Strategy-Proofness and Participation in Collective Decisions

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Abstract

This dissertation examines two important properties of processes of collective decision making: *strategy-proofness* and *participation*. Strategy-proofness is the condition that no individual can obtain a better result through strategic voting than by truth-telling. Participation is the property that individuals prefer to take part in the decision making process over abstaining. There is extensive research in particular on strategy-proofness, yet there is only little work that examines both conditions and their compatibility.

The first part of this thesis focuses on strategy-proofness. We divide this condition into two parts: (i) the existence of a (very) weakly dominant strategy and (ii) truth-telling being a (very) weakly dominant strategy. We show that the existence of a (very) weakly dominant strategy in general does not imply, that truth-telling is such a strategy, yet this implication can be obtained when requiring the voting rule to satisfy additional mild conditions. We show this for the universal domain as well as for two different approaches to restricted domains.

The second part examines the compatibility of strategy-proofness and participation. We consider a model in which agents with single-peaked preferences may take part in a decision making process to determine the value of a one-dimensional variable and in which participation is costly. We show that all rules that are strategy-proof and satisfy two additional mild conditions possess a unique equilibrium with only one participant.

The third part revisits the model of the previous chapter and considers a sequential voting game with the mean and the symmetric median rule as aggregation rules. For both rules we characterize all equilibria. Then, we present the design of a laboratory experiment aiming to determine whether the impact of these rules on the participation decision differs.

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1 Introduction

It is estimated that every adult makes about 35,000 remotely conscious decisions per day (Sahakian and LaBuzetta, 2013). As humans are social beings, it is evident that a substantial number of these decisions concern other individuals and thus in a democratic society these decisions are often taken collectively. The task of social choice theory is to examine processes of collective decision making of all sorts. There is an abundance of different applications, ranging from small groups to entire societies, and with many potential and different properties, e.g. 'dictatorial' decisions such as a parent deciding on what food to prepare for the newborn, or federal elections with equal voting rights for all citizens.

In times when trust in governments and democracy is shrinking,[1] it is even more important that the processes of collective decision making - such as elections - meet certain standards. In order to maintain these standards, social choice theory proposes and analyzes all kinds of desirable properties, for example monotonicity, non-manipulability (or strategy-proofness), applicability to all situations (also known as the universal domain condition), Condorcet-consistency, the incentive to take part in the decision making process and much more.

All of these conditions are desirable in their own way, some of them are even considered necessary for collective decision processes; the German Constitutional Court ruled that a violation of monotonicity in the federal election of the Bundestag was unconstitutional (Bundesverfassungsgericht, 2008). Unfortunately, there is no perfect decision process and social choice theory is shaped by impossibility results such as the theorems of Arrow (1951/63), Gibbard (1973) and Satterthwaite (1975) and Muller and Satterthwaite (1977). Hence, it is vital to examine the conditions and their relation in detail to know if and when they are compatible. In this thesis, we focus on two properties and their relation: (i) strategy-proofness, i.e. the absence of the possibility to vote strategically and (ii) participation, i.e. the incentive to take part in the decision making process.

In Part I of this thesis, we take a close look at strategy-proofness and decompose it into two conditions: (i) the existence of a very weakly dominant strategy (which following Blin and Satterthwaite (1977) we call belief-independence) and (ii) truth-telling being such a strategy. We examine if the second part is implied by the first, by requiring the voting rules to satisfy additional mild and desirable properties and come to a positive conclusion. Moreover, we show by means of a variety of examples that the implication does not hold without the additional conditions. We provide equivalence results of belief-independence and strategy-proofness for the universal domain as well as for restricted preference domains. When considering restricted preference domains, we have a look at the standard approach in the literature, but we additionally consider the *unrestricted message space approach*, which has not yet been systematically studied in the literature. Using this approach, meaningful versions of strategy-proofness and belief-independence are more restrictive but also more robust to beliefs and actions of the others. We also find natural and mild conditions under which this robustness translates to the classical version of strategy-proofness.

In Part II of the thesis, we consider a more concrete setting: a group decision on a one-dimensional variable with costly voting. We assume single-peaked preferences and examine whether strategy-proofness and participation are compatible. We find that all strategy-proof rules that satisfy some additional and

[1] According to the United Nations, Department of Economic and Social Affairs (2021), the trust in the national government has declined from 73 percent in 1958 to 24 percent in 2021 in the United States.

natural conditions imply minimal participation, i.e. all equilibria of strategy-proof voting rules have exactly one participant. We also show that this tension can be resolved if one considers probabilistic instead of deterministic rules. Part II is based on a joint paper with Clemens Puppe.

In the third and final part of this thesis, we revisit the collective decision problem of Part II in a sequential voting game in which first a participation decision is taken, and then, after observing all participation decisions, the votes are cast. We characterize all equilibria for two commonly used rules for this problem: the symmetric median rule and the mean rule. We show that the symmetric median rule only possesses single or (almost) full participation equilibria. We then present the design of a laboratory experiment to test whether individuals understand and play the equilibrium strategies, and to see whether the rules themselves influence the participation rate in situations that are designed to be strategically identical.

The pilot experiment was developed with the help of Toni Heusch, who provided excellent assistance in its preparation and conduction.

Part I

Belief-Independence and (Robust) Strategy-Proofness

2 Motivation

Ever since the celebrated Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975), strategy-proofness ('non-manipulability') is one of the focal properties in voting theory. Strategy-proofness requires that truth-telling is a weakly dominant strategy. This is a desirable property for several reasons. One important aspect of strategy-proofness is the robustness that it implies: under a strategy-proof voting rule, every individual has an optimal (i.e. very weakly dominant) strategy independently of the behavior of all other voters. In particular, optimal play is robust with respect to the beliefs voters may have about the type and the behavior of the other voters. Following Blin and Satterthwaite (1977), we call this property belief-independence of a voting rule. Strategy-proofness adds to this the requirement that one of the optimal strategies for each voter be truth-telling. Thus, strategy-proofness is formally a strictly stronger condition than belief-independence.

In this chapter, we give a number of examples of voting rules (social choice functions) that are belief-independent but not strategy-proof. However, we also show - much in the spirit of Blin and Satterthwaite (1977) - that belief-independence implies strategy-proofness under a few natural additional conditions, and we present a number of natural sufficient conditions that guarantee the equivalence of belief-independence and strategy-proofness. Our results pertain both to the universal domain and to restricted preference domains. We distinguish two different approaches to restricting preferences in voting theory. First, in what we call the *common domain approach*, one assumes that both the actual ('true') preferences of all voters and the voting rule are restricted to the same subset of preferences; this is in fact the standard approach to strategy-proofness taken in the literature (see e.g. Barberà, 2001). On the other hand, and perhaps conceptually more attractive, one may restrict only the actual preferences of voters while keeping the voting rule defined on the universal domain. We call this the *unrestricted message space approach*. The important difference is that under the latter assumption, voters can report any preference from the unrestricted domain even when their factual preferences are restricted to a subdomain. While we assume for the common domain approach that the domain restriction is common knowledge, we only need to assume in this approach that the social planner knows the domain restriction. In this setup, meaningful (and stronger) definitions of strategy-proofness and belief-independence would require robustness with respect to the reporting of these additional preference orderings. We call the corresponding concepts *robust strategy-proofness* and *robust belief-independence*. Again, we (i) give examples, demonstrating that the robust versions of strategy-proofness and belief-independence are indeed stronger, and (ii) provide equivalence results under natural additional conditions.

Relation to the Literature

There is quite some literature dealing with beliefs - in particular in the mechanism design literature - as beliefs play a crucial role when collective decisions are taken under incomplete information. Strategic considerations of the individuals then depend on the actions the other individuals take, and thus on the beliefs on the actions of the other individuals, which on their own depend on their beliefs. Hence, the strategy choice also depends on the beliefs about the beliefs of the others and on the belief about the belief about the belief and so on. This infinite hierarchy of beliefs was first addressed by Harsanyi (1967/1968), who introduced type spaces to deal with that issue. This idea was later formalized by

Mertens and Zamir (1985) and further developed by Bergemann and Morris (2005). In their sophisticated approach, beliefs are implicitly contained in the individual's type. In this chapter, we consider belief-independence as introduced by Blin and Satterthwaite (1977), which is meant to be understood in a literal way: independence of all possible kinds and forms of beliefs via weakly dominant strategies. Blin and Satterthwaite (1977) introduced this condition and provide an impossibility result similar to the theorem of Gibbard (1973) and Satterthwaite (1975). Additionally, they illustrate in how far beliefs might be manipulated in case of non-belief-independent rules. While weaker than strategy-proofness, belief-independence is still a very restrictive condition and one may want to weaken it. The 'strategically simple mechanisms' introduced by Börgers and Li (2019), that only depend on first-order beliefs, can be considered a step in that direction but are out of the scope of this work.

In this thesis, we are interested in the interplay of the 'pure' notions of strategy-proofness and belief-independence; we thus focus on Blin and Satterthwaite's original and simple concept. Importantly, we approach the problem of comparing strategy-proofness and belief-independence from the perspective of voting theory and not from the perspective of mechanism design. Indeed, by the revelation principle for dominant strategies introduced by Gibbard (1973) (see also Mas-Colell et al., 1995) we know that every belief-independent voting rule can be truthfully implemented in dominant strategies. In the present work, we only consider *direct* mechanisms and give natural conditions under which belief-independent voting rules can be truthfully implemented by a direct mechanism. Hence, one way of interpreting our results is that we give conditions under which one does not need to resort to the revelation principle in order to get from a belief-independent to a strategy-proof voting rule.

To the best of our knowledge, the notion of robust strategy-proofness has not yet been systematically analyzed in the literature. However, Maskin and Dasgupta (2020) use this property under the name of *strategy-proofness** in their axiomatic analysis of Condorcet consistent rules.

Structure of this Chapter

The chapter is organized as follows: Section 3 introduces the general model and the basic definitions, presents the theorem of Blin and Satterthwaite (1977) and gives a result, that demonstrates the equivalence of belief-independence and strategy-proofness for sovereign, positive responsive and tops-only voting rules (Proposition 3.2.4). We also show by means of examples that none of these conditions can be dropped in this result.

Section 4 extends the analysis to two different approaches to restricted preference domains: the *common domain approach* and the *unrestricted message space approach*. The common domain approach is the standard approach in the literature restricting all preferences and the domain of the voting rule to the same restricted preference domain for all individuals. In the unrestricted message space approach, the domain of the 'true' preference orderings is also restricted to the same restricted preference domain for all voters, but the voting rule is defined on the universal domain and thus - and in contrast to the common domains approach - all preference orderings are feasible as voters' reports.

For the common domain approach, several examples illustrate the problems that come with generalizing the equivalence result (Proposition 3.2.4) to restricted preference domains. It is shown that, when adding anonymity and replacing positive responsiveness by unanimity, the equivalence of strategy-proofness and belief-independence is restored on restricted common preference domains (Proposition 4.1.6).

In the unrestricted message space approach, we adapt the concepts of strategy-proofness and belief-independence to *robust belief-independence* and *robust strategy-proofness*. We provide an equivalence result for strategy-proof and robust strategy-proof voting rules for tops-only rules on minimally rich domains (Proposition 4.2.5). Moreover, we demonstrate how the equivalence between (robust) strategy-proofness and (robust) belief-independence for the common domain approach can be translated to the unrestricted message space approach (Proposition 4.2.7). An additional equivalence result is provided for tops-only, positive responsive and neutral rules (Proposition 4.2.9).

3 Belief-Independence and Strategy-Proofness on the Universal Domain

3.1 The Model

We consider a situation in which a finite set of individuals $N = \{1, \dots, n\}$ faces a finite set of alternatives $A = \{a_1, \dots, a_m\}$ that contains at least three elements. Every individual has linear preferences over the alternatives. The set of all linear preferences over A is denoted by $\mathcal{P}$ and the preference ordering of individual i by $\succ_i \in \mathcal{P}$. By $\tau^{(k)}(\succ_i)$, we denote the k-th placed alternative within the preference ordering $\succ_i$ but - to simplify notation - we omit the superscript when denoting the most preferred alternative, i.e. $\tau^{(1)}(\succ_i) = \tau(\succ_i)$. A profile of preferences is denoted by $\succ = (\succ_1, \dots, \succ_n) \in \mathcal{P}^n$. We examine deterministic voting rules (or social choice functions) $f : \mathcal{P}^n \to A$ that map profiles of preferences of the individuals to one alternative in A. We will assume throughout this chapter that the voting rules satisfy sovereignty, i.e. for all alternatives $a \in A$ there exists a profile $\succ \in \mathcal{P}^n$ such that $f(\succ) = a$. This is a natural assumption, as the violation of sovereignty implies that some alternatives are never chosen by the voting rule. In this case, they should not be considered alternatives in the first place.

We assume that all individuals are equipped with beliefs towards the other individuals. In general, beliefs may take a variety of forms. As we focus on the independence of beliefs, we keep beliefs as general as possible. In fact, following Blin and Satterthwaite (1977), we say that a voting rule satisfies belief-independence if every individual possesses a very weakly dominant strategy, i.e. if for every individual there exists one strategy that is always at least as good as any other strategy. This dominance concept is the only dominance concept that we will use in this chapter. From now on, to keep the wording simpler, we will omit the ‘very’ when referring to it. Note that this concept varies from the definition of weakly dominant strategies in game theory textbooks, as these require the dominant strategy to be strictly better sometimes.

A weakly dominant strategy is optimal given all possible behavior of the other individuals, and hence independent of whatever belief one might have about the other individuals and moreover independent of the precise shape of the belief. Thus, a weakly dominant strategy really is ‘belief-independent’.

Definition 3.1.1. *A voting rule $f : \mathcal{P}^n \to A$ is called* ***belief-independent*** *if for all $i \in N$ and all $\succ_i \in \mathcal{P}$ there exists $\sigma_i(\succ_i) \in \mathcal{P}$ such that*

$$f(\sigma_i(\succ_i), \succ_{-i}) \;\succcurlyeq_i\; f(\succ'_i, \succ_{-i}) \text{ for all } \succ_{-i} \in \mathcal{P}^{n-1} \text{ and all } \succ'_i \in \mathcal{P}.$$

In this chapter, we examine belief-independence and compare it to a very similar but different condition: strategy-proofness. A voting rule is strategy-proof if truth-telling is a weakly dominant strategy. It is straightforward to see that belief-independence is directly implied by strategy-proofness of a voting rule, as belief-independence requires the existence of a weakly dominant strategy and strategy-proofness guarantees that truth-telling is such a weakly dominant strategy.[2]

[2] Note that due to our interpretation of weakly dominant strategies, such a strategy is not necessarily unique as one can see e.g. in Example 3.1.5.

Definition 3.1.2. *A voting rule $f : \mathcal{P}^n \to A$ is called* ***strategy-proof*** *if for all $i \in N$ and all $\succ_i, \succ'_i \in \mathcal{P}$*

$$f(\succ_i, \succ_{-i}) \succcurlyeq_i f(\succ'_i, \succ_{-i}) \text{ for all } \succ_{-i} \in \mathcal{P}^{n-1}.$$

On the universal domain we know by the theorem of Gibbard (1973) and Satterthwaite (1975) that every strategy-proof and sovereign rule is dictatorial. As belief-independence is a weakening of strategy-proofness, this result does not directly translate to belief-independent rules and in fact, one can find examples of non-dictatorial rules that are belief-independent.

To start, let us define dictatorship. A voting rule is dictatorial if there exists one individual (the dictator) such that the voting rule chooses the top alternative of the submitted preference ordering of the dictator independently of the submitted preferences of the other individuals.

Definition 3.1.3. *A voting rule $f : \mathcal{P}^n \to A$ is called* ***dictatorial*** *if there exists $i \in N$ such that $f(\succ_i, \succ_{-i}) = \tau(\succ_i)$ for all $\succ_i \in \mathcal{P}$ and all $\succ_{-i} \in \mathcal{P}^{n-1}$.*

The following examples illustrate that there are non-dictatorial and belief-independent voting rules.

Example 3.1.4. Consider the following voting rule:

$$f(\succ) := \begin{cases} \tau(\succ_1) & \text{if } \tau^{(m)}(\succ_1) = a \\ a & \text{else.} \end{cases}$$

Every player possesses a weakly dominant strategy: individual 1 submits her true preference if she likes alternative a the most, else she submits her true preference but with alternative a moved down to the last (and therefore m-th) position. This ensures that her most preferred alternative is chosen. No other individual influences the outcome, hence all strategies are weakly dominant strategies. Thus, this voting rule is belief-independent but not strategy-proof.

Example 3.1.5. Let $A = \{a, b, c\}$ and consider the following voting rule:

$$f(\succ) := \begin{cases} a & \text{if } \tau(\succ_1) = b \\ b & \text{if } \tau(\succ_1) = c \\ c & \text{if } \tau(\succ_1) = a. \end{cases}$$

Again, all players have weakly dominant strategies. Individual 1 will adjust her submitted preference ordering - and thus deviate from truth-telling - assuring her most preferred alternative to be chosen. Since no other individual influences the outcome, submitting any preference ordering is a weakly dominant strategy for them. As all individuals possess a weakly dominant strategy, this rule is belief-independent. However, as truth-telling is not a weakly dominant strategy for individual 1, f is not strategy-proof.

3.2 Result of Blin and Satterthwaite (1977) and Extension

The previous examples illustrate non-dictatorial rules; nevertheless, both have a strong dictatorial feel, as there exists one individual who can guarantee her preferred alternative to be the outcome. In order to capture this, Blin and Satterthwaite (1977) use a suitable adaption of the definition of dictatorship, which we renamed to dictatorship with respect to σ in order to avoid confusion.[3]

[3] In addition, our definition is slightly altered but equivalent.

Definition 3.2.1. *Let $f : \mathcal{P}^n \to A$ be a belief-independent voting rule. Then we call f* ***dictatorial with respect to $\boldsymbol{\sigma}$****, if there exists an individual $i \in N$ such that for all $\succ_i \in \mathcal{P}$ and all $\succ_i \in \mathcal{P}^{n-1}$ there exists $\sigma_i(\succ_i) \in \mathcal{P}$ with*

$$f(\sigma_i(\succ_i), \succ_{-i}) = \tau(\succ_i).$$

With this concept of dictatorship, Blin and Satterthwaite (1977) are able to establish a Gibbard-Satterthwaite-type result for belief-independent voting rules.

Theorem 3.2.2 (Blin, Satterthwaite, 1977)**.** *Let $f : \mathcal{P}^n \to A$ be a belief-independent and sovereign voting rule. Then f is dictatorial with respect to σ.*

The original theorem is even stronger, only assuming an effective range of at least three elements rather than sovereignty, but for our purposes, the version above is more suitable.

In the following part, we want to further examine the relation between strategy-proofness and belief-independence on the universal domain. As mentioned before, it can be seen directly from the definitions that strategy-proofness implies belief-independence. We show that, if we require the voting rule to satisfy some mild and natural conditions, belief-independence also implies strategy-proofness.

Definition 3.2.3.

- *Given preference profiles $\succ, \succ' \in \mathcal{P}^n$ and an alternative $a \in A$, we say that a keeps or improves its relative position from $\succ$ to $\succ'$ if for all $b \in A$ and all $i \in N$: $a \succ_i b \Rightarrow a \succ'_i b$.*
- *A voting rule $f : \mathcal{P}^n \to A$ is called* ***positive responsive*** *if for all $\succ, \succ' \in \mathcal{P}^n$ and all $a \in A$:*

$$\left\{ \begin{array}{l} f(\succ) = a, \quad \succ' = \succ \ \text{ on } A \setminus \{a\} \ \text{ and } a \text{ keeps} \\ \text{or improves its relative position from } \succ \text{ to } \succ' \end{array} \right\} \Rightarrow f(\succ') = a.$$

- *A social choice function $f : \mathcal{P}^n \to A$ is called* ***tops-only*** *if for all preference profiles $\succ, \succ' \in \mathcal{P}^n$ with $\tau(\succ_i) = \tau(\succ'_i)$ for all $i \in N$: $f(\succ) = f(\succ')$.*

These conditions are not restrictive. Positive responsiveness is a minimal monotonicity condition, therefore also known as elementary monotonicity (see e.g. Majumdar and Sen, 2004), and tops-onlyness is a property most of the commonly used voting rules share. Moreover, as Chatterji and Sen (2011) find, tops-only and strategy-proof rules are often closely related.

The proposition below establishes the equivalence of strategy-proofness and belief-independence given that the voting rule is tops-only and positive responsive. The idea of adding tops-onlyness stems from Example 3.1.4, the idea of adding positive responsiveness from Example 3.1.5, as these examples exploit the absence of these conditions. Note that while the rule in Example 3.1.4 is not tops-only, it is positive responsive, and the rule in Example 3.1.5 does not satisfy positive responsiveness but is tops-only.

Proposition 3.2.4. *Let $f : \mathcal{P}^n \to A$ be a sovereign voting rule. Then f is belief-independent, positive responsive and tops-only if and only if f is strategy-proof.*

Proof. By Theorem 3.2.2 we know that f is dictatorial with respect to σ. Hence, there exists $i \in N$ such that for all $\succ_i \in \mathcal{P}$ and all $\succ_{-i} \in \mathcal{P}^{n-1}$ we get $f(\sigma_i(\succ_i), \succ_{-i}) = \tau(\succ_i)$.
Consider the preference ordering $\succ'_i$ that

(i) coincides with $\sigma_i(\succ_i)$ on $A \setminus \{\tau(\succ_i)\}$

and for which

(ii) $\tau(\succ_i') = \tau(\succ_i)$.

By positive responsiveness we obtain $f(\succ_i', \succ_{-i}) = \tau(\succ_i)$ for all $\succ_i \in \mathcal{P}$ and all $\succ_{-i} \in \mathcal{P}^{n-1}$.
As f is tops-only we have $f(\succ_i, \succ_{-i}) = \tau(\succ_i)$ for all $\succ_i \in \mathcal{P}$ and all $\succ_{-i} \in \mathcal{P}^{n-1}$.
Thus, f is dictatorial and thus in particular strategy-proof.
Let f be strategy-proof. Then, f is dictatorial and thus in particular positive responsive and tops-only. $\square$

We are assuming voting rules to be sovereign throughout this chapter but it is worth mentioning that this condition cannot be dropped in Proposition 3.2.4 as the following example illustrates.

Example 3.2.5. Let $A = \{a, b, c\}$ and consider the following voting rule:

$$f(\succ) := \begin{cases} a & \text{if } \#\{i \in N : \tau(\succ_i) = a\} \ \geq \ \#\{i \in N : \tau(\succ_i) = b\} \\ b & \text{else.} \end{cases}$$

This rule is belief-independent but not strategy-proof. If an individual's top alternative is c, she has an incentive to deviate from truth-telling and vote for a or b, depending on which alternative she prefers, as there exist cases in which she can shift the outcome to her preferred alternative. Moreover, this rule is positive responsive and tops-only but obviously not sovereign.

4 Restricted Preference Domains

In this section, we extend our analysis to restricted preference domains. We use two different approaches of restricting preference domains and give sufficient conditions for the equivalence of belief-independence and strategy-proofness on restricted domains in both cases.

4.1 Common Domain Approach

We start with the common domain approach, which is the standard approach in the literature on restricted domains. Here, the preferences, as well as the domain of the voting rule, are restricted to a commonly known domain, respectively its n-fold Cartesian product. Formally, we denote a restricted preference domain by $\mathcal{D} \subset \mathcal{P}$ and consider the case where all individuals $i \in N$ have preferences $\succ_i \in \mathcal{D}$ in that domain and the voting rule $f : \mathcal{D}^n \to A$ is defined on that domain only. We adjust the definition of belief-independence to $\mathcal{D}$ by assuming that in $\mathcal{D}$ every individual has a weakly dominant strategy. The definition of strategy-proofness on $\mathcal{D}$ requires that truth-telling is a weakly dominant strategy in the domain $\mathcal{D}$.

Definition 4.1.1.

- *A voting rule $f : \mathcal{D}^n \to A$ is called **belief-independent on $\mathcal{D}$** if for all $i \in N$ and all $\succ_i \in \mathcal{D}$ there exists $\sigma_i(\succ_i) \in \mathcal{D}$ such that*

$$f(\sigma_i(\succ_i), \succ_{-i}) \ \succcurlyeq_i \ f(\succ'_i, \succ_{-i}) \ \text{ for all } \ \succ_{-i} \in \mathcal{D}^{n-1}, \succ'_i \in \mathcal{D}.$$

- *A voting rule $f : \mathcal{D}^n \to A$ is called **strategy-proof on $\mathcal{D}$** if for all $i \in N$ and all $\succ_i \in \mathcal{D}$*

$$f(\succ_i, \succ_{-i}) \ \succcurlyeq_i \ f(\succ'_i, \succ_{-i}) \ \text{ for all } \ \succ_{-i} \in \mathcal{D}^{n-1}, \succ'_i \in \mathcal{D}.$$

Again, by construction, strategy-proofness on a restricted domain implies belief-independence on that domain. The converse is obviously not true in general. Hence, we look to extend Proposition 3.2.4 to restricted preferences. Unfortunately, this turns out to be more subtle than one might expect. The following example describes a rule that is tops-only, positive responsive, sovereign and belief-independent on a restricted preference domain but not strategy-proof on that domain.

Example 4.1.2. Let $A = \{a, b, c\}$ and consider the (restricted) preference domain $\mathcal{D} = \{abc, acb, bca, cba\}$ and the following voting rule:

$$f(\succ) := \begin{cases} a, & \text{if } \tau(\succ_2) = a \\ b, & \text{if } \tau(\succ_1) = b \text{ and } \tau(\succ_2) \in \{b, c\} \\ c, & \text{if } \tau(\succ_1) = c \text{ and } \tau(\succ_2) \in \{b, c\} \\ \tau(\succ_2), & \text{if } \tau(\succ_1) = a \text{ and } \tau(\succ_2) \in \{b, c\}. \end{cases}$$

It can be easily seen that this rule is sovereign and tops-only. The voting rule is also positive responsive. To show this, we need to go through all possible cases.

Case 1: $f(\succ) = a$.
Then, $\tau(\succ_2) = a$ and in all profiles in which a keeps or improves its relative position, a remains the top alternative of individual 2, thus a remains the outcome.

Case 2: Let $f(\succ) = b$.
Then, (i) $\tau(\succ_1) = b$ and $\tau(\succ_2) \in \{b, c\}$ or (ii) $\tau(\succ_1) = a$ and $\tau(\succ_2) = b$. In the first case, only individual 2 can change the top alternative to b, which does not change the outcome. In case (ii), individual 1 can change the top alternative to b, resulting in case (i), but still with outcome b.

Case 3: Let $f(\succ) = c$.
The argument for $f(\succ) = c$ is analogous to the one for b.

Furthermore, f is belief-independent on $\mathcal{D}$. Individual 1 can never change the outcome to a but can decide on whether b or c are chosen, given that individual 2 did not submit a as her top alternative. Hence, individual 1's optimal strategy is putting b or c on top of her submitted preference ordering, depending on which alternative she prefers. Individual 2's optimal strategy is truth-telling as she always likes a the most or the least in all preferences in $\mathcal{D}$. Hence, if she prefers a, she gets her best alternative by truth-telling. If she prefers b and/or c, then by truth-telling she can ensure the outcome to be within b and c, and if individual 2 submits a as her top alternative, she can even determine the outcome. Thus, the voting rule is sovereign, tops-only, positive responsive and belief-independent on $\mathcal{D}$ but not strategy-proof on $\mathcal{D}$ as individual 2's optimal strategy differs from truth-telling.

There are further problems that can arise when restricted preference domains are rather sparse. Then, the condition of positive responsiveness can turn into a vacuous requirement as the following example illustrates.

Example 4.1.3. Let $A = \{a, b, c, d\}$ and let $\mathcal{D} = \{abcd, badc, cbda, dbca\}$. Consider

$$f(\succ) := \begin{cases} a, & \text{if } \tau(\succ_1) = b \\ b, & \text{if } \tau(\succ_1) = c \\ c, & \text{if } \tau(\succ_1) = d \\ d & \text{else.} \end{cases}$$

This rule is sovereign and tops-only. As all players have a weakly dominant strategy, and as truth-telling is not a weakly dominant strategy for individual 1, this rule is belief-independent on $\mathcal{D}$ but not strategy-proof on $\mathcal{D}$. Moreover, for all $\succ_i \in \mathcal{D}$ there exists no $\succ_i' \in \mathcal{D} \setminus \{\succ_i\}$ such that any alternative keeps or improves its relative position from $\succ_i$ to $\succ_i'$. Hence, any voting rule defined on $\mathcal{D}$ is positive responsive (on $\mathcal{D}$).

The natural hope is that all these problems can be solved when choosing appropriate preference domains that are in some sense 'well-behaved'. The first candidate that comes to mind is the domain of single-peaked preferences, which is well-studied in the literature and known to have nice properties, e.g. the existence of non-dictatorial and strategy-proof voting rules. Unfortunately, Proposition 3.2.4 cannot be extended to the domain of single-peaked preferences as the following example demonstrates.

Example 4.1.4. Let $A = \{a_1, \ldots, a_m\}$, let n be odd and let $\mathcal{D}$ be the single-peaked domain with respect to the linear order $a_1 < \cdots < a_m$. Denote by $\underline{\text{med}}(\succ_{-1})$ the lower median and by $\overline{\text{med}}(\succ_{-1})$ the upper median of the top alternatives of the preference profile $\succ_{-1} \in \mathcal{D}^{n-1}$. Note that $\succ_{-1}$ has $n-1$ elements which is an even number since n is odd. Consider

$$f(\succ) := \begin{cases} a_2, & \text{if } \underline{\text{med}}(\succ_{-1}) = a_2,\ \overline{\text{med}}(\succ_{-1}) = a_3,\ \tau(\succ_1) = a_2 \\ \\ \overline{\text{med}}(\succ_{-1}), & \text{else.} \end{cases}$$

This rule is sovereign and tops-only. Moreover, one can show that f is positively responsive.
Let $f(\succ) = a_j$ for $j \neq 2$. Then, $a_j = \overline{\text{med}}(\succ_{-1})$. As f is tops-only, the outcome can only change if an individual submits a different peak. If an individual changes her top alternative to a_j, then a_j remains the upper median and thus the outcome remains unchanged.
Let $f(\succ) = a_2$. If $a_2 = \overline{\text{med}}(\succ_{-1})$, then by the same argument we can deduce that the outcome does not change when we consider a profile in which a_j keeps or improves its relative position. Assume that $\underline{\text{med}}(\succ_{-1}) = a_2$, $\overline{\text{med}}(\succ_{-1}) = a_3$ and $\tau(\succ_1) = a_2$. Then, if an individual with top alternative a_1 changes her top alternative to a_2, nothing changes, as the lower and the upper median remain the same. If an individual with a top alternative different from a_1 and a_2 changes her top alternative to a_2, the upper median changes to a_2. Thus, the outcome is defined by the second instance in the definition of f. As a_2 corresponds to the new upper median, the outcome remains the same. Hence, f is positive responsive.
Furthermore, f is belief-independent on $\mathcal{D}$. Individual 1 only influences the outcome in one instance, where she can choose between a_2 and a_3. If her top alternative is a_1, then she prefers a_2 over a_3 as she has single-peaked preferences. However, she obtains a_2 only by submitting a_2 as her top alternative, hence she has a strict incentive to not submit her true preference ordering in this case. For all other individuals truth-telling is a weakly dominant strategy. Thus, f is belief-independent on $\mathcal{D}$ but not strategy-proof on $\mathcal{D}$.

The previous examples illustrate that there is little to no hope to easily extend Proposition 3.2.4 to restricted domains. Hence, we impose further conditions to establish the equivalence between belief-independence and strategy-proofness on restricted domains.

Definition 4.1.5.
- *For every permutation $p : N \to N$ of N into itself, denote by $\succ_p$ the profile $\succ_P = (\succ_{p(i)})_{i \in N}$. We say that a voting rule $f : \mathcal{D}^n \to A$ is* ***anonymous*** *if for all $\succ \in \mathcal{D}^n$ and all p, $f(\succ_p) = f(\succ)$.*
- *A voting rule $f : \mathcal{D}^n \to A$ is called* ***unanimous*** *if for all $\succ = (\succ_1, \dots, \succ_n) \in \mathcal{D}^n$ with $\succ_i = \succ_j$ for all $i, j \in N$, we have $f(\succ) = \tau(\succ_1)$.*

As the previous examples treat individuals differently, we additionally require the voting rule to satisfy anonymity, and find this to be sufficient to establish the equivalence. In order to avoid problems with positive responsiveness on sparse preference domains, we replace it with unanimity, a condition that is implied by positive responsiveness, sovereignty and tops-onlyness on sufficiently rich domains.[4]

Proposition 4.1.6. *Let $f : \mathcal{D}^n \to A$ be a tops-only, unanimous and anonymous voting rule. Then f is strategy-proof on $\mathcal{D}$ if and only if f is belief-independent on $\mathcal{D}$.*

Proof. Strategy-proofness on $\mathcal{D}$ implies belief-independence on $\mathcal{D}$ by construction. Hence, we only need to show that tops-onlyness, unanimity, anonymity and belief-independence on $\mathcal{D}$ imply strategy-proofness on $\mathcal{D}$.

[4] This implication holds true for example for fully adjacent domains as defined later in Definition 4.2.12.

Let $f : \mathcal{D}^n \to A$ be a tops-only, unanimous and anonymous voting rule. Moreover, let f be belief-independent on $\mathcal{D}$ but not strategy-proof on $\mathcal{D}$.
Then there exists $i \in N$, $\succ_i \in \mathcal{D}$ and $\sigma_i(\succ_i) \in \mathcal{D}$ such that $f(\sigma_i(\succ_i), \succ_{-i}) \not\prec_i f(\succ_i', \succ_{-i})$ for all $\succ_i' \in \mathcal{D}$, $\succ_{-i} \in \mathcal{D}^{n-1}$ and $f(\sigma_i(\succ_i), \succ_{-i}') \succ_i f(\succ_i, \succ_{-i}')$ for some $\succ_{-i}' \in \mathcal{D}^{n-1}$. As f is tops-only, we obtain $\tau(\sigma_i(\succ_i)) \neq \tau(\succ_i)$, as otherwise the outcome would be identical when submitting both preference orderings, while keeping everything else fixed.
Consider $\overline{\succ}_{-i} \in \mathcal{D}^{n-1}$ with $\overline{\succ}_j = \succ_i$ for all $j \in N \setminus \{i\}$. By unanimity, we obtain $f(\succ_i, \overline{\succ}_{-i}) = \tau(\succ_i)$ and by belief- independence we get $f(\sigma_i(\succ_i), \overline{\succ}_{-i}) = \tau(\succ_i)$.
Consider $\overline{\succ}_{-i}'$ with $\overline{\succ}_j' = \overline{\succ}_j$ for all $j \in N \setminus \{i, k\}$ for some $k \neq i$ and $\overline{\succ}_k' = \sigma_i(\succ_i)$.
Since f is anonymous, we get $f(\succ_i, \overline{\succ}_{-i}') = \tau(\succ_i)$, and due to the belief-independence of f we obtain $f(\sigma_i(\succ_i), \overline{\succ}_{-i}') = \tau(\succ_i)$. We repeat this argument $n-1$ times, while replacing all preference orderings in $\overline{\succ}_{-i}$ one by one by $\sigma_i(\succ_i)$. With the resulting preference profile $\overline{\succ}_{-i}^{(n-1)}$, we obtain $f(\succ_i, \overline{\succ}_{-i}^{(n-1)}) = \tau(\succ_i)$ and $f(\sigma_i(\succ_i), \overline{\succ}_{-i}^{(n-1)}) = \tau(\succ_i) \neq \tau(\sigma_i(\succ_i))$ which contradicts the unanimity of f. □

Only dictatorial voting rules satisfy the conditions of Proposition 3.2.4 and thus these are the only rules that are strategy-proof and belief-independent on the universal domain. Hence, as dictatorial rules are excluded here by anonymity, one might wonder whether there are rules that satisfy all conditions of Proposition 4.1.6. Such rules exist, as the following example shows.

Example 4.1.7. Let $\mathcal{D}$ be the domain of all single-peaked preferences and let f be the median rule (both with respect to the order $a_1 < \cdots < a_m$). This rule is tops-only, unanimous, anonymous and non-dictatorial.

4.2 Unrestricted Message Space Approach

When taking the basic idea of belief-independence seriously, it makes sense to adopt a different approach of restricting preferences. If we want a voting rule to be belief-independent, it should not depend on any possible belief, even if it seems implausible. For example one might believe that other individuals, even with preferences from a commonly known restricted preference domain $\mathcal{D}$, submit a preference ordering outside of $\mathcal{D}$. We explicitly want to allow for such beliefs, and thus only restrict the domain of the 'true' preferences of the individuals but not the domain of the voting rule, hence explicitly also allowing for preferences outside of $\mathcal{D}$ to be submitted. We call this the **unrestricted message space approach**. This approach could be considered closer to reality in at least some instances, as voting rules used in practice usually do not exclude preference orderings from being submitted, even if all meaningful preferences follow a certain structure. In this approach we only assume that the restricted domain is known to the social planner, whereas in the common domain approach it is assumed that the restricted domain is common knowledge. A similar idea, but without the focus on belief-independence, can be recently found in Maskin and Dasgupta (2020).
We therefore need to adjust the definitions of strategy-proofness and belief-independence to this setup. Strategy-proofness and belief-independence on a preference domain in this setting are stronger conditions than in the previous section, which is why we call strategy-proofness and belief-independence in this setup robust strategy-proofness[5] and robust belief-independence.

Definition 4.2.1. *A voting rule $f : \mathcal{P}^n \to A$ is called **robustly belief-independent on $\mathcal{D}$** if for all $i \in N$ and all $\succ_i \in \mathcal{D}$ there exists $\sigma_i(\succ_i) \in \mathcal{P}$ such that*

$$f(\sigma_i(\succ_i), \succ_{-i}) \not\prec_i f(\succ_i', \succ_{-i}) \quad \text{for all} \quad \succ_{-i} \in \mathcal{P}^{n-1}, \succ_i' \in \mathcal{P}.$$

[5] Maskin and Dasgupta (2020) call this property Strategy-Proofness*.

Definition 4.2.2. *A voting rule $f : \mathcal{P}^n \to A$ is **robustly strategy-proof on $\mathcal{D}$** if for all $i \in N$ and all $\succ_i \in \mathcal{D}$*

$$f(\succ_i, \succ_{-i}) \;\succcurlyeq_i\; f(\succ_i', \succ_{-i}) \quad \text{for all} \quad \succ_{-i} \in \mathcal{P}^{n-1}, \succ_i' \in \mathcal{P}.$$

It is straightforward to see that robust strategy-proofness on $\mathcal{D}$ implies robust belief-independence on $\mathcal{D}$. Moreover, for $\mathcal{D} = \mathcal{P}$ robust strategy-proofness coincides with strategy-proofness. The following example illustrates that the two concepts are in general not equivalent.

Example 4.2.3. Let $\mathcal{D} = \{abc, cba\}$ and consider the following voting rule:

$$f(\succ) := \begin{cases} \tau(\succ_1), & \text{if } \tau(\succ_1) = \tau(\succ_2) \\ b, & \text{if } \tau(\succ_1) = a \text{ and } \tau(\succ_2) = c \\ b, & \text{if } \tau(\succ_1) = c \text{ and } \tau(\succ_2) = a \\ a, & \text{else.} \end{cases}$$

If f is restricted to $\mathcal{D}$, the last case can never occur and no individual has profitable deviations from truth-telling within $\mathcal{D}$. Nevertheless, there are profitable deviations if all preference orderings are admissible. If the true preference ordering of individual 1 is $\succ_1 = (cba)$, then she has a strict incentive to deviate from truth-telling given that individual 2 submits b as her top alternative. Thus, f is strategy-proof on $\mathcal{D}$ but not robustly strategy-proof on $\mathcal{D}$.

Even if robust strategy-proofness and strategy-proofness on a restricted preference domain are different conditions in general, they coincide under mild conditions. Whenever a voting rule is tops-only and the preference domain satisfies a minimal richness condition, the two definitions are equivalent.

Definition 4.2.4. *A preference domain $\mathcal{D}$ is called **minimally rich** if for all $a \in A$ there exists $\succ_i \in \mathcal{D}$ such that $\tau(\succ_i) = a$.*

The following proposition establishes the equivalence between strategy-proofness on $\mathcal{D}$ and robust strategy-proofness on $\mathcal{D}$ given these mild conditions.

Proposition 4.2.5. *Let $f : \mathcal{P}^n \to A$ be a tops-only voting rule and let $\mathcal{D}$ be minimally rich. Then f is strategy-proof on $\mathcal{D}$ if and only if f is robustly strategy-proof on $\mathcal{D}$.*

Proof. Robust strategy-proofness on $\mathcal{D}$ always implies strategy-proofness on $\mathcal{D}$. Therefore we only need to show that for a minimally rich preference domain $\mathcal{D}$ and a tops-only social choice function f strategy-proofness on $\mathcal{D}$ implies robust strategy-proofness on $\mathcal{D}$.

As $\mathcal{D}$ is minimally rich, we can find for every preference profile $\succ_{-i} \in \mathcal{P}^{n-1}$ a preference profile $\succ_{-i}^\star \in \mathcal{D}_{-i}$ with $\tau(\succ_{-i}) = \tau(\succ_{-i}^\star)$, and for every preference ordering $\succ_i \in \mathcal{P}$ a corresponding preference ordering $\succ_i^\star \in \mathcal{D}$ with $\tau(\succ_i) = \tau(\succ_i^\star)$.
Since f is tops-only the outcome of f does not change when replacing a preference ordering $\succ_i$ by $\succ_i^\star$ or when replacing a preference profile $\succ_{-i}$ by $\succ_{-i}^\star$.
Thus, we obtain

(1) $f(\succ_i, \succ_{-i}) = f(\succ_i, \succ_{-i}^\star)$ for all $\succ_i \in \mathcal{D}_i, \succ_{-i} \in \mathcal{P}^{n-1}$

and

(2) $f(\succ_i', \succ_{-i}) = f(\succ_i'^\star, \succ_{-i}^\star)$ for all $\succ_i' \in \mathcal{P}, \succ_{-i} \in \mathcal{P}^{n-1}$.

As f is strategy-proof on $\mathcal{D}$, we have

(3) $f(\succ_i, \succ^\star_{-i}) \;\succcurlyeq_i\; f(\succ'^\star_i, \succ^\star_{-i})$

as we ensured that all preferences are in $\mathcal{D}$ by using $\succ^\star$ instead of $\succ$. Thus,

$$f(\succ_i, \succ_{-i}) \overset{(1)}{=} f(\succ_i, \succ^\star_{-i}) \overset{(3)}{\succcurlyeq_i} f(\succ'^\star_i, \succ^\star_{-i}) \overset{(2)}{=} f(\succ'_i, \succ_{-i})$$

for all $\succ_i \in \mathcal{D}_i, \succ'_i \in \mathcal{P}, \succ_{-i} \in \mathcal{P}^{n-1}$, which is exactly the condition of robust strategy-proofness. $\square$

The conditions of Proposition 4.2.5 are only sufficient conditions, as for example dictatorship is both strategy-proof and robustly strategy-proof even on non-minimally rich domains. However, Example 4.2.3 shows that tops-onlyness is a minimal condition in the sense that the equivalence does not hold in general when the condition is dropped.

The following example, which is inspired by Barbie et al. (2006), illustrates that the minimal richness of the preference domain is also such a minimal condition.

Example 4.2.6. Let $\mathcal{D} = \{abc, bca, cab\}$ and consider the Borda rule. The Borda rule is strategy-proof on $\mathcal{D}$ as deviations within the domain attribute relatively more points only to the least favorite alternative. Hence, deviations (within $\mathcal{D}$) are never profitable. But if all preferences in $\mathcal{P}$ are admissible as submitted preference orderings, then there are profitable deviations and thus the Borda rule is not robustly strategy-proof on $\mathcal{D}$.

Example 4.1.4 can also be used to illustrate that Proposition 3.2.4 cannot be easily extended in this setup. Thus, we add further conditions that help establish the equivalence of robust strategy-proofness (on $\mathcal{D}$) and robust belief-independence (on $\mathcal{D}$). We start by adding anonymity like in the previous section and obtain an analogous result.

Proposition 4.2.7. *Let $f : \mathcal{P}^n \to A$ be a tops-only, unanimous and anonymous voting rule. Then, f is robustly strategy-proof on $\mathcal{D}$ if and only if f is robustly belief-independent on $\mathcal{D}$.*

Proof. The idea of the proof is identical to the proof of Proposition 4.1.6.
Robust strategy-proofness on $\mathcal{D}$ implies robust belief-independence on $\mathcal{D}$ by construction. Hence, we only need to show that tops-onlyness, unanimity, anonymity and robust belief-independence on $\mathcal{D}$ imply robust strategy-proofness on $\mathcal{D}$.
Let $f : \mathcal{D}^n \to A$ be a tops-only, unanimous and anonymous voting rule. Moreover, let f be robustly belief-independent on $\mathcal{D}$ but not robustly strategy-proof on $\mathcal{D}$.
Then there exists $i \in N$, $\succ_i \in \mathcal{D}$ and $\sigma_i(\succ_i) \in \mathcal{P}$ such that $f(\sigma_i(\succ_i), \succ_{-i}) \succcurlyeq_i f(\succ'_i, \succ_{-i})$ for all $\succ'_i \in \mathcal{P}, \succ_{-i} \in \mathcal{P}^{n-1}$ and $f(\sigma_i(\succ_i), \succ'_{-i}) \succ_i f(\succ_i, \succ'_{-i})$ for some $\succ'_{-i} \in \mathcal{P}^{n-1}$. As f is tops-only, we obtain $\tau(\sigma_i(\succ_i)) \neq \tau(\succ_i)$, as otherwise the outcome would be identical when submitting both preference orderings while keeping everything else fixed.
Consider $\bar{\succ}_{-i} \in \mathcal{P}^{n-1}$ with $\bar{\succ}_j = \succ_i$ for all $j \in N \setminus \{i\}$. By unanimity, we obtain $f(\succ_i, \bar{\succ}_{-i}) = \tau(\succ_i)$ and by robust belief- independence we get $f(\sigma_i(\succ_i), \bar{\succ}_{-i}) = \tau(\succ_i)$.
Consider $\bar{\succ}'_{-i}$ with $\bar{\succ}'_j = \bar{\succ}_j$ for all $j \in N \setminus \{i, k\}$ for some $k \neq i$ and $\bar{\succ}'_k = \sigma_i(\succ_i)$.
Since f is anonymous, we get $f(\succ_i, \bar{\succ}'_{-i}) = \tau(\succ_i)$, and due to the robust belief-independence of f, we obtain $f(\sigma_i(\succ_i), \bar{\succ}'_{-i}) = \tau(\succ_i)$. We repeat this argument $n-1$ times, while replacing all preference orderings in $\bar{\succ}_{-i}$ one by one by $\sigma_i(\succ_i)$. With the corresponding preference profile $\bar{\succ}^{(n-1)}_{-i}$, we obtain $f(\succ_i, \bar{\succ}^{(n-1)}_{-i}) = \tau(\succ_i)$ and $f(\sigma_i(\succ_i), \bar{\succ}^{(n-1)}_{-i}) = \tau(\succ_i) \neq \tau(\sigma_i(\succ_i))$ which contradicts the unanimity of f. $\square$

Again, the median rule on the domain of single-peaked preferences serves as an example of a non-dictatorial rule that satisfies all conditions of the proposition. By assuring equal treatments of individuals, one obtains the equivalence; the same holds true if one requires alternatives to be treated equally, as the following proposition shows.

Definition 4.2.8. *For every permutation p of A into itself, denote by $\succ^p$ the profile $\succ^p = (\succ_i^p)_{i \in N}$, where $\succ_i^p (a) = \succ_i (p(a))$ for all $i \in N$ and all $a \in A$. A voting rule is called **neutral** if for all $\succ \in \mathcal{P}^n$ and all p, $f(\succ)^p = p^{-1}(f(\succ))$.*

Another advantage of this way of restricting preferences is that the problems that arise with neutrality on restricted domains do not play a role here, as the voting rule is defined on the universal domain.

Proposition 4.2.9. *Let $f : \mathcal{P}^n \to A$ be a tops-only, positive responsive and neutral voting rule. Then f is robustly strategy-proof on $\mathcal{D}$ if and only if f is robustly belief-independent on $\mathcal{D}$.*

Proof. By definition, robust strategy-proofness on $\mathcal{D}$ implies robust belief-independence on $\mathcal{D}$.
Let $f : \mathcal{P}^n \to A$ be a tops-only, positive responsive and neutral voting rule. Moreover, let f be robustly belief-independent on $\mathcal{D}$ but not robustly strategy-proof on $\mathcal{D}$. Then, there exists $i \in N, \succ_i \in \mathcal{D}$ and $\sigma_i(\succ_i) \in \mathcal{P}$ such that

$$f(\sigma_i(\succ_i), \succ_{-i}) \not\prec_i f(\succ_i, \succ_{-i}) \text{ for all } \succ_{-i} \in \mathcal{P}^{n-1}$$

and

$$f(\sigma_i(\succ_i), \succ'_{-i}) \succ_i f(\succ_i, \succ'_{-i}) \text{ for some } \succ'_{-i} \in \mathcal{P}^{n-1}.$$

As f is tops-only, we obtain $\tau(\sigma_i(\succ_i)) \neq \tau(\succ_i)$ as otherwise the outcome would be identical when submitting both preference orderings while keeping everything else fixed. Then individual i can never be pivotal between $\tau(\sigma_i(\succ_i))$ and $\tau(\succ_i)$. If she were pivotal, then there exists $\overline{\succ}_{-i} \in \mathcal{P}^{n-1}$ and $\succ_i^*, \succ_i^{**} \in \mathcal{P}$ such that $f(\succ_i^*, \overline{\succ}_{-i}) = \tau(\sigma_i(\succ_i))$ and $f(\succ_i^{**}, \overline{\succ}_{-i}) = \tau(\succ_i)$.
We now apply positive responsiveness and tops-onlyness and obtain $f(\sigma_i(\succ_i), \overline{\succ}_{-i}) = \tau(\sigma_i(\succ_i))$ and $f(\succ_i, \overline{\succ}_{-i}) = \tau(\succ_i)$, which is a contradiction, since $\sigma_i(\succ_i)$ weakly dominates $\succ_i$ by robust belief-independence.

Since individual i is never pivotal between $\tau(\sigma_i(\succ_i))$ and $\tau(\succ_i)$, we obtain by neutrality that individual i is never pivotal between any two alternatives. But then $\sigma_i(\succ_i)$ cannot weakly dominate $\succ_i$, hence we reach a contradiction, which concludes the proof. ☐

Extending Proposition 4.2.9 to the Common Domain Approach

Using the unrestricted message space approach, one can avoid the problems with neutrality on restricted domains, but nevertheless we can deduce a similar result to Proposition 4.2.9 for voting rules defined on common domains, yet not without imposing some adjacency condition on the preference domain as the following example illustrates.

Example 4.2.10. Let $\mathcal{D} = \{abc, cba\}$ and consider the voting rule $f : \mathcal{D}^n \to A$ defined by

$$f(\succ) := \begin{cases} a, & \text{if } \tau(\succ_1) = \tau(\succ_2) = c \\ c, & \text{if } \tau(\succ_1) = \tau(\succ_2) = a \\ b, & \text{else.} \end{cases}$$

This voting rule is tops-only, positive responsive (as it is a vacuous requirement here) and neutral on $\mathcal{D}$, as only permutations that switch alternatives a and c result in a preference ordering within $\mathcal{D}$. Moreover, this rule is belief-independent on $\mathcal{D}$. For $i \in \{1,2\}$ if the true preference ordering is $\succ_i = (abc)$, then submitting $\succ_i' = (cba)$ results in the outcome a if $j \in \{1,2\} \setminus \{i\}$ also submits c as her top alternative and b else. As she prefers a over c, there are no profitable deviations for i. We can apply an analogous argument to $\succ_i = (cba)$ and deduce that there are no profitable deviations from submitting (abc). As all other individuals do not influence the outcome, the rule is belief-independent on $\mathcal{D}$. Moreover, this rule is not strategy-proof on $\mathcal{D}$.
One can find simpler counterexamples of rules that are positive responsive, neutral on $\mathcal{D}$ and tops-only if one allows for non-sovereign rules.

The following definition of adjacency is taken from Chatterji and Zeng (2020).

Definition 4.2.11. *Two alternatives $a, b \in A$ are called **adjacent** in $\mathcal{D}$ if there exists $\succ_i, \succ_i' \in \mathcal{D}$ such that $\tau(\succ_i) = \tau^{(2)}(\succ_i') = a$, $\tau^{(2)}(\succ_i) = \tau(\succ_i') = b$ and $\tau^{(k)}(\succ_i) = \tau^{(k)}(\succ_i')$ for all $k = 3, \dots, m$.*

With this notion of adjacency, we can describe a richness property for restricted preference domains, which prevents positive responsiveness from being a vacuous requirement for voting rules defined on that domain.

Definition 4.2.12. *We say that a preference domain $\mathcal{D}$ is **fully adjacent** if all pairs of alternatives $a, b \in A$ are adjacent in $\mathcal{D}$.*

Adding this richness condition to Proposition 4.2.9 allows to extend the result to common domains.

Proposition 4.2.13. *Let $f : \mathcal{D}^n \to A$ be a tops-only, positive responsive and neutral voting rule and let $\mathcal{D}$ be fully adjacent. Then f is strategy-proof on $\mathcal{D}$ if and only if f is belief-independent on $\mathcal{D}$.*

Proof. The proof works similar to the proof of Proposition 4.2.9:

By definition, strategy-proofness on $\mathcal{D}$ implies belief-independence on $\mathcal{D}$.
Let $f : \mathcal{P}^n \to A$ be a tops-only, positive responsive and neutral voting rule. Moreover, let f be belief-independent on $\mathcal{D}$ but not strategy-proof on $\mathcal{D}$. Then, there exist $i \in N, \succ_i \in \mathcal{D}$ and $\sigma_i(\succ_i) \in \mathcal{D}$ such that

$$f(\sigma_i(\succ_i), \succ_{-i}) \;\succcurlyeq_i\; f(\succ_i, \succ_{-i}) \text{ for all } \succ_{-i} \in \mathcal{D}^{n-1}$$

and

$$f(\sigma_i(\succ_i), \succ_{-i}') \;\succ_i\; f(\succ_i, \succ_{-i}') \text{ for some } \succ_{-i}' \in \mathcal{D}^{n-1}.$$

As f is tops-only, we obtain $\tau(\sigma_i(\succ_i)) \neq \tau(\succ_i)$, as otherwise the outcome would be identical when submitting both preference orderings while keeping everything else fixed. Then individual i can never be pivotal between $\tau(\sigma_i(\succ_i))$ and $\tau(\succ_i)$. If she were pivotal, then there exists $\overline{\succ}_{-i} \in \mathcal{D}^{n-1}$ and $\succ_i^*, \succ_i^{**} \in \mathcal{D}$ such that $f(\succ_i^*, \overline{\succ}_{-i}) = \tau(\sigma_i(\succ_i))$ and $f(\succ_i^{**}, \overline{\succ}_{-i}) = \tau(\succ_i)$.
Since $\mathcal{D}$ is fully adjacent, we can now use the positive responsiveness and the tops-onlyness of f to obtain $f(\sigma_i(\succ_i), \overline{\succ}_{-i}) = \tau(\sigma_i(\succ_i))$ and $f(\succ_i, \overline{\succ}_{-i}) = \tau(\succ_i)$, which is a contradiction, since $\sigma_i(\succ_i)$ weakly dominates $\succ_i$ by belief-independence. Since $\mathcal{D}$ is fully adjacent, we can apply neutrality to deduce that individual i is never pivotal between any two alternatives. But then $\sigma_i(\succ_i)$ cannot weakly dominate $\succ_i$, hence we reach a contradiction, which concludes the proof.

□

5 Conclusion

The concept of belief-independence was originally introduced by Blin and Satterthwaite (1977). In this chapter, we provided examples showing that strategy-proofness and belief-independence are generally not equivalent. We then demonstrated that for sovereign, positive responsiveness and tops-only voting rules both conditions are equivalent. We extended the concept of belief-independence on restricted domains with two different approaches: the common domain and the unrestricted message space approach. For both cases we deduced that (robust) strategy-proofness and (robust) belief-independence are equivalent for (i) tops-only, unanimous and anonymous, and for (ii) tops-only, positive responsive and neutral voting rules, with an additional adjacency condition for common domains in case (ii). Additionally, we demonstrated that for tops-only voting rules and minimally rich preference domains the conditions of robust strategy-proofness and strategy-proofness coincide.

Part II

The Interplay of Strategy-Proofness and Participation

6 Motivation

The celebrated Gibbard-Satterthwaite Theorem (Gibbard, 1973; Satterthwaite, 1975) states that, under very general circumstances, every strategy-proof social choice function on an unrestricted preference domain is dictatorial. In particular, if voting comes at some cost, all voters except one will prefer not to participate in the election process. One interpretation of the Gibbard-Satterthwaite Theorem is thus that two fundamental incentive properties are incompatible on an unrestricted domain: strategy-proofness, i.e. the incentive to reveal preferences truthfully, and participation, i.e. the incentive to invest the cost of casting one's vote.

In this paper, we show that a similar conclusion holds if all voters' preferences are single-peaked, which is one of the paradigmatic cases in which there exist non-dictatorial and strategy-proof voting rules (Black, 1948; Moulin, 1980). Concretely, the following is our main result (Theorem 8.1.1 in Section 8): Suppose that a group of agents collectively decides on the level of a one-dimensional variable and that all agents have single-peaked preferences over that level. Also, assume that each agent prefers not to participate whenever such unilateral abstention does not change the outcome of the election. Moreover, suppose that the voting rule employed is strategy-proof, anonymous and responsive in the sense that the outcome reacts to any unanimous shift of all peaks in one direction. Then, generically there exists a unique equilibrium with a single participant, which is either the agent with the lowest peak or the agent with the highest peak.[6]

Our analysis brings together two different strands of the literature. On the one hand, we rely on Moulin's path-breaking characterization of all strategy-proof social choice functions on the single-peaked domain as the *generalized medians*, i.e. medians with a set of 'phantom voters' (Moulin, 1980); on the other hand, we employ a version of the *pivotal voter model* (Downs, 1957; Palfrey and Rosenthal, 1983), which posits that rational agents will engage in a voting procedure if and only if the expected benefit of doing so exceeds its costs. The main departure from the bulk of the literature on the pivotal voter model is that we assume a rich one-dimensional space of alternatives in which individuals generically have different top alternatives. As we shall see, this radically changes the properties of equilibria. Of course, our assumption requires that there are more alternatives than voters, therefore our model applies to decisions in small committees and not to 'large' elections in which there are many more voters than alternatives.

In our complete information setting, agents take two decisions: whether or not to participate in the voting procedure, and if so, which vote to cast. We consider two versions of this general set-up: a sequential model and a simultaneous model. The sequential model has two stages: in the first stage agents simultaneously decide whether or not to participate in a committee, and in the second stage a simultaneous vote is cast by the committee members on the level of a one-dimensional variable. In the simultaneous model, the participation and voting decisions are made simultaneously by all agents. In either model, we assume that agents incur positive cost if and only if they in fact cast a vote. We shall see that while the two models may lead to different predictions in general, for our main result the timing of decisions is in fact not relevant.

[6] The genericity assumption is that no two agents have the same peak, because otherwise there may evidently exist multiple equilibria. Moreover, as explained in detail below, we assume that in equilibrium each voter employs her unique weakly dominant strategy, i.e. truth-telling, *given* any fixed set of participants.

As a simple example, think of a faculty meeting on a Friday afternoon at which the yearly expenditure shares, say, for research and teaching, have to be determined (given a fixed budget). Each faculty member deliberates about whether or not to participate in the voting procedure. In the sequential model, one can think of the participation decision as being taken before the actual meeting. In the simultaneous model, there could be an announcement during the meeting that a vote would be taken after extensive discussion, and committee members may decide to leave early, thereby abstaining from the collective decision. The assumption of complete information is strong but does not seem to be unrealistic in such a scenario; in fact, since all strategy-proof voting rules (for a fixed set of participants) only depend on the top alternative of each voter, it is sufficient to know each colleagues' top choice.

The conclusion of our main result – that under strategy-proofness, anonymity and responsiveness, a single individual casts her vote in equilibrium – stands in stark contrast to other voting rules that are not strategy-proof. For instance, if the collective outcome is determined to be the *average* of the individual votes (Renault and Trannoy, 2005) on a cardinal scale, every voter can shift the outcome by a positive amount, and full participation is indeed an equilibrium if participation costs are sufficiently small. But while anonymous and responsive, taking the average vote does evidently not define a strategy-proof rule.

The intuition behind the single participation equilibrium under strategy-proofness can be explained by looking at the case of two voters. Strategy-proofness and the responsiveness condition jointly imply that in the case of two voters, the outcome must coincide with one of the two voters' top alternative; anonymity implies that it cannot depend on the identity of the voter, hence it must be either the lower or the higher top alternative. In the first case, if the agent with the lowest top alternative participates, no other agent has an incentive to participate (since no other agent can change the result by unilateral participation); in the latter case, the same holds if the agent with the highest top alternative participates. The proof that there are no other equilibria is more involved (see Section 3). Ultimately, our 'impossibility' result can be traced back to the fact that strategy-proofness necessitates the use of an asymmetric tie-breaking rule in the case of an even number of participants; and the counterfactual situation of an even number of participants is of course also relevant for equilibrium, even if the potential and actual number of voters is odd. This observation suggests that the incompatibility of strategy-proofness and (non-minimal) participation may vanish under a *probabilistic* and symmetric tie-breaking rule in the case of an even number of participants, and we show by means of a simple example that this is indeed true.

A full-fledged analysis of the probabilistic case is beyond the scope of the present work. But we investigate if the standard *symmetric median*, i.e. the average of the two middle tops in case of an even number of voters, can solve the participation problem in the deterministic case at the expense of loosing the strategy-proofness property for an even number of voters. We answer this in the negative and provide a complete characterization of the subgame perfect equilibria of the sequential game if participation costs are sufficiently small but positive (Theorem 2 in Section 4). Taking without loss of generality the unit interval as outcome space, we show that there are only two types of equilibria: the single participation of the voter whose peak is closest to the midpoint 0.5, or (almost) full participation with the fixed outcome 0.5.[7]

[7] In the latter case, if there is a voter with top alternative at 0.5 that voter will be the only one to abstain.

Relation to the Literature

The question of participation when voting is costly has been extensively discussed in the literature. The first contributions to this discussion were made by Downs (1957) and Tullock (1967). They argue that an individual should vote, if the expected gained utility from voting exceeds its costs. As the influence on the outcome is small, at least in larger electorates, they find that voting is often irrational. Ferejohn and Fiorina (1974, 1975) try to rationalize the act of voting. They argue that, due to uncertainty of the situation, individuals try to minimize the maximal regret rather than to maximize the expected utility. Since voting for the preferred candidate is often minimizing the maximal regret, it then is rational to vote. Riker and Ordeshook (1968) argue that voting itself not only causes costs, but can also positively influence ones utility, i.e. due to the sentiment of fulfilled civic duty. Thus voting can be rationalized, even if the individual's impact on the outcome is very small. The first game-theoretic formulation of the so-called pivotal voter model can be found in Palfrey and Rosenthal (1983, 1985) and Ledyard (1984). In their model, the probability of being pivotal is endogenized and depends on the participation decisions.

The vast majority of the contributions in the literature studies the case of majority voting among two alternatives (see e.g. Börgers, 2004). Under complete information, a key issue is to analyze the equilibrium consequences of the fact that a large fraction of voters shares the same preferences. The resulting coordination problem is typically addressed by an analysis of mixed equilibria, see Nöldeke and Peña (2016); Mavridis and Serena (2018) for recent contributions. In the present paper, we set this issue aside by assuming that voters have distinct preference tops, which is the generic case in our framework with a rich set of alternatives.

The two contributions in the literature closest to ours are Osborne et al. (2000) and Cohensious et al. (2017). These authors also study costly voting in committees in a complete information environment with individuals that have single-peaked preferences. Osborne et al. (2000) *assume* that agents vote truthfully and show that 'extreme' voters are more likely to participate than 'moderate' voters. However, in the relevant results of Osborne et al. (2000), the outcome with an even number of participants is determined by the symmetric median in which case sincere voting does not generally constitute an equilibrium. While our main result does not contradict the intuition put forward by Osborne et al. (2000), it qualifies it in an important way. Under the responsiveness condition, the single participant is indeed always an 'extreme' voter: either the agent with the lowest or the agent with the highest top alternative. But as explained above, the rationale is not that the moderate voters cancel each other out; more importantly, without the responsiveness property, equilibria can occur in which only 'moderate' agents participate, see Section 8.2 below.

In independent work, Cohensious et al. (2017) observe that participation by a single voter is the only equilibrium in the special case of the 'lower median' voting rule (and by symmetry also for the 'upper median' rule), but they do not offer a general impossibility result for the class of all strategy-proof voting rules akin to the main result

Structure of this Chapter

The chapter is organized as follows: Section 7 introduces the general model and the basic definitions. Section 8 presents the main result (Theorem 8.1.1) of this chapter that reveals the tension between strategy-proofness and participation. We also discuss all conditions of Theorem 8.1.1 in detail. Section 9 considers probabilistic rules and shows that in at most half of the instances, a deterministic voting rule can be used if one wants to obtain an incentive to participate, given that a strong Nash equilibrium is played in the second stage. Section 10 shows that the symmetric median rule, that is not strategy-proof, cannot induce full participation as an equilibrium in general, either.

7 The Model

We denote the set of agents by $N = \{1, 2, ..., n\}$. Each agent i is characterized by a single-peaked (ordinal) preference relation $\succcurlyeq_i$ over a compact interval in the reals which we assume to be normalized to unity, i.e. $[0, 1] \subseteq \mathbb{R}$. One possible interpretation is that each $0 \leq x \leq 1$ represents the expenditure share for a public project; but there are other, purely ordinal interpretations as well (e.g. potential positions on a political spectrum). In fact, none of our results depends on the assumption of a *continuous* space of alternatives; what is important is that there are sufficiently many more alternatives than agents. Single-peakedness means that each agent i has a unique top alternative $0 \leq p_i \leq 1$ (the 'peak') such that, for all $x, y \in [0, 1]$, we have $x \succ_i y$ whenever $y < x \leq p_i$ or $p_i \leq x < y$.[8]

Agents decide whether or not to participate in a committee that decides on the level of the one-dimensional variable $x \in [0, 1]$ by a voting procedure. Each agent i faces a positive participation cost $c_i > 0$. This cost may vary from agent to agent, it may depend on the finally chosen outcome, and even on the set of the other participating agents; in fact, all that matters for our purpose is that these costs are strictly positive for all agents. In particular, we could allow the cost to be private information. For each possible non-empty set $K \subseteq N$ of participants, there is a social choice function $F^K(\succcurlyeq_1, ..., \succcurlyeq_{\#K}) \in [0, 1]$ that maps every profile of preferences of the agents in K to an outcome in $[0, 1]$. The collection $F = \{F^K\}_{\emptyset \neq K \subseteq N}$ of these social choice functions is referred to as a *voting rule*. The employed voting rule is common knowledge among the agents.

In our model with endogenous participation, we need to specify agents' preferences $\widehat{\succcurlyeq}_i$ over pairs (x, K) of outcomes and sets of participants K who actually cast a vote. We denote by x_0 the (exogenously determined) outcome if nobody participates in the voting process, and will make the following assumptions: For all $i \in N$,

(i) the outcome x_0 is strictly worse than the most preferred outcome with single own participation, i.e.

$$(p_i, \{i\}) \mathrel{\widehat{\succ}_i} (x_0, \emptyset),$$

(ii) for every fixed set $K \neq \emptyset$, the preference over outcomes is given by $\succcurlyeq_i$, i.e.

$$(x, K) \mathrel{\widehat{\succcurlyeq}_i} (y, K) \iff x \succcurlyeq_i y,$$

(iii) for every fixed $x \in [0, 1]$, non-participation is strictly preferred to participation (and indifference with respect to the composition of the set of participants otherwise), i.e. for all $K, K' \neq \emptyset$,

$$\begin{aligned} \{i \notin K \quad \text{or } i \in K'\} &\implies (x, K) \mathrel{\widehat{\succcurlyeq}_i} (x, K'), \\ \{i \notin K \text{ and } i \in K'\} &\implies (x, K) \mathrel{\widehat{\succ}_i} (x, K'). \end{aligned}$$

Observe that, except for the 'non-triviality' condition (i), no assumptions are made about how agents compare an outcome without participation to a *strictly better* outcome with own participation; indeed,

[8] Note that we do not need to make any assumptions about the comparisons of alternatives on different sides of the peak; in fact, the preference relation may even be incomplete and refrain from making such comparisons.

such trade-offs would determine the particular magnitude of participation cost which plays no role in our present analysis.

We will consider two variants of the model, a simultaneous and a sequential version. In the sequential version, agents first simultaneously decide whether or not to participate and then vote simultaneously in a second stage after having observed who the other participants are. By contrast, in the simultaneous version, both the voting and participation decisions are made simultaneously. While the equilibria in general differ in the two models (see the discussion section below), the main conclusions of the present paper are robust with respect to the timing of decision.

In our main result, Theorem 8.1.1 in Section 8, we will require the voting rule to be anonymous and strategy-proof. The *anonymity* condition has two components: first, for each given set of participants K, the outcome under F^K is invariant with respect to permutations of the agents in K; secondly, the employed social choice function F^K should depend only on the number of agents in K. Using the latter condition, we can write F^k for all social choice functions F^K with $\#K = k$, and describe the voting rule $F = \{F^k\}_{1 \leq k \leq n}$ in terms of n social choice functions, one for each number of participants.

Strategy-proofness requires that truth-telling be a (weakly) dominant strategy for all participating agents: for all K, $i \in K$, $\succcurlyeq_i$, $\succcurlyeq'_i$, $\succcurlyeq_{K-i}$,

$$F^k(\succcurlyeq_i, \succcurlyeq_{K-i}) \ \succcurlyeq_i \ F^k(\succcurlyeq'_i, \succcurlyeq_{K-i}),$$

where $k = \#K$ and $\succcurlyeq_{K-i}$ denotes any profile of preferences for the agents in K other than i.

By a well-known result of Moulin (1980), the conditions of anonymity and strategy-proofness jointly imply that all social choice functions F^k are 'generalized medians' with $k+1$ so-called 'phantom voters'. Specifically, for each $k \in \{2, ..., n\}$, F^k only depends on the individual peaks, i.e. for some function $f^k : [0,1]^k \to [0,1]$

$$F^k(\succcurlyeq_1, ..., \succcurlyeq_k) \ = \ f^k(p_1, ..., p_k),$$

and there exist fixed values $\alpha_1^k, \alpha_2^k, ..., \alpha_{k+1}^k \in [0,1]$ such that

$$f^k(p_1, ..., p_k) \ = \ med\{p_1, ..., p_k, \alpha_1^k, \alpha_2^k, ..., \alpha_{k+1}^k\}, \tag{7.1}$$

where *med* denotes the usual median operator and the p_i are the peaks of $\succcurlyeq_i$ for each participating agent i; note that there are $2k+1$, i.e. an odd number, of values in (7.1). An important example is the standard median rule with an odd number of participants; in this case, half of the phantom voters are placed at 0 and half are placed at 1.

We will say that F^k, respectively f^k, satisfies *responsiveness* if for all $p_1, ..., p_k$, $p'_1, ..., p'_k$,

$$p'_i > p_i \text{ for all } i \in K \quad \Rightarrow \quad f^k(p'_1, ..., p'_k) \neq f^k(p_1, ..., p_k).$$

Responsiveness can be viewed as a condition of 'local non-imposition': if *every* agent desires a strictly higher (lower) outcome, the chosen alternative should move at least minimally.[9]

While arguably a weak and plausible condition, responsiveness does restrict the set of admissible voting rules, as follows:

Observation. *The generalized median functions f^k in (7.1) satisfy responsiveness if and only if all 'phantom voters' are either at 0 or at 1, i.e. for all $j = 1, ..., k+1$, $\alpha_j^k \in \{0, 1\}$, and neither are all*

[9] Intuitively, it should clearly move in the *same* direction; in the present context, this slightly stronger requirement is redundant because it follows from responsiveness plus strategy-proofness.

phantom voters located at 0 *nor at* 1*. In particular, in this case the generalized median always coincides with one of the peaks of the agents and the corresponding voting rule is efficient.*

Proof. To verify this, suppose that, for some k and $j_0 \leq k+1$, one has $0 < \alpha^k_{j_0} < 1$. Clearly, for any given set of the other phantom voters α^k_j, $j \neq j_0$, one can choose peaks $p_i \in (0,1)$ all distinct from $\alpha^k_{j_0}$ such that $med\{p_1, ..., p_k, \alpha^k_1, ..., \alpha^k_{k+1}\} = \alpha^k_{j_0}$. But then the generalized median does not react to a sufficiently small uniform move of all peaks. The same is evidently true if all $k+1$ phantom voters are located either at 0 or at 1. Conversely, if all $k+1$ phantom voters are either at 0 or at 1, but not all of them at the same location, the generalized median must be one of the k peaks of the real agents; hence the underlying voting rule is efficient and must react to a uniform move of all peaks. $\square$

It is well-known (Moulin, 1980) that under efficiency, the generalized median functions f^k in (7.1) can be assumed to have $k-1$ instead of $k+1$ phantom voters. Generalized medians for which all $k-1$ phantom voters are at one of the two extremes are also known as the *order statistics*. Specifically, the choice of the i-th lowest value of the $\{p_1, ..., p_k\}$ is referred to as the i-th order statistic, and corresponds to the generalized median in which $k-i$ phantom voters are at 0 and $i-1$ phantom voters are at 1, see Caragiannis et al. (2016) for further discussion.

In the present work, we are not focusing on the coordination problem that arises if several agents have the same top alternative. We therefore assume in all that follows that agents' peaks are in generic position, i.e. that no two peaks coincide: $p_i \neq p_j$ for all pairs $i, j \in N$ of distinct agents. If all social choice functions F^k are strategy-proof, voting truthfully is the unique (weakly) dominant strategy for every participant in the simultaneous game, and in every second-stage voting subgame of the sequential game. We will therefore assume that all participants who actually cast a vote submit their true peak. This assumption could be further justified by an appeal to an equilibrium refinement concept such as perfectness (Selten, 1975) or strong equilibrium (Aumann, 1959).[10]

[10] In the implementation literature, there has been some discussion on the fact that the median rule (as well as generalized medians) may have other Nash equilibria in which agents do not follow their unique weakly dominant strategy; see, e.g., Yamamura and Kawasaki (2013). For instance, if $k \geq 3$ and all agents cast exactly the same (non-truthful) vote, nobody is pivotal, and hence such vote profile constitutes a Nash equilibrium. However, such equilibria are evidently neither robust against trembles nor against deviations by coalitions of agents.

8 The Effect of Strategy-Proofness on Participation

In this section, we present the main theorem (Theorem 8.1.1) that reveals a strong tension between strategy-proofness and participation, and discuss all its components.

8.1 Main Result

The following is our main result:

Theorem 8.1.1. Suppose that the voting rule is anonymous, strategy-proof and responsive, and that every individuals' voting costs are positive.

a) The simultaneous voting game has a unique perfect equilibrium in which exactly one agent participates.

b) The sequential voting game has a unique subgame perfect equilibrium in which all agents choose their unique (weakly) dominant strategy in the second stage; in this equilibrium, again exactly one agent participates.

In either model, the participating agent is either the individual with the highest peak or the individual with the lowest peak.

Proof. The assumptions on the voting rule imply that, for all non-empty sets $K \subseteq N$ of participating agents, the outcome is determined by a generalized median with $\#K - 1$ phantom voters. Moreover, by anonymity, the set of phantom voters only depends on $k = \#K$.

We first show that this implies the existence of an equilibrium with a single participant. As before, we denote by $p_1, ..., p_n$ the preference peaks of the agents, and assume without loss of generality that $p_1 < p_2 < ... < p_n$. For $\#K = 2$, there is one phantom voter α_1^2, and by the Observation in the previous section, we have either $\alpha_1^2 = 0$ or $\alpha_1^2 = 1$. Suppose the former, then the single participation of agent 1 (who reports truthfully) constitutes an equilibrium. Indeed, the outcome then is p_1, which by assumption is preferred by agent 1 to x_0 (the outcome if nobody participates). Every other agent has a higher peak and can thus not unilaterally change the outcome because $\alpha_1^2 = 0$; hence, if costs are positive, every other agent prefers not to participate. The argument is completely symmetric if $\alpha_1^2 = 1$, in which case single participation of individual n is an equilibrium. Note that the argument applies to the dynamic model in the same way as to the simultaneous game.[11]

It remains to show that there are no other equilibria. By contradiction, suppose we have an equilibrium with the set $K \subseteq N$ of participants where $\#K > 1$. If this situation constitutes an equilibrium, it is optimal for all $i \in K$ to participate; we will show that this is not possible. By re-numbering agents, we may assume that $K = \{1, ..., k\}$ and $p_1 < p_2 < ... < p_k$. By the Observation above, there exists $j \in K$ with $f^k(p_1, ..., p_k) = p_j$. First assume that $j = 1$, i.e. that the voter with the lowest peak gets her most preferred alternative. Then, voter k (the one with the highest peak among the participants) has an incentive to abstain; indeed, by the efficiency of f^{k-1}, the outcome without voter k cannot be smaller than p_1. By a similar argument, one can show that $j \neq k$.

[11] Of course the complete strategy in the dynamic game also specifies (for each non-participant) truth-telling in all counterfactual participation situations.

Thus, it must hold that $1 < j < k$ for the individual j who receives her peak p_j. In this case, the assumed optimality of participation by agent 1 implies that

$$f^{k-1}(p_2,...,p_k) \;=\; med\{p_2,...,p_k,\alpha_1^{k-1},...,\alpha_{k-2}^{k-1}\} > p_j, \tag{8.1.1}$$

since otherwise agent 1 would prefer not to participate, thereby saving the associated cost. Similarly, the assumed participation of agent k implies that

$$f^{k-1}(p_1,...,p_{k-1}) \;=\; med\{p_1,...,p_{k-1},\alpha_1^{k-1},...,\alpha_{k-2}^{k-1}\} < p_j. \tag{8.1.2}$$

Without agent 1, there are $j-1$ peaks that are below or equal to p_j. By (8.1.1), the generalized median $f^{k-1}(p_2,...,p_k)$ with $k-1$ participants (i.e. agents 1 to $k-1$) is strictly above p_j; this implies that at most $(k-1-j)$ of the $k-2$ phantom voters $\{\alpha_1^{k-1},...,\alpha_{k-2}^{k-1}\}$ can be located at 0. Similarly, without agent k there are $k-j$ peaks above or equal to p_j. By (8.1.2), the generalized median $f^{k-1}(p_1,...,p_{k-1})$ with $k-1$ participants (i.e. agents 2 to k) is strictly below p_j; this implies that at most $j-2$ of the $k-2$ phantom voters $\{\alpha_1^{k-1},...,\alpha_{k-2}^{k-1}\}$ can be located at 1. By the responsiveness, all of the $k-2$ phantom voters have to be located either at 0 or at 1. However, we have just shown that under conditions (8.1.1) and (8.1.2) this is not possible since

$$(k-1-j) \;+\; (j-2) = k-3 \;<\; k-2.$$

Thus, there can be no equilibrium in which more that one agent participates. This concludes the proof of Theorem 8.1.1.

$\square$

8.2 Discussion

In order to assess the scope and the robustness of Theorem 8.1.1, we now consider each of its assumptions. We explain why they are necessary for the conclusion and we discuss what happens if they were dropped.

Anonymity

Our anonymity condition has two components. First, it requires the voting rule not to depend on the 'names' of voters *for any given set of participants*; secondly, it requires that the same voting rule is employed for all subsets with the *same number* of participants. Arguably, both conditions are natural in the present context. The first part is a standard assumption in voting theory, and in fact Moulin's characterization of all strategy-proof rules for single-peaked preferences in terms of phantom voters needs this assumption. In our present variable electorate context, the second part also appears to be highly plausible. Importantly, it also guarantees the existence of an equilibrium in pure strategies (and its uniqueness). We show this by means of two simple examples, as follows.

Assume that all conditions of Theorem 8.1.1 are satisfied except the second part of the anonymity condition, and consider the following examples.

Example 8.2.1. Let $N = \{1,2,3\}$ and suppose that if the set of participants consists of agents 1 and 2, the outcome function $f^{\{1,2\}}$ chooses the higher peak, i.e. we have $\alpha_1^{\{1,2\}} = 1$ for the corresponding phantom voter; if the set of participants consists of agents 1 and 3, the outcome function $f^{\{1,3\}}$ chooses the lower peak, i.e. $\alpha_1^{\{1,3\}} = 0$ for the corresponding phantom voter; and finally, if the set of participants consists of agents 2 and 3, the outcome function $f^{\{2,3\}}$ chooses again the higher peak, i.e. $\alpha_1^{\{2,3\}} = 1$ for the corresponding phantom voter. Evidently, this specification violates the (second part of the) anonymity condition. Suppose that agents are ordered so that $p_1 < p_2 < p_3$. For no agent single participation is an equilibrium: if agent 1 is the single voter, agent 2 has an incentive to join; if agent 2 is the single voter, agent 3 has an incentive to join; and if agent 3 is the single voter, agent 1 has an incentive to join. A situation with two participants cannot be an equilibrium either because, by the responsiveness condition, one of the two gets her peak in which case the other has an incentive to abstain and save the voting costs. Finally, full participation cannot be an equilibrium either. Indeed, suppose that all agents participate; again by the responsiveness condition, one of the agents must receive her peak. If agent 1 receives her peak, agent 2 has an incentive to abstain because this would not change the outcome and agent 2 would save the participation cost; similarly, if agent 2 receives her peak, agent 3 has an incentive to abstain, and if agent 3 receives her peak, agent 1 has an incentive to abstain. Hence, in this example, there is no equilibrium in pure strategies.

The following is an example in which there are several equilibria, including one with *full participation*.

Example 8.2.2. Let $N = \{1, 2, 3\}$. If all agents participate, the social choice function $f^{\{1,2,3\}}$ chooses the standard median, in other words, the corresponding phantom voters are given by $\alpha_1^{\{1,2,3\}} = 0$ and $\alpha_2^{\{1,2,3\}} = 1$; if agents 1 and 2 participate, the social choice function $f^{\{1,2\}}$ chooses the lower peak, i.e. the corresponding phantom voter is given by $\alpha_1^{\{1,2\}} = 0$; if agents 2 and 3 participate, the social choice function $f^{\{2,3\}}$ chooses the higher peak, i.e. the corresponding phantom voter is given by $\alpha_1^{\{2,3\}} = 1$. No matter how we specify the outcome in the case that the set of participants consists of agents 1 and 3, this already implies that for sufficiently small participation costs, full participation is an equilibrium. Indeed, if all agents participate, the agent with the median peak gets her peak and hence has no incentive to abstain if her participation costs are sufficiently small; for either of the other two agents, unilateral non-participation would move the outcome further away from their respective peak, so neither of them has an incentive to abstain, either. There also exists an additional single participation equilibrium. Indeed, for the set of participants $\{1, 3\}$ we either have $\alpha_1^{\{1,3\}} = 0$ or $\alpha_1^{\{1,3\}} = 1$. In the first case, single participation of the agent with the lowest peak is an equilibrium (since none of the other two agents can unilaterally change the outcome); in the second case, single participation of the agent with the highest peak is an equilibrium.

Responsiveness

Above, we have justified the responsiveness condition by an appeal to a 'local non-imposition' property: if all agents uniformly move in one direction, the outcome should not remain unchanged. We have also shown that this condition is equivalent to the property that all phantom voters should be at the two extreme points 0 or 1. There may exist an even deeper justification for the responsiveness condition in purely ordinal contexts. Indeed, if the set of alternatives is linearly ordered but in a purely ordinal way, any specific location of a phantom voter in the interior of the interval $[0, 1]$ seems arbitrary. On the other hand, if cardinal information is available, such as in the example of dividing a fixed budget, phantoms may be placed at the midpoint (at 0.5) or distributed uniformly in $[0, 1]$ (the latter specification is also known as the 'uniform' or 'linear' median, see Jennings et al. (2020)). In any case, Theorem 8.1.1 fails without the responsiveness condition. As a simple example, consider the case of an even number of agents $N = \{1, 2, ..., 2m\}$ and suppose that the phantom voters are located as follows: $m - 1$ phantom voters are at 0, $m - 1$ phantom voters are at 1, and one phantom voter is located in the interior, say at $x \in (0, 1)$. Also, suppose that for any set of $2m - 1$ participants the standard median is chosen as the outcome. Consider any distribution such that m agents have their peak below x and m agents have their peak above x. Then, if every individuals' costs are sufficiently small, full participation is an equilibrium. Indeed, the outcome under full participation is x, and for any agent unilateral abstention moves the outcome further away from her peak.

But even without the responsiveness condition, there often also exist profiles of peaks such that only one single agent participates in equilibrium. Specifically, let $p_1 < p_2 < ... < p_N$ be such that

(i) $\{\alpha_j^k\}_{j=1,...,k+1}^{k=1,...,N} \cap [p_1, p_N] = \emptyset$, i.e. all phantom voters are either below the smallest peak or above the highest peak,

and

(ii) $\min_j\{\alpha_j^k\} < p_1$ and $p_N < \max_j\{\alpha_j^k\}$ for all $k = 1, ..., N$.

(Note that condition (ii) is implied by efficiency of the voting rule.) Then, we obtain single participation as the unique equilibrium by the same logic as in the proof of Theorem 8.1.1.

Strategy-proofness

The assumption of strategy-proofness is essential for the conclusion of Theorem 8.1.1. Strategy-proofness together with a suitable refinement concept, such as robustness against small trembles ('perfectness'), implies that all participants will cast their vote truthfully. Evidently, the proof of Theorem 8.1.1 hinges on that property. In particular, the fact that the equilibrium does not depend on individual costs (as long as they are all positive) crucially depends on the strategy-proofness (together with the responsiveness condition), as well.

To illustrate this, suppose we can use cardinal information and employ the following *symmetric* version of the median rule: Let the peaks of the participants be ordered such that $p_1 < p_2 < ... < p_k$; for an odd number $k = 2m - 1$ the outcome is the standard median p_m, and for an even number $k = 2m$ the outcome is the midpoint between the two middle peaks $(p_m + p_{m+1})/2$. This rule is not strategy-proof, and therefore we cannot assume that the reported peaks coincide with the true peaks.[12] Consider the peak distribution $p_1 = 0.1$, $p_2 = 0.45$ and $p_3 = 0.9$. Let $\tilde{p}_i$ be the reported peak by agent i, and assume that the participation costs of the two extreme agents 1 and 3 are small but positive. Then, the equilibrium depends, among other things, on the magnitude of the participation cost (i.e. the precise shape of the preferences) of the median agent. If the participation cost of agent 2 is sufficiently small, full participation and truth-telling is an equilibrium in the simultaneous game; on the other hand, if agent 2 prefers the outcome 0.5 without own participation to the outcome 0.45 while participating, $\tilde{p}_1 = 0$ and $\tilde{p}_3 = 1$ is a (non-truthful) equilibrium.

Moreover, it is because of truthful voting in equilibrium that the conclusion of Theorem 8.1.1 is robust with respect to the timing of decisions. Indeed, in the sequential model participants' voting strategy may depend on the set of other participants which becomes common knowledge after all agents have made their participation decision. To illustrate this point, consider again the peak distribution $p_1 = 0.1$, $p_2 = 0.45$ and $p_3 = 0.9$ and the symmetric median rule, but now assume that agents move sequentially. In this case, full participation is no longer an equilibrium; indeed, if agent 3 does not participate, agents 1's and 2's optimal votes are $\tilde{p}_1 = 0$ and $\tilde{p}_2 = 0.9$, respectively, with the outcome 0.45. Since this is the same outcome as under full participation, agent 3 prefers to abstain whenever she has positive participation costs. However, participation of agents 1 and 2 with outcome 0.45 cannot be an equilibrium either, since then agent 1 has an incentive to abstain. In this example, if all agents have sufficiently small (but strictly

[12] It is well-known that there exist no strategy-proof, anonymous and symmetric ('neutral') social choice functions on the domain of single-peaked preferences for an even number of individuals, see Moulin (1980, 1988).

positive) participation cost, the unique subgame perfect equilibrium of the sequential voting game is given by participation of agents 2 and 3 with votes $\tilde{p}_2 = 0$ and $\tilde{p}_3 = 1$, resulting in the outcome 0.5. A complete characterization of the equilibria under the symmetric median rule in the sequential model is provided in the next section under the assumption that in the second stage a strong Nash equilibrium is played.

9 Probabilistic Versus Deterministic Voting Rules

One may interpret Theorem 8.1.1 as saying that if voting is costly no anonymous and deterministic voting rule can implement the median if all participants vote according to their unique dominant strategy. Specifically, we have the following corollary of Theorem 8.1.1:

Corollary. Suppose that the number of agents is odd. There does not exist an anonymous and strategy-proof voting rule that yields the median peak for all distributions of individual peaks if voting is costly and all actual participants vote according to their unique dominant strategy.

To verify this simply note that by the remarks in Section 8.2, even without the responsiveness condition there always exist (generic) peak distributions for which the unique dominant strategy equilibrium involves the single participation either of the agent with the highest peak or the agent with the lowest peak.[13] An important implicit assumption of our analysis is that we require voting rules to be *deterministic*. One may argue that this is a strong assumption; and in fact, together with anonymity and strategy-proofness it forces an asymmetric tie-breaking rule in the case of an even number of participants.

In this section we want to briefly consider probabilistic rules and see in how far our results translate to this setting.

9.1 Basic Definitions

As before, we consider individuals with single-peaked preferences on $[0, 1]$. We denote the domain of these preferences by $\mathcal{S}$. The model we use is taken from Ehlers et al. (2002).

Let $\mathcal{L}$ denote the Borel σ-algebra on $[0, 1]$. Elements of $\mathcal{L}$ are called Borel sets. A probability distribution $\phi : \mathcal{L} \to [0, 1]$ is a function that satisfies the Kolmogorov axioms, i.e. that

(i) assigns to every Borel set a non-negative value, i.e. $\phi(X) \geq 0$ for all $X \in \mathcal{L}$,

(ii) assigns the value 1 to the interval $[0, 1]$, i.e. $\phi([0, 1]) = 1$ and

(iii) is σ-additive, i.e. $\phi(\bigcup_{j=1}^{\infty} X_j) = \sum_{j=1}^{\infty} \phi(X_j)$ for all countable sequences of pairwise disjoint Borel sets $X_1, X_2, \ldots$.

A probabilistic social choice function $\Phi : \mathcal{S}^n \times \mathcal{L} \to [0, 1]$ is a function that selects for all $\succcurlyeq \in \mathcal{S}^n$ and all Borel sets $X \in \mathcal{L}$ a probability in $[0, 1]$, denoted by $\Phi(\succcurlyeq, X)$ or $\Phi(\succcurlyeq)(X)$. For a given preference profile $\succcurlyeq \in \mathcal{S}^n$, $\Phi(\succcurlyeq)$ is a probability distribution over $[0, 1]$. A deterministic rule can be seen as a probabilistic rule that selects for each $\succcurlyeq \in \mathcal{S}$ a distribution placing probability 1 on a single value in $[0, 1]$.

We extend preferences over alternatives in $[0, 1]$ to preferences over distributions ordinally (Gibbard, 1977; Bogomolnaia and Moulin, 2001). This ordinal extension is based on the concept of upper contour sets.

[13] In independent work, Cohensious et al. (2017) have shown that participation of a single agent is the unique equilibrium under the 'lower median' rule, i.e. under the rule that chooses the median if the number of participants is odd and the lower of the two middle votes if the number of participants is even.

Given $x \in [0,1]$ and $\succcurlyeq_i \in \mathcal{S}$, the weak upper contour set of x at $\succcurlyeq_i$ is defined by $B(x, \succcurlyeq_i) = \{y \in [0,1] : y \succcurlyeq_i x\}$. The single-peakedness of $\succcurlyeq_i$ implies that these upper contour sets are intervals which are either closed, open or half open. In particular, they are Borel sets.

In the probabilistic model, we extend preferences in the following way: Given a preference relation $\succcurlyeq_i \in \mathcal{S}$ and two probability distributions Q, Q' over $[0,1]$, Q is weakly preferred to Q' under $\succcurlyeq_i$ if Q assigns to each weak upper contour set of $\succcurlyeq_i$ at least the probability that is assigned by Q' to this set. Abusing notation, we use the same symbols to denote preferences over probability distributions and preferences over values.

Formally, for all $\succcurlyeq_i \in \mathcal{S}$ and all distributions Q, Q' over $[0,1]$.

- $Q \succcurlyeq_i Q'$ if and only if for all $x \in [0,1]$, $\quad Q(B(x, \succcurlyeq_i)) \geq Q'(B(x, \succcurlyeq_i))$.
- $Q \succ_i Q$ if and only if $Q \succcurlyeq_i Q'$ and for some $y \in [0,1]$, $\quad Q(B(y, \succcurlyeq_i)) > Q'(B(y, \succcurlyeq_i))$.

The first inequality is a first order stochastic dominance condition; in particular, it requires that the distributions Q and Q' are comparable in that respect. Therefore, the extension is not complete over the set of all distributions over $[0,1]$. However, note that for preferences over distributions completeness is a demanding requirement.

Ehlers et al. (2002) show that one can equivalently extend preferences with respect to strict upper contour sets.

We now consider for each possible non-empty set $K \subseteq N$ of participants a probabilistic social choice function Φ^K that specifies a probability distribution over $[0,1]$, denoted by $\Phi^K(\succcurlyeq_1, \ldots, \succcurlyeq_{\#K})$ for every profile of preferences of the agents in K. The collection $\Phi = \{\Phi^K\}_{\emptyset \neq K \subseteq N}$ of these probabilistic social choice functions is referred to as a *probabilistic voting mechanism*.

As in our deterministic model, we consider anonymous voting mechanisms. Again we require that for each given set of participants K, the outcome under Φ^K is invariant with respect to permutations of the agents in K, and that the employed (probabilistic) social choice function Φ^K depends only on the number of agents in K. We sometimes write Φ^k for all (probabilistic) social choice functions Φ^K with $\#K = k$.

The definition of strategy-proofness also translates in a natural way to probabilistic voting mechanisms. A probabilistic voting mechanism Φ^K is strategy-proof if for all K, $i \in K$, $\succcurlyeq_i, \succcurlyeq_i', \succcurlyeq_{K-i}$

$$\Phi^K(\succcurlyeq_i, \succcurlyeq_{K-i}) \;\succcurlyeq_i\; \Phi^K(\succcurlyeq_i', \succcurlyeq_{K-i}),$$

where $\succcurlyeq_{K-i}$ denotes any profile of preferences for the agents in K other than i.

We say that Φ^K satisfies *responsiveness* if for all $\succcurlyeq_K = \succcurlyeq_1, \ldots, \succcurlyeq_{\#K}$ with tops $p_1, \ldots, p_{\#K}$ and all $\succcurlyeq_K' = \succcurlyeq_1', \ldots, \succcurlyeq_{\#K}'$ with tops $p_1', \ldots, p_{\#K}'$

$$p_i' > p_i \text{ for all } i \in K \quad \Rightarrow \quad \Phi^K(\succcurlyeq_K) \neq \Phi^K(\succcurlyeq_K').$$

We say that a probabilistic voting mechanism satisfies *participation* if for all $i \in N$ and all $\succcurlyeq_i$ there exists $\succcurlyeq_i'$ such that for all $\succcurlyeq_{K-i}$ and all $K \subseteq N$ with $i \in K$

$$\Phi^K(\succcurlyeq_i', \succcurlyeq_{K-i}) \;\widehat{\succ}_i\; \Phi^{K-i}(\succcurlyeq_{K-i}),$$

where $\widehat{\succ}_i$ denotes agent i's preference over pairs of outcomes and participants who actually cast a vote, as in the deterministic model. Hence, a probabilistic voting mechanism satisfies participation if every individual always prefers to vote over abstaining. Moreover, it is taken into account that the individual might want to misrepresent her true preference ranking to obtain a better result.

As Brandl et al. (2015) point out, this condition is hard to satisfy. For example, in cases in which all submitted preferences in the profile $\succcurlyeq_{K-i}$ are identical, every non-veto rule will choose the top of that alternative, independently of the submitted preference of individual i. Moreover, if individual i abstains, the outcome remains unchanged which results in a violation of participation. In order to rule out such cases, we consider *strong equilibrium participation* which adds to the condition of participation that, given the participation decision, the submitted preference orderings constitute a strong Nash equilibrium.

By Theorem 8.1.1 we know that no deterministic, strategy-proof, anonymous and responsive voting mechanism satisfies strong equilibrium participation. Hence, one needs to consider probabilistic voting mechanisms. One of the most famous probabilistic voting rules is random dictatorship. Random dictatorship attributes to every submitted top alternative the probability $1/\#K$. Thus, random dictatorship randomly chooses one of the participants to be the "dictator" and chooses her top alternative. Hence, this rule is strategy-proof as one either determines the outcome, in which case truth-telling is optimal, or one does not influence the outcome. Moreover, it is easy to see that random dictatorship is anonymous. Furthermore, this rule is responsive, as a shift of all peaks in the same direction shifts the lottery over the top alternatives in the same direction and thus the outcome changes.

Hence indeed, there are probabilistic voting mechanisms that satisfy all the conditions of Theorem 8.1.1 apart from being deterministic. Nevertheless, we do not want to settle for random dictatorship as 'the' rule to use. While this rule satisfies a lot of desirable properties, it excludes ex-ante the possibility of achieving a compromise (Börgers, 1991).

The following proposition describes the minimal amount of probability that one needs to add in order to obtain voting mechanisms that are anonymous, strategy-proof, responsive and that satisfy strong equilibrium participation.

Proposition 9.1.1. *Let $\Phi = \{\Phi^k\}_{1\leq k\leq n}$ be an anonymous, strategy-proof and responsive probabilistic voting mechanism that satisfies strong equilibrium participation. Then there exists no $k \in \{2, \dots, n\}$ such that Φ^k and Φ^{k-1} are deterministic social choice functions.*

Proof. If Φ^k and Φ^{k-1} are deterministic rules, then by Moulin (1980) we know that both are generalized median functions. As the conditions of Theorem 8.1.1 are satisfied, one can apply the same logic as in the proof to argue that there is no equilibrium with K being the participants. Hence, there exist individuals who have no incentive to participate, thus violating participation. By Barberà et al. (2010) we know that strategy-proofness on the single-peaked domain is equivalent to group strategy-proofness. Thus, truth-telling constitutes a strong equilibrium. Yet, individuals have an incentive to abstain; hence, strong equilibrium participation is violated. □

Proposition 9.1.1 states that in order to satisfy all required conditions, one needs to have a probabilistic social choice function in at least half of the instances. The following example shows that there are probabilistic voting mechanisms that satisfy all conditions of Proposition 9.1.1 and that use a deterministic social choice function for all odd number of participants.

Example 9.1.2. Let Φ be the probabilistic voting mechanism that is defined to choose the median of the top alternatives of the preference orderings of the participants if k is odd, and that chooses a 50:50 lottery over the lower and the upper median of the top alternatives of the preference orderings of the participants if k is even. This rule is strategy-proof, responsive, anonymous and satisfies strong equilibrium participation.

9.2 The Result of Peters et al. (2014)

Peters et al. (2014) examine probabilistic and strategy-proof rules on single-peaked domains and show that - given tops-onlyness or unanimity - every probabilistic and strategy-proof rule is a probability mixture of strategy-proof deterministic rules. A probabilistic rule is unanimous if $\Phi(\succcurlyeq, a) = 1$ for all $\succcurlyeq \in \mathcal{S}^n$ with $\tau(\succcurlyeq_i) = a$ for all $i \in N$.

Theorem 9.2.1 (Peters et al. (2014))**.** Let A' be a finite subset of $[0,1]$ and let $\Delta(A')$ denote the set of probability distributions on A'.

- Every strategy-proof, tops-only probabilistic rule $\varphi : \mathcal{S}^n \to \Delta(A')$ is a probability mixture of strategy-proof, tops-only deterministic rules $f : \mathcal{S}^n \to A'$.
- Every strategy-proof, unanimous probabilistic rule $\varphi : \mathcal{S}^n \to \Delta(A')$ is a probability mixture of strategy-proof, unanimous deterministic rules $f : \mathcal{S}^n \to A'$.

While this result shows that probabilistic rules that satisfy strategy-proofness and tops-onlyness and/or unanimity can be composed into deterministic rules that satisfy the same properties, the same does not hold true for other conditions as for example for anonymity. To see this consider the following example:

Example 9.2.1. Let $A' = \{a_1, \dots, a_m\}$ be a finite subset of $[0,1]$ and consider the set of voters $K \subseteq N$ with $\#K = k$ even. Denote by $\underline{med}(\succcurlyeq_K)$ the lower median of $\succcurlyeq_K$ and by $\overline{med}(\succcurlyeq_K)$ the upper median of $\succcurlyeq_K$. Furthermore, consider the rule Φ^k that is defined by choosing a uniformly distributed lottery over the set of alternatives in $A' \cap \left[\underline{med}(\succcurlyeq_K)\ ,\ \overline{med}(\succcurlyeq_K)\right]$. This rule is strategy-proof and anonymous (and both tops-only and unanimous) but there exists no collection of strategy-proof and anonymous deterministic rules such that Φ^k is a probability mixture over these rules.
By contradiction, let us assume that there exists a collection of strategy-proof and anonymous rules such that Φ^k is a probability mixture over these rules. By Moulin (1980) we know that all strategy-proof and anonymous deterministic rules are generalized median rules. Thus, the outcome of the deterministic rules corresponds to one of the peaks or to a phantom voter. Consider $\succcurlyeq'_K$ such that there exists $a \in A'$ with $\underline{med}(\succcurlyeq'_K) < a < \overline{med}(\succcurlyeq'_K) \neq a_m$. Then $\Phi^k(\succcurlyeq'_K, a) > 0$. Hence, since $a \neq \tau(\succcurlyeq_i)$ for all $i \in K$, there exists a deterministic rule that chooses the phantom voter a given $\succcurlyeq'_K$, i.e. the median of the peaks of $\succcurlyeq'_K$ and the specific phantom voters of this rule is a. Consider $\succcurlyeq''_K$ with $\succcurlyeq'_i = \succcurlyeq''_i$ for all $i \in K$ with $\tau(\succcurlyeq'_i) < a$ and $\tau(\succcurlyeq''_i) = a_m$ for all $i \in K$ with $\tau(\succcurlyeq'_i) > a$. Then, all deterministic rules that choose a under $\succcurlyeq'_K$ also choose a under $\succcurlyeq''_K$. Thus $\Phi^k(\succcurlyeq'_K, a) \leq \Phi^k(\succcurlyeq''_K, a)$. But as the interval $A' \cap \left[\underline{med}(\succcurlyeq'_K)\ ,\ \overline{med}(\succcurlyeq'_K)\right]$ is strictly smaller than $A' \cap \left[\underline{med}(\succcurlyeq''_K)\ ,\ \overline{med}(\succcurlyeq''_K)\right]$, all elements within that interval are assigned a lower probability by Φ^k. Hence, we reach a contradiction.

This example illustrates that one cannot describe the class of all anonymous and strategy-proof probabilistic rules as a probability mixture over anonymous and strategy-proof deterministic rules, and thus an easy extension of our results is unfortunately not possible.

10 Symmetric Median Rule

The fair lottery over the two middle peaks in the case of an even number of votes yields the symmetric median in expectation. One may wonder whether the deterministic symmetric median[14] can 'mimic' the randomized median rule and induce full participation (perhaps under additional assumptions) if costs are sufficiently small. The following result provides a complete characterization of all subgame-perfect equilibria of the sequential game if costs are sufficiently small and answers this question in the negative. Note that in the case of an even number of participants, truth-telling no longer constitutes a dominant strategy (cf. Section 8.2 above). To avoid the discussion of an artificial multiplicity of equilibria, we assume that in the second stage a strong Nash equilibrium is played (i.e. an equilibrium such that no subgroup of agents can profitably deviate); for every fixed number k of participants, a strong equilibrium exists (see 13.1.1 in Part III). Say that a participant with peak $p_i \neq 0.5$ exhibits *extreme reporting* if she reports 0 if $p_i < 0.5$ and 1 if $p_i > 1$.

Theorem 10.1. *Consider the symmetric median rule and assume that individuals have sufficiently small but positive costs of participation. Then, all subgame-perfect equilibria, such that in the second stage a strong Nash equilibrium is played, are of the following four types:*

> ***Case 1.*** *There exists an individual $j \in N$ with peak $p_j = 0.5$. Then, single participation of individual j (who reports truthfully) is an equilibrium.*
>
> ***Case 2.*** *The peaks of all individuals are on the same side of 0.5, i.e. either $p_i \geq 0.5$ or $p_i \leq 0.5$ for all $i \in N$. Then, single participation of the individual whose peak is closest to 0.5 (who reports truthfully) is an equilibrium.*
>
> ***Case 3.*** *The number of potential voters n is even with half of them having a peak strictly below 0.5 and half of them a peak strictly above 0.5. Then, full participation with extreme reporting is an equilibrium and the outcome is 0.5.*
>
> ***Case 4.*** *The number of potential voters n is odd; there exists an individual j with peak $p_j = 0.5$ and the peaks of the other $n-1$ individuals are evenly split to both sides of 0.5. Then, it is an equilibrium that all individuals in $N \setminus \{j\}$ participate with extreme reporting; the resulting outcome is 0.5.*

In all other cases, there does not exist a pure strategy equilibrium such that in the second stage a strong Nash equilibrium is played.

The proof and a much more detailed analysis of all equilibria of the symmetric median rule can be found in Part III.

Note that there are cases in which the equilibrium outcome of the symmetric median rule is very far from the median peak. For instance, in Case 2, the median peak could be close to 0 while the equilibrium outcome is the highest peak which could be even at 0.5. The intuition behind this equilibrium is as follows: Suppose that only the voter with the highest peak participates and votes truthfully; clearly,

[14] Recall that the 'symmetric median' selects the standard median in the case of an odd number of votes, and the average of the two middle votes in the case of an even number of votes.

this gives the best outcome for that voter (given her participation). If any one of the remaining voters decides to participate, the voter with the highest peak can adjust her voting behavior in the sequential model and receive her peak again by optimally responding to the vote of the other participant. Thus, the outcome does not change, hence none of the remaining voters has an incentive to participate. In Case 3, the outcome is always 0.5 while the symmetric median (i.e. the midpoint between the two middle peaks) could be arbitrarily close to 0.25 (resp. 0.75).

Finally, let us compare the findings of Theorem 10.1 with the intuition put forward by Osborne et al. (2000) that more extreme voters are more likely to participate. Cases 1, 3 and 4 do not confirm this intuition: In Case 1, the single participant is not determined be her being 'moderate' or 'extreme' but simply by the fact that her peak is at 0.5; in Cases 3 and 4 we have (almost) full participation (albeit with extreme reporting). On the other hand, Case 2 comes closer to the intuition since the single participant is either the voter with the left-most or the right-most peak; however, within the spectrum of *conceivable* positions, this is also the voter with the most 'moderate' view among all potential voters since, by assumption, all of them are on the same side of 0.5.

11 Conclusion

Our main result reveals a strong tension between two kinds of incentive properties if preferences are single-peaked and voting is costly: participation and truthful reporting. If a voting rule is anonymous, strategy-proof and responsive, the only equilibrium in which all participants follow their unique dominant strategy consists of the single participation of either the agent with the highest or the agent with the lowest peak. In particular, there is no anonymous and strategy-proof deterministic voting rule that yields the median peak if participants vote according to their unique dominant strategy. While in this result the need for a tie-breaking rule in the case of an even number of actual participants plays an important role, it is remarkable that the result holds for any number of potential voters.

A possible way to avoid this problem is to consider probabilistic rules and indeed, we have shown by means of example that a simple and natural strategy-proof probabilistic rule implements the median with full participation if voting costs are sufficiently small. A full fledged analysis of probabilistic rules in the context of costly voting in committees appears to be a worthwhile subject for future work.

Part III

An Experiment on Voting with Costly Participation

12 Motivation

In this chapter, we revisit the collective decision problem with single-peaked preferences and costly voting of Part II. While our analysis in the previous chapter covers both, the simultaneous and the sequential voting game, we will turn our attention in this chapter to the sequential voting game. An analysis of the simultaneous voting game can be found in Rollmann (2020). Theorem 8.1.1 states that all strategy-proof rules in this setting induce single-participation in equilibrium, assuming the natural condition of sovereignty and responsiveness. This is bad news, as strategy-proofness is a desirable property due to an abundance of reasons and participation is also seen as a desirable property in democracy, as this gives the decision process and its winner more legitimacy. As we wish to examine settings in which there exist equilibria with more than one participant, we turn our attention to two rules that are not strategy-proof but often used to determine the value of a one-dimensional variable e.g. in allocation problems: the mean rule and the symmetric median rule. We provide a full characterization of all equilibria for both rules for the sequential voting game for small costs of participation and find that there exist equilibria with more than one participant. As before we will consider strong Nash equilibria on the second stage of the sequential game to avoid artificial multiplicity of equilibria. Nevertheless, the equilibria for the entire game for these rules are not necessarily unique. In order to avoid problems with coordination on equilibria and to simplify (i) the strategic environment for the individuals and (ii) the analysis of their behavior we mainly consider peak distributions with unique equilibria.

In order to test whether individuals recognize and play these unique equilibria, we present the design of a laboratory experiment. Furthermore, the experiment is designed to examine in how far the rules themselves influence the participation and strategic behavior of the individuals. To do that, we consider peak distributions such that the equilibria are unique and identical, i.e. the outcome, as well as the votes the participants submit, are the same in equilibrium for both rules. In such cases, a difference in behavior cannot depend on the strategic environment, but stems from the rule itself.

Moreover, in section 10 we discovered that our results do not confirm the intuition of Osborne et al. (2000) that extreme voters are more likely to participate. In the experiment we are keen to see whether we see extreme voters participate.

Our experiment is one of a number of experiments that were conducted to examine aspects of the pivotal voter model. There are quite some experiments, for example on bandwagon and underdog effects (see Agranov et al. (2018); Levin and Palfrey (2007) amongst others). However, similar to the theoretical contributions, most of the literature focuses on the case of two alternatives. The work closest to ours is Rollmann (2020), who conducted a field experiment on a simultaneous voting game with costly participation but without complete information.

Unfortunately, due to the ongoing COVID-19 pandemic, a realization of the lab experiment has not yet been possible. Hence, this chapter will present the equilibrium analysis, the design of the experiment and the design and the results of the pilot that was conducted in 2019.

Structure of this Chapter

The chapter is organized as follows: Section 13 gives a full characterization of all subgame-perfect equilibria such that a strong Nash equilibrium is played in the second stage of the sequential voting game for the symmetric median rule and the mean rule. In Section 13.3 we give conditions such that these conditions are unique given fixed costs of participation. Section 14 presents the design, the hypotheses and the result of the pilot experiment and Section 15 presents the design adaptions for the follow-up experiment.

13 Equilibrium Characterization Results for the Symmetric Median and the Mean Rule in the Sequential Voting Game

In this chapter we revisit the collective decision problem of determining a one-dimensional variable as analyzed in Part II. Hence, we apply the same model as in Section 7. We focus on the sequential voting game and the symmetric median and the mean rule. We characterize all subgame perfect equilibria such that in the second stage of the game a strong Nash equilibrium is played for small but strictly positive cost of participation.

To simplify notation, we assume throughout this section that the individuals are ordered according to their peaks, which we assume to be generic. Moreover, we denote by $K \subseteq N$ the set of participants and by k the number of participants. To further simplify notation, we also sort the participants according to their peak, such that $i \in K$ is the participant with the i-th lowest peak. Note that this participant is most likely not the individual whose peak is the i-th lowest peak among all individuals (including non-participants).

As in Section 10 in Part II, we assume that in the second stage a strong Nash equilibrium is played (i.e. an equilibrium such that no subgroup of agents can profitably deviate) to avoid the discussion of an artificial multiplicity of equilibria. This also guarantees robustness against trembles and thus describes the more interesting equilibria. In the following, we will only consider strong Nash equilibria in the second stage of the game, but might omit the 'strong'. When talking about equilibria for the entire (sequential) game, we will only focus on subgame-perfect Nash equilibria such that a strong Nash equilibrium is played in the second stage. We will also call this an 'equilibrium of the (entire) game'.

13.1 Symmetric Median Rule

We start our analysis with the symmetric median rule in the sequential voting game. The following theorem (which is identical to Theorem 10.1 in Part II) provides a full characterization of all subgame-perfect equilibria such that in the second stage of the game a strong Nash equilibrium is played.

Theorem 13.1.1. Consider the symmetric median rule and assume that individuals have sufficiently small but positive costs of participation. Then, all subgame-perfect equilibria such that in the second stage a strong Nash equilibrium is played are of the following four types:

Case 1. There exists an individual $j \in N$ with peak $p_j = 0.5$. Then, single participation of individual j (who reports truthfully) is an equilibrium.

Case 2. The peaks of all individuals are on the same side of 0.5, i.e. either $p_i \geq 0.5$ or $p_i \leq 0.5$ for all $i \in N$. Then, single participation of the individual whose peak is closest to 0.5 (who reports truthfully) is an equilibrium.

Case 3. The number of potential voters n is even with half of them having a peak strictly below 0.5 and half of them a peak strictly above 0.5. Then, full participation with extreme reporting is an equilibrium and the outcome is 0.5.

Case 4. The number of potential voters n is odd; there exists an individual j with peak $p_j = 0.5$ and the peaks of the other $n - 1$ individuals are evenly split to both sides of 0.5. Then, it is an equilibrium that all individuals in $N \setminus \{j\}$ participate with extreme reporting; the resulting outcome is 0.5.

In all other cases, there does not exist a pure strategy equilibrium such that in the second stage a strong Nash equilibrium is played.

We split the proof of the theorem in two parts. Proposition 13.1.3 deals with the first two cases, while Proposition 13.1.5 proves the claim for the remaining two cases.

We start our analysis with the following lemma, that shows (i) that there always exists a strong Nash equilibrium in the second stage of the game, and (ii) that, if there are several strong Nash equilibria, all of them are outcome-equivalent.

Lemma 13.1.1. *Consider the symmetric median rule with a fixed number of participants k with ordered peaks $p_1, ..., p_k$. There always exists a strong Nash equilibrium. If k is even the outcome is the median of $\{p_{\frac{k}{2}}, 0.5, p_{\frac{k}{2}+1}\}$ in all strong Nash equilibria, if k is odd the outcome is the median of the peaks of the participants.*

Proof. If k is odd, then the symmetric median rule coincides with the median rule that is well-known to be strategy-proof. Thus, truth-telling is a weakly dominant strategy which implies that there exists a strong Nash equilibrium in which all participants tell the truth. There are further strong Nash equilibria in which participants, that are not the median participant, report a different peak than their true peak but on the same side of the peak of the median participant. In all those strong equilibria the outcome is the median of the peaks of the participants.

Let k be even.
Case 1: $p_{\frac{k}{2}} < 0.5 < p_{\frac{k}{2}+1}$:

In this case, it is easily verified that there is a unique strong Nash equilibrium: the $k/2$ agents with peak lower than 0.5 vote for 0, while the $k/2$ agents with peak larger than 0.5 vote for 1, resulting in the outcome 0.5.

Case 2: $p_{\frac{k}{2}} < p_{\frac{k}{2}+1} \leq 0.5$:

In this case, the strong Nash equilibrium is not unique (unless $p_{\frac{k}{2}+1} = 0.5$) but all strong equilibria in fact result in the same outcome. In all strong equilibria the $k/2$ agents with the lowest peaks vote for 0, agent $\frac{k}{2}+1$ votes for $2 \cdot p_{\frac{k}{2}+1}$ (≤ 1), and all other agents submit a vote between $2 \cdot p_{\frac{k}{2}+1}$ and 1, resulting in the outcome $p_{\frac{k}{2}+1}$.

Case 3: $50 \leq p_{\frac{k}{2}} < p_{\frac{k}{2}+1}$:

Analogously to Case 2, in all strong equilibria the $k/2$ individuals with the highest peaks vote for 1, individual $k/2$ votes for $2 \cdot p_{\frac{k}{2}} - 1$ (≥ 0), and all other individuals submit a vote between 0 and $2 \cdot p_{\frac{k}{2}} - 1$ resulting in the outcome $p_{\frac{k}{2}}$. $\square$

As the symmetric median rule differentiates between odd and even number of participants, so do we in our equilibrium analysis, starting with odd number of participants.

Lemma 13.1.2. *Consider the symmetric median rule. There are no subgame-perfect equilibria in which a strong Nash equilibrium is played in the second stage with an odd number of participants greater than 1.*

Proof. By contradiction, let $k > 1$ be the number of participants and let k be odd. Then, in every (strong) Nash equilibrium, the median participant $i = \frac{k+1}{2}$ determines the outcome by truthfully revealing her peak $p_{\frac{k+1}{2}}$.

Case 1: $p_{\frac{k+1}{2}} = 0.5$:

Then, there are $\frac{k-1}{2}$ participants with a peak below 0.5 and $\frac{k-1}{2}$ participants with a peak above 0.5. Hence, if agent $\frac{k+1}{2}$ abstains, the outcome will be the median of $p_{\frac{k-1}{2}}$, 0.5 and $p_{\frac{k+3}{2}}$ by Lemma 13.1.1, that is the outcome will be 0.5. Thus, since the outcome would not change, agent $\frac{k+1}{2}$ has an incentive to abstain.

Case 2: $p_{\frac{k+1}{2}} < 0.5$:

If an agent $i > \frac{k+1}{2}$ (i.e. an agent with peak above the outcome) abstains, the outcome becomes the median of $p_{\frac{k-1}{2}}$, 0.5 and $p_{\frac{k+1}{2}}$ by Lemma 13.1.1. However, since $p_{\frac{k-1}{2}} < p_{\frac{k+1}{2}} < 0.5$, this means that the outcome does not change; hence, agent i will rather abstain.

Case 3: $p_{\frac{k+1}{2}} > 0.5$:

This case is symmetric to Case 2. □

With the help of this lemma, we can characterize all equilibria with an odd number of participants.

Proposition 13.1.3. *Consider the symmetric median rule. There exists a single participation equilibrium (with a (strong) Nash equilibrium played on the second stage) for all $c > 0$ if and only if one of the following two conditions holds true:*

1. *There exists one individual $j \in N$ with a peak of $p_j = 0.5$.*
2. *The peaks of all individuals are on the same side of 0.5, i.e. either $p_i \geq 0.5$ for all $i \in N$ or $p_i \leq 0.5$ for all $i \in N$.*

In Case 1, the single participant is individual j, i.e. the individual with peak $p_j = 0.5$. In Case 2, the single participant is the individual whose peak is the closest to 0.5.

Proof. Case 1: It is easy to see that the existence of an individual j with peak $p_j = 0.5$ leads to a single participation equilibrium with this individual being the single participant. If j is the single participant, the outcome is $p_j = 0.5$. Whenever one individual joins, individual j will adapt her vote such that the outcome remains 0.5. Hence, no individual (apart from j) has an incentive to participate. Moreover, there is no other single participant equilibrium for all $c > 0$, as individual j could move the outcome to 0.5 by joining which is profitable for her for sufficiently small participation costs.

Case 2: Assume that all peaks are on the same side of 0.5, w.l.o.g. assume $p_i \geq 0.5$ for all $i \in N$. Then, by a similar argument, one can show that there is only one single participation equilibrium in which individual 1 is the only participant. As before, if another individual decides to join her, she can adapt her vote such that the outcome doesn't change. Thus, no individual has an incentive to join her. Moreover, there is no other single participation equilibrium as individual 1 has an incentive to join since she can move the outcome to her peak which again is profitable for sufficiently small costs of participation. The case of all peaks below 0.5 can be dealt with by replacing individual 1 by individual n, who in this case is the individual with peak closest to 0.5.

It remains to show that in all other cases there exists no single participation equilibrium for all $c > 0$. As neither case 1 nor 2 apply, there is no individual with a peak of 0.5 and given ordered peaks we have

$p_1 < 0.5 < p_n$, i.e. at least one individual with a peak strictly below 0.5 and at least one individual with a peak strictly above 0.5. Assume that there is a single participation equilibrium for all $c > 0$ with an individual with a peak below 0.5 participating. Then, individual n has an incentive to join, as she could move the outcome to 0.5 by participating, which is profitable for sufficiently small costs of participation. Analogously, if there were a single participation equilibrium with an individual with peak above 0.5 participating, then individual 1 has an incentive to join for sufficiently small costs of participation. Hence, there exists no single participation equilibrium for all $c > 0$ if case 1 or 2 do not apply. □

One can easily see that if there exists a single-participation equilibrium for some cost c, then this equilibrium is also an equilibrium for $c' > c$. Even with lower costs of participation all non-participants prefer to abstain and the single-participant will never want to abstain. Thus, we deduce that there are no other single-participation equilibria for small but strictly positive costs of participation, than the ones identified in Proposition 13.1.3.

Moreover, note that the result of Proposition 13.1.3 holds without the strengthening of the equilibrium concept to strong Nash equilibria. As we consider at most two participants, all Nash equilibria in this setting are also strong Nash equilibria.

We now turn our attention to the analysis of even number of participants, starting with a lemma stating that in a strong Nash equilibrium with a fixed even number of participants the outcome is 0.5. Moreover, the peaks of the participants need to differ from 0.5 and need to be evenly distributed to both sides of 0.5.

Lemma 13.1.4. *All equilibria of the sequential participation game under the symmetric median rule with an even number of participants have the outcome* 0.5 *provided that in the second stage participants play a strong Nash equilibrium. Moreover, in all equilibria half of the participants have a peak strictly above and half of the participants have a peak strictly below 0.5.*

Proof. Assume, by way of contradiction, that there exists an equilibrium of the required sort with an even number of participants k and an outcome that is different from 0.5. We will show that this is not possible. By Lemma 13.1.1, the outcome is the median of $p_{\frac{k}{2}}$, 0.5 and $p_{\frac{k}{2}+1}$. As the outcome is assumed to be different from 0.5, we must have either $p_{\frac{k}{2}} < p_{\frac{k}{2}+1} < 0.5$ or $0.5 < p_{\frac{k}{2}} < p_{\frac{k}{2}+1}$.

Case 1: $p_{\frac{k}{2}} < p_{\frac{k}{2}+1} < 0.5$:

In this case, Lemma 13.1.1 implies that the outcome is $p_{\frac{k}{2}+1}$. If an agent $i < \frac{k}{2} + 1$ (i.e. with a peak below $p_{\frac{k}{2}+1}$) abstains, then there are $\frac{k}{2} - 1$ participants with a peak below $p_{\frac{k}{2}+1}$ and $\frac{k}{2} - 1$ participants with a peak above $p_{\frac{k}{2}+1}$. Hence, $j = \frac{k}{2} + 1$ is the median participant and the outcome is $p_{\frac{k}{2}+1}$. Since the outcome is unchanged, agent i has an incentive to abstain whenever her participation costs are positive.

Case 2: $0.5 < p_{\frac{k}{2}} < p_{\frac{k}{2}+1}$:

By a completely symmetric argument, one shows that in this case every agent $i > \frac{k}{2}$ (i.e. with a peak above $p_{\frac{n}{2}}$) has an incentive to abstain since this would again not change the outcome.

Hence, we showed that the outcome is 0.5. This implies directly that the number of individuals whose peak is below 0.5 needs to be same as the number of those individuals with a peak above 0.5. Furthermore, there cannot be an individual with peak 0.5. Otherwise some individual will find it profitable to abstain as the outcome will not change as 0.5 will be the median vote after the abstention. □

With the result of Lemma 13.1.4 we obtain a necessary condition for equilibria with an even number of participants, but without conditions for the existence of such equilibria. The following lemma shows that such equilibria exists and that these equilibria are (almost) full participation equilibria, i.e. an equilibrium with at least $n-1$ participants. To be more precise, we find full participation equilibria when the number of individuals is even, and almost full full participation equilibria when the number of individuals is odd.

Proposition 13.1.5. *Consider the symmetric median rule. Then, the only subgame-perfect equilibria (with a strong Nash equilibrium played on the second stage) for small but strictly positive costs of participation with an even number of participants are (almost) full participation equilibria. Such equilibria exist if and only if there are as many individuals with a peak strictly below 0.5 as there are individuals with a peak strictly above 0.5.*

Proof. As peaks are generic we get $\#\{i \in N : p_i < 0.5\} = \#\{i \in N : p_i > 0.5\} = \lfloor \frac{n}{2} \rfloor$, as there is at most one individual with a peak of 0.5. If n is odd, then $p_{(n+1)/2} = 0.5$.

Case 1: n is even.
By Lemma 13.1.4 we deduce that the outcome is 0.5 if all individuals participate. If an individual i with peak below 0.5 abstains, the outcome shifts to $p_{(n/2)+1}$ which is worse for individual i for sufficiently small costs of participation. Similarly, if an individual i' with peak above 0.5 abstains, the outcome changes to $p_{n/2}$ which again is worse for individual i' for sufficiently small costs of participation. Hence, full participation is an equilibrium given this distribution of peaks.

Case 2: n is odd.
By Lemma 13.1.4 we get that the outcome is 0.5 if all individuals participate except for individual $(n+1)/2$. By a similar argument one can show that no participant has an incentive to abstain for small but strictly positive costs of participation. As the outcome corresponds already to her peak, there is no incentive for individual $(n+1)/2$ to participate, hence, we found an almost full participation equilibrium.

It remains to show that the existence of a (almost) full participation equilibrium implies the even distribution of the peaks required in the proposition.

Assume that there exists a (almost) full participation equilibrium. As the number of participants is even, we know by Lemma 13.1.4 that there need to be as many individuals with a peak strictly below 0.5 as there are with peak strictly above 0.5. If n is odd, there exists one individual who did not participate and even for very small costs has no incentive to participate. This implies that the peak of this individual has to be 0.5.

As a last step, we show that there are no other equilibria with an even number of participants. Assume that, for small costs of participation, there exists an equilibrium with an even number of participants, that is not a (almost) full participation equilibrium. By Lemma 13.1.4 we know that the outcome is 0.5. As we do not consider (almost) full participation equilibria there are at least 2 individuals who do not participate. As peaks are generic at least one of those individuals has a peak that is different from 0.5. This individual can shift the outcome closer to her peak by participating, as there is no participant with peak 0.5. For (sufficiently) small costs of participation this is profitable for her. Hence, this cannot constitute an equilibrium. □

These results imply that the symmetric median rule chooses the median of the peaks of the participants in subgame perfect equilibria such that a strong Nash equilibrium is played in the second stage of the game only if the median of the peaks is 0.5. It may even choose one peak that is furthest away from the median if all peaks are on the same side of 0.5.

Corollary 13.1.6. Consider the symmetric median rule. In every subgame-perfect equilibrium with a strong Nash equilibrium played on the second stage, a participant's peak is the outcome if and only if this participant is the only participant.

Proof. Due to Lemma 13.1.2 we know that there are no equilibria with an odd number of participants greater than 1. By Proposition 13.1.5 we know that all equilibria with an even number of participants are (almost) full participation equilibria with outcome 0.5 and without a participant with a peak of 0.5. □

13.2 Mean Rule

The equilibria for a fixed number of participants are characterized by Renault and Trannoy (2005): For all generic peak distributions there is a unique Nash equilibrium. In this equilibrium at most one voter submits a non-extreme vote, i.e. a vote that is different from 0 and 1. If there exists such a voter, then the outcome corresponds to her peak.

Using this result, we deduce that the outcome in a subgame-perfect equilibrium corresponds only in single participation equilibria to a participants peak.

Proposition 13.2.1. *Consider the mean rule. In every subgame-perfect equilibrium a participant's peak is the outcome if and only if this participant is the only participant.*

Proof. As peaks are generic, there exists at most one individual whose peak corresponds to the outcome in equilibrium. Block (2014) provides conditions to identify such an individual. Every participant i is associated with an interval $A_i = \left[\frac{1}{k}(k-i)\ ,\ \frac{1}{k}(k-i+1)\right]$ and the outcome in equilibrium corresponds to p_i if and only if $p_i \in A_i$.
Consider a Nash equilibrium (given a fixed number of $k > 1$ voters) in which there exists a participant i whose peak corresponds to the outcome. Let $i \notin \{1, k\}$. By Block (2014)

$$p_i \in A_i \quad = \quad \left[\frac{1}{k}(k-i)\ ,\ \frac{1}{k}(k-i+1)\right].$$

If participant $1 < i$ abstains, then individual i is now on position $i-1$ in the new set of participants. The corresponding interval $A_i^{K-\{1\}}$ is

$$\begin{aligned} A_i^{K-\{1\}} \quad &= \quad \left[\frac{1}{k-1}(k-1-(i-1))\ ,\ \frac{1}{k-1}(k-1-(i-1)+1)\right] \\ &= \quad \left[\frac{1}{k-1}(k-i)\ ,\ \frac{1}{k-1}(k-i+1)\right]. \end{aligned}$$

If participant $k > i$ abstains, then individual i remains on position i in the new set of participants and the corresponding interval $A_i^{K-\{k\}}$ is

$$\begin{aligned} A_i^{K-\{k\}} \quad &= \quad \left[\frac{1}{k-1}(k-1-i)\ ,\ \frac{1}{k-1}(k-1-i+1)\right] \\ &= \quad \left[\frac{1}{k-1}(k-1-i)\ ,\ \frac{1}{k-1}(k-i)\right]. \end{aligned}$$

We consider

$$A_i^{K-\{1\}} \cup A_i^{K-\{k\}} = \left[\frac{1}{k-1}(k-1-i)\ ,\ \frac{1}{k-1}(k-i+1)\right].$$

and since

$$\frac{1}{k-1}(k-1-i) = 1 - \frac{i}{k-1} \quad \leq \quad 1 - \frac{i}{k} = \frac{1}{k}(k-i)$$

and

$$\frac{1}{k-1}(k-i+1) \quad \geq \quad \frac{1}{k}(k-i+1)$$

we obtain $A_i \subseteq \left(A_i^{K-\{1\}} \; \cup \; A_i^{K-\{k\}} \right)$.

Hence, if $p_i \in A_i$, then $p_i \in A_i^{K-\{1\}}$ or $p_i \in A_i^{K-\{k\}}$. Thus, either individual 1 or individual k will prefer to abstain, as the outcome remains p_i with and without her vote.

It remains to show that there is no equilibrium for $i \in \{1, k\}$.

If $i = 1$, then $A_1 = \left[\frac{k-1}{k} \;,\; 1\right]$ and $A_1^{K-\{k\}} = \left[\frac{(k-1)-1}{k-1} \;,\; 1\right]$. As $\frac{k-2}{k-1} \leq \frac{k-1}{k}$, if $p_1 \in A_1$, then $p_1 \in A_1^{K-\{k\}}$ and thus abstention is profitable for individual k.

If $i = k$, then $A_k = \left[0 \;,\; \frac{1}{k}\right]$ and $A_k^{K-\{1\}} = \left[0 \;,\; \frac{1}{k-1}\right]$. As $\frac{1}{k-1} \geq \frac{1}{k}$, if $p_k \in A_k$, then $p_k \in A_k^{K-\{1\}}$ and thus abstention is profitable for individual 1.

Thus, there exists no subgame-perfect equilibrium in which a participant's peak is the outcome in a situation with more than one participant. If there is only one participant, then obviously her peak is the outcome. The characterization of such equilibria is given in Proposition 13.1.3, as the mean and the symmetric median rule coincide for one and two participants, and those are the only cases that need to be considered for that result. □

The direct implication of this proposition is that in every subgame-perfect equilibrium (except for single-participation) all votes are extreme.

Proposition 13.2.2. *Consider the mean rule. Then, all subgame-perfect equilibria for small but strictly positive cost of participation with more than one participant are (almost) full participation equilibria.*

- *Full participation equilibria exist if there exists $\ell \in \{1, \ldots, n-1\}$ such that $p_\ell < \frac{n-\ell}{n} < p_{\ell+1}$. The corresponding outcome in this equilibrium is $\frac{n-\ell}{n}$.*
- *Almost full participation equilibria exist if there exists $\ell \in \{1, \ldots, k-1\}$ such that $p_\ell < \frac{k-\ell}{k} < p_{\ell+1}$ and $p_i = \frac{k-\ell}{k}$ for $i \in N \setminus \{K\}$.*

Proof. Consider a subgame-perfect equilibrium with more than one participant. Due to Proposition 13.2.1 we know that there exist no participants whose peak corresponds to the outcome and all votes are extreme. Hence, if the outcome does not correspond to the peak, the individuals who did not participate can shift the outcome towards their peak by participating, which is profitable for sufficiently small cost of participation. As peaks are generic there exists at most one individual whose peak corresponds to the peak and thus in equilibrium there exists at most one individual who abstains, i.e. we have either a full participation or an almost full participation equilibrium.
It remains to show that such equilibria exist. Since all votes are extreme all possible outcomes for k

participants are in $\left\{\frac{\ell}{k} : \ell \in \{1, \ldots, k-1\}\right\}$. Note that outcomes of 0 and 1 are not possible since peaks are generic. More precise, the outcome is $\frac{\ell}{k}$ if the peak of ℓ participants is below $\frac{\ell}{k}$ and $k-j$ participants have a peak above $\frac{\ell}{k}$. Due to Proposition 13.2.1, all peaks differ from $\frac{\ell}{k}$ and thus all participants have no incentive to abstain for sufficiently small cost of participation, as a abstention moves the outcome further away from their respective peak. Almost full participation equilibria exist only if the abstaining individual's peak corresponds to $\frac{\ell}{k}$. Then, this individual has no incentive to join while all others have no incentive to abstain. □

Note that for the mean rule we would not need additional requirements on the equilibria, as the existing Nash equilibria are already unique (Renault and Trannoy, 2005). But the equilibria are also strong Nash equilibria.

13.3 Uniqueness of Equilibria

The usage of strong Nash equilibria in the second stage of the game implies that in this stage there exists a unique equilibrium. Nevertheless, this does neither ensure the existence nor the uniqueness of equilibria of the entire game. As multiplicity of equilibria may lead to a problem of equilibrium selection, we will turn our attention to conditions such that the equilibria of the entire game, and thus the subgame-perfect equilibria such that a strong Nash equilibrium is played in the second stage, are unique. This now depends on (i) the concrete utility function of the individuals and (ii) the magnitude of the costs. One could think of a lot of possible and logical utility functions, but for the sake of simplicity of the experiment we decided for linear utility functions that depend on the distance between the peak of the individual and the outcome (and on the cost of participation given a positive participation decision). Furthermore, we assume that the structure of the utility function as well as the cost of participation is identical for all individuals.

Moreover, we want to focus on the impact of the rule itself in situations that are strategically identical. Hence, we focus on equilibria that are identical for the mean and the symmetric median rule. We say that an equilibrium is identical for both rules if the participants and the outcome are identical for these two rules. Note that, while in theory votes would be allowed to differ, this never occurs.

Due to Lemma 13.1.4 we know that the equilibria in question have either a single participant or an even number of participants. In both these cases, the equilibrium strategies are identical for both rules: For 1 participant both rules are identical, and for an even number of participants in equilibrium all participants vote for 0 or 1 in both rules.

The following proposition shows that there are two kind of equilibria such that the equilibria are unique and identical: (i) full-participation equilibria and (ii) single participation equilibria.

Proposition 13.3.1. *All equilibria for the symmetric median rule in the sequential voting game that are unique for given costs c are either single-participation or full-participation equilibria.*

Proof. Consider an equilibrium for $1 < k < n$ participants. Then, due to Lemma 13.1.2, k is even and by Lemma 13.1.4 we know that the outcome is 0.5. Moreover, as $k < n$ there exists $i \in N \setminus K$.
Case 1: There exists $i \in N \setminus K$ with $p_{\frac{k}{2}} < p_i < p_{\frac{k}{2}+1}$.
If $p_i = 0.5$, then single-participation of individual i is an equilibrium for all $c \geq 0$ as no other individual can change the outcome by joining i.
Let $p_i \neq 0.5$. If there are several individuals in $N \setminus K$ with peak between $p_{\frac{k}{2}}$ and $p_{\frac{k}{2}+1}$, then denote by i the individual whose peak is closest to 0.5.
As we consider an equilibrium in which individual i prefers to abstain, we know that $c \geq |0.5 - p_i|$

as individual i could shift the outcome to her peak by voting. However, for all $c \geq |0.5 - p_i|$ single-participation of individual i is also an equilibrium. All individual on the same side of 0.5 are further away from 0.5 as i is closest to 0.5 and thus cannot influence the outcome. All individuals on the other side can change the outcome to 0.5 which is not profitable.

Case 2: There exists no $i \in N \setminus K$ such that $p_i \in [p_{\frac{k}{2}}, p_{\frac{k}{2}+1}]$.
Take $i \in N \setminus K$ and without loss of generality assume that $p_i < 0.5$. Then, $c \geq |0.5 - p_{\frac{k}{2}}|$ as individual i could change the outcome to $p_{\frac{k}{2}}$ but prefers not to. As all $j \in K$ with $p_j > 0.5$ prefer to vote, $c \leq |0.5 - p_{\frac{k}{2}}|$ as abstaining would yield $p_{\frac{k}{2}}$ as the outcome. Hence, $c = |0.5 - p_{\frac{k}{2}}|$. However, for $c = |0.5 - p_{\frac{k}{2}}|$, single participation of $\frac{k}{2}$ is an equilibrium, as she is closer to 0.5 than all other individuals on the same side of 0.5. Hence, all those individuals cannot influence the outcome and all other individuals can change it to 0.5 which is not profitable.

We have shown, that in both cases there exists other (single-participation) equilibria. Hence, there exist no unique equilibria with $1 < k < n$ participants.

□

This proposition states that there are only a few potential equilibria to consider. Note that there are no unique almost full participation equilibria, as they require the existence of an individual with peak 0.5 and single-participation of that individual is an equilibrium for all $c > 0$. We will now examine under which conditions such equilibria exist, starting by describing all full-participation equilibria that are unique and identical for the symmetric median and the mean rule.

Proposition 13.3.2. *There exists a unique full participation equilibrium if and only if there are as many individuals with a peak strictly below 0.5 as there are individuals with a peak strictly above 0.5.*
This equilibrium exists for all $c < \min\{\min_{i \in N} |p_i - 0.5| \ ; \ 0.1\overline{6}\}$.

Proof. Consider a (almost) full participation equilibrium. By Proposition 13.1.5 we know that there are $\lfloor \frac{k}{2} \rfloor$ individuals with a peak strictly below 0.5 and $\lfloor \frac{k}{2} \rfloor$ individuals with a peak strictly above 0.5.

Case 1: There is no individual with a peak in $[0.\overline{3}, 0.\overline{6}]$
Then, full-participation is an equilibrium for the mean rule for $c < 0.1\overline{6}$ and for the symmetric median rule for $c < \min_{i \in N} |p_i - 0.5| (> 0.1\overline{6})$ as abstention moves the outcome further away by that margins. Hence, for $c < 0.1\overline{6}$ both rules have a full-participation equilibrium.

Case 2: There exists an individual with a peak in $[0.\overline{3}, 0.\overline{6}]$
Then, full-participation is an equilibrium for both rules for $c < \min_{i \in N} |p_i - 0.5|$. Note that due to propositions 13.2.1 and 13.1.5 there exists no participant i with $p_i = 0.5$.

It remains to show that these equilibria are unique (given the fixed value c). Due to Proposition 13.3.1 we know that it is sufficient to show that there are no single participation equilibria for that value c.

Single participation of an individual i is an equilibrium if it is not profitable for all other individuals to join individual i. As there are individuals on both side of 0.5, there exists in particular an individual j whose peak is on the different side of 0.5 than individual i's peak. If individual j joins, the outcome changes to 0.5. In equilibrium this is not profitable; hence, the cost of participation exceeds $|p_i - 0.5|$. Thus, there exist no single-participation equilibria for $c < \min_i |p_i - 0.5|$ and hence, the full-participation equilibria are unique. □

We use the single-participation equilibria described in Theorem 13.1.1 and examine for which values of c these equilibria are unique. We will make a case distinction depending on whether or not there exists an individual with peak 0.5.

Proposition 13.3.3. *Let there be an individual i with $p_i = 0.5$. Then, there exists a unique single-participation equilibrium for $c < \min_{j \neq i} |0.5 - p_j|$.*

Proof. Due to Proposition 13.3.1 we know, that it is sufficient to consider single- and full-participation equilibria. Due to Proposition 13.3.2 there exist no full-participation equilibria since $p_i = 0.5$.
Take any other individual j and consider a single-participation equilibrium of j. Then, in order for individual i to abstain, the cost of participation needs to exceed $|0.5 - p_j|$ since individual i could shift the outcome to 0.5. Any other individual ℓ cannot shift the outcome further, as 0.5 is the most the outcome can be shifted to, if p_ℓ is on the other side of 0.5 as p_j. If p_ℓ is on the same side of 0.5 as p_j, then the outcome either does not change if p_ℓ is further away to 0.5 than p_j, or it changes to p_ℓ which is less than a change to 0.5. Hence, for $c > |0.5 - p_j|$ no individual wants to join individual j, if $c \leq |0.5 - p_j|$ individual i (and potentially others) prefer to join. Hence, there exist no single-participation equilibria different from single-participation of individual i, for $c > \min_{j \neq i} |0.5 - p_j|$. □

Proposition 13.3.4. *Let there be no individual i with $p_i = 0.5$.*
Then, there exists a unique single participation equilibrium for small but strictly positive costs of participation if and only if either

(i) $p_k < 0.5$

(ii) $0.5 < p_1$

These equilibria are unique for $c < p_k - p_{k-1}$ in case (i) and for $c < p_2 - p_1$ in case (ii). In both cases, the individual closest to 0.5 is the single participant.

Proof. First, we show that, given there exists no individual i with $p_i = 0.5$, the only single-participation equilibria for small but strictly positive costs of participation are if all peaks are on the same side of 0.5. Assume not, then there exists individuals i and j with $p_i < 0.5$ and $p_j > 0.5$. For all individuals ℓ with peak $p_\ell < 0.5$, individual j prefers to join for all $c < 0.5 - p_\ell > 0$, hence, for $c < 0.5 - p_\ell$ this is not an equilibrium. Analogously, for $p_\ell > 0.5$, individual i prefers to join for all $c < p_\ell - 0.5 > 0$, hence, for $c < p_\ell - 0.5$ this is not an equilibrium.

Let $p_k < 0.5$. Then, for all $c > 0$, single participation of individual k is an equilibrium, since all other individuals do not change the outcome by joining. In order to obtain a single-participation equilibrium for one of the other individuals, call her j, it cannot be profitable for individual k to participate, even though she obtains her peak as the outcome by joining. Hence, $c > p_k - p_j$. Note that again all individuals with a peak even further away from 0.5 do not influence the outcome by joining. Hence, for $c > p_k - p_{k-1}$ there exists the next single-participation equilibrium (in which individual $k - 1$ is the single participant).

Due to symmetry, the argument for case (ii) works analogously.

□

In our analysis we focus on equilibria that exist for small but strictly positive costs of participation. There are further equilibria that only appear once a certain threshold value is surpassed. We exclude these equilibria as they heavily depend on the form of the utility functions.

14 Pilot

In a laboratory experiment, we analyze in how far individuals play Nash equilibria strategies in a costly voting setting for two different rules: the mean rule and the symmetric median rule.

14.1 Setup

The pilot experiment was conducted at the KD2Lab of the Karlsruhe Institute of Technology (KIT) in August 2019 in 4 sessions of about 75 minutes each. There were 12 participants in each session. The experiment was organized and recruited with the software hroot (Bock et al., 2012). 85 % of the participants were students, 48% of the participants were female and the average age of the participants was 25.3 years. Most experiments conducted at the KD2Lab have a higher share of students, and the share of females in this pilot experiment is also clearly exceeding the overall share of female students at the KIT at that time, which was 29.5% (Karlsruher Institut für Technologie (KIT), 2019). The high share of non-students can be explained by the timing of the experiment, as it took place outside of the lecture period.

14.2 Laboratory Procedure

For each session 17 individuals were invited, to ensure that at least 12 individuals show up in time in order to properly run the experiment. 61 of the 68 individuals showed up (at least 14 to each session), and all individuals who arrived in time received a show-up fee of 2.00 Euros. All individuals, who showed up on time, but only after all 12 places were already given away, received a show-up fee of 5.00 Euros and could not participate. Upon arrival, individuals were randomly allocated to their place, where they waited for the start of the experiment without possibility of interaction with the other individuals. At each place, the individuals found the instructions for the experiment, blank paper and a pen to take notes. At the start of the experiment, the instructions (see Appendix A.1) were read aloud by audiotape, to ensure identical conditions across sessions. The experiment was conducted as a pen and paper experiment, as the implementation on a computer was not possible at that time. Due to this setup, a round took significantly longer, than it would have taken if it were implemented on a computer. Due to this time constraint, a control round to ensure that all individuals understood the procedure was omitted. This turned out to be a mistake later on, as two individuals clearly did not understand the procedure.

The individuals then played 3 rounds as described in Section 14.3 below.

After the experiment, the individuals were given a questionnaire on demographic data and questions concerning their strategic reasoning behind their decision. They also took a risk-attitude test similar to the one proposed by Holt and Laury (2002).

Then, individuals were brought to a separate room individually to receive their payoff without anyone else observing. The payoff corresponded to the sum of the show-up fee, the payoff of one randomly selected case of the risk-attitude test and the payoff of one randomly selected round that was played in the experiment. For randomization, the individuals themselves threw a dice. The average earning of the individuals was 12.40 Euros.

14.3 Design

The treatment variable in this experiment is the voting rule used. We applied a between-subject design, i.e. in every session only one voting rule was used.

The individuals were told in the instructions that they participate in a two-staged voting process, in which they first decide on their participation, which comes with a cost. Then, if they chose to participate, they submit a vote in the second stage. In each round, they take the decision in a group of 6 individuals that changes for each round via stranger matching, i.e. individuals might play with some individuals they already played with in an earlier round, but they have no way of identifying the other individuals.

We apply the strategy method of Selten (1967). Every individual is handed out a list of all peaks that will be distributed in this round. Every individual decides for each peak, whether she wants to submit a vote in the second stage or not. After all individuals have made their choices, every individual is assigned to a peak. This assignment depends on the place of the individuals and the assignment of the peaks of the first round to the places was randomized before the experiment, but kept the same across sessions. Moreover, the peaks in the following rounds were also identical for individuals on the same place in different sessions. This was done to simplify the procedure of the experiment and additionally we tried to assure that the peak distributions are ‘fair’ in the sense that no individual only receives extreme peaks, that tend to result in a lower payoff. After the assignment, the participation decision is made according to the list of choices they submitted. Then, every individual is informed of which peak was assigned to her, reminded of her corresponding participation decision and is informed about which other participation decisions were realized. As the individuals cannot identify the other players, all they know is which ‘peaks’ will now submit a vote in the second stage. Every individual, who chose to participate given the assigned peak, is asked to submit a vote and all individuals are asked to give a prediction of the result. The latter has no influence on the outcome and on the payoff and is meant to obtain a hint on whether the individuals understand the Nash equilibrium, or whether they expect it to be played. We decided to use the strategy method to obtain more participation decisions of the individuals which also enable us to understand the strategies of the individuals better.

Next, the result of the vote is computed and communicated to the individuals and the next round starts.

14.3.1 Payoff Function

To simplify the voting process for the participants of the individuals, we now consider the interval $[0, 100]$ instead of the interval $[0, 1]$. We use a linear payoff function, that depends on the distance of the assigned peak p_i of individual i to the outcome x, the participation decision $\delta_i \in \{0, 1\}$ of individual i and the (fixed) cost of participation c. In theory, p_i can take on any value from 0 to 100, but in our pilot experiment the lowest peak was 30 and the highest peak was 85. Nevertheless, we designed the utility function as if the peaks could take on all values, to keep it simple and to not give away potential hints to the individuals. To ensure that the utility function is positive, we subtract the distance between the peak and the outcome and the participation cost, from the maximal absolute value of their sum, which is $100 + c$. We model the case that no individual chooses to cast a vote, as the worst case scenario. Then, the utility function of all individuals is 0, and by design smaller than all cases with someone voting. Hence, the utility function has the following form

$$u_i(x, p_i, c, \delta_i) = \begin{cases} (100 + c) - |x - p_i| - \delta_i \cdot c & \text{if } \sum_{j \in N} \delta_j > 0 \\ 0 & \text{else.} \end{cases} \tag{14.3.1}$$

The unit of the utility was called “points” and 10 points corresponded to 1 Euro.

For the pilot experiment we choose $c = 9$. This choice stems from the peak distributions (see below), as further equilibria occurred, e.g. for $c = 10$. Moreover, for $c = 10$ the equilibria are less stable as some individuals are indifferent between participating and abstaining (e.g. in Peak Distribution 1), yet the equilibrium requires them to chose the ‘correct’ participation decision.

14.3.2 Peak Distributions

Every treatment consists of the same peak distributions in the same order to keep potential learning effect the same across treatments and to enable a comparison between the treatments for the same peak distributions. Peak distribution 1 is played in the first round, followed by Peak Distribution 2 and Peak Distribution 3 was played in the last round.

Peak Distribution 1

The first peak distribution contains the peaks 30, 40, 55, 60, 65 and 70.

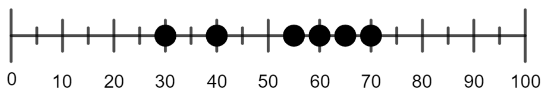

Figure 14.1: Peak Distribution 1

All peaks are rather centralized with 4 peaks in the upper half and 2 peaks in the lower half. For the cost level $c = 9$ that was chosen, there exists a unique Nash equilibrium which is single participation of the individual with peak 55. This is an equilibrium, as no individual with a peak above 55 can profitably influence the outcome, since the individual with peak 55 will adjust her vote and still obtain her peak as the outcome. The two individuals with a peak in the lower half can move the outcome to 50, but this is not profitable as the cost of participation exceeds the resulting improvement.

Peak Distribution 2

The second peak distribution contains the peaks 30, 35, 40, 60, 65 and 70.

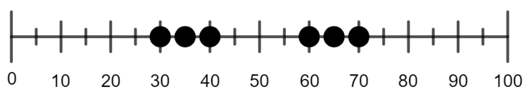

Figure 14.2: Peak Distribution 2

All peaks remain rather centralized, but now they are evenly split in both halves. Given $c = 9$, there exists a unique Nash equilibrium which is full participation. Abstention of any individual moves the peak away by 10, which is not profitable as the cost of participation is 9.

Peak Distribution 3

The third peak distribution contains the peaks 60, 65, 70, 75, 80 and 85.

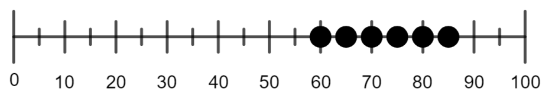

Figure 14.3: Peak Distribution 3

The peaks are not centralized any more and all are placed in the upper half. For $c = 9$, there exist two Nash equilibria, both of which are single participation equilibria. The first of these equilibria is single participation of the individual with peak 60, the second equilibrium is single participation of the individual with peak 65. In both case no individual with a peak above the single participant can profitably influence the outcome as the adjusted vote of the original participant will ensure that the outcome remains the same. In the second equilibrium, the individual with peak 60 can shift the outcome to her peak, but this improvement is strictly smaller than the participation costs.

14.4 Hypotheses

Hypothesis (H1). The individuals play Nash equilibrium strategies on the second stage of the game.

Rollmann (2020) consider a budget allocation problem without participation on a 3-dimensional space of alternatives. They find that individuals struggle finding the Nash equilibrium strategies. Their setup is similar to the game on the second stage, but with a more complex alternative space. We want to examine whether, in our setup, participants manage to identify the strategy that leads to the unique (strong) Nash equilibrium, and whether they play that strategy.

Hypothesis (H2). The individuals play Nash equilibrium strategies of the entire game more often, when it involves participation.

This hypothesis has two parts. First, we suspect that individuals will rarely play the Nash equilibrium strategy of the entire game. Due to the complexity of the game we do not expect individuals to find those strategies. Second, we suspect that if a Nash equilibrium strategy of the entire game is played, then this is driven by a tendency to prefer voting over abstention. As Riker and Ordeshook (1968) argue, there may exist a positive influence on ones utility by voting, that is driven for example by the sentiment of fulfilled civic duty. This may drive the participation rate and cannot be captured as it is hard to quantify and observe. We suspect that individuals, if unable to fully understand the strategic dimension of this game, tend to choose participation over abstention. Another element that could also increase participation rates in these scenarios is the desire to “do something” during the experiment and the feeling of being able to influence the outcome in the second stage of the game. Additionally, risk-aversion might lead to participation to avoid the worst case of non-participation of all individuals.

Hypothesis (H3). The rules themselves do not influence whether Nash equilibrium strategies are played.

In all scenarios, the equilibria are identical for both rules. Hence, strategically there is no difference between the rules and thus we do not expect to observe higher participation for one rule, even though there are differences between the rules. The mean rule guarantees a small, but strictly positive effect on the outcome (which might be negated by strategic voting of the others), whereas the median rule has a lot of instances where a vote has no impact, but if a voter is pivotal, one has the possibility to change the outcome by a great margin.

14.5 Results

Unfortunately, there were two participants (in the same session) who did not understand the rules and correspondingly chose “random” participation decisions and votes. One of these participants seemed to confuse the mean of the votes with the sum of the votes. The other participant openly admitted, that she did not know what to do and chose “random” decisions. Both participants were excluded from the analysis. As one cannot exclude that this behavior influenced the other individuals in later rounds, the results of their session have to be treated carefully.

When talking about participation, we will consider the participation decisions as listed in the table (see Appendix A.2) using the strategy method and not the realized participation decision, as the latter is only a random realization of the first. We thus have 144 participation decisions for the symmetric median rule and 132 participation decisions for the mean rule for each peak distribution. As a side-note, the realized level of participation was 59.72 % (86 of 144 instances) and in average 3.58 votes were cast in the second stage of the game. Only once, all individuals cast a vote in the second stage of the game. The minimum level of participation was two participants, which occurred three times. Hence, the worst case by design, that no individuals participates never occurred.

We tested whether the demographic data has an influence on the participation decision and found that neither age, nor experience in game theory, risk-attitude and educational background had a significant effect. The only effect we found was that female participants were slightly more likely to participate, but only on a 10% significance level.

We did not find significant differences of the participation level of the mean and the symmetric median rule neither for any of the treatments, nor for all of them combined. When looking at the participation levels that can be found in the following table, this comes as no surprise, as all of them are very similar and the number of observations is rather small.

Peak Distribution	Mean Rule	Symmetric Median Rule
Peak Distribution 1	60.61 % (80 of 132)	63.19 % (91 of 144)
Peak Distribution 2	62.12 % (82 of 132)	64.58 % (93 of 144)
Peak Distribution 3	64.39 % (85 of 132)	63.89 % (92 of 144)
Peak Distribution 1-3	62.37 % (247 of 396)	63.89 % (276 of 432)

Table 14.1: Participation Levels

We will now have a closer look at the participation decisions for each peak distribution.

Peak Distribution 1

The participation levels in Peak Distribution 1 can be found in Table 14.2 below.

Peak	Mean Rule	Symmetric Median Rule
30	31.82 % (7 of 22)	25.00 % (6 of 24)
40	40.91 % (9 of 22)	58.33 % (14 of 24)
55	86.36 % (19 of 22)	83.33 % (20 of 24)
60	86.36 % (19 of 22)	79.17 % (19 of 24)
65	59.09 % (13 of 22)	75.00 % (18 of 24)
70	59.09 % (13 of 22)	58.33 % (14 of 24)
All peaks	60.61 % (80 of 132)	63.19 % (91 of 144)

Table 14.2: Participation Levels for Peak Distribution 1

The participation levels for both rules have a great range, from 25% to 83.33% for the symmetric median rule and from 31.82% to 86.36% for the mean rule. For both rules there were 3 individuals each, who chose to participate for all peaks.

Note that the unique equilibrium is single participation of the individual with peak 55. Even though there is much more participation than in equilibrium, the individuals seemed to be able to identify - at least in some sense - the special position of peak 55, as the participation given that peak is the highest. Nevertheless, there was only one individual who chose to only participate for the peak 55. The participation levels are visualized in Figure 14.4.

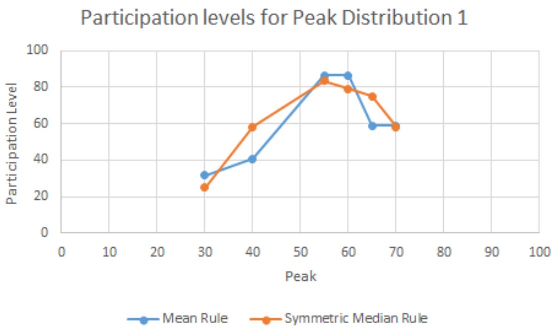

Figure 14.4: Participation Levels for Peak Distribution 1

In Figure 14.4 we observe for both rules an inverse U-shaped distribution of the participation levels. Due to the small sample size, we can only say that the participation levels for the mean rule differ significantly (on a 10% significance level) from a uniform distribution (chi-squared test, $p = 0.0995$). This is in particular interesting, as it states that individuals with extreme peaks are less likely to participate, which is contradicting the intuition put forward by Osborne et al. (2000) and thus is an important aspect to examine in the follow-up experiment.

Peak Distribution 2

The participation levels in Peak Distribution 2 can be found in Table 14.3 below.

Peak	Mean Rule	Symmetric Median Rule
30	36.36 % (8 of 22)	33.33 % (8 of 24)
35	59.09 % (13 of 22)	58.33 % (14 of 24)
40	54.55 % (12 of 22)	83.33 % (20 of 24)
60	72.73 % (16 of 22)	79.17 % (19 of 24)
65	77.27 % (17 of 22)	79.17 % (19 of 24)
70	72.73 % (16 of 22)	54.17 % (13 of 24)
All peaks	62.12 % (82 of 132)	64.58 % (93 of 144)

Table 14.3: Participation Levels for Peak Distribution 2

The unique equilibrium for this peak distribution is full participation. Interestingly, in comparison to Peak Distribution 1, the minimal participation levels per peak increased to 36.36% for the mean rule and to 33.33% for the symmetric median rule, while the maximal participation levels per peak decreased to 77.27% for the mean rule and 83.33% for the symmetric median rule. The overall participation rate increased slightly but non-significantly in comparison to Peak Distribution 1. Here, there were 4 individuals for the mean rule and 5 individuals for the symmetric rule, who chose to participate for all peaks. Even though the peak distribution is symmetric, only 21 of the 46 players chose a symmetric participation decision. Moreover, almost all peaks in the upper half have a higher participation rate than their symmetric counterparts in the lower half. Some individuals later stated, that - as a principle - they did not participate for small peaks. We have no plausible explanation for such a behavior. The participation levels are visualized in Figure 14.5.

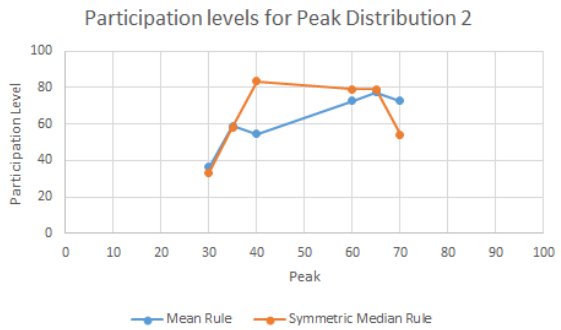

Figure 14.5: Participation Levels for Peak Distribution 2

As in Figure 14.4, we observe in Figure 14.5 an inverse U-shaped distribution of the participation levels for the symmetric median rule, but we do not find that the distribution significantly differs from a uniform distribution.

Peak Distribution 3

The participation levels in Peak Distribution 3 can be found in Table 14.4 below.

Peak	Mean Rule	Symmetric Median Rule
60	77.27 % (17 of 22)	58.33 % (14 of 24)
65	63.64 % (14 of 22)	58.33 % (14 of 24)
70	59.09 % (13 of 22)	58.33 % (14 of 24)
75	54.55 % (12 of 22)	62.50 % (15 of 24)
80	63.64 % (14 of 22)	75.00 % (18 of 24)
85	68.18 % (15 of 22)	70.83 % (17 of 24)
All peaks	64.39 % (85 of 132)	63.89 % (92 of 144)

Table 14.4: Participation Levels for Peak Distribution 3

This peak distribution has the most narrow range of per peak participation rates. In comparison to Peak Distribution 2, the minimal participation rate per peak increased to 54.55% for the mean rule and 58.33% for the symmetric median rule. The maximal participation rate per peak remained at 77.27% for the mean rule and decreased to 75% for the symmetric median rule. The overall participation rate slightly increased for the mean rule and slightly decreased for the symmetric median rule, but all differences are non-significant. For this peak distributions there were 9 individuals for the mean rule and 7 individuals for the symmetric median rule that chose to participate for all peaks, even though the only two equilibria are single participation equilibria of the two lowest peaks. At least for the symmetric median rule, the individuals did not play according to these equilibria as the two lowest peaks have the lowest participation rates. Moreover, there was no individual who chose to only participate for one of the two lowest peaks. The participation levels are visualized in Figure 14.6.

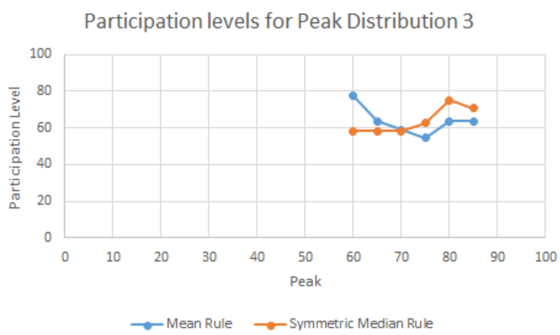

Figure 14.6: Participation Levels for Peak Distribution 3

In contrast to figures 14.4 and 14.5, we do not observe the inverse U-shaped participation levels for neither of the two rules, but rather nearly constant participation levels.

Evaluating the Hypotheses

Let us revisit out hypotheses and see how our results compare to them.

Hypothesis (H1). The individuals play Nash equilibrium strategies on the second stage of the game.

We find that the participants find and play the Nash equilibrium strategies on the second stage of the game, only if there is an odd number of participants in the symmetric median rule treatment. 7 times, there was an odd number of participants in the symmetric median rule treatment and 6 times the equilibrium was played. The remaining 5 times, there was an even number of participants, with a total of 20 votes cast altogether in these cases, and only 7 of these 20 votes corresponded to the equilibrium strategies.

In the mean treatment, there were a total of 38 votes cast in the second stage of the game, and only 5 of these votes corresponded to the equilibrium strategies.

Hence, the individuals play the equilibrium strategy only for the symmetric median rule with an odd number of individuals. We asked the individuals (also the non-participating individuals) for their prediction of the outcome and found that most predictions, even for the symmetric median rule with an odd number of participants, did not correspond to the equilibrium outcome: For the mean rule 11 of 66 predictions corresponded to the equilibrium outcome, for the symmetric median rule with an even number of participants 9 of 30 predictions and for an odd number of participants 18 of 42 predictions correspond to the equilibrium outcome. Hence, either the individuals did not find the equilibrium or they did not expect the equilibrium to be played. Due to the answers in the questionnaire we suspect that most individuals could not find the equilibrium.

Hypothesis (H2). The individuals play Nash equilibrium strategies of the entire game more often, when it involves participation.

We find that individuals do not play the Nash equilibrium strategies of the entire game. In Peak Distribution 1 there is only one individual of the 46 participants who only participates given the peak 55, which is the single participant in the unique equilibrium. In Peak Distribution 2 there are 9 players in both treatments combined that participate for all peaks, which corresponds to the full participation equilibrium. But most of them also participate for all peaks given the other peak distributions with single participation equilibria. Hence, it seems that it is not necessarily the equilibrium that drives the participation decision. In Peak Distribution 3 no individuals' choice corresponds to a Nash equilibrium of the entire game.

Clearly, there are more Nash equilibrium strategies played when the equilibrium is full participation, but it is not clear that it is the equilibrium itself that drives this effect. In the questionnaire some individuals (31 %) stated that they participated because they wanted to "influence the outcome". This seems to play a stronger role here.

Hypothesis (H3). The rules themselves do not influence whether Nash equilibrium strategies are played.

The participation levels are very similar for both rules and in fact all differences are non-significant. Thus, there seems to be no impact of the rule itself on the participation levels.

In the analysis of all hypotheses we see that the number of observations is to small to obtain significant results. Moreover, it seems that the individuals were mostly unable to find the Nash equilibria on the second stage of the game (except in part for the symmetric median rule with an odd number of individuals). Furthermore, the equilibria of the entire game were not played and we suspect that they were not discovered by the participants of the experiment. We believe that the understanding of the equilibria can be facilitated if the group size is reduced, which is a change we implement for the follow-up experiment.

15 Experiment

The following section will present the design of the 'main' experiment.

15.1 Design

The design of the experiment is - in principle - identical to the design of the pilot experiment. The main difference is that the group size is reduced to 4 individuals. In the pilot experiment we saw that individuals failed to recognize the Nash equilibria of the game. We therefore decided to reduce the group size and with that the complexity of the game. We reduce the group size to the minimal size that is still of interest. For 1 and 2 participants, the mean and the symmetric median rule are identical. For three individuals, there exists no full participation equilibrium due to Lemma 13.1.2. Hence, $n = 4$ is the smallest group size, such that all rules differ and such that there potentially exist non-single-participation equilibria. Obviously, the peak distributions have to change with the reduced group size. The peak distributions that we consider have the additional property (in contrast to Peak Distribution 3 in the pilot) that all equilibria are unique. This simplifies the interpretation of the actions and avoids issues of coordination on equilibria, assuming that the individuals found all equilibria. Furthermore, we only consider equilibria that remain equilibria if the cost of participation were reduced. This was not the case in the pilot in peak distributions 1 and 3. In our opinion, equilibria that only occur once the participation cost exceed a certain threshold are less stable, more difficult to detect and depend even stronger on the precise shape of the utility functions. The equilibria that we will consider are equilibria that exist for sufficiently small cost of participation, and thus those we already focused on in Part II.

Another difference is that, while the pilot experiment was conducted as a pen-and-paper experiment, this experiment will be implemented as a computer based experiment, which greatly reduces the time per round and enables us to play more rounds. Moreover, this simplifies the introduction of a control round, to ensure that everybody understands the rules used in the experiment.

We also change the cost of participation to $c = 10$, which seems more natural than the value of 9 that was chosen in the pilot. The peak distributions are chosen such that the equilibria are nevertheless not dependent on specific participation decisions of individuals that are indifferent between participation and abstention. We aim for 'high' costs of participation to make voting really costly and to make individuals really consider abstention as a (strategically) viable option. However, as argued before, we only consider equilibria that keep on existing if the cost of participation is reduced. Hence, there is a limit to how much c can be increased while preserving these properties.

15.2 Peak Distributions

Peak Distribution 1

The first peak distribution consists of the peaks 5, 10, 90 and 95 and the unique equilibrium is full participation of all individuals.

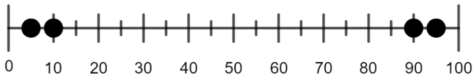

Figure 15.1: Peak Distribution 1

In this symmetric peak distribution, all peaks take on rather extreme values. Therefore, abstention comes with the risk that the outcome is very far away from one's peak and thus we suspect high participation rates.

Peak distribution 2

The second peak distribution consists of the peaks 5, 30, 70, 95 and the unique equilibrium is again full participation.

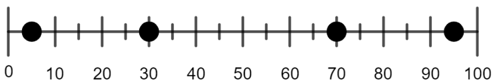

Figure 15.2: Peak Distribution 2

In comparison to Peak Distribution 1, two votes are much more centralized, but the equilibrium and the equilibrium strategies remain unchanged. This peak distribution together with Peak Distribution 1 aims to identify whether there are different participation rates for moderate and extreme voters, even if the equilibrium considerations are identical.

Peak Distribution 3

The third peak distribution consists of the peaks 25, 30, 70 and 75. Once more, the unique equilibrium is full participation.

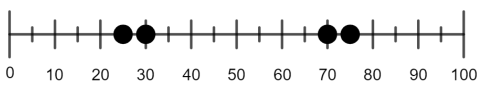

Figure 15.3: Peak Distribution 3

In comparison to Peak Distribution 2, another two peaks are more centralized which might make abstention a potential consideration for all individuals. Again, the equilibrium and its strategies are identical to both previous peak distributions.

Peak Distribution 4

The fourth peak distribution is the first asymmetric peak distribution and consists of the peaks 5, 10, 65 and 70.

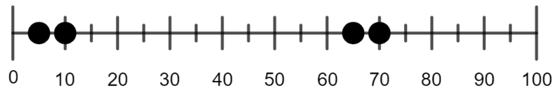

Figure 15.4: Peak Distribution 4

The unique equilibrium is full participation. Here, there are only rather extreme peaks in the lower half, whereas the peaks in the upper half are rather centralized. Abstention of the individuals with peaks in the lower half (starting from full participation) will lead to a smaller shift in outcome, than abstention of the individuals with peak in the upper half. Hence, even though abstention is not profitable for anybody, we expect that, if there is a difference in participation levels, individuals with the lower peaks are more likely to abstain.

Peak Distribution 5

The fifth peak distribution consists of the peaks 5, 10, 65 and 95.

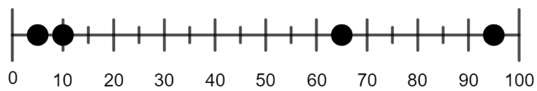

Figure 15.5: Peak Distribution 5

This asymmetric peak distribution's unique Nash equilibrium is full participation, too. All strategic considerations are identical here to those in Peak Distribution 4, at least those starting at full participation. Hence, one would not expect a different behavior, even though one vote is more extreme than before. However, if the individuals with a lower peak show a smaller level of participation, this would oppose the intuition of Osborne et al. (2000).

Peak Distribution 6

The next peak distribution consists of the peaks 5, 35, 90 and 95 and is identical to Peak Distribution 5 but reversed.

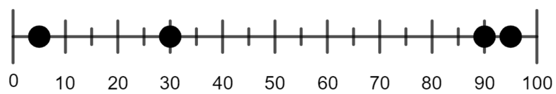

Figure 15.6: Peak Distribution 6

We add this peak distribution, as in the pilot the participants treated higher peaks differently to their symmetric counterparts. This peak distribution is meant to deliver insights on whether there is a systematic difference in the treatment of peaks in the upper and lower halves.

Peak Distribution 7

Next is a peak distribution with peaks 5, 7, 10 and 25.

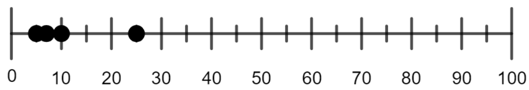

Figure 15.7: Peak Distribution 7

This peak distribution is the first peak distribution, in which full participation is not an equilibrium; the unique Nash equilibrium is single participation of the individual with the highest peak. This peak distribution is ideal to check whether individuals are able to identify the optimal strategies of all individuals for all possible levels of participation, as in all cases the individual with the highest peak can obtain her peak.[15]

Peak Distribution 8

The eighth and last peak distribution consists of the peaks 5, 25, 30 and 45 and the unique equilibrium is again single participation of the individual with the highest peak.

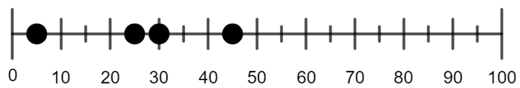

Figure 15.8: Peak Distribution 8

Here, the strategic considerations are more complex. As argued in section 13, in all cases there exists one individual whose peak is chosen, given 'optimal' play of the others for the median rule. But, in contrast to Peak Distribution 6 there is no individual who always takes on this role, given her participation. If we consider full participation, then the individual with peak 30 gets her peak as the outcome. Note that for the mean rule, the outcome in equilibrium, given fixed participation decisions might differ from the peaks of the participants, e.g. if exactly the individuals with peaks 5, 25 and 45 participate. In this case, the outcome is $33.\overline{3}$, which nevertheless is not an equilibrium for $c = 10$ as the cost exceed the minimal distance to the next peak.

[15] To be precise, if one takes into account that individuals might play suboptimal strategies, the individual with peak 25 can assure the outcome to be at or greater than 25.

15.3 Hypotheses

For the experiment we reconsider the hypotheses of the pilot experiment. Moreover, using the insights from the pilot experiment, we add two more hypotheses (Hypothesis H2 and Hypothesis H5) .

Hypothesis (H1). The individuals play Nash equilibrium strategies on the second stage of the game.

We expect the participants to discover and play the strategies that are part of the unique (strong) Nash equilibrium. We reduced the group size in comparison to the pilot. With that, the number of participants should also decline compared to the pilot experiment. As the complexity of the game is very low for 1 and 2 participants and as participants were able to identify the Nash equilibrium for three participant at least for the (symmetric) median rule in the pilot, we expect that most participants detect and play the (strong) Nash equilibrium strategies.

Hypothesis (H2). The individuals play the Nash equilibrium strategies of the entire game.

In the pilot experiment we found that the individuals were unable to identify the Nash equilibria on the second stage of the game, with the exception of the symmetric median rule with an odd number of participants. This implied that they were also unable to find the Nash equilibria of the entire game. As we reduced the group size and with it the complexity of the game and its equilibria, we are interested in seeing if the individuals are able to identify and play the Nash equilibrium strategies of the entire game.

Hypothesis (H3). The individuals play Nash equilibrium strategies of the entire game more often, when it involves participation.

In the pilot we found that the individuals played the Nash equilibrium strategy more often, when the equilibrium was a full participation equilibrium. However, as the participation levels were almost the same for all peak distributions it seems that - in the pilot - the equilibrium structure was not the driving force for this observation. With the reduced complexity of the game, we hope for an improved understanding of the Nash equilibria of the entire game and are interested in observing its impact. We still expect that the factors that drove participation in the pilot play an important role here. We try to understand the motives better by putting a greater emphasis on the reasons of participation in the questionnaire.

Hypothesis (H4). The rules themselves do not influence whether Nash equilibrium strategies are played.

In the pilot experiment, there was no significant difference between the two rules. This might be influenced by the insufficient understanding of the rules by some of the individuals. By implementing a control question and with the group-size induced reduction in complexity, we hope to be able to more clearly identify the potential impact of the rule. As the peak distributions are chosen to be strategically equivalent, we suspect that the rule does not play an important role, whether individuals play the Nash equilibrium on the second stage and the equilibrium of the entire game. Driving factors that could lead to different behavior for both rules are that the mean rule is better known to the individuals and the different potential impact on the outcome (see Rollmann (2020) for an analysis on the different impact of the mean and the symmetric median rule).

Hypothesis (H5). The participation level per peak is lower for more extreme peaks.

We observed in the pilot experiment in peak distributions 1 and 2 that the participation levels per peak take the form of an inverse U-shape. We only found significant difference for the mean rule in one peak distribution, but this might be due to the low number of observations. We are interested to see if this structure remains visible in an experiment with a larger scale. While we do not have a theoretical explanation based on strategic considerations, this finding is in contrast to the intuition put forward by Osborne et al. (2000) that extreme voters are more likely to participate.

16 Conclusion

We classified all subgame-perfect Nash equilibria such that a strong Nash equilibrium is played in the second stage for the mean and the symmetric median rule. Next, we presented a laboratory experiment that is designed to test whether individuals are able to identify the Nash equilibrium strategies and whether they actually play them. Moreover, the experiment is examining whether the mean and the symmetric median rule influence participation behavior. In order to be able to identify this influence, we consider several peak distributions in which there exists a unique Nash equilibrium that is strategically identical for both rules. Due to the COVID-19 pandemic, the experiment itself could not yet be conducted. Therefore, we can only present the results of the pilot experiment. Due to the small sample size, and due to the complex strategic setting, we do not find significant results. The only significant result (on a relatively low significance level) is that the observed inverse U-shaped distribution of participation levels of the mean rule for one peak distributions differ from a uniform distribution. Interestingly, this finding - while without theoretical backup - opposes the intuition of Osborne et al. (2000) that extreme voters are more likely to participate.

In order to obtain clearer and hopefully significant results, we present some changes on the design to reduce the complexity of the game. Moreover, we hope that an increase in the number of observations paints a better picture.

Appendix

A Pilot Experiment

A.1 Instructions

A.1.1 Mean Rule Treatment

Anleitung

Herzlich Willkommen zum Experiment!

Dieses Experiment erfolgt anonym, keiner der anderen Teilnehmer erhält Informationen über Ihre Identität. In diesem Experiment können Sie Geld verdienen, welches Ihnen im Anschluss an das Experiment bar ausgezahlt wird.

Sie erhalten eine **Grundauszahlung von 2,00 €**. Zusätzlich zu diesem Betrag können Sie weiteres Geld verdienen. Dabei ist Ihre spätere Auszahlung von Ihren eigenen, sowie den Entscheidungen Ihrer Mitspieler abhängig. Das Experiment besteht aus drei Runden. Jede Runde besteht aus einer Abstimmung mit einer vorgeschalteten Teilnahmeentscheidung. In jeder Runde entsteht durch die Berechnung des Durchschnitts der abgegebenen Stimmen ein Ergebnis. Sie haben in jeder Runde die Möglichkeit **Punkte** zu gewinnen. Für die spätere Auszahlung wird die gewonnene Punktzahl einer der drei Runden zufällig ausgewählt, in Geldeinheiten (Euro) umgerechnet und Ihnen ausgezahlt.

Für die Auszahlung gilt folgende Umrechnung: **10 Punkte = 1,00 €**.

In Ihrer Kabine befindet sich ein leeres Blatt Papier, auf dem Sie sich gerne Notizen machen dürfen. Falls Sie während des Experiments Fragen haben, öffnen Sie bitte Ihre Kabinentür. Bitte bleiben Sie während des gesamten Experiments auf Ihrem Platz sitzen.

Spielidee

Das Spiel wird **3 Runden** in Folge gespielt. Dabei ist jede Runde komplett unabhängig von den anderen Runden. Keine Entscheidung in einer Runde hat also einen Einfluss auf eine andere der drei Runden. In diesem Spiel spielen Sie immer in einer Gruppe von **6 Personen**. Eine Gruppe setzt sich in jeder Runde aus einem Pool von zwölf Teilnehmern neu zusammen. Am Ende jeder Runde kommt dabei durch die getroffenen Entscheidungen aller Gruppenmitglieder ein Ergebnis zustande. Dieses Ergebnis ist ein Wert in dem Intervall $[0, 100]$. Zur Berechnung des Ergebnisses wird der **Durchschnitt** aller abgegebenen Stimmen berechnet.

Bevor Sie Ihre Stimme abgeben, haben Sie die Möglichkeit zu entscheiden, ob Sie an der Stimmabgabe teilnehmen wollen. Während des Spiels erhalten Sie einen ganzzahligen Wert im Intervall $[0, 100]$, dieser Wert stellt Ihr persönlich präferiertes Ergebnis (Präferenz) dar. Wie viele Punkte Sie erreichen hängt davon ab, wie nah das allgemeine Ergebnis Ihrer Gruppe an Ihrem präferiertem Ergebnis liegt.

Auszahlung

Sie erhalten am Ende jeder Runde eine Punktzahl. Die Anzahl der Punkte, die Sie in einer Runde gewinnen, kann dabei nicht negativ werden. Am Ende des Experiments, wenn Sie Ihre Auszahlung in Bargeld erhalten, wird dabei eine der drei Runden zufällig ausgewählt und die erspielten Punkte in Bargeld umgerechnet. Ihre erreichte Punktzahl wird in jeder Runde auf einen ganzzahligen Wert gerundet.
Für eine einzelne Runde im Spiel setzt sich die Auszahlung wie folgt zusammen:

Punktzahl = Grundanzahl - Abweichung - Teilnahmekosten

Sie erhalten im Voraus eine Grundanzahl von **110 Punkten**. Von diesen Punkten wird der absolute Betrag der **Abweichung Ihrer Präferenz zum Ergebnis** abgezogen. Außerdem werden Ihnen die **Teilnahmekosten in Höhe von 9 Punkten** abgezogen, falls Sie teilgenommen haben. Falls Sie nicht teilnehmen, werden Ihnen keine Teilnahmekosten abgezogen, eine Stimmabgabe wäre Ihnen aber somit verwehrt. Ihre Punktzahl berechnet sich dann allein aus der Differenz von Grundanzahl und Abweichung zum Ergebnis.

Achtung! Nimmt in einer Runde kein Spieler teil, erhalten alle Spieler in dieser Runde die gleiche Punktzahl in Höhe von **0 Punkten**!

Ablauf

Teilnahmeentscheidung

Auf der ersten Stufe erhalten Sie einen Zettel mit einer Tabelle, in der alle sechs Präferenzen aufgelistet sind, die es in dieser Runde in Ihrer Gruppe gibt. Jede Präferenz wird genau einem Spieler der Gruppe zugeteilt. Bitte geben Sie für jede Präferenz an, ob Sie teilnehmen wollen, gegeben der Annahme, dass Sie diese Präferenz im nächsten Schritt zugeteilt bekommen. Dabei können Sie sich für jede einzelne Präferenz neu entscheiden, ob Sie teilnehmen wollen. **Nach** Ihrer Entscheidung wird Ihnen eine dieser Präferenzen zugeteilt und je nachdem, ob Sie sich für diese Ihnen zugeteilte Präferenz für oder gegen eine Teilnahme entschieden haben, können Sie im nächsten Schritt teilnehmen, oder eben nicht.
Diese Tabelle dient als Veranschaulichung:

Spieler	Präferenz	Teilnahme	
1	10	Ja	Nein
2	25	Ja	Nein
3	50	Ja	Nein
4	60	Ja	Nein
5	80	Ja	Nein
6	95	Ja	Nein

Gleich im Experiment kreisen Sie bitte für die Teilnahmeentscheidung die Wörter „**Ja**“ oder „**Nein**“ ein, um kenntlich zu machen, ob Sie teilnehmen wollen, oder nicht. Wenn Sie Ihre Entscheidung(en) getroffen haben, öffnen Sie bitte Ihre Kabinentür. Wir werden den Zettel dann einsammeln.

Stimmabgabe

Auf der zweiten Stufe des Experiments erhalten Sie einen Zettel mit einer Tabelle. Aus dieser Tabelle erhalten Sie die Information, welcher Spieler, und somit auch welche Präferenz, Ihnen für diese Runde zugeteilt wurde. Ihre Rolle entspricht dabei der **grau hinterlegten Zeile**. Es wird Ihnen mitgeteilt, ob Sie sich, gegeben dieser Präferenz, für oder gegen eine Teilnahme entschieden haben. Je nachdem, ob Sie sich für oder gegen eine Teilnahme entschieden haben, nehmen Sie nun teil, oder eben nicht. Außerdem erhalten Sie die Information, welche anderen Spieler, und somit auch welche anderen Präferenzen, sich für eine Teilnahme entschieden haben. Bei den teilnehmenden Spielern ist in der Spalte der Teilnahmeentscheidung das Wort „**Ja**“ eingetragen. Bei den Spielern, die nicht teilnehmen, das Wort „**Nein**“.

Spieler	**Präferenz**	**Teilnahme**
1	10	Nein
2	25	Ja
3	50	Ja
4	60	Nein
5	80	Ja
6	95	Ja

In diesem Beispiel übernehmen Sie die Rolle des Spielers **5**. Somit beträgt Ihre Präferenz „**80**“ und Sie haben sich dafür entschieden teilzunehmen. Die Spieler mit den Präferenzen „10“ und „60“ dürfen in diesem Beispiel nicht mehr abstimmen.

Unter der Tabelle befinden sich zwei leere Felder.
In das ersten Feld „**Stimmabgabe**“ tragen Sie bitte einen **ganzzahligen Wert im Intervall [0,100]** ein, dieser Wert entspricht Ihrer Stimmabgabe und wird in die Berechnung des Ergebnisses Ihrer Gruppe in dieser Runde einbezogen. Falls Sie nicht teilnehmen, lassen Sie das Feld bitte frei.

In das zweite Feld „**Einschätzung des Ergebnisses**“ tragen Sie bitte einen **Wert im Intervall [0,100]** ein. Dieser Wert soll Ihrer persönlichen Einschätzung entsprechen, welchen Wert das Ergebnis Ihrer Gruppe in dieser Runde annehmen wird. Dieses Feld füllen Sie bitte auch dann aus, wenn Sie nicht teilnehmen.

Wenn Sie Ihre Entscheidung(en) getroffen haben, öffnen Sie bitte Ihre Kabinentür. Wir werden den Zettel dann einsammeln.

Ergebnis

Im letzten Schritt erhalten Sie einen Zettel, der Ihnen die Resultate der Runde mitteilt. Als Information erhalten Sie nach jeder Runde zur Übersicht:

1. Ihre persönliche Präferenz
2. Ihre Teilnahmeentscheidung
3. Ihre Stimmabgabe
4. Ergebnis Ihrer Gruppe
5. Ihre persönlich erreichte Punktzahl dieser Runde

Bevor Ihnen der neue Zettel zur Teilnahmeentscheidung für die nächste Runde ausgeteilt wird, sammeln wir den Zettel mit dem Ergebnis wieder ein.

Beispiele

1. **Berechnung Ergebnis**
   Angenommen es nehmen vier Spieler einer Gruppe teil, die in diesem Beispiel folgende Werte als Stimmabgabe abgegeben haben: 20, 50, 90, 100

   $\text{Ergebnis} = \frac{20+50+90+100}{4} = 65$

2. **Auszahlung**
   Ihre Präferenz ist beispielhaft der Wert „50". Falls das Ergebnis 65 beträgt und Sie teilgenommen haben, erhalten Sie folgende Punktzahl:

   $\text{Punktzahl} = 110 - (65 - 50) - 9 = 86.$

   Für diese Runde erhalten Sie 86 Punkte. Zuerst werden Ihnen 15 Punkte abgezogen, da Ihre Präferenz („50") um 15 von dem Ergebnis („65") Ihrer Gruppe abweicht, zusätzlich werden Ihnen die Teilnahmekosten in Höhe von 9 Punkten abgezogen.

Fragebogen

Zum Abschluss erhalten Sie einen Fragebogen, den Sie bitte sorgfältig, wahrheitsgemäß und in Ruhe ausfüllen. Wenn alle Teilnehmer den Fragebogen vollständig ausgefüllt haben, ist das Experiment beendet und Sie werden nacheinander zur Auszahlung gebeten.

Bitte schalten Sie Ihr Mobiltelefon für die Dauer des Experiments aus.
Vielen Dank für Ihre Teilnahme am Experiment!

A.1.2 Symmetric Median Rule Treatment

Anleitung

Herzlich Willkommen zum Experiment!

Dieses Experiment erfolgt anonym, keiner der anderen Teilnehmer erhält Informationen über Ihre Identität. In diesem Experiment können Sie Geld verdienen, welches Ihnen im Anschluss an das Experiment bar ausgezahlt wird.

Sie erhalten eine **Grundauszahlung von 2,00 €**. Zusätzlich zu diesem Betrag können Sie weiteres Geld verdienen. Dabei ist Ihre spätere Auszahlung von Ihren eigenen, sowie den Entscheidungen Ihrer Mitspieler abhängig. Das Experiment besteht aus drei Runden. Jede Runde besteht aus einer Abstimmung mit einer vorgeschalteten Teilnahmeentscheidung. In jeder Runde entsteht durch die Berechnung des Medians der abgegebenen Stimmen ein Ergebnis. Sie haben in jeder Runde die Möglichkeit **Punkte** zu gewinnen. Für die spätere Auszahlung wird die gewonnene Punktzahl einer der drei Runden zufällig ausgewählt, in Geldeinheiten (Euro) umgerechnet und Ihnen ausgezahlt.

Für die Auszahlung gilt folgende Umrechnung: **10 Punkte = 1,00 €**.

In Ihrer Kabine befindet sich ein leeres Blatt Papier, auf dem Sie sich gerne Notizen machen dürfen. Falls Sie während des Experiments Fragen haben, öffnen Sie bitte Ihre Kabinentür. Bitte bleiben Sie während des gesamten Experiments auf Ihrem Platz sitzen.

Spielidee

Das Spiel wird **3 Runden** in Folge gespielt. Dabei ist jede Runde komplett unabhängig von den anderen Runden. Keine Entscheidung in einer Runde hat also einen Einfluss auf eine andere der drei Runden. In diesem Spiel spielen Sie immer in einer Gruppe von **6 Personen**. Eine Gruppe setzt sich in jeder Runde aus einem Pool von zwölf Teilnehmern neu zusammen. Am Ende jeder Runde kommt dabei durch die getroffenen Entscheidungen aller Gruppenmitglieder ein Ergebnis zustande. Dieses Ergebnis ist ein Wert in dem Intervall $[0, 100]$. Zur Berechnung des Ergebnisses wird der **Median** aller abgegebenen Stimmen berechnet. Der Median ist der mittlere Wert einer geordneten Datenreihe. Bei einer geraden Anzahl von Werten ist der Median der Durchschnitt der beiden mittleren Werte (einer geordneten Datenreihe). Beispiele zur Berechnung des Medians finden Sie am Ende dieser Anleitung.

Bevor Sie Ihre Stimme abgeben, haben Sie die Möglichkeit zu entscheiden, ob Sie an der Stimmabgabe teilnehmen wollen. Während des Spiels erhalten Sie einen ganzzahligen Wert im Intervall $[0, 100]$, dieser Wert stellt Ihr persönlich präferiertes Ergebnis (Präferenz) dar. Wie viele Punkte Sie erreichen hängt davon ab, wie nah das allgemeine Ergebnis Ihrer Gruppe an Ihrem präferiertem Ergebnis liegt.

Auszahlung

Sie erhalten am Ende jeder Runde eine Punktzahl. Die Anzahl der Punkte, die Sie in einer Runde gewinnen, kann dabei nicht negativ werden. Am Ende des Experiments, wenn Sie Ihre Auszahlung in Bargeld erhalten, wird dabei eine der drei Runden zufällig ausgewählt und die erspielten Punkte in Bargeld umgerechnet. Ihre erreichte Punktzahl wird in jeder Runde auf einen ganzzahligen Wert gerundet.
Für eine einzelne Runde im Spiel setzt sich die Auszahlung wie folgt zusammen:

Punktzahl = Grundanzahl - Abweichung - Teilnahmekosten

Sie erhalten im Voraus eine Grundanzahl von **110 Punkten**. Von diesen Punkten wird der absolute Betrag der **Abweichung Ihrer Präferenz zum Ergebnis** abgezogen. Außerdem werden Ihnen die **Teilnahmekosten in Höhe von 9 Punkten** abgezogen, falls Sie teilgenommen haben. Falls Sie nicht teilnehmen, werden Ihnen keine Teilnahmekosten abgezogen, eine Stimmabgabe wäre Ihnen aber somit verwehrt. Ihre Punktzahl berechnet sich dann allein aus der Differenz von Grundanzahl und Abweichung zum Ergebnis.

Achtung! Nimmt in einer Runde kein Spieler teil, erhalten alle Spieler in dieser Runde die gleiche Punktzahl in Höhe von **0 Punkten**!

Ablauf

Teilnahmeentscheidung

Auf der ersten Stufe erhalten Sie einen Zettel mit einer Tabelle, in der alle sechs Präferenzen aufgelistet sind, die es in dieser Runde in Ihrer Gruppe gibt. Jede Präferenz wird genau einem Spieler der Gruppe zugeteilt. Bitte geben Sie für jede Präferenz an, ob Sie teilnehmen wollen, gegeben der Annahme, dass Sie diese Präferenz im nächsten Schritt zugeteilt bekommen. Dabei können Sie sich für jede einzelne Präferenz neu entscheiden, ob Sie teilnehmen wollen. **Nach** Ihrer Entscheidung wird Ihnen eine dieser Präferenzen zugeteilt und je nachdem, ob Sie sich für diese Ihnen zugeteilte Präferenz für oder gegen eine Teilnahme entschieden haben, können Sie im nächsten Schritt teilnehmen, oder eben nicht.
Diese Tabelle dient als Veranschaulichung:

Spieler	Präferenz	Teilnahme	
1	10	Ja	Nein
2	25	Ja	Nein
3	50	Ja	Nein
4	60	Ja	Nein
5	80	Ja	Nein
6	95	Ja	Nein

Gleich im Experiment kreisen Sie bitte für die Teilnahmeentscheidung die Wörter „**Ja**“ oder „**Nein**“ ein, um kenntlich zu machen, ob Sie teilnehmen wollen, oder nicht. Wenn Sie Ihre Entscheidung(en) getroffen haben, öffnen Sie bitte Ihre Kabinentür. Wir werden den Zettel dann einsammeln.

Stimmabgabe

Auf der zweiten Stufe des Experiments erhalten Sie einen Zettel mit einer Tabelle. Aus dieser Tabelle erhalten Sie die Information, welcher Spieler, und somit auch welche Präferenz, Ihnen für diese Runde zugeteilt wurde. Ihre Rolle entspricht dabei der **grau hinterlegten Zeile**. Es wird Ihnen mitgeteilt, ob Sie sich, gegeben dieser Präferenz, für oder gegen eine Teilnahme entschieden haben. Je nachdem, ob Sie sich für oder gegen eine Teilnahme entschieden haben, nehmen Sie nun teil, oder eben nicht. Außerdem erfahren Sie, welche anderen Spieler, und somit auch welche anderen Präferenzen, sich für eine Teilnahme entschieden haben. Bei den teilnehmenden Spielern ist in der Spalte der Teilnahmeentscheidung das Wort „**Ja**“ eingetragen. Bei den Spielern, die nicht teilnehmen, das Wort „**Nein**“.

Spieler	Präferenz	Teilnahme
1	10	Nein
2	25	Ja
3	50	Ja
4	60	Nein
5	80	Ja
6	95	Ja

In diesem Beispiel übernehmen Sie die Rolle des Spielers **5**. Somit beträgt Ihre Präferenz „**80**“ und Sie haben sich dafür entschieden teilzunehmen. Die Spieler mit den Präferenzen „10“ und „60“ dürfen in diesem Beispiel nicht mehr abstimmen.

Unter der Tabelle befinden sich zwei leere Felder. In das ersten Feld „**Stimmabgabe**“ tragen Sie bitte einen **ganzzahligen Wert im Intervall [0,100]** ein, dieser Wert entspricht Ihrer Stimmabgabe und wird in die Berechnung des Ergebnisses Ihrer Gruppe in dieser Runde einbezogen. Falls Sie nicht teilnehmen, lassen Sie das Feld bitte frei.
In das zweite Feld „**Einschätzung des Ergebnisses**“ tragen Sie bitte einen **Wert im Intervall [0,100]** ein. Dieser Wert soll Ihrer persönlichen Einschätzung entsprechen, welchen Wert das Ergebnis Ihrer Gruppe in dieser Runde annehmen wird. Dieses Feld füllen Sie bitte auch dann aus, wenn Sie nicht teilnehmen.

Wenn Sie Ihre Entscheidung(en) getroffen haben, öffnen Sie bitte Ihre Kabinentür. Wir werden den Zettel dann einsammeln.

Ergebnis

Im letzten Schritt erhalten Sie einen Zettel, der Ihnen die Resultate der Runde mitteilt. Als Information erhalten Sie nach jeder Runde zur Übersicht:

1. Ihre persönliche Präferenz
2. Ihre Teilnahmeentscheidung
3. Ihre Stimmabgabe
4. Ergebnis Ihrer Gruppe
5. Ihre persönlich erreichte Punktzahl dieser Runde

Bevor Ihnen der neue Zettel zur Teilnahmeentscheidung für die nächste Runde ausgeteilt wird, sammeln wir den Zettel mit dem Ergebnis wieder ein.

Beispiele

1. **Berechnung Ergebnis bei ungerader Anzahl Teilnehmer**
   Angenommen es nehmen fünf Spieler einer Gruppe teil, die in diesem Beispiel folgende Werte als Stimmabgabe abgegeben haben: 20, 50, 60, 90, 100

   Ergebnis $= 60$
   Der Median (mittlere Stimmabgabe) ist in diesem Fall genau der mittlere Wert.

2. **Berechnung Ergebnis bei gerade Anzahl Teilnehmer**
   Angenommen es nehmen vier Spieler einer Gruppe teil, die in diesem Beispiel folgende Werte als Stimmabgabe abgegeben haben: 20, 50, 90, 100

   Ergebnis $= \frac{50+90}{2} = 70$
   Der Median (mittlere Stimmabgabe) ist in diesem Fall der Durchschnitt der beiden mittleren Werte, da es keine alleinige mittlere Stimmabgabe gibt.

3. **Auszahlung**
   Ihre Präferenz ist beispielhaft der Wert „50“. Falls das Ergebnis 60 beträgt und Sie teilgenommen haben, erhalten Sie folgende Punktzahl:

   Punktzahl $= 110 - (60 - 50) - 9 = 91.$

   Für diese Runde erhalten Sie 91 Punkte. Zuerst werden Ihnen 10 Punkte abgezogen, da Ihre Präferenz („50“) um 10 von dem Ergebnis („60“) Ihrer Gruppe abweicht, zusätzlich werden Ihnen die Teilnahmekosten in Höhe von 9 Punkten abgezogen.

Fragebogen

Zum Abschluss erhalten Sie einen Fragebogen, den Sie bitte sorgfältig, wahrheitsgemäß und in Ruhe ausfüllen. Wenn alle Teilnehmer den Fragebogen vollständig ausgefüllt haben, ist das Experiment beendet und Sie werden nacheinander zur Auszahlung gebeten.

Bitte schalten Sie Ihr Mobiltelefon für die Dauer des Experiments aus.
Vielen Dank für Ihre Teilnahme am Experiment!

A.2 Pen-and-Paper Documents

Following we show how the documents of the pen-and-paper experiment look like, for the example of Round 1 and the individual in booth A. The first document was used to take the participation decision, the second document to cast the vote (given a positive participation decision for the assigned peak) and the third document was used to inform the individual about the result.

Teilnahmeentscheidung

1. Runde - Kabine A

In dieser Tabelle sind sechs Spieler mit sechs verschiedenen Präferenzen (präferiertes Ergebnis) aufgelistet. Diese sechs Präferenzen sind die Präferenzen dieser Runde. Im nächsten Schritt übernehmen Sie die Rolle eines dieser Spieler und somit auch seine Präferenz. Bitte geben Sie in der Tabelle in der Spalte „Teilnahme" an, ob Sie für die gegebene Präferenz des Spielers teilnehmen wollen, oder nicht. Für eine Teilnahme kreisen Sie bitte „Ja" ein, für Nicht-Teilnahme bitte „Nein" einkreisen.

Spieler	Präferenz	Teilnahme	
1	30	Ja	Nein
2	40	Ja	Nein
3	55	Ja	Nein
4	60	Ja	Nein
5	65	Ja	Nein
6	70	Ja	Nein

Wenn Sie Ihre Entscheidung(en) getroffen haben, öffnen Sie bitte Ihre Kabinentür. Wir werden den Zettel dann einsammeln.

Stimmabgabe

1. Runde - Kabine A

Sie nehmen in dieser Runde die Rolle des Spielers ein, dessen Felder grau hinterlegt sind. Die grau hinterlegte Präferenz entspricht also Ihrem präferierten Ergebnis. In der Spalte „Teilnahme“ erfahren Sie, ob Sie teilnehmen („Ja“), oder ob Sie nicht teilnehmen („Nein“). Ebenso können Sie aus der Tabelle ablesen, welche Spieler mit welchen Präferenzen sich für eine Teilnahme entschieden haben.

Spieler	Präferenz	Teilnahme
1	30	
2	40	
3	55	
4	60	
5	65	
6	70	

Bitte geben Sie im Feld „Stimmabgabe“ Ihre Stimmabgabe ein, falls Sie teilnehmen. Im Falle einer Nicht-Teilnahme lassen Sie das Feld bitte frei. Bitte geben Sie in dem Feld „Einschätzung des Ergebnisses“ eine Prognose darüber ab, was das Ergebnis dieser Runde ist, egal ob Sie teilgenommen haben, oder nicht.

Stimmabgabe:

(**ganze** Zahl im Intervall [0,100])

Einschätzung des Ergebnisses:

Wenn Sie Ihre Entscheidung(en) getroffen haben, öffnen Sie bitte Ihre Kabinentür. Wir werden den Zettel dann einsammeln.

Auswertung

1. Runde - Kabine A

In dieser Tabelle finden Sie die Auswertung dieser Runde.

Präferenz	Teilnahme	Stimmabgabe	Ergebnis

Persönlich erreichte Punktzahl:

A.3 Questionnaire

Fragebogen (Teil 1)

Sie nehmen nun an einer Lotterie teil. Bei dieser Lotterie können Sie zusätzliches Geld verdienen. Es werden Ihnen insgesamt 10 Situationen vorgestellt. In jeder Situation haben Sie die Möglichkeit entweder die „**Lotterie A**“ oder die „**Lotterie B**“ auszuwählen. Während die zu gewinnende Summe an Geld in den beiden Lotterien unverändert bleibt, ändern sich die Wahrscheinlichkeiten von Situation zu Situation.

Zum **Beispiel** liegt in der **Situation 4** bei der Lotterie A die Wahrscheinlichkeit, dass Sie 1,00 € gewinnen, bei 40%. Die Wahrscheinlichkeit, dass Sie 0,80 € erhalten, beträgt 60%. Entscheiden Sie sich in Situation 4 für die Lotterie B, haben Sie eine Chance von 40% auf 2,00 € und eine Chance von 60% auf 0,10 €.

Sie müssen sich nun entscheiden an welcher Lotterie Sie für welche Situation teilnehmen wollen. Nachdem Sie für alle Situationen Ihre Entscheidung getroffen haben wird eine der 10 Situationen zufällig ausgewählt. Abhängig von Ihrer Entscheidung wird entweder die Lotterie A oder die Lotterie B gespielt. Entsprechend der Wahrscheinlichkeiten der Situation wird Ihre Gewinnsumme gelost. Bitte tragen Sie in der Spalte „Entscheidung“ ein, ob Sie sich für die gegebene Situation für die Lotterie A oder B entscheiden, indem Sie entweder „A“ oder „B“ eintragen.

Situation	Lotterie A	Lotterie B	Entscheidung
1	mit 10% Gewinn von 1,00 € mit 90% Gewinn von 0,80 €	mit 10% Gewinn von 2,00 € mit 90% Gewinn von 0,10 €	
2	mit 20% Gewinn von 1,00 € mit 80% Gewinn von 0,80 €	mit 20% Gewinn von 2,00 € mit 80% Gewinn von 0,10 €	
3	mit 30% Gewinn von 1,00 € mit 70% Gewinn von 0,80 €	mit 30% Gewinn von 2,00 € mit 70% Gewinn von 0,10 €	
4	mit 40% Gewinn von 1,00 € mit 60% Gewinn von 0,80 €	mit 40% Gewinn von 2,00 € mit 60% Gewinn von 0,10 €	
5	mit 50% Gewinn von 1,00 € mit 50% Gewinn von 0,80 €	mit 50% Gewinn von 2,00 € mit 50% Gewinn von 0,10 €	
6	mit 60% Gewinn von 1,00 € mit 40% Gewinn von 0,80 €	mit 60% Gewinn von 2,00 € mit 40% Gewinn von 0,10 €	
7	mit 70% Gewinn von 1,00 € mit 30% Gewinn von 0,80 €	mit 70% Gewinn von 2,00 € mit 30% Gewinn von 0,10 €	
8	mit 80% Gewinn von 1,00 € mit 20% Gewinn von 0,80 €	mit 80% Gewinn von 2,00 € mit 20% Gewinn von 0,10 €	
9	mit 90% Gewinn von 1,00 € mit 10% Gewinn von 0,80 €	mit 90% Gewinn von 2,00 € mit 10% Gewinn von 0,10 €	
10	mit 100% Gewinn von 1,00 € mit 0% Gewinn von 0,80 €	mit 100% Gewinn von 2,00 € mit 0% Gewinn von 0,10 €	

Fragebogen (Teil 2)

Zum Abschluss des Experiments bitten wir Sie den folgenden Fragebogen auszufüllen. Antworten Sie bitte sorgfältig und wahrheitsgemäß, vielen Dank!

Fragen zum Experiment

1. *Haben Sie sich verpflichtet gefühlt teilzunehmen?*

☐ ja ☐ eher ja ☐ teils-teils ☐ eher nein ☐ nein

2. *Hatten Sie das Gefühl Macht zu besitzen?*

☐ ja ☐ eher ja ☐ teils-teils ☐ eher nein ☐ nein

3. *Hatten Sie das Gefühl benachteiligt zu sein?*

☐ ja ☐ eher ja ☐ teils-teils ☐ eher nein ☐ nein

4. *Haben Sie befürchtet, dass kein Spieler teilnimmt?*

☐ ja ☐ eher ja ☐ teils-teils ☐ eher nein ☐ nein

5. *Haben Sie sich über Ihre Mitspieler geärgert?*

☐ ja ☐ eher ja ☐ teils-teils ☐ eher nein ☐ nein

6. *Hatten Sie das Gefühl das Ergebnis beeinflussen zu können?*

☐ ja ☐ eher ja ☐ teils-teils ☐ eher nein ☐ nein

7. *Bitte kreuzen Sie „eher nein“ an?*

☐ ja ☐ eher ja ☐ teils-teils ☐ eher nein ☐ nein

8. *Hätten Sie Ihre Stimmabgabe im Nachhinein gerne geändert?*

☐ ja ☐ eher ja ☐ teils-teils ☐ eher nein ☐ nein

9. *Hatten Sie das Gefühl Ihrer Präferenz Gewicht verleihen zu müssen?*

☐ ja ☐ eher ja ☐ teils-teils ☐ eher nein ☐ nein

10. *Haben Sie Ihren Mitspielern vertraut?*

☐ ja ☐ eher ja ☐ teils-teils ☐ eher nein ☐ nein

Die beiden folgenden Fragen dürfen Sie gerne ausführlich beantworten.

1. *Was waren Ihre Beweggründe für oder gegen eine Teilnahme?*

2. *Weshalb haben Sie so abgestimmt, wie Sie abgestimmt haben?*

Persönliche Eigenschaften

1. Ich bin ein vorsichtiger Mensch.

☐ trifft zu ☐ trifft eher zu ☐ teils-teils ☐ trifft eher nicht zu ☐ trifft nicht zu

2. Auf mich kann man sich verlassen.

☐ trifft zu ☐ trifft eher zu ☐ teils-teils ☐ trifft eher nicht zu ☐ trifft nicht zu

3. Ich gehe gerne und oft Risiko ein.

☐ trifft zu ☐ trifft eher zu ☐ teils-teils ☐ trifft eher nicht zu ☐ trifft nicht zu

4. Ich zeige oft meine Gefühle und Emotionen.

☐ trifft zu ☐ trifft eher zu ☐ teils-teils ☐ trifft eher nicht zu ☐ trifft nicht zu

5. Meine Entscheidungen sind gut durchdacht.

☐ trifft zu ☐ trifft eher zu ☐ teils-teils ☐ trifft eher nicht zu ☐ trifft nicht zu

6. Ich bin ein ängstlicher Mensch.

☐ trifft zu ☐ trifft eher zu ☐ teils-teils ☐ trifft eher nicht zu ☐ trifft nicht zu

7. Ich würde mich als gelassen bezeichnen.

☐ trifft zu ☐ trifft eher zu ☐ teils-teils ☐ trifft eher nicht zu ☐ trifft nicht zu

8. In wichtigen Situationen entscheide ich rational.

☐ trifft zu ☐ trifft eher zu ☐ teils-teils ☐ trifft eher nicht zu ☐ trifft nicht zu

9. Sicherheit ist für mich ein entscheidender Faktor.

☐ trifft zu ☐ trifft eher zu ☐ teils-teils ☐ trifft eher nicht zu ☐ trifft nicht zu

10. Mich kann man leicht provozieren.

☐ trifft zu ☐ trifft eher zu ☐ teils-teils ☐ trifft eher nicht zu ☐ trifft nicht zu

11. Ich habe die Fragen wahrheitsgemäß beantwortet.

☐ Ja ☐ Nein

Sagt Ihnen ein Nash-Gleichgewicht etwas?

☐ Ja ☐ Nein

Wie gut würden Sie Ihre Vorkenntnisse in der Spieltheorie bewerten? ____________________

Wie viele spieltheoretische Experimente haben Sie ungefähr schon besucht? ______________

Fragen zur Demografie

Geschlecht ☐ männlich ☐ weiblich ☐ divers

Alter ____________

Höchster Bildungsabschluss

☐ Kein Schulabschluss
☐ Realschulabschluss
☐ Fachhochschulreife
☐ Hochschulreife/Abitur
☐ Berufsausbildung
☐ Bachelor
☐ Master
☐ Promotion
☐ Habilitation
☐ Sonstiges: ____________________________

Aktuelle Tätigkeit

☐ Schüler
☐ Student
☐ Berufstätig
☐ Arbeitssuchend
☐ Rente
☐ Sonstiges: ____________________________

Studiengang (falls Student) ____________________________

Platz für Anmerkungen und Fragen:

Wenn Sie den Fragebogen vollständig ausgefüllt haben, öffnen Sie bitte Ihre Kabinentür und bleiben Sie an Ihrem Platz sitzen, bis wir Sie zur Auszahlung abholen.

Vielen Dank für Ihre Teilnahme!

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