Incompressible Inhomogeneous Viscous Fluid Flows: Existence, Uniqueness and Regularity

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Abstract

This thesis is devoted to the study of the solvability and regularity problems for the motion of incompressible inhomogeneous viscous fluid flows in the presence of variable viscosity coefficients.

Chapter 2 is devoted to the existence and the regularity properties of (a class of) weak solutions to the two-dimensional *stationary* incompressible inhomogeneous Navier–Stokes equations with density-dependent viscosity coefficients. The three-dimensional case under special symmetry assumptions is also considered.

Chapter 3 proves the existence, uniqueness, and regularity results of the two-dimensional *evolutionary* incompressible Boussinesq equations with temperature-dependent thermal and viscosity diffusion coefficients in general Sobolev spaces.

In addition to the above results in the domain of fluid mechanics, we study the turbulence cascades for a two-parameter family of damped Szegő equations in Chapter 4.

Keywords

Fluid mechanics, Navier–Stokes equations, Boussinesq equations, solvability, uniqueness, regularity, variable viscosity coefficient, variable thermal diffusivity, Sobolev spaces, Szegő equation, turbulence cascade

Zusammenfassung

Diese Arbeit widmet sich der Untersuchung der Lösbarkeits- und Regularitätsprobleme für die Bewegung inkompressibler inhomogener viskoser Fluidströmungen in Gegenwart variabler Viskositätskoeffizienten.

Kapitel 2 widmet sich der Existenz und den Regularitätseigenschaften (einer Klasse) von schwachen Lösungen der zweidimensionalen *stationären* inkompressiblen inhomogenen Navier–Stokes-Gleichungen mit dichteabhängigen Viskositätskoeffizienten. Der dreidimensionale Fall unter speziellen symmetrischen Annahmen wird betrachtet.

Kapitel 3 beweist die Existenz-, Eindeutigkeits- und Regularitätsergebnisse der zweidimensionalen *evolutionären* inkompressiblen Boussinesq-Gleichungen mit temperaturabhängigen thermischen und Viskositäts-Diffusionskoeffizienten in allgemeinen Sobolev-Räumen.

Zusätzlich zu den obigen Ergebnissen im Bereich der Strömungsmechanik untersuchen wir in Kapitel 4 die Turbulenzkaskaden für eine zweiparametrige Familie gedämpfter Szegő-Gleichungen.

Schlüsselwörter

Strömungsmechanik, Navier–Stokes-Gleichungen, Boussinesq-Gleichungen, Lösbarkeit, Eindeutigkeit, Regularität, variabler Viskositätskoeffizient, variable Temperaturleitfähigkeit, Sobolev-Räume, Szegő-Gleichung, Turbulenzkaskade

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Contents

Chapter 1

Introduction

In this thesis, we study the motion of the incompressible inhomogeneous viscous fluid flows through two essential models in the domain of fluid mechanics: the incompressible inhomogeneous *stationary* Navier–Stokes equations and the incompressible *evolutionary* Boussinesq equations. We are in particular interested in the case when the viscosity coefficient is variable and depends on density or temperature.

The incompressible inhomogeneous Navier–Stokes equations are composed of the incompressibility condition, the mass and momentum conservation laws. We consider the solvability and regularity problems of the two- and three-dimensional *stationary* Navier–Stokes equations

$$\begin{cases} \operatorname{div} u = 0, \quad x \in \Omega \subset \mathbb{R}^d, \quad d = 2, 3, \\ \operatorname{div}(\rho u) = 0, \\ \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu S u) + \nabla \Pi = f. \end{cases} \tag{Navier–Stokes}$$

The unknowns are the density function $\rho : \Omega \to \mathbb{R}_+$, the velocity vector field $u = (u^1, ..., u^d)^T : \Omega \to \mathbb{R}^d$, and the pressure $\Pi : \Omega \to \mathbb{R}$. The external force $f : \Omega \to \mathbb{R}^d$ is given. We denote $u \otimes u = (u_i u_j)_{1 \leqslant i,j \leqslant d}$ and $Su = \nabla u + (\nabla u)^T$ with $\nabla u = (\partial_{x_j} u_i)_{1 \leqslant i,j \leqslant d}$. We assume that the viscosity coefficient μ depends smoothly on the density function ρ as follows

$$\mu = b(\rho),$$

where $b \in C(\mathbb{R}_+; [\mu_*, +\infty))$ is a given function, and $\mu_* > 0$ is a positive constant.

We also study the well-posedness and regularity issues of the two-dimensional incompressible *evolutionary* Boussinesq equations as the nonlinear coupling between the Navier-Stokes type of equations and the thermodynamic equations for temperature

$$\begin{cases} \operatorname{div}_x u = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^2, \\ \partial_t \theta + u \cdot \nabla_x \theta - \operatorname{div}_x(\kappa \nabla_x \theta) = 0, \\ \partial_t u + u \cdot \nabla_x u - \operatorname{div}_x(\mu S_x u) + \nabla_x \Pi = \theta \vec{e_2}. \end{cases} \tag{Boussinesq}$$

The unknowns are the temperature function $\theta = \theta(t,x) : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$, the velocity vector field $u = u(t,x) : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ and the pressure $\Pi = \Pi(t,x) : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$. The vector field $\vec{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ denotes the unit vector in the vertical direction, and $\theta \vec{e_2}$ is the buoyancy force. The thermal diffusivity κ and the viscosity coefficient μ depend smoothly on the temperature function θ as follows

$$\begin{aligned} \kappa &= a(\theta) \quad \text{with} \quad a \in C(\mathbb{R}; [\kappa_*, \kappa^*]) \quad \text{given}, \\ \mu &= b(\theta) \quad \text{with} \quad b \in C(\mathbb{R}; [\mu_*, \mu^*]) \quad \text{given}, \end{aligned}$$

where $0 < \kappa_* \leqslant \kappa^*$, $0 < \mu_* \leqslant \mu^*$ are positive constants.

The introduction chapter consists of three parts. Section 1.1 is devoted to studying the two-dimensional *evolutionary* incompressible Navier–Stokes equations, whose existence, uniqueness and regularity properties have been widely considered in the literature. We start this chapter with the evolutionary Navier–Stokes system as a nice background of our main topics.

We state our main results and give explanations on the boundary value problem of the two- and three-dimensional *stationary* incompressible Navier–Stokes equations in Section 1.2. Section 1.3 is devoted to the results of the two-dimensional *evolutionary* incompressible Boussinesq equations.

1.1 Evolutionary Navier–Stokes equations

This section is mainly devoted to stating the results of the two-dimensional evolutionary homogeneous/inhomogeneous incompressible Navier–Stokes equations. The evolutionary Navier–Stokes equation is an essential model in the domain of fluid mechanics and there are plenty of work devoting to its well-posedness problem since 19th century. Though the evolutionary Navier–Stokes equations are not directly related to the main results of this thesis, we start with it to present a nice background and motivation for the topics we are going to consider.

This section consists of three parts. Subsection 1.1.1 is devoted to presenting the models and the existence results of the two-dimensional evolutionary incompressible homogeneous and inhomogeneous Navier–Stokes equations.

In Subsection 1.1.2, a remarkable open problem concerning the uniqueness and regularity properties of the inhomogeneous flows will be introduced. In Subsection 1.1.3, we summarize some (partial) results concerning the existence, uniqueness and regularity of the homogeneous and inhomogeneous incompressible Navier–Stokes equations.

1.1.1 Presentation of the equations and existemce results

Homogeneous incompressible Navier–Stokes equations

The Cauchy problem of the two-dimensional homogeneous incompressible Navier–Stokes equations can be written as

$$\begin{cases} \operatorname{div}_x u = 0, \quad (t,x) \in [0,+\infty) \times \mathbb{R}^2, \\ \partial_t u + \operatorname{div}_x (u \otimes u) - \nu \Delta_x u + \nabla_x \Pi = 0, \\ u\big|_{t=0} = u_0. \end{cases} \tag{1.1}$$

The unknowns are velocity vector field $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} : [0,+\infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ and pressure $\Pi : [0,+\infty) \times \mathbb{R}^2 \to \mathbb{R}$. The viscosity coefficient ν is a positive non-zero constant. We write

$$u \otimes u = \begin{pmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{pmatrix}.$$

The *energy* of the system (1.1) is defined as

$$E(t) = \|u(t)\|^2_{L^2(\mathbb{R}^2)} + 2\nu \int_0^t \|\nabla u(\tau)\|^2_{L^2(\mathbb{R}^2)} \, d\tau, \quad t \geqslant 0.$$

We formally have the following energy estimate for smooth enough solutions. If $u \in (C_c^\infty([0,+\infty) \times \mathbb{R}^2))^2$ satisfies the equation (1.1), then we take the $L^2(\mathbb{R}^d)-$inner product of u and (1.1) to derive

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |u|^2 \, dx + \int_{\mathbb{R}^2} \operatorname{div}(u \otimes u) \cdot u \, dx \\ &- \nu \int_{\mathbb{R}^2} \Delta u \cdot u \, dx + \int_{\mathbb{R}^2} \nabla \Pi \cdot u \, dx = 0. \end{aligned} \tag{1.2}$$

By incompressibility condition, one has

$$\operatorname{div}(u \otimes u) = u \cdot \nabla u, \quad \nabla u = \begin{pmatrix} \partial_1 u_1 & \partial_2 u_1 \\ \partial_1 u_2 & \partial_2 u_2 \end{pmatrix}.$$

By integration by parts, one has

$$\begin{aligned}\int_{\mathbb{R}^2} (u\cdot\nabla u)\cdot u\,dx &= \frac{1}{2}\int_{\mathbb{R}^2} u\cdot\nabla|u|^2\,dx\\ &= -\frac{1}{2}\int_{\mathbb{R}^2}(\operatorname{div} u)|u|^2\,dx = 0,\end{aligned}$$

$$\int_{\mathbb{R}^2}\nabla\Pi\cdot u\,dx = -\int_{\mathbb{R}^2}\Pi(\operatorname{div} u)\,dx = 0,$$

and

$$-\nu\int_{\mathbb{R}^2}\Delta u\cdot u\,dx = \nu\|\nabla u\|^2_{L^2(\mathbb{R}^2)}.$$

We integrate (1.2) over time from 0 to t to derive

$$E(t) = E(0) = \|u_0\|^2_{L^2(\mathbb{R}^2)}.$$

The above energy estimate also holds for regular enough solutions of the equations (1.1) on $\mathbb{R}^d$ with $d \geqslant 3$.

The weak solutions of (1.1) with finite energy is called Leray-Hopf weak solutions, which is defined as following.

Definition 1.1.1 (Leray-Hopf weak solutions)**.** *We say that u is a Leray-Hopf weak solution of the homogeneous Navier–Stokes equation* (1.1) *with the given initial data* $u_0 \in (L^2(\mathbb{R}^2))^2$ *if*

$$u(t,x) \in C([0,+\infty);(L^2(\mathbb{R}^2))^2) \cap L^2([0,+\infty);(\dot{H}^1(\mathbb{R}^2))^2)$$

satisfies the initial condition $u|_{t=0} = u_0$*, the incompressibility condition* $div_x u = 0$ *in the distribution sense and the weak formulation*

$$\begin{aligned}-\int_{\mathbb{R}^2} u_0\cdot\varphi(x,0)\,dx + \int\int_{\mathbb{R}^2\times[0,+\infty)} \big\{ -u\cdot\partial_t\varphi\\ -(u\otimes u):\nabla\varphi + \nu\nabla u:\nabla\varphi\big\}\,dxdt = 0,\end{aligned} \tag{1.3}$$

where $div_x\varphi = 0$ *and* $\varphi \in C_c^\infty([0,+\infty)\times\mathbb{R}^2)^2$*. Here we define the notation* $A : B = \sum_{j,k=1,2} a_{jk}b_{jk}$ *for two matrices* $A = (a_{jk})_{j,k=1,2}$ *and* $B = (b_{jk})_{j,k=1,2}$.

Furthermore, the following energy estimate holds

$$\|u(t)\|^2_{L^2(\mathbb{R}^2)} + 2\nu\int_0^t \|\nabla u(\tau)\|^2_{L^2(\mathbb{R}^2)}\,d\tau \leqslant \|u_0\|^2_{L^2(\mathbb{R}^2)}, \quad \forall t > 0.$$

The existence of the Leray-Hopf solution was given by celebrated work of Leray [Ler33] .

Theorem 1.1.1 (Existence of Leray-Hopf weak solutions, [Ler33])**.** *For the two-dimensional case, there exists a unique global-in-time Leray-Hopf weak solution of* (1.1)*.*

The existence result also holds for the three-dimensional case. However, the uniqueness of weak solution fails. More discussions on the existence and uniqueness problems of the equation (1.1) can be found in Subsection 1.1.3.

Remark 1.1.1. *We discuss the regularity of the pressure term* $\nabla\Pi$ *corresponding to the Leray-Hopf weak solutions, which was eliminated in the weak formulation* (1.3)*.*

We first introduce the Leray-Helmholtz projector $\mathbb{P} : L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2)$, $p \in (1, \infty)$*, which projects a vector-valued tempered distribution* $v \in \mathcal{S}'(\mathbb{R}^2; \mathbb{R}^2)$ *into its div-free part defined as*

$$v = \nabla^{\perp} V_1 + \nabla V_2,$$

and

$$\nabla^{\perp} V_1 = \nabla^{\perp}\Delta^{-1}\nabla^{\perp} \cdot v =: \mathbb{P}v, \quad V_2 = \nabla\Delta^{-1}\nabla \cdot v = (I - \mathbb{P})v.$$

We apply the Leray-Helmholtz project $\mathbb{P}$ *to* $(1.1)_2$ *to derive*

$$\partial_t u = -\mathbb{P}(div_x(u \otimes u)) + \nu\Delta_x u.$$

Notice that $\Delta u \in L^2_{loc}([0, \infty); (H^{-1}(\mathbb{R}^2))^2)$ *and* $u \in L^4_{loc}([0, \infty); (L^4(\mathbb{R}^2))^2)$*. As a consequence,* $\mathbb{P}(div_x(u\otimes u))$*,* $\partial_t u \in L^2_{loc}([0, \infty); (H^{-1}(\mathbb{R}^2))^2)$*. Now we recover* $\nabla\Pi$ *in terms of* u

$$\nabla_x \Pi = -\partial_t u - div_x(u \otimes u) + \nu\Delta_x u \in L^2_{loc}([0, \infty); (H^{-1}(\mathbb{R}^2))^2).$$

The pressure function can be normalised as $\Pi \in L^2_{loc}([0, \infty) \times \mathbb{R}^2)$ *by assuming* $\int_{B_1} \Pi \,\mathrm{d}x = 0$*, where* $B_1 \subset \mathbb{R}^2$ *is the unit disk.*

For the incompressible homogeneous Navier–Stokes equation (1.1), the density function is constant. In next paragraph, we will discuss the inhomogeneous involving variable density.

Inhomogeneous incompressible Navier–Stokes equations

The evolutionary two-dimensional incompressible inhomogeneous Navier–Stokes equations with variable viscosity coefficient can be written as

$$\begin{cases} \operatorname{div}_x u = 0, \quad (t,x) \in [0,+\infty) \times \mathbb{R}^2, \\ \partial_t \rho + \operatorname{div}_x(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}_x(\rho u \otimes u) - \operatorname{div}_x(\mu S_x u) + \nabla_x \Pi = 0, \\ \rho\big|_{t=0} = \rho_0, \quad (\rho u)\big|_{t=0} = m_0. \end{cases} \tag{1.4}$$

The unknowns are density function $\rho : \mathbb{R}^2 \times [0,+\infty) \to [0,+\infty)$, velocity vector field $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} : \mathbb{R}^2 \times [0,+\infty) \to \mathbb{R}^2$ and pressure $\Pi : \mathbb{R}^2 \times [0,+\infty) \to \mathbb{R}$.

The variable viscosity coefficient μ may depend smoothly on the density function ρ with the form

$$\mu = b(\rho), \quad b \in C([0,\infty); [\mu_*, +\infty))$$

where $\mu_* > 0$ is the positive lower bound of b.

We write

$$\nabla u = \begin{pmatrix} \partial_{x_1} u_1 & \partial_{x_2} u_1 \\ \partial_{x_1} u_2 & \partial_{x_2} u_2 \end{pmatrix},$$

and

$$Su = \nabla u + (\nabla u)^T = \begin{pmatrix} 2\partial_{x_1} u_1 & \partial_{x_2} u_1 + \partial_{x_1} u_2 \\ \partial_{x_1} u_2 + \partial_{x_2} u_1 & 2\partial_{x_2} u_2 \end{pmatrix}.$$

Notice that $\frac{1}{2} Su$ is the symmetric part of ∇u. We write div $= \nabla \cdot = \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \end{pmatrix} \cdot$.

We recover the homogeneous system (1.1) from the inhomogeneous system (1.4) by setting

$$\rho = 1 \quad \text{and} \quad \mu = \nu > 0 \quad \text{a positive constant.}$$

In particular, the incompressibility condition $\operatorname{div} u = 0$ ensures that

$$\operatorname{div}(\nu S u) = \nu \Delta u.$$

We define the weak solutions of (1.4) as in [Lio96].

Definition 1.1.2 (Weak solutions of evolutionary inhomogeneous Navier–Stokes equations)**.** *We say that a pair (ρ, u) is a weak solution of the Navier–Stokes equation* (1.4) *with the given initial data (ρ_0, m_0) satisfying*

$$\begin{aligned} &\rho_0 \geqslant 0 \quad a.e. \ in \quad \mathbb{R}^2, \quad \rho_0 \in L^\infty(\mathbb{R}^2), \\ &m_0 \in (L^2(\mathbb{R}^2))^2, \quad m_0 = 0 \quad a.e. \ on \quad \{\rho_0 = 0\}, \\ &|m_0|^2/\rho_0 \in L^1(\mathbb{R}^2), \end{aligned} \tag{1.5}$$

if the following statements hold:

- *The density function*

$$\rho = \rho(t,x) \in L^\infty([0,+\infty) \times \mathbb{R}^2) \cap C([0,+\infty); L^p_{loc}(\mathbb{R}^2)), \quad 1 \leqslant p < \infty$$

satisfies the mass conservation law

$$\partial_t \rho + \operatorname{div}(\rho u) = 0$$

in the sense of distribution.

- *The velocity vector field*

$$\nabla u(t,x) \in (L^2([0,+\infty) \times \mathbb{R}^2))^4, \quad \rho|u|^2 \in L^\infty([0,+\infty); L^1(\mathbb{R}^2))$$

satisfies the initial condition $\rho u|_{t=0} = m_0$*, the incompressibility condition* $div_x u = 0$ *in the distribution sense and the weak formulation*

$$-\int_{\mathbb{R}^2} m_0 \cdot \varphi(0,x)\, dx + \int\int_{\mathbb{R}^2 \times [0,+\infty)} \{ -\rho u \cdot \partial_t \varphi - \rho(u \otimes u) : \nabla \varphi + \frac{1}{2} \mu S u : S\varphi \} \, dxdt = 0,$$

where $div_x \varphi = 0$ *and* $\varphi \in C_c^\infty([0,+\infty) \times \mathbb{R}^2)^2$.

Notice that for the non-vacuum case ($0 < \rho_* \leqslant \rho_0 \leqslant \rho^*$), the assumptions on the initial values (1.5) can be reduced to

$$\rho|_{t=0} = \rho_0 \in L^\infty(\mathbb{R}^2), \quad u|_{t=0} = u_0 \in (L^2(\mathbb{R}^2))^2.$$

We formally derive the energy estimate for the inhomogeneous equation (1.4). Let $(\rho, u) \in L^\infty(\mathbb{R}^2) \times (C_c^\infty([0,+\infty) \times \mathbb{R}^2))^2$ satisfy the system (1.4). By integration by parts, we take L^2-inner product of (1.4) and u to derive

$$\int_{\mathbb{R}^2} \rho |u|^2 \, dx + \int_0^t \int_{\mathbb{R}^2} \mu |Su|^2 \, dxd\tau = \int_{\mathbb{R}^2} \frac{|m_0|^2}{\rho_0} \, dx, \quad \forall t > 0,$$

where we use integration by parts to obatin

$$\int_{\mathbb{R}^2} \operatorname{div}(\mu S u) \cdot u \, dx = \frac{1}{2} \int_{\mathbb{R}^2} \mu |Su|^2 \, dx.$$

As a consequence of the energy estimate, Lions [Lio96] proved the following existence theorem.

Theorem 1.1.2 (Existence of weak solutions, [Lio96])**.** *There exists at least one weak solution of* (1.4)*. Furthermore, we have*

- *The following energy estimate holds*

$$\int_{\mathbb{R}^2} \rho|u|^2\,dx + \int_0^t \int_{\mathbb{R}^2} \mu|Su|^2\,dxd\tau \leqslant \int_{\mathbb{R}^2} \frac{|m_0|^2}{\rho_0}\,dx, \quad \forall t > 0.$$

- *For any* $0 \leqslant \alpha \leqslant \beta < \infty$

$$\textit{meas}\ \{x \in \mathbb{R}^2 | \alpha \leqslant \rho(t,x) \leqslant \beta\} \quad \textit{is independent of } t \in [0, +\infty). \tag{1.6}$$

Remark 1.1.2. *The above existence results allows the vanish of the density function ρ on a subset of $\mathbb{R}^2$, which is related to the vacuum in the fluids. In particular, the case that a bubble of homogeneous fluids embeds into vacuum is allowed. The vacuum case is always more difficult than the non-vacuum case, since the system is degenerated in the vacuum region.*

1.1.2 Open problems and transport equations revisited

Lions [Lio96] asked about the density-patch problem for inhomogeneous fluid flows. Suppose that the initial density has the form

$$\rho_0 = \mathbb{1}_D,$$

where $D \subset \mathbb{R}^2$ is a smooth domain. Theorem 1.1.2 provides at least one corresponding weak solution

$$\rho(t) = \mathbb{1}_{D(t)} \quad \text{and} \quad \nabla u(t,x) \in L^2([0,+\infty) \times \mathbb{R}^2; \mathbb{R}^4),$$

where the property (1.6) ensures that $\text{meas}(D) = \text{meas}(D(t))$. However, it is unclear whether the regularity of ∂D is preserved by the time evolution. This is so-called the density patch problem. It can also be seen as a free boundary problem for the homogeneous incompressible Navier–Stokes equation (1.1).

The density-patch problem is strongly related to the uniqueness and regularity properties of the fluid flows, which is still open for the equation (1.4) with density dependent viscosity coefficient μ even in dimension two, see for example [Lio96]. We will summarize some (partial) results for the homogeneous and inhomogeneous flows in Subsection 1.1.3.

If we assume the Lipschitz continuity in the spatial direction on u, i.e.

$$\nabla u \in L^1_{\text{loc}}([0,+\infty); (L^\infty(\mathbb{R}^2))^4), \tag{1.7}$$

then one can obtain the uniqueness and furthermore regularity results on the corresponding weak solutions. However, with the only bounded density function ρ it is very hard to obtain the Lipschitz continuity on u. The constant viscosity coefficient case was solved by Danchin and Mucha [DM19a], see Theorem 1.1.4. However the variable viscosity coefficient case is still open.

On the one hand, it is natural to ask whether one can obtain uniqueness and regularity results if we assume appropriated smallness or smoothness on the density function ρ or viscosity coefficient μ. We will summarize some results towards this direction in Subsection 1.1.3.

On the other hand, it is also interesting to investigate the maximal regularity of u with only bounded density function ρ. We will discuss this problem for the stationary Navier-Stokes equations in Chapter 2.

In the following, we will focus on the transport equation $(1.4)_2$ to discuss the necessity of the Lipschitz continuity on u (1.7) for the uniqueness and regularity properties of density function ρ.

Transport equation: Lipschitz framework

The transport equation $(1.4)_2$ reads as

$$\begin{cases} \partial_t \rho + u \cdot \nabla_x \rho = 0, \\ \rho|_{t=0} = \rho_0, \end{cases} \tag{1.8}$$

where $u : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ and $\operatorname{div}_x u = 0$. In the Lipschitz framework, namely

$$u \in L^1_{\mathrm{loc}}([0, +\infty); (W^{1,\infty}(\mathbb{R}^2))^2), \tag{1.9}$$

the transport equation (1.8) is strongly related to the ordinary differential equation

$$\frac{d}{dt} X(t) = u(t, X(t)), \quad X(0) = x \in \mathbb{R}^2 \tag{1.10}$$

as the solution can be expressed explicitly as $\rho(t, X(t)) = \rho_0(x)$. The Cauchy-Lipschitz Theorem ensures the existence and uniqueness of the solution for the equation (1.10) and hence the equation (1.8). Concerning the regularity problem, we have the following a priori estimate, see for example [BCD11].

- In the low regularity regime with $(p, r) \in [1, \infty]^2$ and

$$-1 - \max\{\frac{2}{p}, \frac{2}{p'}\} < s < 1 + \frac{2}{p} \quad \text{with} \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

  the following estimate holds

$$\frac{d}{dt} \|\rho\|_{B^s_{p,r}} \leqslant C \|\nabla v\|_{B^{\frac{2}{p}}_{p,\infty} \cap L^\infty} \|\rho\|_{B^s_{p,r}}.$$

- In the high regularity regime with $(p, r) \in [1, \infty]^2$ and $s > 1 + \frac{2}{p}$ or $(s, p, r) = (1 + \frac{2}{p}, p, 1)$, the following estimate holds

$$\frac{d}{dt}\|\rho\|_{B^s_{p,r}} \leqslant C\|\nabla v\|_{B^{s-1}_{p,r}}\|\rho\|_{B^s_{p,r}}.$$

Notice that even to propagate the low regularity, the Lipschitz continuity is necessary.

It is natural to ask whether the uniqueness and regularity results hold without Lipschitz continuity assumption on u. In the following, we discuss some work considering less regular velocity field, for example, u has Sobolev regularity (which can not be embedded into $W^{1,\infty}(\mathbb{R}^2)$) or BV (bounded variation) regularity with respect to the spatial variable and integrable over time.

Transport equation: Uniqueness

DiPerna and Lions [DL89] relaxed the Lipschitz continuity condition to Sobolev regularity assumption on u. They assumed the Sobolev regularity

$$\begin{aligned} &\rho_0(x) \in L^p(\mathbb{R}^2), \quad 1 \leqslant p \leqslant \infty, \ \frac{1}{p} + \frac{1}{p'} = 1, \\ &u(t,x) \in L^1_{\mathrm{loc}}([0, +\infty); (W^{1,p'}_{\mathrm{loc}}(\mathbb{R}^2))^2), \end{aligned} \tag{1.11}$$

and that u additionally satisfies

$$\frac{u}{1+|x|} \in L^1_{\mathrm{loc}}([0, +\infty); (L^1(\mathbb{R}^2))^2) + L^1_{\mathrm{loc}}([0, +\infty); (L^\infty(\mathbb{R}^2))^2)$$

then the equation (1.8) has a unique solution $\rho \in L^\infty([0, +\infty); L^p(\mathbb{R}^2))$. In particular, for $p = \infty$, the Sobolev regularity $u \in L^1_{\mathrm{loc}}([0, +\infty); (W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2))^2)$ can be relaxed to

$$u(t,x) \in L^1_{\mathrm{loc}}([0, +\infty); (BV_{\mathrm{loc}}(\mathbb{R}^2))^2),$$

see [Amb04].

However, for high dimensional cases $d \geqslant 3$, Modena and Székelyhidi [MS18] constructed non-unique examples with

$$\begin{aligned} &\rho(t,x) \in L^\infty_{\mathrm{loc}}([0, +\infty); L^p(\mathbb{T}^d)), \quad \frac{1}{p} + \frac{1}{\tilde{p}} \geqslant 1 + \frac{1}{d-1}, \\ &u(t,x) \in L^1_{\mathrm{loc}}([0, +\infty); (W^{1,\tilde{p}} \cap L^{p'}(\mathbb{T}^d))^d), \end{aligned}$$

for the equation (1.8). Notice that, in their example, the index $\tilde{p}$ is smaller than p' comparing to (1.11)

$$\frac{1}{\tilde{p}} > \frac{1}{p'} + \frac{1}{d-1}.$$

Transport equation: Propagation of regularity

The Lipschitz continuity assumption on u seems to be necessary for the propagation of regularity (comparing to the uniqueness result). Danchin [Dan05] showed the regularity results with u in some "almost Lipschitz" Besov spaces. However, Colombini, Luo, and Rauch [CLR04] showed that even for the time independent

$$u(x) \in \bigcap_{1 \leqslant p < \infty} (W^{1,p}(\mathbb{R}^2))^2,$$

neither continuity nor BV regularity of ρ_0 can be propagated. Alberti, Crippa, and Mazzucato [ACM19] constructed counterexamples with

$$u(t,x) \in \bigcap_{1 \leqslant p < \infty} L^\infty_{\mathrm{loc}}([0,+\infty); (W^{1,p}(\mathbb{R}^2))^2),$$

and the corresponding solution $\rho(t,\cdot)$ associated to a compactly supported smooth initial data ρ_0 does not belong to $\dot{H}^s(\mathbb{R}^2)$, for any $s > 0$ and $t > 0$.

1.1.3 Uniqueness and regularity results

Homogeneous evolutionary flow

In this paragraph, we focus on the (classical) incompressible Navier–Stokes equation (1.1)

$$\begin{cases} \operatorname{div}_x u = 0, \quad (t,x) \in [0,+\infty) \times \mathbb{R}^2, \\ \partial_t u + \operatorname{div}_x (u \otimes u) - \nu \Delta_x u + \nabla_x \Pi = 0, \\ u\big|_{t=0} = u_0. \end{cases}$$

We observe the following scaling invariance property: If (u,Π) is a pair of solution of (1.1) on $[0,T] \times \mathbb{R}^2$, then the scaling pair (u_λ, Π_λ) with $\lambda \in \mathbb{R}$ defined as

$$(u_\lambda(t,x), \Pi_\lambda(t,x)) = (\lambda u(\lambda^2 t, \lambda x), \lambda^2 \Pi(\lambda^2 t, \lambda x)) \tag{1.12}$$

is also a solution of (1.1) on $[0, \frac{T}{\lambda^2}] \times \mathbb{R}^2$. The functional spaces, whose norms are invariant under the transformation (1.12), are so-called *critical* spaces. For example, $\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$, $L^d(\mathbb{R}^d)$ and $\dot{B}^{\frac{d}{p}-1}_{p,q}(\mathbb{R}^d)$ are critical spaces for the system (1.1). The existence and uniqueness problems in critical spaces are widely studied.

Fujita and Kato [FK64] showed the following unique local-in-time solutions in the general Sobolev spaces.

Theorem 1.1.3 ([FK64]). *Let $u_0 \in (\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d))^d$. Then there exists a unique local-in-time solution on $[0,T]$ for some $T>0$ with*

$$u \in C([0,T];(\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d))^d) \cap L^2([0,T];(\dot{H}^{\frac{d}{2}}(\mathbb{R}^d))^d) \tag{1.13}$$

Furthermore, if u_0 is small comparing to viscosity coefficient ν

$$\|u_0\|_{\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)} \lesssim \nu, \tag{1.14}$$

then the solution (1.13) *exists globally. Moreover, the following estimate holds*

$$\|u(t)\|^2_{\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)} + \int_0^t \|\nabla u(\tau)\|^2_{\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)}\, d\tau \lesssim_\nu \|u_0\|^2_{\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)}.$$

If the lifespan T is finite, then we have

$$\int_0^T \|u(\tau)\|^4_{\dot{H}^{\frac{d-1}{2}}(\mathbb{R}^d)}\, d\tau = \infty. \tag{1.15}$$

For the two-dimensional case, $L^2(\mathbb{R}^2)$ is a critical space, and the solution holds globally without the smallness assumption (1.14), which is a consequence of the energy estimate

$$\|u(t)\|^2_{L^\infty_T L^2_x} + 2\nu \|\nabla u(\tau)\|^2_{L^2_T L^2_x} \leqslant \|u_0\|^2_{L^2_x}, \quad \forall T \in [0,+\infty). \tag{1.16}$$

Indeed, we consider the blow-up condition (1.15), which is globally bounded for the two dimensional case since

$$\int_0^T \|u(\tau)\|^4_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)}\, d\tau \leqslant \|u\|^2_{L^\infty_T L^2_x} \|\nabla u\|^2_{L^2_T L^2_x} \lesssim_\nu \|u_0\|^4_{L^2_x}, \quad \forall T \in [0,+\infty),$$

where we use the interpolation inequality $\|u\|_{\dot{H}^{\frac{1}{2}}} \leqslant \|u\|^{\frac{1}{2}}_{L^2} \|\nabla u\|^{\frac{1}{2}}_{L^2}$.

For the two dimensional case, one can show uniqueness of the solution by energy method in Ladyženskaja and Solonnikov [LS75]. We sketch the idea here. We consider u_1, $u_2 \in L^\infty([0,+\infty);(L^2(\mathbb{R}^2))^2) \cap L^2([0,+\infty);(H^1(\mathbb{R}^2))^2)$ as two different solutions of (1.18) with the same initial value, then the differences $\dot{u} = u_1 - u_2$ and $\dot{\Pi} = \Pi_1 - \Pi_2$ satisfy

$$\begin{cases} \operatorname{div}_x \dot{u} = 0, \\ \partial_t \dot{u} + \dot{u} \cdot \nabla_x u_1 + u_2 \cdot \nabla_x \dot{u} - \nu \Delta_x \dot{u} + \nabla_x \dot{\Pi} = 0, \\ \dot{u}|_{t=0} = 0. \end{cases}$$

We take L^2-inner product of the difference equation and $\dot{u}$ to derive

$$\frac{1}{2} \int_{\mathbb{R}^2} |\dot{u}|^2\, dx + \nu \int_{\mathbb{R}^2} |\nabla \dot{u}|\, dx \leqslant \int_{\mathbb{R}^2} |\dot{u} \cdot \nabla u_1 \cdot \dot{u}|\, dx, \tag{1.17}$$

where we use the fact $\int_{\mathbb{R}^2} u_2 \cdot \nabla \dot{u} \cdot \dot{u}\, dx = 0$. The right-hand side of (1.17) can be bounded by

$$\begin{aligned}\|\dot{u}\|_{L^4}^2 \|\nabla u_1\|_{L^2} &\leqslant C\|\dot{u}\|_{L^2}\|\nabla \dot{u}\|_{L^2}\|\nabla u_1\|_{L^2} \\ &\leqslant \varepsilon \|\nabla \dot{u}(\tau)\|_{L^2}^2\, d\tau + C_\varepsilon \|\nabla u_1\|_{L^2}^2 \|\dot{u}\|_{L^2}^2, \quad \varepsilon > 0,\end{aligned}$$

where we used Gagliardo-Nirenberg's inequality and Young's inequality. Then we have the following estimate

$$\frac{d}{dt}\|\dot{u}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \dot{u}\|_{L^2}^2 \leqslant C\|\nabla u_1\|_{L^2}^2 \|\dot{u}\|_{L^2}^2.$$

Since $\nabla u_1 \in L^2([0,T];(L^2(\mathbb{R}^2))^4)$ and $\dot{u}_0 = 0$, we obtain $\dot{u} \equiv 0$ by Gronwall's inequality.

However, for the three-dimensional case, the existence and uniqueness of the global-in-time smooth solutions without the smallness assumption (1.14) are still unknown (Millennium problem).

In the L^p frame work, Kato [Kat84] showed the local well-posedness for the d-dimensional equation (1.1) in $L^d(\mathbb{R}^d)$ and this solution exists globally if the initial value u_0 is small. As a consequence of the energy estimate (1.16), the smallness condition can be removed in dimension two.

There are some work studying the equation (1.1) in the Besov space framework. Chemin [Che99] and Kozono and Yamazaki [KY94] showed the existence and uniqueness of the local-in-time solution $u \in L^\infty([0,T];(\dot{B}_{p,\infty}^{\frac{d}{p}-1}(\mathbb{R}^d))^d) \cap L^1([0,T];(\dot{B}_{p,\infty}^{\frac{d}{p}+1}(\mathbb{R}^d))^d)$, $d < p < \infty$; Furthermore, under a smallness assumption on the initial value u_0, the corresponding solution exists globally. For the two-dimensional case, the global-in-time unique solution $u \in L_{\text{loc}}^\infty([0,+\infty);(\dot{B}_{p,q}^{\frac{2}{p}-1}(\mathbb{R}^2))^2$ with $2 \leqslant p < \infty$ and $2 < q < \infty$ was shown in [GP02]. For the three-dimensional case, an ill-posed example of the equation (1.1) in the functional space $L_{\text{loc}}^\infty([0,+\infty);(\dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^3))^3)$ was provided in [BP08] in the sense that the solution corresponding to a small initial value can grow arbitrarily large in an arbitrarily small time. In the smaller functional space $\text{BMO}^{-1}(\mathbb{R}^d) \subset \dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^d)$, Koch and Tataru [KT01] showed the global well-posedness result with the small initial data u_0.

Inhomogeneous evolutionary flow with constant viscosity coefficient

The Cauchy problem of the two-dimensional inhomogeneous incompressible Navier–Stokes equations (1.4) with constant viscosity coefficient $\nu > 0$ can be written as

$$\begin{cases} \operatorname{div}_x u = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\ \partial_t \rho + \operatorname{div}_x(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}_x(\rho u \otimes u) - \nu \Delta_x u + \nabla_x \Pi = 0, \\ \rho\big|_{t=0} = \rho_0, \quad (\rho u)\big|_{t=0} = m_0. \end{cases} \tag{1.18}$$

The existence of the global-in-time weak solutions of the above system was given by Simon [Sim90], which is compatible with Theorem 1.1.2.

The Navier–Stokes equation (1.4) (and (1.18)) is invariant under the translation with $\lambda \in \mathbb{R}$

$$(\rho_\lambda(t,x), u_\lambda(t,x)) = (\rho(\lambda^2 t, \lambda x), \lambda u(\lambda^2 t, \lambda x), \lambda^2 \Pi(\lambda^2 t, \lambda x)).$$

There are some work considering the non-vacuum fluid in the critical Besov space framework. For the strictly positive density $\rho > 0$, we define

$$a = \frac{1}{\rho} - 1 \quad \text{and} \quad a_0 = \frac{1}{\rho_0} - 1.$$

Then the system (1.4) can be written as

$$\begin{cases} \operatorname{div}_x u = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\ \partial_t a + u \cdot \nabla_x a = 0, \\ \partial_t u + u \cdot \nabla_x u + (1+a)(-\nu \Delta_x u + \nabla_x \Pi) = 0, \\ (a,u)|_{t=0} = (a_0, u_0). \end{cases} \tag{1.19}$$

Danchin [Dan03] showed the global well-posedness of the equation (1.19) with

$$\begin{aligned} &a \in C_b([0,+\infty); \dot{B}^1_{2,1}(\mathbb{R}^2)) \cap L^\infty([0,+\infty) \times \mathbb{R}^2), \\ &u \in C_b([0,+\infty); (\dot{B}^0_{2,1}(\mathbb{R}^2))^2) \cap L^1([0,+\infty); (\dot{B}^2_{2,1}(\mathbb{R}^2))^2) \end{aligned} \tag{1.20}$$

under the smallness assumption

$$\|a_0\|_{\dot{B}^1_{2,1}} + \nu^{-1} \|u_0\|_{\dot{B}^0_{2,1}(\mathbb{R}^d)} < \varepsilon, \quad \varepsilon \text{ sufficiently small.}$$

In general Sobolev spaces, he [Dan04] showed local well-posedness for the smooth solutions with $(a_0, u_0) \in H^{\alpha+1}(\mathbb{R}^2) \times (H^\beta(\mathbb{R}^2))^2$ (if $\alpha = 1$ the Lipschitz continuity on a_0 is assumed), where α, $\beta > 0$ and $\alpha - 1 < \beta \leqslant \alpha + 1$.

Remark 1.1.3. • *In general, for the velocity vector field, thanks to the viscosity term, there is a gain of regularity on spatial direction of order* 1 *when taking* L^2*-norm in the time variable and of order* 2 *when taking* L^1*-norm in the time variable.*

- *We have the following embedding onto Lipschitz space*

$$B^s_{p,r}(\mathbb{R}^2) \hookrightarrow W^{1,\infty}(\mathbb{R}^2), \quad s > 1 + \frac{2}{p} \text{ or } (s,r) = (1 + \frac{2}{p}, 1).$$

Notice that the Besov space $\dot{B}^1_{2,1}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ *in* (1.20)*, which coincides with Lipschitz condition* (1.9) *for the transport equation.*

Under the smallness assumptions on (a_0, ρ_0), the global well-posedness result with the initial value $(a_0, u_0) \in \dot{B}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2) \times (\dot{B}^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2))^2$, $1 < p \leqslant 2$, was shown in [Abi07]; The case of different regularity exponent $(a_0, u_0) \in \dot{B}^{\frac{2}{p_1}}_{p_1,1}(\mathbb{R}^2) \times (\dot{B}^{\frac{2}{p_2}-1}_{p_2,1}(\mathbb{R}^2))^2$, $|\frac{1}{p_1} - \frac{1}{p_2}| \leqslant \frac{1}{2}$ and $1 \leqslant \frac{1}{p_1} + \frac{1}{p_2}$, was studied in [AP07]. For the three-dimensional case, there are some work studying the global well-posedness with only smallness assumption on u_0, see [AGZ12; AGZ13].

There are some work studying the density-patch problem for the non-vacuum case. The equation (1.18) with the density function

$$\rho(t,x) = (1-\varepsilon)\mathbb{1}_{D(t)}(x) + \mathbb{1}_{D^c(t)}(x), \quad \varepsilon \quad \text{sufficiently small},$$

was studied in [DM12; HPZ13b; DM13; LZ16]; The smallness jump assumption was moved in [DM13; LZ19a; LZ19b; DM19a].

Danchin and Mucha [DM19a] answered the density-patch problem of the vacuum case with

$$\rho(t,x) = \mathbb{1}_{D(t)}(x),$$

in the following theorem.

Theorem 1.1.4 ([DM19a])**.** *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded* C^2-*domain. Let* $(\rho_0, u_0) \in L^\infty(\Omega) \times (H^1_0(\Omega))^2$*. Then there exists a unique global-in-time solution of* (1.1) *satisfying*

$$\rho \in L^\infty([0,+\infty) \times \Omega), \quad u \in L^\infty([0,+\infty); (H^1_0(\Omega))^2) \cap L^2([0,+\infty); L^2(\Omega)).$$

Furthermore,

$$\sqrt{\rho} u \in C([0,+\infty); L^2(\Omega)),$$

and for any $p < +\infty$

$$\rho \in C([0,+\infty); L^p(\Omega)), \quad u \in H^\eta_{loc}([0,+\infty); L^p(\Omega)), \quad \eta < \frac{1}{2}.$$

They used a time-weighted estimate to obtain the Lipschitz continuity $\nabla u \in L^1_{\text{loc}}([0,+\infty); (L^\infty(\Omega))^2)$, and they did not assume any regularity or positivity conditions on the density function.

Inhomogeneous evolutionary flow with variable viscosity coefficient

In this subsection, we consider the incompressible inhomogeneous Navier–Stokes equation (1.4) with density-dependent viscosity coefficient

$$\mu = b(\rho), \quad b \in C(\mathbb{R}_+; [\mu_*, +\infty)) \quad \text{given,}$$

where $\mu_* > 0$ is the positive lower bound of b. Comparing to the constant viscosity coefficient case (1.18), more difficulties aroused in the presence of the density-dependent viscosity coefficient $\mu = b(\rho)$.

There are some partial results on the well-posedness of the equation (1.4) under the smallness assumptions

$$\|b(\rho_0) - 1\|_{L^\infty} < \varepsilon, \quad \varepsilon \quad \text{sufficiently small.} \tag{1.21}$$

Under this assumption, Desjardins [Des97] obtained global-in-time weak solution $(\rho, u) \in L^\infty_{\mathrm{loc}}([0, +\infty) \times \mathbb{T}^2) \times L^\infty_{\mathrm{loc}}([0, +\infty); (H^1(\mathbb{T}^2))^2)$, furthermore,

$$\nabla u \in L^2_{\mathrm{loc}}([0, +\infty); (L^p(\mathbb{T}^2))^4), \quad p \in [4, p^*),\, p^* \sim \|\mu - 1\|^{-1}_{L^\infty}.$$

However, neither uniqueness nor regularity is ensured for such weak solutions.

Concerning the case $\|\rho - 1\|_{L^\infty} < \varepsilon$ (as a consequence (1.21) also holds), the global-in-time unique solutions were obtained in general Sobolev spaces, see for example [Abi07; GZ09]. With the assumption that $|\rho_0 - 1|$ and u_0 are both small, Abidi [Abi07] showed the existence of the global-in-time unique solution in Besov space $\dot{B}^1_{p,1}(\mathbb{R}^2) \times (\dot{B}^0_{p,1}(\mathbb{R}^2))^2$, $1 < p \leqslant 2$; Huang, Paicu, and Zhang [HPZ13a] showed the unique global-in-time solution in the critical space $B^{\frac{2}{q}}_{q,1}(\mathbb{R}^2) \times (B^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2))^2$ with $1 < q \leqslant p < 4$ and $1 - \frac{1}{p} \leqslant \frac{1}{q} \leqslant \frac{1}{p} + \frac{1}{2}$.

There are some works only assume the small oscillations on μ (1.21). Abidi and Zhang [AZ15a] proved the existence and uniqueness of the global-in-time solution with $(\rho_0 - 1, u_0) \in (L^2(\mathbb{R}^2) \cap L^\infty \cap \dot{W}^{1,r}(\mathbb{R}^2)) \times (\dot{H}^{-2\delta} \cap H^1(\mathbb{R}^2))^2$, with $r > 2$ and $0 < \delta < \frac{1}{2}$. Furthermore, they established the global-in-time regularity of the solutions in the Sobolev setting $(\rho_0 - 1, u_0) \in H^{1+s}(\mathbb{R}^2) \times (H^s(\mathbb{R}^2))^2$, $s > 1$. Paicu and Zhang [PZ20] additionally assumed that μ is Lipschitz continuous in one spatial direction, where the unique global-in-time solution was obtained with $u \in L^\infty([0, +\infty); (H^1(\mathbb{R}^2))^2) \cap L^1([0, +\infty); (\mathrm{Lip}(\mathbb{R}^2))^2)$. Furthermore, the global-in-time unique solution corresponding to the piecewise-constant density function

$$\rho(t, x) = (1 - \varepsilon)\mathbb{1}_{D(t)}(x) + \mathbb{1}_{D(t)^c}(x), \quad \varepsilon \quad \text{sufficiently small}$$

was obtained, and the $H^3(\mathbb{R}^2)$ regularity of $\partial D(t)$ is persevered.

1.2 Stationary Navier–Stokes equations

In this section, we will study the two- and three-dimensional stationary Navier–Stokes equations. We will extensively show the results of homogeneous flows in Subsection 1.2.1. The main results of this thesis on the stationary inhomogeneous Navier–Stokes equations will be discussed in Subsection 1.2.2.

1.2.1 Homogeneous stationary flows

The two- and three-dimensional homogeneous stationary Navier–Stokes equations in a domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, can be written as

$$\begin{cases} \operatorname{div}(u \otimes u) - \nu \Delta u + \nabla \Pi = f, \\ \operatorname{div} u = 0. \end{cases} \tag{1.22}$$

The velocity field $u : \Omega \to \mathbb{R}^d$ and the pressure $\Pi : \Omega \to \mathbb{R}$ are unknown. The external force $f : \Omega \to \mathbb{R}^d$ is given.

We follow [Gal11] to define the weak solutions of the stationary Navier–Stokes equation (1.22).

Definition 1.2.1 (Weak solutions of the stationary Navier–Stokes equations on a bounded domain)**.** *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain. We say that $u \in H_0^1(\Omega; \mathbb{R}^d)$ is a weak solution of the Navier–Stokes equation* (1.22)*, if* $\operatorname{div} u = 0$ *holds in the distribution sense, and the following integral identity*

$$\nu \int_\Omega \nabla u : \nabla v \, dx = \int_\Omega u \otimes u : \nabla v \, dx + \int_\Omega f \cdot v,$$

holds for all $v \in H_0^1(\Omega; \mathbb{R}^d)$ with $\operatorname{div} v = 0$.

Then we have the following existence result.

Theorem 1.2.1 (Existence of weak solutions, Leray [Ler33])**.** *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded Lipschitz domain. Let $f \in H^{-1}(\Omega; \mathbb{R}^d)$. Then there exists at least one weak solution $u \in H_0^1(\Omega; \mathbb{R}^d)$ for the stationary Navier–Stokes equation* (1.22)*.*

Remark 1.2.1.
- *This existence result can be extended to the non-homogeneous boundary value cases*

$$u|_{\partial\Omega} = u_0,$$

  and also to the unbounded domains. More detailed discussions can be found in the following paragraphs.

- *In general, we do not expect the uniqueness of weak solution in the above theorem. There are some uniqueness results under smallness or symmetric assumptions, see for example [Gal11]. Several non-unique examples were provided in [Gal11]. We here mention the non-unique example of Hamel flows on the two-dimensional exterior domain* $B_1(0)^c \subset \mathbb{R}^2$*. We consider the polar coordinate with*

$$e_r = \begin{pmatrix} \frac{x_1}{r} \\ \frac{x_2}{r} \end{pmatrix}, \quad e_\theta = \begin{pmatrix} \frac{x_2}{r} \\ -\frac{x_1}{r} \end{pmatrix},$$

and the velocity field reads

$$u(r,\theta) = u_r e_r + u_\theta e_\theta, \quad (r,\theta) \in [0,+\infty) \times [0,2\pi).$$

Then the equation (1.22) *with* $0 < \nu < \frac{1}{2}$ *combined with the boundary conditions*

$$u|_{\partial B_1(0)^c} = -\frac{1}{\nu} e_r, \quad \lim_{|x|\to\infty} u = 0$$

has a family of solutions

$$u = -\frac{1}{\nu r} e_r + \frac{c\nu}{(1-2\nu)r}(1 - r^{-\frac{1}{\nu}+2}) e_\theta, \quad \textit{for any } c \in \mathbb{R},$$

and

$$\Pi = -\int \frac{1}{\nu^3 r^2} + \frac{c^2 \nu}{(1-2\nu^2) r^3}(1 - r^{-\frac{1}{\nu}+2})^2.$$

Non-homogeneous boundary value and fluxes assumptions

Theorem 1.2.1 can be generalised to the non-homogeneous boundary value case. On the bounded domain Ω, we combine the stationary Navier–Stokes equation (1.22) with the boundary value

$$u|_{\partial\Omega} = u_0. \tag{1.23}$$

For the compatibility of the incompressibility condition $\operatorname{div} u = 0$, we assume that there is no flux through the boundary $\partial\Omega$

$$\mathcal{F} = \int_{\partial\Omega} u_0 \cdot n \, ds = 0. \tag{1.24}$$

In the above, $n = (n_1, n_2)$ or $n = (n_1, n_2, n_3)$ denotes the exterior normal vector to the boundary $\partial\Omega$. Furthermore, Leray's method requires if $\Omega =$

$\cup_{i=1}^{l}\Omega_i$, $\Omega_i \cap \Omega_j = \varnothing$ is multi-connected, then there is no flux through the boundary of each connected component

$$\mathcal{F}_i = \int_{\partial\Omega_i} u_0 \cdot n \, ds = 0, \quad \forall\, 1 \leqslant i \leqslant l. \tag{1.25}$$

In another word, this model can be seen as a fix piece of an incompressible flow on the whole space . So it is natural to ask about the case without the strict zero fluxes assumption for each $\partial\Omega_i$ (1.25), which allows sink/source outside of the domain. Galdi [Gal11] and Finn [Fin61] relaxed the zero sub-fluxes zero condition (1.25) in dimension two and three respectively to the small fluxes assumption , namely, $\sum_{i=1}^{l} |\mathcal{F}_i|$ is sufficiently small. Korobkov, Pileckas, and Russo [KPR15] proved the existence results under only zero total flux (1.24) on bounded two-dimensional multi-connected domains and three-dimensional axially symmetric domains.

Theorem 1.2.1 can be also generalised to the exterior domains under the assumption $\mathcal{F}_i = 0$, $1 \leqslant i \leqslant l$ (and hence the total flux $\mathcal{F} = 0$) by an approximation argument, which will be detailed explained in the next subsection. The total flux condition (1.24) was removed for the three-dimensional axially symmetric exterior domains by Korobkov, Pileckas, and Russo in [KPR18] and for two-dimensional exterior domains in [KPR14].

Solvability on the unbounded domains

Leray's solution of the boundary value problem (1.22)-(1.23) can be extended to the exterior domain $\Omega = (\cup_{i=1}^{l}\Omega_i)^c$ under the additional zero flux assumption on each Ω_i (1.25). In addition, we assume the boundary value at infinity

$$\lim_{|x|\to\infty} u = u_\infty.$$

The solvability can be shown by an approximation method. Let $N \in \mathbb{N}$ such that $\Omega^C \subset B_N(0) = \{x \in \mathbb{R}^d \,|\, |x| < N\}$. Let $\Omega_n = \Omega \cap B_{N+n}(0) \subset \mathbb{R}^d$, then $\{\Omega_n\}$ is a monotonically increasing sequence which has Ω as its limit. We consider the approximation boundary value problem

$$\begin{cases} \operatorname{div} u_n = 0, \\ \operatorname{div}(u_n \otimes u_n) - \nu\Delta u_n + \nabla\Pi_n = f, \\ u|_{\partial\Omega} = u_0, \\ u|_{\partial B_{N+n}(0)} = u_\infty. \end{cases} \tag{1.26}$$

The existence of the weak solutions for the approximation system (1.26) are given by Theorem 1.2.1, and furthermore, there exits a constant C such that

$$\int_\Omega |\nabla u_n|^2 \leqslant C.$$

Then there exists a weakly convergence subsequence

$$u_{n_k} \rightharpoonup u,$$

where u is a weak solution of (1.22)-(1.23) and satisfying

$$\int_\Omega |\nabla u|^2 \leqslant C.$$

For the three-dimensional case, thanks to the Hardy inequality

$$\left\| \frac{u - u_\infty}{|x|} \right\|_{L^2(\Omega)} \leqslant C \|\nabla u\|_{L^2(\Omega)}, \quad \Omega \subset \mathbb{R}^3, \tag{1.27}$$

which guarantees the limit at large distance of the weak solution u to be u_∞, in the sense that

$$\int_{S^2} |u - u_\infty|^2 = O(|x|^{-1}).$$

Finn [Fin59] showed the pointwise convergence of the Leray's approximation solution u to u_∞ as $|x| \to \infty$. As a consequence of the Hardy inequality (1.27), Leray's method works for the case $\Omega = \mathbb{R}^3$.

Hardy inequality for the two-dimensional case reads as

$$\left\| \frac{u - u_\infty}{|x| \log |x|} \right\|_{L^2(\Omega)} \leqslant C \|\nabla u\|_{L^2(\Omega)}, \quad \Omega \subset \mathbb{R}^2.$$

This, however, leaves the limit of Leary's approximation solutions at infinity as well as the solvability on whole plane $\mathbb{R}^2$ open. We have more discussion on this problem in the next paragraph. Guillod and Wittwer [GW18] showed the solvability of (1.22) on the whole plane $\mathbb{R}^2$, where they showed that for any given vector $d \in \mathbb{R}^2$ and a bounded positive measure set $D \subset \mathbb{R}^2$, there exist at least one weak solution u satisfying the prescribed mean value $d = \frac{1}{\text{meas}(D)} \int_D u \in \mathbb{R}^2$. The half plane case was considered in [GW16].

Asymptotic behavior for the two-dimensional case

The D-solutions of the stationary Navier–Stokes equations (1.22) are defined as the solutions with finite Dirichlet integral

$$\int_\Omega |\nabla u|^2 \, dx < \infty, \quad \Omega \subset \mathbb{R}^2. \tag{D-solution}$$

Notice that Lerary's solutions are also D-solutions.

In this paragraph, we summarize some results considering the asymptotic behaviors of D-solutions of two-dimensional stationary Navier–Stokes equation (1.22) with zero boundary value and zero external force on the unbounded domain $\Omega \subset \mathbb{R}^2$

$$\begin{cases} \operatorname{div} u = 0, \\ \operatorname{div}(u \otimes u) - \nu \Delta u + \nabla \Pi = 0, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1.28}$$

We assume the boundary value at infinity

$$\lim_{|x| \to \infty} u = u_\infty, \quad u_\infty \in \mathbb{R}^2. \tag{1.29}$$

If $u_\infty = 0$, we linearise the equation (1.28) around infinity, and the linear equation is the Stokes equations

$$\begin{cases} \operatorname{div} u = 0, \\ -\Delta u + \nabla \Pi = 0. \end{cases} \tag{1.30}$$

We consider the Stokes equations (1.30) on the two-dimensional exterior domains. There is only trivial solution if the boundary value $u_0 = 0$. In other words, there is no solution of (1.30)-(1.29) provided with $u_\infty \neq 0$. This is the *Stokes paradox*, which is opposed to the three-dimensional case. In 3D, a non-trivial solution of (1.30)-(1.29) was give by Stokes, see for example [Gal11]

$$\begin{aligned} u(x) &= -\frac{3}{4} \nabla \times \left[|x|^2 \nabla \times \left(\frac{u_\infty}{|x|} \right) \right] - \frac{1}{4} \nabla \times \nabla \times \left(\frac{u_\infty}{|x|} \right) + u_\infty, \\ \Pi(x) &= -\frac{3}{2} u_\infty \cdot \nabla \left(\frac{1}{|x|} \right), \quad u_\infty \in \mathbb{R}^3, \end{aligned}$$

where $u|_{\partial B_1(0)} = 0$ and $\lim_{|x| \to \infty} u = u_\infty$.

It is natural to ask whether the Stokes paradox happens in the nonlinear Navier–Stokes equation (1.28). Finn and Smith [FS67] showed the existence of non-trivial weak solutions of (1.28) provided with $|u_\infty| \neq 0$ sufficiently small, which is "contradiction" to the Stoke paradox. Moreover, they showed the leading term of this weak solution coincides with the fundamental solution of the Oseen equation. The Oseen equation can be seen as the linearisation of (1.28) around infinity ($u_\infty \neq 0$)

$$\begin{cases} \operatorname{div} u = 0, \\ -\Delta u + u_\infty \cdot \nabla u + \nabla \Pi = 0. \end{cases}$$

Gilbarg and Weinberger [GW74; GW78] showed that for the D-solutions of the general stationary Navier–Stokes equation (1.22) either there exists $u_{inf} \in \mathbb{R}^2$ such that

$$\lim_{r\to\infty} \int_0^{2\pi} |u(r,\theta) - u_{inf}|^2 \, d\theta = 0 \tag{1.31}$$

or

$$\lim_{r\to\infty} \int_0^{2\pi} |u(r,\theta)|^2 \, d\theta = \infty.$$

Amick [Ami88a; Ami91] showed that for the equation (1.28) the first case (1.31) happens and $u \to u_{inf}$ uniformly in θ.

We recall the Leray's approximation method in the last paragraph, where we construct a sequence of approximation solutions $\{u_n\}$ satisfying the approximation system (1.26), where u_n satisfies

$$u_n(x) = u_\infty, \quad x \in \overline{B_n(0)}^c.$$

However, it is still unknown for Leray's solutions, whether u_{inf} and u_∞ in (1.29) are coincident. There are some work studying Leray's solutions of (1.28)-(1.29) provided with $u_\infty \neq 0$. Korobkov, Pileckas, and Russo [KPR19] showed that Leray's solutions converge uniformly to u_{inf}. Later on, they [KPR20] showed that Leray's solutions of (1.22) with non-zero boundary value u_0 are uniformly bounded. They [KPR21] proved that Leray's solution of (1.28) with $u_\infty \neq 0$ is always non-trivial.

There are some work studying the existence of decaying solutions of (1.28), see for example [Gal04; PR12; Yam11; HW13]. The problem concerning the decay rate for u is more complicated. We mention a remarkable work for the three-dimensional problem by Korolev and Šverák [KŠ11]. They showed that for the solutions decaying like $|x|^{-1}$, the leading terms are given by the Landau solution, which is the scaling invariant solution of (1.22) satisfying $u(x) = \lambda u(\lambda x)$, $\lambda \in \mathbb{R}$. There are some work discussing this problem on $\mathbb{R}^2$, see for example [GW15b; GW15a; HW13].

Liouville problem

Liouville problem asks whether all (D-solution) of the system (1.28)-(1.29) provided $u_\infty = 0$ on $\mathbb{R}^2$ or $\mathbb{R}^3$ are trivial. Similar to the asymptotic behavior, the Liouville problem also dramatically relies on the dimensions. On $\mathbb{R}^2$, Gilbarg and Weinberger [GW78] gave a positive answer. This problem is much more complicated on $\mathbb{R}^3$ and the complete answer is still open. Galdi [Gal11] showed if $u \in L^{\frac{9}{2}}(\mathbb{R}^3)$, then u is a trivial solution. More partial results are given under regularity or decay rate assumptions, see for example [CW16; Cha20; Ser16].

1.2.2 Inhomogeneous stationary flows

The stationary inhomogeneous incompressible Navier–Stokes equations read as

$$\begin{cases} \operatorname{div} u = 0, \quad x \in \Omega, \\ \operatorname{div}(\rho u) = 0, \\ \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu S u) + \nabla \Pi = f, \end{cases} \tag{1.32}$$

where $Su = \nabla u + (\nabla u)^T$. We assume the boundary value condition

$$u|_{\partial\Omega} = u_0, \tag{1.33}$$

which satisfies the zero flux condition (1.25). The density-dependent viscosity coefficient depends continuously on the density function

$$\mu = b(\rho), \quad b \in C(\mathbb{R}_+; [\mu_*, +\infty)) \quad \text{given},$$

where $\mu_* > 0$ is the positive lower bound.

We define weak solutions as follows.

Definition 1.2.2 (Weak solutions of the inhomogeneous stationary Navier–Stokes equations)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded connected $C^{1,1}$ domain. We say that a pair $(\rho, u) \in L^\infty(\Omega; [0,\infty)) \times H_0^1(\Omega; \mathbb{R}^2)$ is a weak solution of the inhomogeneous stationary Navier–Stokes equation* (1.32) *with the given data $f \in H^{-1}(\Omega; \mathbb{R}^2)$, if $div\, u = 0$, $div\,(\rho u) = 0$ hold in Ω in the distribution sense, $u_0 = u|_{\partial\Omega}$ is the trace of u on $\partial\Omega$ and the following integral identity*

$$\frac{1}{2}\int_\Omega \mu S u : S v\, dx = \int_\Omega \rho (u \otimes u) : \nabla v\, dx + \int_\Omega f \cdot v,$$

holds for all $v \in H_0^1(\Omega; \mathbb{R}^2)$ with $div\, v = 0$.

If $\Omega \subset \mathbb{R}^2$ is a simply connected domain, then for any solenoidal vector field $u \in H_0^1(\Omega)$, there exists a stream function $\Phi : \mathbb{R}^2 \to \mathbb{R}$, such that

$$u = \nabla^\perp \Phi \stackrel{\text{def}}{=} \begin{pmatrix} \partial_{x_2} \Phi \\ -\partial_{x_1} \Phi \end{pmatrix}.$$

Frolov [Fro93] introduced the weak solutions for (1.32) of the form

$$(\rho, u) = \big(\eta(\Phi),\ \nabla^\perp \Phi\big), \tag{1.34}$$

where η is any given positive bounded scalar function. Under this assumption, the incompressibility condition and the density equation hold automatically

$$\operatorname{div}(u) = 0, \quad \operatorname{div}(\rho u) = 0.$$

Indeed, we take a sequence of mollifiers $(\phi^\varepsilon)_\varepsilon$ on $\mathbb{R}$, with $\phi^\varepsilon = \frac{1}{\varepsilon}\phi(\frac{\cdot}{\varepsilon})$, $\phi \in C_0^\infty(\mathbb{R})$, $\int_{\mathbb{R}} \phi = 1$. We regularize η in the following way

$$\eta^\varepsilon = \phi^\varepsilon * \eta \in C_b^\infty(\mathbb{R}; [0, \rho^*]),$$

such that

$$\rho^\varepsilon = \eta^\varepsilon(\Phi) \xrightarrow{*} \rho = \eta(\Phi) \quad \text{in} \quad L^\infty(\Omega), \quad \text{as} \quad \varepsilon \to 0,$$

and

$$\operatorname{div}(\rho^\varepsilon u) = (\eta^\varepsilon)' \nabla\Phi \cdot \nabla^\perp \Phi = 0.$$

The following convergence holds

$$0 = -\int_{\mathbb{R}^2} \operatorname{div}(\rho^\varepsilon u)\psi\, dx = \int_{\mathbb{R}^2} \rho^\varepsilon u \cdot \nabla\psi\, dx \to \int_{\mathbb{R}^2} \rho u \cdot \nabla\psi\, dx, \quad \forall \psi \in C_c^\infty(\mathbb{R}^2),$$

which implies that the density equation

$$\operatorname{div}(\rho u) = 0$$

holds at least in the distribution sense.

Constant viscosity coefficient

We consider the stationary Navier–Stokes equation of (1.32) with constant viscosity coefficient $\nu > 0$

$$\begin{cases} \operatorname{div} u = 0, \quad x \in \Omega \\ \operatorname{div}(\rho u) = 0, \\ \operatorname{div}(\rho u \otimes u) - \nu\Delta u + \nabla\Pi = f. \end{cases} \tag{1.35}$$

Frolov [Fro93] showed the solvability of (1.35)-(1.33) with the representation (1.34) on the bounded and exterior domain for the Hölder-continuous density function ρ. Later on, Santos [San02] generalised this result to the only bounded $\rho \in L^\infty(\Omega)$. There are also results on the domains with unbounded boundaries, see for example [AS06; AS05].

Inhomogeneous flow with variable viscosity coefficient

One of the main results of this thesis is to show the solvability and regularity property of the stationary inhomogeneous incompressible Navier–Stokes equation (1.32) with the variable viscosity coefficient

$$\mu = b(\rho), \quad b \in C(\mathbb{R}_+; [\mu_*, +\infty)) \quad \text{given},$$

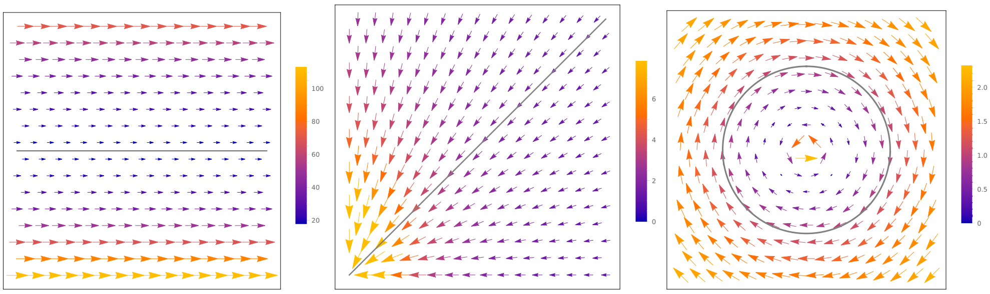

Figure 1.1: Explicit examples

where $\mu_* > 0$ is the positive lower bound. We are interested in the influence of the large variation of the density-dependent viscosity coefficient.

Two-dimensional case

We are going to show the existence and regularity properties of the Forolov type weak solutions

$$(\rho, u) = \big(\eta(\Phi), \nabla^{\perp}\Phi\big),$$

of the system (1.32) by studying the fourth-order elliptic equations of the stream functions. More precisely, we substitute the Frolov-type solutions to the velocity equation $(1.32)_3$ and apply $\nabla^{\perp}\cdot$ to the equation to derive

$$L_{\mu}\Phi = -\nabla^{\perp} \cdot f + \nabla^{\perp} \cdot \operatorname{div}(\rho\nabla^{\perp}\Phi \otimes \nabla^{\perp}\Phi), \tag{1.36}$$

where L_{μ} denotes the fourth-order elliptic operator

$$L_{\mu} = (\partial_{x_2x_2} - \partial_{x_1x_1})\mu(\partial_{x_2x_2} - \partial_{x_1x_1}) + (2\partial_{x_1x_2})\mu(2\partial_{x_1x_2}),$$

such that the identity $\nabla^{\perp} \cdot \operatorname{div}(\mu S u) = L_{\mu}\Phi$ holds. In particular, for $\mu = \nu \in \mathbb{R}_+$, we have

$$L_{\nu} = \nu\Delta^2.$$

We have the following existence and regularity results.

Theorem 1.2.2 (Existence and regularity results, [HL20]). *1. (Existence of Forlov-type weak solutions). Let $\eta \in L^{\infty}(\mathbb{R}; [0, \infty))$, $b \in C(\mathbb{R}; [\mu_*, +\infty))$, $\mu_* > 0$ be given. Let $\Omega \subset \mathbb{R}^2$ be a bounded connected Lipschitz domain. Let $f \in H^{-1}(\Omega; \mathbb{R}^2)$ be given. Then there exists at least one weak solution of (1.32) with the form*

$$(\rho, u) = \big(\eta(\Phi), \nabla^{\perp}\Phi\big) \in L^{\infty}(\Omega) \times H_0^1(\Omega),$$

where $\Phi \in H_0^2(\Omega)$ is a weak solution of the elliptic equation (1.36).

2. *(Regularity of the weak solutions). Let Ω be a bounded Lipschitz domain. For any solution $(\rho, u) \in L^\infty(\Omega) \times H_0^1(\Omega; \mathbb{R}^2)$ to the stationary Navier–Stokes equation* (1.32) *with zero external force $f = 0$, we have*

$$\mathbb{P} div (\mu S u) \in L^p(\Omega; \mathbb{R}^2), \quad \forall p \in (1, 2),$$

where $\mathbb{P}$ denotes the Leray-Helmholtz projector.

Furthermore, if the viscosity coefficient μ are variably partially VMO, namely, one direction is VMO while the other direction is only measurable, then we have

$$\nabla u, \, \mathbb{P} div (\mu S u) \in L^p(\Omega), \quad \forall p \in [2, \infty).$$

The above existence and regularity results can also be generalised to non-homogeneous boundary value problem (1.32)-(1.33) with smooth enough boundary value u_0. And similar to the classical case (1.22), the existence result can also be generalised to exterior domains by Leray's approximation method.

We discuss on the maximal regularity of the weak solutions. We consider the fourth-order elliptic equation (1.36) with vanishing right-hand side

$$L_\mu \Phi = (\partial_{x_2 x_2} - \partial_{x_1 x_1}) \mu (\partial_{x_2 x_2} - \partial_{x_1 x_1}) \Phi + (2\partial_{x_1 x_2}) \mu (2 \partial_{x_1 x_2}) \Phi = 0. \quad (1.37)$$

We define $\Psi : \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$\begin{pmatrix} \mu(\partial_{22} - \partial_{11})\Phi \\ \mu 2 \partial_{12} \Phi \end{pmatrix} = \begin{pmatrix} -2\partial_{12}\Psi \\ (\partial_{22} - \partial_{11})\Psi \end{pmatrix},$$

then the complex-valued function $\Lambda = \Phi + i\Psi$ solves the following second-order Beltrami-type equation

$$\partial_{\bar{z}}^2 \Lambda = \frac{1-\mu}{1+\mu} \overline{\partial_z^2 \Lambda}, \quad z = x_1 + i x_2.$$

Following the convex integration method for the first order Beltrami equation in [AFS08], we can show that there exists a measurable function $\mu : \Omega \mapsto \{\frac{1}{K}, K\}$, $K > 1$ such that the solutions $\Phi \in H^2(\Omega)$ of the equation (1.37) satisfies

$$\int_B |\nabla^2 \Phi|^{\frac{2K}{K-1}} = \infty, \quad \text{for any disk } B \subset \Omega.$$

Although it is not clear whether this constructed solution solves the stationary Navier-Stokes equation (1.32) or not, we expect in general that the solutions for (1.32) with only bounded viscosity coefficient μ (without any smoothness assumption)

$$\nabla u \notin L^p(\Omega), \text{ for any } p \geqslant p_*,$$

where $p_* < \infty$ depends on the deviation $|\mu - 1|$.

We formulate the parallel, concentric, and radial flows of the stationary Navier-Stokes system (1.32) by assuming certain symmetries on the density function in the following theorem. Notice that they are all solutions of Frolov's type.

Theorem 1.2.3 (Examples of paralle, concentric and radial flows, [HL20]). *If the density function*

$$\rho = \rho(x_2) \textit{ in } \mathbb{R}^2, \quad \textit{or} \quad \rho(r) \textit{ in } \mathbb{R}^2\backslash\{0\}, \quad \textit{or} \quad \rho(\theta) \textit{ in } \mathbb{R}^2\backslash\{0\}, \quad \textit{with } \rho' \neq 0,$$

where (r,θ) *are polar coordinates in* $\mathbb{R}^2$, *then the velocity vector field* u *of the stationary Navier–Stokes equations* (1.32) *reads correspondingly as*

$$u = u_1(x_2)\, e_1 \textit{ in } \mathbb{R}^2, \quad \textit{or} \quad rg(r)\, e_\theta \textit{ in } \mathbb{R}^2\backslash\{0\}, \quad \textit{or} \quad \frac{h(\theta)}{r} e_r \textit{ in } \mathbb{R}^2\backslash\{0\},$$

where $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_r = \begin{pmatrix} \frac{x_1}{r} \\ \frac{x_2}{r} \end{pmatrix}$, $e_\theta = \begin{pmatrix} \frac{x_2}{r} \\ -\frac{x_1}{r} \end{pmatrix}$.

Let the external force $f = 0$ *in the system* (1.32), *then the scalar functions* u_1, g, h *above should satisfy the following three ordinary differential equations of second order respectively*

$$\begin{aligned} &\partial_{x_2}(\mu \partial_{x_2} u_1) = C, \\ &\partial_r(\mu r^3 \partial_r g) = -Cr, \\ &\rho h^2 + \partial_\theta(\mu \partial_\theta h) + 4(\mu h) = C, \end{aligned}$$

where $C \in \mathbb{R}$ *can be arbitrarily chosen.*

We consider the explicit examples of the above symmetric solutions with piecewise constant viscosity coefficients as following

$$\begin{aligned} &\mu(x_2) = \mathbb{1}_{\{x_2<0\}} + 2\mathbb{1}_{\{x_2 \geqslant 0\}}, \\ \text{or} \quad &\mu(r) = \mathbb{1}_{\{0<r<1\}} + 2\mathbb{1}_{\{r \geqslant 1\}}, \\ \text{or} \quad &\mu(\theta) = \mathbb{1}_{\{0<r<\frac{\pi}{4}\}} + 2\mathbb{1}_{\{\frac{\pi}{4} \leqslant r \leqslant \frac{\pi}{2}\}}. \end{aligned} \tag{1.38}$$

Then the motion of the corresponding flows are described in Figure 1.1. We observe the results in Theorem 1.2.2 for the symmetric solutions with the piece-wise constant viscosity coefficients (1.38).

Remark 1.2.2. *1. These explicit solutions indeed fulfil the Frolov form* (1.34).

2. *On the one hand the piece-wise constant viscosity coefficients are variably partially VMO, since they are smooth in the tangential directions of the boundary $\partial\Omega$ and have jumps in the normal directions. Indeed, the corresponding solutions satisfy*

$$\nabla u \in L^p_{loc}(\mathbb{R}^2 \backslash \{0\}), \quad \forall p \in [1, \infty].$$

On the other hand, we verify

$$\mathbb{P}\, div\, (\mu S u) \in L^p_{loc}(\mathbb{R}^2 \backslash \{0\}), \quad \forall p \in [1, \infty].$$

3. *We have the following irregular properties*

$$\Delta u \notin L^1_{loc}(\mathbb{R}^2 \backslash \{0\}) \text{ and } \nabla\omega \notin L^1_{loc}(\mathbb{R}^2 \backslash \{0\}),$$

where $\omega = \partial_{x_2} u_1 - \partial_{x_1} u_2$ is defined as the vorticity of u.

In particular, for the radial flows with piece-wise constant viscosity coefficient (1.38)*, we have*

$$\mathrm{div}(\mu S u) \notin L^1_{loc}(\mathbb{R}^2 \backslash \{0\})$$

while

$$\mathbb{P}\, div\, (\mu S u) \in L^p_{loc}(\mathbb{R}^2 \backslash \{0\}), \quad \forall p \in [1, \infty].$$

Three-dimensional case

To our best knowledge, there is no existence results on the three-dimensional inhomogeneous equation (1.32). We show the existence of weak solutions of the three-dimensional stationary incompressible inhomogeneous Navier–Stokes equations with density-dependent viscosity coefficient in axially symmetric case.

In the cylindrical coordinate (r, z, θ), we write e_r, e_z, e_θ as the coordinate axis

$$e_r = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \quad e_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad e_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}.$$

We consider the *axially symmetric* velocity field

$$u = u_r e_r + u_\theta e_\theta + u_z e_z,$$

where u_r, u_θ, u_z are independent of θ. We define the functional spaces for axially symmetric functions

$$H(\Omega) = \{u \in H_0^1(\Omega) \mid u \text{ is axially symmetric}, \operatorname{div} u = 0\}.$$

The incompressibility condition of $u \in H(\Omega)$ reads as

$$r\operatorname{div} u = \partial_r(ru_r) + \partial_z(ru_z) = 0.$$

For $u \in H(\Omega)$, there exists a stream function $\varphi = \varphi(r,z) \in H_0^2(\Omega)$ such that

$$ru_r = \partial_z\varphi, \quad ru_z = -\partial_r\varphi.$$

For any fixed scalar function $\eta \in L^\infty(\mathbb{R};[0,\infty))$, if we assume the density function to be

$$\rho = \eta(\varphi),$$

then the mass conservation law

$$r\operatorname{div}(\rho u) = \partial_r\rho\partial_z\varphi - \partial_z\rho\partial_r\varphi = 0$$

holds at least in the distribution sense.

We have the following existence results.

Theorem 1.2.4 (Existence of weak solutions in the three dimensional axially symmetric case)**.** *Let $\eta \in L^\infty(\mathbb{R};[0,\infty))$ and $b \in C(\mathbb{R};[\mu_*,+\infty))$, $\mu_* > 0$ be given. Let Ω be a bounded connected axially symmetric Lipschitz domain. Let $f \in H^{-1}(\Omega;\mathbb{R}^3)$ be axially symmetric function. Then there exists at least one axially symmetric weak solution*

$$(\rho, u) = (\eta(\varphi), \frac{1}{r}\partial_z\varphi e_r - \frac{1}{r}\partial_r\varphi e_z + u_\theta e_\theta) \in L^\infty(\Omega) \times H(\Omega)$$

of the three-dimensional system (1.32)*, where $\varphi \in H_0^2(\Omega)$ is a stream function of u.*

This existence result can be generalised to non-homogeneous boundary value problem (1.32)-(1.33) on bounded or exterior domains or $\mathbb{R}^3$.

We can obtain the existence of weak solutions under another symmetric assumption

$$(\rho, u) = (\eta(\theta), u_r e_r + u_z e_z) \in L^\infty(\Omega) \times H^1(\Omega).$$

In this case, the mass conservation law also holds immediately

$$r\operatorname{div}(\rho u) = r\partial_r\rho u_r + r\partial_z\rho u_z + \partial_\theta\rho u_\theta = 0.$$

The analogue existence results in spherical and Cartesian coordinates will be given in Section 2.3.

1.3 Evolutionary Boussinesq equations

The Boussinesq system is the nonlinear coupling between the Navier-Stokes type of equations and the thermodynamic equations for the temperature (or density) functions. In this subsection, we consider the Cauchy problem of the two-dimensional incompressible evolutionary Boussinesq equations

$$\begin{cases} \operatorname{div}_x u = 0, \\ \partial_t \theta + u \cdot \nabla_x \theta - \operatorname{div}_x(\kappa \nabla_x \theta) = 0, \\ \partial_t u + u \cdot \nabla_x u - \operatorname{div}_x(\mu S_x u) + \nabla_x \Pi = \theta \vec{e_2}, \\ (\theta, u) \mid_{t=0} = (\theta_0, u_0). \end{cases} \tag{1.39}$$

The unknowns are the temperature function $\theta = \theta(t,x) : [0,+\infty) \times \mathbb{R}^2 \to \mathbb{R}$, the velocity vector field $u = u(t,x) : [0,+\infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ and the pressure $\Pi = \Pi(t,x) : [0,+\infty) \times \mathbb{R}^2 \to \mathbb{R}$. The vector field $\vec{e_2}$ denotes the unit vector in the vertical direction with $\vec{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $\theta \vec{e_2}$ is the buoyancy force.

The thermal diffusivity κ and the viscosity coefficient μ may depend on the temperature function θ smoothly as follows

$$\kappa = a(\theta), \quad \mu = b(\theta), \quad \text{with } a \in C_b^1(\mathbb{R}; [\kappa_*, \kappa^*]),\ b \in C_b^1(\mathbb{R}; [\mu_*, \mu^*]),$$

where $0 \leqslant \kappa_* \leqslant \kappa^*$, $0 \leqslant \mu_* \leqslant \mu^*$ are positive constants.

1.3.1 Constant thermal and viscosity coefficients

When k, $\nu \geqslant 0$ are constant thermal diffusivity and viscosity coefficient, the Cauchy problem (1.39) reads as

$$\begin{cases} \operatorname{div}_x u = 0, \\ \partial_t \theta + u \cdot \nabla_x \theta - k \Delta_x \theta = 0, \\ \partial_t u + u \cdot \nabla_x u - \nu \Delta_x u + \nabla_x \Pi = \theta \vec{e_2}, \\ (\theta, u)|_{t=0} = (\theta_0, u_0), \end{cases} \tag{1.40}$$

If k, $\nu > 0$, the strong diversities lead to the global well-posed smooth solutions, see for example [CD80]. If $k > 0$, $\nu = 0$ or $k = 0$, $\nu > 0$, the unique global-in-time unique smooth solutions were obtained, see for example [Cha06; HL05]. In their results, the key point is to use the sharp Sobolev embedding in dimension two with a logarithm correction

$$\begin{aligned} \|\nabla u\|_{L^\infty(\mathbb{R}^2)} \leqslant & C(\|\Delta u\|_{L^2(\mathbb{R}^2)} + \|\nabla u\|_{L^2(\mathbb{R}^2)} + 1) \\ & \times (\log(\|\Delta \nabla u\|_{L^2(\mathbb{R}^2)} + \|\nabla u\|_{L^2(\mathbb{R}^2)} + e))^{\frac{1}{2}}, \end{aligned}$$

which provides the Lipschitz continuity on u, and hence the global well-posedness results. There are some work concerning horizontal and vertical dissipation, see for example [DP11; LLT13; ACW10; ACW11; CW13]. For $\kappa = \mu = 0$, the two-dimensional inviscid Boussinesq equations (1.40) can be compared with the three-dimensional incompressible axisymmetric Euler equations with swirl. The local-in-time wellposedness as well as some blowup criteria have been well known for decades, see e.g. [CN97; Dan13; ES94; Dan13]. The global-in-time regularity problem for the Euler equation is a remarkable open problem, for which the partial results are given in [EJ20; Elg21].

1.3.2 Temperature-dependent thermal and viscosity coefficients

In this subsection, we consider the Boussinesq equation (1.39) with temperature-dependent κ and μ as following

$$\kappa = a(\theta), \quad \mu = b(\theta), \qquad \text{with } a \in C_b^1(\mathbb{R}; [\kappa_*, \kappa^*]),\ b \in C_b^1(\mathbb{R}; [\mu_*, \mu^*]), \tag{1.41}$$

where $0 \leqslant \kappa_* \leqslant \kappa^*$, $0 \leqslant \mu_* \leqslant \mu^*$ are positive constants.

Lorca and Boldrini [LB99] proved the global-in-time weak solutions on the smooth bounded domain with

$$(\theta, u) \in (L^\infty_{\text{loc}}([0, \infty); L^2(\Omega)))^3,$$

and they also showed the local-in-time strong solutions. Wang and Zhang [WZ11] showed the existence of global unique smooth solutions

$$(\theta, u) \in (L^\infty_{\text{loc}}([0, \infty); H^s(\mathbb{R}^2)) \cap L^2_{\text{loc}}([0, \infty); H^{s+1}(\mathbb{R}^2)))^3, \quad s > 2.$$

Notice that $H^s(\mathbb{R}^2) \hookrightarrow W^{1,\infty}(\mathbb{R}^2)$, for $s > 2$. Sun and Zhang [SZ13] showed the global well-posedness result with $(\theta_0, u_0) \in (H^2(\mathbb{R}^2))^3$ on the bounded smooth domain.

Existence, uniqueness and regularity results in general Sobolev spaces

The author and Liao [HL22] showed the existence, uniqueness and regularity results in the (optimal) regularity exponent ranges respectively, in particular in the low regularity region $s < 2$.

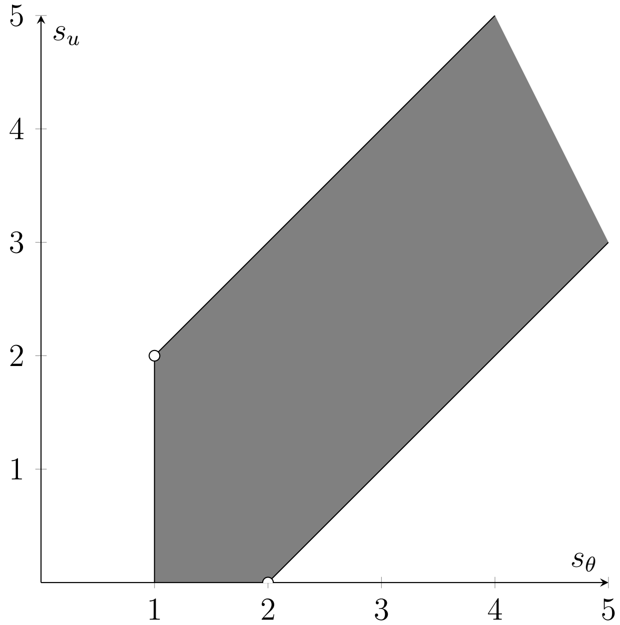

Figure 1.2: Admissible regularity exponents of (1.39)

Theorem 1.3.1 (Existence, uniqueness & regularity, [HL22])**.** *For any initial data $\theta_0 \in L^2(\mathbb{R}^2)$ and $u_0 \in (L^2(\mathbb{R}^2))^2$, there exists a global-in-time weak solution*

$$(\theta, u) \in C([0,\infty); (L^2(\mathbb{R}^2))^3) \cap L^2_{loc}([0,\infty); (H^1(\mathbb{R}^2))^3)$$

of the initial value problem (1.39)-(1.41)*.*

If $\theta_0 \in H^1(\mathbb{R}^2)$, $u_0 \in (L^2(\mathbb{R}^2))^2$ and the functions $a \in C^2_b(\mathbb{R}; [\kappa_*, \kappa^*])$, $b \in C^2_b(\mathbb{R}; [\mu_*, \mu^*])$ have finite first and second derivatives, then the weak solution is indeed unique, and satisfies

$$\theta \in C([0,\infty); H^1(\mathbb{R}^2)) \cap L^2_{loc}([0,\infty); H^2(\mathbb{R}^2)).$$

Furthermore, the general H^s-regularities can be propagated globally in time in the following sense: For any initial data

$$(\theta_0, u_0) \in H^{s_\theta}(\mathbb{R}^2) \times (H^{s_u}(\mathbb{R}^2))^2 \text{ with}$$
$$(s_\theta, s_u) \in \{(s_\theta, s_u) \subset [1,\infty) \times [0,\infty) \mid s_u - 1 \leqslant s_\theta \leqslant s_u + 2\} \backslash \{(2,0),(1,2)\}$$

and the functions $a \in C^2_b \cap C^{[s_\theta]+1}$, $b \in C^2_b \cap C^{[s_u]+1}$, the unique solution (θ, u) stays in

$$C([0,\infty); H^{s_\theta}(\mathbb{R}^2) \times (H^{s_u}(\mathbb{R}^2))^2) \cap L^2_{loc}([0,\infty); H^{s_\theta+1}(\mathbb{R}^2) \times (H^{s_u+1}(\mathbb{R}^2))^2).$$

The grey area in Figure 1.2 represent the admissible regularity exponent range. In the following, we will give some explanations and discussions on the optimality of the regularity exponents for the solutions of the initial value problem (1.39)-(1.41) in Theorem 1.3.1. We will also discuss the influence of the variation of the variable coefficients κ and μ on the uniqueness and regularity properties.

Existence and uniqueness results.

The global-in-time weak solutions can be seen as a consequence of energy estimates. Indeed, we take L^2-inner product of $(1.39)_2$ and θ, $(1.39)_3$ and u, then the following energy estimates hold for any given $T > 0$,

$$\begin{aligned}
&\|\theta\|^2_{L^\infty_T L^2_x} + \|\nabla\theta\|^2_{L^2_T L^2_x} \leqslant C\|\theta_0\|^2_{L^2_x},\\
&\|u\|^2_{L^\infty_T L^2_x} + \|\nabla u\|^2_{L^2_T L^2_x} \leqslant C\big(T\|\theta_0\|^2_{L^2} + \|u_0\|^2_{L^2}\big),
\end{aligned}$$

where the positive constant C depends only on κ_*, μ_*.

Concerning the uniqueness result with $(\theta_0, u_0) \in H^1(\mathbb{R}^2) \times (L^2(\mathbb{R}^2))^2$, we have the following H^1-estimate of θ:

$$\begin{aligned}
&\|\theta\|^2_{L^\infty_T H^1_x} + \|(\partial_t\theta, \nabla^2\theta)\|^2_{L^2_T L^2_x}\\
&\leqslant C\|\theta_0\|^2_{H^1}(1 + \|\nabla\theta_0\|^2_{L^2})\exp\big(C(T^2\|\theta_0\|^4_{L^2} + \|u_0\|^4_{L^2})\big).
\end{aligned}$$

Based on the above energy estimates, one can show the difference $(\dot\theta, \dot u) \in (L^\infty_T H^1_x \cap L^2_T H^2_x) \times (L^\infty_T L^2_x \cap L^2_T H^1_x)$ of two weak solutions with the same initial value is indeed trivial in the Sobolev space $H^{1+\delta}(\mathbb{R}^2) \times (H^\delta(\mathbb{R}^2))^2$ with $\delta \in (-1, 0)$. On the other hand, non-uniqueness is expected in the lower-regularity region with $\theta_0 \in H^s(\mathbb{R}^2) \not\hookrightarrow L^\infty(\mathbb{R}^2)$, $0 < s < 1$, since the coefficients κ, μ are in general not continuous uniformly in time.

Regularity results.

To show the propagation of regularity, in the lower regularity region θ_0 or $u_0 \in H^s(\mathbb{R}^2) \not\hookrightarrow W^{1,\infty}(\mathbb{R}^2)$, with $s \leqslant 2$, we use the commutator estimate

$$\big\|\big(2^{js}\|[\phi, \Delta_j]\nabla\psi\|_{L^2(\mathbb{R}^2)}\big)_{j\geqslant 1}\big\|_{l^1} \leqslant C\|\nabla\phi\|_{H^\nu(\mathbb{R}^2)}\|\nabla\psi\|_{H^{s-\nu}(\mathbb{R}^2)},$$

where the exponents $(s, \nu) \in \mathbb{R}^2$ satisfies $-1 < s < \nu + 1$ and $-1 < \nu < 1$, and the positive constant C depends only on s, ν. In the high regularity region $\theta_0, u_0 \in H^s(\mathbb{R}^2) \hookrightarrow W^{1,\infty}(\mathbb{R}^2)$, with $s > 2$, we use the commutator estimate

$$\begin{aligned}
&\big\|\big(2^{js}\|[\phi, \Delta_j]\nabla\psi\|_{L^2(\mathbb{R}^2)}\big)_{j\geqslant 1}\big\|_{l^2}\\
&\leqslant C\big(\|\nabla\phi\|_{L^\infty(\mathbb{R}^2)}\|\nabla\psi\|_{H^{s-1}(\mathbb{R}^2)} + \|\nabla\phi\|_{H^{s-1}(\mathbb{R}^2)}\|\nabla\psi\|_{L^\infty(\mathbb{R}^2)}\big).
\end{aligned}$$

On the other hand, to propagate the H^{s_θ}, $s_\theta \geqslant 2$-regularity of θ, we require the transport term $u\cdot\nabla\theta$ in the θ-equation to be at least in $L^2_{\mathrm{loc}}([0,\infty); H^{s_\theta-1}_x)$,

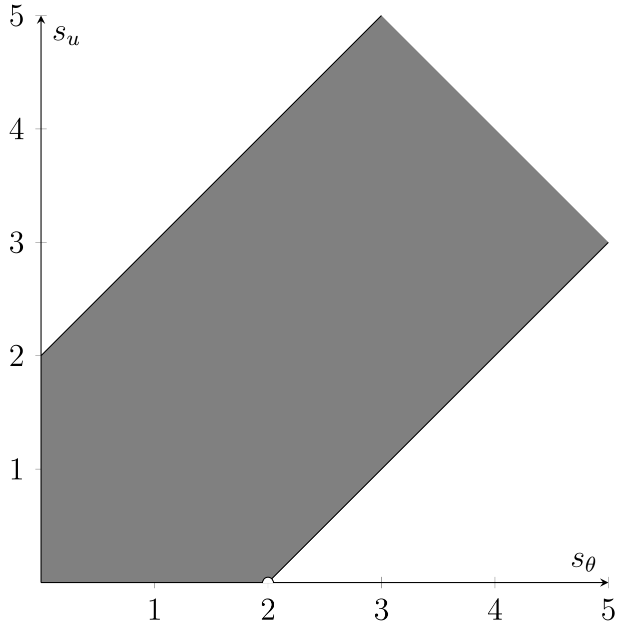

Figure 1.3: Admissible regularity exponents of (1.40)

which requires $u \in L^2_{\mathrm{loc}}([0,\infty); H^{s_\theta-1}_x)$ and hence the initial assumption $u_0 \in H^{s_u}$ with the restriction $s_u \geqslant s_\theta - 2$. Similarly, in order the propagate the H^{s_u}, $s_u \geqslant 2$-regularity of u, we require the viscosity term $\operatorname{div}(\mu S u)$ in the u-equation to be at least in $L^2_{\mathrm{loc}}([0,\infty); H^{s_u-1}_x)$, which requires $\mu S u \in L^2_{\mathrm{loc}}([0,\infty); H^{s_u}_x)$ and hence the initial assumption $\theta_0 \in H^{s_\theta}$ with the restriction $s_\theta \geqslant s_u - 1$.

Remark 1.3.1 (Excepted admissible regularity exponent ranges for the constant coefficient case (1.40))**.** *We expect that a similar argument as for the equation* (1.39) *can also show the optimal optimal regularity exponent range for the equation* (1.40) *with constant thermal and viscous coefficients. More precisely, the H^s-regularity of the equation* (1.40) *is expected to propagated in the following range (see the grey area in Figure 1.3)*

$$(\theta_0, u_0) \in H^{s_\theta}(\mathbb{R}^2) \times (H^{s_u}(\mathbb{R}^2))^2 \text{ with}$$
$$(s_\theta, s_u) \in \{(s_\theta, s_u) \subset [0,\infty) \times [0,\infty) \mid s_u - 1 \leqslant s_\theta \leqslant s_u + 2\} \backslash \{(2,0)\}.$$

We compare the Figure 1.2 and the Figure 1.3 to see that the temperature-dependent coefficients κ and μ indeed influence the admissible regularity exponent range.

For the equation (1.40)*, the coupling between θ and u happens only through the transport term $u \cdot \nabla_x \theta$ in the equation* $(1.40)_2$ *and the force term θe_2 in* *the equation* $(1.40)_3$. *The solution is unique with* $(\theta_0, u_0) \in (L^2(\mathbb{R}^2))^3$. *Similar to the variable thermal and viscosity coefficients case, to propagate the* H^{s_θ}, $s_\theta \geqslant 2$*-regularity of* θ, *we require the transport term* $u \cdot \nabla_x \theta$ *to be at least in* $L^2_{loc}([0,\infty); H_x^{s_\theta - 1})$, *and hence the initial value* $u_0 \in H^{s_u}$ *with* $s_u \geqslant s_\theta - 2$ *is required. To propagate the* H^{s_u}, $s_u \geqslant 2$*-regularity of* u, *we require* θe_2 *to be at least in* $L^2_{loc}([0,\infty); H_x^{s_u - 1})$, *which requires* $\theta_0 \in H^{s_\theta}$ *with the restriction* $s_\theta \geqslant s_u - 2$.

Partial diffusion case.

Concerning the case with thermal diffusion $\kappa = a(\theta) > 0$ while no viscous diffusion $\mu = 0$, Li and Xu [LX13] and Chen and Jiang [CJ14] showed the existence of the unique global-in-time smooth solution

$$\begin{aligned} &\theta \in C([0,+\infty); H^s(\mathbb{R}^2)) \cap L^2_{\mathrm{loc}}([0,+\infty); H^{s+1}(\mathbb{R}^2)), \\ &u \in C([0,+\infty); (H^s(\mathbb{R}^2))^2), \quad s > 0. \end{aligned}$$

We remark that the structure of the system (1.39) lacking of heat diffusion (i.e. $\kappa = 0$, $\mu = b(\theta)$) is similar but simpler compared to the two-dimensional inhomogeneous incompressible Navier-Stokes equations (1.4) with density-dependent viscosity coefficient. The similar difficulties of the regularity and uniqueness issues arise, and it is still not clear whether there will be finite time singularity.

There are some work approaching this case, where less heat diffusion is assumed, namely,

$$\partial_t \theta + u \cdot \nabla \theta + \nu |D|^\alpha \theta = 0, \quad \nu > 0,$$

where $|D|^\alpha$ is the Fourier multiplier defined as

$$\widehat{|D|^\alpha f}(\xi) = |\xi|^\alpha \hat{f}(\xi).$$

Notice that $\alpha = 2$ recovers the heat equation in (1.39). Under the small viscosity variation assumption (1.21)

$$\|\mu - 1\|_{L^\infty} < \varepsilon,$$

Abidi and Zhang [AZ17] studied the cases when $\alpha = 1$; Dong, Ye, and Zhai [DYZ20] considered the case when $0 < \alpha < 1$.

Chapter 2

Two- and three-dimensional incompressible inhomogeneous Navier–Stokes equations with variable viscosity coefficient

In this chapter, Section 2.1 and Section 2.2 are devoted to showing the existence and the regularity properties of (a class of) weak solutions to the two-dimensional stationary incompressible inhomogeneous Navier–Stokes equations with variable viscosity coefficient, by analyzing a fourth-order nonlinear elliptic equation for the stream function. The density function and the viscosity coefficient may have large variations. In addition, we formulate the solutions for the parallel, concentric and radial flows respectively, and we give some irregularity results as well as some explicit examples in the case of piecewise-constant viscosity coefficients. The regularity of the divergence-free part of the viscous term is discussed separately.

In Section 2.3, we show the existence of (a class of) weak solutions to the three-dimensional stationary incompressible inhomogeneous Navier–Stokes equations with density-dependent viscosity coefficient in the axially symmetric case. More symmetric solutions in cylindrical coordinate, spherical coordinate and cartesian coordinate are also discussed.

Section 2.1 and Section 2.2 are based on the joint work with JProf. Xian Liao in [HL20].

2.1 Main results in two-dimensional case

In Section 2.1 and Section 2.2, we are going to study the two-dimensional stationary inhomogeneous incompressible Navier–Stokes equations

$$\begin{cases} \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu S u) + \nabla \Pi = f, \\ \operatorname{div} u = 0, \quad \operatorname{div}(\rho u) = 0. \end{cases} \tag{2.1}$$

The unknown density function $\rho \geqslant 0$, the unknown velocity vector field $u = (u_1, u_2) \in \mathbb{R}^2$ and the unknown pressure $\Pi \in \mathbb{R}$ depend on the spatial variable $x = (x_1, x_2) \in \mathbb{R}^2$. The variable viscosity coefficient depends continuously on the density function

$$\mu = b(\rho), \quad b \in C(\mathbb{R}_+; [\mu_*, \mu_*]) \quad \text{given},$$

where μ_*, μ_* are positive constants.[1] The external force $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is given.

Let $\nabla = \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \end{pmatrix}$, then $\nabla u = \begin{pmatrix} \partial_{x_1} u_1 & \partial_{x_2} u_1 \\ \partial_{x_1} u_2 & \partial_{x_2} u_2 \end{pmatrix}$ and the deformation strain tensor in (2.1) reads as

$$S u \stackrel{\text{def}}{=} (\nabla + \nabla^T) u = \begin{pmatrix} 2\partial_{x_1} u_1 & \partial_{x_2} u_1 + \partial_{x_1} u_2 \\ \partial_{x_2} u_1 + \partial_{x_1} u_2 & 2\partial_{x_2} u_2 \end{pmatrix}.$$

Let div $= \nabla \cdot$ and $u \otimes u = \begin{pmatrix} {u_1}^2 & u_1 u_2 \\ u_1 u_2 & {u_2}^2 \end{pmatrix}$, then the convection term in (2.1) reads as

$$\operatorname{div}(\rho u \otimes u) = \begin{pmatrix} \partial_{x_1}(\rho {u_1}^2) + \partial_{x_2}(\rho u_1 u_2) \\ \partial_{x_1}(\rho u_1 u_2) + \partial_{x_2}(\rho {u_2}^2) \end{pmatrix}.$$

We will give the existence and the regularity properties of the weak solutions, of Frolov's form (2.14) below $(\rho, u) = (\eta(\Phi), \nabla^\perp \Phi)$, to the stationary Navier–Stokes equations (2.1) on a two-dimensional domain $\Omega \subset \mathbb{R}^2$ in Subsection 2.1.2 (the proof of which is postponed in Section 2.2). Here Ω could be a bounded (simply) connected $C^{1,1}$ domain, or the exterior domain of a bounded connected $C^{1,1}$ set, or the whole plane $\mathbb{R}^2$. If Ω has a boundary $\partial\Omega$, we associate the system (2.1) with the following boundary value condition

$$u|_{\partial\Omega} = u_0, \tag{2.2}$$

and assume no flux through the boundary $\partial\Omega$

$$\int_{\partial\Omega} u_0 \cdot n \, ds = 0. \tag{2.3}$$

[1] It is sufficient to assume $b \in C(\mathbb{R}_+; [\mu_*, +\infty))$, since we are going to consider bounded density functions ρ. For convenient, we fix an upper bound μ^* for b here.

In the above, $n = (n_1, n_2)$ denotes the exterior normal to the boundary $\partial\Omega$. We will consider indeed the stream function Φ, such that the velocity vector field

$$u = \nabla^\perp \Phi \stackrel{\text{def}}{=} \begin{pmatrix} \partial_{x_2}\Phi \\ -\partial_{x_1}\Phi \end{pmatrix}.$$

We will solve the fourth-order nonlinear elliptic equation which is derived by applying $\nabla^\perp\cdot$ to the velocity equation $(2.1)_3$

$$L_\mu \Phi = -\nabla^\perp \cdot f + \nabla^\perp \cdot \operatorname{div}(\rho \nabla^\perp \Phi \otimes \nabla^\perp \Phi),$$

where L_μ denotes the fourth-order elliptic operator

$$L_\mu = (\partial_{x_2x_2} - \partial_{x_1x_1})\mu(\partial_{x_2x_2} - \partial_{x_1x_1}) + (2\partial_{x_1x_2})\mu(2\partial_{x_1x_2}),$$

and the identity $\nabla^\perp \cdot \operatorname{div}(\mu S u) = L_\mu \Phi$ holds. Then for any given nonnegative bounded measurable function $\eta \in L^\infty([0,\infty))$, the pair $(\rho, u) = (\eta(\Phi), \nabla^\perp\Phi)$ is a weak solution of the stationary Navier–Stokes system (2.1).

In Subsection 2.1.3 we are interested in the symmetric solutions to the stationary Navier–Stokes equations (2.1), and in particular we will formulate the solutions for the parallel, concentric and radial flows respectively:

$$(\rho, u) = \Big(\rho(x_2),\, u_1(x_2)\, e_1\Big), \text{ or } \Big(\rho(r),\, rg(r)\, e_\theta\Big), \text{ or } \Big(\rho(\theta),\, \frac{h(\theta)}{r} e_r\Big).$$

Here (r, θ) are the polar coordinates on $\mathbb{R}^2$, $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_r = \begin{pmatrix} \frac{x_1}{r} \\ \frac{x_2}{r} \end{pmatrix}$, $e_\theta = \begin{pmatrix} \frac{x_2}{r} \\ -\frac{x_1}{r} \end{pmatrix}$, and u_1, g, h are scalar functions satisfying three different second-order ordinary differential equations respectively (see Theorem 2.1.3 below for more details). In particular for the case of piecewise-constant viscosity coefficients, we derive some irregularity results such as (see Corollary 2.1.1 below)

$$\Delta u \notin L^1_{\text{loc}}(\mathbb{R}^2 \backslash \{0\}), \quad \operatorname{div}(\mu \nabla u) \notin L^1_{\text{loc}}(\mathbb{R}^2 \backslash \{0\}),$$

and we also calculate some explicit examples (see Examples 2.1.1 below).

On the other side, in Subsection 2.1.4, we will show the L^p-boundedness of the divergence-free part of the viscous term (see Theorem 2.1.4)

$$\mathbb{P}\operatorname{div}(\mu S u) \in L^p(\Omega)$$

for the given solutions (e.g. the solutions given in Theorem 2.1.2 or in Theorem 2.1.3). Here $\mathbb{P}$ is the Leray-Helmholtz projector on a bounded C^1 domain

Ω. We observe here also the following (formal) one-to-one correspondence between $\mathbb{P}\operatorname{div}(\mu Su)$ and $L_\mu\Phi$:

$$L_\mu\Phi = \nabla^\perp \cdot \mathbb{P}\operatorname{div}(\mu Su), \quad \mathbb{P}\operatorname{div}(\mu Su) = \nabla^\perp\Delta^{-1}L_\mu\Phi.$$

We will conclude this introduction part with some further discussions on the fourth-order elliptic operator L_μ.

We first start with some related known works for the incompressible Navier–Stokes equations in Subsection 2.1.1 below.

2.1.1 Related works

There are a few works in the literature contributing to the study of the *evolutionary* two-dimensional incompressible inhomogeneous Navier–Stokes equations with variable viscosity coefficient

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \quad (t,x) \in \mathbb{R}^+ \times \Omega, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu Su) + \nabla\Pi = 0, \\ \operatorname{div} u = 0, \quad (t,x) \in \mathbb{R}^+ \times \Omega, \\ \rho\big|_{t=0} = \rho_0, \quad (\rho u)\big|_{t=0} = m_0. \end{cases} \tag{2.4}$$

In [Lio96], P. L. Lions showed the global-in-time existence of weak solutions $(\rho, u) \in \big(L^\infty(\mathbb{R}^+ \times \Omega), (L^2(\mathbb{R}^+; H^1(\Omega)))^2\big)$ of the system (2.4) under the initial condition $\rho_0 \in L^\infty(\Omega)$, $\frac{m_0}{\rho_0} \in (L^2(\Omega))^2$. The uniqueness and the regularity properties of such weak solutions are still open, even in dimension two. There are some partial results toward this issue, but to our best knowledge they are all limited to the case where the viscosity coefficient $\mu(x)$ is close to some positive constant $\nu \in \mathbb{R}^+$:

$$\|\mu(x) - \nu\|_{L^\infty(\Omega)} < \varepsilon, \tag{2.5}$$

where ε is some small enough positive constant. B. Desjardins in [Des97] showed the regularity property of the velocity vector field $u \in L^\infty(\mathbb{R}^+; (H^1(\mathbb{T}^2))^2)$ for initial data $u|_{t=0} \in (H^1(\mathbb{T}^2))^2$, if the smallness condition (2.5) holds. H. Abidi and P. Zhang [AZ15a] proved the existence and uniqueness of the solution under (2.5) and further smoothness assumptions on the initial density function $\rho_0 - 1 \in L^2(\mathbb{R}^2) \cap L^\infty \cap \dot{W}^{1,r}(\mathbb{R}^2)$, $r > 2$. M. Paicu and P. Zhang in [PZ20] considered the so-called density-patch problem with piecewise-constant density function $\rho_0 = \eta_1\mathbb{1}_\Omega(x) + \eta_2\mathbb{1}_{\Omega^C}(x)$, $\eta_1, \eta_2 \in \mathbb{R}^+$, and showed that the $H^3(\mathbb{R}^2)$-boundary regularity of the domain is propagated by time evolution provided with (2.5). The case where $\mu(x) = \nu$ is a positive constant has been intensively studied in the past two decades, see e.g. [Dan04; DM19b; LS75] and the references therein. It is also worth mentioning the work [VK97] for the study of the *compressible* Navier–Stokes equations with variable viscosity coefficient.

If we consider the stationary *homogeneous* incompressible flow where the density function $\rho = 1$ and the viscosity coefficient $\mu = \nu$ is a positive constant, then the system (2.1) becomes the following classical stationary Navier–Stokes equations

$$\begin{cases} \operatorname{div}(u \otimes u) - \nu \Delta u + \nabla \Pi = f, \quad x \in \Omega, \\ \operatorname{div} u = 0, \\ u|_{\partial\Omega} = u_0. \end{cases} \tag{2.6}$$

It has been studied extensively in the literature, whenever the underlying domain is a connected bounded domain Ω , or a multi-connected domain $\cup_{i=1}^n \Omega_i$, or the exterior of a multi-connected set $U = (\cup_{i=1}^n \Omega_i)^C$, or the whole plane $\mathbb{R}^2$, see the celebrated books [Gal11; Lad69]. J. Leray [Ler33] showed the existence of weak solutions $u \in (H^1(\Omega))^2$ on a simply connected bounded domain Ω under the zero flux condition (2.3). This solvability result can be generalized straightforward to a multi-connected domain case $\cup_{i=1}^n \Omega_i$, if we assume no flux through the boundary of each connected component

$$\mathcal{F}_i = \int_{\partial\Omega_i} u_0 \cdot n \, ds = 0, \quad \forall 1 \leqslant i \leqslant n. \tag{2.7}$$

If we assume only the smallness of the fluxes $\mathcal{F}_i$ or assume some further symmetric properties, the solvability of the system (2.6) was also obtained, cf. [Gal91]. On a multi-connected domain with only the zero *total flux* condition (2.3), the solvability was shown by M. Korobkov, K. Pileckas and R. Russo in [KPR15]. J. Leray in [Ler33] studied the system (2.6) also on the exterior domain of a multi-connected set $U = (\cup_{i=1}^n \Omega_i)^C$ under the boundary condition (2.7), and obtained the weak solutions $u \in (\dot{H}^1(U))^2$ by constructing a sequence of weak solutions on the bounded domains which converge to U. If the fluxes $\mathcal{F}_i$ are small, the solvability of (2.6) on U was established by R. Finn in [Fin59]. Concerning the whole plane $\mathbb{R}^2$ case, J. Guillod and P. Wittwer [GW18] showed that for any given vector $d \in \mathbb{R}^2$ and a bounded positive measure set $D \subset \mathbb{R}^2$, there exist solutions $u \in (\dot{H}^1(\mathbb{R}^2))^2$ satisfying the prescribed mean value on D: $d = \frac{1}{\operatorname{meas}(D)} \int_D u \in \mathbb{R}^2$. However, the existence of decaying solutions, as well as the uniqueness and the asymptotic behaviour of the solutions on the unbounded domains are still open, see e.g. [GW15b; Gui17; Rus09] for further related discussions. We also mention that J. Leray in [Ler33] studied also (2.6) in dimension three, as well as the *evolutionary* classical Navier-Stokes equations (i.e. (2.4) with $\rho = 1$ and $\mu = \nu$, see also the celebrated books [CF88; Tem77]).

The stationary inhomogeneous incompressible flow with *constant* viscosity coefficient is described by

$$\begin{cases} \operatorname{div}(\rho u \otimes u) - \nu \Delta u + \nabla \Pi = f, \quad x \in \Omega, \\ \operatorname{div} u = 0, \operatorname{div}(\rho u) = 0, \\ u|_{\partial\Omega} = u_0. \end{cases}$$

On a simply connected domain in dimension two, by using the incompressibility condition $\operatorname{div} u = 0$ and the zero flux condition (2.3), the velocity vector field u can be written as

$$u = \nabla^{\perp}\Phi,$$

where Φ is the stream function of u. If

$$\rho = \eta(\Phi),$$

for some well-chosen function $\eta : \mathbb{R} \mapsto [0, \infty)$, then the density equation $\operatorname{div}(\rho u) = 0$ should be automatically satisfied. N.N. Frolov showed in [Fro93] the existence and regularity results for the solutions of the following form

$$(\rho, u) = (\eta(\Phi), \nabla^{\perp}\Phi), \tag{2.8}$$

where η is a *Hölder continuous* function. From now on, we call the form (2.8) as Frolov's form. M. Santos in [San02] improved this existence result to only *bounded* η-functions. M. Santos and F. Ammar-Khodja in [AS05; AS06] considered the unbounded Y-shape domain.

However, as far as we know, there are neither existence nor regularity results of solutions to the two-dimensional stationary Navier-Stokes system (2.1) with *variable* viscosity coefficient. The rest of this introduction part is organised as follows. We are going to give some existence and regularity results for the solutions of Frolov's form to the system (2.1) in Subsection 2.1.2, whose proof is postponed in Section 2.2. We will formulate the solutions with certain symmetry properties in Subsection 2.1.3, where some irregularity results as well as some explicit examples with piecewise-constant viscosity coefficients will also be given. Finally we will discuss further the regularity issues in Subsection 2.1.4.

2.1.2 Existence and regularity results

We study here the two dimensional stationary inhomogeneous incompressible Navier-Stokes equation (2.1) with general variable viscosity coefficient

$$\mu = b(\rho),$$

where $b \in C(\mathbb{R}; [\mu_*, \mu_*])$, $\mu_*, \mu_* > 0$ is a given function. We will search for the weak solutions of Frolov's form (2.8) above. We are going to consider the stream function Φ, which satisfies a fourth-order nonlinear elliptic equation (see (2.10) below). To our best knowledge, this is the first time such elliptic equation for the stream function has been found in the case of variable viscosity coefficient.

If $\Omega \subset \mathbb{R}^2$ is a simply connected C^1 domain, by the zero flux condition (2.3) and the divergence free condition $\operatorname{div} u = 0$, there exists a stream function $\Phi : \overline{\Omega} \to \mathbb{R}$ such that

$$u = \nabla^\perp \Phi \stackrel{\text{def}}{=} \begin{pmatrix} \partial_{x_2}\Phi \\ -\partial_{x_1}\Phi \end{pmatrix},$$

and Φ satisfies the boundary value condition

$$\frac{\partial \Phi}{\partial n}\Big|_{\partial\Omega} = u_0 \cdot \tau, \quad \frac{\partial \Phi}{\partial \tau}\Big|_{\partial\Omega} = -u_0 \cdot n,$$

where $\tau = (n_2, -n_1)$ denotes the tangential vector field on the boundary $\partial\Omega$. If we parameterize the boundary $\partial\Omega$ by $\gamma : [0, 2\pi) \mapsto \partial\Omega$ such that $\gamma'(s) = \tau(\gamma(s))$, then with a constant $C_0 \in \mathbb{R}$,

$$\begin{aligned} &\Phi|_{\partial\Omega}(\gamma(s)) = \Phi_0(\gamma(s)) \stackrel{\text{def}}{=} -\int_0^s u_0 \cdot n \, d\theta + C_0, \quad s \in [0, 2\pi), \\ &\frac{\partial \Phi}{\partial n}\Big|_{\partial\Omega}(\gamma(s)) = \Phi_1(\gamma(s)) \stackrel{\text{def}}{=} (u_0 \cdot \tau)(\gamma(s)), \quad s \in [0, 2\pi). \end{aligned} \tag{2.9}$$

We fix this constant $C_0 = 0$ from now on.

We apply $\nabla^\perp \cdot = \begin{pmatrix} \partial_{x_2} \\ -\partial_{x_1} \end{pmatrix} \cdot$ to the first equation in (2.1) to arrive at

$$\nabla^\perp \cdot \operatorname{div}(\mu S u) = -\nabla^\perp \cdot f + \nabla^\perp \cdot \operatorname{div}(\rho u \otimes u),$$

where the left-hand side reads as a fourth-order elliptic operator with positive variable coefficient $\mu \geqslant \mu_* > 0$ on Φ:

$$\begin{aligned} \nabla^\perp \cdot \operatorname{div}(\mu S u) &= \nabla^\perp \cdot \operatorname{div}\left(\mu \begin{pmatrix} 2\partial_{x_1x_2}\Phi & (\partial_{x_2x_2} - \partial_{x_1x_1})\Phi \\ (\partial_{x_2x_2} - \partial_{x_1x_1})\Phi & -2\partial_{x_1x_2}\Phi \end{pmatrix}\right) \\ &= (\partial_{x_2x_2} - \partial_{x_1x_1})\big(\mu(\partial_{x_2x_2} - \partial_{x_1x_1})\Phi\big) + 2\partial_{x_1x_2}(\mu 2\partial_{x_1x_2}\Phi). \end{aligned}$$

That is, the first equation in (2.1) becomes

$$L_\mu \Phi = -\nabla^\perp \cdot f + \nabla^\perp \cdot \operatorname{div}(\rho \nabla^\perp \Phi \otimes \nabla^\perp \Phi), \tag{2.10}$$

where L_μ denotes the fourth-order elliptic operator

$$L_\mu = (\partial_{x_2x_2} - \partial_{x_1x_1})\mu(\partial_{x_2x_2} - \partial_{x_1x_1}) + (2\partial_{x_1x_2})\mu(2\partial_{x_1x_2}).$$

In particular, if $\mu(x) = \nu$ is a positive constant, then $L_\nu = \nu\Delta^2$.

We recall here the definition of elliptic operators in divergence form of order $2m$, $m \in \mathbb{N}$ (see e.g. [ADN59; ADN64; DK11]) for readers' convenience. Let $\mathcal{L}u = \sum_{|\alpha|,|\beta|\leqslant m} D^\alpha(a_{\alpha\beta}D^\beta u)$ where α and β are multi-indices, $u : \mathbb{R}^d \to \mathbb{R}^n$ is a vector-valued function and $a_{\alpha\beta} = \left[a^{ij}_{\alpha\beta}(x)\right]^n_{i,j=1}$, $|\alpha|, |\beta| \leqslant m$, are $n \times n$ matrix-valued functions. We say that $\mathcal{L}$ is an elliptic operator of $2m$th-order if there exists a constant $\delta \in (0,1)$ such that

$$\delta|\xi|^2 \leqslant \sum_{|\alpha|=|\beta|=m} \mathrm{Re}\Big(a_{\alpha\beta}(x)\xi_\beta, \xi_\alpha\Big) \leqslant \delta^{-1}|\xi|^2,$$

for any $x \in \mathbb{R}^d$ and $\xi = (\xi_\alpha)_{|\alpha|=m}\,, \xi_\alpha \in \mathbb{R}^n$. Here we can rewrite L_μ as

$$\begin{aligned} L_\mu =&\partial_{x_1x_1}\mu\partial_{x_1x_1} + \partial_{x_2x_2}\mu\partial_{x_2x_2} - \partial_{x_1x_1}(\mu - \frac{\mu_*}{2})\partial_{x_2x_2} - \partial_{x_2x_2}(\mu - \frac{\mu_*}{2})\partial_{x_1x_1} \\ &+ 2\partial_{x_1x_2}(\mu - \frac{\mu_*}{2})\partial_{x_1x_2} + 2\partial_{x_2x_1}\mu\partial_{x_2x_1} \overset{\text{def}}{=} \sum_{|\alpha|=|\beta|=2} D^\alpha(a^\mu_{\alpha\beta}D^\beta), \end{aligned}$$

where μ_*, $\mu^* > 0$ are the positive lower and upper bounds for the function μ. Then for any $\xi = (\xi_\alpha)_{|\alpha|=2}\,, \xi_\alpha \in \mathbb{R}^2$ the following inequality holds:

$$\begin{aligned} &\frac{\mu_*}{2}|\xi|^2 \leqslant \sum_{|\alpha|=|\beta|=2} a^\mu_{\alpha\beta}(x)\xi_\beta\xi_\alpha \\ &= \frac{\mu_*}{2}(\xi_{11}^2 + \xi_{22}^2) + (\mu - \frac{\mu_*}{2})(\xi_{11} - \xi_{22})^2 + 2(\mu - \frac{\mu_*}{2})\xi_{12}^2 + 2\mu\xi_{21}^2 \leqslant 2\mu^*|\xi|^2. \end{aligned}$$

Hence, L_μ is a fourth-order elliptic operator as we can simply take $\delta = \min\{\frac{\mu_*}{2}, \frac{1}{2\mu^*}, \frac{1}{2}\}$.

Following Frolov's idea in [Fro93], we make an Ansatz

$$\rho = \eta(\Phi),$$

where the nonnegative function $\eta \in L^\infty(\mathbb{R}; [\rho_*, \rho^*])$ with $0 \leqslant \rho_* \leqslant \rho^*$ can be *arbitrarily* chosen, such that

$$\mathrm{div}\,(\rho u) = \mathrm{div}\,(\eta(\Phi)\nabla^\perp\Phi) = 0$$

holds in the distribution sense provided with e.g. $\Phi \in H^2_{\mathrm{loc}}(\Omega)$. Notice that, in this case we assume a priori that ρ satisfies the boundary condition $\rho|_{\partial\Omega} = \eta(\Phi_0)$ where Φ_0 is given in (2.9).

We then aim to verify that if Φ solves the boundary value problem for the fourth-order elliptic equation

$$\begin{cases} L_\mu\Phi = -\nabla^\perp \cdot f + \nabla^\perp \cdot \mathrm{div}\,(\rho\nabla^\perp\Phi \otimes \nabla^\perp\Phi), \\ \Phi|_{\partial\Omega} = \Phi_0, \quad \dfrac{\partial\Phi}{\partial n}\Big|_{\partial\Omega} = \Phi_1, \end{cases} \tag{2.11}$$

where $\rho = \eta(\Phi)$, $\mu = (b \circ \eta)(\Phi)$, then the pair of Frolov's form (2.8):

$$(\rho, u) = \big(\eta(\Phi),\, \nabla^{\perp}\Phi\big)$$

solves the boundary value problem for the stationary Navier-Stokes equations

$$\begin{cases} \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu S u) + \nabla \Pi = f, \\ \operatorname{div} u = 0,\ \operatorname{div}(\rho u) = 0, \\ u|_{\partial\Omega} = u_0, \end{cases} \tag{2.12}$$

where the boundary value u_0 satisfies the zero flux condition (2.3): $\int_{\partial\Omega} u_0 \cdot n\, ds = 0$ if Ω is a simply connected domain. We first take into account the functional framework to define the weak solutions.

Definition 2.1.1 (Weak solutions of the Navier–Stokes equations on a bounded connected domain). *Let $\Omega \subset \mathbb{R}^2$ be a bounded connected C^1-domain. We say that a pair $(\rho, u) \in L^\infty(\Omega; [0, \infty)) \times H^1(\Omega; \mathbb{R}^2)$ is a weak solution of the boundary value problem (2.12) with the given data $u_0 \in H^{\frac{1}{2}}(\partial\Omega; \mathbb{R}^2)$, $f \in H^{-1}(\Omega; \mathbb{R}^2)$, if $\operatorname{div} u = 0$, $\operatorname{div}(\rho u) = 0$ hold in Ω in the distribution sense, $u_0 = u|_{\partial\Omega}$ is the trace of u on $\partial\Omega$ and the following integral identity*

$$\frac{1}{2}\int_\Omega \mu S u : S v\, dx = \int_\Omega \rho (u \otimes u) : \nabla v\, dx + \int_\Omega f \cdot v, \tag{2.13}$$

holds for all $v \in H^1_0(\Omega; \mathbb{R}^2)$ with $\operatorname{div} v = 0$. Here $A : B \overset{def}{=} \sum_{i,j=1}^2 A_{ij} B_{ij}$ for the matrices $A = (A_{ij})_{1 \leqslant i,j \leqslant 2}$ and $B = (B_{ij})_{1 \leqslant i,j \leqslant 2}$.

In the above, $g \in H^{\frac{1}{2}}(\partial\Omega)$ the fractional Sobolev space means that $g \in L^2(\partial\Omega)$ with the following norm

$$\|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \overset{\text{def}}{=} \|g\|_{L^2(\partial\Omega)} + \Big(\int_{\partial\Omega}\int_{\partial\Omega} \frac{|g(s) - g(s')|^2}{|s - s'|^2} ds ds'\Big)^{\frac{1}{2}}$$

being finite. The Sobolev space $H^1_0(\Omega)$ is defined as $H^1_0(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\|_{H^1(\Omega)}}$ with the norm $\|g\|_{H^1(\Omega)} = \|g\|_{L^2(\Omega)} + \|\nabla g\|_{(L^2(\Omega))^2}$, and $H^{-1}(\Omega)$ is the dual space of $H^1_0(\Omega)$ with respect to the $L^2(\Omega)$ inner product. We recall the trace theorem and inverse trace theorem (see e.g. [HW08, Section 4.2]) below.

Theorem 2.1.1 (Trace theorem & Inverse trace theorem). *1. Let Ω be a C^1-domain. Then there exists a linear continuous trace operator*

$$\gamma_0 : H^1(\Omega) \mapsto H^{\frac{1}{2}}(\partial\Omega),$$

which is an extension of $\gamma_0 u = u|_{\partial\Omega}$ for $u \in C^0(\overline{\Omega})$, and there exists a linear continuous right inverse Γ_0 to γ_0 with

$$\Gamma_0 : H^{\frac{1}{2}}(\partial\Omega) \mapsto H^1(\Omega) \text{ and } \gamma_0(\Gamma_0(u_0)) = u_0, \text{ for all } u_0 \in H^{\frac{1}{2}}(\partial\Omega).$$

2. *Let Ω be a $C^{1,1}$ -domain. Then there exist two linear continuous trace operators*

$$\gamma_0 : H^2(\Omega) \mapsto H^{\frac{3}{2}}(\partial\Omega), \quad \gamma_1 : H^2(\Omega) \mapsto H^{\frac{1}{2}}(\partial\Omega),$$

which are extensions of

$$\gamma_0\Phi = \Phi|_{\partial\Omega} \text{ for } \Phi \in C^0(\overline{\Omega}), \quad \gamma_1\Phi = \frac{\partial\Phi}{\partial n}\Big|_{\partial\Omega} \text{ for } \Phi \in C^3(\overline{\Omega}).$$

Inversely, there exists a linear continuous right inverse Γ_1 to (γ_0, γ_1) with

$$\Gamma_1 : H^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \mapsto H^2(\Omega) \text{ and } \gamma_j(\Gamma_1(\Phi_0, \Phi_1)) = \Phi_j, \quad j = 0, 1,$$

for all $(\Phi_0, \Phi_1) \in H^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

In the above, $g \in H^{\frac{3}{2}}(\partial\Omega)$ the fractional Sobolev space means that $g \in L^2(\partial\Omega)$ with the following norm

$$\|g\|_{H^{\frac{3}{2}}(\partial\Omega)} \overset{\text{def}}{=} \|g\|_{L^2(\partial\Omega)} + \Big(\int_{\partial\Omega}\int_{\partial\Omega} \frac{|g(s) - g(s')|^2}{|s - s'|^4} ds ds'\Big)^{\frac{1}{2}}$$

being finite.

We now define the weak solutions of the elliptic equation (2.11).

Definition 2.1.2 (Weak solutions of the elliptic equation on a bounded connected domain)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded connected $C^{1,1}$-domain. Let $\eta \in L^\infty(\mathbb{R}; [0, \infty))$ and $b \in C(\mathbb{R}; [\mu_*, \mu^*])$, $\mu_*, \mu^* > 0$ be two given functions.*

We say that $\Phi \in H^2(\Omega)$ is a weak solution of the boundary value problem (2.11) *with the given data $\Phi_0 \in H^{\frac{3}{2}}(\partial\Omega)$, $\Phi_1 \in H^{\frac{1}{2}}(\partial\Omega)$, $f \in H^{-1}(\Omega; \mathbb{R}^2)$, if $\Phi_0 = \Phi|_{\partial\Omega}$ and $\Phi_1 = \frac{\partial\Phi}{\partial n}\Big|_{\partial\Omega}$ in the trace sense and the following integral identity*

$$\begin{aligned}&\int_\Omega \mu\Big((\partial_{x_2x_2}\Phi - \partial_{x_1x_1}\Phi)(\partial_{x_2x_2}\psi - \partial_{x_1x_1}\psi) + (2\partial_{x_1x_2}\Phi)(2\partial_{x_1x_2}\psi)\Big)\,dx \\ &= \int_\Omega f \cdot \nabla^\perp\psi\,dx + \int_\Omega \rho(\nabla^\perp\Phi \otimes \nabla^\perp\Phi) : \nabla\nabla^\perp\psi\,dx,\end{aligned}$$

holds for all $\psi \in H_0^2(\Omega)$.

Let Ω be an exterior domain or the whole plane. We define the functional spaces we are going to use

$$D^k(\Omega) := \dot{H}^k(\Omega) \cap \big(\cap_{n\geqslant 1} H^k(\Omega \cap B_n(0))\big), \quad k = 1, 2,$$

where $B_n(0)$ denotes the disk centered at 0 with radius n and the homogeneous Sobolev space $\dot{H}^k(\Omega)$, $k \in \mathbb{N}$ is defined as

$$\dot{H}^k(\Omega) = \{g \in L^1_{\text{loc}}(\Omega) : \partial^\alpha g \in L^2(\Omega), |\alpha| = k\}.$$

We define the corresponding weak solutions as follows.

Definition 2.1.3. *(i)* (Weak solutions of the Navier–Stokes equations on an exterior domain). *Let $\Omega \subset \mathbb{R}^2$ be the exterior domain of a bounded simply connected $C^{1,1}$ set. We say that a pair $(\rho, u) \in L^\infty(\Omega; [0, \infty)) \times D^1(\Omega; \mathbb{R}^2)$ is a weak solution of the boundary value problem* (2.12) *with the given data $u_0 \in H^{\frac{1}{2}}(\partial\Omega; \mathbb{R}^2)$, $f \in H^{-1}(\Omega; \mathbb{R}^2)$, if $div\, u = 0$, $div\,(\rho u) = 0$ hold in Ω in the distribution sense, $u_0 = u|_{\partial\Omega}$ is the trace of u on $\partial\Omega$ and the integral identity* (2.13) *holds for all $v \in C_c^\infty(\Omega; \mathbb{R}^2)$ with $div\, v = 0$.*

(ii) (Weak solutions of the elliptic equation on an exterior domain). *Let $\Omega \subset \mathbb{R}^2$ be the exterior domain of a bounded connected $C^{1,1}$ set. Let $\eta \in L^\infty(\mathbb{R}; [0, \infty))$ and $b \in C(\mathbb{R}; [\mu_*, \mu^*])$, $\mu_*, \mu^* > 0$ be two given functions. We say that $\Phi \in D^2(\Omega)$ is a weak solution of the boundary value problem* (2.11) *with the given data $\Phi_0 \in H^{\frac{3}{2}}(\partial\Omega)$, $\Phi_1 \in H^{\frac{1}{2}}(\partial\Omega)$, and $f \in H^{-1}(\Omega; \mathbb{R}^2)$, if $\Phi_0 = \Phi|_{\partial\Omega}$ and $\Phi_1 = \frac{\partial\Phi}{\partial n}\Big|_{\partial\Omega}$ in the trace sense, and the identity* (2.1.2) *holds true for all $\Psi \in C_c^\infty(\Omega; \mathbb{R})$.*

(iii) (Weak solutions on $\mathbb{R}^2$). *We define the weak solutions of the equations* (2.1) *(resp.* (2.10)*) on $\mathbb{R}^2$ as in* (i) *(resp. in (ii)) without any boundary condition.*

Since for any $v \in H_0^1(\Omega; \mathbb{R}^2)$ (resp. $v \in C_c^\infty(\Omega; \mathbb{R}^2)$) defined on a bounded connected $C^{1,1}$-domain Ω (resp. on the exterior domain of a bounded connected $C^{1,1}$ set or on the whole space $\mathbb{R}^2$) with $\operatorname{div} v = 0$ there exists a corresponding stream function $\psi \in H_0^2(\Omega)$ (resp. $\psi \in C_c^\infty(\Omega; \mathbb{R}^2)$) such that $v = \nabla^\perp\psi$, the equality (2.1.2) implies the equality (2.13) with $u = \nabla^\perp\Phi$. Therefore we have the following fact which transfers the solvability of the Navier-Stokes system (2.1) to the solvability of the elliptic equation (2.10).

Fact. *Let $\Omega \subset \mathbb{R}^2$ be a bounded connected $C^{1,1}$-domain. Let $\eta \in L^\infty(\mathbb{R}; [0, \infty))$, $b \in C(\mathbb{R}; [\mu_*, \mu^*])$ with $\mu_*, \mu^* > 0$ and $f \in H^{-1}(\Omega; \mathbb{R}^2)$ be given. Let $u_0 \in H^{\frac{1}{2}}(\partial\Omega; \mathbb{R}^2)$ satisfy* (2.3) *and let $\Phi_0 \in H^{\frac{3}{2}}(\partial\Omega)$ be given in* (2.9) *in terms of u_0 and some fixed constant $C_0 \in \mathbb{R}$.*

If $\Phi \in H^2(\Omega)$ is a weak solution of the boundary value problem (2.11)*, then the pair of Frolov's form* (2.8)*: $(\rho, u) = \big(\eta(\Phi), \nabla^\perp\Phi\big)$ is a weak solution of the boundary value problem* (2.12)*.*

The above holds true for the exterior domain of a bounded connected $C^{1,1}$ set or for the whole plane domain $\mathbb{R}^2$.

Our main theorem concerning the existence and the regularity properties of the weak solutions to the Navier-Stokes system (2.1) as well as to the elliptic equation (2.10) reads as follows.

Theorem 2.1.2 (Existence and regularity results for the weak solutions of Frolov's form, [HL20])**.**
Let $\eta \in L^\infty(\mathbb{R}; [0, \infty))$, $b \in C(\mathbb{R}; [\mu_*, \mu^*])$, $\mu_*, \mu^* > 0$ be given.

(i) Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected $C^{1,1}$-domain (resp. the exterior domain of a simply connected $C^{1,1}$ set). Let $f \in H^{-1}(\Omega; \mathbb{R}^2)$ (resp. $f = \operatorname{div} F$, where $F \in L^2(\Omega; \mathbb{R}^2 \times \mathbb{R}^2)$) be given. Then for any $\Phi_0 \in H^{\frac{3}{2}}(\partial\Omega)$ and $\Phi_1 \in H^{\frac{1}{2}}(\partial\Omega)$, there exists at least one weak solution $\Phi \in H^2(\Omega)$ (resp. $\Phi \in D^2(\Omega)$) of the boundary value problem (2.11).

Let $C_0 \in \mathbb{R}$ and $u_0 \in H^{\frac{1}{2}}(\partial\Omega; \mathbb{R}^2)$ satisfy (2.3). If $\Phi_0 \in H^{\frac{3}{2}}(\partial\Omega)$ is given by (2.9) and $\Phi \in H^2(\Omega)$ (resp. $\Phi \in D^2(\Omega)$) is a weak solution of (2.11) given above, then the pair of Frolov's form

$$(\rho, u) = \big(\eta(\Phi), \nabla^\perp\Phi\big) \tag{2.14}$$

is a weak solution of the boundary value problem (2.12) with $u \in H^1(\Omega; \mathbb{R}^2)$ (resp. $u \in D^1(\Omega; \mathbb{R}^2)$).

(ii) Let $\Omega = \mathbb{R}^2$ and $D \subset \Omega$ be a bounded subset of positive Lebesgue measure. Let $f = \operatorname{div} F$, where $F \in L^2(\mathbb{R}^2; \mathbb{R}^2 \times \mathbb{R}^2)$. Then for any fixed vector $d \in \mathbb{R}^2$, there exists at least one weak solution $\Phi \in D^2(\mathbb{R}^2)$ of the elliptic equation (2.10) on $\mathbb{R}^2$, such that $u = \nabla^\perp\Phi \in D^1(\mathbb{R}^2; \mathbb{R}^2)$ is a weak solution of the equation (2.1) on $\mathbb{R}^2$ and $\frac{1}{meas(D)} \int_D u = d$.

Furthermore, we have the following regularity results under additional smoothness assumptions.

(1) If Ω is a connected $C^{2,1}$-domain, the function η is taken to be continuous and $f \in L^2(\Omega; \mathbb{R}^2)$, then for any $\Phi_0 \in H^{\frac{5}{2}}(\partial\Omega)$, $\Phi_1 \in H^{\frac{3}{2}}(\partial\Omega)$ (resp. $u_0 \in H^{\frac{3}{2}}(\partial\Omega; \mathbb{R}^2)$) the weak solution Φ (resp. u) given in (i) belongs to $W^{2,p}(\Omega)$ (resp. $W^{1,p}(\Omega; \mathbb{R}^2)$), for all $p \in [1, \infty)$.

(2) Let $k \geqslant 2$ be an integer. If Ω is a connected $C^{k+1,1}$-domain, the functions $\eta, b \in C_b^k(\mathbb{R}) = \{h \in C^k(\mathbb{R}) \,|\, \|h^{(j)}\|_{L^\infty} < \infty, \forall 0 \leqslant j \leqslant k\}$ and $f \in H^{k-1}(\Omega; \mathbb{R}^2)$, then for any $\Phi_0 \in H^{k+\frac{3}{2}}(\partial\Omega)$, $\Phi_1 \in H^{k+\frac{1}{2}}(\partial\Omega)$ (resp. $u_0 \in H^{k+\frac{1}{2}}(\partial\Omega)$), the weak solution Φ (resp. u) given in (i) belongs to $W^{k+1,p}(\Omega)$ (resp. $W^{k,p}(\Omega; \mathbb{R}^2)$) for all $1 \leqslant p < \infty$. In particular, if $k = 2$, then $u \in W^{2,p}(\Omega; \mathbb{R}^2)$, $p > 2$ is Lipschitz continuous.

Theorem 2.1.2 will be proved in Section 2.2, where we will follow J. Leray's approach in [Ler33] for the resolution of the classical stationary Navier-Stokes equation. By virtue of the above Fact, it remains to study the fourth-order nonlinear elliptic equation (2.10) for the stream function Φ. Compared to the classical case, we here have to pay more attention on the nonlinear dependence of the density ρ and the viscosity coefficient μ on Φ.

We give here some remarks on the results in Theorem 2.1.2.

Remark 2.1.1. *(i) (Recover the classical results) If η, b are positive constant functions, then the system* (2.1) *becomes the classical stationary Navier-Stokes equations* (2.6) *and the above theorem recover the classical existence and regularity results in [Lad69].*

(ii) (Relaxation on the regularity assumption) For $k \geqslant 2$, we can relax the hypotheses on the data f, Φ_0, Φ_1 (resp. u_0) to $f \in W^{k-2,p_0}(\Omega; \mathbb{R}^2)$, $\Phi_0 \in W^{k+\frac{1}{2},p_0}(\partial\Omega)$, $\Phi_1 \in W^{k-\frac{1}{2},p_0}(\partial\Omega)$ (resp. $u_0 \in W^{k-\frac{1}{2},p_0}(\partial\Omega; \mathbb{R}^2)$), $p_0 > 2$, and show that the weak solutions have the regularity properties $\Phi \in W^{k+1,p_0}(\Omega)$ (resp. $u \in W^{k,p_0}(\Omega; \mathbb{R}^2)$).

(iii) (Domains of other types) We can also consider the system (2.1) *in domains of other types, following the arguments for the classical Navier–Stokes equations* (2.6)*.*

For example, it is obvious that the existence and regularity results in Theorem 2.1.2 hold true on a bounded multi-connected domain $\cup_{i=1}^n \Omega_i$, under zero flux assumption on the boundary of each connected component (2.7)*.*

The existence result in Theorem 2.1.2 can also be easily extended to the strip domain $\mathbb{R} \times [0, 1]$ by use of Poincaré inequality.

We can follow the idea in [GW16] by J. Guillod and P. Wittwer for (2.6) *on the half plane, to show the solvability of* (2.1)-(2.2)-(2.3) *on the half plane $\mathbb{R} \times [0, \infty)$ by assuming small boundary value $\|u_0\|_{L^\infty}$ on the unbounded boundary $\mathbb{R} \times \{0\}$.*

(iv) (Boundary conditions on unbounded domains) If Ω is an unbounded domain, we denote the "boundary condition" of the solutions u at infinity by u_∞

$$\lim_{|x|\to\infty} u(x) = u_\infty, \quad u_\infty \in \mathbb{R}^2.$$

The existence result in Theorem 2.1.2 does not give the information of u_∞. We even don't know the existence of decaying solutions of the Navier-Stokes system (2.1) *on the exterior domain or the whole plane.*

The solvability of the classical stationary Navier-Stokes equation (2.6) *on the exterior domain with* $u_\infty = 0$ *(under some symmetric assumptions) was established in e.g. [HW13; Yam11]. There are also some works considering the asymptotic behaviors of the (general) weak solutions: In [VG74; GW78], D. Gilbarg and H.F. Weinberger showed that the solutions of* (2.6) *satisfy* $\lim_{|x|\to\infty} \int_{\mathbb{S}^1} |u|^2 = \infty$ *or* $\lim_{|x|\to\infty} \int_{\mathbb{S}^1} |u-\bar{u}|^2 = 0$ *for some* $\bar{u} \in \mathbb{R}^2$*, and J. Amick discussed the relation between* u_∞ *and* $\bar{u}$ *in [Ami88b].*

In [Gal11], Galidi showed the non uniqueness of the solutions to the classic Navier–Stokes equation (2.6) *with certain boundary condition* u_0 *and* $u_\infty = 0$*. Hence the weak solutions of the system* (2.1) *are also not unique, at least in the case without any smallness or symmetric assumptions.*

2.1.3 Symmetric solutions

We turn to study the stationary Navier-Stokes equations (2.1) under some symmetry assumptions on the density function in this subsection.

We give first an observation when we write the velocity vector field $u = \nabla^\perp \Phi$ in terms of the stream function Φ. Let $U \subset \mathbb{R}^2$ be an open set and we consider another coordinate system (y_1, y_2) on it. We suppose that the Jacobian $\nabla_x y = (\frac{\partial y_i}{\partial x_j})_{1\leqslant i,j\leqslant 2}$ is not degenerate and we consider the stationary Navier-Stokes system (2.1) on U. If the density function depends only on y_1

$$\rho = \alpha(y_1),$$

and $\alpha' \neq 0$ does not vanish, then, by formal calculations, the equation

$$0 = \operatorname{div}(\rho u) = \operatorname{div}(\rho \nabla^\perp \Phi) = \alpha'(\nabla_x y_1 \cdot \nabla_x^\perp y_2)\partial_{y_2}\Phi = \alpha' \det\big(\nabla_x y\big)\, \partial_{y_2}\Phi$$

implies that $\Phi = \beta(y_1)$ depends also only on y_1 on U. Nevertheless it is not necessary that there exists a function η such that $\rho = \eta(\Phi)$. Similarly, if Φ depends only on y_1

$$\Phi = \beta(y_1),$$

and $\beta' \neq 0$ does not vanish, then $\rho = \alpha(y_1)$ depends also only on y_1 and $\rho = \eta(\Phi)$ with $\eta = \alpha \circ \beta^{-1}$. In this case the pair $(\rho, u) = (\alpha(y_1), \nabla_x^\perp(\beta(y_1)))$ is a solution of the form (2.14) to the stationary Navier-Stokes system (2.1) if it further satisfies the first equation in (2.1).

We formulate the solutions to the stationary Navier-Stokes system (2.1) when assuming certain symmetries on the density function in the following theorem. In particular, the Couette flow between a parallel channel, the concentric flow between concentric rotating circles, and the radial flow (also called the Jeffery-Hamel flow) between two nonparallel converging/diverging lines are described.

Theorem 2.1.3 (Formulation for the parallel, concentric and radial flows, [HL20]). *If the density function*

$$\rho = \rho(x_2) \text{ in } \mathbb{R}^2, \quad \text{or} \quad \rho(r) \text{ in } \mathbb{R}^2\backslash\{0\}, \quad \text{or} \quad \rho(\theta) \text{ in } \mathbb{R}^2\backslash\{0\}, \quad \text{with } \rho' \neq 0,$$

where (r, θ) *are polar coordinates in* $\mathbb{R}^2$*, then the velocity vector field* u *of the stationary Navier–Stokes equations* (2.1) *reads correspondingly as*

$$u = u_1(x_2)\, e_1 \text{ in } \mathbb{R}^2, \quad \text{or} \quad rg(r)\, e_\theta \text{ in } \mathbb{R}^2\backslash\{0\}, \quad \text{or} \quad \frac{h(\theta)}{r} e_r \text{ in } \mathbb{R}^2\backslash\{0\}, \tag{2.15}$$

where $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_r = \begin{pmatrix} \frac{x_1}{r} \\ \frac{x_2}{r} \end{pmatrix}$, $e_\theta = \begin{pmatrix} \frac{x_2}{r} \\ -\frac{x_1}{r} \end{pmatrix}$.

Let the external force $f = 0$ *in the system* (2.1)*, then the scalar functions* u_1, g, h *above satisfy the following three ordinary differential equations of second order respectively*

$$\begin{aligned} &\partial_{x_2}(\mu \partial_{x_2} u_1) = C, \\ &\partial_r(\mu r^3 \partial_r g) = -Cr, \\ &\rho h^2 + \partial_\theta(\mu \partial_\theta h) + 4(\mu h) = C, \end{aligned} \tag{2.16}$$

where $C \in \mathbb{R}$ *can be arbitrarily chosen. Correspondingly the stream function*

$$\Phi = \Phi(x_2) \quad \text{or} \quad \Phi(r) \quad \text{or} \quad \Phi(\theta)$$

satisfies the following elliptic equations of fourth order respectively

$$\begin{aligned} &\partial_{x_2 x_2}(\mu \partial_{x_2 x_2} \Phi) = 0, \\ &\partial_{rr}\Big(\mu r^3 \partial_r(\frac{1}{r}\partial_r \Phi)\Big) = -C, \\ &\partial_{\theta\theta}(\mu \partial_{\theta\theta}\Phi) + \partial_\theta\big(\rho(\partial_\theta \Phi)^2 + 4\mu \partial_\theta \Phi\big) = 0. \end{aligned}$$

Remark 2.1.2. *(i) (Recover the classical results) If* ρ, μ *are positive constants, then the solutions* (2.15)-(2.16) *are solutions to the classical stationary Navier-Stokes equations* (2.6).

(ii) (Resolution of the ODEs) In the case $\rho = \rho(x_2)$ *or* $\rho = \rho(r)$*, the velocity vector field* u *is related only to the viscosity coefficient* μ *(while not* ρ*). Under some Dirichlet boundary conditions the above ODEs* (2.16) *with given functions* ρ, μ *can be solved up to a real constant, and hence there are uncountably many solutions to the corresponding boundary value problems of the system* (2.1).

Proof of Theorem 2.1.3. We are going to consider the cases $\rho = \rho(x_2)$, $\rho = \rho(r)$ and $\rho = \rho(\theta)$ separately. We notice that if we take the polar coordinate (r, θ) on the plane $\mathbb{R}^2$, with

$$(x_1, x_2) = (r\cos\theta, r\sin\theta),$$

then

$$\nabla_x = e_r\partial_r - \frac{e_\theta}{r}\partial_\theta, \quad \nabla_x^\perp = \frac{e_r}{r}\partial_\theta + e_\theta\partial_r, \text{ with } e_r = \begin{pmatrix} \frac{x_1}{r} \\ \frac{x_2}{r} \end{pmatrix}, \quad e_\theta = \begin{pmatrix} \frac{x_2}{r} \\ -\frac{x_1}{r} \end{pmatrix}.$$

Case $\rho = \rho(x_2)$

If $\rho = \rho(x_2)$ with $\rho' \neq 0$, then the equations $\operatorname{div}(\rho u) = 0$ and $\operatorname{div} u = 0$ imply that $u_2 = 0$ and $\partial_{x_1} u_1 = 0$. Thus $u_1 = u_1(x_2)$. Hence

$$\rho(u\cdot\nabla)u = 0 \in \mathbb{R}^2, \quad \operatorname{div}(\mu(Su)) = \partial_{x_2}(\mu\partial_{x_2}u_1)\, e_1, \quad \Delta u = (\partial_{x_2x_2}u_1)\, e_1. \tag{2.17}$$

If $f = 0$, then the system (2.1) reads as

$$\begin{pmatrix} -\partial_{x_2}(\mu\partial_{x_2}u_1) + \partial_{x_1}\Pi \\ \partial_{x_2}\Pi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The equation $\partial_{x_2}\Pi = 0$ implies $\Pi = \Pi(x_1)$. Thus there exists a constant $C \in \mathbb{R}$ such that

$$\partial_{x_2}(\mu\partial_{x_2}u_1) = -\partial_{x_1}\Pi = C.$$

Case $\rho = \rho(r)$

If $\rho = \rho(r)$ with $\rho' \neq 0$, then the equations $\operatorname{div}(\rho u) = 0$ and $\operatorname{div} u = 0$ imply that $u \cdot e_r = 0$ and hence $u = g_1(r,\theta)e_\theta$ for some scalar function g_1. The incompressibility $\operatorname{div} u = 0$ then implies $(\partial_r g_1)e_r \cdot e_\theta - (\partial_\theta g_1)\frac{e_\theta}{r} \cdot e_\theta = 0$, that is, $\partial_\theta g_1 = 0$. Thus $u = g_1(r)e_\theta$.

Let

$$g(r) = \frac{g_1(r)}{r}, \text{ such that } u = rg(r)e_\theta,$$

then it is straightforward to calculate

$$\nabla u = \begin{pmatrix} rg'\frac{x_1x_2}{r^2} & g + rg'\frac{x_2^2}{r^2} \\ -g - rg'\frac{x_1^2}{r^2} & -rg'\frac{x_1x_2}{r^2} \end{pmatrix}, \; Su = \nabla u + \nabla^T u = rg' \begin{pmatrix} 2\frac{x_1x_2}{r^2} & \frac{x_2^2 - x_1^2}{r^2} \\ \frac{x_2^2 - x_1^2}{r^2} & -2\frac{x_1x_2}{r^2} \end{pmatrix},$$

and

$$\rho(u\cdot\nabla)u = -r\rho g^2 e_r, \quad \operatorname{div}(\mu(Su)) = \frac{\partial_r(r^3\mu\partial_r g)}{r^2}e_\theta, \quad \Delta u = (r\partial_{rr}g + 3\partial_r g)e_\theta. \tag{2.18}$$

If $f = 0$, then the system (2.1) reads as

$$\big(-r\rho g^2 + \partial_r \Pi\big)e_r + \big(-\frac{\partial_r(r^3\mu\partial_r g)}{r^2} - \frac{1}{r}\partial_\theta \Pi\big)e_\theta = 0. \tag{2.19}$$

Since $\mu = \mu(r)$ and $g = g(r)$, we derive from the above equation (2.19) in the e_θ-direction that $\partial_\theta \Pi = \alpha(r)$, where α is a function depending only on r. Then Π has the form $\Pi(r,\theta) = \alpha(r)\theta + \beta(r)$, where β is a function depending only on r. The above equation (2.19) in the e_r-direction implies that $\partial_r \Pi$ depends only on r and hence $\alpha(r) = C$ is a constant, such that

$$\Pi(r,\theta) = C\theta + \beta(r).$$

We substitute $\partial_\theta \Pi = C$ into the equation (2.19) to obtain $(2.16)_2$.

Case $\rho = \rho(\theta)$

If $\rho = \rho(\theta)$ with $\rho' \neq 0$, then the equations $\operatorname{div}(\rho u) = 0$ and $\operatorname{div} u = 0$ imply that $u \cdot e_\theta = 0$ and hence $u = h_1(r,\theta)e_r$ for some scalar function h_1. The incompressibility $\operatorname{div} u = 0$ then implies

$$\partial_r h_1 + \frac{1}{r}h_1 = 0.$$

Thus $h_1(r,\theta) = \frac{h(\theta)}{r}$ and $u = \frac{h(\theta)}{r}e_r$. It is straightforward to calculate

$$\begin{aligned}
\nabla u &= \frac{1}{r^4}\begin{pmatrix} -(x_1^2 - x_2^2)h - x_1x_2h' & -2x_1x_2h + x_1^2h' \\ -2x_1x_2h - x_2^2h' & (x_1^2 - x_2^2)h + x_1x_2h' \end{pmatrix}, \\
Su &= \nabla u + \nabla^T u \\
&= \frac{1}{r^4}\begin{pmatrix} -2(x_1^2 - x_2^2)h - 2x_1x_2h' & -4x_1x_2h + (x_1^2 - x_2^2)h' \\ -4x_1x_2h + (x_1^2 - x_2^2)h' & 2(x_1^2 - x_2^2)h + 2x_1x_2h' \end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
\rho(u\cdot\nabla)u &= -\rho\frac{h^2}{r^3}e_r, \\
\operatorname{div}(\mu(Su)) &= \frac{\partial_\theta(\mu\partial_\theta h)}{r^3}e_r - 2\frac{\partial_\theta(\mu h)}{r^3}e_\theta, \\
\Delta u &= \frac{\partial_{\theta\theta}h}{r^3}e_r - 2\frac{\partial_\theta h}{r^3}e_\theta.
\end{aligned} \tag{2.20}$$

Thus the system (2.1) with $f = 0$ reads as

$$\Big(-\rho\frac{h^2}{r^3} - \frac{\partial_\theta(\mu\partial_\theta h)}{r^3} + \partial_r\Pi\Big)e_r + \Big(2\frac{\partial_\theta(\mu h)}{r^3} - \frac{1}{r}\partial_\theta\Pi\Big)e_\theta = 0. \tag{2.21}$$

We derive from the above equation (2.21) in the e_θ-direction that $\partial_\theta \Pi = 2r^{-2}\partial_\theta(\mu h)$. Since $\mu = \mu(\theta)$ and $h = h(\theta)$, Π has the form

$$\Pi(r,\theta) = 2r^{-2}(\mu h) + \alpha(r),$$

where α is a function depending only on r. We substitute $\partial_r \Pi = -\frac{4}{r^3}(\mu h) + \alpha'(r)$ into (2.21) to derive

$$\rho h^2 + \partial_\theta(\mu \partial_\theta h) + 4(\mu h) = r^3 \alpha'(r),$$

where the left-hand side depends only on θ and the right-hand side depends only on r. Hence there exists $C \in \mathbb{R}$ such that $(2.16)_3$ holds.

□

We have the following irregularity results, as a straightforward consequence from Theorem 2.1.3.

Corollary 2.1.1 (Irregularity results in the case of piecewise-constant viscosity coefficients). *For the parallel, concentric and radial flows formulated in Theorem 2.1.3 above, if we assume that the viscosity coefficient*

$$\begin{aligned}&\mu = \mu(x_2), \quad \text{or} \quad \mu(r), \quad \text{or } \mu(\theta)\\ &\text{is a step function with the jump point at } a \in (0, 2\pi),\end{aligned} \tag{2.22}$$

ρ, μ have positive lower and upper bounds, and that

$$\begin{aligned}&\partial_{x_2} u_1 \in L^1_{loc}(\mathbb{R}), \quad \text{or} \quad \partial_r g \in L^1_{loc}(\mathbb{R}^+), \quad \text{or} \quad h \text{ and } \partial_\theta h \in L^1_{loc}([0,2\pi))\\ &\text{do not vanish in a neighborhood } U_a \text{ of } a,\end{aligned} \tag{2.23}$$

then

$$\begin{aligned}\Delta u &= (\partial_{x_2 x_2} u_1)\, e_1 \notin L^1_{loc}(\mathbb{R}^2),\\ \text{or} \quad &(r\partial_{rr} g + 3\partial_r g) e_\theta \notin L^1_{loc}(\mathbb{R}^2 \backslash \{0\}),\\ \text{or} \quad &\frac{\partial_{\theta\theta} h}{r^3} e_r - 2\frac{\partial_\theta h}{r^3} e_\theta \notin L^1_{loc}(\mathbb{R}^2 \backslash \{0\}).\end{aligned}$$

In the case of radial flow $(\rho, u) = (\rho(\theta), \frac{h(\theta)}{r} e_r)$, we also have

$$\operatorname{div}(\mu S u) = \frac{\partial_\theta(\mu \partial_\theta h)}{r^3} e_r - 2\frac{\partial_\theta(\mu h)}{r^3} e_\theta \notin L^1_{loc}(\mathbb{R}^2 \backslash \{0\}).$$

Proof. If the viscosity coefficient $\mu = \mu(x_2)$ or $\mu(r)$ or $\mu(\theta)$ is a step function with the jump point at a, then μ' is the delta distribution δ_a (up to a constant) which does not belong to $L^1(U_a)$, with U_a a neighborhood of a. The expressions for Δu, $\operatorname{div}(\mu S u)$ in Corollary 2.1.1 can be found in (2.17), (2.18) and (2.20) above. We are going to discuss the cases $\rho = \rho(x_2)$, $\rho = \rho(r)$ and $\rho = \rho(\theta)$ separately, by contradiction argument.

We assume by contradiction that

$$\begin{aligned}
&\Delta u = (\partial_{x_2 x_2} u_1) e_1 \in L^1_{\text{loc}}(\mathbb{R}^2),\\
\text{or}\quad &(r\partial_{rr} g + 3\partial_r g) e_\theta \in L^1_{\text{loc}}(\mathbb{R}^2 \backslash \{0\}),\\
\text{or}\quad &\frac{\partial_{\theta\theta} h}{r^3} e_r - 2\frac{\partial_\theta h}{r^3} e_\theta \in L^1_{\text{loc}}(\mathbb{R}^2 \backslash \{0\}),
\end{aligned}$$

then by the assumptions (2.23) we have

$$\begin{aligned}
&\partial_{x_2} u_1 \in W^{1,1}_{\text{loc}}(\mathbb{R}) \subset L^\infty_{\text{loc}}(\mathbb{R}),\\
\text{or}\quad &\partial_r g \in W^{1,1}_{\text{loc}}(\mathbb{R}^+) \subset L^\infty_{\text{loc}}(\mathbb{R}^+),\\
\text{or}\quad &h, \partial_\theta h \in W^{1,1}_{\text{loc}}([0, 2\pi)) \subset L^\infty_{\text{loc}}([0, 2\pi)).
\end{aligned}$$

Thus by the ODEs (2.16), in the neighborhood U_a,

$$\begin{aligned}
&\partial_{x_2}\mu = \frac{1}{\partial_{x_2} u_1}(C - \mu\partial_{x_2 x_2} u_1) \in L^1_{\text{loc}}(U_a),\\
\text{or}\ &\partial_r \mu = \frac{1}{r^3 \partial_r g}\big(-Cr - \mu\partial_r(r^3\partial_r g)\big) \in L^1_{\text{loc}}(U_a),\\
\text{or}\quad &\partial_\theta \mu = \frac{1}{\partial_\theta h}\big(C - 4\mu h - \rho h^2 - \mu \partial_{\theta\theta} h\big) \in L^1_{\text{loc}}(U_a).
\end{aligned}$$

This is a contradiction to (2.22).

Similarly, in the case of radial flow $(\rho, u) = (\rho(\theta), \frac{h(\theta)}{r} e_r)$, if we assume by contradiction that

$$\operatorname{div}(\mu S u) = \frac{\partial_\theta(\mu\partial_\theta h)}{r^3} e_r - 2\frac{\partial_\theta(\mu h)}{r^3} e_\theta \in L^1_{\text{loc}}(\mathbb{R}^2 \backslash \{0\}),$$

then by the ODE $(2.16)_3$ and the assumptions (2.23) we have

$$\partial_\theta(\mu\partial_\theta h) = C - 4\mu h - \rho h^2 \in L^1_{\text{loc}}([0, 2\pi)), \text{ and hence } \partial_\theta(\mu h) \in L^1_{\text{loc}}([0, 2\pi)),$$

which implies the following which is a contradiction to (2.22):

$$\partial_\theta \mu = \frac{1}{h}(\partial_\theta(\mu h) - \mu\partial_\theta h) \in L^1_{\text{loc}}(U_a).$$

□

We can indeed calculate explicitly the solutions to the Navier-Stokes system 2.1 in the case of piecewise-constant viscosity coefficients. We will see that they can indeed be of Frolov's form (2.14) in some particular cases.

Example 2.1.1 (Explicit solutions in the case of piecewise-constant viscosity coefficients)**.** *We give examples of parallel, concentric and radial flows with piecewise-constant viscosity coefficients respectively.*

- ***Example of parallel flows***

  If $(\rho, u) = \big(\rho(x_2),\ u_1(x_2)\, e_1\big)$ *(not necessarily* $\rho' \neq 0$*) solves the system* (2.1) *with* $f = 0$*, then* u *satisfies* $(2.16)_1$*:* $\partial_{x_2}(\mu \partial_{x_2} u_1) = C \in \mathbb{R}$*. In particular, with the following viscosity coefficient* μ

  $$\mu = \mu(x_2) = 2\mathbb{1}_{\{x_2>0\}} + \mathbb{1}_{\{x_2\leqslant 0\}},$$

  we have for some constant $C_1 \in \mathbb{R}$ *that*

  $$\partial_{x_2} u = {u_1}'(x_2)\, e_1 = \Big(\big(\frac{C}{2}x_2 + \frac{C_1}{2}\big)\mathbb{1}_{\{x_2>0\}} + \big(Cx_2 + C_1\big)\mathbb{1}_{\{x_2\leqslant 0\}}\Big) e_1,$$

  and hence

  $$u_1''(x_2) = \frac{C}{2}\mathbb{1}_{\{x_2>0\}} + C\mathbb{1}_{\{x_2<0\}} - \frac{C_1}{2}\delta_0(x_2).$$

  There exists a real constant $C_2 \in \mathbb{R}$ *such that* $u \in H^1_{loc}(\mathbb{R}^2)$ *reads as*

  $$u = \Big(\big(\frac{C}{4}x_2^2 + \frac{C_1}{2}x_2 + C_2\big)\mathbb{1}_{\{x_2>0\}} + \big(\frac{C}{2}x_2^2 + C_1 x_2 + C_2\big)\mathbb{1}_{\{x_2\leqslant 0\}}\Big) e_1. \quad (2.24)$$

  If we consider the Couette flow on the strip $\mathbb{R}\times[-1,1]$ *with the boundary conditions*

  $$u|_{\mathbb{R}\times\{\pm 1\}} = a_\pm\, e_1 \in \mathbb{R}^2, \quad (2.25)$$

  then there hold only two equations for the three constants C, C_1, C_2

  $$C = 4(a_- - a_+) + 6C_1, \quad C_2 = 2a_+ - a_- - 2C_1, \quad C_1 \in \mathbb{R}.$$

  Hence there are uncountably many solutions with the density function

  $$\rho(x_2) = b^{-1}(2)\mathbb{1}_{\{x_2>0\}} + b^{-1}(1)\mathbb{1}_{\{x_2\leqslant 0\}}, \quad (2.26)$$

  and the velocity vector field (2.24) *to the boundary value problem* (2.1)-(2.25). [2]

[2] For the homogeneous flow $\mu = 1$, the velocity vector field in the form of $u_1(x_2)\, e_1$ reads as $u = \big(\frac{C}{2}x_2^2 + C_1 x_2 + C_2\big)e_1$ with $C_1 = \frac{a_+ - a_-}{2}$, $\frac{C}{2} + C_2 = \frac{a_+ + a_-}{2}$.

It is easy to see that if $a_+ < a_- < 2a_+$ *and* $0 < C_1 < \frac{2a_+ - a_-}{2}$*, then* $C, C_2 > 0$ *and* $u_1(x_2) > 0$ *for* $x_2 \in [-1, 1]$*. Hence* $\partial_{x_2}\Phi = u_1 > 0$ *and there exists a constant* $C_3 \in \mathbb{R}$ *such that the stream function*

$$\begin{aligned}\Phi = &\big(\frac{C}{12}x_2^3 + \frac{C_1}{4}x_2^2 + C_2x_2 + C_3\big)\mathbb{1}_{\{x_2>0\}}\\ &+ \big(\frac{C}{6}x_2^3 + \frac{C_1}{2}x_2^2 + C_2x_2 + C_3\big)\mathbb{1}_{\{x_2\leqslant 0\}}\end{aligned}$$

is a strictly increasing function from $[-1, 1]$ *to* $[\Phi_-, \Phi_+]$*, where*

$$\begin{aligned}\Phi_- &= \Phi(-1) = \frac{3}{2}C_1 + C_3 - \frac{4}{3}a_+ + \frac{1}{3}a_-,\\ \Phi_+ &= \Phi(1) = -\frac{5}{4}C_1 + C_3 + \frac{5}{3}a_+ - \frac{2}{3}a_-.\end{aligned}$$

Then the pair (2.26)-(2.24) *is a solution of the system* (2.1) *in the Frolov's form* (2.14) *with*

$$\eta(y) = \begin{cases} b^{-1}(2) & \text{if } y \in (C_3, \Phi_+],\\ b^{-1}(1) & \text{if } y \in [\Phi_-, C_3].\end{cases}$$

- ***Examples of concentric flows***

If $(\rho, u) = (\rho(r), rg(r)e_\theta)$ *(not necessarily* $\rho' \neq 0$*) solves the system* (2.1) *with* $f = 0$*, then* u *satisfies* $(2.16)_2$*:* $\partial_r(\mu r^3 \partial_r g) = -Cr$*. In particular, with the following viscosity coefficient* μ

$$\mu = \mu(r) = 2\mathbb{1}_{\{0<r<1\}} + \mathbb{1}_{\{r\geqslant 1\}}, \tag{2.27}$$

we have for some real constant $C_1 \in \mathbb{R}$ *that*

$$\begin{aligned}\partial_r g &= \frac{-\frac{C}{2}r^2 + C_1}{\mu r^3}\\ &= \big(-\frac{C}{4}\frac{1}{r} + \frac{C_1}{2}\frac{1}{r^3}\big)\mathbb{1}_{\{0<r<1\}} + \big(-\frac{C}{2}\frac{1}{r} + C_1\frac{1}{r^3}\big)\mathbb{1}_{\{r\geqslant 1\}},\ C, C_1 \in \mathbb{R}.\end{aligned}$$

There exists a constant $C_2 \in \mathbb{R}$ *such that (for* $u \in H^1_{loc}(\mathbb{R}^2\backslash\{0\})$*)*

$$\begin{aligned}g(r) = &\big(-\frac{C}{4}\ln r - \frac{C_1}{4}(\frac{1}{r^2} - 1) + C_2\big)\mathbb{1}_{\{0<r<1\}}\\ &+ \big(-\frac{C}{2}\ln r - \frac{C_1}{2}(\frac{1}{r^2} - 1) + C_2\big)\mathbb{1}_{\{r\geqslant 1\}}.\end{aligned} \tag{2.28}$$

If we consider the concentric flow on the annulus $\{x \in \mathbb{R}^2 \mid \frac{1}{2} \leqslant |x| \leqslant 2\}$ *and suppose the boundary conditions*

$$u|_{\{x||x|=\frac{1}{2}\}} = \frac{1}{2}g_-e_\theta|_{\{x||x|=\frac{1}{2}\}}, \quad u|_{\{x||x|=2\}} = 2g_+e_\theta|_{\{x||x|=2\}}, \tag{2.29}$$

then

$$C = \left(\frac{3\ln 2}{4}\right)^{-1}(\frac{9}{8}C_1 - g_+ + g_-), \quad C_2 = \frac{1}{3}(\frac{9}{8}C_1 + g_+ + 2g_-), \quad C_1 \in \mathbb{R}.$$

Hence the density function

$$\rho = \rho(r) = b^{-1}(2)\mathbb{1}_{\{0<r<1\}} + b^{-1}(1)\mathbb{1}_{\{r\geqslant 1\}}$$

and the velocity vector field $u = rge_\theta$ *with* g *given in* (2.28) *is a solution of the boundary value problem* (2.1)-(2.29)*. We can follow the argument at the end of Case* $\rho = \rho(x_2)$ *to find the function* η *such that* $\rho = \eta(\Phi)$*, provided with more restrictions on* g_-, g_+, C_1*. We leave this to interested readers.*

- ***Examples of radial flows***

If $(\rho, u) = (\rho(\theta), \frac{h(\theta)}{r}e_r)$ *(not necessarily* $\rho' \neq 0$*) solves the system* (2.1) *with* $f = 0$*, then* u *satisfies* $(2.16)_3$*:* $\rho h^2 + \partial_\theta(\mu\partial_\theta h) + 4(\mu h) = C \in \mathbb{R}$*. Let the viscosity coefficient* μ *be*

$$\mu = \mu(\theta) = 2\mathbb{1}_{[0,\frac{\pi}{4})} + \mathbb{1}_{[\frac{\pi}{4},\frac{\pi}{2}]}.$$

Then $(\rho, u) = (\rho(\theta), \frac{h(\theta)}{r}e_r)$ *with* $h(\theta)$ *satisfying the following*

$$\partial_\theta(\mu\partial_\theta h) = 0, \quad \rho h + 4\mu = 0$$

is a solution of (2.1) *with* $f = 0$*, and in particular* h, ρ *can be taken as follows*

$$\begin{aligned} \partial_\theta h &= \frac{-2}{\mu} = -\mathbb{1}_{[0,\frac{\pi}{4})} - 2\mathbb{1}_{[\frac{\pi}{4},\frac{\pi}{2}]}, \\ h &= (-\frac{\pi}{2} - \theta)\mathbb{1}_{[0,\frac{\pi}{4})} + (-\frac{\pi}{4} - 2\theta)\mathbb{1}_{[\frac{\pi}{4},\frac{\pi}{2}]}, \quad \rho = -\frac{4\mu}{h} \end{aligned}$$

such that $\mu = b(\rho)$ *holds.*

This radial flow moves toward the origin and moves faster when closer to the origin. There are obviously other solutions of form $(\rho, u) = (\rho(\theta), \frac{h(\theta)}{r}e_r)$ *to the system* (2.1) *with* $f = 0$ *and the viscosity coefficient* (2.1.1)*, and we do not go to details here.*

2.1.4 Further discussions on the regularity issues

In contrast to the irregularity results for the solutions of the stationary Navier-Stokes system (2.1) with piecewise-constant viscosity coefficients (see Corollary 2.1.1)

$$\Delta u \notin L^1_{\mathrm{loc}}(\mathbb{R}^2\backslash\{0\}), \quad \mathrm{div}\,(\mu S u) \notin L^1_{\mathrm{loc}}(\mathbb{R}^2\backslash\{0\}),$$

we should have some regularity results for the divergence-free part of the viscous term $\mathrm{div}\,(\mu S u)$

$$\mathbb{P}\mathrm{div}\,(\mu S u),$$

where $\mathbb{P}$ is the Leray-Helmholtz projector. On the *whole plane* $\mathbb{R}^2$, by use of Fourier transform, any vector-valued tempered distribution $v \in \mathcal{S}'(\mathbb{R}^2;\mathbb{R}^2)$ can be decomposed into its div-free and curl-free parts separately

$$\begin{aligned}
&v = \nabla^\perp V_1 + \nabla V_2,\\
&\text{with } \nabla^\perp V_1 = \nabla^\perp \Delta^{-1}\nabla^\perp \cdot v = \mathbb{P}v, \quad \nabla V_2 = \nabla\Delta^{-1}\nabla\cdot v = (1-\mathbb{P})v,
\end{aligned}$$

and the Leray-Helmholtz projector $\mathbb{P}$ (as Calderón-Zygmund operator) maps $L^p(\mathbb{R}^2;\mathbb{R}^2)$ into itself, for any $p \in (1,\infty)$. We can also define $\mathbb{P}$ on $L^p(\Omega;\mathbb{R}^2)$, $1 < p < \infty$ where Ω is a *bounded* C^1 *domain* and we recall here briefly a possible definition (see [FMM98] for more details). Let $v \in L^p(\Omega;\mathbb{R}^2)$ and let $\mathbb{N}_\Omega : \mathcal{E}'(\Omega) \mapsto \mathcal{D}'(\Omega)$ be the Newtonian potential operator which acts component-wise on vector fields. We define the Leray-Helmholtz projector as follows:

$$\mathbb{P}v = v - \nabla\mathrm{div}\,\mathbb{N}_\Omega(v) - \nabla V,$$

where $V \in W^{1,p}(\Omega)$ solves the following Laplacian equation with Neumann boundary condition

$$\begin{cases} \Delta V = 0 \text{ in } \Omega,\\ \frac{\partial V}{\partial n} = \big(v - \nabla\mathrm{div}\,\mathbb{N}_\Omega(v)\big)\cdot n \text{ on } \partial\Omega. \end{cases}$$

By the results in Section 11 in [FMM98], we have the following Helmholtz-decomposition

$$L^p(\Omega;\mathbb{R}^2) = L^p_{\mathrm{div},0}(\Omega) \oplus \mathrm{grad}W^{1,p}(\Omega),$$

where

$$\begin{aligned}
&L^p_{\mathrm{div},0}(\Omega) \overset{\mathrm{def}}{=} \{v \in L^p(\Omega;\mathbb{R}^2) \,|\, \mathrm{div}\, v = 0, \quad v\cdot n|_{\partial\Omega} = 0\},\\
&\mathrm{grad}W^{1,p}(\Omega) \overset{\mathrm{def}}{=} \{\nabla V \,|\, V \in W^{1,p}(\Omega)\},
\end{aligned}$$

and the orthogonal Leray-Helmholtz projector $\mathbb{P} : L^p(\Omega) \mapsto L^p_{\mathrm{div},0}(\Omega)$ is bounded and onto.

In this subsection we always consider the stationary Navier-Stokes system (2.1) on a bounded C^1 domain Ω, with zero external force $f = 0$ (noticing $\operatorname{div}(\rho u \otimes u) = \rho u \cdot \nabla u$ by the density equation $\operatorname{div}(\rho u) = 0$)

$$\begin{cases} \rho u \cdot \nabla u - \operatorname{div}(\mu S u) + \nabla \Pi = 0, \\ \operatorname{div} u = 0, \ \operatorname{div}(\rho u) = 0. \end{cases} \tag{2.30}$$

We apply the Leray-Helmholtz projector $\mathbb{P}$ to the first equation of the stationary Navier-Stokes system (2.30) to derive (whenever one side is well-defined)

$$\mathbb{P}\operatorname{div}(\mu S u) = \mathbb{P}(\rho u \cdot \nabla u). \tag{2.31}$$

We can then analyze $\mathbb{P}\operatorname{div}(\mu S u)$ by use of the given information on u and ∇u.

Theorem 2.1.4 (L^p-boundedness for $\mathbb{P}\operatorname{div}(\mu S u)$, [HL20])**.** *Let Ω be a bounded C^1-domain.*

For any weak solution $(\rho, u) \in L^\infty(\Omega) \times H^1(\Omega; \mathbb{R}^2)$ to the boundary value problem of the stationary Navier-Stokes system (2.12) *with zero external force $f = 0$ (e.g. the solutions given in Theorem 2.1.2), we have*

$$\mathbb{P}\,div\,(\mu S u) \in L^p(\Omega; \mathbb{R}^2), \quad \forall p \in (1, 2). \tag{2.32}$$

If furthermore the boundary value $u_0 \in W^{1,\infty}(\partial\Omega)$ and the viscosity coefficient $\mu \in [\mu_*, \mu^*]$, $\mu_*, \mu^* > 0$ is a variably partially BMO coefficient *(e.g. all the solutions in Theorem 2.1.3 which are continuous in one direction), i.e. there exist $R_0 \in (0, 1]$ and $\gamma = \gamma(p, \mu_*, \mu^*) \in (0, 1/20)$ such that for any $x \in \Omega$ and any $r \in (0, \min\{R_0, \operatorname{dist}(x, \partial\Omega)/2\})$ there exists a coordinate system (y_1, y_2) depending on x and r such that*

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} \left| \mu(y_1, y_2) - \frac{1}{2r} \int_{y_2 - r}^{y_2 + r} \mu(y_1, s)\, ds \right| dy \leqslant \gamma,$$

then we have

$$\nabla u \in L^p(\Omega; \mathbb{R}^4) \text{ and } \mathbb{P}\,div\,(\mu S u) \in L^p(\Omega; \mathbb{R}^2), \quad \forall p \in [2, \infty). \tag{2.33}$$

Proof. By Sobolev embedding and Hölder's inequality, we have for the solutions $(\rho, u) \in L^\infty(\Omega) \times H^1(\Omega; \mathbb{R}^2)$ that

$$u \in L^s(\Omega; \mathbb{R}^2) \text{ for any } s \in [1, \infty), \text{ and hence } \rho u \cdot \nabla u \in L^p(\Omega; \mathbb{R}^2) \text{ for any } p \in [1, 2).$$

By the L^p-estimate for the Leray-Helmholtz projector $\mathbb{P}$ and the equality (2.31), we have (2.32).

Recall the fourth-order elliptic equation (2.10) for the stream function Φ (we assume $f = 0$)

$$L_\mu \Phi = \nabla^\perp \cdot \operatorname{div}(\rho u \otimes u),$$

with $L_\mu = (\partial_{x_2x_2} - \partial_{x_1x_1})\mu(\partial_{x_2x_2} - \partial_{x_1x_1}) + (2\partial_{x_1x_2})\mu(2\partial_{x_1x_2})$. By Sobolev embedding and Hölder's inequality again, for any solutions $(\rho, u) \in L^\infty(\Omega) \times H^1(\Omega; \mathbb{R}^2)$ we have

$$\rho u \otimes u \in L^p(\Omega; \mathbb{R}^4), \quad \forall p \in [2, \infty).$$

For any boundary value $u_0 \in W^{1-\frac{1}{p},p}(\partial\Omega)$, $p \in [2,\infty)$, we may assume $\Phi_0 \in W^{2,p}(\Omega)$ to be the extension of the boundary value defined in (2.9) with $\|\Phi_0\|_{W^{2,p}(\Omega)} \lesssim \|u_0\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}$. By the L^p-Estimate for the fourth-order elliptic equation with variably partially BMO coefficient in Theorem 8.6 in [DK11], we have for any $p \in [2, \infty)$

$$\|\Phi - \Phi_0\|_{W^{2,p}(\Omega)} \leqslant C(p, \mu_*, \mu^*, R_0, |\Omega|)\big(\|\rho u \otimes u\|_{L^p(\Omega)} + \|u_0\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}\big).$$

As $u \in L^s(\Omega; \mathbb{R}^2)$, $\forall s \in [1, \infty)$ by Sobolev embedding, we have

$$\Phi \in W^{2,p}(\Omega) \text{ and hence } u \in W^{1,p}(\Omega) \subset L^\infty(\Omega), \quad \forall p \in [2, \infty).$$

Thus (2.33) follows from the equation (2.31). □

Remark 2.1.3 (Symmetric flows in Theorem 2.1.3 revisited)**.** *Notice that in the parallel flow case and in the concentric flow case, we have*

$$\mathbb{P}\operatorname{div}(\mu S u) = \operatorname{div}(\mu S u),$$

which is smooth by view of (2.16), (2.17) *and* (2.18).

In the radial flow case $\rho = \rho(\theta)$, we assume that

$$\operatorname{div}(\mu S u) = \nabla^\perp\Big(\frac{\alpha(\theta)}{r^2}\Big) + \nabla\Big(\frac{\beta(\theta)}{r^2}\Big),$$

where $\alpha = \alpha(\theta)$, $\beta = \beta(\theta)$ are scalar functions depending only on θ. Then by (2.20) *and* $(2.16)_3$, *α and β satisfy*

$$\begin{cases} -2\beta + \alpha' = (\mu h')' \\ \beta' + 2\alpha = 2(\mu h)', \end{cases} \text{ that is,}$$

$$\begin{cases} \alpha'' + 4\alpha = 4(\mu h)' + (\mu h')'' = -(\rho h^2)', \\ \beta'' + 4\beta = -2(\mu h')' + 2(\mu h)'' = 2(\rho h^2 + 4\mu h - C) + 2(\mu h)''. \end{cases}$$

We calculate straightforward (in the sense of distribution) that

$$\begin{aligned}\alpha =& -\frac{\sin(2\theta)}{2}\int_0^\theta \cos(2s)(\rho h^2)'(s)\,ds + \frac{\cos(2\theta)}{2}\int_0^\theta \sin(2s)(\rho h^2)'(s)\,ds \\ &+ C_1\sin(2\theta) + C_2\cos(2\theta), \\ \partial_\theta\alpha =& -\cos(2\theta)\int_0^\theta \cos(2s)(\rho h^2)'(s)\,ds - \sin(2\theta)\int_0^\theta \sin(2s)(\rho h^2)'(s)\,ds \\ &+ 2C_1\cos(2\theta) - 2C_2\sin(2\theta).\end{aligned}$$

for some real constants $C_1, C_1 \in \mathbb{R}$. It is then easy to see that if $\rho h^2 \in L^p([0,2\pi))$ then $\alpha, \partial_\theta\alpha \in L^p([0,2\pi))$ and hence

$$\mathbb{P}\,\mathrm{div}(\mu S u) = \nabla^\perp(\frac{\alpha}{r^2}) = -\frac{2\alpha}{r^3}e_\theta + \frac{\partial_\theta\alpha}{r^3}e_r \in L^p_{loc}(\mathbb{R}^2\backslash\{0\}).$$

We conclude this introduction part with some further discussions on the fourth-order elliptic equation (2.10). If the right-hand of the equation (2.10) simply vanishes, that is,

$$L_\mu\Phi = (\partial_{x_2x_2} - \partial_{x_1x_1})\mu(\partial_{x_2x_2} - \partial_{x_1x_1})\Phi + (2\partial_{x_1x_2})\mu(2\partial_{x_1x_2})\Phi = 0, \quad (2.34)$$

then with the function $\Psi : \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$\begin{pmatrix}\mu(\partial_{22} - \partial_{11})\Phi \\ \mu 2\partial_{12}\Phi\end{pmatrix} = \begin{pmatrix}-2\partial_{12}\Psi \\ (\partial_{22} - \partial_{11})\Psi\end{pmatrix},$$

the complex value function $\Lambda = \Phi + i\Psi$ solves the following second-order Beltrami-type equation

$$\partial_{\bar z}^2\Lambda = \frac{1-\mu}{1+\mu}\overline{\partial_z^2\Lambda}, \quad z = x_1 + ix_2.$$

This description can be compared with the first-order Beltrami equation

$$\partial_{\bar z}\tilde w = \frac{1-\sigma}{1+\sigma}\overline{\partial_z \tilde w}.$$

Here $\tilde w = \tilde u + i\tilde v$ is a complex value function, where the real part $\tilde u$ satisfies a second-order elliptic equation of divergence form

$$\mathrm{div}(\sigma(x)\nabla\tilde u) = 0, \quad (2.35)$$

and the imaginary part $\tilde v$ is related by $\sigma(x)\nabla\tilde u = \nabla^\perp\tilde v$. According to [AIM09], on a bounded domain $\Omega \subset \mathbb{R}^2$, there exists a measurable function $\sigma : \Omega \mapsto$

$\{\frac{1}{K}, K\}$, $K > 1$ such that the solutions $\tilde{u} \in H^1(\Omega)$ to the equation (2.35) with the boundary condition $\tilde{u}|_{\partial\Omega} = x_1$ satisfies

$$\int_B |\nabla\tilde{u}|^{\frac{2K}{K-1}} = \infty,$$

for any disk $B \subset \Omega$. That is, $\tilde{u} \notin W^{1,p}(\Omega)$ for any $p \geqslant \frac{2K}{K-1}$.

Following the convex integration method in [AIM09], we can show that there exists a measurable function $\mu : \Omega \mapsto \{\frac{1}{K}, K\}$, $K > 1$ such that the solutions $\Phi \in H^2(\Omega)$ of the equation (2.34) satisfies

$$\int_B |\nabla^2\Phi|^{\frac{2K}{K-1}} = \infty,$$

for any disk $B \subset \Omega$. Although it is not clear whether this constructed solution $(\rho, u) = (b^{-1}(\mu), \nabla^\perp\Phi)$ solves the stationary Navier-Stokes equation (2.1), we expect in general that the solutions for (2.1) with only bounded viscosity coefficient μ (without any smoothness assumption)

$$\nabla u \notin L^p(\Omega), \text{ for any } p \geqslant p_*,$$

where $p_* < \infty$ depends on the deviation $|\mu - 1|$.

2.2 Proofs in two-dimensional case

In this section we are going to prove Theorem 2.1.2.

By virtue of the Fact and Definitions in Subsection 2.1.2, in order to prove (i) in Theorem 2.1.2, it suffices to show the existence of the weak solutions $\Phi \in H^2(\Omega)$ (resp. $\Phi \in \dot{H}^2(\Omega)$) of the boundary value problem (2.11)

$$\begin{cases} L_\mu\Phi = -\nabla^\perp \cdot f + \nabla^\perp \cdot \operatorname{div}(\rho\nabla^\perp\Phi \otimes \nabla^\perp\Phi), \\ \rho = \eta(\Phi), \quad \mu = (b \circ \eta)(\Phi), \\ \Phi|_{\partial\Omega} = \Phi_0, \quad \dfrac{\partial\Phi}{\partial n}|_{\partial\Omega} = \Phi_1, \end{cases} \tag{2.36}$$

where L_μ denotes the following fourth-order elliptic operator

$$L_\mu = (\partial_{x_2x_2} - \partial_{x_1x_1})\mu(\partial_{x_2x_2} - \partial_{x_1x_1}) + (2\partial_{x_1x_2})\mu(2\partial_{x_1x_2}).$$

Here the functions $\eta \in L^\infty(\mathbb{R}; [0, \rho^*])$, $0 < \rho^*$, $b \in C(\mathbb{R}; [\mu_*, \mu^*])$, $\mu_*, \mu^* > 0$ and $f \in H^{-1}(\Omega; \mathbb{R}^2)$ (resp. $f = \operatorname{div} F$, $F \in L^2(\Omega; \mathbb{R}^2 \times \mathbb{R}^2)$) are given. We will focus on the solvability on a bounded simply connected $C^{1,1}$-domain in Subsection 2.2.1, and the solvability on the exterior domain of a simply connected $C^{1,1}$ set will be achieved by an approximation argument in Subsection 2.2.2.

In Subsection 2.2.3 we will follow the method in [GW18] to show the existence of weak solutions to the system (2.1) on the whole plane, taking the prescribed mean value $d = \frac{1}{\text{meas}(D)} \int_D u$ on some set D of positive Lebesgue measure.

Finally more regularity results will be proved in Subsection 2.2.4.

2.2.1 The bounded domain case

Let Ω be a bounded connected $C^{1,1}$ domain on $\mathbb{R}^2$.

We first treat the boundary condition $\Phi|_{\partial\Omega} = \Phi_0 \in H^{\frac{3}{2}}(\partial\Omega)$ and $\frac{\partial\Phi}{\partial n}|_{\partial\Omega} = \Phi_1 \in H^{\frac{1}{2}}(\partial\Omega)$. By the inverse trace Theorem 2.1.1 and the Whitney's extension Theorem, we extend Φ_0 on the whole plane $\mathbb{R}^2$ (still denoted by Φ_0) such that $\Phi_0 \in H^2(\mathbb{R}^2)$ and $\frac{\partial\Phi_0}{\partial n}|_{\partial\Omega} = \Phi_1$. We then take a sequence of truncated functions $\zeta(x;\delta)$ on the boundary $\partial\Omega$ and define

$$\Phi_0^\delta(x) = \Phi_0(x)\zeta(x;\delta) \in H^2(\mathbb{R}^2). \tag{2.37}$$

Here $\zeta(x;\delta)$ is a smooth function, with $\zeta(x;\delta) = 1$ near $\partial\Omega$ and $\zeta(x;\delta) = 0$ if $\text{dist}(x,\partial\Omega) \geqslant \delta$, such that

$$|\zeta(x;\delta)| \leqslant C, \quad |\nabla\zeta(x;\delta)| \leqslant C\delta^{-1}, \quad \forall\delta \in (0,\delta_1],$$

for some fixed constants $C > 0$ and $\delta_1 > 0$. Then

$$\Phi_0^\delta|_{\partial\Omega} = \Phi_0|_{\partial\Omega}, \quad \frac{\partial\Phi_0^\delta}{\partial n}|_{\partial\Omega} = \frac{\partial\Phi_0}{\partial n}|_{\partial\Omega}.$$

Fix $\delta > 0$. If $\Phi \in H^2(\Omega)$ is a weak solution of the elliptic problem (2.36), then

$$\varphi^\delta \stackrel{\text{def}}{=} \Phi - \Phi_0^\delta \in H_0^2(\Omega)$$

satisfies

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \mu\Big((\partial_{22}\varphi^\delta - \partial_{11}\varphi^\delta)(\partial_{22}\psi - \partial_{11}\psi) + (2\partial_{12}\varphi^\delta)(2\partial_{12}\psi)\Big)\,dx \\
&= \int_\Omega \rho(\nabla^\perp(\Phi_0^\delta + \varphi^\delta) \otimes \nabla^\perp(\Phi_0^\delta + \varphi^\delta)) : \nabla\nabla^\perp\psi\,dx + \int_\Omega f\cdot\nabla^\perp\psi\,dx \\
&\quad - \frac{1}{2}\int_\Omega \mu\Big((\partial_{22}\Phi_0^\delta - \partial_{11}\Phi_0^\delta)(\partial_{22}\psi - \partial_{11}\psi) + (2\partial_{12}\Phi_0^\delta)(2\partial_{12}\psi)\Big)\,dx, \\
&\quad \text{with } \rho = \eta(\Phi_0^\delta + \varphi^\delta) \text{ and } \mu = b(\rho), \quad \forall\psi \in H_0^2(\Omega),
\end{aligned} \tag{2.38}$$

and vice versa. We hence search for $\varphi^\delta \in H_0^2(\Omega)$ satisfying (2.38).

Fix $\tilde{\varphi} \in H_0^2(\Omega)$ and set

$$\tilde{\rho}^\delta = \eta(\Phi_0^\delta + \tilde{\varphi}), \quad \tilde{\mu}^\delta = (b \circ \eta)(\Phi_0^\delta + \tilde{\varphi}).$$

We take a sequence of mollifiers $(\sigma^\varepsilon)_\varepsilon$ on $\mathbb{R}^2$, with $\sigma^\varepsilon = \frac{1}{\varepsilon^2}\sigma(\frac{\cdot}{\varepsilon})$, $\sigma \in C_0^\infty(\mathbb{R}^2)$, $\int_{\mathbb{R}^2} \sigma = 1$, and a sequence of mollifiers $(\phi^\varepsilon)_\varepsilon$ on $\mathbb{R}$, with $\phi^\varepsilon = \frac{1}{\varepsilon}\phi(\frac{\cdot}{\varepsilon})$, $\phi \in C_0^\infty(\mathbb{R})$, $\int_{\mathbb{R}} \phi = 1$. We regularize $\Phi_0^\delta, \tilde{\rho}^\delta, \tilde{\mu}^\delta, f$ (we simply extend f trivially on the whole plane)

$$\begin{aligned}
&\Phi_0^{\delta,\varepsilon} = \sigma^\varepsilon * \Phi_0^\delta \in H^3(\mathbb{R}^2), \quad f^\varepsilon = \sigma^\varepsilon * f \in L^2(\Omega;\mathbb{R}^2),\\
&\eta^\varepsilon = \phi^\varepsilon * \eta \in C_b^2(\mathbb{R};[0,\rho^*]), \quad b^\varepsilon = \phi^\varepsilon * b \in C_b^2(\mathbb{R};[\mu_*,\mu^*]),\\
&\tilde{\rho}^{\delta,\varepsilon} = \eta^\varepsilon(\Phi_0^{\delta,\varepsilon} + \tilde{\varphi}) \leqslant \rho^*, \quad \tilde{\mu}^{\delta,\varepsilon} = b^\varepsilon\big(\tilde{\rho}^{\delta,\varepsilon}\big) \in H^2(\Omega;[\mu_*,\mu^*]),
\end{aligned}$$

such that

$$\begin{aligned}
&\Phi_0^{\delta,\varepsilon} \to \Phi_0^\delta \text{ in } H^2(\mathbb{R}^2), \quad f^\varepsilon \to f \text{ in } H^{-1}(\Omega;\mathbb{R}^2),\\
&\tilde{\rho}^{\delta,\varepsilon} \overset{*}{\rightharpoonup} \tilde{\rho}^\delta \text{ and } \tilde{\mu}^{\delta,\varepsilon} \overset{*}{\rightharpoonup} \tilde{\mu}^\delta \text{ in } L^\infty(\Omega) \text{ as } \varepsilon \to 0.
\end{aligned} \tag{2.39}$$

In the following we are going to find $\varphi^\delta \in H_0^2(\Omega)$ satisfying (2.38) in three steps. In Step 1 we will search for the unique $\varphi \in H_0^2(\Omega)$ satisfying

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \tilde{\mu}^{\delta,\varepsilon}\Big((\partial_{22}\varphi - \partial_{11}\varphi)(\partial_{22}\psi - \partial_{11}\psi) + (2\partial_{12}\varphi)(2\partial_{12}\psi)\Big)\,dx\\
&= \int_\Omega \tilde{\rho}^{\delta,\varepsilon}(\nabla^\perp(\Phi_0^{\delta,\varepsilon} + \tilde{\varphi}) \otimes \nabla^\perp(\Phi_0^{\delta,\varepsilon} + \varphi)) : \nabla\nabla^\perp\psi\,dx + \int_\Omega f^\varepsilon \cdot \nabla^\perp\psi\,dx\\
&\quad - \frac{1}{2}\int_\Omega \tilde{\mu}^{\delta,\varepsilon}\Big((\partial_{22}\Phi_0^{\delta,\varepsilon} - \partial_{11}\Phi_0^{\delta,\varepsilon})(\partial_{22}\psi - \partial_{11}\psi) + (2\partial_{12}\Phi_0^{\delta,\varepsilon})(2\partial_{12}\psi)\Big)\,dx,\\
&\forall \psi \in H_0^2(\Omega).
\end{aligned} \tag{2.40}$$

This unique solution will be denoted by $\varphi^{\delta,\varepsilon}$.

Similarly, let $\lambda \in [0,1]$ be a parameter. Then there exists a unique solution $\varphi_\lambda^{\delta,\varepsilon} \in H_0^2(\Omega)$ satisfying

$$\begin{aligned}
&\int_\Omega \tilde{\mu}_\lambda^{\delta,\varepsilon}\Big((\partial_{22}\varphi_\lambda^{\delta,\varepsilon} - \partial_{11}\varphi_\lambda^{\delta,\varepsilon})(\partial_{22}\psi - \partial_{11}\psi) + (2\partial_{12}\varphi_\lambda^{\delta,\varepsilon})(2\partial_{12}\psi)\Big)\,dx\\
&= \lambda\int_\Omega \tilde{\rho}_\lambda^{\delta,\varepsilon}(\nabla^\perp(\lambda\Phi_0^{\delta,\varepsilon} + \tilde{\varphi}) \otimes \nabla^\perp(\lambda\Phi_0^{\delta,\varepsilon} + \varphi_\lambda^{\delta,\varepsilon})) : \nabla\nabla^\perp\psi\,dx + \lambda\int_\Omega f^\varepsilon \cdot \nabla^\perp\psi\,dx\\
&\quad - \lambda\int_\Omega \tilde{\mu}_\lambda^{\delta,\varepsilon}\Big((\partial_{22}\Phi_0^{\delta,\varepsilon} - \partial_{11}\Phi_0^{\delta,\varepsilon})(\partial_{22}\psi - \partial_{11}\psi) + (2\partial_{12}\Phi_0^{\delta,\varepsilon})(2\partial_{12}\psi)\Big)\,dx,\\
&\quad \text{with } \tilde{\rho}_\lambda^{\delta,\varepsilon} = \eta^\varepsilon(\lambda\Phi_0^{\delta,\varepsilon} + \tilde{\varphi}) \text{ and } \tilde{\mu}^{\delta,\varepsilon} = b^\varepsilon(\tilde{\rho}_\lambda^{\delta,\varepsilon}), \quad \forall \psi \in H_0^2(\Omega).
\end{aligned} \tag{2.41}$$

Notice that (2.40) is (2.41) with $\lambda = 1$.

In Step 2 we will define the map

$$T^{\delta,\varepsilon} : [0,1] \times H_0^2(\Omega) \ni (\lambda, \tilde{\varphi}) \mapsto \varphi_\lambda^{\delta,\varepsilon} \in H_0^2(\Omega).$$

Notice that, if $\varphi_\lambda^{\delta,\varepsilon}$ satisfies $\varphi_\lambda^{\delta,\varepsilon} = T^{\delta,\varepsilon}(\lambda, \varphi_\lambda^{\delta,\varepsilon})$, then (2.41) can be seen as an equality for $\Phi_\lambda^\varepsilon = \lambda\Phi_0^{\delta,\varepsilon} + \varphi_\lambda^{\delta,\varepsilon}$

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \mu_\lambda^\varepsilon \Big((\partial_{x_2x_2}\Phi_\lambda^\varepsilon - \partial_{x_1x_1}\Phi_\lambda^\varepsilon)(\partial_{x_2x_2}\psi - \partial_{x_1x_1}\psi) + (2\partial_{x_1x_2}\Phi_\lambda^\varepsilon)(2\partial_{x_1x_2}\psi) \Big)\, dx \\
&= \int_\Omega f \cdot \nabla^\perp\psi\, dx + \int_\Omega \rho_\lambda^\varepsilon (\nabla^\perp\Phi_\lambda^\varepsilon \otimes \nabla^\perp\Phi_\lambda^\varepsilon) : \nabla\nabla^\perp\psi\, dx, \quad \forall\psi \in H_0^2(\Omega),
\end{aligned} \tag{2.42}$$

where $\rho_\lambda^\varepsilon = \eta^\varepsilon(\Phi_\lambda^\varepsilon)$, $\mu_\lambda^\varepsilon = b^\varepsilon(\rho_\lambda^\varepsilon)$. We observe that Φ_λ^ε is independent of δ. This fact is going to be used to show a uniform bound on the sequence $(\varphi_\lambda^{\delta,\varepsilon})$.

We are going to show that the map $T^{\delta,\varepsilon}$ has a fixed point with $\lambda = 1$ (denoted by $\varphi^{\delta,\varepsilon}$) satisfying (2.40) with $\tilde{\varphi} = \varphi^{\delta,\varepsilon}$:

$$\begin{aligned}
&\int_\Omega \mu^{\delta,\varepsilon} \Big((\partial_{22}\varphi^{\delta,\varepsilon} - \partial_{11}\varphi^{\delta,\varepsilon})(\partial_{22}\psi - \partial_{11}\psi) + (2\partial_{12}\varphi^{\delta,\varepsilon})(2\partial_{12}\psi) \Big)\, dx \\
&= \int_\Omega \rho^{\delta,\varepsilon} (\nabla^\perp(\Phi_0^{\delta,\varepsilon} + \varphi^{\delta,\varepsilon}) \otimes \nabla^\perp(\Phi_0^{\delta,\varepsilon} + \varphi^{\delta,\varepsilon})) : \nabla\nabla^\perp\psi\, dx + \int_\Omega f^\varepsilon \cdot \nabla^\perp\psi\, dx \\
&\quad - \int_\Omega \mu^{\delta,\varepsilon} \Big((\partial_{22}\Phi_0^{\delta,\varepsilon} - \partial_{11}\Phi_0^{\delta,\varepsilon})(\partial_{22}\psi - \partial_{11}\psi) + (2\partial_{12}\Phi_0^{\delta,\varepsilon})(2\partial_{12}\psi) \Big)\, dx, \\
&\quad \text{with } \rho^{\delta,\varepsilon} = \eta^\varepsilon(\Phi_0^{\delta,\varepsilon} + \varphi^{\delta,\varepsilon}) \text{ and } \mu^{\delta,\varepsilon} = b^\varepsilon(\rho_\lambda^{\delta,\varepsilon}), \quad \forall\psi \in H_0^2(\Omega).
\end{aligned} \tag{2.43}$$

To show the existence of the fixed point, we will apply the following Leray-Schauder's fixed point theorem, after checking the conditions (LS1), (LS2) and (LS3) one by one.

Theorem 2.2.1 (Leray-Schauder's fixed point theorem, [MPS00])**.** *Let B be a Banach space. If*

(LS1) $T(0,u) = 0$, for all $u \in B$,

(LS2) T is a compact map from $B \times [0,1]$ to B,

(LS3) The solutions of $u = T(u,\lambda)$ for all $\lambda \in [0,1]$ are uniformly bounded in B.

Then there exists $u \in B$ such that $u = T(1,u)$.

In Step 3 we will take $\varepsilon \to 0$ in the sequence $\{\varphi^{\delta,\varepsilon}\}$ such that the limit φ^δ satisfies (2.38), and hence $\Phi = \Phi_0^\delta + \varphi^\delta$ solves the boundary value problem (2.36) on the bounded domain Ω.

Step 1 Unique solvability of (2.40)**.**
Let $\tilde{\varphi} \in H_0^2(\Omega)$ be given. We are going to search for $\varphi \in H_0^2(\Omega)$ satisfying (2.40) under the following assumptions on the given functions:

$$\tilde{\rho}^{\delta,\varepsilon} = \eta^\varepsilon(\Phi_0^{\delta,\varepsilon} + \tilde{\varphi}) \leqslant \rho^*, \quad \tilde{\mu}^{\delta,\varepsilon} \in [\mu_*, \mu^*], \quad \Phi_0^{\delta,\varepsilon} \in H^2(\mathbb{R}^2), \quad f^\varepsilon \in H^{-1}(\Omega; \mathbb{R}^2). \tag{2.44}$$

For notational simplicity we do not indicate the upper indices δ, ε explicitly in this step.

We define the inner product $\langle \cdot, \cdot \rangle$ on the Hilbert space $H_0^2(\Omega)$ as follows:

$$\langle \varphi, \psi \rangle \stackrel{\text{def}}{=} \int_\Omega \tilde{\mu}\Big((\partial_{22}\varphi - \partial_{11}\varphi)(\partial_{22}\psi - \partial_{11}\psi) + (2\partial_{12}\varphi)(2\partial_{12}\psi) \Big)\, dx.$$

Then the corresponding norm $\langle \cdot, \cdot \rangle^{\frac{1}{2}}$ is equivalent to the H^2-norm on $H_0^2(\Omega)$. Indeed,

$$\mu_* a \leqslant \langle \varphi, \varphi \rangle = \int_\Omega \tilde{\mu}\Big((\partial_{22}\varphi - \partial_{11}\varphi)^2 + (2\partial_{12}\varphi)^2 \Big)\, dx \leqslant \mu^* a,$$

where

$$a \stackrel{\text{def}}{=} \int_\Omega \Big((\partial_{22}\varphi - \partial_{11}\varphi)^2 + (2\partial_{12}\varphi)^2 \Big)\, dx \geqslant 0.$$

By integration by parts, for $\varphi \in H_0^2(\Omega)$ there holds

$$\begin{aligned} a &= \int_\Omega \Big((\partial_{11}\varphi)^2 + (\partial_{22}\varphi)^2 - 2\partial_{11}\varphi\partial_{22}\varphi + (2\partial_{12}\varphi)^2 \Big)\, dx \\ &= \int_\Omega \Big((\partial_{11}\varphi)^2 + (\partial_{22}\varphi)^2 + 2\partial_{11}\varphi\partial_{22}\varphi \Big)\, dx \\ &= \int_\Omega (\partial_{11}\varphi + \partial_{22}\varphi)^2\, dx = \|\Delta\varphi\|_{L^2(\Omega)}^2. \end{aligned}$$

Thus

$$\sqrt{\mu_*}\|\Delta\varphi\|_{L^2(\Omega)} \leqslant \langle \varphi, \varphi \rangle^{\frac{1}{2}} \leqslant \sqrt{\mu^*}\|\Delta\varphi\|_{L^2(\Omega)},$$

and hence by virtue of the equivalence of the norms $\|\Delta \cdot\|_{L^2(\Omega)} \sim \|\cdot\|_{H^2(\Omega)}$ on $H_0^2(\Omega)$, we have the equivalence of the norms

$$\langle \cdot, \cdot \rangle^{\frac{1}{2}} \sim \|\cdot\|_{H^2(\Omega)}, \qquad \text{on } H_0^2(\Omega).$$

Notice that the left-hand side of (2.40) reads as $\langle \varphi, \psi \rangle$. We are going to show that the right-hand side of (2.40) is a linear functional on $H_0^2(\Omega)$

$$l(\psi),$$

which by Lax-Milgram theorem defines a unique element (denoted by $A\varphi$) in $H_0^2(\Omega)$ such that

$$l(\psi) = \langle A\varphi, \psi \rangle.$$

Then we will verify the conditions (LS1), (LS2) and (LS3) in Leray-Schauder's fixed point Theorem 2.2.1 for the map

$$\alpha A : [0,1] \times H_0^2(\Omega) \mapsto H_0^2(\Omega),$$

to show the existence of the unique solution for the equation

$$\varphi = A\varphi$$

and hence (2.40).

Definition of the operator A**.** By virtue of (2.44), the right-hand side of (2.40) depends linearly on ψ and can be bounded by

$$\begin{aligned}&\Big(\rho^* \|\Phi_0 + \tilde{\varphi}\|_{W^{1,4}} \|\Phi_0 + \varphi\|_{W^{1,4}} + \|f\|_{H^{-1}} + 8\mu^* \|\Phi_0\|_{H^2}\Big) \|\psi\|_{H^2} \\ &\leqslant C(\rho^* + \mu^* + 1)(\|\Phi_0\|_{H^2} + \|\tilde{\varphi}\|_{H^2} + \|f\|_{H^{-1}})\big(1 + \|\Phi_0\|_{H^2} + \|\varphi\|_{H^2}\big)\|\psi\|_{H^2},\end{aligned}$$

for some constant $C > 0$. Here we used the Sobolev's inequality

$$\|g\|_{L^4(\Omega)} \leqslant C\|g\|_{H^1(\Omega)}, \quad \forall g \in H_0^1(\Omega).$$

Hence the right-hand side of (2.40) defines a linear functional $l(\psi)$ on $H_0^2(\Omega)$, which defines correspondingly by Lax-Milgram theorem an element (denoted by $A\varphi$) such that $l(\psi) = \langle A\varphi, \psi \rangle$.

Verification of Condition (LS1). If $\alpha = 0$, then the map $\alpha A = 0$.

Verification of Condition (LS2). In order to show the compactness of the operator αA, we take a weak convergent sequence $(\alpha_n, \varphi_n) \subset [0,1] \times H_0^2(\Omega)$. By virtue of the compact embedding $H_0^2(\Omega) \hookrightarrow W^{1,4}(\Omega)$, there exists a subsequence (still denoted by (α_n, φ_n)) converging strongly in $[0,1] \times W^{1,4}(\Omega)$, and hence

$$\begin{aligned}
&\|\alpha_n A\varphi_n - \alpha_m A\varphi_m\|_{H^2} \\
&\leqslant \sup_{\|\psi\|_{H^2}=1} \big(|\alpha_n\langle A\varphi_n - A\varphi_m, \psi\rangle| + |(\alpha_n - \alpha_m)\langle A\varphi_m, \psi\rangle|\big) \\
&\leqslant \sup_{\|\psi\|_{H^2}=1} \Big|\int_\Omega \tilde{\rho}\big(\nabla^\perp(\Phi_0 + \tilde{\varphi}) \otimes \nabla^\perp(\varphi_n - \varphi_m)\big) : \nabla\nabla^\perp\psi \, dx\Big| \\
&\quad + |\alpha_n - \alpha_m|\|A\varphi_m\|_{H^2} \\
&\leqslant C(\|\varphi_n - \varphi_m\|_{W^{1,4}} + |\alpha_m - \alpha_n|) \\
&\quad \times \big(\rho^*\|\Phi_0 + \tilde{\varphi}\|_{W^{1,4}}(1 + \|\varphi_m\|_{W^{1,4}}) + \|f\|_{H^{-1}} + \mu^*\|\Phi_0\|_{H^2}\big) \\
&\to 0 \text{ as } n, m \to \infty.
\end{aligned}$$

Verification of Condition (LS3). The solutions of $\varphi = \alpha A\varphi$ are uniformly bounded in $H_0^2(\Omega)$. Indeed, if $\varphi = \alpha A\varphi \in H_0^2(\Omega)$, then $\langle\varphi, \psi\rangle = \alpha\langle A\varphi, \psi\rangle = \alpha l(\psi)$ for any $\psi \in H_0^2(\Omega)$, and in particular when $\psi = \varphi$,

$$\begin{aligned}
\langle\varphi, \varphi\rangle = \alpha \int_\Omega \tilde{\rho}(\nabla^\perp(\Phi_0 + \tilde{\varphi}) \otimes \nabla^\perp(\Phi_0 + \varphi)) : \nabla\nabla^\perp\varphi \, dx + \alpha \int_\Omega f \cdot \nabla^\perp\varphi \, dx \\
- \alpha \int_\Omega \tilde{\mu}\Big((\partial_{22}\Phi_0 - \partial_{11}\Phi_0)(\partial_{22}\varphi - \partial_{11}\varphi) + (2\partial_{12}\Phi_0)(2\partial_{12}\varphi)\Big) \, dx.
\end{aligned}$$

Notice the equality

$$\begin{aligned}
&\int_\Omega \tilde{\rho}(\nabla^\perp(\Phi_0 + \tilde{\varphi}) \otimes \nabla^\perp\varphi) : \nabla\nabla^\perp\varphi \, dx \\
&= \int_\Omega \tilde{\rho}\nabla^\perp(\Phi_0 + \tilde{\varphi}) \cdot \nabla\nabla^\perp\varphi \cdot \nabla^\perp\varphi \, dx \qquad (2.45) \\
&= -\frac{1}{2}\int_\Omega \operatorname{div}\big(\tilde{\rho}\nabla^\perp(\Phi_0 + \tilde{\varphi})\big)|\nabla^\perp\varphi|^2 \, dx = 0,
\end{aligned}$$

where we used $\tilde{\rho} = \eta(\Phi_0 + \tilde{\varphi})$ in the last equality. We hence derive from $\langle\varphi, \varphi\rangle = \alpha l(\varphi)$ above and $\|g\|_{L^4(\Omega)} \leqslant C\|g\|_{H^1(\Omega)}$ that

$$\langle\varphi, \varphi\rangle \leqslant C\alpha(\rho^* + 1 + \mu^*)(\|\Phi_0\|_{H^2} + \|\tilde{\varphi}\|_{H^2} + \|f\|_{H^{-1}})\big(1 + \|\Phi_0\|_{H^2}\big)\|\varphi\|_{H^2}.$$

Since the norm $\langle\cdot, \cdot\rangle^{\frac{1}{2}} \geqslant \sqrt{\mu_*}\|\Delta \cdot \|_{L^2(\Omega)}$ is equivalent to $\| \cdot \|_{H^2(\Omega)}$ on $H_0^2(\Omega)$, there is a uniform bound for all $\varphi \in H_0^2(\Omega)$ such that $\varphi = \alpha A\varphi$, $\alpha \in [0, 1]$:

$$\|\varphi\|_{H^2} \leqslant C\mu_*^{-1}(\rho^* + 1 + \mu^*)(\|\Phi_0\|_{H^2} + \|\tilde{\varphi}\|_{H^2} + \|f\|_{H^{-1}})\big(1 + \|\Phi_0\|_{H^2}\big). \quad (2.46)$$

By Leray-Schauder's Theorem 2.2.1, there exists a solution of $\varphi = A\varphi$ in $H_0^2(\Omega)$. This solution solves (2.40): $\langle\varphi, \psi\rangle = \langle A\varphi, \psi\rangle = l(\psi)$ for all $\psi \in H_0^2(\Omega)$.

This solution is unique. Indeed, if there exist two solutions $\varphi_1, \varphi_2 \in H_0^2(\Omega)$ of (2.40), then their difference $\dot{\varphi} = \varphi_1 - \varphi_2 \in H_0^2(\Omega)$ satisfies

$$\langle \dot{\varphi}, \psi \rangle = \int_\Omega \tilde{\rho} \nabla^\perp (\Phi_0 + \tilde{\varphi}) \cdot \nabla \nabla^\perp \psi \cdot \nabla \dot{\varphi}, \quad \forall \psi \in H_0^2(\Omega).$$

Take $\psi = \dot{\varphi}$, then by the calculation in (2.45) the right-hand side above vanishes and hence $\dot{\varphi} = 0$, i.e., $\varphi_1 = \varphi_2$.

Step 2 Solvability of (2.43)**.**
By the procedure in Step 1 above, we can solve (2.41) uniquely for any $\lambda \in [0,1]$, and we denote this unique solution satisfying (2.41) by $\varphi_\lambda^{\delta,\varepsilon}$.

We are going to check the conditions (LS1), (LS2) and (LS3) for the map $T^{\delta,\varepsilon} : (\lambda, \tilde{\varphi}) \mapsto \varphi_\lambda^{\delta,\varepsilon}$, in order to show the existence of the fixed point of $T^{\delta,\varepsilon}$ with $\lambda = 1$ by the Leray-Schauder fixed point Theorem 2.2.1.

Verification of Condition (LS1). Let $\lambda = 0$ in (2.41) and let $\varphi_0^{\delta,\varepsilon}$ satisfy $\varphi_0^{\delta,\varepsilon} = T^{\delta,\varepsilon}(0, \tilde{\varphi})$. We take $\psi = \varphi_0^{\delta,\varepsilon}$ in (2.41), which implies

$$\|\Delta(\varphi_0^{\delta,\varepsilon})\|_{L^2} = 0.$$

Since $\varphi_0^{\delta,\varepsilon} \in H_0^2(\Omega)$, $\varphi_0^{\delta,\varepsilon} = 0$.

Verification of Condition (LS2). The map

$$T^{\delta,\varepsilon} : [0,1] \times H_0^2(\Omega) \ni (\lambda, \tilde{\varphi}) \mapsto \varphi_\lambda^{\delta,\varepsilon} \in H_0^2(\Omega)$$

is compact, where $\varphi_\lambda^{\delta,\varepsilon}$ is the solution of (2.41), under the following assumptions on the regularized data:

$$\Phi_0^{\delta,\varepsilon} \in H^3(\mathbb{R}^2), \quad \tilde{\rho}_\lambda^{\delta,\varepsilon} = \eta^\varepsilon(\lambda \Phi_0^{\delta,\varepsilon} + \tilde{\varphi}) \leqslant \rho^*, \quad \tilde{\mu}_\lambda^{\delta,\varepsilon} \in H^2(\Omega), \quad f^\varepsilon \in L^2(\Omega; \mathbb{R}^2).$$

Indeed, let $(\lambda_n, \tilde{\varphi}_n)$ be a bounded sequence in $[0,1] \times H_0^2(\Omega)$. Then there exists a subsequence (still denote by $(\lambda_n, \tilde{\varphi}_n)$), such that

$$|\lambda_m - \lambda_n| \to 0, \quad \|\tilde{\varphi}_m - \tilde{\varphi}_n\|_{W^{1,4}} \to 0, \text{ as } m, n \to \infty.$$

We denote $\varphi_n^{\delta,\varepsilon} = T^{\delta,\varepsilon}(\lambda_n, \tilde{\varphi}_n)$, $\tilde{\rho}_n^{\delta,\varepsilon} = \eta^\varepsilon(\lambda_n \Phi_0^{\delta,\varepsilon} + \tilde{\varphi}_n)$ and $\tilde{\mu}_n^{\delta,\varepsilon} = b^\varepsilon(\tilde{\rho}_n^{\delta,\varepsilon})$. We take the difference between (2.41) with $(\lambda_m, \tilde{\varphi}_m)$ and (2.41) with $(\lambda_n, \tilde{\varphi}_n)$. Let

$\psi = \dot{\varphi}^{\delta,\varepsilon} = \varphi_m^{\delta,\varepsilon} - \varphi_n^{\delta,\varepsilon}$, then (noticing (2.45) again and $\|\cdot\|_{L^\infty(\Omega)} \lesssim \|\cdot\|_{W^{1,4}(\Omega)}$)

$$
\begin{aligned}
&\|\Delta(\varphi_m^{\delta,\varepsilon} - \varphi_n^{\delta,\varepsilon})\|_{L^2}^2 \leqslant C(\mu_*)\Big(\Big|\int_\Omega (\tilde{\mu}_m^{\delta,\varepsilon} - \tilde{\mu}_n^{\delta,\varepsilon})\times \\
&\times \Big((\partial_{22}\varphi_m^{\delta,\varepsilon} - \partial_{11}\varphi_m^{\delta,\varepsilon})(\partial_{22}\dot{\varphi}^{\delta,\varepsilon} - \partial_{11}\dot{\varphi}^{\delta,\varepsilon}) + (2\partial_{12}\varphi_m^{\delta,\varepsilon})(2\partial_{12}\dot{\varphi}^{\delta,\varepsilon})\Big)\,dx\Big| \\
&+ |\lambda_m - \lambda_n|\Big|\int_\Omega \tilde{\rho}_m^{\delta,\varepsilon}(\nabla^\perp(\lambda_m\Phi_0^{\delta,\varepsilon} + \tilde{\varphi}_m) \otimes \nabla^\perp(\lambda_m\Phi_0^{\delta,\varepsilon} + \varphi_m^{\delta,\varepsilon})) : \nabla\nabla^\perp\dot{\varphi}^{\delta,\varepsilon}\,dx\Big| \\
&+ \Big|\int_\Omega \lambda_n(\tilde{\rho}_m^{\delta,\varepsilon} - \tilde{\rho}_n^{\delta,\varepsilon})(\nabla^\perp(\lambda_m\Phi_0^{\delta,\varepsilon} + \tilde{\varphi}_m) \otimes \nabla^\perp(\lambda_m\Phi_0^{\delta,\varepsilon} + \varphi_m^{\delta,\varepsilon})) : \nabla\nabla^\perp\dot{\varphi}^{\delta,\varepsilon}\,dx\Big| \\
&+ \Big|\int_\Omega \lambda_n\tilde{\rho}_n^{\delta,\varepsilon}(\nabla^\perp((\lambda_m - \lambda_n)\Phi_0^{\delta,\varepsilon}) \otimes \nabla^\perp(\lambda_m\Phi_0^{\delta,\varepsilon} + \varphi_m^{\delta,\varepsilon})) : \nabla\nabla^\perp\dot{\varphi}^{\delta,\varepsilon}\,dx\Big| \\
&+ \Big|\int_\Omega \lambda_n\tilde{\rho}_n^{\delta,\varepsilon}(\nabla^\perp(\tilde{\varphi}_m - \tilde{\varphi}_n) \otimes \nabla^\perp(\lambda_m\Phi_0^{\delta,\varepsilon} + \varphi_m^{\delta,\varepsilon})) : \nabla\nabla^\perp\dot{\varphi}^{\delta,\varepsilon}\,dx\Big| \\
&+ \Big|\int_\Omega \lambda_n\tilde{\rho}_n^{\delta,\varepsilon}(\nabla^\perp(\lambda_n\Phi_0^{\delta,\varepsilon} + \tilde{\varphi}_n) \otimes \nabla^\perp((\lambda_m - \lambda_n)\Phi_0^{\delta,\varepsilon})) : \nabla\nabla^\perp\dot{\varphi}^{\delta,\varepsilon}\,dx\Big| \\
&+ \Big|\int_\Omega (\tilde{\mu}_m^{\delta,\varepsilon} - \tilde{\mu}_n^{\delta,\varepsilon})\Big((\partial_{22}\Phi_0^{\delta,\varepsilon} - \partial_{11}\Phi_0^{\delta,\varepsilon})(\partial_{22}\dot{\varphi}^{\delta,\varepsilon} - \partial_{11}\dot{\varphi}^{\delta,\varepsilon}) + (2\partial_{12}\Phi_0^{\delta,\varepsilon})(2\partial_{12}\dot{\varphi}^{\delta,\varepsilon})\Big)\,dx\Big|\Big) \\
&\leqslant C(\rho^*, \mu_*, \mu^*)(1 + \|\Delta\varphi_m^{\delta,\varepsilon}\|_{L^2} + \|\Delta\Phi_0^{\delta,\varepsilon}\|_{L^2}) \\
&\times (|\lambda_m - \lambda_n| + \|\tilde{\varphi}_m - \tilde{\varphi}_n\|_{W^{1,4}})(1 + \|b^\varepsilon(\eta^\varepsilon)\|_{W^{1,\infty}} + \|\eta^\varepsilon\|_{W^{1,\infty}}) \\
&\times (1 + \|\Phi_0^{\delta,\varepsilon}\|_{W^{1,4}} + \|\tilde{\varphi}_m\|_{W^{1,4}} + \|\tilde{\varphi}_n\|_{W^{1,4}})(1 + \|\Phi_0^{\delta,\varepsilon}\|_{W^{1,4}} + \|\varphi_m^{\delta,\varepsilon}\|_{W^{1,4}})\|\Delta\dot{\varphi}^{\delta,\varepsilon}\|_{L^2}.
\end{aligned} \tag{2.47}
$$

Notice that, since $\{\tilde{\varphi}_n\}$ is uniformly bounded in H^2, the uniform bound of $\{\|\Delta\varphi_n^{\delta,\varepsilon}\|_{L^2}\}$ can be derived similarly to (2.46). Hence, the following strong convergence holds

$$\|\Delta(\varphi_m^{\delta,\varepsilon} - \varphi_n^{\delta,\varepsilon})\|_{L^2} \leqslant C(\|\tilde{\varphi}_m - \tilde{\varphi}_n\|_{W^{1,4}} + |\lambda_m - \lambda_n|) \to 0 \text{ as } m, n \to \infty.$$

The map $T^{\delta,\varepsilon} : (\lambda, \tilde{\varphi}) \mapsto \varphi_\lambda^{\delta,\varepsilon}$ is compact.

Verification of Condition (LS3). Let $\varphi_\lambda^{\delta,\varepsilon}$ denote the fixed point of $\varphi = T^{\delta,\varepsilon}(\lambda, \varphi)$ satisfying (2.41). We are going to derive a uniform bound on $\|\varphi_\lambda^{\delta,\varepsilon}\|_{H^2}$ by a contradiction argument. Suppose by contradiction that there exists a subsequence $(\varphi_{\lambda_n}^{\delta,\varepsilon}) \subset (\varphi_\lambda^{\delta,\varepsilon})$ such that

$$\|\varphi_{\lambda_n}^{\delta,\varepsilon}\|_{H^2} \to \infty.$$

Then we drive from (2.41) with $\psi = \varphi_{\lambda_n}^{\delta,\varepsilon}$ that (noticing again the equality (2.45))

$$
\begin{aligned}
\mu_*\|\Delta\varphi_{\lambda_n}^{\delta,\varepsilon}\|_{L^2}^2 \leqslant C(\rho^*, \mu_*, \mu^*)\Big(\Big(\|\Phi_0^{\delta,\varepsilon}\|_{H^2}^2 + \|f^\varepsilon\|_{H^{-1}} + \|\Phi_0^{\delta,\varepsilon}\|_{H^2}\Big)\|\varphi_{\lambda_n}^{\delta,\varepsilon}\|_{H^2} \\
+ \int_\Omega \rho_{\lambda_n}^{\delta,\varepsilon}\nabla^\perp\varphi^{\delta,\varepsilon} \cdot \nabla\nabla^\perp\varphi^{\delta,\varepsilon} \cdot \nabla^\perp\Phi_0^{\delta,\varepsilon}\,dx\Big),
\end{aligned} \tag{2.48}
$$

Let us denote $g_{\lambda_n}^{\delta,\varepsilon} = \frac{\varphi_{\lambda_n}^{\delta,\varepsilon}}{\|\varphi_{\lambda_n}^{\delta,\varepsilon}\|_{H^2}}$, then we drive from the above inequality that

$$1 \leqslant \frac{C(\rho^*, \mu_*, \mu^*, \|\Phi_0\|_{H^2}, \|f\|_{H^{-1}})}{\|\varphi_{\lambda_n}^{\delta,\varepsilon}\|_{H^2}} + C\int_\Omega \left|\nabla^\perp g_{\lambda_n}^{\delta,\varepsilon} \cdot \nabla\nabla^\perp g_{\lambda_n}^{\delta,\varepsilon} \cdot \nabla^\perp \Phi_0^{\delta,\varepsilon}\right| dx.$$

Since $\|g_{\lambda_n}^{\delta,\varepsilon}\|_{H^2} = 1$, there exist subsequences (still denoted by $(g_{\lambda_n}^{\delta,\varepsilon})$) such that

$$g_{\lambda_n}^{\delta,\varepsilon} \rightharpoonup g \text{ in } H_0^2(\Omega), \quad g_{\lambda_n}^{\delta,\varepsilon} \to g \text{ in } W^{1,4}(\Omega).$$

Here the limit g does not depend on δ. Indeed, notice that the δ-independent function $\Phi_\lambda^\varepsilon = \lambda\Phi_0^{\delta,\varepsilon} + \varphi_\lambda^{\delta,\varepsilon} = \lambda\Phi_0^{\delta',\varepsilon} + \varphi_\lambda^{\delta',\varepsilon}$ satisfies (2.42). Then $\|\varphi_{\lambda_n}^{\delta',\varepsilon}\|_{H^2} \to \infty$, and

$$\begin{aligned}
\lim_{n\to\infty} \frac{\varphi_{\lambda_n}^{\delta',\varepsilon}}{\|\varphi_{\lambda_n}^{\delta',\varepsilon}\|_{H^2}} &= \lim_{n\to\infty} \frac{\varphi_{\lambda_n}^{\delta,\varepsilon} + \lambda\Phi_0^{\delta,\varepsilon} - \lambda\Phi_0^{\delta',\varepsilon}}{\|\varphi_{\lambda_n}^{\delta',\varepsilon}\|_{H^2}} \\
&= \lim_{n\to\infty} \frac{\varphi_{\lambda_n}^{\delta,\varepsilon}}{\|\varphi_{\lambda_n}^{\delta',\varepsilon}\|_{H^2}} = \lim_{n\to\infty} \frac{\varphi_{\lambda_n}^{\delta,\varepsilon}}{\|\varphi_{\lambda_n}^{\delta,\varepsilon}\|_{H^2}} = g.
\end{aligned}$$

Then taking $n \to \infty$ in the above inequality we arrive at

$$1 \leqslant C\int_\Omega \left|\nabla^\perp g \cdot \nabla\nabla^\perp g \cdot \nabla^\perp \Phi_0^\delta\right| dx.$$

Recall the definition of Φ_0^δ in (2.37), such that

$$|\nabla^\perp \Phi_0^\delta| = |\nabla^\perp(\Phi_0(x)\zeta(x;\delta))| \leqslant C(\delta^{-1}|\Phi_0| + |\nabla\Phi_0|). \tag{2.49}$$

Hence with Ω^δ denoting the boundary strip of width δ, we derive from the above inequality that

$$\begin{aligned}
1 &\leqslant C\int_{\Omega^\delta} \left|\nabla^\perp g \cdot \nabla\nabla^\perp g\right| (\delta^{-1}|\Phi_0| + |\nabla\Phi_0|)\, dx \\
&\leqslant C\delta^{-1}\|\nabla g\|_{L^2(\Omega^\delta)}\|\nabla^2 g\|_{L^2(\Omega^\delta)}\|\Phi_0\|_{L^\infty} \\
&\quad + C\|\nabla g\|_{L^4(\Omega^\delta)}\|\nabla^2 g\|_{L^2(\Omega^\delta)}\|\nabla\Phi_0\|_{L^4(\Omega^\delta)}.
\end{aligned} \tag{2.50}$$

Since by Poincaré's inequality and $g \in H_0^2(\Omega)$ we have

$$\|\nabla g\|_{L^2(\Omega^\delta)} \leqslant C\delta\|\nabla^2 g\|_{L^2(\Omega^\delta)}, \tag{2.51}$$

the above inequality yields

$$1 \leqslant C\|\nabla^2 g\|_{L^2(\Omega^\delta)}^2 \|\Phi_0\|_{H^2(\Omega^\delta)},$$

where the right-hand side tends to 0 as $\delta \to 0$. This is a contradiction. Thus there is a constant C independent on λ such that

$$\|\varphi_\lambda^{\delta,\varepsilon}\|_{H^2(\Omega)} \leqslant C.$$

By Leray-Schauder's fixed point theorem, the map $T^{\delta,\varepsilon}(1,\cdot)$ has a fixed point $\varphi^{\delta,\varepsilon}$ satisfying (2.43).

Step 3 Passing to the limit $\varepsilon \to 0$.
Let $(\varphi^{\delta,\varepsilon}) \in H_0^2(\Omega)$ be the solution of (2.43) given in Step 2. We can follow exactly the argument to verify Condition (LS3) in Step 2 to show the uniform bound

$$\|\varphi^{\delta,\varepsilon}\|_{H^2(\Omega)} \leqslant C,$$

where C is independent of ε. Hence there exists a subsequence (still denoted by $\varphi^{\delta,\varepsilon}$) such that

$$\varphi^{\delta,\varepsilon} \to \varphi^\delta \text{ in } W^{1,4}(\Omega).$$

Thus up to a subsequence $\Phi_0^{\delta,\varepsilon} + \varphi^{\delta,\varepsilon} \to \Phi_0^\delta + \varphi^\delta$ in $L^\infty(\Omega)$ and

$$\begin{aligned}
&\rho^{\delta,\varepsilon} = \eta^\varepsilon(\Phi_0^{\delta,\varepsilon} + \varphi^{\delta,\varepsilon}) \overset{*}{\rightharpoonup} \rho^\delta = \eta(\Phi_0^\delta + \varphi^\delta),\\
&\mu^{\delta,\varepsilon} = b^\varepsilon(\rho^{\delta,\varepsilon}) \overset{*}{\rightharpoonup} \mu^\delta = b(\rho^\delta) \quad \text{in } L^\infty(\Omega), \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

Similar to (2.47), we take the difference between $(2.43)^\varepsilon$ and $(2.43)^{\varepsilon'}$ to derive the inequality for $\dot\varphi^{\delta,\varepsilon} = \varphi^{\delta,\varepsilon} - \varphi^{\delta,\varepsilon'}$

$$\begin{aligned}
&\|\Delta\dot\varphi^{\delta,\varepsilon}\|_{L^2}^2 \leqslant C\Big(\Big|\int_\Omega (\mu^{\delta,\varepsilon} - \mu^{\delta,\varepsilon'})\\
&\times \Big((\partial_{22}\varphi^{\delta,\varepsilon} - \partial_{11}\varphi^{\delta,\varepsilon})(\partial_{22}\dot\varphi^{\delta,\varepsilon} - \partial_{11}\dot\varphi^{\delta,\varepsilon}) + (2\partial_{12}\varphi^{\delta,\varepsilon})(2\partial_{12}\dot\varphi^{\delta,\varepsilon})\Big)\,dx\Big|\\
&+ \Big|\int_\Omega (\rho^{\delta,\varepsilon} - \rho^{\delta,\varepsilon'})(\nabla^\perp(\Phi_0^{\delta,\varepsilon} + \varphi^{\delta,\varepsilon}) \otimes \nabla^\perp(\Phi_0^{\delta,\varepsilon} + \varphi^{\delta,\varepsilon})) : \nabla\nabla^\perp\dot\varphi^{\delta,\varepsilon}\,dx\Big|\\
&+ \Big(\|\nabla(\Phi_0^{\delta,\varepsilon} + \varphi^{\delta,\varepsilon} - \Phi_0^{\delta,\varepsilon'} - \varphi^{\delta,\varepsilon'})\|_{L^4}\|\nabla(\Phi_0^{\delta,\varepsilon} + \varphi^{\delta,\varepsilon})\|_{L^4} + \|f^\varepsilon - f^{\varepsilon'}\|_{H^{-1}}\Big)\|\dot\varphi^{\delta,\varepsilon}\|_{H^2}\\
&+ \Big|\int_\Omega (\mu^{\delta,\varepsilon} - \mu^{\delta,\varepsilon'})\Big((\partial_{22}\Phi_0^{\delta,\varepsilon} - \partial_{11}\Phi_0^{\delta,\varepsilon})(\partial_{22}\dot\varphi^{\delta,\varepsilon} - \partial_{11}\dot\varphi^{\delta,\varepsilon}) + (2\partial_{12}\Phi_0^{\delta,\varepsilon})(2\partial_{12}\dot\varphi^{\delta,\varepsilon})\Big)\,dx\Big|\Big).
\end{aligned} \tag{2.52}$$

Therefore by view of the above convergence results

$$\varphi^{\delta,\varepsilon} \to \varphi^\delta \text{ in } H^2(\Omega).$$

Finally we take $\varepsilon \to 0$ in (2.43), then the limit φ^δ satisfies (2.38). Hence $\Phi = \varphi^\delta + \Phi_0^\delta$ is a weak solution of (2.36).

2.2.2 The exterior domain case

Let Ω be the exterior domain of a simply connected $C^{1,1}$ set. Let $N \in \mathbb{N}$ such that $\Omega^C \subset B_N(0) = \{x \in \mathbb{R}^2 \,|\, |x| < N\}$. Let $\Omega_n = \Omega \cap B_{N+n}(0) \subset \mathbb{R}^2$, then $\{\Omega_n\}$ is a monotonically increasing sequence which has the exterior domain Ω as its limit. By the solvability result in Subsection 2.2.1, for any given $\eta \in L^\infty(\mathbb{R}; [0, \infty))$, $b \in C(\mathbb{R}; [\mu_*, \mu^*])$, $\mu_*, \mu^* > 0$, and $f = \operatorname{div} F \in H^{-1}(\Omega; \mathbb{R}^2)$, there exists a weak solution $\Phi_n \in H^2(\Omega_n)$ of the boundary value problem (2.36) on Ω_n with the boundary condition $\Phi_n|_{\partial\Omega} = \Phi_0 \in H^{\frac{3}{2}}(\partial\Omega)$, $\frac{\partial \Phi_n}{\partial n}|_{\partial\Omega} = \Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$, and $\Phi_n|_{\partial B_{N+n}(0)} = 0$. Furthermore, for any fixed small enough $\delta > 0$, we can write

$$\Phi_n = \varphi_n^\delta + \Phi_0^\delta, \text{ with } \varphi_n^\delta \in H_0^2(\Omega_n) \text{ satisfying (2.38)},$$

and $\Phi_0^\delta(x) = \Phi_0(x)\zeta(x;\delta)$ is defined in (2.37). We extend φ_n^δ from Ω_n to Ω by simply taking $\varphi_n^\delta|_{\Omega\setminus\Omega_n} = 0$ (still denoted by φ_n^δ).

We are going to show that $\|\varphi_n^\delta\|_{\dot{H}^2(\Omega)}$ is uniformly bounded. We take $\psi = \varphi_n^\delta$ in the equation (2.38) for φ_n^δ, to derive

$$\begin{aligned}
&\int_\Omega \mu_n \Big((\partial_{22}\varphi_n^\delta - \partial_{11}\varphi_n^\delta)^2 + (2\partial_{12}\varphi_n^\delta)^2 \Big)\, dx \\
&= \int_\Omega \rho_n (\nabla^\perp(\Phi_0^\delta + \varphi_n^\delta) \otimes \nabla^\perp(\Phi_0^\delta + \varphi_n^\delta)) : \nabla\nabla^\perp \varphi_n^\delta \, dx - \int_\Omega F \cdot \nabla\nabla^\perp \varphi_n^\delta \, dx \\
&\quad - \int_\Omega \mu_n \Big((\partial_{22}\Phi_0^\delta - \partial_{11}\Phi_0^\delta)(\partial_{22}\varphi_n^\delta - \partial_{11}\varphi_n^\delta) + (2\partial_{12}\Phi_0^\delta)(2\partial_{12}\varphi_n^\delta) \Big)\, dx,
\end{aligned}$$

where $\rho_n = \eta(\Phi_n) = \eta(\varphi_n^\delta + \Phi_0^\delta)$ and $\mu_n = b(\rho_n)$. Similarly as in the derivation of (2.48), we have

$$\begin{aligned}
\|\Delta\varphi_n^\delta\|_{L^2(\Omega)}^2 \leqslant C(\rho^*, \mu_*, \mu^*) \Big(&\big(\|\Phi_0^\delta\|_{H^2}^2 + \|F\|_{L^2} + \|\Phi_0^\delta\|_{H^2} \big) \|\Delta\varphi_n^\delta\|_{L^2(\Omega)} \\
&+ \int_\Omega \rho_n \nabla^\perp \varphi_n^\delta \cdot \nabla\nabla^\perp \varphi_n^\delta \cdot \nabla^\perp \Phi_0^\delta \, dx \Big).
\end{aligned} \tag{2.53}$$

By the Riesz inequality (cf. [DD12]), we have $\|\Delta\varphi_n^\delta\|_{L^2} \sim \|\varphi_n^\delta\|_{\dot{H}^2}$. We are going to follow exactly the contradiction argument in Step 3 in Subsection 2.2.1 to show the uniform boundedness of $\|\varphi_n^\delta\|_{\dot{H}^2(\Omega)}$ and hence we will just sketch the proof and emphasize the difference for the exterior domain case. Suppose by contradiction that there exists a subsequence $(\varphi_{k_n}^\delta) \subset (\varphi_n^\delta)$ such that

$$\|\Delta\varphi_{k_n}^\delta\|_{L^2(\Omega)} \to \infty, \quad \text{as} \quad k_n \to \infty.$$

Denote $g_{k_n}^\delta = \frac{\varphi_{k_n}^\delta}{\|\Delta\varphi_{k_n}^\delta\|_{L^2(\Omega)}}$, then $\|\Delta g_{k_n}^\delta\|_{L^2(\Omega)} = 1$, $\mathrm{tr}(g_{k_n}^\delta)|_{\partial\Omega} = 0$ and there exist a subsequence (still denoted by $(g_{k_n}^\delta)$) and $g \in \dot{H}^2(\Omega)$ with $\mathrm{tr}(g)|_{\partial\Omega} = 0$ such that

$$g_{k_n}^\delta \rightharpoonup g \text{ in } \dot{H}^2(\Omega), \quad \text{as} \quad k_n \to \infty.$$

Here the limit function g does not depend on δ. Recall that Ω^δ is the boundary strip of width δ. By Poincaré's inequality we obtain $g_{k_n}^\delta|_{\Omega^\delta} \rightharpoonup g|_{\Omega^\delta}$ in $H^2(\Omega^\delta)$ and by Sobolev embedding $g_{k_n}^\delta|_{\Omega^\delta} \to g|_{\Omega^\delta}$ in $W^{1,4}(\Omega^\delta)$. We take $k_n \to \infty$ in (2.53) to derive that

$$1 \leqslant C \int_{\Omega^\delta} \left| \nabla^\perp g \cdot \nabla\nabla^\perp g \cdot \nabla^\perp \Phi_0^\delta \right| dx.$$

By using the same estimates (2.49)-(2.50)-(2.51) we arrive at

$$1 \leqslant C\|\Delta g\|_{L^2(\Omega^\delta)}^2 \|\Phi_0\|_{H^2},$$

where the right-hand side tends to 0 as $\delta \to 0$. This is a contradiction. Hence there exists a constant C independent of n such that

$$\|\varphi_n^\delta\|_{\dot{H}^2(\Omega)} \leqslant C.$$

Then there exists a subsequence (still denote by (φ_n^δ)) converging weakly to a limit φ^δ in $\dot{H}^2(\Omega)$, with $\mathrm{tr}\,|_{\partial\Omega}(\varphi^\delta) = 0$. Let

$$\Phi = \Phi_0^\delta + \varphi^\delta,$$

then $\Phi_n = \Phi_0^\delta + \varphi_n^\delta \rightharpoonup \Phi$ in $\dot{H}^2(\Omega)$. By Poincaré's inequality and a Cantor diagonal argument, there exists a subsequence (still denoted by (Φ_n)) such that

$$\Phi_n \to \Phi \text{ a.e. in } \Omega \text{ and } \rho_n \overset{*}{\rightharpoonup} \rho = \eta(\Phi),\ \mu_n \overset{*}{\rightharpoonup} b(\rho) = \mu \text{ in } L^\infty(\Omega) \text{ as } n \to \infty.$$

We are going to show that Φ is a weak solution of the equation (2.36) on the exterior domain Ω. Fix any test function $\Psi \in C_c^\infty(\Omega)$. Then there exists a ball containing $\Omega^C \cup \mathrm{Supp}(\Psi)$ and without loss of generality we suppose it to be $B_1(0)$. Let $V = B_1(0) \cap \Omega$, then, up to a subsequence,

$$\varphi_n^\delta \to \varphi^\delta \text{ in } H^2(V).$$

Indeed, we take a smooth cutoff function χ with $\chi = 1$ on $B_1(0)$ and $\chi = 0$ outside $B_2(0)$. We take the difference between the equation (2.38) for φ_n^δ and the equation (2.38) for φ_m^δ and then take $\psi = \chi\varphi_{n,m}^\delta$, $\varphi_{n,m}^\delta = \varphi_n^\delta - \varphi_m^\delta$. We arrive at the following inequality similar as (2.52)

$$\begin{aligned}
&\int_{B_2(0)\cap\Omega} \mu_n\Big((\partial_{22}-\partial_{11})\varphi_{n,m}^\delta(\partial_{22}-\partial_{11})(\chi\varphi_{n,m}^\delta) + 2\partial_{12}(\varphi_{n,m}^\delta)2\partial_{12}(\chi\varphi_{n,m}^\delta)\Big)\\
&\leqslant \Big|\int_{B_2(0)\cap\Omega} (\mu_n-\mu_m)\Big((\partial_{22}-\partial_{11})\varphi_m^\delta(\partial_{22}-\partial_{11})(\chi\varphi_{n,m}^\delta) + 2\partial_{12}(\varphi_m^\delta)2\partial_{12}(\chi\varphi_{n,m}^\delta)\Big)\,dx\Big|\\
&+\Big|\int_{B_2(0)\cap\Omega}\Big(\rho_n\nabla^\perp\Phi_n\otimes\nabla^\perp\Phi_n - \rho_m\nabla^\perp\Phi_m\otimes\nabla^\perp\Phi_m\Big) : \nabla\nabla^\perp(\chi\varphi_{n,m}^\delta)\,dx\\
&-\int_{B_2(0)\cap\Omega}(\mu_n-\mu_m)\Big((\partial_{22}-\partial_{11})\Phi_0^\delta(\partial_{22}-\partial_{11})(\chi\varphi_{n,m}^\delta) + (2\partial_{12}\Phi_0^\delta)2\partial_{12}(\chi\varphi_{n,m}^\delta)\Big)\,dx\Big|.
\end{aligned}$$

The left-hand side above is bigger than

$$\begin{aligned}
&\int_V \mu_n\Big(\big((\partial_{22}-\partial_{11})\varphi_{n,m}^\delta\big)^2 + \big(2\partial_{12}(\varphi_{n,m}^\delta)\big)^2\Big)\\
&-\Big|\int_{B_2(0)\setminus B_1(0)}\mu_n\Big((\partial_{22}-\partial_{11})\varphi_{n,m}^\delta\big((\partial_{22}-\partial_{11})\chi\varphi_{n,m}^\delta\\
&\quad + 2(\partial_2\chi\partial_2\varphi_{n,m}^\delta - \partial_1\chi\partial_1\varphi_{n,m}^\delta)\big)\\
&\quad + 2\partial_{12}\varphi_{n,m}^\delta(2\partial_{12}\chi\varphi_{n,m}^\delta + 2\partial_1\chi\partial_2\varphi_{n,m}^\delta + 2\partial_2\chi\partial_1\varphi_{n,m}^\delta)\Big)\Big|.
\end{aligned}$$

As up to a subsequence we may assume

$$\begin{aligned}
&\varphi_{n,m}^\delta \to 0 \text{ in } H^1(B_2(0)\cap\Omega),\\
&\Phi_n - \Phi_m \to 0 \text{ in } W^{1,4}(B_2(0)\cap\Omega) \text{ as } n,m\to\infty,
\end{aligned}$$

we have $\varphi_{n,m}^\delta \to 0$ in $H^2(V)$. Therefore $\Phi_n \to \Phi$ in $H^2(V)$, and the limit Φ (together with the limits ρ,μ) satisfies the integral equality (2.1.2). As $\Psi \in C_c^\infty(\Omega)$ has been chosen arbitrarily, Φ is a weak solution of equation (2.36) on Ω.

2.2.3 The whole plane case

We follow the idea in [GW18] to prove (ii) in Theorem 2.1.2. We will denote $⨍_D = \frac{1}{\text{meas}(D)}\int_D$. We take a bounded simply connected $C^{1,1}$ domain $U \supset D$ and we make an Ansatz

$$u = d + w - \bar{w},$$

where $w \in H_0^1(U)$, $\operatorname{div} w = 0$ and $\bar{w} = ⨍_D w$. In other words, if γ is the stream function of w, then

$$u = \nabla^\perp\tilde{\gamma},$$

where we take

$$\tilde{\gamma} = \gamma + (d - \bar{w}) \cdot (x_2 \ \ - x_1)^T + C, \quad \bar{w} = \fint_D \nabla^\perp \gamma \in \mathbb{R}^2$$

with any fixed constant $C \in \mathbb{R}$. We can typically choose

$$C = C_\gamma := - \fint_D \gamma \, dx - \fint_D (d - \overline{w}) \cdot \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} dx,$$

such that $\fint_D \tilde{\gamma} = 0$. We then search for $\gamma \in H_0^2(U)$ satisfying

$$\begin{aligned} &\int_U \mu \Big((\partial_{22}\gamma - \partial_{11}\gamma)(\partial_{22}\psi - \partial_{11}\psi) + (2\partial_{12}\gamma)(2\partial_{12}\psi) \Big) \, dx \\ &= \int_U \rho(\nabla^\perp \tilde{\gamma} \otimes \nabla^\perp \tilde{\gamma}) : \nabla\nabla^\perp \psi \, dx - \int_U F \cdot \nabla\nabla^\perp \psi \, dx, \quad \forall \psi \in H_0^2(U; \mathbb{R}), \end{aligned}$$

where $\rho = \eta(\tilde{\gamma})$ and $\mu = b(\rho)$. Such γ exists by Subsection 2.2.1, and hence there exists $w \in H_0^1(U)$ satisfying

$$\frac{1}{2} \int_U \mu S w : S v \, dx = \int_U \rho(w + d - \bar{w}) \otimes (w + d - \bar{w}) : \nabla v \, dx - \int_U F \cdot \nabla v \, dx \quad (2.54)$$

for any $v \in H_0^1(U; \mathbb{R}^2)$ with $\operatorname{div} v = 0$. By taking $v = w$ in (2.54), we obtain

$$\|w\|_{\dot{H}^1(U)} \leqslant C(\mu_*) \|F\|_{L^2(\mathbb{R}^2)}.$$

And we arrive at a weak solution $u = d + w - \bar{w}$ of the system (2.1) on the set U.

As in Subsection 2.2.2, we take the approximation argument to show the existence of the solution on the whole plane $\mathbb{R}^2$. Indeed, if we take $U = B_n(0)$ in the above, then we have arrived at a weak solution of (2.1) in $B_n(0)$:

$$\begin{aligned} &u_n = d + w_n - \overline{w_n} \in \dot{H}^1(B_n(0)), \\ &\text{with } w_n \in H_0^1(B_n(0); \mathbb{R}^2) \text{ and } \fint_D u_n = d. \end{aligned}$$

We extend w_n trivially to $\mathbb{R}^2$ (that is, we simply take $w_n = 0$ outside $B_n(0)$) and take $u_n = d - \overline{w_n}$ outside $B_n(0)$. Let

$$\tau_n = w_n - \overline{w_n} \text{ with } \fint_D \tau_n = 0, \text{ such that } u_n = d + \tau_n,$$

then

$$\|\tau_n\|_{\dot{H}^1(\mathbb{R}^2)} = \|w_n\|_{\dot{H}^1(\mathbb{R}^2)} \leqslant C(\mu_*) \|F\|_{L^2(\mathbb{R}^2)}.$$

Let $v \in C_c^\infty(\mathbb{R}^2;\mathbb{R}^2)$ with $\operatorname{div} v = 0$ be any test function, then there exists $N \in \mathbb{N}$ such that $\operatorname{Supp}(v) \cup D \subset B_N(0)$. By the above uniform bound on (τ_n), there exists a subsequence (still denoted by (τ_n)) such that $\tau_n \rightharpoonup \tau$ in $\dot{H}^1(\mathbb{R}^2)$ as $n \to \infty$, and in $H^1(B_N(0))$ by Poincaré's inequality. Thus $u_n \rightharpoonup u$ in $H^1(B_N(0))$. Since $⨍_D \tilde{\gamma}_n = 0$, by Poincaré inequality again, $\{\tilde{\gamma}_n\}$ is uniformly bounded in $H^2(B_N(0))$, and up to a subsequence $\tilde{\gamma}_n \rightharpoonup \tilde{\gamma}$ in $H^2(B_N(0))$, with $\nabla^\perp \tilde{\gamma} = u$ and $⨍_D \tilde{\gamma} = 0$. Thus $\tilde{\gamma}_n \to \tilde{\gamma}$ in $W^{1,4}(B_N(0)) \subset C^{1/2}(B_N(0))$, and $\rho_n = \eta(\tilde{\gamma}_n) \overset{*}{\rightharpoonup} \rho = \eta(\tilde{\gamma})$, $\mu_n = b(\rho_n) \overset{*}{\rightharpoonup} \mu = b(\rho)$ in $L^\infty(B_N(0))$. Exactly as the end of Subsection 2.2.2, $u_n \to u$ in $H^1(B_N(0))$. Thus the limits u, ρ, μ satisfy the integral equality (2.54) for given test function v, and hence u is a weak solution of equation (2.1) on $\mathbb{R}^2$.

2.2.4 More regularity results

In this subsection we prove the regularity results in Theorem 2.1.2 in the cases when η is continuous and when $\eta \in C_b^k$, $k \geqslant 2$, respectively.

Case when η is continuous
If Ω is a connected $C^{2,1}$-domain, $\Phi_0 \in H^{\frac{5}{2}}(\partial\Omega)$ and $\Phi_1 \in H^{\frac{3}{2}}(\partial\Omega)$, then we can extend the function Φ_0 to the whole plane (still denoted by Φ_0) such that $\Phi_0 \in H^3(\mathbb{R}^2)$ with compact support and $\frac{\partial\Phi_0}{\partial n}|_{\partial\Omega} = \Phi_1$. Since the weak solution obtained in Subsection 2.2.1 $\Phi \in H^2(\Omega) \subset C^\alpha(\Omega)$, $\forall \alpha \in (0,1)$, then

$$\rho = \eta(\Phi) \text{ and } \mu = b(\rho) \in C_b(\Omega),$$

if η is continuous. Since $f \in L^2(\Omega;\mathbb{R}^2)$ and $H^1(\Omega) \hookrightarrow L^p(\Omega)$, $\forall p \in [2,\infty)$, we can rewrite the elliptic equation (2.36) as the fourth-order elliptic equation for $\varphi = \Phi - \Phi_0 \in H_0^2(\Omega)$:

$$L_\mu \varphi = -\nabla^\perp \cdot f + \nabla^\perp \cdot \operatorname{div}(\rho \nabla^\perp \Phi \otimes \nabla^\perp \Phi) - L_\mu(\Phi_0).$$

By the L^p estimate for the above fourth-order elliptic equation in Theorem 8.6 in [DK11] again, we have $\varphi \in W_0^{2,p}(\Omega)$ and hence $\Phi = \Phi_0 + \varphi \in W^{2,p}(\Omega)$ for all finite p.

Case when $\eta \in C_b^k$, $k \geqslant 2$
If Ω is a connected $C^{k+1,1}$ domain and we assume the boundary condition $\Phi_0 \in H^{k+\frac{3}{2}}(\partial\Omega)$, $\Phi_1 \in H^{k+\frac{1}{2}}(\partial\Omega)$, then the above extended function $\Phi_0 \in H^{k+2}(\mathbb{R}^2) \subset W^{k+1,p}(\mathbb{R}^2)$, $\forall p \geqslant 2$ with compact support and $\frac{\partial\Phi_0}{\partial n}|_{\partial\Omega} = \Phi_1$. We assume also smoothness in the data $\eta, b \in C_b^k$ and $f \in H^{k-1}(\Omega)$ for $k \geqslant 2$.

As $\Phi \in W^{2,p}(\Omega)$ is proved in the case when η is continuous, we first prove that $\Phi \in W^{3,p}(\Omega)$ under the assumptions $\Phi_0 \in H^4(\mathbb{R}^2) \hookrightarrow W^{3,p}(\Omega)$,

$f \in H^1(\mathbb{R}^2) \hookrightarrow L^p(\Omega)$ and $\eta, b \in C_b^2$. We rewrite the elliptic equation (2.36) as follows:

$$\begin{aligned}\Delta^2\varphi = \mu^{-1}\Big(&-\big((\partial_{22}-\partial_{11})\mu\big)\big((\partial_{22}-\partial_{11})\varphi\big) - \big((2\partial_{12})\mu\big)\big((2\partial_{12})\varphi\big)\\ &-2(\partial_2\mu)\big((\partial_{222}-\partial_{112})\varphi\big) + 2(\partial_1\mu)\big((\partial_{122}-\partial_{111})\varphi\big)\\ &-2(\partial_1\mu)(2\partial_{122})\varphi - 2(\partial_2\mu)(2\partial_{112})\varphi - L_\mu(\Phi_0)\\ &-\nabla^\perp\cdot f + \nabla^\perp\cdot \mathrm{div}\,(\rho\nabla^\perp\Phi\otimes\nabla^\perp\Phi)\Big),\end{aligned} \tag{2.55}$$

where $\varphi = \Phi - \Phi_0 \in H_0^2(\Omega)\cap W^{2,p}(\Omega)$. Notice that $\rho = \eta(\Phi)$ and $\mu = (b\circ\eta)(\Phi)$ belong to $W^{2,p}(\Omega)$ for any $p \in (2,\infty)$. Then for any fixed $\psi \in W_0^{1,q}(\Omega)$, $1 < q < 2$ we have

$$\begin{aligned}&\nabla^2\mu\,\psi \in L^q(\Omega), \quad \nabla\mu\,\psi \in W_0^{1,q}(\Omega),\\ &\rho\nabla^2\psi \in W^{-1,q}(\Omega), \quad \mu^{-1}\psi \in W_0^{1,q}(\Omega),\end{aligned}$$

and hence the righthand side of (2.55) is in $W^{-1,q'}(\Omega)$, the dual space of $W_0^{1,q}(\Omega)$. Therefore by (2.55), $\varphi \in W^{3,p}(\Omega)$ for all $p \in (2,\infty)$ and the same holds for $\Phi = \Phi_0 + \varphi$.

We assume inductively $\eta, b \in C_b^k$ and $\Phi \in W^{k,p}(\Omega)$, for $k \geqslant 3$, $\forall p \in (2,\infty)$, then $\rho = \eta(\Phi)$, $\mu = b(\rho)$ and φ belong to $W^{k,p}(\Omega)$ for any $p \in (2,\infty)$. Thus the righthand side of (2.55) belongs to $W^{k-3,p}(\Omega)$, and hence $\varphi \in W^{k+1,p}(\Omega)$, which implies $\Phi = \Phi_0 + \varphi \in W^{k+1,p}(\Omega)$.

2.3 Three-dimensional axially symmetric case

The three-dimensional stationary inhomogeneous incompressible Navier–Stokes equations read as

$$\begin{cases}\mathrm{div}(\rho u\otimes u) - \mathrm{div}(\mu S u) + \nabla\Pi = f, \quad x \in \Omega \subset \mathbb{R}^3,\\ \mathrm{div}\,u = 0,\ \mathrm{div}(\rho u) = 0.\end{cases} \tag{2.56}$$

The velocity field $u : \Omega \to \mathbb{R}^3$, the density function $\rho : \Omega \to \mathbb{R}_+$ and the pressure $\Pi : \Omega \to \mathbb{R}$ are unknown. The external force $f : \Omega \to \mathbb{R}^3$ is given. We write $\nabla u = (\partial_j u_i)_{1\leqslant i,j\leqslant 3}$, $Su = \nabla u + (\nabla u)^T$ and $\frac{1}{2}Su$ is the symmetric part of ∇u. We denote $v\otimes w = (v_i w_j)_{1\leqslant i,j\leqslant 3}$ for vectors $v = (v_1, v_2, v_3)^T$ and $w = (w_1, w_2, w_3)^T$.

The viscosity coefficient μ depends smoothly on the density function ρ with the form

$$\mu = b(\rho), \quad b \in C(\mathbb{R}_+; [\mu_*, \mu^*]) \quad \text{given},$$

where $\mu_*, \mu^* > 0$ are the positive lower and upper bounds.

On the bounded domain Ω, we consider the boundary value problem of (2.56) under the boundary condition

$$u|_{\partial\Omega} = u_0 \tag{2.57}$$

satisfying the zero flux condition

$$\int_{\partial\Omega} u_0 \cdot \vec{n}\, ds = 0. \tag{2.58}$$

Leray [Ler33] showed the solvability of the classical stationary incompressible Navier–Stokes equation

$$\begin{cases} \operatorname{div}(u \otimes u) - \nu\Delta u + \nabla\Pi = f, \\ \operatorname{div} u = 0. \end{cases}$$

on some bounded, exterior domains or $\mathbb{R}^3$. There are some work devoted to considering the asymptotic behavior of Leray's solutions, see for example [Fin59; Ami91]. We mention a celebrated book on the stationary fluid flows by Galdi [Gal11].

However, to our knowledge, there are not so many works on the stationary inhomogeneous Navier–Stokes equations (2.56). For the equation (2.56) with *constant* viscosity coefficient

$$\begin{cases} \operatorname{div}(\rho u \otimes u) - \nu\Delta u + \nabla\Pi = f, \\ \operatorname{div} u = 0,\ \operatorname{div}(\rho u) = 0, \end{cases}$$

Frolov [Fro93] showed the existence of the weak solutions with the form

$$(\rho, u) = (\eta(\Phi), \nabla^\perp\Phi), \quad \nabla^\perp = \begin{pmatrix} \partial_{x_2} \\ -\partial_{x_1} \end{pmatrix},$$

where Φ is the stream function of u and η is any given Hölder continuous function. Under this assumption, the density equation holds immediately

$$\operatorname{div}(\rho u) = \nabla\eta(\Phi) \cdot \nabla^\perp\Phi = 0.$$

Later on Santos [San02] generalised this existence result to the only bounded function η. Concerning the *density-dependent* viscosity coefficient, the author and Liao [HL20] showed the existence and regularity results of the equation (2.56). We mention a celebrated book on the *evolutionary* incompressible inhomogeneous Navier–Stokes equations by Lions [Lio96]. To our best knowledge, there is no existence results on the three-dimensional *stationary* inhomogeneous incompressible Navier–Stokes equation (2.56).

Organization of this section

Subsection 2.3.1 is devoted to stating our main existence result Theorem 2.3.1 for the equation (2.56) in the axially symmetric case. Subsection 2.3.2 is devoted to showing more symmetric solutions in cylindrical, spherical and cartesian coordinates. We sketch the proof of Theorem 2.3.1 in Subsection 2.3.3.

2.3.1 Main result

We consider the cylindrical coordinate $(r, z, \theta) \in [0, \infty) \times \mathbb{R} \times [0, 2\pi)$ and write e_r, e_z, e_θ as the coordinate axis with

$$e_r = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \quad e_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad e_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}. \tag{2.59}$$

We consider the system (2.56) on the axially symmetric simply connected domain

$$\Omega = [0, r_1) \times (z_1, z_2) \times [0, 2\pi), \tag{2.60}$$

where $0 < r_1 < +\infty$ and $-\infty < z_1 < z_2 < +\infty$. The velocity field

$$u = u_r e_r + u_\theta e_\theta + u_z e_z$$

is called *axially symmetric*, if u_r, u_θ, u_z are independent of θ. We define the functional spaces for axially symmetric functions

$$H^1_\sigma(\Omega) = \{v \in H^1(\Omega) \mid v \text{ is axially symmetric}, \operatorname{div} v = 0\}.$$

The incompressibility condition of $u \in H^1_\sigma(\Omega)$ reads

$$\operatorname{div} u = \frac{1}{r}\partial_r(ru_r) + \frac{1}{r}\partial_z(ru_z) = 0, \quad r \neq 0. \tag{2.61}$$

If u also satisfies the non-flux assumption then there exists an axially symmetric stream function $\varphi = \varphi(r, z)$ such that

$$ru_r = \partial_z\varphi, \quad ru_z = -\partial_r\varphi.$$

We take any fixed scalar function $\eta \in L^\infty(\mathbb{R}; [0, \infty))$. If we take the density function as $\rho = \eta(\varphi)$, then the mass conservation law

$$\operatorname{div}(\rho u) = \frac{1}{r}\partial_r\rho\partial_z\varphi - \frac{1}{r}\partial_z\rho\partial_r\varphi = 0, \quad r \neq 0$$

holds in the distribution sense.

Our main result states as following and we will sketch the proof in Subection 2.3.3.

Theorem 2.3.1. *Let $\eta \in L^\infty(\mathbb{R}; [0, \infty))$ and $b \in C(\mathbb{R}; [\mu_*, \mu^*])$, $\mu_*, \mu^* > 0$ be given. Let Ω be a bounded connected axially symmetric domain defined as in* (2.60). *Let $u_0 \in H_\sigma^{1/2}(\partial\Omega) = \{tr(u) \mid u \in H_\sigma^1(\mathbb{R}^3)\}$ and $f \in H^{-1}(\Omega; \mathbb{R}^3)$ be axially symmetric functions satisfying* (2.58). *Then there exists at least one axially symmetric weak solution*

$$(\rho, u) = (\eta(\varphi), \frac{1}{r}\partial_z \varphi e_r - \frac{1}{r}\partial_r \varphi e_z + u_\theta e_\theta) \in L^\infty(\Omega) \times H_\sigma^1(\Omega)$$

of the boundary value problem (2.56)-(2.57), *where $\varphi \in H^2(\Omega)$ is a stream function of u, in the sense that $div(\rho u) = 0$ holds in Ω, $u_0 = u|_{\partial\Omega}$ is the trace of u on $\partial\Omega$, and the integral identity*

$$\frac{1}{2}\int_\Omega \mu S u : S v \, dx = \int_\Omega (\rho u \otimes u) : \nabla v \, dx + \int_\Omega f \cdot v \, dx$$

holds for all $v \in H_\sigma^1(\Omega) \cap H_0^1(\Omega)$. Here $A : B \stackrel{def}{=} \sum_{i,j=1}^3 A_{ij} B_{ij}$ for the matrices $A = (A_{ij})_{1 \leqslant i,j \leqslant 3}$ and $B = (B_{ij})_{1 \leqslant i,j \leqslant 3}$.

Remark 2.3.1.
- *The domain* (2.60) *can be relaxed to any $C^{1,1}$-symmetric domains with respect to the coordinate axis. The $C^{1,1}$-regularity is necessary to extend $u_0 \in H^{1/2}(\partial\Omega)$ to a H^1-regularity function on $\mathbb{R}^3$.*
- *One can generalise the above solvability theorem to axially symmetric multi-connected domain $\partial\Omega = \cup_{i=1}^k \Gamma_i$ and there is no flux through each component*

$$\int_{\Gamma_i} u_0 \cdot \vec{n} \, ds = 0, \quad i = 1, \dots, k. \tag{2.62}$$

- *Following the Leray's approximation method in [Ler33], one can generalise the existence result to the exterior domains and whole space $\mathbb{R}^3$.*

2.3.2 Other symmetric solutions

In this section, we will show the existence of symmetric solutions of (2.56)-(2.57) in cylindrical coordinate, spherical coordinate and Cartesian coordinate. The key point is to choose the structure of ρ carefully such that the density equation

$$\operatorname{div}(\rho u) = 0 \tag{2.63}$$

holds automatically. We will also formulate explicit examples in the cartesian coordinate.

More precisely, we consider the following two types of symmetries

- *Symmetry type I:* We assume that u is a three-dimensional vector-valued function depending only on a two-dimensional variable, and hence there exits a stream function φ of u. We take any bounded positive function η and the density function of the form
$$\rho = \eta(\varphi).$$
This is the case in Theorem 2.3.1.

- *Symmetry type II:* We assume that u is a two-dimensional vector-valued function and vanishing in the spatial direction e_w with respect to the variable x_w. The density function ρ has the form
$$\rho = \eta(x_w).$$

Then the mass conservation law (2.63) holds immediately for the above symmetric types.

We recall the coordinate axis e_r, e_z, e_θ as in (2.59) in the cylindrical coordinate. We denote the unit standard vectors in the cartesian coordinate $(x_1, x_2, x_3) \in \mathbb{R}^3$ by

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and in the spherical coordinate $(\tilde{r}, \alpha, \theta) \in [0, \infty) \times [0, \pi] \times [0, 2\pi)$ by

$$e_{\tilde{r}} = \begin{pmatrix} \sin\alpha\cos\theta \\ \sin\alpha\sin\theta \\ \cos\alpha \end{pmatrix}, \quad e_\alpha = \begin{pmatrix} \cos\alpha\cos\theta \\ \cos\alpha\sin\theta \\ -\sin\alpha \end{pmatrix}, \quad e_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}.$$

Then we have the following existence theorem with respect to Symmetric type I and II in cylindrical coordinate, spherical coordinate and cartesian coordinate.

Theorem 2.3.2. *Let $\eta \in L^\infty(\mathbb{R}; [0, \infty))$ and $b \in C(\mathbb{R}; [\mu_*, \mu^*])$, $\mu_*, \mu^* > 0$ be given. In the cylindrical, cartesian, and spherical coordinates, let Ω be a bounded connected symmetric $C^{1,1}$-domain with respect to coordinate axis. Let $u_0 \in H^{1/2}(\partial\Omega)$ satisfy the no-flux assumption (2.62) and let $f \in H^{-1}(\Omega; \mathbb{R}^3)$.*

- *Symmetry type I: If u_0 and f depend only on r, z, or x_1, x_2, or $\tilde{r}, \alpha$. Then there exists at least one solution in the weak sense as in Theorem 2.3.1*
$$(\rho, u) \in L^\infty(\Omega) \times H^1(\Omega)$$

with the form

$$
\begin{aligned}
& (\rho, u) = (\eta(\varphi), \frac{1}{r}\partial_z\varphi e_r - \frac{1}{r}\partial_r\varphi e_z + u_\theta e_\theta) && (2.64)\\
& \varphi,\ u_\theta \text{ depending only on } r,\ z, \\
\text{or} \quad & (\rho, u) = (\eta(\varphi), \partial_2\varphi e_1 - \partial_1\varphi e_2 + u_3 e_3) && (2.65)\\
& \varphi,\ u_3 \text{ depending only on } x_1,\ x_2, \\
\text{or} \quad & (\rho, u) = (\eta(\varphi), \frac{1}{\tilde r^2 \sin\alpha}\partial_\alpha\varphi e_{\tilde r} - \frac{1}{\tilde r \sin\alpha}\partial_{\tilde r}\varphi e_\alpha + u_\theta e_\theta) \\
& \varphi,\ u_\theta \text{ depending only on } \tilde r,\ \alpha && (2.66)
\end{aligned}
$$

of the boundary value problem (2.56)-(2.57).

- *Symmetry type II: If u_0 and f have the form*

$$
\begin{aligned}
& u_0 = u_{0,r}e_r + u_{0,z}e_z, \quad f = f_r e_r + f_z e_z, \\
\text{or} \quad & u_0 = u_{0,1}e_1 + u_{0,2}e_2, \quad f = f_1 e_1 + f_2 e_2, \\
\text{or} \quad & u_0 = u_{0,\tilde r}e_{\tilde r} + u_{0,\alpha}e_\alpha, \quad f = f_{\tilde r} e_{\tilde r} + f_\alpha e_\alpha.
\end{aligned}
$$

Then there exists at least one weak solution

$$(\rho, u) \in L^\infty(\Omega) \times H^1(\Omega)$$

with the form

$$
\begin{aligned}
& (\rho, u) = (\eta(\theta), u_r e_r + u_z e_z), \\
\text{or} \quad & (\rho, u) = (\eta(x_3), u_1 e_1 + u_2 e_2), \\
\text{or} \quad & (\rho, u) = (\eta(\theta), u_{\tilde r} e_{\tilde r} + u_\alpha e_\alpha)
\end{aligned}
$$

of the boundary value problem (2.56)-(2.57).

Proof. For the symmetric solutions of Symmetric type I, the cylindrical case was shown in Theorem 2.3.1, analogously we can show the Cartesian and spherical cases. The proof of Symmetry type II is similar to type I, see the solvability in Remark 2.3.3. We omit the detailed proof here. □

Remark 2.3.2. *The stream functions* (2.64) *and* (2.66) *are called the Stokes stream functions. The above existence results hold also for solutions of the Symmetric type I and II with respect to other axis. We write down the rest of solutions of Symmetric type I and II in the cylindrical and spherical* *coordinates*

$$\begin{aligned}
&(\rho, u) = (\eta(\varphi), u_r e_r - \frac{1}{r}\partial_\theta \varphi e_z + \partial_z \varphi e_\theta)\\
&\varphi,\ u_r \text{ depending only on } z,\ \theta,\\
\textit{or}\quad &(\rho, u) = (\eta(\varphi), \frac{1}{r}\partial_\theta \varphi e_r + u_z e_z - \partial_r \varphi e_\theta)\\
&\varphi,\ u_z \text{ depending only on } r,\ \theta,
\end{aligned}$$

and

$$\begin{aligned}
&(\rho, u) = (\eta(\varphi), u_{\tilde r} e_{\tilde r} + \frac{1}{\sin\alpha}\partial_\theta \varphi e_\alpha - \partial_\alpha \varphi e_\theta)\\
&\varphi,\ u_{\tilde r} \text{ depending only on } \alpha,\ \theta,\\
\textit{or}\quad &(\rho, u) = (\eta(\varphi), \frac{1}{\tilde r^2 \sin\alpha}\partial_\theta \varphi e_{\tilde r} + u_\alpha e_\alpha - \frac{1}{\tilde r}\partial_{\tilde r} \varphi e_\theta)\\
&\varphi,\ u_\alpha \text{ depending only on } \tilde r,\ \theta.
\end{aligned}$$

In the following, we verify the mass conservation law for the solutions in Theorem 2.3.2. In cylindrical, cartesian, and spherical coordinates, the gradient operators can be written as

$$\nabla = e_r \partial_r + e_z \partial_z + \frac{e_\theta}{r}\partial_\theta,$$

$$\nabla = e_1 \partial_1 + e_2 \partial_2 + e_3 \partial_3,$$

and

$$\nabla = e_{\tilde r}\,\partial_{\tilde r} + \frac{e_\alpha}{\tilde r}\,\partial_\alpha + \frac{e_\theta}{\tilde r \sin\alpha}\partial_\theta.$$

For the solutions of Symmetric type II, the mass conservation laws hold immediately since

$$\operatorname{div}(\rho u) = \nabla\rho \cdot u.$$

Concerning the solutions of Symmetric type I, the case of Cylindrical coordinate was shown in Section 2.3.1. In Cartesian and spherical coordinates we consider

$$u = u_1 e_1 + u_2 e_2 + u_3 e_3, \quad u = u_{\tilde r} e_{\tilde r} + u_\alpha e_\alpha + u_\theta e_\theta,$$

where u_1, u_2, u_3 depend only on x_1, x_2 and $u_{\tilde r}, u_\alpha, u_\theta$ depend only on $\tilde r, \alpha$. Then the incompressibility conditions can be written as

$$\operatorname{div} u = \partial_1 u_1 + \partial_2 u_2 = 0, \quad \operatorname{div} u = \frac{1}{\tilde r^2}\partial_{\tilde r}(\tilde r^2 u_{\tilde r}) + \frac{1}{\tilde r \sin\alpha}\partial_\alpha(\sin\alpha u_\alpha) = 0. \tag{2.67}$$

For u satisfies (2.67) and the zero-flux assumption (2.57), there exists stream functions $\varphi = \varphi(x_1, x_2)$ and $\varphi = \varphi(\tilde{r}, \alpha)$ such that

$$u_1 = \partial_2\varphi, \quad u_2 = -\partial_1\varphi,$$

and

$$u_{\tilde{r}} = \frac{1}{\tilde{r}^2 \sin\alpha}\partial_\alpha\varphi, \quad u_\alpha = -\frac{1}{\tilde{r}\sin\alpha}\partial_{\tilde{r}}\varphi.$$

Then the pair

$$(\rho, u) = (\eta(\varphi), \partial_2\varphi e_1 - \partial_1\varphi e_2),$$

and

$$(\rho, u) = (\eta(\varphi), \frac{1}{\tilde{r}^2 \sin\alpha}\partial_\alpha\varphi e_{\tilde{r}} - \frac{1}{\tilde{r}\sin\alpha}\partial_{\tilde{r}}\varphi e_\alpha + u_\theta e_\theta)$$

satisfies the mass conservation laws, since

$$\operatorname{div}(\rho u) = \partial_1\rho\partial_2\varphi - \partial_2\rho\partial_1\varphi = 0,$$

and

$$\operatorname{div}(\rho u) = \partial_{\tilde{r}}\rho\frac{1}{\tilde{r}^2 \sin\alpha}\partial_\alpha\varphi - \frac{1}{\tilde{r}}\partial_\alpha\rho\frac{1}{\tilde{r}\sin\alpha}\partial_{\tilde{r}}\varphi = 0.$$

In the following, we formulate explicit solutions of (2.56) of the Symmetry type I and II in the Cartesian coordinate. The similar explicit solutions for the two-dimensional stationary Navier–Stokes equation (2.56) were given in [HL20].

- *Symmetry type I:* We consider ρ and u depending only on x_1, x_2, then the system (2.56) with $f = 0$ reads as

$$\begin{pmatrix} u_1\partial_1 u_1 + u_2\partial_2 u_1 \\ u_1\partial_1 u_2 + u_2\partial_2 u_2 \\ u_1\partial_1 u_3 + u_2\partial_2 u_3 \end{pmatrix} - \begin{pmatrix} 2\partial_1(\mu\partial_1 u_1) + \partial_2(\mu(\partial_1 u_2 + \partial_2 u_1)) \\ \partial_1(\mu(\partial_1 u_2 + \partial_2 u_1)) + 2\partial_2(\mu\partial_2 u_2) \\ \partial_1(\mu\partial_1 u_3) + \partial_2(\mu\partial_2 u_3) \end{pmatrix} + \begin{pmatrix} \partial_1\Pi \\ \partial_2\Pi \\ \partial_3\Pi \end{pmatrix} = 0.$$

  We consider $\Pi = \Pi(x_1, x_2)$, then $(\rho, u_1e_1 + u_2e_2)$ satisfies the two-dimensional stationary Navier–Stokes equation (2.56). We base on the radial solutions of the two-dimensional equation (2.56) given in [HL20] to derive a solution satisfy

$$\begin{aligned} &\rho = \rho(r), \quad (r, \theta) = (\sqrt{x_1^2 + x_2^2}, \arctan(x_2/x_1)), \\ &u_1 = (rg\sin\theta), \quad u_2 = -(rg\sin\theta), \quad \partial_r(\mu\partial_r u_3) = 0, \end{aligned}$$

where g satisfies the ODE

$$\partial_r(\mu r^3 \partial_r g) = -Cr, \quad C \in \mathbb{R}.$$

The corresponding stream function Φ satisfies

$$\partial_{rr}\Big(\mu r^3 \partial_r(\frac{1}{r}\partial_r \Phi)\Big) = -C.$$

- *Symmetry type II:* We assume tat

$$(\rho, u) = (\rho(x_3), u_1(x_3)e_1 + u_2(x_3)e_2),$$

then the equation (2.56) with $f = 0$ reads as

$$\begin{pmatrix} \partial_3(\mu\partial_3 u_1) \\ \partial_3(\mu\partial_3 u_2) \\ 0 \end{pmatrix} = \begin{pmatrix} \partial_1\Pi \\ \partial_2\Pi \\ \partial_3\Pi \end{pmatrix}.$$

The equation $\partial_3\Pi = 0$ implies that Π is independent of x_3. Then the following there exit constants C_1, $C_2 \in \mathbb{R}$ such that

$$\partial_1\Pi = \partial_3(\mu\partial_3 u_1) = C_1, \quad \partial_2\Pi = \partial_3(\mu\partial_3 u_2) = C_2.$$

2.3.3 Proof of existence

We sketch the proof of Theorem 2.3.1, which is based the method in [Ler33]. We define the functional space

$$H(\Omega) = H^1_\sigma(\Omega) \cap H^1_0(\Omega).$$

For any $u \in H(\Omega)$, as a consequence of (2.61)-(2.58), there exists a stream function $\varphi \in H^2_0(\Omega)$ such that

$$u = \frac{1}{r}\partial_z\varphi e_r - \frac{1}{r}\partial_r\varphi e_z + u_\theta e_\theta.$$

We are going to prove Theorem 2.3.1 in the following three steps.

Boundary condition

We extended the axially symmetric boundary value u_0 to $\mathbb{R}^3$ (still denote by $u_0 \in H^1_\sigma(\mathbb{R}^3)$). Then there exists a axially symmetric stream function $\varphi_0 \in H^2(\mathbb{R}^3)$ such that

$$u_0 := \frac{1}{r}\partial_z\varphi_0 e_r - \frac{1}{r}\partial_r\varphi_0 e_z + u_{0,\theta}e_\theta, \quad r > 0.$$

Recall the axially symmetric domain $\Omega = [0, r_1) \times (z_1, z_2) \times [0, 2\pi)$. We define the smooth cut-off function $\zeta(r, z; \delta)$ such that $\zeta(r, z; \delta) = 1$ if $\min\{|r - r_1|, |z - z_i|\} \leqslant \frac{\delta}{2}$, $i = 1, 2$ and $\zeta(r, z; \delta) = 0$ if $\min\{|r - r_1|, |z - z_i|\} \geqslant \delta$, and there exists a constant such that

$$|\zeta(r, z; \delta)| \leqslant C, \quad |\nabla\zeta(r, z; \delta)| \leqslant C\delta^{-1}.$$

We write

$$\varphi_0^\delta(r, z) = \varphi_0(r, z)\zeta(r, z; \delta), \quad u_0^\delta = \frac{1}{r}\partial_z\varphi_0^\delta e_r - \frac{1}{r}\partial_r\varphi_0^\delta e_z + u_{0,\theta}e_\theta.$$

Then for a fixed $\delta > 0$, and we only need to search for the weak solutions

$$u^\delta = u - u_0^\delta \in H(\Omega)$$

satisfying

$$\begin{aligned} \frac{1}{2}\int_\Omega \mu^\delta Su^\delta : Sv\,dx = \int_\Omega \rho^\delta(u_0^\delta + u^\delta) \otimes (u_0^\delta + u^\delta)) : \nabla v\,dx + \int_\Omega f \cdot v\,dx \\ - \frac{1}{2}\int_\Omega \mu^\delta Su_0^\delta : Sv\,dx, \quad \forall v \in H(\Omega), \end{aligned} \tag{2.68}$$

where $\rho^\delta = \eta(\varphi_0^\delta + \varphi^\delta)$, $\mu^\delta = b(\rho^\delta)$, and $\varphi^\delta \in H_0^2(\Omega)$ is a stream function of u^δ.

Linearised system

We fix an axially symmetric $\tilde{u} \in H(\Omega)$ and write its stream function as $\tilde{\varphi} \in H_0^2(\Omega)$. Then we define the density function and viscosity coefficient correspondingly as

$$\tilde{\rho}^\delta = \eta(\tilde{\varphi} + \varphi_0^\delta), \quad \tilde{\mu}^\delta = b(\tilde{\rho}^\delta).$$

In the proof, we will need the Lipschitz regularity of μ^δ. We can regularize the given quantities to be obtain a sequence of approximation solutions and pass to the limit in the last step. For simplicity, we omit these two steps here.

We consider the linearised problem with a parameter $\lambda \in [0, 1]$

$$\begin{aligned} \frac{1}{2}\int_\Omega \tilde{\mu}_\lambda^\delta Su : Sv\,dx = \lambda\int_\Omega \tilde{\rho}_\lambda^\delta(\lambda u_0^\delta + \tilde{u}) \otimes (\lambda u_0^\delta + u) : \nabla v\,dx \\ + \lambda\int_\Omega f \cdot v\,dx - \frac{\lambda}{2}\int_\Omega \tilde{\mu}_\lambda^\delta Su_0^\delta : Sv\,dx, \quad \forall v \in H(\Omega), \end{aligned} \tag{2.69}$$

where $\tilde{\rho}^\delta_\lambda = \eta(\tilde{\varphi} + \lambda\varphi^\delta_0)$ and $\tilde{\mu}^\delta = b(\tilde{\rho}^\delta_\lambda)$. Notice that if $\lambda = 1$ and $u^\delta_1 = \tilde{u}$, then u^δ_1 satisfies the weak formulation (2.68).

The left-hand side of (2.69) defines an inner product $\langle\cdot,\cdot\rangle$ on $H(\Omega)$ through

$$\langle u, v\rangle \overset{\text{def}}{=} \frac{1}{2}\int_\Omega \tilde{\mu}^\delta_\lambda Su : Sv\, dx,$$

moreover,

$$\sqrt{\mu_*/2}\|\nabla u\|_{L^2(\Omega)} \leqslant \langle u, u\rangle^{\frac{1}{2}} \leqslant \sqrt{\mu^*/2}\|\nabla u\|_{L^2(\Omega)},$$
$$\langle\cdot,\cdot\rangle^{\frac{1}{2}} \sim \|\cdot\|_{H^1(\Omega)} \quad \text{on } H(\Omega),$$

where μ_* and μ^* are the positive lower and upper bound of b. The right-hand side of (2.69) defines a bounded linear functional for $v \in H(\Omega)$. By using Leray-Schauder's Principle, there exists a unique weak solution $u^\delta_\lambda \in H(\Omega)$ of the linear problem (2.69).

Nonlinear problem

We define the map

$$T^\delta : [0,1] \times H(\Omega) \ni (\lambda, \tilde{u}) \mapsto u^\delta_\lambda \in H(\Omega).$$

One can show the existence of the fixed point $u^\delta_1 = T(1, u^\delta_1)$ by Leray-Shauder principle. The uniform bound of $\|u^\delta_\lambda\|_{H^1}$ with $u^\delta_\lambda = T(\lambda, u^\delta_\lambda)$ can be shown by a contraction argument as in [Ler33].

Notice that the foxed point $u^\delta = u^\delta_1 \in H(\Omega)$ satisfying (2.68). And the pair $(\rho^\delta, u^\delta_0 + u^\delta) \in L^\infty(\Omega) \times H^1_\delta(\Omega)$ is a weak solution of (2.56).

Remark 2.3.3. *We can follow the above proof to show the sovability under the assumption Symmetry type II. In this case, ρ and μ are fixed and independent of u. As a consequence, we do not need to fix $\tilde{u}$ as in Subsection 2.3.3. The solvability can be obtained by directly applying Leray-Schauder principle.*

Chapter 3

Two-dimensional Boussinesq equations with temperature-dependent thermal and viscosity diffusions in general Sobolev spaces

In this chapter, we study the existence, uniqueness as well as regularity issues for the two-dimensional incompressible Boussinesq equations with temperature-dependent thermal and viscosity diffusion coefficients in general Sobolev spaces. The optimal regularity exponent ranges are considered.

This chapter is based on the joint work with JProf. Xian Liao in [HL22].

3.1 Introduction and main results

In the present chapter we consider the two-dimensional incompressible Boussinesq equations

$$\begin{cases} \partial_t\theta + u\cdot\nabla_x\theta - \mathrm{div}_x(\kappa\nabla_x\theta) = 0, \\ \partial_t u + u\cdot\nabla_x u - \mathrm{div}_x(\mu S_x u) + \nabla_x\Pi = \beta\theta\vec{e_2}, \\ \mathrm{div}_x\, u = 0, \end{cases} \tag{3.1}$$

where $(t,x)\in[0,\infty)\times\mathbb{R}^2$ denote the time and space variables respectively. The unknown temperature function $\theta=\theta(t,x):[0,\infty)\times\mathbb{R}^2\to\mathbb{R}$ satisfies the parabolic-type equation $(3.1)_1$, and the unknown velocity vector field $u=u(t,x):[0,\infty)\times\mathbb{R}^2\to\mathbb{R}^2$ together with the unknown pressure term

$\Pi = \Pi(t,x) : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the incompressible Navier-Stokes type equations $(3.1)_2-(3.1)_3$ respectively. We are going to study the well-posedness and regularity problems for the Boussinesq system (3.1) together with the initial data

$$(\theta, u)\mid_{t=0} = (\theta_0, u_0). \tag{3.2}$$

We write $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ with x_1, x_2 denoting the horizontal and vertical components respectively. Let $u = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}^2$, and let

$$\frac{1}{2} S_x u := \frac{1}{2}(\nabla_x u + (\nabla_x u)^T), \text{ with } \nabla_x u = (\partial_{x_j} u^i)_{1 \leqslant i,j \leqslant 2}$$

denote the symmetric deformation tensor in the second equation $(3.1)_2$ above. The vector field $\vec{e_2}$ denotes the unit vector in the vertical direction: $\vec{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $\beta\theta\vec{e_2}$ stands for the buoyancy force, with the constant parameter $\beta > 0$ denoting the thermodynamic dilatation coefficient which will be assumed to be 1 in the following context for simplicity.

We consider the cases when the heat diffusion and the viscosity in the fluids are sensitive to the change of temperatures, that is, the thermal diffusivity κ and the viscosity coefficient μ may depend on the temperature function θ as follows

$$\kappa = a(\theta), \quad \mu = b(\theta), \quad \text{with } \kappa_* \leqslant a \leqslant \kappa^*, \ \mu_* \leqslant b \leqslant \mu^*, \tag{3.3}$$

where $\kappa_* \leqslant \kappa^*$, $\mu_* \leqslant \mu^*$ are positive constants. We will not assume any smallness conditions on $\kappa^* - \kappa_*$ or $\mu^* - \mu_*$, and large variations in these diffusivity coefficients are permitted.

The Boussinesq system (3.1) arises from the zero order approximation to the corresponding inhomogeneous hydrodynamic systems, which are nonlinear coupling between the Navier-Stokes equations or Euler equations and the thermodynamic equations for the temperature or density functions: The Boussinesq approximation [Bou72] ignores density differences except when they appear in the buoyancy term. They are common geophysical models describing the dynamics from large scale atmosphere and ocean flows to solar and plasma inner convection, where density stratification is a typical feature [Gil82; Maj03].

The temperature or density differences in the inhomogeneous fluids may cause *density gradients*. When the thermodynamical coefficients such as the heat conducting coefficients and the viscosity coefficients are assumed to be constant in the Boussinesq approximation (i.e. κ, μ are constants in (3.1)), density gradients influence the motion of the flows only through the buoyancy force, which may lead to finite time singularity in the flows (the formation of the finite time singularity is sensitive to the thermal and viscous dissipation and see Subsection 3.1.1 below for more references on this topic).

However, the temperature variations do influence the thermal conductivity and the viscosity coefficients effectively, even for simple fluids such as pure water [Lid05, Section 6][1] [2]. In many applications in the engineering one also aims for effective thermal conductivities in building thermal energy storage materials [Gae+20]. Therefore in plenty of physical models density gradients would influence the motion of the fluids not only through buoyancy force, but also through the variations of the diffusion coefficients. It is then interesting to study the wellposedness and regularity problems of the Boussinesq system (3.1)-(3.3).

3.1.1 Known results

The wellposedness and regularity problems on the two - dimensional Boussinesq equations have attracted considerable attention from the PDE community. Many interesting mathematical results have been established in the past two decades, mainly in the cases with constant thermal diffusivity coefficient κ and viscosity coefficient μ:

$$\begin{cases} \partial_t \theta + u \cdot \nabla_x \theta - \kappa \Delta_x \theta = 0, \\ \partial_t u + u \cdot \nabla_x u - \mu \Delta_x u + \nabla_x \Pi = \theta \vec{e_2}, \\ \operatorname{div}_x u = 0, \\ (\theta, u) \mid_{t=0} = (\theta_0, u_0). \end{cases} \tag{3.4}$$

If $\kappa = \mu = 0$, the two-dimensional inviscid Boussinesq equations (3.4) can be compared with the three-dimensional incompressible axisymmetric Euler equations with swirl, where the buoyancy force corresponds to the vortex stretching mechanism [MB02]. The local-in-time wellposedness as well

[1] The absolute viscosity of the water under nominal atmospheric pressure in units of millipascal seconds is given by 1.793 (0 °C), 0.547 (50 °C), 0.282 (100 °C) respectively [Lid05, Page 6-186]. The thermal conductivity of the water under nominal atmospheric pressure in units of watt per meter kelvin is given by 0.5562 (0 °C), 0.6423 (50 °C), 0.6729 (100 °C) respectively [Lid05, Page 6-214].

[2] It is common to adapt the exponential viscosity law $\mu(T) = C_1 \exp(C_2/(C_3+T))$ and quasi-constant heat conductivity law $\kappa(T) = C_4$ for the liquids, while the viscosity law $\mu(T) = (\mu(T_m))\frac{T}{T_m}\frac{T_m+C_5}{T+C_6}$ and the thermal conductivity law $\kappa(T) = C_6\mu(T)$ for the gases, where T denotes the absolute temperature, T_m denotes the reference temperature and C_j, $1 \leqslant j \leqslant 6$ are positive constants [PTBC08, p. I].

as some blowup criteria have been well known for decades, see e.g. [CN97; Dan13; ES94]. We mention that an (improved) lower bound for the lifespan which tends to infinity as the initial temperature tends to a constant (and correspondingly, as the initial swirl tends to zero for the 3D axisymmetric Euler equations) was given in [Dan13]. The fundamental global regularity problem for the 2D inviscid Boussinesq equations remains still open. Recently an interesting example of finite-energy strong solutions with a finite weighted Hölder norm in a wedge-shaped domain, which become singular at the origin in finite time, has given in [EJ20] (see also an interesting example of solutions in Hölder-type spaces with finite-time singularity for 3D axisymmetric Euler equations in [Elg21]).

If $\kappa > 0$ and $\mu > 0$ are positive constants, on the contrary, the convection terms can be controlled thanks to the strong diffusion effects, and the global-in-time existence and regularity results can be established (see e.g. [CD80]). Particular interests then raised if only partial dissipation is present, that is, either $\kappa = 0$ whereas $\mu > 0$ or $\kappa > 0$ whereas $\mu = 0$ (see e.g. H.K. Moffatt's list of the 21st Century PDE problems [Mof01]). The global-in-time results continue to hold, thanks to a priori estimates in the L^p-framework as well as the sharp Sobolev embedding inequality in dimension two with a logarithm correction, which help the partial diffusion terms to control the demanding term $\partial_{x_1}\theta$ successfully (see [Cha06; HL05] and see [HK09] for less regular cases). Further developments were made for horizontal dissipation cases (see e.g. [DP11]), for vertical dissipation cases (see e.g. [CW11]), and for the fractional dissipation cases (see e.g. [HKR10; HKR11]). See the review notes [Wu] and the references therein for more interesting results and sketchy proofs.

There also have been remarkable progresses in solving the two dimensional Boussinesq equations (3.1)-(3.3) when the thermal and viscosity diffusion coefficients κ, μ are variable and depend smoothly on the unknown temperature function θ. In the variational formulation framework, the global-in-time existence of a solution of (3.1)-(3.3) has been established in [DL72] (see [FM06] for a similar formulation of (3.1)-(3.3)) for the motion of the so-called Bingham fluid (as a non-Newtonian fluid), where κ is a positive constant, $\beta = 0$ and μ depends not only on θ but also on $Su/|Su|$. The Boussinesq-Stefan model has been investigated in [Rod92], where the phase transition was taken into account. The global-in-time existence as well as the uniqueness of the solutions for (3.1)-(3.3) have been shown in [DG98; Gon02; LB99] under Dirichlet boundary conditions and in [PTBC08] under generalized outflow boundary conditions. We remark that the resolution of the nonhomogeneous Boussinesq system under more physical boundary conditions (e.g. with Dirichlet boundary conditions only on the inflow part of the boundary while with no prescribed assumptions on the outflow part) remains unsolved.

S. Lorca and J. Boldrini [LB99] (see also [DG98; Gon02]) studied the initial-boundary value problem of the Boussinesq system (3.1)-(3.3) in dimension two and three under the initial condition (3.2) and Dirichlet boundary conditions, and obtained a global-in-time weak solution

$$(\theta, u) \in (L^\infty_{\mathrm{loc}}([0,\infty); L^2(\Omega)))^3$$

as well as a local-in-time unique strong solution

$$(\theta, u) \in L^\infty_{\mathrm{loc}}([0,\infty); H^2(\Omega)) \times (L^\infty_{\mathrm{loc}}([0,\infty); H^1(\Omega))^2. \tag{3.5}$$

The remarkable global-in-time existence and uniqueness results of the smooth solutions

$$(\theta, u) \in (L^\infty_{\mathrm{loc}}([0,\infty); H^s(\mathbb{R}^2)) \cap L^2_{\mathrm{loc}}([0,\infty); H^{s+1}(\mathbb{R}^2)))^3, \quad s > 2 \tag{3.6}$$

have been successfully established by C. Wang and Z. Zhang [WZ11], which affirms the propagation of high regularities (without finite time singularity) of the two dimensional Boussinesq flow in the presence of viscosity variations (see [SZ13] for the case $s = 2$). We remark that the L^2_x-norm of the velocity vector field may grow in time due to the buoyancy forcing term, even provided with constant diffusion coefficients and smooth and fast decaying small initial data [BS12], and hence the norm with respect to the time variable in (3.5) and (3.6) is only locally in time.

It is still not clear whether there will be finite time singularity for the two dimensional Boussinesq flow (3.1)-(3.3) in the presence of viscosity variations while no heat diffusion (i.e. $\kappa = 0$, $\mu = \mu(\theta)$), and we mention a recent work [AZ17] toward this direction in the case of less heat diffusion (with $\mathrm{div}\,(\kappa\nabla\theta)$ replaced by $(-\Delta)^{1/2}$) and the small viscosity variation assumption: $|\mu - 1| \leqslant \varepsilon$. A closely related question would pertain to the global-in-time wellposedness problem of the two-dimensional inhomogeneous incompressible Navier-Stokes equations with density-dependent viscosity coefficient

$$\begin{cases} \partial_t\rho + u\cdot\nabla_x\rho = 0, \\ \partial_t(\rho u) + \mathrm{div}_x(\rho u \otimes u) - \mathrm{div}_x(\mu S_x u) + \nabla_x \Pi = 0, \\ \mathrm{div}_x\, u = 0, \\ (\rho,\, \rho u)\big|_{t=0} = (\rho_0, m_0). \end{cases} \tag{3.7}$$

The global-in-time existence results of weak solutions of (3.7) (see e.g. [AKM90; Lio96]) as well as the local-in-time well-posedness results (see e.g. [LS75]) have been well known, while the global-in-time regularities still remain open (see e.g. [AZ15b; Des97] for some interesting results under the assumption on the weak inhomogeneity).

To the best of our knowledge, there are no global-in-time regularity propagation results by the two-dimensional Boussinesq flow with temperature-dependent diffusion coefficients (3.1)-(3.2)-(3.3) in the low regularity regime

$$H^s, \quad s < 2,$$

or in the general Sobolev setting

$$\theta_0 \in H_x^{s_\theta}(\mathbb{R}^2), \quad u_0 \in (H_x^{s_u}(\mathbb{R}^2))^2$$

with different regularity indices s_θ and s_u. In this chapter we are going to investigate the existence, uniqueness as well as the regularity problems in these general Sobolev functional settings.

To conclude this subsection let us just mention some recent interesting progresses on the stability of the stationary shear flow solutions (together with the corresponding striated temperature function) to the Boussinesq equations (3.4), with full dissipation or partial dissipation, in e.g. [DWZ21; TWZZ20; Zil21] and references therein. It should also be interesting to investigate the stability of the stationary striated solutions of the Boussinesq equations with variable diffusion coefficients (3.1). We mention a recent work in this direction on the incompressible Navier-Stokes equations with constant density function but with variable viscosity coefficient [LZ21].

3.1.2 Main results

We are going to show the global-in-time existence of weak solutions to the Cauchy problem for the Boussinesq system (3.1)-(3.2)-(3.3) in the whole two-dimensional space $\mathbb{R}^2$ under the low-regularity initial condition $(\theta_0, u_0) \in L^2(\mathbb{R}^2) \times (L^2(\mathbb{R}^2))^2$. The uniqueness result holds true if the initial temperature function becomes smoother $(\theta_0, u_0) \in H^1(\mathbb{R}^2) \times (L^2(\mathbb{R}^2))^2$. Finally we will establish the global-in-time regularity of the solutions in the general Sobolev setting $(\theta_0, u_0) \in H^{s_\theta}(\mathbb{R}^2) \times (H^{s_u}(\mathbb{R}^2))^2 \subset H^1(\mathbb{R}^2) \times (L^2(\mathbb{R}^2))^2$ with the restriction $s_u - 1 \leqslant s_\theta \leqslant s_u + 2$. These regularity exponent ranges are optimal for the existence, uniqueness and regularity results respectively, by view of the formulations of the Boussinesq equations (3.1) with temperature-dependant diffusion coefficients (see Remark 3.1.2 below for more details).

We first define the weak solutions as follows.

Definition 3.1.1 (Weak solutions)**.** *We say that a pair* (θ, u) *is a weak solution of the Boussinesq equations* (3.1)-(3.3) *with the given initial data* $(\theta_0, u_0) \in (L^2(\mathbb{R}^2))^3$ *if the following statements hold:*

- *The temperature function*

$$\theta = \theta(t,x) \in C([0,\infty); L^2_x(\mathbb{R}^2)) \cap L^2_{loc}([0,\infty); H^1_x(\mathbb{R}^2))$$

satisfies the initial condition $\theta|_{t=0} = \theta_0$*, the energy equality*

$$\frac{1}{2}\|\theta(T,\cdot)\|^2_{L^2_x(\mathbb{R}^2)} + \int_0^T \int_{\mathbb{R}^2} (\kappa|\nabla\theta|^2)(t,x)\,\mathrm{d}x\ \mathrm{d}t\ = \frac{1}{2}\|\theta(0,\cdot)\|^2_{L^2_x(\mathbb{R}^2)}, \quad (3.8)$$

for all positive times $T > 0$*, and the equation*

$$\partial_t\theta + u\cdot\nabla\theta - div_x(\kappa\nabla\theta) = 0$$

in $L^2_{loc}([0,\infty); H^{-1}_x(\mathbb{R}^2))$*.*

- *The velocity vector field*

$$u = u(t,x) \in C([0,\infty); (L^2_x(\mathbb{R}^2))^2) \cap L^2_{loc}([0,\infty); (H^1_x(\mathbb{R}^2))^2)$$

satisfies the initial condition $u|_{t=0} = u_0$*, the divergence-free condition* $div_x u = 0$*, the energy equality*

$$\begin{aligned}&\frac{1}{2}\|u(T,\cdot)\|^2_{L^2_x(\mathbb{R}^2)} + \frac{1}{2}\int_0^T \int_{\mathbb{R}^2} (\mu|Su|^2)(t,x)\,\mathrm{d}x\ \mathrm{d}t \\ &= \frac{1}{2}\|u(0,\cdot)\|^2_{L^2_x(\mathbb{R}^2)} + \int_0^T \int_{\mathbb{R}^2} (\theta u_2)(t,x)\,\mathrm{d}x\ \mathrm{d}t\,, \quad \forall T > 0,\end{aligned} \quad (3.9)$$

and the equation

$$\partial_t u + u\cdot\nabla_x u - \mathrm{div}_x(\mu S_x u) + \nabla_x \Pi = \theta\vec{e_2} \quad (3.10)$$

in $L^2_{loc}([0,\infty); (H^{-1}_x(\mathbb{R}^2))^2)$ *for some scalar function* $\Pi \in L^2_{loc}([0,\infty)\times\mathbb{R}^2)$ *with* $\nabla\Pi \in L^2_{loc}([0,\infty); (H^{-1}_x(\mathbb{R}^2))^2)$ *and* $\int_{B_1}\Pi\,\mathrm{d}x = 0$ *a.e.* t *(with* B_1 *denoting the unit disk in* $\mathbb{R}^2$*).*

For any fixed $T > 0$, $p \geqslant 1$, $q \geqslant 1$, $s \geqslant 0$ and for any fixed (vector-valued) function $f : [0,T]\times\mathbb{R}^2 \mapsto \mathbb{R}^m$, $m \geqslant 1$, we denote

$$\|f\|_{L^p_T X_x} := \big\|\|f(t)\|_{X_x(\mathbb{R}^2;\mathbb{R}^m)}\big\|_{L^p_t([0,T])} \text{ with } X = H^s \text{ or } L^q. \quad (3.11)$$

The functional space $L^p([0,T]; H^s(\mathbb{R}^2;\mathbb{R}^m))$ consists of all functions $f : [0,\infty)\times\mathbb{R}^2 \to \mathbb{R}^m$ satisfying $\|f\|_{L^p_T H^s_x} < \infty$. We have the following existence, uniqueness as well as global-in-time regularity results for the solutions of the Cauchy problem for the Boussinesq equations (3.1)-(3.2)-(3.3) on the whole two dimensional space $\mathbb{R}^2$.

Theorem 3.1.1 (Existence, uniqueness & global-in-time regularity, [HL22]). *For any initial data $\theta_0 \in L^2(\mathbb{R}^2)$ and $u_0 \in (L^2(\mathbb{R}^2))^2$, there exists a global-in-time weak solution*

$$(\theta, u) \in C([0,\infty); (L^2(\mathbb{R}^2))^3) \cap L^2_{loc}([0,\infty); (H^1(\mathbb{R}^2))^3)$$

of the initial value problem (3.1)-(3.2)-(3.3).

If $\theta_0 \in H^1(\mathbb{R}^2)$, $u_0 \in (L^2(\mathbb{R}^2))^2$ and the functions $a \in C^2_b(\mathbb{R}; [\kappa_*, \kappa^*])$, $b \in C^2_b(\mathbb{R}; [\mu_*, \mu^*])$ have finite first and second derivatives, then the weak solution is indeed unique, and satisfies

$$\theta \in C([0,\infty); H^1(\mathbb{R}^2)) \cap L^2_{loc}([0,\infty); H^2(\mathbb{R}^2)),$$

as well as the following energy estimates for any given $T > 0$,

$$\|u\|^2_{L^\infty_T L^2_x} + \|\nabla u\|^2_{L^2_T L^2_x} \leqslant C\big(T\|\theta_0\|^2_{L^2} + \|u_0\|^2_{L^2}\big), \tag{3.12}$$

and

$$\begin{aligned} &\|\theta\|^2_{L^\infty_T H^1_x} + \|(\partial_t\theta, \nabla^2\theta)\|^2_{L^2_T L^2_x} \\ &\leqslant C\|\theta_0\|^2_{H^1}(1 + \|\nabla\theta_0\|^2_{L^2}) \exp\big(C(T^2\|\theta_0\|^4_{L^2} + \|u_0\|^4_{L^2})\big), \end{aligned} \tag{3.13}$$

where C is a positive constant depending only on $\|a\|_{\mathrm{Lip}}, \kappa_*, \kappa^*, \mu_*$.

Furthermore, the general H^s-regularities can be propagated globally in time in the following sense: For any initial data (see the grey unbounded quadrangle in Figure 3.1 for the admissible regularity exponent range)

$$\begin{aligned} &(\theta_0, u_0) \in H^{s_\theta}(\mathbb{R}^2) \times (H^{s_u}(\mathbb{R}^2))^2 \text{ with } (s_\theta, s_u) \in D, \\ &D = \{(s_\theta, s_u) \subset [1,\infty) \times [0,\infty) \mid s_u - 1 \leqslant s_\theta \leqslant s_u + 2\} \backslash \{(2,0), (1,2)\} \end{aligned} \tag{3.14}$$

and the functions $a \in C^2_b \cap C^{[s_\theta]+1}$, $b \in C^2_b \cap C^{[s_u]+1}$, the unique solution (θ, u) stays in

$$C([0,\infty); H^{s_\theta}(\mathbb{R}^2) \times (H^{s_u}(\mathbb{R}^2))^2) \cap L^2_{loc}([0,\infty); H^{s_\theta+1}(\mathbb{R}^2) \times (H^{s_u+1}(\mathbb{R}^2))^2). \tag{3.15}$$

Theorem 3.1.1 will be proved in Section 3.2. The proof of the existence of weak solutions is rather standard, and we are going to sketch the proof in Subsection 3.2.1 for the reason of completeness, as we did not find the proof in the literature. As mentioned before, some well-posedness results have already been established for smooth data in the bounded domain case (see (3.5) above in e.g. [DG98; DL72; Gon02; LB99]) or in smoother functional frameworks in the whole space case (see (3.6) above in e.g. [WZ11]). We are going to focus on the proofs of the uniqueness result and the global-in-time regularity

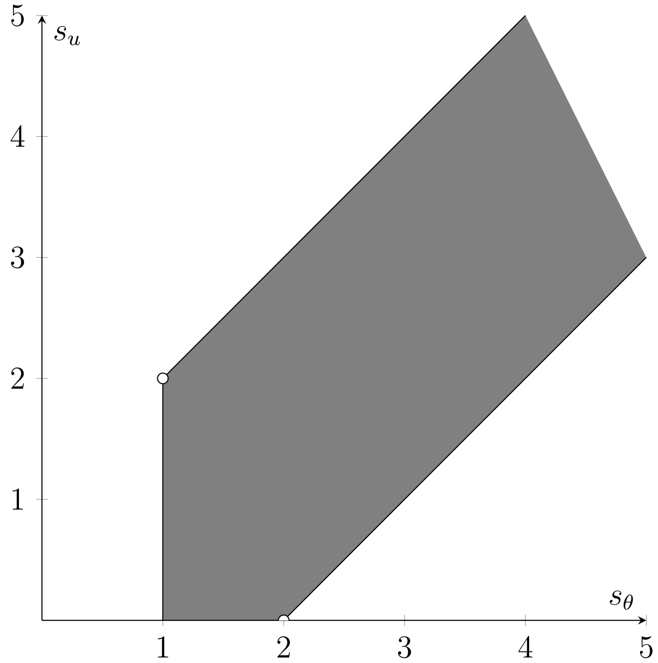

Figure 3.1: Admissible regularity exponents

result (in the low regularity regimes) in Subsection 3.2.3 and Subsection 3.2.4 respectively, where different regularity exponents for different unknowns are permitted. The commutator estimates as well as the composition estimates in Lemma 3.2.1 will play an important role, and the a priori estimates for a general linear parabolic equation in Lemma 3.2.2 will be of independent interest.

We conclude this introduction part with several remarks on the results in Theorem 3.1.1.

Remark 3.1.1. *To show the regularity results in the admissible exponent range D as in* (3.14)*, we only need to prove on*

$$\partial D \backslash \{(2,0),(1,2)\} \quad \textit{and} \quad \{(2,s),(s+1,2) | s \in (0,1)\},$$

since the admissible exponent range D (the grey area in Figure 3.1) is convex.

Remark 3.1.2 (Optimality of the regularity exponent ranges in Theorem 3.1.1)**.** *We are going to follow the standard procedure to show the existence of weak solutions for L^2-initial data by use of the a priori energy (in)equalities* (3.8) *and* (3.9) *(see Subsection 3.2.1 below).*

Under the lower-regularity assumption $\theta_0 \in H^s_x$ with $0 < s < 1$, the coefficients κ, μ are not expected to be continuous uniformly in time, and *hence no uniqueness or* H^s*-regularity results for* θ *or* H^{s_1}*,* $s_1 > 0$*-regularity results for* u *are expected. Nevertheless with constant diffusion coefficients (e.g.* $\kappa = \mu = 1$*), the uniqueness result for the weak solutions holds true by virtue of the* L^2_x*-energy (in)equalities (similar as the classical global-in-time well-posedness result for the classical two dimensional incompressible Navier-Stokes equations). Furthermore, if* $\kappa = 1$ *is a positive constant, then the* H^s_x*,* $s \in (0,1)$*-Estimate for* θ *holds true, provided with* $u \in L^4_{loc}(L^4_x(\mathbb{R}^2))^2$ *(or with* $u_0 \in (L^2(\mathbb{R}^2))^2$*), simply by an interpolation argument between* (3.8) *and* (3.13)*. Similarly if* $\mu = 1$ *is a positive constant, then the* H^s_x*,* $s > 0$*-Estimate for* u *holds true, provided with* $\theta \in L^2_{loc}(H^{s-1}_x(\mathbb{R}^2))$*. Thus with constant diffusion coefficients (e.g.* $\kappa = \mu = 1$*), the Sobolev regularities*

$$(\theta_0, u_0) \in (H^s(\mathbb{R}^2)) \times (L^2(\mathbb{R}^2))^2 \ \textit{or}\ (L^2(\mathbb{R}^2)) \times (H^s(\mathbb{R}^2))^2, \quad 0 < s \leqslant 1$$

can be propagated globally in time, and the admissible regularity exponent set (3.14) *extends itself indeed to the closed set consisting of all non-negative admissible regularity exponents:*

$$(s_\theta, s_u) \in \{(s_\theta, s_u) \subset [0,\infty) \times [0,\infty) \,|\, s_u - 1 \leqslant s_\theta \leqslant s_u + 2\}\backslash\{(2,0)\}.$$

In order to propagate the H^{s_θ}*,* $s_\theta \geqslant 2$*-regularity of* θ*, we require the transport term* $u \cdot \nabla\theta$ *in the* θ*-equation to be at least in* $L^2_{loc}([0,\infty); H^{s_\theta-1}_x)$*, which requires* $u \in L^2_{loc}([0,\infty); H^{s_\theta-1}_x)$ *and hence the initial assumption* $u_0 \in H^{s_u}$ *with the restriction* $s_u \geqslant s_\theta - 2$ *(as there is a gain of regularity of oder* 1 *when taking* L^2*-norm in the time variable in general). Similarly, in order the propagate the* H^{s_u}*,* $s_u \geqslant 2$*-regularity of* u*, we require the viscosity term* $div(\mu Su)$ *in the* u*-equation to be at least in* $L^2_{loc}([0,\infty); H^{s_u-1}_x)$*, which requires* $\mu Su \in L^2_{loc}([0,\infty); H^{s_u}_x)$ *and hence the initial assumption* $\theta_0 \in H^{s_\theta}$ *with the restriction* $s_\theta \geqslant s_u - 1$*.*

Concerning the endpoints $(2,0)$ *or* $(1,2)$ *in the Figure 3.1, in general we can not show the regularity results. Because the boundedness of*

$$\int_0^T \int_{\mathbb{R}^2} |\nabla\Delta\eta \cdot \nabla u \cdot \nabla\eta| \,\mathrm{d}x \,\mathrm{d}t \ \ \textit{or}\ \ \int_0^T \int_{\mathbb{R}^2} |\nabla^2\mu \cdot Su \cdot \nabla\Delta u| \,\mathrm{d}x \,\mathrm{d}t\,,$$

is lacking with

$$\begin{aligned} &(\eta, u) \in L^\infty_{loc}(H^2_x \times (L^2_x)^2) \cap L^2_{loc}(H^3_x \times (H^1_x)^2) \\ \textit{or}\quad &(\eta, u) \in L^\infty_{loc}(H^1_x \times (H^2_x)^2) \cap L^2_{loc}(H^2_x \times (H^3_x)^2), \end{aligned}$$

as a consequence of the failure of the Sobolev embedding $H^1(\mathbb{R}^2) \not\hookrightarrow L^\infty(\mathbb{R}^2)$*.*

Remark 3.1.3 (Precise H^s_x-Estimates in the high regularity regime)**.** *The global-in-time regularity in the high regularity regime* (3.14)-(3.15) *follows immediately from the following* borderline *a priori estimates:*

- *If* $\theta_0 \in H^s(\mathbb{R}^2)$, $u_0 \in (L^2(\mathbb{R}^2))^2$ *with* $s \in (1,2)$ *and the function* $a \in C_b^2(\mathbb{R})$, *then*

$$\begin{aligned}&\|\theta\|^2_{L^\infty_T H^s_x} + \|\nabla\theta\|^2_{L^2_T H^s_x} \leqslant C(\kappa_*)\|\theta_0\|^2_{H^s_x} \times \\ &\times \exp\Big(C(\kappa_*, s, \|a\|_{C^2}, \|\theta\|_{L^\infty_T H^1_x})\big(\|\nabla u\|^2_{L^2_T L^2_x} + \|\nabla\theta\|^2_{L^2_T H^1_x}\big)\Big).\end{aligned} \tag{3.16}$$

- *If* $\theta_0 \in H^1(\mathbb{R}^2)$, $u_0 \in (H^s(\mathbb{R}^2))^2$ *with* $s \in (0,2)$ *and the function* $b \in C_b^2(\mathbb{R})$, *then*

$$\begin{aligned}&\|u\|^2_{L^\infty_T H^s_x} + \|\nabla u\|^2_{L^2_T H^s_x} \leqslant C(\mu_*)(\|u_0\|^2_{H^s_x} + T\|\theta_0\|^2_{L^2_x} + \|\theta\|^2_{L^2_T H^{s-1}_x}) \\ &\times \exp\Big(C(\mu_*, s, \|b\|_{C^2}, \|\theta\|_{L^\infty_T H^1_x})\big(\|\nabla u\|^2_{L^2_T L^2_x} + \|\nabla\theta\|^2_{L^2_T H^1_x}\big)\Big).\end{aligned} \tag{3.17}$$

- *If* $\theta_0 \in H^2(\mathbb{R}^2)$, $u_0 \in (H^\varepsilon(\mathbb{R}^2))^2$ *with* $\varepsilon \in (0,1)$ *and the function* $a \in C_b^2(\mathbb{R})$, *then*

$$\begin{aligned}&\|\theta\|^2_{L^\infty_T H^2_x} + \|\nabla\theta\|^2_{L^2_T H^2_x} \leqslant C(\kappa_*, \|a\|_{C^2}, \kappa^*)\|\theta_0\|^2_{H^2}(1 + \|\nabla\theta_0\|^2_{L^2}) \\ &\times \exp\Big(C(\kappa_*, \varepsilon, \|a\|_{\mathrm{Lip}})(\|u\|^2_{L^2_T H^{1+\varepsilon}_x} + \|u\|^4_{L^4_T L^4_x} + \|\nabla\theta\|^4_{L^4_T L^4_x})\Big).\end{aligned} \tag{3.18}$$

- *If* $\theta_0 \in H^{1+\varepsilon}(\mathbb{R}^2)$ *and* $u_0 \in (H^2(\mathbb{R}^2))^2$ *with* $\varepsilon \in (0,1)$ *and the function* $b \in C_b^2(\mathbb{R})$, *then*

$$\begin{aligned}&\|u\|^2_{L^\infty_T H^2_x} + \|\nabla u\|^2_{L^2_T H^2_x} \leqslant (\|u\|^2_{L^\infty_T H^1_x} + \|\nabla u\|^2_{L^2_T H^1_x}) \\ &+ C\Big(\|\Delta u_0\|^2_{L^2_x} + \|u\|^2_{L^\infty_T H^1_x \cap L^2_T \dot H^2_x}(\|u\|^2_{L^\infty_T H^1_x \cap L^2_T \dot H^2_x} + \|\nabla\theta\|^2_{L^2_T H^{1+\varepsilon}_x}) \\ &\quad + \|\Delta\theta\|_{L^2_T L^2_x}\|\Delta u\|_{L^2_T L^2_x}\Big) \times \exp\Big(C\big(\|(u,\nabla\theta)\|^4_{L^4_T L^4_x} + \|\nabla^2\theta\|^2_{L^2_T H^\varepsilon_x}\big)\Big).\end{aligned} \tag{3.19}$$

where the constant C *depends on* $\mu_*, \varepsilon, \|b\|_{C^2}, \|\theta\|_{L^\infty_T H^{1+\varepsilon}_x}, \|\nabla\theta\|_{L^2_T H^1_x}$.

- *If* $\theta_0 \in H^s(\mathbb{R}^2)$, $u_0 \in (H^{s-2}(\mathbb{R}^2))^2$ *with* $s > 2$ *and the function* $a \in C^{[s]+1}$, *then for* $s \in (2,3)$ *it holds*

$$\begin{aligned}&\|\theta\|^2_{L^\infty_T H^s_x} + \|\nabla\theta\|^2_{L^2_T H^s_x} \leqslant C(\kappa_*)\|\theta_0\|^2_{H^s_x} \times \\ &\quad \times \exp\Big(C(\kappa_*, s, a, \|\theta\|_{L^\infty_T L^\infty_x})(\|u\|^2_{L^2_T H^{s-1}_x} + \|\nabla\theta\|^2_{L^2_T L^\infty_x})\Big),\end{aligned} \tag{3.20}$$

and for $s \geqslant 3$ *it holds*

$$\begin{aligned}&\|\theta\|^2_{L^\infty_T H^s_x} + \|\nabla\theta\|^2_{L^2_T H^s_x} \leqslant C(\kappa_*, s)(\|\theta_0\|^2_{H^s_x} + \|\nabla\theta\|^2_{L^\infty_T L^\infty_x}\|\nabla u\|^2_{L^2_T H^{s-2}_x}) \\ &\quad \times \exp(C(\kappa_*, s, a, \|\theta\|_{L^\infty_T L^\infty_x})(\|\nabla u\|^2_{L^2_T L^\infty_x} + \|\nabla\theta\|^2_{L^2_T L^\infty_x})).\end{aligned} \tag{3.21}$$

- *If $\theta_0 \in H^{s-1}(\mathbb{R}^2)$, $u_0 \in (H^s(\mathbb{R}^2))^2$ with $s > 2$ and the function $b \in C^{[s]+1}$, then for $s \in (2,3)$ it holds*

$$\begin{aligned}&\|u\|^2_{L^\infty_T H^s_x} + \|\nabla u\|^2_{L^2_T H^s_x} \leqslant C(\mu_*)(\|u_0\|^2_{H^s_x} + T\|\theta\|^2_{L^\infty_T H^{s-1}_x})\\ &\times \exp(C(\mu_*, s, b, \|\theta\|_{L^\infty_T L^\infty_x})(\|\nabla u\|^2_{L^2_T H^1_x} + \|\nabla\theta\|^2_{L^2_T H^{s-1}_x})),\end{aligned} \tag{3.22}$$

  and for $s \geqslant 3$ it holds

$$\begin{aligned}&\|u\|^2_{L^\infty_T H^s_x} + \|\nabla u\|^2_{L^2_T H^s_x}\\ &\leqslant C(\mu_*)(\|u_0\|^2_{H^s_x} + T\|\theta\|^2_{L^\infty_T H^{s-1}_x} + \|\nabla u\|^2_{L^\infty_T L^\infty_x}\|\nabla\theta\|^2_{L^2_T H^{s-1}_x})\times\\ &\times \exp(C(\mu_*, s, b, \|\theta\|_{L^\infty_T L^\infty_x})(\|\nabla u\|^2_{L^2_T L^\infty_x} + \|\nabla\theta\|^2_{L^2_T L^\infty_x})).\end{aligned} \tag{3.23}$$

We are going to prove the above borderline estimates one by one in Subsection 3.2.4 below.

Remark 3.1.4 (L^2-in time Estimate V.S. L^1-in time Estimate)**.** *Instead of the classical $L^\infty_t H^s_x \cap L^1_t H^{s+2}_x$-type estimate in the literature, we derive $L^\infty_t H^s_x \cap L^2_t H^{s+1}_x$-type estimate here, since e.g. only the $L^2_t \dot{H}^1_x$-a priori estimate for the velocity vector field is available from the energy estimates (roughly speaking, the L^2_t-in time norm asks less spacial regularity on the coefficients). See Lemma 3.2.2 below for the a priori H^s_x, $s \in (0,2)$-estimates for a general linear parabolic equation with divergence-free $L^2_t H^1_x$-velocity vector field, which is of independent interest.*

It is in general not true that $\theta \in L^1_t H^{s+2}_x$ (or $u \in L^1_t H^{s+2}_x$) in the low regularity regime, although it holds straightforward in the high regularity regime.

Remark 3.1.5 (Remarks on the smoothness assumptions on the functions a, b)**.** *It is common to assume smooth heat conductivity law and viscosity law [PTBC08, p. I] in fluid models.*

The Lipschitz continuity assumption $a, b \in \mathrm{Lip}$ is enough for the $H^1 \times L^2$-Estimates (3.12)-(3.13) *in Theorem 3.1.1. As for the uniqueness result, due to the following $\dot{H}^1_x$-Estimate for the difference of the diffusion coefficinets*

$$\begin{aligned}\|a(\theta_1) - a(\theta_2)\|_{\dot{H}^1_x} &\leqslant \|(a'(\theta_1) - a'(\theta_2))\nabla\theta_1\|_{L^2_x} + \|a(\theta_2)\nabla(\theta_1 - \theta_2)\|_{L^2_x}\\ &\leqslant \|a'\|_{\mathrm{Lip}}\|\nabla\theta_1\|_{L^4_x}\|\theta_1 - \theta_2\|_{L^4_x} + \|a\|_{L^\infty}\|\nabla(\theta_1 - \theta_2)\|_{L^2_x},\end{aligned}$$

the Lipschitz continuity assumptions $a', b' \in \mathrm{Lip}$ are required.

The dependance on the function a of the constants C in (3.20)-(3.21) *reads precisely as (similarly for the constants in* (3.22)-(3.23)*)*

$$\sup_{k=0,\cdots,[s]+1} \sup_{|y| \leqslant c\|\theta\|_{L^\infty_T L^\infty_x}} \Big|\frac{d}{dy^k} a(y)\Big|,$$

and hence only $a \in C^{[s]+1}$ instead of $a \in C_b^{[s]+1}$ is required.

For the integer regularity exponents, we can simply derive the energy estimates by integration by parts (instead of the application of the commutator estimates or the composition estimates in Lemma 3.2.1 below), such that the requirement for $a \in C^{[s_\theta]+1}$ and $b \in C^{[s_u]+1}$ can be relaxed.

3.2 Proofs of existence, uniqueness and regularity

Recall the Cauchy problem for the two dimensional Boussinesq equations (3.1)-(3.3)

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta - \operatorname{div}(\kappa\nabla\theta) = 0, \\ \partial_t u + u\cdot\nabla u - \operatorname{div}(\mu S u) + \nabla\Pi = \theta\vec{e_2}, \\ \operatorname{div} u = 0, \\ (\theta, u)\mid_{t=0} = (\theta_0, u_0), \end{cases} \tag{3.24}$$

where $\kappa = a(\theta) \in C_b^1(\mathbb{R}; [\kappa_*, \kappa^*])$, $\mu = b(\theta) \in C_b^1(\mathbb{R}; [\mu_*, \mu^*])$ with $\kappa_*, \kappa^*, \mu_*, \mu^*$ four positive constants.

We are going to show the existence result in Theorem 3.1.1 in Subsection 3.2.1. We derive of the a priori H_x^s, $s \in (0, 2)$-Estimate for a general linear parabolic equation in Subsection 3.2.2. The uniqueness as well as the global-in-time regularity results in Theorem 3.1.1 in Subsection 3.2.3, and Subsection 3.2.4 respectively.

Recall the definition of the $\|\cdot\|_{L_T^q X_x}$-norm in (3.11). The Gagliardo-Nirenberg's inequality

$$\|f\|_{L_T^4 L_x^4(\mathbb{R}^2)} \leqslant C\|f\|_{L_T^2 L_x^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla f\|_{L_T^2 L_x^2(\mathbb{R}^2)}^{\frac{1}{2}} \tag{3.25}$$

as well as the equivalence relations between the norms

$$\begin{aligned} &\|Su\|_{L_x^2(\mathbb{R}^2)}^2 = 2\|\nabla u\|_{L_x^2(\mathbb{R}^2)}^2 \text{ if } \operatorname{div} u = 0, \\ &\|\Delta\eta\|_{L_x^2(\mathbb{R}^2)} \sim \|\nabla^2\eta\|_{L_x^2(\mathbb{R}^2)} \end{aligned} \tag{3.26}$$

will be used freely in the proof.

3.2.1 Existence of weak solutions

We will follow the standard procedure to show the existence of the weak solutions under the initial condition

$$(\theta_0, u_0) \in L^2(\mathbb{R}^2) \times (L^2(\mathbb{R}^2))^2,$$

namely

Step 1 We construct a sequence of approximate solutions, which satisfy the energy estimates uniformly.

Step 2 We show the convergence of this approximate solution sequence to a weak solution and study the property of the weak solution.

We are going to sketch the proof and pay attention to the low-regulartiy assumptions.

Step 1: Construction of approximate solutions with uniform bounds

We use the Friedrich's method to construct a sequence of approximate solutions. We consider the following system of (θ_n, u_n)

$$\begin{cases} \partial_t \theta_n + P_n(u_n \cdot \nabla\theta_n) - P_n \operatorname{div}(\kappa_n \nabla\theta_n) = 0, \\ \partial_t u_n + P_n\mathbb{P}(u_n \cdot \nabla u_n) - P_n\mathbb{P}\operatorname{div}(\mu_n S u_n) = \mathbb{P}(\theta_n \vec{e_2}), \\ u_n(0,x) = P_n u_0(x), \quad \theta_n(0,x) = P_n\theta_0(x), \end{cases} \tag{3.27}$$

where $\kappa_n = a(\theta_n)$ and $\mu_n = b(\theta_n)$. The operator P_n, $n \in \mathbb{N}$, is the low-frequency cut-off operator which is defined as follows

$$P_n f(x) = \mathcal{F}^{-1}(\mathbb{1}_{B_n}(\xi)\mathcal{F}f(\xi))(x),$$

where $B_n \subset \mathbb{R}^2$ is the disk with center at 0 and radius n, and $\mathcal{F}, \mathcal{F}^{-1}$ are the standard Fourier and inverse Fourier transformations. The operator $\mathbb{P}$ in (3.27) denotes the Leray-Helmholtz projector on $\mathbb{R}^2$, which decomposes the tempered distributions $v \in \mathcal{S}'(\mathbb{R}^2;\mathbb{R}^2)$ into div-free and curl-free parts as follows

$$v = \nabla^\perp V_1 + \nabla V_2, \tag{3.28}$$

where

$$\nabla^\perp V_1 = -\nabla^\perp(-\Delta)^{-1}\nabla^\perp \cdot v =: \mathbb{P}v, \quad \nabla V_2 = -\nabla(-\Delta)^{-1}\nabla \cdot v = (1-\mathbb{P})v$$

with $\nabla^\perp = (\partial_{x_2}, -\partial_{x_1})^T$. Notice that $\mathbb{P}$ maps $L^p(\mathbb{R}^2;\mathbb{R}^2)$ into itself for any $p \in (1,\infty)$ and it is commutative with the projection operator P_n.

We define the Banach spaces L^2_n and $L^{2,\sigma}_n$ as following

$$\begin{aligned} L^2_n(\mathbb{R}^2) &= \{f \in L^2(\mathbb{R}^2) \mid f = P_n f\}, \\ L^{2,\sigma}_n(\mathbb{R}^2) &= \{f \in (L^2_n(\mathbb{R}^2))^2 \mid \operatorname{div}_x(f) = 0\}. \end{aligned}$$

The system (3.27) turns out to be an ordinary differential equation system in $L^2_n(\mathbb{R}^2) \times L^{2,\sigma}_n(\mathbb{R}^2)$. Indeed, the following estimates hold

$$\begin{aligned} \|P_n(u_n \cdot \nabla\theta_n) - P_n \operatorname{div}(\kappa_n\nabla\theta_n)\|_{L^2_x} &\leqslant Cn^3(\|u_n\|_{L^2_x} + \kappa^*)\|\theta_n\|_{L^2_x}, \\ \|P_n\mathbb{P}(u_n \cdot \nabla u_n) - P_n\mathbb{P}\operatorname{div}(\mu_n S u_n)\|_{L^2_x} &\leqslant Cn^3(\|u_n\|_{L^2_x} + \mu^*)\|u_n\|_{L^2_x}. \end{aligned}$$

Hence, for any $n \in \mathbb{N}$, there exists $T_n > 0$ such that the system (3.27) has a solution $(\theta_n, u_n) \in C([0,T_n]; L^2_n(\mathbb{R}^2)) \times C([0,T_n]; L^{2,\sigma}_n(\mathbb{R}^2))$.

We take the $L^2(\mathbb{R}^2)$-inner product of the equation $(3.27)_1$ and θ_n to derive

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} \theta_n^2 + \int_{\mathbb{R}^2} \kappa_n |\nabla\theta_n|^2 = 0.$$

Then the following uniform estimate for (θ_n) holds

$$\frac{1}{2}\|\theta_n\|^2_{L^\infty_T L^2_x} + \kappa_* \|\nabla\theta_n\|^2_{L^2_T L^2_x}\, dt \leqslant \frac{1}{2}\|P_n\theta_0\|^2_{L^2_x} \leqslant \frac{1}{2}\|\theta_0\|^2_{L^2_x}, \quad \forall T > 0. \tag{3.29}$$

Similarly we take the $L^2(\mathbb{R}^2)$-inner product of the equation $(3.27)_2$ and u_n to derive

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{L^2_x} + \frac{1}{2}\|\mu_n S u_n\|^2_{L^2_x} \leqslant \|\theta_n\|_{L^2_x}\|u_n\|_{L^2_x} \leqslant \frac{1}{2}(T\|\theta_n\|^2_{L^2_x} + \frac{1}{T}\|u_n\|^2_{L^2_x}),$$

for all positive times $T > 0$, and thus by Gronwall's inequality we arrive at the following uniform estimate for (u_n) (noticing $\|Su_n\|^2_{L^2_x} = 2\|\nabla u_n\|^2_{L^2_x}$)

$$\frac{1}{2}\|u_n\|^2_{L^\infty_T L^2_x} + \mu_* \|\nabla u_n\|^2_{L^2_T L^2_x} \leqslant \frac{e}{2}(T\|\theta_0\|^2_{L^2_x} + \|u_0\|^2_{L^2_x}), \quad \forall T > 0. \tag{3.30}$$

Thus the approximate solutions (θ_n, u_n) exist for all positive times.

Step 2: Passing to the limit

By the above uniform bounds (3.29)-(3.30) there exists a subsequence, still denote by (θ_n, u_n), converging weakly to a limit $(\theta, u) \in L^\infty_{\mathrm{loc}}([0,\infty); (L^2_x)^3) \cap L^2_{\mathrm{loc}}([0,\infty); (H^1_x)^3)$:

$$\begin{aligned}
&\theta_n \overset{*}{\rightharpoonup} \theta \quad \text{in } L^\infty_{\mathrm{loc}}([0,\infty); L^2(\mathbb{R}^2)), \quad \nabla\theta_n \rightharpoonup \nabla\theta \quad \text{in } L^2_{\mathrm{loc}}([0,\infty); (L^2(\mathbb{R}^2))^2),\\
&u_n \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty_{\mathrm{loc}}([0,\infty); (L^2(\mathbb{R}^2))^2), \quad \nabla u_n \rightharpoonup \nabla u \quad \text{in } L^2_{\mathrm{loc}}([0,\infty); (L^2(\mathbb{R}^2))^4).
\end{aligned}$$

Since by the Gagliardo-Nirenberg's inequality (θ_n, u_n) is a bounded sequence in $L^4_T L^4_x$ for any $T > 0$, the sequence of the time derivatives $(\partial_t\theta_n, \partial_t u_n)$ is bounded in $L^2_T(H^{-1}_x)$ (by use of the equations in (3.27)), and hence $\{(\theta_n, u_n)\}$ is relatively compact in $L^p_T L^2_x(B_R)$ for any fixed disk $B_R \subset \mathbb{R}^2$ and $p \in [1,\infty)$, which implies the pointwise convergence (up to a subsequence)

$$\theta_n \to \theta, \quad u_n \to u \text{ for almost every } t \in \mathbb{R}^+, \quad x \in \mathbb{R}^2,$$

as well as the convergence of the nonlinear terms (noticing e.g. $u_n\varphi \to u\varphi$ in $L^4_T L^4_x$ for fixed $\varphi \in C^\infty_c((0,T) \times \mathbb{R}^2)$)

$$u_n\theta_n \to u\theta, \quad u_n \otimes u_n \to u \otimes u \text{ in } \mathcal{D}'((0,T) \times \mathbb{R}^2) \text{ and hence weakly in } L^2_T L^2_x.$$

Consequently, $\kappa_n = a(\theta_n) \to \kappa = a(\theta)$ and $\mu_n = b(\theta_n) \to \mu = b(\theta)$ almost everywhere and

$$\kappa_n \nabla\theta_n \rightharpoonup \kappa\nabla\theta, \quad \mu_n S u_n \rightharpoonup \mu S u \text{ in } L^2_T L^2_x.$$

Thus the equation (noticing $P_n \to \mathrm{Id}$ as an operator from $H^s(\mathbb{R}^2)$ to itself)

$$\partial_t \theta + \mathrm{div}\,(u\theta) - \mathrm{div}\,(\kappa\nabla\theta) = 0 \text{ holds in } L^2_{\mathrm{loc}}((0,\infty); H^{-1}_x(\mathbb{R}^2)),$$

and we can test it by $\theta \in L^2_{\mathrm{loc}}([0,\infty); H^1_x)$ to arrive at the energy equality (3.8) for θ, such that $\theta|_{t=0} = \theta_0$ and $\theta \in C([0,\infty); L^2_x)$ hold true.

Similarly, the equation

$$\partial_t u + \mathbb{P}\mathrm{div}\,(u \otimes u - \mu S u) = \mathbb{P}(\theta\vec{e}_2) \text{ holds in } L^2_{\mathrm{loc}}((0,\infty); (H^{-1}_x(\mathbb{R}^2))^2), \tag{3.31}$$

and we can test it by the divergence-free velocity field $u \in L^2_{\mathrm{loc}}((0,\infty); (H^1_x(\mathbb{R}^2))^2)$ to arrive at the energy equality (3.9), which implies $u \in C([0,\infty); (L^2_x(\mathbb{R}^2))^2)$ and $u|_{t=0} = u_0$. We take the solution Π of the Poisson equation

$$\Delta\Pi = \mathrm{div}\,(1-\mathbb{P})(\theta\vec{e}_2 - \mathrm{div}\,(u\otimes u - \mu S u)) \tag{3.32}$$

under the renormalisation condition $\int_{B_1} \Pi\,\mathrm{d}x = 0$, such that

$$\nabla\Pi = (1-\mathbb{P})(\theta\vec{e}_2 - \mathrm{div}\,(u\otimes u - \mu S u)) \in L^2_{\mathrm{loc}}((0,\infty); (H^{-1}_x(\mathbb{R}^2))^2),$$

and the equation (3.10) holds in $L^2_{\mathrm{loc}}((0,\infty); (H^{-1}_x(\mathbb{R}^2))^2)$.

3.2.2 Estimates for the general parabolic equations

In this subsection, we will derive a priori H^s_x, $s \in (0,2)$-Estimate for a general linear parabolic equation. We are going to use these a priori estimates to establish the uniqueness and regularity results of the Boussinesq equation (3.1) in Subsection 3.2.3, and Subsection 3.2.4 respectively.

For readers' convenience we recall here briefly the Littlewood-Paley dyadic decomposition and the definition of the $H^s(\mathbb{R}^n)$-norms (see e.g. Chapter 2 in the book [BCD11] for more details). We fix a nonincreasing radial function $\chi \in C^\infty_c(B_{\frac{4}{3}})$ with $\chi(x) = 1$ for $x \in B_1$, where $B_r \subset \mathbb{R}^n$ denotes the ball centered at 0 with radius r. We define the function $\varphi(\xi) = \chi(\frac{\xi}{2}) - \chi(\xi)$ and $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ with $j \geqslant 0$. We do the Littlewood-Paley decomposition in the following way

$$g = \Delta_{-1} g + \sum_{j\geqslant 0} \Delta_j g, \tag{3.33}$$

where

$$\mathcal{F}(\Delta_{-1}g)(\xi) = \chi(\xi)\mathcal{F}(g)(\xi), \quad \mathcal{F}(\Delta_j g)(\xi) = \varphi_j(\xi)\mathcal{F}(g)(\xi), \quad j \geqslant 0,$$

and $\mathcal{F}$ denotes the Fourier transform. We have the following Bernstein's inequalities for some universal constant C (depending only on n)

$$\begin{aligned} &\|\Delta_{-1}g\|_{L^2(\mathbb{R}^n)} \leqslant C\|g\|_{L^2(\mathbb{R}^n)}, \\ &C^{-1}2^j\|\Delta_j g\|_{L^2(\mathbb{R}^n)} \leqslant \|\nabla(\Delta_j g)\|_{L^2(\mathbb{R}^n)} \leqslant C2^j\|\Delta_j g\|_{L^2(\mathbb{R}^n)}, \quad \forall j \geqslant 0. \end{aligned} \tag{3.34}$$

Let $s \geqslant 0$ and p, $r \geqslant 1$. We define the nonhomogeneous Besov spaces $B^s_{p,r}(\mathbb{R}^n)$ as the spaces consisting of all tempered distributions $g \in \mathcal{S}'(\mathbb{R}^n)$ satisfying

$$\|g\|_{B^s_{p,r}(\mathbb{R}^n)} = \left\|(2^{js}\|\Delta_j g\|_{L^p(\mathbb{R}^n)})_{j\geqslant -1}\right\|_{l^r} < \infty.$$

The inhomogeneous Sobolev spaces $H^s(\mathbb{R}^n) = B^s_{2,2}(\mathbb{R}^n)$ can be defined by

$$H^s(\mathbb{R}^n) = \{g \in \mathcal{S}'(\mathbb{R}^n) \mid \|g\|_{H^s(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} (1+|\xi|^2)^{\frac{s}{2}}|\mathcal{F}(g)(\xi)|^2\,d\xi\Big)^{1/2} < \infty\},$$

where the $H^s(\mathbb{R}^n)$-norm reads in terms of Littlewood-Paley decomposition as follows

$$\|g\|_{H^s(\mathbb{R}^n)} \sim \|g\|_{L^2(\mathbb{R}^n)} + \Big(\sum_{j\geqslant 0} 2^{2js}\|\Delta_j g\|^2_{L^2(\mathbb{R}^n)}\Big)^{\frac{1}{2}}. \tag{3.35}$$

It is straightforward to derive the following interpolation inequality

$$\|u\|_{H^{t_\sigma}} \leqslant C\|u\|^{1-\sigma}_{H^{t_0}}\|u\|^{\sigma}_{H^{t_1}}, \text{ where } t_\sigma = (1-\sigma)t_0 + \sigma t_1, \sigma \in [0,1]. \tag{3.36}$$

We are going to use the following known estimates to control the nonlinear terms in the Boussinesq system (3.1).

Lemma 3.2.1. *We have the following commutator, product and composition estimates.*

(1) [DL12, Proposition 2.4] In the low regularity regime where $(s,\nu) \in \mathbb{R}^2$ satisfy

$$-1 < s < \nu + 1, \quad \textit{and } -1 < \nu < 1,$$

the following commutator estimate holds true (in $\mathbb{R}^2$):

$$\|\big(2^{js}\|[\phi,\Delta_j]\nabla\psi\|_{L^2(\mathbb{R}^2)}\big)_{j\geqslant -1}\|_{l^1} \leqslant C\|\nabla\phi\|_{H^\nu(\mathbb{R}^2)}\|\nabla\psi\|_{H^{s-\nu}(\mathbb{R}^2)}, \tag{3.37}$$

where C is a constant depending only on s,ν.

(2) *[BCD11, Lemma 2.100] For any $s > 0$, the following commutator estimate holds true*

$$\begin{aligned} &\|\big(2^{js}\|[\phi, \Delta_j]\nabla\psi\|_{L^2(\mathbb{R}^n)}\big)_{j\geqslant -1}\|_{l^1} \\ &\leqslant C\big(\|\nabla\phi\|_{L^\infty(\mathbb{R}^n)}\|\nabla\psi\|_{H^{s-1}(\mathbb{R}^n)} + \|\nabla\phi\|_{H^{s-1}(\mathbb{R}^n)}\|\nabla\psi\|_{L^\infty(\mathbb{R}^n)}\big). \end{aligned} \tag{3.38}$$

(3) *[DL12, Proposition 2.3] In the low regularity regime if $s_1, s_2 < 1$, and $s_1 + s_2 > 0$, the following product estimate holds true*

$$\|\phi\psi\|_{H^{s_1+s_2-1}(\mathbb{R}^2)} \leqslant C\|\phi\|_{H^{s_1}(\mathbb{R}^2)}\|\psi\|_{H^{s_2}(\mathbb{R}^2)}. \tag{3.39}$$

(4) *[BCD11, Corollary 2.86] For any $s > 0$, the following product estimate holds true*

$$\|\phi\psi\|_{H^s(\mathbb{R}^n)} \leqslant C\big(\|\phi\|_{L^\infty(\mathbb{R}^n)}\|\psi\|_{H^s(\mathbb{R}^n)} + \|\phi\|_{H^s(\mathbb{R}^n)}\|\psi\|_{L^\infty(\mathbb{R}^n)}\big).$$

(5) *[BCD11, Theorem 2.87 & Theorem 2.89] For any $s > 0$ and $g \in C^{k+1}$ with $k = [s] \in \mathbb{N}$, the following composition estimate holds true*

$$\|\nabla(g\circ\theta)\|_{H^{s-1}(\mathbb{R}^n)} \leqslant C(g, \|\theta\|_{L^\infty(\mathbb{R}^n)})\|\nabla\theta\|_{H^{s-1}(\mathbb{R}^n)}. \tag{3.40}$$

If $g \in C_b^{k+1}$ with $k = [s] \in \mathbb{N}$, then the above estimate can be improved in the spacial dimension two as follows

$$\|\nabla(g\circ\theta)\|_{H^{s-1}(\mathbb{R}^2)} \leqslant C(\|g\|_{C^{k+1}}, \|\theta\|_{H^1(\mathbb{R}^2)})\|\nabla\theta\|_{H^{s-1}(\mathbb{R}^2)}. \tag{3.41}$$

The commutator estimate (3.37) will present its power in the low regularity regime, and the classical commutator estimate (3.38) will help in the high regularity regime (see Subsection 4 below).

The composition estimate (3.41) will help to bound the diffusion coefficients κ, μ in terms of θ in the low regularity regime, where only $H^1(\mathbb{R}^2)$-norm (instead of L^∞_x-norm) of θ is available, which will be used in Subsection 3.2.3 and Subsection 3.2.4 intensively.

We derive in this paragraph a priori H^s, $s \in (0,2)$-Estimates for a general linear parabolic equation, which should be of independent interest.

Lemma 3.2.2. *Let $\psi = \psi(t,x) : [0,\infty) \times \mathbb{R}^2 \mapsto \mathbb{R}^m$, $m \geqslant 1$ be a smooth solution with sufficiently decay of the following linear parabolic equation*

$$\begin{cases} \partial_t\psi + u\cdot\nabla_x\psi - \mathrm{div}_x(\kappa\nabla_x\psi) = f, \\ \psi|_{t=0} = \psi_0, \end{cases} \tag{3.42}$$

where

- $u = u(t,x) : \mathbb{R}^+ \times \mathbb{R}^2 \mapsto \mathbb{R}^2$ *is a given divergence-free vector field:* $div_x u = 0$*;*
- $\kappa = \kappa(t,x) : \mathbb{R}^+ \times \mathbb{R}^2 \to [\kappa_*, \kappa^*]$ *with* $\kappa_*, \kappa^* \in (0,\infty)$*;*
- $f = f(t,x) : \mathbb{R}^+ \times \mathbb{R}^2 \mapsto \mathbb{R}^m$ *denotes the given external force.*

Then the following a priori H^s_x-Estimates for (3.42) holds true:

$$\begin{aligned}&\|\psi\|^2_{L^\infty_T H^s_x} + \|\nabla\psi\|^2_{L^2_T H^s_x} \leqslant C(\kappa_*)\Big(\|\psi_0\|^2_{H^s_x} + \|f\|^2_{L^2_T H^{s-1}_x}\Big)\times\\ &\times \exp\Big(C(\kappa_*, s, \nu)(\|\nabla u\|^2_{L^2_T L^2_x} + \|\nabla\kappa\|^{2/\nu}_{L^{\frac{2}{\nu}}_T H^\nu_x} + \|f\|_{L^1_T H^{-s}_x})\Big)\end{aligned} \tag{3.43}$$

for any $s \in (0,2)$ and $\nu \in (s-1, 1) \subset (-1,1)$.

Proof. It is straightforward to derive the following L^2_x-Estimate by simply taking the $L^2(\mathbb{R}^2)$ inner product of the equation (3.42) and ψ itself

$$\|\psi\|^2_{L^\infty_T L^2_x} + \|\nabla\psi\|^2_{L^2_T L^2_x} \leqslant C(\kappa_*)\Big(\|\psi_0\|^2_{L^2_x} + \int_0^T \langle\psi, f\rangle_{H^s_x, H^{-s}_x}\, dt\Big), \quad \forall s \in \mathbb{R}. \tag{3.44}$$

We next consider the a priori estimates for the $H^s(\mathbb{R}^2)$-norm. By virtue of the description (3.35) of the $H^s(\mathbb{R}^2)$-norm, we consider the dyadic piece of ψ:

$$\psi_j := \Delta_j\psi, \quad j \geqslant 0. \tag{3.45}$$

where the operator Δ_j is defined in (3.33). We apply Δ_j to the linear ψ-equation to derive the equation for ψ_j:

$$\partial_t\psi_j + u\cdot\nabla\psi_j - \mathrm{div}(\kappa\nabla\psi_j) = [u, \Delta_j]\cdot\nabla\psi - \mathrm{div}\,([\kappa,\Delta_j]\nabla\psi) + f_j, \quad j \geqslant 0. \tag{3.46}$$

We take the L^2 inner product of the equation (3.46) and ψ_j and make use of $\mathrm{div}\, u = 0$ and $\kappa \geqslant \kappa_*$ to derive

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\psi_j\|^2_{L^2_x} + \kappa_*\|\nabla\psi_j\|^2_{L^2_x} &\leqslant \|\psi_j\|_{L^2_x}\|[u,\Delta_j]\cdot\nabla\psi\|_{L^2_x}\\ &+ \|\nabla\psi_j\|_{L^2_x}\|[\kappa,\Delta_j]\nabla\psi\|_{L^2_x} + \|f_j\|_{L^2_x}\|\psi_j\|_{L^2_x},\ j \geqslant 0.\end{aligned}$$

By use of Bernstein's inequality (3.34) we have

$$\begin{aligned}&\frac{d}{dt}\|\psi_j\|^2_{L^2_x} + 2^{2j}\|\psi_j\|^2_{L^2_x}\\ \leqslant& C(\kappa_*)\|\psi_j\|_{L^2_x}\left(\|[u,\Delta_j]\cdot\nabla\psi\|_{L^2_x} + 2^j\|[\kappa,\Delta_j]\nabla\psi\|_{L^2_x} + \|f_j\|_{L^2_x}\right),\end{aligned}$$

that is,

$$\begin{aligned}&\frac{d}{dt}\|\psi_j\|_{L^2_x}+2^{2j}\|\psi_j\|_{L^2_x}\\ \leqslant &C(\kappa_*)\left(\|[u,\Delta_j]\cdot\nabla\psi\|_{L^2_x}+2^j\|[\kappa,\Delta_j]\nabla\psi\|_{L^2_x}+\|f_j\|_{L^2_x}\right),\quad j\geqslant 0.\end{aligned}\tag{3.47}$$

We make use of the commutator estimate (3.37) in Lemma 3.2.1 to estimate the commutators $\|[u,\Delta_j]\cdot\nabla\psi\|_{L^2_x}$ and $2^j\|[\kappa,\Delta_j]\nabla\psi_j\|_{L^2_x}$ in the above inequality in the following way. Let $(l_j)_{j\geqslant 0}$ be a normalised sequence in $\ell^1(\mathbb{N})$ such that $l_j\geqslant 0$ and $\sum_{j\geqslant 0}l_j=1$. Then we have

$$\begin{aligned}&\|[u,\Delta_j]\nabla\psi\|_{L^2}\leqslant C(s)l_j2^{j(1-s)}\|\nabla u\|_{L^2_x}\|\nabla\psi\|_{H^{s-1}_x},\ \text{for } s\in(0,2),\\ &2^j\|[\kappa,\Delta_j]\nabla\psi\|_{L^2_x}\leqslant C(s,\nu)l_j2^{j(1-s)}\|\nabla\kappa\|_{H^\nu_x}\|\nabla\psi\|_{H^{s-\nu}_x}\\ &\qquad\qquad\qquad\qquad\text{for } \nu\in(-1,1),\ s\in(-1,\nu+1).\end{aligned}\tag{3.48}$$

Therefore we have

$$\begin{aligned}&\frac{d}{dt}\|\psi_j\|_{L^2_x}+2^{2j}\|\psi_j\|_{L^2_x}\\ &\leqslant C(\kappa_*,s,\nu)l_j2^{j(1-s)}\left(\|\nabla u\|_{L^2_x}\|\nabla\psi\|_{H^{s-1}_x}+\|\nabla\kappa\|_{H^\nu_x}\|\nabla\psi\|_{H^{s-\nu}_x}\right)+C(\kappa_*)\|f_j\|_{L^2_x}\\ &\qquad\qquad\qquad\text{for } \nu\in(-1,1),\ s\in(0,\nu+1),\ j\geqslant 0.\end{aligned}$$

We use Duhamel's Principle to derive

$$\begin{aligned}\|\psi_j\|_{L^2_x}&\leqslant e^{-t2^{2j}}\|(\psi_0)_j\|_{L^2_x}+C(\kappa_*)\int_0^t e^{-(t-\tau)2^{2j}}\|f_j(\tau)\|_{L^2_x}\,d\tau\\ &+C(\kappa_*,s,\nu)2^{j(1-s)}l_j\int_0^t e^{-(t-\tau)2^{2j}}\left(\|\nabla u(\tau)\|_{L^2_x}\|\nabla\psi(\tau)\|_{H^{s-1}_x}\right.\\ &\qquad\left.+\|\nabla\kappa(\tau)\|_{H^\nu_x}\|\nabla\psi(\tau)\|_{H^{s-\nu}_x}\right)d\tau,\quad j\geqslant 0.\end{aligned}\tag{3.49}$$

We multiply the inequality (3.49) by 2^{js} to derive

$$\begin{aligned}2^{js}\|\psi_j\|_{L^2_x}&\leqslant 2^{js}e^{-t2^{2j}}\|(\psi_0)_j\|_{L^2_x}+C(\kappa_*)2^{js}\int_0^t e^{-(t-\tau)2^{2j}}\|f_j\|_{L^2_x}\,d\tau\\ &+C(\kappa_*,s,\nu)2^jl_j\int_0^t e^{-(t-\tau)2^{2j}}\left(\|\nabla u(\tau)\|_{L^2_x}\|\nabla\psi(\tau)\|_{H^{s-1}_x}\right.\\ &\qquad\left.+\|\nabla\kappa(\tau)\|_{H^\nu_x}\|\nabla\psi(\tau)\|_{H^{s-\nu}_x}\right)d\tau,\quad j\geqslant 0.\end{aligned}\tag{3.50}$$

We take $L^\infty([0,T])$-norm in t of (3.50) and the $L^2([0,T])$-norm in t of $2^j\cdot$(3.50), to derive by use of Young's inequality that

$$\begin{aligned}&2^{js}\|\psi_j\|_{L^\infty_TL^2_x}+2^{j(s+1)}\|\psi_j\|_{L^2_TL^2_x}\leqslant 2^{js}\|(\psi_0)_j\|_{L^2_x}+C(\kappa_*)2^{j(s-1)}\|f_j\|_{L^2_TL^2_x}\\ &\quad+C(\kappa_*,s,\nu)l_j\Big\|\|\nabla u\|_{L^2_x}\|\nabla\psi\|_{H^{s-1}_x}+\|\nabla\kappa\|_{H^\nu_x}\|\nabla\psi\|_{H^{s-\nu}_x}\Big\|_{L^2_T}.\end{aligned}\tag{3.51}$$

We take square of (3.51) and sum them up for $j \in \mathbb{N}$ to derive

$$\begin{aligned}
&\sum_{j\geqslant 0}\Big(2^{2js}\|\psi_j\|^2_{L^\infty_T L^2_x} + 2^{2j(s+1)}\|\psi_j\|^2_{L^2_T L^2_x}\Big) \\
&\lesssim_{\kappa_*,s,\nu} \sum_{j\geqslant 0}\Big(2^{2js}\|(\psi_0)_j\|^2_{L^2_x} + 2^{2j(s-1)}\|f_j\|^2_{L^2_T L^2_x}\Big) \\
&+ \int_0^T \|\nabla u\|^2_{L^2_x}\|\nabla\psi\|^2_{H^{s-1}_x} + \|\nabla\kappa\|^2_{H^\nu_x}\|\nabla\psi\|^2_{H^{s-\nu}_x} dt, \quad j \geqslant 0,
\end{aligned}$$

that is, by virtue of the L^2-estimate (3.44),

$$\begin{aligned}
\|\psi\|^2_{L^\infty_T H^s_x} + \|\nabla\psi\|^2_{L^2_T H^s_x} &\lesssim_{\kappa_*,s,\nu} \Big(\|\psi_0\|^2_{H^s_x} + \|f\|^2_{L^2_T H^{s-1}_x} + \int_0^T \|\psi\|_{H^s_x}\|f\|_{H^{-s}_x} dt \\
&+ \int_0^T \|\nabla u\|^2_{L^2_x}\|\nabla\psi\|^2_{H^{s-1}_x} + \|\nabla\kappa\|^2_{H^\nu_x}\|\nabla\psi\|^2_{H^{s-\nu}_x} dt\Big).
\end{aligned} \tag{3.52}$$

We next consider the norm $\|\nabla\psi\|_{H^{s-\nu}_x}$. By the interpolation inequality (3.36) we have

$$\|\nabla\psi\|_{H^{s-\nu}_x} \leqslant C\|\nabla\psi\|^\nu_{H^{s-1}_x}\|\nabla\psi\|^{1-\nu}_{H^s_x}, \quad \nu \in (0,1),$$

which implies by Young's inequality that

$$\begin{aligned}
\int_0^T \|\nabla\kappa\|^2_{H^\nu_x}\|\nabla\psi\|^2_{H^{s-\nu}_x} dt &\leqslant \int_0^T \|\nabla\kappa\|^2_{H^\nu_x}\|\nabla\psi\|^{2\nu}_{H^{s-1}_x}\|\nabla\psi\|^{2(1-\nu)}_{H^s_x} dt \\
&\leqslant \varepsilon\|\nabla\psi\|^2_{L^2_T H^s_x} + C_\varepsilon \int_0^T \|\nabla\kappa\|^{2/\nu}_{H^\nu_x}\|\nabla\psi\|^2_{H^{s-1}_x}\, dt.
\end{aligned}$$

To conclude, by taking ε small enough and Gronwall's inequality, we derive the H^s-Estimate (3.43). □

3.2.3 Energy estimates & Uniqueness of the weak solutions

We first introduce a scalar function η, which is given in terms of the temperature function as follows (recalling $\kappa = a(\theta) \in C^1_b(\mathbb{R}; [\kappa_*, \kappa^*])$)

$$\eta = A(\theta), \text{ with } A(z) := \int_0^z a(\alpha)\, d\alpha \text{ the primitive function of } a. \tag{3.53}$$

As $A'(\theta) = a(\theta) \geqslant \kappa_* > 0$, the function A is invertible and we can write

$$\theta = A^{-1}(\eta), \tag{3.54}$$

where $(A^{-1})'(\eta) = \frac{1}{a(A^{-1}(\eta))} \leqslant \frac{1}{\kappa^*}$. We have the following equivalence relations [3]

$$
\begin{aligned}
&\kappa_* \|\theta\|_{L^2_x} \leqslant \|\eta\|_{L^2_x} \leqslant \kappa^* \|\theta\|_{L^2_x},\\
&\kappa_* \|\nabla\theta\|_{L^2_x} \leqslant \|\nabla\eta\|_{L^2_x} = \|a(\theta)\nabla\theta\|_{L^2_x} \leqslant \kappa^* \|\nabla\theta\|_{L^2_x},\\
&\kappa_* \|\partial_t\theta\|_{L^2_x} \leqslant \|\partial_t\eta\|_{L^2_x} = \|a(\theta)\partial_t\theta\|_{L^2_x} \leqslant \kappa^* \|\partial_t\theta\|_{L^2_x},\\
&\|\nabla^2\eta\|_{L^2_x} \leqslant \|a\|_{\mathrm{Lip}} \|\nabla\theta\|^2_{L^4_x} + \kappa^* \|\nabla^2\theta\|_{L^2_x} \leqslant (C\|a\|_{\mathrm{Lip}} \|\nabla\theta\|_{L^2_x} + \kappa^*)\|\nabla^2\theta\|_{L^2_x},\\
&\|\nabla^2\theta\|_{L^2_x} \leqslant \frac{\|a\|_{\mathrm{Lip}}}{\kappa_*^3} \|\nabla\eta\|^2_{L^4_x} + \frac{1}{\kappa_*}\|\nabla^2\eta\|_{L^2_x} \leqslant (C\frac{\|a\|_{\mathrm{Lip}}}{\kappa_*^3}\|\nabla\eta\|_{L^2_x} + \frac{1}{\kappa_*})\|\nabla^2\eta\|_{L^2_x}.
\end{aligned}
\tag{3.55}
$$

That is,

$$\theta(t,\cdot) \in H^k_x(\mathbb{R}^2) \Leftrightarrow \eta(t,\cdot) \in H^k_x(\mathbb{R}^2), \quad k = 0,1,2. \tag{3.56}$$

Let $(\theta, u) \in C([0,\infty); (L^2(\mathbb{R}^2))^3) \cap L^2_{\mathrm{loc}}([0,\infty); (H^1(\mathbb{R}^2))^3)$ be a weak solution of the Cauchy problem (3.24) in the sense of Definition 3.1.1 with

$$\partial_t\theta + u\cdot\nabla\theta - \mathrm{div}\,(\kappa\nabla\theta) = 0 \text{ holding in } L^2_{\mathrm{loc}}([0,\infty); H^{-1}_x(\mathbb{R}^2)). \tag{3.57}$$

Since $Y := L^\infty_{t,x}([0,\infty)\times\mathbb{R}^2) \cap L^2_{\mathrm{loc}}([0,\infty); H^1_x(\mathbb{R}^2))$ is an algebra (in the sense that the product of any two elements in Y still belongs to Y), we can multiply the above θ-equation by $\kappa = a(\theta)$ (with $a(\theta) - a(0) \in Y$), to arrive at the parabolic equation for $\eta = A(\theta) \in C([0,\infty); L^2(\mathbb{R}^2)) \cap L^2_{\mathrm{loc}}([0,\infty); H^1(\mathbb{R}^2))$:

$$\partial_t\eta + u\cdot\nabla\eta - \kappa\Delta\eta = 0 \text{ holding in the dual space } Y'. \tag{3.58}$$

We are going to derive the H^1-Estimate for η (and hence for θ [4]) as well as the L^2-Estimate for u first. Then we will show the uniqueness result of the weak solutions by considering the difference of two possible weak solutions in $H^{1+\delta}(\mathbb{R}^2) \times (H^\delta(\mathbb{R}^2))^2 \subset H^1(\mathbb{R}^2) \times (L^2(\mathbb{R}^2))^2$ with $-1 < \delta < 0$.

$H^1 \times L^2$-Estimate for (θ, u)

By virtue of the energy equalities (3.8) and (3.9) and the derivation of the uniform estimates (3.29) and (3.30), we have the L^2-Estimate

$$\|\theta\|^2_{L^\infty_T L^2_x} + \|\nabla\theta\|^2_{L^2_T L^2_x} \leqslant C(\kappa_*)\|\theta_0\|^2_{L^2}, \tag{3.59}$$

---

[3] We can easily compute

$$
\begin{aligned}
&\nabla\eta = a(\theta)\nabla\theta, \quad \nabla\theta = \frac{1}{a(A^{-1}(\eta))}\nabla\eta,\\
&\nabla^2\eta = a'(\theta)\nabla\theta\otimes\nabla\theta + a(\theta)\nabla^2\theta, \quad \nabla^2\theta = -\frac{a'(A^{-1}(\eta))}{a^3(A^{-1}(\eta))}\nabla\eta\otimes\nabla\eta + \frac{1}{a(A^{-1}(\eta))}\nabla^2\eta.
\end{aligned}
$$

[4] The introduction of the η-function makes the derivation of the H^1-Estimate for θ straightforward (and possible).

and the L^2-Estimate (3.12) for u. By Gagliardo-Nirenberg's inequality (3.25) it holds

$$\|u\|_{L^4_T L^4_x} \leqslant C(\mu_*)\big(\sqrt{T}\|\theta_0\|_{L^2} + \|u_0\|_{L^2}\big). \tag{3.60}$$

We assume a priori that the function η is smooth and decay sufficiently fast at infinity. We test the η-equation (3.58) by $\Delta\eta$ to derive by integration by parts that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\nabla\eta|^2 \,\mathrm{d}x + \int_{\mathbb{R}^2} \kappa|\Delta\eta|^2 \,\mathrm{d}x = \int_{\mathbb{R}^2} u\cdot\nabla\eta\Delta\eta \,\mathrm{d}x \leqslant \|u\|_{L^4_x}\|\nabla\eta\|_{L^4_x}\|\Delta\eta\|_{L^2_x}.$$

By Gagliardo-Nirenberg's inequality (3.25), the equivalence $\|\Delta\eta\|_{L^2_x(\mathbb{R}^2)} \sim \|\nabla^2\eta\|_{L^2_x(\mathbb{R}^2)}$ and Young's inequality we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\nabla\eta|^2 \,\mathrm{d}x + \frac{\kappa_*}{2}\int_{\mathbb{R}^2} |\Delta\eta|^2 \,\mathrm{d}x \leqslant C(\kappa_*)\|u\|^4_{L^4_x}\|\nabla\eta\|^2_{L^2_x}.$$

Gronwall's inequality gives

$$\|\nabla\eta(T)\|^2_{L^2_x} + \|\nabla^2\eta\|^2_{L^2_T L^2_x} \leqslant C(\kappa_*)\|\nabla\eta_0\|^2_{L^2_x}\exp(C(\kappa_*)\|u\|^4_{L^4_T L^4_x})$$

for any positive time $T > 0$. Thus by the η-equation

$$\begin{aligned}\|\partial_t\eta\|_{L^2_T L^2_x} &= \|u\cdot\nabla\eta - \kappa\Delta\eta\|_{L^2_T L^2_x} \leqslant \|u\|_{L^4_T L^4_x}\|\nabla\eta\|_{L^4_T L^4_x} + \kappa^*\|\Delta\eta\|_{L^2_T L^2_x} \\ &\leqslant C(\kappa_*,\kappa^*)\|\nabla\eta_0\|_{L^2_x}\exp(C(\kappa_*)\|u\|^4_{L^4_T L^4_x}).\end{aligned}$$

By virtue of the equivalence relation (3.55):

$$\|\nabla\theta\|^2_{L^\infty_T L^2_x} + \|\nabla^2\theta\|^2_{L^2_T L^2_x} \leqslant C(\kappa_*,\|a\|_{\mathrm{Lip}})(\|\nabla\eta\|^2_{L^\infty_T L^2_x} + (1+\|\nabla\eta\|^2_{L^\infty_T L^2_x})\|\nabla^2\eta\|^2_{L^2_T L^2_x})$$

and (3.59)-(3.60), we have the a priori H^1-Estimate (3.13) for θ:

$$\begin{aligned}&\|\theta\|^2_{L^\infty_T H^1_x} + \|\nabla\theta\|^2_{L^2_T H^1_x} + \|\partial_t\theta\|^2_{L^2_T L^2_x} \\ &\leqslant C(\kappa_*,\|a\|_{\mathrm{Lip}},\kappa^*)\|\theta_0\|^2_{H^1}(1+\|\nabla\theta_0\|^2_{L^2})\exp(C(\kappa_*)\|u\|^4_{L^4_T L^4_x}).\end{aligned} \tag{3.61}$$

Therefore both the parabolic equations (3.57) and (3.58) for θ and η hold in $L^2_{\mathrm{loc}}([0,\infty); L^2(\mathbb{R}^2))$. A standard density argument ensures the H^1-Estimate (3.13) for θ, and hence $\theta \in C([0,\infty); H^1_x(\mathbb{R}^2))$.

Proof of uniqueness

Let (θ_1, u_1, Π_1) and (θ_2, u_2, Π_2) be two weak solutions of the Cauchy problem (3.24) with the same initial data $(\theta_0, u_0) \in H^1(\mathbb{R}^2) \times (L^2(\mathbb{R}^2))^2$, which satisfy the energy estimates (3.12)-(3.13). Recall (3.53) for the definition of the function A, and we set

$$\eta_1 = A(\theta_1), \quad \eta_2 = A(\theta_2).$$

We consider the difference

$$(\dot{\eta}, \dot{u}, \nabla\dot{\Pi}) = (\eta_1 - \eta_2, u_1 - u_2, \nabla\Pi_1 - \nabla\Pi_2),$$

which lies in

$$\begin{aligned}&\big(C([0,\infty); H^1(\mathbb{R}^2)) \cap L^2_{\mathrm{loc}}([0,\infty); H^2(\mathbb{R}^2))\big)\\ &\times \big(C([0,\infty); (L^2(\mathbb{R}^2))^2) \cap L^2_{\mathrm{loc}}([0,\infty); (H^1(\mathbb{R}^2))^2)\big)\\ &\times L^2_{\mathrm{loc}}([0,\infty); (H^{-1}(\mathbb{R}^2))^2)\big).\end{aligned}$$

The goal of this paragrah is to derive the $H^{\delta+1} \times H^{\delta}$-Estimate for $(\dot{\eta}, \dot{u})$ with $-1 < \delta < 0$ to show $(\dot{\eta}, \dot{u}) = (0,0)$.

Notice that for a divergence free fled v we have $\mathrm{div}\,(u \otimes v) = v \cdot \nabla u$, where $u \otimes v = (u_i v_j)_{1 \leqslant i,j \leqslant 2}$. In this paragraph, we write $\mathrm{div}\,(u \otimes u)$ instead of $u \cdot \nabla u$. Then $(\dot{\eta}, \dot{u}, \nabla\dot{\Pi})$ satisfies the following Cauchy problem

$$\begin{cases} \partial_t \dot{\eta} + u_1 \cdot \nabla\dot{\eta} - \kappa_1 \Delta\dot{\eta} = \dot{\kappa}\Delta\eta_2 - \dot{u} \cdot \nabla\eta_2, \\ \partial_t \dot{u} + \mathrm{div}\,(\dot{u} \otimes u_1) - \mathrm{div}(\mu_1 S\dot{u}) + \nabla\dot{\Pi} = \dot{\theta}\vec{e_2} - \mathrm{div}\,(u_2 \otimes \dot{u}) + \mathrm{div}(\dot{\mu} S u_2), \\ \mathrm{div}\,\dot{u} = 0, \\ (\dot{\eta}_0, \dot{u}_0) = (0,0), \end{cases} \tag{3.62}$$

where

$$\kappa_1 = a(\theta_1),\ \mu_1 = b(\theta_1),\ \dot{\theta} = \theta_1 - \theta_2,\ \dot{\kappa} = a(\theta_1) - a(\theta_2),\ \dot{\mu} = b(\theta_1) - b(\theta_2).$$

Similarly as in (3.55) we have the following equivalence relationships

$$\begin{aligned} &\kappa_* \|\dot{\theta}\|_{L^2_x} \leqslant \|\dot{\eta}\|_{L^2_x} \leqslant \kappa^* \|\dot{\theta}\|_{L^2_x}, \\ &\|\nabla\dot{\eta}\|_{L^2_x} \leqslant \|a\|_{\mathrm{Lip}} \|\nabla\theta_1\|_{L^4_x} \|\dot{\theta}\|_{L^4_x} + \kappa^* \|\nabla\dot{\theta}\|_{L^2_x}, \\ &\|\nabla\dot{\theta}\|_{L^2_x} \leqslant \frac{\|a\|_{\mathrm{Lip}}}{\kappa_*^3} \|\nabla\eta_1\|_{L^4_x} \|\dot{\eta}\|_{L^4_x} + \frac{1}{\kappa_*} \|\nabla\dot{\eta}\|_{L^2_x}. \end{aligned} \tag{3.63}$$

Moreover, we have the equivalence estimate of $\dot{\theta}$ and $\dot{\eta}$ in the Sobolev space $H^{\delta+1}(\mathbb{R}^2)$, $-1 < \delta < 0$. We first use the commutator estimate (3.37) to derive the following product inequality with $-1 < s < 1$, $p, q \in [1, \infty]$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$

$$\begin{aligned} &\|\varphi\psi\|_{H^s(\mathbb{R}^2)} \leqslant \big\| \big(2^{js} \|\Delta_j(\varphi\psi)\|_{L^2(\mathbb{R}^n)}\big)_{j \geqslant -1} \big\|_{l^2} \\ \leqslant &\big\| \big(2^{js} \|\varphi\Delta_j\psi\|_{L^2(\mathbb{R}^2)}\big)_{j \geqslant -1} \big\|_{l^2} + \big\| \big(2^{js} \|[\varphi, \Delta_j]\psi\|_{L^2(\mathbb{R}^2)}\big)_{j \geqslant -1} \big\|_{l^2} \\ \lesssim &\|\varphi\|_{L^p(\mathbb{R}^2)} \|\psi\|_{B^s_{q,2}(\mathbb{R}^2)} + \|\nabla\varphi\|_{L^2(\mathbb{R}^2)} \|\psi\|_{H^s(\mathbb{R}^2)}. \end{aligned} \tag{3.64}$$

Now we are ready to estimate $\dot{\theta}$ and $\dot{\eta}$. We recall the definition of η as in (3.53) and write

$$\dot{\eta} = \dot{\theta} \int_0^1 a(\theta_2 + \tau\dot{\theta})\, d\tau \quad \text{and} \quad \dot{\theta} = \dot{\eta} \int_0^1 \frac{1}{(a \circ A^{-1})(\eta_2 + \tau\dot{\eta})}\, d\tau. \tag{3.65}$$

By using the product estimate (3.64) with $(p, q) = (\infty, 2)$, we have

$$C^{-1}\|\dot{\theta}\|_{H^s(\mathbb{R}^2)} \leqslant \|\dot{\eta}\|_{H^s(\mathbb{R}^2)} \leqslant C\|\dot{\theta}\|_{H^s(\mathbb{R}^2)}, \quad -1 < s < 1, \tag{3.66}$$

where $C = C(\kappa^*, \kappa_*, \|a\|_{\mathrm{Lip}}, \|(\theta_1, \theta_2)\|_{H^1(\mathbb{R}^2)})$. Correspondingly by using the composition estimate (3.41), we have

$$\begin{aligned} \|(\dot{\kappa}, \dot{\mu})\|_{L^2_x} &\leqslant C(\|(a,b)\|_{\mathrm{Lip}}, \kappa_*)\|\dot{\eta}\|_{L^2_x}, \\ \|\nabla(\dot{\kappa}, \dot{\mu})\|_{H^{s-1}_x} &\leqslant C(\kappa_*, \kappa^*, \|(a,b)\|_{\mathrm{Lip}}, \|(\theta_1, \theta_2)\|_{H^1(\mathbb{R}^2)})\|\nabla\dot{\eta}\|_{H^{s-1}_x}. \end{aligned} \tag{3.67}$$

We are going to follow exactly the procedure in Subsection 3.2.2 to derive the $H^{\delta+1} \times H^{\delta}$-Estimate for $(\dot{\eta}, \dot{u})$ with $-1 < \delta < 0$.

(i) **$H^{\delta+1}$−estimate of $\dot{\eta}$, $-1 < \delta < 0$.**
Similarly as (3.47), we have the following preliminary estimate for $\dot{\eta}_j = \Delta_j \dot{\eta}$ with $j \geqslant 0$:

$$\begin{aligned} \frac{d}{dt}\|\eta_j\|^2_{L^2_x} + 2^{2j}\|\eta_j\|^2_{L^2_x} \lesssim_{\kappa_*} \|\dot{\eta}_j\|_{L^2_x}\big(&\|\Delta_j(\dot{\kappa}\Delta\eta_2)\|_{L^2_x} + \|\Delta_j(\dot{u}\cdot\nabla\eta_2)\|_{L^2_x} \\ &+ \|[u_1, \Delta_j]\nabla\dot{\eta}\|_{L^2_x} + \|[\kappa_1, \Delta_j]\Delta\dot{\eta}\|_{L^2_x}\big) + \int_{\mathbb{R}^2} |\dot{\eta}_j \nabla\kappa_1 \cdot \nabla\dot{\eta}_j|\, dx. \end{aligned} \tag{3.68}$$

By using Gagliardo-Nirenberg's inequality (3.25) and Young's inequality, we have

$$\begin{aligned} \int_{\mathbb{R}^2} |\dot{\eta}_j \nabla\kappa_1 \cdot \nabla\dot{\eta}_j|\, dx \leqslant& \|\nabla\dot{\eta}_j\|_{L^2_x}\|\dot{\eta}_j\|_{L^4_x}\|\nabla\kappa_1\|_{L^4_x} \\ \leqslant& C\|\nabla\dot{\eta}_j\|^{\frac{3}{2}}_{L^2_x}\|\dot{\eta}_j\|^{\frac{1}{2}}_{L^2_x}\|\nabla\kappa_1\|_{L^4_x} \\ \leqslant& \varepsilon\|\nabla\dot{\eta}_j\|^2_{L^2_x} + C(\varepsilon)\|\nabla\dot{\eta}_j\|_{L^2_x}\|\dot{\eta}_j\|_{L^2_x}\|\nabla\kappa_1\|^2_{L^4_x}, \end{aligned}$$

where $\varepsilon > 0$ is a sufficient small constant. By using the commutator estimate (3.37) we have the following estimate with $\delta \in (-1, 0)$

$$\begin{aligned} \|[u_1, \Delta_j]\nabla\dot{\eta}\|_{L^2_x} &\leqslant C l_j 2^{-j\delta}\|\nabla u_1\|_{L^2_x}\|\nabla\dot{\eta}\|_{H^{\delta}_x}, \\ \|[\kappa_1, \Delta_j]\Delta\dot{\eta}\|_{L^2_x} &\leqslant C l_j 2^{-j\delta}\|\nabla\kappa_1\|_{H^{\frac{1}{2}}_x}\|\Delta\dot{\eta}\|_{H^{\delta-\frac{1}{2}}_x}. \end{aligned}$$

We follow a similar argument as (3.49)-(3.52), and similarly bound $\|\Delta_{-1}\dot\eta\|_{L^\infty_T L^2_x} + \|\Delta_{-1}\nabla\dot\eta\|_{L^2_T L^2_x}$ to arrive at

$$\begin{aligned}
&\|\dot\eta\|^2_{L^\infty_T H^{\delta+1}_x} + \|\nabla\dot\eta\|^2_{L^2_T H^{\delta+2}_x} \\
&\leqslant C(\kappa_*)\int_0^T \|\eta_j\|^2_{H^\delta_x}\|\nabla\kappa_1\|^4_{L^4_x} + \|\dot\kappa\Delta\eta_2\|^2_{H^\delta_x} + \|\dot u\cdot\nabla\eta_2\|^2_{H^\delta_x}\,\mathrm{d}t \\
&+ C(k_*,\delta)\int_0^T \|\nabla u_1\|^2_{L^2_x}\|\nabla\dot\eta\|^2_{H^\delta_x} + \|\nabla\kappa_1\|^2_{H^{\frac12}_x}\|\Delta\dot\eta\|^2_{H^{\delta-\frac12}_x}\,\mathrm{d}t\,.
\end{aligned} \tag{3.69}$$

By interpolation inequality (3.36) we have

$$\begin{aligned}
\|\nabla\kappa_1\|^2_{H^{\frac12}_x}\|\Delta\dot\eta\|^2_{H^{\delta-\frac12}_x} &\leqslant C\|\nabla\kappa_1\|^2_{H^{\frac12}_x}\|\dot\eta\|_{H^{\delta+1}_x}\|\dot\eta\|_{H^{\delta+2}_x} \\
&\leqslant \varepsilon\|\nabla\dot\eta\|^2_{H^{\delta+1}_x} + C(\varepsilon)\|\dot\eta\|^2_{H^{\delta+1}_x}(\|\nabla\kappa_1\|^4_{H^{\frac12}_x}+1).
\end{aligned}$$

We use the product estimate (3.39) and the equivalence relation (3.66) to show

$$\|\dot\kappa\Delta\eta_2\|^2_{H^\delta_x} \leqslant C\|\dot\kappa\|^2_{H^{\delta+1}_x}\|\Delta\eta_2\|^2_{L^2_x} \leqslant C\|\dot\eta\|^2_{H^{\delta+1}_x}\|\Delta\eta_2\|^2_{L^2_x},$$

and the product estimate (3.64) with $(p,q)=(4,4)$ implies

$$\begin{aligned}
\|\dot u\cdot\nabla\eta_2\|^2_{H^\delta_x} &\lesssim \|\dot u\|^2_{B^\delta_{4,2}}\|\nabla\eta_2\|^2_{L^4_x} + \|\dot u\|^2_{H^\delta_x}\|\nabla\eta_2\|^2_{H^1_x} \\
&\lesssim \|\dot u\|_{H^\delta_x}\|\dot u\|_{H^{\delta+1}_x}\|\nabla\eta_2\|^2_{L^4_x} + \|\dot u\|^2_{H^\delta_x}\|\nabla\eta_2\|^2_{H^1_x} \\
&\leqslant \varepsilon\|\nabla\dot u\|^2_{H^\delta_x} + C(\varepsilon,\delta)\|\dot u\|^2_{H^\delta_x}(1+\|\nabla\eta_2\|^4_{L^4_x}+\|\nabla\eta_2\|^2_{H^1_x}).
\end{aligned}$$

Now we take ε small enough to arrive at

$$\begin{aligned}
&\|\dot\eta\|^2_{L^\infty_T H^{\delta+1}_x} + \|\nabla\dot\eta\|^2_{L^2_T H^{\delta+1}_x} \leqslant C(\kappa_*,\delta,\|a\|_{C^2},\|(\theta_1,\theta_2)\|_{L^\infty_T H^1_x}) \\
&\int_0^T \|\dot u\|^2_{H^\delta_x}\big(1+\|\nabla\eta_2\|^4_{L^4_x}+\|\nabla\eta_2\|^2_{H^1_x}\big)\,dt + \int_0^T \|\dot\eta\|^2_{H^{\delta+1}_x}\big(\|\nabla\eta_1\|^4_{L^4_x} \\
&+\|\nabla\eta_2\|^2_{H^1_x}+\|\nabla u_1\|^2_{L^2_x}+\|\nabla\eta_1\|^4_{H^{\frac12}_x}\big)\,dt + \varepsilon\int_0^T\|\nabla\dot u\|^2_{H^\delta_x}\,dt.
\end{aligned} \tag{3.70}$$

(ii) **$H^\delta-$estimate of $\dot u$, $-1<\delta<0$.**
Recall (3.28) for the definition of the Leray-Helmholtz projector $\mathbb{P}$ such that

$$\mathbb{P}u = u, \quad \mathbb{P}\nabla\Pi = 0.$$

We apply $\mathbb{P}$ to the velocity equation $(3.62)_2$ to arrive at

$$\partial_t\dot u + \mathbb{P}(u_1\cdot\nabla\dot u) - \mathbb{P}\,\mathrm{div}(\mu_1 S\dot u) = \mathbb{P}(\dot\theta\vec{e_2}) - \mathbb{P}(\dot u\cdot\nabla u_2) + \mathbb{P}\,\mathrm{div}(\dot\mu S u_2) \tag{3.71}$$

We apply Δ_j to the above equation (3.71) to arrive at the equation for $\dot{u}_j := \Delta_j \dot{u}$

$$\begin{aligned}&\partial_t \dot{u}_j + \mathbb{P} u_1 \cdot \nabla \dot{u}_j - \mathbb{P}\,\mathrm{div}(\mu_1 S \dot{u}_j) = \mathbb{P}\mathrm{div}\left([u_1, \Delta_j] \otimes \dot{u}\right)\\ &- \mathbb{P}\mathrm{div}\left([\mu_1, \Delta_j] S\dot{u}\right) + \mathbb{P}(\theta_j \vec{e_2}) - \mathbb{P}\mathrm{div}\left(\Delta_j(u_2 \otimes \dot{u})\right) + \mathbb{P}\,\mathrm{div}(\Delta_j(\dot{\mu} S u_2)).\end{aligned} \tag{3.72}$$

We take the $L^2(\mathbb{R}^2)$-inner product between (3.72) and the divergence-free dyadic piece $\dot{u}_j = \mathbb{P}\dot{u}_j$ to arrive at

$$\begin{aligned}&\frac{d}{dt}\|\dot{u}_j\|_{L^2_x} + 2^{2j}\|\dot{u}_j\|_{L^2_x} \lesssim_{\mu_*} \|\dot{u}_j\|_{L^2_x}\left(\|[u_1,\Delta_j]\cdot\nabla\dot{u}\|_{L^2_x} + \|\dot{\theta}_j\|_{L^2_x}\right)\\ &+ 2^j\|\dot{u}_j\|_{L^2_x}\left(\|\Delta_j(\dot{\mu}Su_2)\|_{L^2_x} + \|[\mu_1,\Delta_j]S\dot{u}\|_{L^2_x} + \|\Delta_j(u_2\otimes\dot{u})\|_{L^2_x}\right), \quad j \geqslant 0.\end{aligned} \tag{3.73}$$

By using the commutator estimate (3.37) we have the following estimate for $\delta \in (-1, 0)$

$$\begin{aligned}\|[u_1,\Delta_j]\cdot\nabla\dot{u}\|_{L^2_x} &\leqslant C l_j 2^{j(1-\delta)}\|\nabla u_1\|_{L^2}\|\dot{u}\|_{H^\delta},\\ 2^j\|[\mu_1,\Delta_j]S\dot{u}\|_{H^\delta_x} &\leqslant C l_j 2^{j(1-\delta)}\|\nabla\mu_1\|_{H^{\frac12}_x}\|\nabla\dot{u}\|_{H^{\delta-\frac12}_x}.\end{aligned}$$

We follow the steps (3.49)-(3.52), and similarly estimate $\|\Delta_{-1}\dot{u}\|_{L^\infty_T L^2_x} + \|\Delta_{-1}\nabla\dot{u}\|_{L^2_T L^2_x}$ to derive

$$\begin{aligned}\|\dot{u}\|^2_{L^\infty_T H^\delta_x} + \|\nabla\dot{u}\|^2_{L^2_T H^\delta_x} &\lesssim_{\mu_*,\delta} \int_0^T \|\dot{\theta}\|^2_{H^{\delta-1}_x} + \|\dot{\mu}Su_2\|^2_{H^\delta_x} + \|u_2\otimes\dot{u}\|^2_{H^\delta_x}\,dt\\ &+ \int_0^T \|\nabla u_1\|^2_{L^2}\|\dot{u}\|^2_{H^\delta} + \|\nabla\mu_1\|^2_{H^{\frac12}_x}\|\nabla\dot{u}\|^2_{H^{\delta-\frac12}_x}\,dt.\end{aligned}$$

We use the product estimate (3.39) and the equivalence relation (3.66) to show

$$\|\dot{\mu}Su_2\|_{H^\delta_x} \leqslant C\|\dot{\mu}\|_{H^{\delta+1}_x}\|Su_2\|_{L^2_x} \leqslant C\|\dot{\eta}\|_{H^{\delta+1}_x}\|\nabla u_2\|_{L^2_x},$$

and we use the product estimate (3.64) with $(p,q) = (4,4)$ to show

$$\begin{aligned}\|u_2\otimes\dot{u}\|^2_{H^\delta_x} &\leqslant C(\|u_2\|^2_{L^4_x}\|\dot{u}\|^2_{B^\delta_{4,2}} + \|u_2\|^2_{H^1_x}\|\dot{u}\|^2_{H^\delta_x})\\ &\leqslant \varepsilon\|\nabla\dot{u}\|^2_{H^\delta_x} + C(\varepsilon,\delta)\|\dot{u}\|^2_{H^\delta_x}(1 + \|u_2\|^4_{L^4_x} + \|u_2\|^2_{H^1_x}).\end{aligned}$$

We use the interpolation inequality (3.36) to derive

$$\begin{aligned}\|\nabla\mu_1\|^2_{H^{\frac12}_x}\|\nabla\dot{u}\|^2_{H^{\delta-\frac12}_x} &\leqslant C\|\nabla\eta_1\|^2_{H^{\frac12}_x}\|\dot{u}\|_{H^\delta_x}\|\nabla\dot{u}\|_{H^\delta_x}\\ &\leqslant \varepsilon\|\nabla\dot{u}\|^2_{H^\delta_x} + C(\varepsilon)\|\nabla\eta_1\|^4_{H^{\frac12}_x}\|\dot{u}\|^2_{H^\delta_x}.\end{aligned}$$

Now we take $\varepsilon > 0$ sufficient small to arrive at

$$\begin{aligned}
&\|\dot{u}\|^2_{L^\infty_T H^\delta_x} + \|\nabla \dot{u}\|^2_{L^2_T H^\delta_x} \leqslant C(\kappa_*, \delta, \|b\|_{C^2}, \|(\theta_1, \theta_2)\|_{L^\infty_T H^1_x}) \\
&\times \int_0^T \|\dot{\eta}\|^2_{H^{\delta+1}_x}(1 + \|\nabla u_2\|^2_{L^2_x}) + \|\dot{u}\|^2_{H^\delta_x}\big(1 + \|u_2\|^4_{L^4_x} \\
&+ \|\nabla u_1\|^2_{L^2_x} + \|u_2\|^2_{H^1_x} + \|\nabla \eta_1\|^4_{H^{\frac{1}{2}}_x}\big)\, \mathrm{d}t\,.
\end{aligned} \tag{3.74}$$

To conclude, we add the estimates and (3.70) and (3.74) and take ε sufficient small to derive

$$\begin{aligned}
&\|\dot{\eta}\|^2_{L^\infty_T H^{\delta+1}_x} + \|\dot{u}\|^2_{L^\infty_T H^\delta_x} + \|\nabla \dot{u}\|^2_{L^2_T H^\delta_x} + \|\nabla \dot{\eta}\|^2_{L^2_T H^{\delta+1}_x} \\
&\leqslant C(\delta, \|(a,b)\|_{\mathrm{Lip}}, \|(a',b')\|_{\mathrm{Lip}}, \kappa_*, \mu_*, \|(\theta_1, \theta_2)\|_{L^\infty_T H^1_x}) \\
&\quad \times \int_0^T B(t)(\|\dot{\eta}\|^2_{H^{\delta+1}_x} + \|\dot{u}\|^2_{H^\delta_x})\, dt,
\end{aligned}$$

where

$$B(t) = 1 + \|(u_1, u_2)\|^2_{H^1_x} + \|(\nabla \eta_1, \nabla \eta_2)\|^4_{L^4_x} + \|u_2\|^4_{L^4_x} + \|\Delta \eta_2\|^2_{L^2_x} + \|\nabla \eta_1\|^4_{H^{\frac{1}{2}}_x}.$$

By virtue the energy estimates (3.12) and (3.13) , we have $B(t) \in L^1_{\mathrm{loc}}([0, +\infty))$. In particular, we use the interpolation inequality (3.36) to verify that

$$\int_0^T \|\nabla \eta_1\|^4_{H^{\frac{1}{2}}_x}\, dt \leqslant \int_0^T \|\nabla \eta_1\|^2_{L^2_x} \|\Delta \eta_1\|^2_{L^2_x}\, dt < \infty.$$

At the last step, Gronwall's inequality implies then $\dot{\eta} = 0$ and $\dot{u} = 0$. The uniqueness of the weak solutions follows.

3.2.4 Propagation of the general H^s-regularities

In this Subsection, we are going to derive the precise H^s_x-estimates (3.16)-(3.23) in Remark 3.1.3 in the subsequent paragraphs:

- In Paragraph 1 the global-in-time $H^s_x(\mathbb{R}^2) \times (L^2_x(\mathbb{R}^2))^2$, $s \in (1,2)$ - regularities (i.e. (3.16)) will be established.
- In Paragraph 2 the global-in-time $H^1_x(\mathbb{R}^2) \times (H^s_x(\mathbb{R}^2))^2$, $s \in (0,2)$-regularities (i.e. (3.17)) will be established.
- In Paragraph 3 the global-in-time $H^2_x(\mathbb{R}^2) \times (H^{0+}_x(\mathbb{R}^2))^2$ or $H^{1+}_x(\mathbb{R}^2) \times (L^2_x(\mathbb{R}^2))^2$-regularities will be established.

- In Paragraph 4 the global-in-time $H^s_x(\mathbb{R}^2) \times (H^{s-2}_x(\mathbb{R}^2))^2$ (i.e. (3.20)-(3.21)) and $H^{s-1}_x \times (H^s_x(\mathbb{R}^2))^2$, $s > 2$-regularities (i.e. (3.22)-(3.23)) will be established respectively.

As far as the borderline estimates (3.16)-(3.23) are established, the global-in-time regularity (3.15) follows immediately.

1. **Case** $(\theta_0, u_0) \in H^s(\mathbb{R}^2) \times (L^2(\mathbb{R}^2))^2$**,** $1 < s < 2$

   In this paragraph we are going to prove the H^s-Estimates (3.16) for the unique solution (θ, u) of the Boussinesq equations (3.1) with the initial data $(\theta_0, u_0) \in H^s(\mathbb{R}^2) \times (L^2(\mathbb{R}^2))^2$, $s \in (1, 2)$, following exactly the procedure in Subsection 3.2.2. We will pay more attention on the "nonlinearities" in the equations such as $\kappa = a(\theta)$, $u \cdot \nabla u$ when using the commutator estimates and will sketch the proof.

   Similarly as (3.47), we have the following preliminary estimate for $\theta_j = \Delta_j \theta$ with $j \geqslant 0$:

$$\frac{d}{dt}\|\theta_j\|_{L^2_x} + 2^{2j}\|\theta_j\|_{L^2_x} \leqslant C(\kappa_*)\left(\|[u, \Delta_j] \cdot \nabla\theta\|_{L^2_x} + 2^j\|[\kappa, \Delta_j]\nabla\theta\|_{L^2_x}\right). \tag{3.75}$$

   By use of the commutator estimates (3.48) and the action estimate (3.41):

$$\|\nabla\kappa\|_{H^\nu} \leqslant C(\|a\|_{C^2}, \|\theta\|_{H^1})\|\nabla\theta\|_{H^\nu} \text{ for } \nu \in (0,1),$$

   we derive similar as (3.51)

$$\begin{aligned} 2^{2js}\|\theta_j\|^2_{L^\infty_T L^2_x} + 2^{2j(s+1)}\|\theta_j\|^2_{L^2_T L^2_x} &\leqslant 2^{2js}\|(\theta_0)_j\|^2_{L^2_x} \\ + C(\kappa_*, s, \nu, \|a\|_{C^2}, \|\theta\|_{L^\infty_T H^1_x})(l_j)^2 &\int_0^T \Big(\|\nabla u\|^2_{L^2_x}\|\nabla\theta\|^2_{H^{s-1}_x} \\ &+ \|\nabla\theta\|^2_{H^\nu_x}\|\nabla\theta\|^2_{H^{s-\nu}_x}\Big)dt, \quad 1 < s < \nu + 1 < 2. \end{aligned} \tag{3.76}$$

   By using the interpolation inequality (3.36), we have

$$\|\nabla\theta\|_{H^\nu_x}\|\nabla\theta\|_{H^{s-\nu}_x} \leqslant C\|\nabla\theta\|^{1-\nu}_{L^2_x}\|\nabla\theta\|^\nu_{H^1_x}\|\nabla\theta\|^\nu_{H^{s-1}_x}\|\nabla\theta\|^{1-\nu}_{H^s_x}, \quad 0 < \nu < 1.$$

   Recall the L^2-Estimate (3.59) for θ:

$$\|\theta\|^2_{L^\infty_T L^2_x} + \|\nabla\theta\|^2_{L^2_T L^2_x} \leqslant C(\kappa_*)\|\theta_0\|^2_{L^2_x}. \tag{3.77}$$

   Therefore by Young's inequality we arrive at

$$\begin{aligned} &\|\theta\|^2_{L^\infty_T H^s_x} + \|\nabla\theta\|^2_{L^2_T H^s_x} \leqslant C(\kappa_*)\|\theta_0\|^2_{H^s_x} \\ &+ C(\kappa_*, s, \nu, \|a\|_{C^2}, \|\theta\|_{L^\infty_T H^1_x})\int_0^T \Big(\|\nabla u\|^2_{L^2_x} + \|\nabla\theta\|^2_{H^1_x}\Big)\|\nabla\theta\|^2_{H^{s-1}_x}dt, \end{aligned}$$

   which, together with Gronwall's inequality, implies (3.16).

2. **Case** $(\theta_0, u_0) \in H^1(\mathbb{R}^2) \times (H^s(\mathbb{R}^2))^2$**,** $0 < s < 2$

   In this paragrah we are going to sketch the proof of the H^s, $s \in (0,2)$-Estimate (3.17) for the divergence-free vector field u of the unique solution (θ, u) to the Boussinesq equations (3.1), under the assumption that $\theta_0 \in H^1(\mathbb{R}^2)$, following the procedure in Subsection 3.2.2.

   Recall (3.28) for the definition of the Leray-Helmholtz projector $\mathbb{P}$ such that
$$\mathbb{P}u = u, \quad \mathbb{P}\nabla\Pi = 0.$$
   We apply $\mathbb{P}$ to the velocity equation $(3.1)_2$ to arrive at
$$\partial_t u + \mathbb{P}(u \cdot \nabla u) - \mathbb{P}\,\mathrm{div}(\mu S u) = \mathbb{P}(\theta \vec{e_2}). \tag{3.78}$$
   We apply Δ_j to the above equation (3.78) to arrive at the equation for $u_j := \Delta_j u$
$$\partial_t u_j + \mathbb{P}u\cdot\nabla u_j - \mathbb{P}\,\mathrm{div}(\mu S u_j) = \mathbb{P}[u, \Delta_j]\cdot\nabla u - \mathbb{P}\mathrm{div}\left([\mu, \Delta_j] S u\right) + \mathbb{P}(\theta_j \vec{e_2}). \tag{3.79}$$
   We take the $L^2(\mathbb{R}^2)$-inner product between (3.79) and the divergence-free dyadic piece $u_j = \mathbb{P}u_j$ and follow the similar argument as to arrive at (3.47), to deduce
$$\begin{aligned}&\frac{d}{dt}\|u_j\|_{L^2_x} + 2^{2j}\|u_j\|_{L^2_x}\\ \leqslant &C(\mu_*)\left(\|[u, \Delta_j]\cdot\nabla u\|_{L^2_x} + 2^j\|[\mu, \Delta_j]\nabla u\|_{L^2_x} + \|\theta_j\|_{L^2_x}\right).\end{aligned} \tag{3.80}$$
   By use of the commutator estimate (3.37) in Lemma 3.2.1 again, we have the following commutator estimates as in (3.48):
$$\begin{aligned}&\|[u, \Delta_j]\nabla u\|_{L^2_x} \leqslant C l_j 2^{j(1-s)}\|\nabla u\|_{L^2_x}\|\nabla u\|_{H^{s-1}_x}, \text{ for } s \in (0,2),\\ &2^j\|[\mu, \Delta_j]\nabla u\|_{L^2_x} \leqslant C l_j 2^{j(1-s)}\|\nabla \mu\|_{H^\nu_x}\|\nabla u\|_{H^{s-\nu}_x},\\ &\text{for } \nu \in (-1,1),\ s \in (-1, \nu+1).\end{aligned}$$
   By virtue of the composition estimate (3.41) in Lemma 3.2.1:
$$\|\nabla\mu\|_{H^\nu_x} \leqslant C(\|b\|_{C^{[\nu]+2}}, \|\theta\|_{H^1})\|\nabla\theta\|_{H^\nu_x},$$
   we derive similar as (3.51) that, for $0 < s < \nu + 1 < 2$,
$$\begin{aligned}&2^{2js}\|u_j\|^2_{L^\infty_T L^2_x} + 2^{2j(s+1)}\|u_j\|^2_{L^2_T L^2_x} \leqslant 2^{2js}\|(u_0)_j\|^2_{L^2_x}\\ &+ C(\mu_*)\int_0^T 2^{2j(s-1)}\|\theta_j\|^2_{L^2_x} dt + C(\mu_*, s, \nu, \|b\|_{C^{[\nu]+2}}, \|\theta\|_{L^\infty_T H^1_x})(l_j)^2\times\\ &\quad\times\int_0^T \left(\|\nabla u\|^2_{L^2_x}\|\nabla u\|^2_{H^{s-1}_x} + \|\nabla\theta\|^2_{H^\nu_x}\|\nabla u\|^2_{H^{s-\nu}_x}\right) dt.\end{aligned} \tag{3.81}$$

By the interpolation inequality (3.36):

$$\|\nabla\theta\|_{H^\nu_x}\|\nabla u\|_{H^{s-\nu}_x} \leqslant C\|\nabla\theta\|^{1-\nu}_{L^2_x}\|\nabla\theta\|^{\nu}_{H^1_x}\|\nabla u\|^{\nu}_{H^{s-1}_x}\|\nabla u\|^{1-\nu}_{H^s_x},$$

for $\nu \in (0,1)$, and the L^2-Estimate (3.12):

$$\|u\|^2_{L^\infty_T L^2_x} + \|\nabla u\|^2_{L^2_T L^2_x} \leqslant C(\mu_*)(\|u_0\|^2_{L^2_x} + T\|\theta_0\|^2_{L^2_x}), \tag{3.82}$$

we arrive at the following by Young's inequality

$$\begin{aligned}\|u\|^2_{L^\infty_T H^s_x} + \|\nabla u\|^2_{L^2_T H^s_x} &\leqslant C(\mu_*)\Big(\|u_0\|^2_{H^s_x} + T\|\theta_0\|^2_{L^2_x} + \|\theta\|^2_{L^2_T H^{s-1}_x}\Big)\\ &+ C(\mu_*, s, \nu, \|b\|_{C^2}, \|\theta\|_{L^\infty_T H^1_x})\int_0^T (\|\nabla u\|^2_{L^2_x} + \|\nabla\theta\|^2_{H^1_x})\|\nabla u\|^2_{H^{s-1}_x} dt,\end{aligned}$$

which, together with Gronwall's inequality, implies (3.17).

3. **Case** $(\theta_0, u_0) \in H^2(\mathbb{R}^2) \times (H^\varepsilon(\mathbb{R}^2))^2$ **or** $H^{\varepsilon+1}(\mathbb{R}^2) \times (L^2(\mathbb{R}^2))^2$ **with** $0 < \varepsilon < 1$

In this paragraph, we will show the estimate with $(\theta_0, u_0) \in H^2(\mathbb{R}^2) \times (H^\varepsilon(\mathbb{R}^2))^2$ and $H^{\varepsilon+1}(\mathbb{R}^2) \times (L^2(\mathbb{R}^2))^2$, $0 < \varepsilon < 1$ by using a similar argument as in proof of the H^1-estimate of θ in Subsection 3.2.3. We will sketch the proof here.

- **Case** $(\theta_0, u_0) \in H^2(\mathbb{R}^2) \times (H^\varepsilon(\mathbb{R}^2))^2$**,** $0 < \varepsilon < 1$
  We recall the function $\eta = A^{-1}(\theta)$ defined in (3.54), and the parabolic η-equation (3.58):

$$\partial_t\eta + u\cdot\nabla\eta - \kappa\Delta\eta = 0. \tag{3.83}$$

  We are going to derive the a priori H^2-Estimate for η under the conditions

$$\begin{aligned}&\operatorname{div} u = 0, \quad \nabla u \in L^2_{\text{loc}}([0,\infty); (H^s(\mathbb{R}^2))^4),\\ &\text{and } \nabla\kappa \in L^4_{\text{loc}}([0,\infty); (L^4(\mathbb{R}^2))^2).\end{aligned}$$

  We test the above η-equation (3.83) by $\Delta^2\eta$, to arrive at

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\Delta\eta|^2\,\mathrm{d}x + \int_{\mathbb{R}^2} \kappa|\nabla\Delta\eta|^2\,\mathrm{d}x\\ &= -\int_{\mathbb{R}^2}\Big(u\cdot\nabla\eta\Delta^2\eta + \nabla\kappa\cdot\nabla\Delta\eta\Delta\eta\Big)\,\mathrm{d}x\,.\end{aligned}$$

By integration by parts and $\operatorname{div} u = 0$, we have

$$-\int_{\mathbb{R}^2} u \cdot \nabla\eta \Delta^2\eta \,\mathrm{d}x \;=\; \int_{\mathbb{R}^2} \nabla\Delta\eta \cdot \nabla u \cdot \nabla\eta - \nabla u : \nabla^2\eta\Delta\eta \;\mathrm{d}x\,,$$

where

$$\begin{aligned}
\int_{\mathbb{R}^2} |\nabla u : \nabla^2\eta\Delta\eta| \,\mathrm{d}x \;&\leqslant \|\nabla u\|_{L^2_x}\|\nabla^2\eta\|^2_{L^4_x} \\
&\leqslant \|\nabla u\|_{L^2_x}\|\Delta\eta\|_{L^2_x}\|\nabla\Delta\eta\|_{L^2_x} \\
&\leqslant \frac{\kappa_*}{4}\|\nabla\Delta\eta\|^2_{L^2_x} + C(\kappa_*)\|\nabla u\|^2_{L^2_x}\|\Delta\eta\|^2_{L^2_x}.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\int_{\mathbb{R}^2} |\nabla\kappa \cdot \nabla\Delta\eta\Delta\eta| \,\mathrm{d}x \;&\leqslant \|\nabla\kappa\|_{L^4_x}\|\nabla\Delta\eta\|_{L^2_x}\|\Delta\eta\|_{L^4_x} \\
&\leqslant \frac{\kappa_*}{4}\|\nabla\Delta\eta\|^2_{L^2_x} + C(\kappa_*)\|\nabla\kappa\|^4_{L^4_x}\|\Delta\eta\|^2_{L^2_x}.
\end{aligned}$$

By using the Sobolev embedding $H^\varepsilon(\mathbb{R}^2) \hookrightarrow L^{\frac{2}{1-\varepsilon}}(\mathbb{R}^2)$ and $H^1(\mathbb{R}^2) \hookrightarrow L^{\frac{2}{\varepsilon}}(\mathbb{R}^2)$, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} |\nabla\Delta\eta \cdot \nabla u \cdot \nabla\eta| \, dx &\leqslant \frac{\kappa_*}{4}\|\nabla\Delta\eta\|^2_{L^2_x} + C(\kappa_*)\|\nabla u \cdot \nabla\eta\|^2_{L^2_x} \\
&\leqslant \frac{\kappa_*}{4}\|\nabla\Delta\eta\|^2_{L^2_x} + C(\kappa_*)\|\nabla u\|^2_{L_x^{\frac{2}{1-\varepsilon}}}\|\nabla\eta\|^2_{L_x^{\frac{2}{\varepsilon}}} \\
&\leqslant \frac{\kappa_*}{4}\|\nabla\Delta\eta\|^2_{L^2_x} + C(\kappa_*, \varepsilon)\|\nabla u\|^2_{H^\varepsilon_x}\|\nabla\eta\|^2_{H^1_x}
\end{aligned}$$

To conclude, we have the following a priori $\dot{H}^2_x$-Estimate for η and any positive time $T > 0$ by Gronwall's inequality

$$\begin{aligned}
\|\Delta\eta(T)\|^2_{L^2_x} + \|\nabla\Delta\eta\|^2_{L^2_T L^2_x} &\leqslant C(\kappa_*, \varepsilon)(\|\Delta\eta_0\|^2_{L^2_x} + \|\nabla u\|^2_{L^2_T H^\varepsilon_x}\|\nabla\eta\|^2_{L^\infty_T L^2_x}) \\
&\quad \times \exp\Big(C(\kappa_*, \varepsilon)(\|u\|^2_{L^2_T H^{\varepsilon+1}_x} + \|\nabla\kappa\|^4_{L^4_T L^4_x})\Big).
\end{aligned}$$

By view of the equivalence relation (3.55) as well as[5]

$$\begin{aligned}\|\nabla^3\theta\|_{L^2_T L^2_x} &\leqslant C(\kappa_*, \|a\|_{C^2}) \\ &\times \Big((\|\nabla\eta\|^2_{L^4_T L^4_x} + \|\nabla^2\eta\|_{L^2_T L^2_x})\|\nabla\eta\|_{L^\infty_T L^2_x} + \|\nabla^3\eta\|_{L^2_T L^2_x} \Big),\end{aligned}$$

we derive the H^2-Estimate (3.18) for θ by virtue of the H^1-Estimate (3.61):

$$\begin{aligned}&\|\theta\|^2_{L^\infty_T H^2_x} + \|\nabla\theta\|^2_{L^2_T H^2_x} \leqslant C(\kappa_*, \|a\|_{C^2}, \kappa^*)\|\theta_0\|^2_{H^2}(1 + \|\nabla\theta_0\|^2_{L^2}) \\ &\times \exp\Big(C(\kappa_*, \varepsilon, \|a\|_{\mathrm{Lip}})(\|u\|^2_{L^2_T H^{1+\varepsilon}_x} + \|u\|^4_{L^4_T L^4_x} + \|\nabla\theta\|^4_{L^4_T L^4_x}) \Big),\end{aligned}$$

- **Case** $(\theta_0, u_0) \in H^{\varepsilon+1}(\mathbb{R}^2) \times (L^2(\mathbb{R}^2))^2$**,** $0 < \varepsilon < 1$
  We are going to derive the a priori H^2-Estimate for η under the conditions

$$\begin{aligned}&\operatorname{div} u = 0, \quad \nabla u \in L^2_{\mathrm{loc}}([0,\infty); (H^s(\mathbb{R}^2))^4) \\ &\text{and } \nabla\kappa \in L^4_{\mathrm{loc}}([0,\infty); (L^4(\mathbb{R}^2))^2).\end{aligned}$$

We recall the u-equation (3.78) where

$$\operatorname{div}(\mu S u) = \mu\Delta u + \nabla\mu \cdot S u.$$

We test (3.78) by the divergence-free vector field $\Delta^2 u$, to arrive at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\Delta u|^2 \,\mathrm{d}x + \int_{\mathbb{R}^2} \mu|\nabla\Delta u|^2 \,\mathrm{d}x &= \int_{\mathbb{R}^2} \Big(-u \cdot \nabla u \Delta^2 u \\ &+ \nabla\mu \cdot S u \cdot \Delta^2 u - \nabla\mu \cdot \nabla\Delta u \cdot \Delta u + \Delta\theta\Delta u_2 \Big) \,\mathrm{d}x\,.\end{aligned}$$

By use of the embedding $H^\varepsilon(\mathbb{R}^2) \hookrightarrow L^{\frac{2}{1-\varepsilon}}(\mathbb{R}^2)$ with $0 < \varepsilon < 1$, the righthand side can be bounded by

$$\begin{aligned}&C(\|\nabla u\|^2_{L^4_x} + \|u\|_{L^4_x}\|\nabla^2 u\|_{L^4_x} + \|\nabla^2\mu\|_{H^\varepsilon_x}\|\nabla u\|_{H^1_x} \\ &+ \|\nabla\mu\|_{L^4_x}\|\nabla^2 u\|_{L^4_x})\|\nabla\Delta u\|_{L^2_x} + \|\Delta\theta\|_{L^2_x}\|\Delta u\|_{L^2_x}.\end{aligned}$$

[5] It is also straightforward to calculate

$$\begin{aligned}\partial_{jkl}\eta &= a''(\theta)(\partial_j\theta\partial_k\theta\partial_l\theta) + a'(\theta)(\partial_{jk}\theta\partial_l\theta + \partial_{jl}\partial_k\theta + \partial_{kl}\theta\partial_j\theta) + a(\theta)\partial_{jkl}\theta, \\ \partial_{jkl}\theta &= \Big(-\frac{a''(A^{-1}(\eta))}{a^4(A^{-1}(\eta))} + \frac{3(a'(A^{-1}(\eta)))^2}{a^5(A^{-1}(\eta))} \Big)\partial_j\eta\partial_k\eta\partial_l\eta \\ &- \frac{a'(A^{-1}(\eta))}{a^3(A^{-1}(\eta))}\Big(\partial_{jk}\eta\partial_l\eta + \partial_{jl}\eta\partial_l\eta + \partial_{kl}\eta\partial_j\eta \Big) + \frac{1}{a(A^{-1}(\eta))}\partial_{jkl}\eta.\end{aligned}$$

Thus we have the following a priori $\dot{H}^2_x$-Estimate for u and any positive time $T > 0$ by Young's inequality and Gronwall's inequality

$$\begin{aligned}
&\|\Delta u(T)\|^2_{L^2_x} + \|\nabla\Delta u\|^2_{L^2_T L^2_x} \\
&\leqslant C(\mu_*)(\|\Delta u_0\|^2_{L^2_x} + \|\nabla u\|^4_{L^4_T L^4_x} + \|\nabla^2\mu\|^2_{L^2_T H^\varepsilon_x}\|\nabla u\|^2_{L^\infty_T L^2_x} \\
&\qquad + \|\Delta\theta\|_{L^2_T L^2_x}\|\Delta u\|_{L^2_T L^2_x}) \\
&\qquad \times \exp\Big(C(\mu_*)(\|(u,\nabla\mu)\|^4_{L^4_T L^4_x} + \|\nabla^2\mu\|^2_{L^2_T H^\varepsilon_x})\Big).
\end{aligned} \tag{3.84}$$

Notice that

$$\begin{aligned}
\|\nabla^2\mu\|_{L^2_T H^\varepsilon_x} &\leqslant \|b''(\theta)(\nabla\theta)^2\|_{L^2_T H^\varepsilon_x} + \|b'(\theta)\nabla^2\theta\|_{L^2_T H^\varepsilon_x} \\
&\lesssim C(\|b\|_{C^2})(\|\nabla\theta\|_{L^\infty_T H^\varepsilon_x}\|\nabla^2\theta\|_{L^2_T H^\varepsilon_x} + \|\theta\|_{L^\infty_T H^1_x}\|\nabla\theta\|^2_{L^2_T H^1_x} \\
&\quad + \|\nabla^2\theta\|_{L^2_T H^\varepsilon_x}(1 + \|\theta\|_{L^\infty_T H^1_x}))
\end{aligned}$$

the above inequality and (3.84) give (3.19).

4. **Case** $(\theta_0, u_0) \in H^s(\mathbb{R}^2) \times (H^{s-2}(\mathbb{R}^2))^2$ **or** $H^{s-1} \times (H^s(\mathbb{R}^2))^2$**,** $s > 2$

   We are going to use the estimates in the high regularity regime in Lemma 3.2.1 to derive the H^s-estimates (3.20)-(3.21)-(3.22)-(3.23) in Remark 3.1.3. Let $(l'_j)_{j\geqslant 0}$ be a normalised sequence in $\ell^2(\mathbb{N})$ such that $l'_j \geqslant 0$ and $\sum_{j\geqslant 0}(l'_j)^2 = 1$.

   - **Case** $(\theta_0, u_0) \in H^s(\mathbb{R}^2) \times (H^{s-2}(\mathbb{R}^2))^2$**,** $s > 2$
     We can view the transport term $u \cdot \nabla\theta$ simply as a source term of the θ-equation:

     $$\partial_t\theta - \operatorname{div}(\kappa\nabla\theta) = -u \cdot \nabla\theta$$

     Then the preliminary estimate for $\theta_j = \Delta_j\theta$ in (3.75) can be replaced by

     $$\frac{d}{dt}\|\theta_j\|_{L^2_x} + 2^{2j}\|\theta_j\|_{L^2_x} \leqslant C(\kappa_*)\left(\|(u\cdot\nabla\theta)_j\|_{L^2_x} + 2^j\|[\kappa, \Delta_j]\nabla\theta\|_{L^2_x}\right)$$

     for $j \geqslant 0$. We apply Lemma 3.2.1 to derive the following estimates for $s > 1$:

     $$\begin{aligned}
     &\|(u\cdot\nabla\theta)_j\|_{L^2_x} \leqslant Cl'_j 2^{j(1-s)}(\|u\|_{L^\infty_x}\|\nabla\theta\|_{H^{s-1}_x} + \|u\|_{H^{s-1}_x}\|\nabla\theta\|_{L^\infty_x}), \\
     &2^j\|[\kappa, \Delta_j]\nabla\theta\|_{L^2_x} \leqslant Cl'_j 2^{j(1-s)}(\|\nabla\kappa\|_{L^\infty_x}\|\nabla\theta\|_{H^{s-1}_x} + \|\nabla\kappa\|_{H^{s-1}_x}\|\nabla\theta\|_{L^\infty_x}).
     \end{aligned}$$

Therefore we have the following estimate similarly as in (3.76) for $s \in (2,3)$ by virtue of $H^{s-1}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$:

$$
\begin{aligned}
2^{2js}\|\theta_j\|^2_{L^\infty_T L^2_x} + 2^{2j(s+1)}\|\theta_j\|^2_{L^2_T L^2_x} &\leqslant 2^{2js}\|(\theta_0)_j\|^2_{L^2_x} \\
&+ C(\kappa_*, s)(l'_j)^2 \int_0^T \|u\|^2_{H^{s-1}_x}\|\nabla\theta\|^2_{H^{s-1}_x}\,\mathrm{d}t \\
&+ C(\kappa_*, a, \|\theta\|_{L^\infty_T L^\infty_x})(l'_j)^2 \int_0^T \|\nabla\theta\|^2_{L^\infty_x}\|\nabla\theta\|^2_{H^{s-1}_x} dt, \quad j \geqslant 0,
\end{aligned}
$$

which, together with the L^2-estimate (3.77) and the Gronwall's inequality, implies (3.20).

For $s \geqslant 3$, we make use of the following commutator estimate

$$
\|[u, \Delta_j]\nabla\theta\|_{L^2_x} \leqslant C l'_j 2^{j(1-s)}(\|\nabla u\|_{L^\infty_x}\|\nabla\theta\|_{H^{s-2}_x} + \|\nabla u\|_{H^{s-2}_x}\|\nabla\theta\|_{L^\infty_x}), \tag{3.85}
$$

such that the estimate (3.21) follows.

- **Case** $(\theta_0, u_0) \in H^{s-1}(\mathbb{R}^2) \times (H^s(\mathbb{R}^2))^2$**,** $s > 2$

  We recall the preliminary estimate for u_j in (3.80). We apply Lemma 3.2.1 to derive the following commutator estimates for $s \in (2,3)$ and $\nu \in (s-2, 1) \subset (0,1)$

$$
\begin{aligned}
\|[u, \Delta_j]\nabla u\|_{L^2_x} &\leqslant C l_j 2^{j(1-s)}\|\nabla u\|_{H^\nu_x}\|\nabla u\|_{H^{s-1-\nu}_x}, \\
2^j\|[\mu, \Delta_j]\nabla u\|_{L^2_x} &\leqslant C l'_j 2^{j(1-s)}(\|\nabla\mu\|_{L^\infty_x}\|\nabla u\|_{H^{s-1}_x} \\
&\qquad + \|\nabla\mu\|_{H^{s-1}_x}\|\nabla u\|_{L^\infty_x}),
\end{aligned}
$$

  which implies then

$$
\begin{aligned}
&2^{2js}\|u_j\|^2_{L^\infty_T L^2_x} + 2^{2j(s+1)}\|u_j\|^2_{L^2_T L^2_x} \\
\leqslant &2^{2js}\|(u_0)_j\|^2_{L^2_x} + C(\mu_*)\int_0^T 2^{2j(s-1)}\|\theta_j\|^2_{L^2_x}\,\mathrm{d}t \\
&+ C(\mu_*, s, \nu)(l_j)^2 \int_0^T \|\nabla u\|^2_{H^\nu_x}\|\nabla u\|^2_{H^{s-1-\nu}_x}\,\mathrm{d}t \\
&+ C(\mu_*, s, \|b\|_{C^{[s]+1}}, \|\theta\|_{L^\infty_T H^1_x})(l'_j)^2 \\
&\quad \times \int_0^T \|\nabla\theta\|^2_{L^\infty_x}\|\nabla u\|^2_{H^{s-1}_x} + \|\nabla\theta\|^2_{H^{s-1}_x}\|\nabla u\|^2_{L^\infty_x} dt, \quad j \geqslant 0.
\end{aligned}
$$

  This, together with the L^2-estimate (3.82) and Sobolev's embedding $H^{s-1}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, implies (3.22) where $\nu \in (0,1)$ is taken to be a small constant bigger than $s-2$.

  For $s \geqslant 3$, we use the commutator estimate (3.85) with θ replaced by u, to arrive at (3.23).

Chapter 4

Turbulent Cascades for a family of damped Szegő equations

In this chapter, we study the transfer of energy from low to high frequencies for a family of damped Szegő equations. The cubic Szegő equation has been introduced as a toy model for a totally non-dispersive degenerate Hamiltonian equation. It is a completely integrable system which develops the growth of high Sobolev norms. Here, we consider a two-parameter family of damped Szegő equations, and give a panorama of the dynamics for such equations on a six-dimensional submanifold.

This Chapter is based on the joint work with Prof. Patrick Gérard and Prof. Sandrine Grellier in [GGH21].

4.1 Introduction

An interesting aspect of turbulence is the transfer of energy from long to short-wavelength modes, leading to concentration of energy on small spatial scales. It is usually quantified by growth of Sobolev norms. In this chapter, we study turbulent cascades for the family of damped Szegő equations on the one-dimensional torus

$$i\partial_t u + i\nu(u|\mathbb{1}) = \Pi(|u|^2u) + \alpha(u|\mathbb{1}) - \beta S\Pi(|S^*u|^2S^*u), \tag{4.1}$$

where $\nu > 0$ and $\alpha, \beta \in \mathbb{R}$ are given parameters.

In this chapter, $u : \mathbb{T} \to \mathbb{C}$ is an unknown complex-valued function. We consider the space $L^2(\mathbb{T})$ endowed with the inner product

$$(u|v) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{ix})\overline{v(e^{ix})}\,dx.$$

The Fourier transform on $L^2(\mathbb{T})$ is defined as

$$\hat{u}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)e^{-ikx}\,dx, \quad k \in \mathbb{Z},$$

and

$$u(x) = \sum_{k\in\mathbb{Z}} \hat{u}(k)e^{ikx}.$$

The Szegő projector $\Pi\colon L^2(\mathbb{T}) \to L^2_+(\mathbb{T})$ is the Fourier multiplier defined by

$$\Pi(u) = \sum_{k\geqslant 0} \hat{u}(k)e^{ikx} \in L^2_+(\mathbb{T}).$$

The term ν is the damping term on the smallest Fourier mode

$$(u|\mathbb{1}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{ix})\,dx.$$

The shift operator $S : L^2_+(\mathbb{T}) \to L^2_+(\mathbb{T})$ and its adjoint are defined by

$$Su = e^{ix}u \quad \text{and} \quad S^*u = e^{-ix}(u - (u|\mathbb{1})).$$

When $\nu = \alpha = \beta = 0$, we recover the usual cubic Szegő equation

$$i\partial_t u = \Pi(|u|^2u), \tag{4.2}$$

which was introduced by the first two authors [GG10] as a toy model of a non-dispersive Hamiltonian system. The α-deformation in (4.1) was first introduced by Xu [Xu14] in the α-Szegő equation

$$i\partial_t u = \Pi(|u|^2u) + \alpha(u|\mathbb{1}), \tag{4.3}$$

and the β-deformation in (4.1) was introduced by Biasi and Evnin [BE20] in the β-Szegő equation

$$i\partial_t u = \Pi(|u|^2u) - \beta S\Pi(|S^*u|^2S^*u). \tag{4.4}$$

Observe that, on the Fourier side, this equation corresponds to

$$i\frac{d\hat{u}(n)}{dt} = \sum_{\substack{m,k,l=0\\ n+m=l+k}}^{\infty} C^{(\beta)}_{nmkl}\overline{\hat{u}(m)}\hat{u}(k)\hat{u}(l),\ n \in \mathbb{N}$$

where

$$C^{(\beta)}_{nmkl} = \begin{cases} 1 & \text{if } nmkl = 0 \\ 1-\beta & \text{otherwise.}\end{cases}$$

When $\beta = 1$, the equation (4.4) is so called the truncated Szegő equation and corresponds to the case where most of the interaction of the Fourier coefficients disappeared. Some other variants of the Szegő equation have also been studied, see e.g. ([Poc11; Thi19]).

4.1.1 Promoted turbulence

The Szegő equation is a completely integrable Hamiltonian system with two Lax pair structures displaying some turbulence cascade phenomenon for a generic set of initial data, in spite of infinitely many conservation laws. Namely, there exists a dense G_δ subset of initial data in $L^2_+ \cap C^\infty$, such that the solutions of the cubic Szegő equation satisfy, for every $s > \frac{1}{2}$,

$$\limsup_{t\to+\infty} \|u(t)\|_{H^s} = +\infty \ , \ \liminf_{t\to+\infty} \|u(t)\|_{H^s} < +\infty \ .$$

Furthermore, this subset has an empty interior, since it does not contain any trigonometric polynomial [GG17].

However, there is no explicit examples of such phenomenon, and even less is known about the existence of solutions with high-Sobolev norms tending to infinity.

Later the first two authors [GG20] added a damping term to the cubic Szegő equation

$$i\partial_t u + i\nu(u|\mathbb{1}) = \Pi(|u|^2 u), \quad \nu > 0. \tag{4.5}$$

Paradoxically, the turbulence phenomenon is promoted by the damping term. Indeed, a nonempty open set of initial data generating trajectories with high-Sobolev norms tending to infinity was observed. In comparison, this is not the case for the damped Benjamin–Ono equation, see [Gas22].

The goal of this chapter is to generalise the study of turbulent cascades for the damped Szegő equation to a family of damped Szegő equations (4.1). Biasi and Evnin [BE20] suggested the study of a two-parameter family of equations referred as the (α, β)-Szegő equations given by

$$i\partial_t u = \Pi(|u|^2 u) - \beta S \Pi(|S^* u|^2 S^* u) + \alpha(u|\mathbb{1}), \quad \alpha, \beta \in \mathbb{R}. \tag{4.6}$$

Inspired by this, we study the damped (α, β)-Szegő equations (4.1). The family is constructed in such a way that part of the Lax-pair structure inherited from the cubic Szegő equation is preserved but the damping term breaks the Hamiltonian structure. In our case, similar to the damped Szegő equation, the damping term promotes the existence of unbounded trajectories. We have the following turbulent cascades result.

Theorem 4.1.1 ([GGH21])**.** *There exists an open subset $\Omega \subset H^{\frac{1}{2}}_+ = H^{\frac{1}{2}} \cap L^2_+$ independent of (α, β) such that, for every $s > \frac{1}{2}$, the set $\Omega \cap H^s_+$ is nonempty and, for every $\beta \neq 1$, every solution u of (4.1) with $u(0) \in \Omega \cap H^s_+$ satisfies*

$$\|u(t)\|_{H^s} \underset{t\to\infty}{\longrightarrow} +\infty \ .$$

Furthermore, there exist rational initial data in Ω which generate stationary solutions of (4.1) for $\beta = 1$.

When β is different from 1, the damping term acts on the (α,β)-Szegő equations as on the cubic Szegő equation[GG20]. The case $\beta = 1$ of the damped truncated Szegő equation appears to be more degenerate, and we do not know whether there exists a nonempty open subset of blowing up data.

We define the functional

$$F(u) = \sum_k (-1)^{k-1}\sigma_k^2$$

where $(\sigma_k^2)_k$ is the strictly decreasing sequence of positive eigenvalues of $\tilde{H}_u^2$ (defined in Subsection 4.1.2). We have the following sufficient conditions for the exploding trajectories.

Theorem 4.1.2 ([GGH21])**.** *Assume $\beta \neq 1$. Let $s > \frac{1}{2}$. If $u_0 \in H_+^s$ satisfies*

- *either $\|u_0\|_{L^2}^2 < F(u_0)$,*
- *or $\|u_0\|_{L^2}^2 = F(u_0)$ and $(u_0|\mathbb{1}) \neq 0$,*

then the solution of the damped (α,β)-Szegő equation satisfies

$$\|u(t)\|_{H^s} \underset{t\to\infty}{\longrightarrow} +\infty.$$

The wave turbulence phenomenon for Hamiltonian systems has been actively studied by mathematicians and physicists in the last decades. Bourgain [Bou00] asked whether there is a solution of the cubic defocusing nonlinear Schrödinger equation on the two-dimensional torus $\mathbb{T}^2$ with initial data $u_0 \in H^s(\mathbb{T}^2)$, $s > 1$, such that

$$\limsup_{t\to\infty} \|u(t)\|_{H^s} = \infty.$$

There is still no complete answer to this question. However, the first mathematical evidence of such behaviour has been exhibited in the seminal work [CKSTT10] in which it is proven that, given any initial data with small Sobolev norm, it is possible to find a sufficiently large time for which the Sobolev norm of the solution is larger than any prescribed constant. This phenomenon also occurs for the half-wave equations on the real line or on the one-dimensional torus, see e.g. [Poc13; GG12]. Based on this, the first author, Lenzmann, Pocovnicu, and Raphaël [GLPR18] gave a complete picture for a class of solutions on the real line. Namely, after the transient turbulence, the Sobolev norms of such solutions remains stationary large in time. The turbulence also occurs for two-dimensional incompressible Euler equations, the sharp double exponentially growing vorticity gradient on the disk was constructed by Kiselev and Šverák [KŠ14] and the existence of exponentially growing vorticity gradient solutions on the torus was shown by Zlatoš [Zla15].

4.1.2 Preliminary observations

For the damped (α, β)-Szegő equations (4.1), the momentum

$$\mathcal{M}(u) = (Du|u) = \sum_{k \geqslant 1} k|\hat{u}(k)|^2$$

is preserved by the flow. An easy modification of the arguments in [GG10] shows that (4.1) is globally wellposed on H^s_+ for every $s \geqslant \frac{1}{2}$. Our goal is to study the behaviour of solutions of (4.1) as $t \to +\infty$, in particular the growth of H^s-Sobolev norms for $s > \frac{1}{2}$.

The main property which allows us to do computations, is the existence of a Lax pair. Namely, if u satisfies (4.1) then

$$\frac{d}{dt}\tilde{H}_u = [C_u - \beta B_{S^* u}, \tilde{H}_u],$$

where H_u is the Hankel operator

$$H_u : \begin{cases} L^2_+(\mathbb{T}) & \to & L^2_+(\mathbb{T}) \\ f & \mapsto & \Pi(u\overline{f}) \end{cases},$$

and $\tilde{H}_u = S^* H_u$ is the shifted Hankel operator. The operators B_u and C_u are the anti-self-adjoint operators appearing in the Lax pairs of the cubic Szegő equation, and defined as following

$$B_u = -iT_{|u|^2} + \frac{i}{2}H_u^2 \text{ and } C_u = -iT_{|u|^2} + \frac{i}{2}\tilde{H}_u^2, \tag{4.7}$$

where T_b denotes the Toeplitz operator of symbol b given on L^2_+ by $T_b(f) = \Pi(bf)$.

Thanks to this Lax pair, there are invariant manifolds consisting of the functions u such that $\operatorname{rank}(\tilde{H}_u) = k$, $k \geqslant 0$. From a well known result by Kronecker [Kro81], these manifolds consist of the rational functions

$$u(x) = \frac{P_1(e^{ix})}{P_2(e^{ix})},$$

where P_1 and P_2 are polynomials of degrees at most k with $\deg(P_1) = k$ or $\deg(P_2) = k$, no common roots and P_2 has no roots inside the disk $\{z \in \mathbb{C} \mid |z| < 1\}$.

4.1.3 An invariant six-dimensional submanifold

In this section, we restrict ourselves to the lowest dimensional submanifold where $\tilde{H}_u$ has rank 1

$$\mathcal{W} := \left\{ u(x) = b + \frac{c\, e^{ix}}{1 - pe^{ix}},\ b, c, p \in \mathbb{C}, c \neq 0,\ |p| < 1 \right\}.$$

We will give a complete picture of (4.1) on $\mathcal{W}$, which consists of periodic, blow-up and scattering trajectories.

We consider the trajectories with a fixed momentum $M > 0$, and define

$$\mathcal{E}_M = \{u \in \mathcal{W} \mid \mathcal{M}(u(t)) = M\}, \quad \mathcal{C}_M = \{u(x) = ce^{ix} \mid |c|^2 = M\}.$$

Notice that $\mathcal{C}_M \subset \mathcal{E}_M$ consists of the periodic trajectories. We write

$$\mathbb{S}_{\nu,\alpha,\beta}(t)u_0 = u(t)$$

for the solution of the equation (4.1) with initial data $u_0 \in H_+^{\frac{1}{2}}(\mathbb{T})$. Then we have the following theorem.

Theorem 4.1.3 ([GGH21]). *Let $\nu > 0$ and $\alpha, \beta \in \mathbb{R}$. There exists a three-dimensional submanifold $\Sigma_{M,\nu,\alpha,\beta} \subset \mathcal{E}_M$ disjoint from $\mathcal{C}_M$, invariant under the flow $\mathbb{S}_{\nu,\alpha,\beta}(t)$ and such that $\Sigma_{M,\nu,\alpha,\beta} \cup \mathcal{C}_M$ is closed and*

1. *If $u_0 \in \mathcal{E}_M \backslash (\Sigma_{M,\nu,\alpha,\beta} \cup \mathcal{C}_M)$, then*

$$\|\mathbb{S}_{\nu,\alpha,\beta}(t)u_0\|_{H^s} \sim_{s,\nu,\alpha,\beta,M} t^{s-\frac{1}{2}}, \quad s > \frac{1}{2}. \tag{4.8}$$

2. *If $u_0 \in \Sigma_{M,\nu,\alpha,\beta}$, then dist$(\mathbb{S}_{\nu,\alpha,\beta}(t)u_0, \mathcal{C}_M) \sim e^{-ct}$, for some $c > 0$.*

Let us emphasize that on this submanifold $\mathcal{W}$, unlike on higher dimensional submanifolds, there is no difference between the case $\beta = 1$ and the other cases.

Compared to Theorem 4.1.1, in Theorem 4.1.3 the open set consisting of the initial data generating blow-up trajectories, is dense in $\mathcal{W}$. Furthermore, the Sobolev norms of such generating trajectories grow at a uniform polynomial rate $\sim t^{s-\frac{1}{2}}$, independent of ν, α, β. This is consistent with the blow-up rate for the damped Szegő equation (4.5), see [GG20]. In contrast, the initial data in $\mathcal{W}$ generate only bounded trajectories in the case of the Szegő equation, the α and β-szegő equations with negative α and β, see [GG10; Xu14; BE20]. However, if $\alpha > 0$ and $\beta > 0$, then even faster blow-up solutions occur for the α and β-Szegő equations. In this case, there exist trajectories $u(t)$, whose Sobolev norms grow exponentially in time with

$$\|u(t)\|_{H^s} \sim e^{ct(s-\frac{1}{2})}, \quad s > \frac{1}{2}. \tag{4.9}$$

Moreover, if $\beta > 9$, then there also exists a class of solutions for the β-Szegő equation with the polynomially growing Sobolev norms at the rate (4.8), see [BE20]. In other words, the authors exhibit various strong turbulence phenomena for β-Szegő equations when β is large enough.

One important feature of the damped (α, β)-Szegő equations is the existence of the Lyapunov functional

$$\frac{d}{dt}\|\mathbb{S}_{\nu,\alpha,\beta}(t)u_0\|_{L^2}^2 + 2\nu|(\mathbb{S}_{\nu,\alpha,\beta}(t)u_0|\mathbb{1})|^2 = 0.$$

Together with the conserved momentum, one infers the weak limit points of $u(t)$ as $t \to \infty$ in $H^{\frac{1}{2}}$. In the second part of Theorem 4.1.3, when $\beta = 1$, such weak limit points are also strong limit points. Namely, there exists $u_\infty \in \mathcal{C}_M$ such that

$$\|\mathbb{S}_{\nu,\alpha,1}(t)u_0 - u_\infty\|_{H^s} \to 0, \quad \forall s \geqslant \frac{1}{2}.$$

Let us complete this paragraph by a few more remarks about stationary solutions in the case $\beta = 1$. On $\mathcal{W}$, we already observed that these solutions are of the form ce^{ix} with $c \neq 0$. An elementary calculation shows that such initial data generate periodic solutions in the case $\beta \neq 1$ and arbitrary (α, ν). However, as stated in Theorem 4.1.1, one can construct rational stationary solutions in the case $\beta = 1$ which generate blow up solutions for every $\beta \neq 1$, $\nu > 0$ and arbitrary $\alpha \in \mathbb{R}$.

4.1.4 The Lyapunov functional

As in the case of the damped Szegő equation, an important tool in the study of equation (4.1) is the existence of a Lyapunov functional. Precisely, the following lemma holds.

Lemma 4.1.1. *Let $u_0 \in H_+^{1/2}(\mathbb{T})$. Then, for any $t \in \mathbb{R}$,*

$$\frac{d}{dt}\|\mathbb{S}_{\nu,\alpha,\beta}(t)u_0\|_{L^2}^2 + 2\nu|(\mathbb{S}_{\nu,\alpha,\beta}u_0(t)|\mathbb{1})|^2 = 0. \tag{4.10}$$

As a consequence, if $\nu > 0$, $t \mapsto \|\mathbb{S}_{\nu,\alpha,\beta}(t)u_0\|_{L^2}$ is decreasing, and $|(\mathbb{S}_{\nu,\alpha,\beta}(t)u_0|\mathbb{1})|$ is square integrable on $[0, +\infty)$, tending to zero as t goes to $+\infty$.

Proof. Denote by $u(t) := \mathbb{S}_{\nu,\alpha,\beta}(t)u_0$ the solution of (4.1) with $u(0) = u_0$. We first observe the Lyapunov functional (4.10)

$$\begin{aligned}
&\frac{d}{dt}\|u(t)\|_{L^2}^2 = 2\mathrm{Re}(\partial_t u|u) = 2\mathrm{Im}(i\partial_t u|u) \\
&= 2\mathrm{Im}(\Pi(|u|^2u)|u) - 2\beta\mathrm{Im}(S\Pi(|S^*u|^2S^*u)|u) + 2\mathrm{Im}((\alpha - i\nu)((u|\mathbb{1})|u)) \\
&= -2\nu|(u(t)|\mathbb{1})|^2 \leqslant 0,
\end{aligned}$$

which implies the decreasing of $t \mapsto \|u(t)\|_{L^2}$, and hence, $t \mapsto \|u(t)\|_{L^2}^2$ admits a limit at infinity.

The finiteness of $\int_0^\infty |(u(s)|\mathbb{1})|^2 ds$ is given by

$$2\nu\int_0^t |(u(s)|\mathbb{1})|^2\, ds = \|u_0\|_{L^2}^2 - \|u(t)\|_{L^2}^2 \leqslant \|u_0\|_{L^2}^2, \quad \forall t > 0.$$

Then, to show the decay of $|(u(t)|\mathbb{1})|$, we only need to show the boundedness of $\frac{d}{dt}|(u(t)|\mathbb{1})|^2$. We calculate

$$\begin{aligned}
&\frac{d}{dt}|(u(t)|\mathbb{1})|^2 = 2\mathrm{Re}((\partial_t u|\mathbb{1})(\mathbb{1}|u)) \\
&= 2\mathrm{Im}((\alpha - i\nu)(u|\mathbb{1})(\mathbb{1}|u)) + 2\mathrm{Im}((\Pi(|u|^2u|\mathbb{1})(\mathbb{1}|u))) \\
&\quad - 2\beta\mathrm{Im}((S\Pi(|S^*u|^2S^*u|\mathbb{1})(\mathbb{1}|u))) \\
&= -2\nu|(u|\mathbb{1})|^2 + 2\mathrm{Im}((u^2|u)(\mathbb{1}|u)),
\end{aligned}$$

where

$$|(u(t)|\mathbb{1})| \leqslant \|u\|_{L^2} \leqslant \|u_0\|_{L^2}$$

and

$$|(u^2(t)|u(t))| \leqslant \|u\|_{L^2} \times \|u\|_{L^4}^2 \leqslant \|u\|_{L^2} \times \|u\|_{H^{1/2}}^2 \leqslant \|u_0\|_{L^2}(\mathcal{M}(u_0) + \|u_0\|_{L^2}^2).$$

Now we conclude that $|(u(t)|\mathbb{1})| \to 0$ as $t \to \infty$. □

By Lemma 4.1.1 and the conservation of the momentum, the $H^{1/2}$ norm of $\mathbb{S}_{\nu,\alpha,\beta}(t)u_0$ remains bounded as $t \to +\infty$, hence one can consider limit points u_∞ of $\mathbb{S}_{\nu,\alpha,\beta}(t)u_0$ for the weak topology of $H^{1/2}$ as $t \to +\infty$. The following lemma describes more precisely these limit points, according to LaSalle's invariance principle.

Proposition 1. *Let $u_0 \in H^{1/2}(\mathbb{T})$. Any $H^{1/2}$- weak limit point u_∞ of $(\mathbb{S}_{\nu,\alpha,\beta}(t)u_0)$ as $t \to +\infty$ satisfies $(\mathbb{S}_{\nu,\alpha,\beta}(t)u_\infty|\mathbb{1}) = 0$ for all t. In particular, $\mathbb{S}_{\nu,\alpha,\beta}(t)u_\infty$ solves the (α,β)- Szegő equation, in other words $\mathbb{S}_{\nu,\alpha,\beta}(t)u_\infty = \mathbb{S}_{\alpha,\beta}(t)u_\infty$.*

Proof. We denote $Q := \lim_{t\to\infty} \mapsto \|\mathbb{S}_{\nu,\alpha,\beta}(t)u_0\|_{L^2}^2$. By the weak continuity of the flow in $H_+^{1/2}(\mathbb{T})$, one has

$$u(t+t_n) = \mathbb{S}_{\nu,\alpha,\beta}(t)u(t_n) \to \mathbb{S}_{\nu,\alpha,\beta}(t)u_\infty \quad \text{as } t_n \to \infty$$

weakly in $H^{1/2}$. Hence, thanks to the Rellich theorem

$$\|u(t+t_n)\|_{L^2}^2 \to \|\mathbb{S}_{\nu,\alpha,\beta}(t)u_\infty\|_{L^2}^2 \quad \text{as } t_n \to \infty.$$

On the other hand, Lemma 4.1.1 ensures that

$$\|u(t+t_n)\|_{L^2}^2 \to Q \quad \text{as } t_n \to \infty.$$

We conclude that for any $t \in \mathbb{R}$

$$\|\mathbb{S}_{\nu,\alpha,\beta}(t)u_\infty\|_{L^2}^2 = \|u_\infty\|_{L^2}^2, \quad \text{and} \quad \frac{d}{dt}\|\mathbb{S}_{\nu,\alpha,\beta}(t)u_\infty\|_{L^2}^2 = 0.$$

We recall the Lyapunov functional (4.10), which implies

$$(\mathbb{S}_{\nu,\alpha,\beta}(t)u_\infty|\mathbb{1}) = 0, \quad \forall t \in \mathbb{R}.$$

Hence, $\mathbb{S}_{\nu,\alpha,\beta}(t)u_\infty = \mathbb{S}_{\alpha,\beta}(t)u_\infty$ is a solution to the (α,β)-Szegő equation without damping. □

4.2 The six-dimensional manifold revisited

In this section, we provide a panorama of the dynamics of the damped (α,β)-Szegő equations for any fixed $(\alpha,\beta) \in \mathbb{R}^2$ on the six-dimensional submanifold

$$\mathcal{W} := \left\{ u(x) = b + \frac{c\,e^{ix}}{1-pe^{ix}},\ b,c,p \in \mathbb{C}, c \neq 0,\ |p| < 1 \right\}.$$

Recall that $\mathcal{W}$ is preserved by the damped (α,β)-Szegő flow since it corresponds to the symbol of the shifted Hankel operator of rank 1.

For $u \in \mathcal{W}$, we calculate the mass and the conserved momentum as following

$$\|u\|_{L^2}^2 = |b|^2 + \frac{|c|^2}{1-|p|^2}, \quad \mathcal{M}(u) = \frac{|c|^2}{(1-|p|^2)^2}.$$

We will repeatedly use the relation between the mass and the momentum

$$\|u\|_{L^2}^2 = |b|^2 + \mathcal{M}(u)(1-|p|^2). \tag{4.11}$$

We are going to consider the solutions of (4.1) with a fixed momentum $\mathcal{M}(u(t)) = M > 0$. We define two subsets of $\mathcal{W}$

$$\mathcal{E}_M = \{u \in \mathcal{W} \mid \mathcal{M}(u) = M\}, \quad \mathcal{C}_M = \{u(x) = ce^{ix} \mid |c|^2 = M\}.$$

We observe that $\mathcal{C}_M \subset \mathcal{E}_M$ is invariant by the damped (α, β)-flow (4.1) which consists of the periodic trajectories.

We write the damped (α, β)-Szegő equations on $\mathcal{E}_M$ in the (b, c, p)-coordinate as

$$\begin{cases} ib' + (i\nu - \alpha)b = (|b|^2 + 2M(1 - |p|^2))b + Mc\overline{p}, \\ \qquad ic' = 2|b|^2c + 2M(1 - |p|^2)bp + (1 - \beta)Mc, \\ \qquad ip' = c\overline{b} + (1 - \beta)Mp(1 - |p|^2), \end{cases} \tag{4.12}$$

where $\nu > 0$ is the coefficient of the damping term in (4.1).

We will determine all types of trajectories of the damped (α, β)-Szegő equations on $\mathcal{E}_M \backslash \mathcal{C}_M \subset \mathcal{W}$, which consists of blow-up and scattering trajectories. By Lemma 4.1.1, the L^2 norm of $u(t)$ converges

$$\|u(t)\|_{L^2}^2 = |b(t)|^2 + M(1 - |p(t)|^2) \to Q \quad \text{and} \quad |b(t)| = |(u(t)|\mathbb{1})| \to 0,$$

as $t \to \infty$, which implies $M(1 - |p(t)|^2) \to Q$. As a consequence, $|p(t)|$ admits a limit in $[0, 1]$. We claim that this limit can only be 0 or 1, which corresponds to the scattering trajectories or the blow-up trajectories respectively.

We prove that the limit of $|p(t)|$ can only be 0 or 1 by contradiction. Indeed, if $0 < \lim_{t\to\infty} |p(t)|^2 < 1$, then the trajectory $\{u(t)\}$ is compact in $H^s(\mathbb{T})$. As a consequence, $u(t)$ has a weak limit $u_\infty \in \mathcal{W}$. By Proposition 1,

$$\mathbb{S}_{\nu,\alpha,\beta}(t)u_\infty = b_\infty(t) + \frac{c_\infty(t)e^{ix}}{1 - p_\infty(t)e^{ix}}$$

is a solution of the (α, β)-Szegő equation (4.6) with $(\mathbb{S}_{\nu,\alpha,\beta}(t)u_\infty|\mathbb{1}) = b_\infty(t) = 0$. Moreover, the triplet $(0, c_\infty, p_\infty)$ satisfies the ODE system $(4.12)_1$, which implies

$$Mc_\infty(t)\overline{p_\infty}(t) = 0.$$

On the other hand, the momentum conservation law

$$\mathcal{M}(\mathbb{S}_{\nu,\alpha,\beta}(t)u_\infty) = \mathcal{M}(u_\infty) = \mathcal{M}(u_0) > 0$$

ensures that c_∞ cannot be 0. Therefore, $p_\infty = 0$, which contradicts our assumption.

We will show that all the initial value u_0 with corresponding $|p(t)| \to 1$ form a dense open set of $\mathcal{W}$, on which the growth of the H^s norm of $\mathbb{S}_{\nu,\alpha,\beta}(t)u_0$ is of order $t^{s-\frac{1}{2}}$ as $t \to \infty$. Since in this case$\|\mathbb{S}_{\nu,\alpha,\beta}(t)u_0\|_{L^2}^2 \to 0$, we have

$$\|\mathbb{S}_{\nu,\alpha,\beta}(t)u_0\|_{L^2}^2 < M, \quad \text{for some } t.$$

We remark that the sufficient condition in Theorem 4.1.2 on $\mathcal{W}$ reads

$$\|u_0\|_{L^2}^2 < M,$$

since $F(u_0) = \mathcal{M}(u_0) = M$ with $u_0 \in \mathcal{W}$.

We will show that the case $|p(t)| \to 0$ corresponds to trajectories which exponentially converge to $\mathcal{C}_M$. The conserved momentum $\mathcal{M}(u) = \frac{|c|^2}{(1-|p|^2)^2} = M$ implies that

$$|c(t)|^2 \to M.$$

On the other hand, the decay of $|b(t)|$ (showed in Lemma 4.1.1) implies that

$$\|u(t)\|_{L^2}^2 = |b(t)|^2 + M(1 - |p(t)|^2) \to M.$$

Since $\|u(t)\|_{L^2}^2$ decays monotonically, to study the non-periodic trajectories corresponding to $|p(t)| \to 0$, one only needs to study the trajectories $\{u(t)\}$ disjoint from $\mathcal{C}_M$ and

$$\|u(t)\|_{L^2}^2 \geqslant M, \quad \forall t \geqslant 0.$$

The following theorem of the alternative holds:

Theorem 4.2.1 ([GGH21]). *Let $\nu > 0$ and $\alpha, \beta \in \mathbb{R}$. Then there exists a three dimensional submanifold $\Sigma_{M,\nu,\alpha,\beta} \subset \mathcal{E}_M$, disjoint from $\mathcal{C}_M$ and invariant under the flow $\mathbb{S}_{\nu,\alpha,\beta}(t)$, such that $\Sigma_{M,\nu,\alpha,\beta} \cup \mathcal{C}_M$ is closed and the following holds:*

1. *If $u_0 \in \mathcal{E}_M \backslash (\Sigma_{M,\nu,\alpha,\beta} \cup \mathcal{C}_M)$, then $\|\mathbb{S}_{\nu,\alpha,\beta}(t)u_0\|_{H^s}^2$ blows up with the rate*

$$\|\mathbb{S}_{\nu,\alpha,\beta}(t)u_0\|_{H^s}^2 \sim a^2 t^{2s-1}, \quad s > \frac{1}{2}, \tag{4.13}$$

where

$$a^2(s,\nu,\alpha,\beta,M) = \Gamma(2s+1)M^{4s-1}\left(\frac{\nu^2 + ((1-\beta)M - \alpha)^2}{2\nu}\right)^{1-2s}.$$

2. *If $u_0 \in \Sigma_{M,\nu,\alpha,\beta}$, then $\mathbb{S}_{\nu,\alpha,\beta}(t)u_0$ tends to $\mathcal{C}_M$ as $t \to \infty$, and*

$$\mathit{dist}\,(\mathbb{S}_{\nu,\alpha,\beta}(t)u_0, \mathcal{C}_M) \sim e^{-\frac{\nu+\sigma}{2}t},$$

where $\sigma = \left(\frac{(\nu^2-\alpha^2-4M\alpha)+\sqrt{(\nu^2-\alpha^2-4M\alpha)^2+4\nu^2(2M+\alpha)^2}}{2}\right)^{\frac{1}{2}} \geqslant \nu.$

This alternative behavior holds for all (ν, α, β), which is consistent with the dynamics of the damped Szegő equation ($\alpha = \beta = 0$). Indeed, we will follow a similar argument as for the damped Szegő equation [GG20] to show the above theorem and mainly point out the differences. The first and second parts of the above theorem will be proved in Subsection 4.1 and Subsection 4.2 respectively.

4.2.1 Blow-up trajectories

We introduce a reduced system with

$$\eta = |b|^2, \quad \gamma = M(1-|p|^2), \quad \zeta = Mc\overline{bp},$$

which satisfy the following ODE system

$$\begin{cases} \eta' + 2\nu\eta = 2\mathrm{Im}\zeta, \\ \gamma' = -2\mathrm{Im}\zeta, \\ \zeta' + (\nu + i(1-\beta)M - i\alpha)\zeta = i\zeta((3-\beta)\gamma - \eta) - 2i\eta\gamma M + i\gamma^2(M - \gamma + 3\eta). \end{cases} \tag{4.14}$$

For $u \in \mathcal{W}$, notice that $\hat{u}(k) = cp^{k-1}$, $k \geqslant 1$, then we have

$$\|u(t)\|_{H^s}^2 = \sum_{k=0}^{\infty}(1+k^2)^s|\hat{u}(k)|^2 \sim \frac{\Gamma(2s+1)M}{(1-|p(t)|^2)^{2s-1}} = \Gamma(2s+1)M^{2s}\gamma(t)^{1-2s}.$$

Hence, we only need to show

$$\gamma(t) \sim \frac{\kappa}{t}, \quad \kappa = \frac{\nu^2 + ((1-\beta)M - \alpha)^2}{2\nu M} \tag{4.15}$$

to obtain (4.13)

$$\|u(t)\|_{H^s}^2 \sim a^2 t^{2s-1}$$

with

$$a^2(s, \nu, \alpha, \beta, M) = \Gamma(2s+1)M^{2s}\kappa^{1-2s}.$$

We observe some facts in this case. The conserved momentum implies that $1 - |p|^2$ and $|c|$ decay with the same rate. The integrability and decay of $|b|$ were given in Lemma 4.1.1. As a consequence, in (η, γ, ζ)-coordinate one has

$$\eta \in L^1(\mathbb{R}_+) \quad \text{and} \quad \zeta = o(\gamma). \tag{4.16}$$

To show (4.15), we take the imaginary part of ζ equation and use $\gamma' = -2\mathrm{Im}\zeta$ to derive

$$\frac{\mathrm{Im}\zeta'}{\nu + i((1-\beta)M - \alpha)} - \frac{\gamma'}{2} = \mathrm{Im}f + \mathrm{Im}r, \tag{4.17}$$

where

$$f = i\frac{\gamma^2}{\nu + i((1-\beta)M - \alpha)}(M - \gamma + (3-\beta)\frac{\zeta}{\gamma})$$

and

$$r = -i\frac{\eta}{\nu + i((1-\beta)M - \alpha)}(\zeta + 2\gamma M - 3\gamma^2).$$

The boundedness of η, γ, ζ and the integrability of η ensure that $r \in L^1(\mathbb{R}_+)$. As a consequence of (4.17), one has $\mathrm{Im} f \in L^1(\mathbb{R}_+)$. Furthermore, the structure of f ensures that $\gamma^2 \in L^1(\mathbb{R}_+)$.

Now, one can integrate the both side of (4.17) to derive

$$(1 + o(1))\gamma(t) = 2\int_t^\infty \mathrm{Im} f + 2\int_t^\infty \mathrm{Im} r,$$

where in the left-hand side, we use the fact that $\zeta = o(\gamma)$. Computing the right hand side, we have

$$\int_t^\infty \mathrm{Im} f(s)\, ds = \frac{1}{2\kappa}\Big(\int_t^\infty \gamma(s)^2\, ds\Big)(1 + o(1))$$

and

$$\int_t^\infty \mathrm{Im}\ r(s)\, ds = O\Big(\int_t^\infty \eta(s)\gamma(s)\, ds\Big) = o(\sup_{s\geqslant t} \gamma(s)),$$

where the last equality holds due to the integrability of η.

Now we arrive at

$$\gamma(t) = \frac{1}{\kappa}\Big(\int_t^\infty \gamma(s)^2\, ds\Big)(1 + o(1)) + o(\sup_{s\geqslant t} \gamma(s)). \tag{4.18}$$

We take $\sup_{s\geqslant t}$ on the above equality to get

$$\sup_{s\geqslant t} \gamma(s) = \frac{1}{\kappa}\Big(\int_t^\infty \gamma(s)^2\, ds\Big)(1 + o(1)).$$

Inserting this equality in the equation (4.18) gives

$$\gamma(t) = \frac{1}{\kappa}\Big(\int_t^\infty \gamma(s)^2\, ds\Big)(1 + o(1)).$$

Solving this integral equation gives

$$\gamma(t) = \frac{\kappa}{t}(1 + o(1))$$

as desired.

4.2.2 Scattering trajectories

In this subsection, we the investigate trajectories $\{u(t)\}$ with momentum M, which do not lie in $\mathcal{C}_M$ but converge to it. As a consequence of Lemma 4.1.1 , the L^2 norm of $u(t)$ decays to the momentum M, namely

$$\|u(t)\|_{L^2}^2 = |b(t)|^2 + M(1 - |p(t)|^2) \to M,$$
$$\text{and} \quad \|u(t)\|_{L^2}^2 \geqslant M, \quad \forall t \geqslant 0.$$

We first define the set $\Sigma_{M,\nu,\alpha,\beta}$ as follows

$$\Sigma_{M,\nu,\alpha,\beta} = \{u_0 \in \mathcal{E}_M \backslash \mathcal{C}_M \mid \|\mathbb{S}_{\nu,\alpha,\beta}(t)u_0\|_{L^2}^2 \geqslant M, \, \forall t \geqslant 0\}. \tag{4.19}$$

At the end of this subsection, we will see that $\Sigma_{M,\nu,\alpha,\beta}$ is a three-dimensional submanifold in $\mathcal{E}_M$. We first observe a few facts of the trajectories with $u_0 \in \Sigma_{M,\nu,\alpha,\beta}$:

- Since $\|u(t)\|_{L^2}^2 = |b|^2 + M(1 - |p|^2) \geqslant M$, one has
$$|b(t)|^2 \geqslant M|p(t)|^2. \tag{4.20}$$

- One has
$$\forall t \in \mathbb{R}_+, \quad b(t) \neq 0. \tag{4.21}$$
Otherwise, b and p would cancel at the same time and the trajectory would not be disjoint from $\mathcal{C}_M$.

- From Lemma 4.1.1, (4.20) and the equality $|c(t)|^2 = M(1 - |p(t)|^2)^2$, one has
$$|b(t)|^2, \, |p(t)|^2 \in L^1(\mathbb{R}_+) \quad \text{and} \quad |c(t)|^2 \in L^\infty(\mathbb{R}_+) \tag{4.22}$$

We are going to show the second part of Theorem 4.2.1 in the following four steps: In step 1, we show that $u(t)$ converges to $\mathcal{C}_M$ as $t \to \infty$; In step 2, we establish a scattering property of a reduced system related to b, c, p; In step 3, the asymptotic behavior of $u(t)$ can be recovered on the basis of step 2. The geometric structure of $\Sigma_{M,\nu,\alpha,\beta}$ will be discussed in step 4.

Step 1: Convergence of $c(t)$ and $\frac{\overline{p(t)}}{b(t)}$. We show that there exists $\theta \in \mathbb{T}$ such that

$$e^{itM(1-\beta)}c(t) \to c_\infty = \sqrt{M}e^{i\theta}$$

and

$$e^{-itM(1-\beta)}\sqrt{M}\frac{\overline{p(t)}}{b(t)} \to \left(\frac{\varsigma\rho - \alpha + i(\nu - \sigma)}{2M} - 1\right)e^{-i\theta} \tag{4.23}$$

as $t \to +\infty$. Here

$$\varsigma = \operatorname{sgn}(\alpha + 2M) \in \{-1, 1\} \tag{4.24}$$

and σ, ρ are real non negative numbers satisfying

$$\sigma^2 - \rho^2 = \nu^2 - \alpha^2 - 4\alpha M \text{ and } \varsigma\sigma\rho = \nu(\alpha + 2M). \tag{4.25}$$

In particular

$$\sigma = \left(\frac{(\nu^2 - \alpha^2 - 4M\alpha) + \sqrt{(\nu^2 - \alpha^2 - 4M\alpha)^2 + 4\nu^2(\alpha + 2M)^2}}{2} \right)^{\frac{1}{2}}. \tag{4.26}$$

We first derive the convergence of $c(t)$. The $c(t)$-equation in the ODE system (4.12) implies that

$$i\frac{d}{dt}(e^{itM(1-\beta)}c(t)) = e^{itM(1-\beta)}\big[2|b(t)|^2c(t) + 2M(1 - |p(t)|^2)b(t)p(t)\big]. \tag{4.27}$$

The fact (4.22) ensures the integrability of the right-hand side of the above ODE, together with the conserved momentum, one obtains that

$$e^{itM(1-\beta)}c(t) \to c_\infty = \sqrt{M}e^{i\theta}, \quad \theta \in \mathbb{T}.$$

Since $|b(t)| \to 0$, we choose $\varepsilon = |b(T)|$ for some $T >> 1$. As a consequence of (4.20), one has $|p(T)| \leqslant \frac{\varepsilon}{\sqrt{M}}$. Furthermore, we claim that

$$|c(T) - c_\infty e^{-iTM(1-\beta)}| = O(\int_T^\infty |b(t)|^2\, dt) = O(\varepsilon^2).$$

Indeed, in the above equation, the first equality holds by integration of (4.27). For the second estimate, one integrates the Lyapunov functional (4.10)

$$\frac{d}{dt}(|b(t)|^2 + M(1 - |p(t)|^2)) = -2\nu|b(t)|^2\ ,$$

from T to ∞ to get

$$O(\varepsilon^2) = |b(T)|^2 - M|p(T)|^2 = 2\nu \int_T^\infty |b(t)|^2\, dt.$$

It proves the second estimate.

We combine these estimates of b, c, p at time T and the structure of u on $\mathcal{W}$ to obtain the following estimates in any $H^s(\mathbb{T})$

$$\begin{aligned} &\text{dist } (u(T), c_\infty e^{-iTM(1-\beta)}e^{ix}) = O(\varepsilon), \\ &\text{dist } (u(T), c_\infty e^{-iTM(1-\beta)}e^{ix} + b(T) + c_\infty e^{-iTM(1-\beta)}p(T)e^{i2x}) = O(\varepsilon^2). \end{aligned} \tag{4.28}$$

Now we are ready to show the convergence of $\sqrt{M}\overline{\frac{p(t)}{b(t)}}$ by a linearisation argument. Roughly speaking, we use the convergence (4.28) to linearise the trajectories after time T and check the L^2-norm of the solution. For reader's convenience, we mention that a baby example of the linearisation procedure around solutions with periodic trajectories for the damped Szegő equation was provided in [GG20].

As a consequence of (4.28), we study the following trajectory

$$\mathbb{S}_{\nu,\alpha,\beta}(t)u(T)(x) = e^{-itM(1-\beta)}(c_\infty e^{-iTM(1-\beta)}e^{ix} + \varepsilon v(t,x) + \varepsilon^2 w(t,x)),$$

where w is uniformly bounded in the sense

$$\|w(t)\|_{H^s} \leqslant C_{s,R} \text{ for any } t \in [0,R].$$

To derive the equation v satisfied, we calculate the following quantities

$$\begin{aligned}
&e^{itM(1-\beta)}\partial_t u^\varepsilon(t) = -iM(1-\beta)(c_\infty e^{-iTM(1-\beta)}e^{ix} + \varepsilon v) + \varepsilon\partial_t v + O(\varepsilon^2),\\
&e^{itM(1-\beta)}(u^\varepsilon(t)|\mathbb{1}) = \varepsilon(v|\mathbb{1}) + O(\varepsilon^2),\\
&e^{itM(1-\beta)}\Pi(|u^\varepsilon(t)|^2u^\varepsilon(t)) = Mc_\infty e^{-iTM(1-\beta)}e^{ix} + \varepsilon c_\infty^2\Pi(e^{-i2TM(1-\beta)}e^{i2x}\overline{v})\\
&\qquad\qquad + \varepsilon 2Mv + O(\varepsilon^2),\\
&e^{itM(1-\beta)}S\Pi(|S^*u^\varepsilon(t)|^2S^*u^\varepsilon(t)) = Mc_\infty e^{-iTM(1-\beta)}e^{ix}\\
&\qquad\qquad + \varepsilon c_\infty^2 e^{-i2TM(1-\beta)}e^{ix}\Pi(e^{ix}\overline{v}) + \varepsilon 2Me^{ix}\Pi(e^{-ix}v)\\
&\qquad\qquad - \varepsilon c_\infty^2 e^{-i2TM(1-\beta)}e^{i2x}\overline{(v|\mathbb{1})} + O(\varepsilon^2).
\end{aligned}$$

Then v satisfies the equation

$$\begin{aligned}
&i\partial_t v + (i\nu-\alpha)(v|\mathbb{1}) = c_\infty^2 e^{-i2TM(1-\beta)}\Pi(e^{i2x}\overline{v}) + (\beta+1)Mv\\
&- \beta c_\infty^2 e^{-i2TM(1-\beta)}e^{ix}\Pi(e^{ix}\overline{v}) - 2\beta Me^{ix}\Pi(e^{-ix}v) + \beta c_\infty^2 e^{-i2TM(1-\beta)}e^{i2x}\overline{(v|\mathbb{1})}.
\end{aligned}$$

with the initial value $v(0,x) = \frac{b(T)}{\varepsilon} + c_\infty e^{-iTM(1-\beta)}\frac{p(T)}{\varepsilon}e^{i2x}$. We observe the equation of v and make the ansatz

$$v(t,x) = q_0(t) + q_1(t)e^{ix} + q_2(t)e^{i2x},$$

where

$$\begin{aligned}
iq_0' + (i\nu-\alpha)q_0 &= (1+\beta)Mq_0 + c_\infty^2 e^{-2iTM(1-\beta)}\overline{q}_2,\\
iq_1' &= (1-\beta)(c_\infty^2 e^{-2iTM(1-\beta)}\overline{q}_1 + Mq_1),\\
iq_2' &= (1-\beta)Mq_2 + c_\infty^2 e^{-2iTM(1-\beta)}\overline{q}_0,\\
q_0(0) = \frac{b(T)}{\varepsilon}, q_2(0) &= c_\infty e^{-iTM(1-\beta)}\frac{p(T)}{\varepsilon}.
\end{aligned}$$

We take the derivative of the q_0-equation and substitute the equation of q_2 to derive

$$\begin{aligned}
&q_0'' + (\nu + i(\alpha + 2\beta M))q_0' - ((1-\beta)(i\nu - \alpha)M + \beta^2 M^2)q_0 = 0,\\
&q_0(0) = \frac{b(T)}{\varepsilon},\\
&q_0'(0) = -(\nu + i(\alpha + (1+\beta)M))\frac{b(T)}{\varepsilon} - iMc_\infty e^{-iTM(1-\beta)}\frac{\overline{p(T)}}{\varepsilon}.
\end{aligned}$$

The characteristic equation of this second-order ODE reads

$$\lambda^2 + (\nu + i(\alpha + 2\beta M))\lambda - ((1-\beta)(i\nu - \alpha)M + \beta^2 M^2) = 0.$$

The solutions are given by

$$\lambda_\pm = \frac{-(\nu + i(\alpha + 2\beta M)) \pm (\sigma + i\varsigma\rho)}{2},$$

where ς, ρ, σ are defined by equations (4.24), (4.25).

We will prove in the end of Subsection 4.2.2 that

$$\frac{\sigma - \nu}{\sigma + \nu} > 0$$

so that $\sigma > \nu$. Hence, the real parts of λ_+ and λ_- admits different signs as

$$\mathrm{Re}(\lambda_+) = \sigma - \nu > 0 \quad \text{and} \quad \mathrm{Re}(\lambda_-) = -\sigma - \nu < 0.$$

And hence, the solution q_0 is given by

$$q_0(t) = A_+ e^{\lambda_+ t} + A_- e^{\lambda_- t}, \tag{4.29}$$

where

$$\begin{aligned}
A_+(T) &= \frac{q_0'(0) - \lambda_- q_0(0)}{\lambda_+ - \lambda_-}\\
&= -\frac{(\nu + i(\alpha + (1+\beta)M) + \lambda_-)b(T) + iMc_\infty e^{-iTM(1-\beta)}\overline{p(T)}}{\varepsilon(T)(\sigma + i\sqrt{\sigma^2 - \nu^2 + \alpha^2 + 4M\alpha})},\\
A_-(T) &= \frac{\lambda_+ q_0(0) - q_0'(0)}{\lambda_+ - \lambda_-}\\
&= \frac{(\nu + i(\alpha + (1+\beta)M) + \lambda_+)b(T) + iMc_\infty e^{-iTM(1-\beta)}\overline{p(T)}}{\varepsilon(T)(\sigma + i\sqrt{\sigma^2 - \nu^2 + \alpha^2 + 4M\alpha}))}.
\end{aligned}$$

Now we are ready to check the L^2-norm of $\mathbb{S}_{\nu,\alpha,\beta}(t)u(T)$, especially the two important features: $\|\mathbb{S}_{\nu,\alpha,\beta}(t)u(T)\|_{L^2}^2 \geqslant M$ and the Lyapunov functional.

Indeed, the following estimate holds for any fixed T and $t \in [0, R]$

$$\begin{aligned}
0 &\leqslant \|\mathbb{S}_{\nu,\alpha,\beta}(t)u(T)\|_{L^2}^2 - M \\
&= \|u(T)\|_{L^2}^2 - M - 2\nu \int_0^t |(\mathbb{S}_{\nu,\alpha,\beta}(s)u(T)|\mathbb{1})|^2 \, ds \\
&= |b(T)|^2 - M|p(T)|^2 - 2\nu\varepsilon^2 \int_0^t \big(|(v(s)|\mathbb{1})|^2 + O_R(\varepsilon)\big) \, ds.
\end{aligned}$$

We divide the above inequality both sides by $|b(T)|^2$ to derive

$$1 - \underbrace{\frac{M|p(T)|^2}{|b(T)|^2}}_{\leqslant 1} - 2\nu \int_0^t |\underbrace{(v(s)|\mathbb{1})}_{=q_0(s)}|^2 \, ds \geqslant -c_R\varepsilon(T), \quad t \in [0, R],$$

where c_R is a constant depending only on R. Recall $q_0(t) = A_+(T)e^{\lambda_+ t} + A_-(T)e^{\lambda_- t}$. By computing the integral of q_0 and using the above estimate, we infer the existence of a constant B such that

$$|A_+(T)|^2 e^{(\sigma-\nu)R} \leqslant c_R\varepsilon(T) + B,$$

We first take upper limit in T of the above inequality

$$\limsup_{T\to\infty} |A_+(T)|^2 e^{(\sigma-\nu)R} \leqslant B.$$

Then we take limit in $R \to \infty$, the above inequality holds only if

$$\limsup_{T\to\infty} |A_+(T)|^2 = 0,$$

which implies the convergence (4.23)

$$\begin{aligned}
e^{-iTM(1-\beta)}\sqrt{M}\frac{\overline{p(T)}}{b(T)} &\to i\frac{\nu + \lambda_- + i(\alpha + (1+\beta)M)}{M} e^{-i\theta} \\
&= \Big(\frac{\varsigma\rho - \alpha + i(\nu - \sigma)}{2M} - 1\Big)e^{-i\theta}.
\end{aligned}$$

Step 2: Scattering properties for η, δ, ζ We write

$$\eta = |b|^2,\ \delta = M|p|^2,\ \zeta = Mc\overline{pb},$$

which satisfy the ODE system

$$\begin{cases} \eta' + 2\nu\eta = 2\mathrm{Im}\zeta, \\ \delta' = 2\mathrm{Im}\zeta, \\ \zeta' + (\nu - i(2M+\alpha))\zeta = -i((3-\beta)\delta + \eta)\zeta + i(M-\delta)^2(\delta+\eta) - 2i\eta\delta(M-\delta). \end{cases} \tag{4.30}$$

Due to the decay and boundedness of (b, c, p) in (4.22), we have $(\eta(t), \delta(t), \zeta(t)) \to (0, 0, 0)$ and $\eta \in L^1(\mathbb{R}_+)$. The Lyapunov functional in the (η, δ, ζ)-coordinate can be written as

$$\eta(t) - \delta(t) = 2\nu \int_t^\infty \eta(s)\, ds. \tag{4.31}$$

The convergence (4.23) implies that

$$\frac{\delta(t)}{\eta(t)} \to \left| \frac{\varsigma\rho - \alpha + i(\nu - \sigma)}{2M} - 1 \right|^2, \quad \text{as } t \to +\infty. \tag{4.32}$$

We will prove in the end of Subsection 4.2.2 that

$$\left| \frac{\varsigma\rho - \alpha + i(\nu - \sigma)}{2M} - 1 \right|^2 = \frac{\sigma - \nu}{\sigma + \nu} \tag{4.33}$$

Combining the Lyapunov functional (4.31) and the convergence (4.32), the decay rate of $\eta(t)$ is given by

$$\frac{\eta(t)}{\int_t^\infty \eta(s)\, ds} \to \sigma + \nu, \quad \text{as } t \to +\infty, \tag{4.34}$$
$$\log\Big(\int_t^\infty \eta(s)\, ds\Big) \to -(\sigma+\nu)t(1+o(1)), \quad \text{as } t \to +\infty,$$

and

$$\eta(t) \leqslant C_\varepsilon e^{-(\sigma+\nu-\varepsilon)t}, \quad t \geqslant 0, \text{ for any } \varepsilon > 0. \tag{4.35}$$

We write

$$X = \begin{pmatrix} \eta(t) \\ \delta(t) \\ \zeta_R(t) = \mathrm{Re}\zeta(t) \\ \zeta_I(t) = \mathrm{Im}\zeta(t) \end{pmatrix} \in \mathbb{R}^4,$$

then the ODE system of (η, δ, ζ) can be written as

$$X' + AX = Q(X), \tag{4.36}$$

where

$$A = \begin{pmatrix} 2\nu & 0 & 0 & -2 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & \nu & 2M+\alpha \\ -M^2 & -M^2 & -(2M+\alpha) & \nu \end{pmatrix},$$

and

$$Q(X) = \begin{pmatrix} 0 \\ 0 \\ (\eta + (3-\beta)\delta)\zeta_I \\ -(\eta + (3-\beta)\delta)\zeta_R - 2M\delta^2 - 4M\eta\delta + \delta^3 + 3\eta\delta^2 \end{pmatrix}.$$

The matrix A has the eigenvalues

$$\nu \pm \sigma, \quad \nu \pm i\rho.$$

Now we write the ODE (4.36) as

$$\frac{d}{dt}(e^{tA}X(t)) = e^{tA}Q(X(t)),$$

where the solution $X(t)$ is given by the Duhamel formula

$$X(t) = e^{-tA}X_\infty - \int_t^\infty e^{(s-t)A}Q(X(s))\,ds. \tag{4.37}$$

Since $\sigma + \nu$ is the largest eigenvalue of A, and Q is a quadratic-cubic form of X, one has the following estimates

$$\begin{aligned} |X(t)| &\lesssim_\varepsilon e^{-(\sigma+\nu-\varepsilon)t}, \\ |e^{tA}(Q(X(t)))| &\lesssim_\varepsilon e^{-(\nu+\sigma-2\varepsilon)t}, \\ |e^{(s-t)A}(Q(X(t)))| &\lesssim_\varepsilon e^{(\sigma+\nu)(s-t)-2(\sigma+\nu-\varepsilon)s}, \quad s,\, t \geqslant 0 \end{aligned}$$

and

$$\int_t^\infty |e^{(s-t)A}(Q(X(t)))|\,ds \lesssim_\varepsilon e^{-2(\sigma+\nu-\varepsilon)t}, \quad t \geqslant 0.$$

We substitute the above estimates to the Duhamel formula to obtain

$$e^{-tA}X_\infty = O(e^{-(\sigma+\nu)t}). \tag{4.38}$$

This shows that X_∞ is an eigenvector of A for the eigenvalue $\sigma + \nu$. Consequently, there exists a constant $\eta_\infty \in \mathbb{R}$, such that

$$X_\infty = \eta_\infty \begin{pmatrix} 1 \\ \dfrac{\sigma-\nu}{\sigma+\nu} \\ (2M+\alpha)\dfrac{\nu-\sigma}{2\sigma} \\ \dfrac{\nu-\sigma}{2} \end{pmatrix}.$$

We substitute X_∞ to the Duhamel formula (4.37) to conclude that there exists $\eta_\infty > 0$ (since $\eta_\infty(t) > 0$) such that

$$\begin{aligned} \eta(t) &= \eta_\infty e^{-(\sigma+\nu)t}(1 + O(e^{-(\sigma+\nu)t})), \\ \delta(t) &= \frac{\sigma-\nu}{\sigma+\nu}\eta_\infty e^{-(\sigma+\nu)t}(1 + O(e^{-(\sigma+\nu)t})), \\ \zeta(t) &= \big(\frac{\varsigma\rho - \alpha + i(\nu-\sigma)}{2} - M\big)\eta_\infty e^{-(\sigma+\nu)t}(1 + O(e^{-(\sigma+\nu)t})), \end{aligned} \tag{4.39}$$

where the last equality holds since, using (4.25)

$$(2M+\alpha)\frac{(\nu-\sigma)}{2\sigma} + i\frac{\nu-\sigma}{2} = \frac{\varsigma\rho - \alpha + i(\nu-\sigma)}{2} - M.$$

Conversely, we claim that for every $\eta_\infty > 0$, there exists a unique triple (η, δ, ζ) such that

$$\begin{cases} \eta' + 2\nu\eta = 2\mathrm{Im}\zeta, \\ \delta' = 2\mathrm{Im}\zeta, \\ \zeta' + (\nu - i(2M+\alpha))\zeta = -i((3-\beta)\delta + \eta)\zeta + i(M-\delta)^2(\delta+\eta) - 2i\eta\delta(M-\delta) \end{cases} \tag{4.40}$$

satisfying the asymptotic behavior (4.39). Indeed, by a standard fixed point argument one can solve (4.37) on $[T, \infty)$, where T is large enough such that $|X_\infty| e^{-(\nu+\sigma)T} \lesssim 1$, with the norm

$$\|X\|_T := \sup_{t \geqslant T} e^{(\nu+\sigma)t}|X(t)|.$$

Then the extension to the whole real line is ensured by, say, the identities

$$|\zeta|^2 = (M-\delta)^2\eta\delta \ , \ \delta(t) + 2\nu\int_t^\infty \eta(s)\, ds = \eta(t)$$

which, combined with the first equation, lead to

$$|\dot{\eta}| = O(\eta) \ .$$

Furthermore, following the same argument as for the damped Szegő equation in [GG20], the lower (upper) bounds of initial value $\eta(0)$ ensure the lower (upper) bounds for η_∞. Namely, for every $C > 0$, there exists $C' > 0$ such that

- if $\eta(0) \geqslant C^{-1}$, then $\eta_\infty \geqslant (C')^{-1}$,
- if $\eta(0) \leqslant C$, then $\eta_\infty \leqslant C'$.

Step 3: Asymptotic behavior for b, c, p
We first show that there exists

$$(\eta_\infty, \theta, \varphi) \in (0, \infty) \times \mathbb{T} \times \mathbb{T}$$

such that, as $t \to +\infty$

$$\begin{aligned}
b(t) &\sim \sqrt{\eta_\infty} e^{-\frac{\nu+\sigma}{2}t - it(2M+\alpha)(\frac{\nu+\sigma}{2\sigma}) + i\varphi},\\
c(t) &\sim \sqrt{M} e^{-itM(1-\beta)} e^{i\theta},\\
p(t) &\sim \sqrt{\frac{\eta_\infty}{M}} \left(\frac{\varsigma\rho - \alpha + i(\nu - \sigma)}{2M} - 1 \right) e^{-\frac{\nu+\sigma}{2}t + it\frac{(\alpha+2M\beta)\sigma + (2M+\alpha)\nu}{2\sigma} + i(\theta-\varphi)}.
\end{aligned} \tag{4.41}$$

Notice that, the convergence of $c(t)$ was shown in (4.2.2). We combine the equations for b and η, p and δ to derive

$$\begin{aligned}
i\frac{d}{dt}\Big(\frac{b}{\sqrt{\eta}}\Big) &= \big(\eta - 2\delta + 2M + \alpha + \frac{\mathrm{Re}\zeta}{\eta}\big)\frac{b}{\sqrt{\eta}}\\
&= \big((2M+\alpha)\frac{\nu+\sigma}{2\sigma} + O(e^{-(\nu+\sigma)t})\big)\frac{b}{\sqrt{\eta}}\\
i\frac{d}{dt}\Big(\frac{\sqrt{M}p}{\sqrt{\delta}}\Big) &= \big((1-\beta)(M-\delta) + \frac{\mathrm{Re}\zeta}{\delta}\big)\frac{\sqrt{M}p}{\sqrt{\delta}}\\
&= \big(-\frac{(\alpha+2M\beta)\sigma + (2M+\alpha)\nu}{2\sigma} + O(e^{-(\nu+\sigma)t})\big)\frac{\sqrt{M}p}{\sqrt{\delta}}.
\end{aligned}$$

Then there exist φ, ψ such that

$$\begin{aligned}
b(t) &\sim \sqrt{\eta_\infty} e^{-\frac{\nu+\sigma}{2}t - it(2M+\alpha)\frac{\nu+\sigma}{2\sigma} + i\varphi},\\
p(t) &\sim \sqrt{\frac{\eta_\infty}{M}} \Big(\frac{\sigma-\nu}{\sigma+\nu}\Big)^{\frac{1}{2}} e^{-\frac{\nu+\sigma}{2}t + it\frac{(\alpha+2M\beta)\sigma + (2M+\alpha)\nu}{2\sigma} + i\psi}.
\end{aligned}$$

On the other side, we recall the convergence (4.23)

$$e^{-iTM(1-\beta)}\sqrt{M}\frac{\overline{p(T)}}{b(T)} \to \left(\frac{\varsigma\rho - \alpha + i(\nu-\sigma)}{2M} - 1 \right) e^{-i\theta}$$

and the equality (4.33)

$$\Big(\frac{\sigma-\nu}{\sigma+\nu}\Big)^{\frac{1}{2}} = \Big|\frac{\varsigma\rho - \alpha + i(\nu-\sigma)}{2M} - 1\Big|.$$

This implies that

$$p(t) \sim \sqrt{\frac{\eta_\infty}{M}} \left(\frac{\varsigma\rho - \alpha + i(\nu - \sigma)}{2M} - 1 \right) e^{-\frac{\nu+\sigma}{2}t + it\frac{(\alpha+2M\beta)\sigma + (2M+\alpha)\nu}{2\sigma} + i(\theta-\varphi)}.$$

Now we show that the asymptotic behavior (4.41) holds conversely. Namely, for a fixed $(\eta_\infty, \theta, \varphi) \in (0, \infty) \times \mathbb{T} \times \mathbb{T}$, there exists a unique trajectory

$$u(t,x) = b(t) + \frac{c(t)e^{ix}}{1 - p(t)e^{ix}}$$

satisfying the asymptotic behavior (4.41). From step 2, for such $(\eta_\infty, \theta, \varphi) \in (0, \infty) \times \mathbb{T} \times \mathbb{T}$ there exists a unique trajectory $(\eta, \delta, \zeta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{C}$ such that

$$\begin{aligned}
\eta(t) &= \eta_\infty e^{-(\sigma+\nu)t}(1 + O(e^{-(\sigma+\nu)t})),\\
\delta(t) &= \frac{\sigma - \nu}{\sigma + \nu}\eta_\infty e^{-(\sigma+\nu)t}(1 + O(e^{-(\sigma+\nu)t})),\\
\zeta(t) &= \left(\frac{\varsigma\rho - \alpha + i(\nu - \sigma)}{2} - M\right)\eta_\infty e^{-(\sigma+\nu)t}(1 + O(e^{-(\sigma+\nu)t})).
\end{aligned}$$

Due to the structure of η, δ, ζ, there exists a fixed large enough $T > 0$ such that $M > \delta(T) > 0$, $\zeta(T) \neq 0$ and $\eta(T) > 0$. Then there exists (b_1, c_1, p_1) solving the ODE system (4.12) such that

$$b_1(T) = \sqrt{\eta(T)}, \quad \sqrt{M}p_1(T) = \sqrt{\delta(T)}, \quad Mc_1(T) = \frac{\zeta(T)}{\overline{b_1(T)p_1(T)}}.$$

Furthermore, due to the uniqueness of the Cauchy problem of the ODE system for (η, δ, ζ), the above equations hold for all $t \in \mathbb{R}$. On the other hand, (b_1, c_1, p_1) satisfies the asymptotic behavior (4.41) with a pair $(\theta_1, \varphi_1) \in \mathbb{T} \times \mathbb{T}$, i.e.

$$\begin{aligned}
b_1(t) &\sim \sqrt{\eta_\infty}e^{-\frac{\nu+\sigma}{2}t - it(2M+\alpha)\frac{\nu+\sigma}{2\sigma} + i\varphi_1},\\
c_1(t) &\sim \sqrt{M}e^{-itM(1-\beta)}e^{i\theta_1},\\
p_1(t) &\sim \sqrt{\frac{\eta_\infty}{M}}\left(\frac{\varsigma\rho - \alpha + i(\nu - \sigma)}{2M} - 1\right)e^{-\frac{\nu+\sigma}{2}t + it\frac{(\alpha+2M\beta)\sigma + (2M+\alpha)\nu}{2\sigma} + i(\theta_1 - \varphi_1)}.
\end{aligned}$$

Then the triplet (b, c, p) with

$$b(t) = e^{i(\varphi - \varphi_1)}b_1(t), \quad c(t) = e^{i(\theta - \theta_1)}c_1(t), \quad p(t) = e^{i(\theta - \varphi - \theta_1 + \varphi_1)}p_1(t)$$

satisfies the ODE system (4.12) with the desired asymptotic properties.

At the last step, we show the uniqueness of the solution (b, c, p). We assume that $(\tilde{b}, \tilde{c}, \tilde{p})$ is another solution of the ODE system (4.12) with the same asymptotic properties (4.41). Then by the uniqueness of (η, δ, ζ), for any $t \in \mathbb{R}$

$$|\tilde{b}(t)|^2 = \eta(t), \quad M|\tilde{p}(t)|^2 = \delta(t), \quad M\tilde{c}(t)\overline{\tilde{b}(t)}\overline{\tilde{p}(t)} = \zeta(t)$$

and hence, $b-\tilde{b}$ and $p-\tilde{p}$ satisfy

$$\begin{aligned} i\frac{d}{dt}\Big(\frac{b-\tilde{b}}{\sqrt{\eta}}\Big) &= \Big((2M+\alpha)\frac{\nu+\sigma}{2\sigma} + O(e^{-(\nu+\sigma)t})\Big)\frac{b-\tilde{b}}{\sqrt{\eta}}, \\ i\frac{d}{dt}\Big(\frac{\sqrt{M}(p-\tilde{p})}{\sqrt{\delta}}\Big) &= \Big(-\frac{(\alpha+2M\beta)\sigma+(2M+\alpha)\nu}{2\sigma} + O(e^{-(\nu+\sigma)t})\Big)\frac{\sqrt{M}(p-\tilde{p})}{\sqrt{\delta}}. \end{aligned}$$

The first ODE implies

$$\Big|\frac{b-\tilde{b}}{\sqrt{\eta}}(t)\Big| \leqslant \int_t^{\infty} O(e^{-(\nu+\sigma)s})\Big|\frac{b-\tilde{b}}{\sqrt{\eta}}(s)\Big|\,ds,$$

Combining this estimate with the boundedness of $|\frac{b-\tilde{b}}{\sqrt{\eta}}|$, one can show that $b(t)=\tilde{b}(t)$ for any $t\in\mathbb{R}$ by an iterative argument. By a similar argument, we have $p(t)=\tilde{p}(t)$ for all $t\in\mathbb{R}$. Since $c(t)\overline{b(t)p(t)}=\tilde{c}(t)\overline{\tilde{b}(t)\tilde{p}(t)}=\zeta(t)$, we conclude that $c(t)=\tilde{c}(t)$, for all $t\in\mathbb{R}$.

Step 4: Geometric structure of $\Sigma_{M,\nu,\alpha,\beta}\subset\mathcal{E}_M$. We define the map

$$J:(0,\infty)\times\mathbb{T}\times\mathbb{T}\to\mathcal{E}_M,\quad (\eta_\infty,\theta,\varphi)\mapsto u(0),$$

where u is the unique solution of (4.1) corresponding to the asymptotic behavior (4.41). One can follow a similar argument as in [GG20] to show that J is an one-to-one proper immersion, and hence, $\Sigma_{M,\nu,\alpha,\beta}$ is a three-dimensional submanifold of $\mathcal{E}_M$. We only point out the following different identity as in [GG20]

$$\begin{aligned} &e^{i\varphi}\mathbb{S}_{\alpha,\beta,\nu}(t+T)\left[J(\eta_\infty,\theta_0,\varphi_0)\right](x+\theta-\varphi) \\ =&\mathbb{S}_{\alpha,\beta,\nu}(t)\left[J(\bar{\eta}_\infty,\bar{\theta},\bar{\varphi})\right](x), \end{aligned}$$

where

$$\begin{aligned} &\bar{\eta}_\infty=\eta_\infty e^{-(\sigma+\nu)T}, \\ &\bar{\theta}=\theta_0+\theta-M(1-\beta)T, \\ &\bar{\varphi}=\varphi_0+\varphi-(M+\frac{\alpha}{2})(1+\frac{\nu}{\sigma})T. \end{aligned}$$

A calculation

At the end of this subsection, we give the details of the calculations on

$$Z:=\left|\frac{\varsigma\rho-\alpha+i(\nu-\sigma)}{2M}-1\right|^2=\frac{\sigma-\nu}{\sigma+\nu},$$

where ς, σ, ρ are given by (4.24) and (4.25). We recall the following equalities.

$$\sigma^2 - \rho^2 = \nu^2 - \alpha^2 - 4\alpha M \text{ and } \varsigma\sigma\rho = \nu(\alpha + 2M).$$

Remark first that the case $\sigma = \nu$ is excluded. Indeed, if $\sigma = \nu$ then $\rho^2 = \alpha^2 + 4M\alpha$ which is not compatible with the second condition. We calculate

$$\begin{aligned}
Z &= \left|\frac{\varsigma\rho - \alpha + i(\nu - \sigma)}{2M} - 1\right|^2 \\
&= 1 + \frac{1}{4M^2}\Big((\varsigma\rho - \alpha)^2 + (\nu - \sigma)^2 - 4M(\varsigma\rho - \alpha)\Big) \\
&= 1 + \frac{1}{4M^2}\Big(\rho^2 + \alpha^2 + \nu^2 + \sigma^2 - 2\nu\sigma + 4M\alpha - 2\varsigma\rho(\alpha + 2M)\Big) \\
&= 1 + \frac{1}{2M^2}(\rho^2 + \nu^2)(1 - \frac{\sigma}{\nu}).
\end{aligned}$$

Then as above

$$\begin{aligned}
(1 - Z)(\nu + \sigma) &= \frac{\rho^2 + \nu^2}{2M^2}\frac{\sigma - \nu}{\nu}(\sigma + \nu) \\
&= \frac{1}{2\nu M^2}(\sigma^2\rho^2 + \sigma^2\nu^2 - \nu^2\rho^2 - \nu^4) \\
&= \frac{\nu}{2M^2}((\alpha + 2M)^2 + \sigma^2 - \rho^2 - \nu^2) \\
&= \frac{\nu}{2M^2}((\alpha + 2M)^2 - \alpha^2 - 4M\alpha) \\
&= 2\nu.
\end{aligned}$$

According to the above calculation, one has

$$\frac{2\nu}{1 - Z} = \nu + \sigma \quad \text{and} \quad Z = \frac{\sigma - \nu}{\sigma + \nu}.$$

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