\frac{4}{d-2}}]d(u,v)\leq \frac{1}{2}d(u,v),
\end{aligned}$$

provided δ is chosen small enough depending on A, and T is chosen small enough depending on A and N. This yields uniqueness and analytic dependence of the data-to-solution mapping within B. By standard arguments, uniqueness extends to $X^1([0,T]) \cap C_t([0,T], M^1_{4,2}(\mathbb{R}^d))$. □

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