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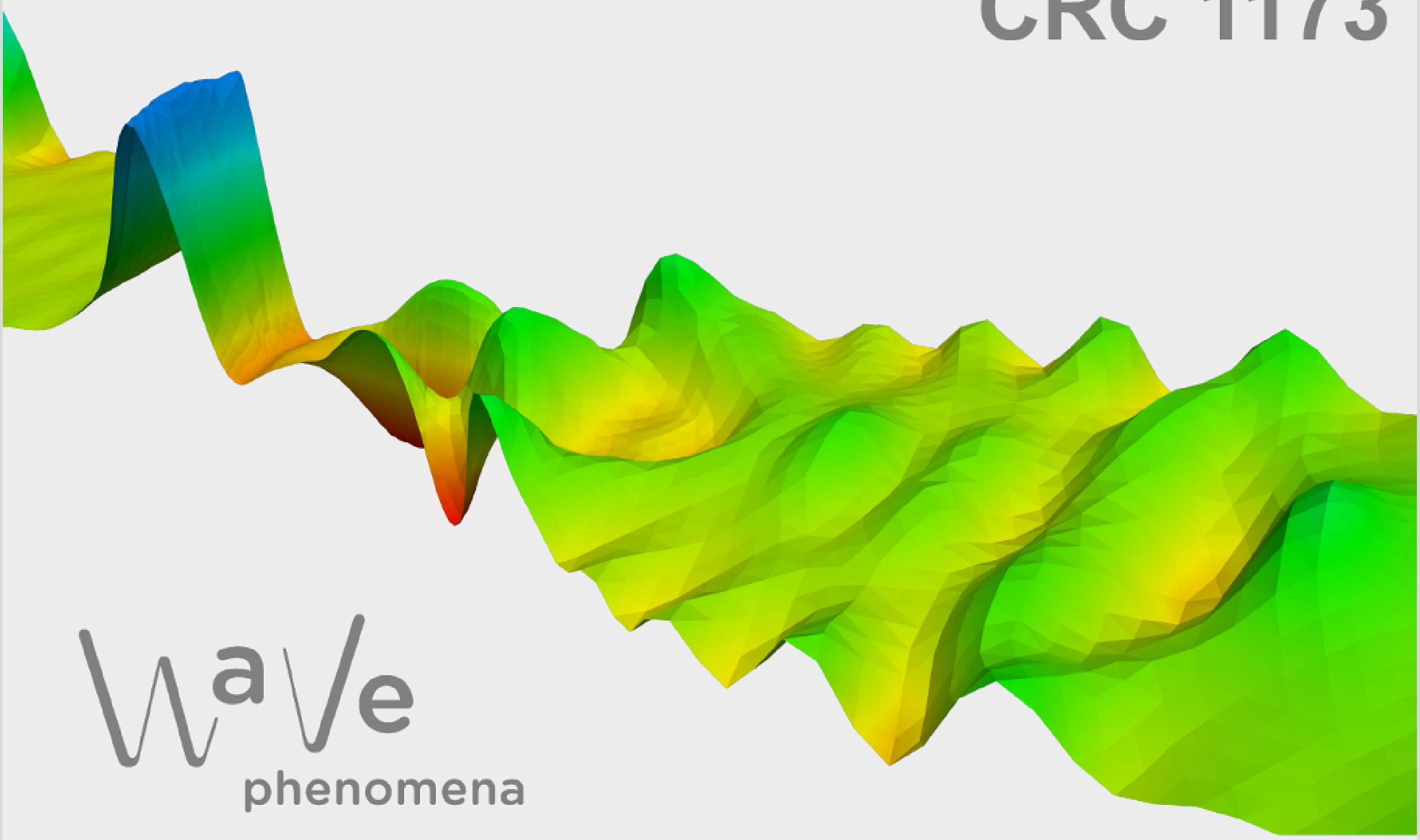

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CONSERVED ENERGIES FOR THE ONE DIMENSIONAL GROSS-PITAEVSKII EQUATION: LOW REGULARITY CASE

HERBERT KOCH AND XIAN LIAO

Abstract. We construct a family of conserved energies for the one dimensional Gross-Pitaevskii equation, but in the low regularity case (in [14] we have constructed conserved energies in the high regularity situation). This can be done thanks to regularization procedures and a study of the topological structure of the finite-energy space. The asymptotic (regularised conserved) phase change on the real line with values in $\mathbb{R}/2\pi\mathbb{Z}$ is studied. We also construct a conserved quantity, the renormalized momentum H_1 (see Theorem 1.3), on the universal covering space of the finite-energy space.

Keywords: Gross-Pitaevskii equation, transmission coefficient, conserved energies, regularisation, asymptotic phase change, renormalized momentum

AMS Subject Classification (2010): 35Q55, 37K10

1. Introduction

We consider the one dimensional Gross-Pitaevskii equation

$$i\partial_t q + \partial_{xx} q = 2q(|q|^2 - 1). \tag{1.1}$$

Here $(t, x) \in \mathbb{R}^2$ denote the one dimensional time and space variables respectively and $q = q(t, x)$ denotes the unknown complex-valued wave function.

The Gross-Pitaevskii equation (1.1) can be viewed as the defocusing cubic nonlinear Schrödinger equation (NLS), but assuming a nonzero boundary condition at infinity

$$|q(t, x)| \to 1, \quad \text{as } |x| \to \infty.$$

It is relevant in the physical contexts of Bose-Einstein condensation, nonlinear optics (e.g. optical vortices) and fluid mechanics (e.g. superfluidity of Helium II). Thanks to this nonzero boundary condition, the Gross-Pitaevskii equation (1.1) possesses the following interesting black (with $c = 0$) and dark (with $c \neq 0$) soliton solutions in nonlinear optics

$$q_c(t, x) = \sqrt{1 - c^2} \tanh\Big(\sqrt{1 - c^2}(x - 2ct)\Big) + ic, \quad -1 < c < 1. \tag{1.2}$$

They are called black resp. dark since their density drops from the background density 1:

$$|q_c(t, x)|^2 = 1 - (1 - c^2)\,\mathrm{sech}^2\big(\sqrt{1 - c^2}(x - 2ct)\big) < 1, \quad -1 < c < 1.$$

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There are no soliton solutions of (1.1) with travelling speed $|2c| \geq 2$. There were many works, e.g. [3, 2, 4, 10, 16], contributing to the stability of these soliton solutions.

The Gross-Pitaevskii equation (1.1) is the Hamiltonian flow with respect to the Hamiltonian function

$$\mathcal{E}(q) = \int_{\mathbb{R}} \big((|q|^2-1)^2 + |\partial_x q|^2\big)\,\mathrm{d}x\,, \tag{1.3}$$

and the Poisson structure (see [7])

$$\omega(F,G) = i\int_{\mathbb{R}} \Big(\frac{\delta F}{\delta q}\frac{\delta G}{\delta \bar q} - \frac{\delta F}{\delta \bar q}\frac{\delta G}{\delta q}\Big)\,\mathrm{d}x\,.$$

Aside from the energy $\mathcal{E}$, the mass

$$\mathcal{M} = \int_{\mathbb{R}} (|q|^2-1)\,\mathrm{d}x\,, \tag{1.4}$$

and the momentum

$$\mathcal{P} = \operatorname{Im}\int_{\mathbb{R}} q\partial_x \bar q\,\mathrm{d}x \tag{1.5}$$

are conserved by the Gross-Pitaevskii flow. One computes straightforward for the soliton solution family (1.2) that

$$\begin{aligned}\mathcal{M}(q_c) &= -2(1-c^2),\\ \mathcal{P}(q_c) &= 2c\sqrt{1-c^2},\\ \mathcal{E}(q_c) &= \frac{8}{3}\sqrt{1-c^2}^{\,3}.\end{aligned}$$

The Gross-Pitaevskii equation (1.1) was shown by P. Zhidkov [18] to be locally-in-time well-posed in the so-called Zhidkov's space

$$\begin{aligned}&Z^k = \{q \in L^\infty(\mathbb{R}) \,|\, \partial_x q \in H^{k-1}(\mathbb{R})\},\\ &\text{associated with the norm } \|q\|_{Z^k} = \|q\|_{L^\infty} + \sum_{1\leq l\leq k}\|\partial_x^l q\|_{L^2},\end{aligned}$$

for $k \geq 1$, and globally-in-time well-posed in Z^1 by use of the conserved energy $\mathcal{E}(q)$ in (1.3). However even the stability with respect to this metric of the trivial solution $q=1$ is expected to be false: We expect that for all $\varepsilon > 0$ and $k \geq 0$ there exists initial data q_0 so that $\|q_0 - 1\|_{Z^k} < \varepsilon$ but the solution $q(t)$ satisfies $\sup_t \|q(t)-1\|_{L^\infty} \geq 2$, with 2 being the diameter of the unit circle.

It is also interesting to mention the global-in-time well-posedness results by P. Gérard [8, 9] in the energy space $Y^1 = \{q \in H^1_{\mathrm{loc}}(\mathbb{R}^n) : |q|^2-1 \in L^2(\mathbb{R}^n),\ \nabla q \in L^2(\mathbb{R}^n)\}$, endowed with the metric distance

$$\begin{aligned}&d_{Y^1}(p,q) = \|p-q\|_{Z^1+H^1} + \big\||p|^2-|q|^2\big\|_{L^2},\\ &\text{with } \|u\|_{A+B} = \inf\{\|u_1\|_A + \|u_2\|_B \,|\, u = u_1+u_2,\ u_1\in A,\ u_2 \in B\}.\end{aligned}$$

Notice that there is no simple relation between the metric space (X^1, d^1) (see Theorem 1.1 below) and the Zhidkov's space $(Z^1, \|\cdot\|_{Z^1})$ or the Gérard's space (Y^1, d_{Y^1}). In higher dimensions there are scattering results in [11, 12], which are not expected to hold in one space dimension by virtue of these soliton solutions (1.2).

We denote the finite-energy space of the Gross-Pitaevskii equation by

$$q(t,\cdot) \in X^1 := \big\{p \in H^1_{\mathrm{loc}}(\mathbb{R}) \,:\, |p|^2-1, \partial_x p \in L^2(\mathbb{R})\big\}/\mathbb{S}^1,$$

where $\mathbb{S}^1$ denotes the unit circle (that is, the functions which differ by a multiplicative constant of modulus 1 are identified in X^1). Let the operator $D = \langle \partial_x \rangle$ be defined by

$$\widehat{Df}(\xi) = \langle \xi \rangle \hat{f}(\xi), \tag{1.6}$$

where $\langle \xi \rangle = \sqrt{2^2 + \xi^2}$, and $\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} f(x)\,\mathrm{d}x$ denotes the Fourier transform of the function $f(x)$. Here the factor 2 appears due to the mismatch between the usual Fourier transform and the (inverse) scattering transform terminology. We define

$$\|q\|_{H^s} = \|D^s q\|_{L^2} \tag{1.7}$$

and introduce the energy "norm" in X^s

$$E^s(q) = \Big(\|D^{s-1}(|q|^2 - 1)\|_{L^2}^2 + \|D^{s-1}(\partial_x q)\|_{L^2}^2 \Big)^{\frac{1}{2}}, \tag{1.8}$$

and in particular

$$(E^1(q))^2 = \mathcal{E}(q), \quad (E^0(q))^2 = \|D^{-1}(|q|^2 - 1)\|_{L^2}^2 + \|D^{-1}(\partial_x q)\|_{L^2}^2.$$

Obviously the energy norm $E^1(q)$ is conserved by the Gross-Pitaevskii flow. We aim to show (almost) conservation of the energy norm $E^s(q)$ for all $s \geq 0$, and hence the global-in-time wellposedness for all $s \geq 0$. In the present paper we consider more general finite-energy spaces

$$X^s = \big\{ q \in H^s_{\mathrm{loc}}(\mathbb{R}) : |q|^2 - 1 \in H^{s-1}(\mathbb{R}), \quad \partial_x q \in H^{s-1}(\mathbb{R}) \big\} / \mathbb{S}^1, \quad s \geq 0. \tag{1.9}$$

We observe that neither mass $\mathcal{M}$ (see (1.4)) nor momentum $\mathcal{P}$ (see (1.5)) can be defined on any X^s, $s \geq 0$.

In the following we are going to state properties of the metric space (X^s, d^s) in Subsection 1.1 together with a conserved quantity, the renormalized momentum H_1, on the universal covering space of the energy space. Then we will state the main results Theorem 1.5 on conserved energies for $s \geq 0$ in Subsection 1.2, which follows from more general properties of the (renormalized) transmission coefficients associated to the Gross-Pitaevskii equation in Theorem 1.6 in the low regularity regime. Some of the proofs and the ideas will be sketched in Subsection 1.3. The proofs of the central results are found in Section 2 and Section 3.

1.1. The metric space (X^s, d^s).

We first recall some properties of the generalized finite-energy space X^s defined in (1.9) from [14].

Theorem 1.1 (Properties of the metric space, [14]). *The distance function $d^s(\cdot,\cdot)$ on the space X^s is defined by* [1]

$$d^s(p, q) = \Big(\int_{\mathbb{R}} \inf_{|\lambda(y)|=1} \big\| \operatorname{sech}(\cdot - y)(\lambda p - q) \big\|_{H^s(\mathbb{R})}^2 \,\mathrm{d}y \Big)^{\frac{1}{2}}. \tag{1.10}$$

Then the metric space (X^s, d^s), $s \geq 0$ has the following properties:

- *The space $(X^s, d^s(\cdot,\cdot))$ is a complete metric space.*
- *The subset $\{q \,|\, q - 1 \in C_0^\infty(\mathbb{R})\}$ is dense in X^s.*
- *Any set $\{q \in H^s_{\mathrm{loc}}(\mathbb{R}) : \|\partial_x q\|_{H^{s-1}(\mathbb{R})} + \||q|^2 - 1\|_{H^{s-1}(\mathbb{R})} < C\}$ is contained in some ball $B^s_r(1)$ with r depending on C.*

[1] Here the weight function $\operatorname{sech}(x) = \frac{2}{e^x + e^{-x}}$ can be equivalently replaced by any other strictly positive and smooth function with fast decay at infinity.

- *Any closed ball* $\overline{B_r^s(q)}$ *in* X^s, $s>0$ *is weakly sequentially compact.*
- *There is an analytic structure on* X^s *which is compatible with the metric.*
- *There exists an absolute constant* c *such that*

$$d^s(1,q) \le cE^s(q).$$

If $p \in X^s$ *and* $q \in H^s_{loc}$ *so that* $d^s(p,q)<\infty$, *then* $q \in X^s$ *and* [2]

$$E^s(q) \le E^s(p) + c(1+\|p'\|_{H^{s-1}} + \||p|^2-1\|^{\frac12}_{H^{s-1}})d^s(p,q) + c(d^s(p,q))^2.$$

The analytic structure defined in [14] can be described as follows. We fix a function $\eta \in C_0^\infty([-4,4])$ with $\eta = 1$ on $[-2,2]$. We fix a partition of unity

$$1 = \sum_{n\in\mathbb{Z}} \rho(x-n).$$

We define the Hilbert space of real valued sequences l^2_d equipped with the norm

$$\|(a_n)_n\|_{l^2_d} = \big(\sum_n |a_n - a_{n-1}|^2\big)^{\frac12}$$

and, given $q\in X^s$ and given $L>0$, we define

$$\tilde H^s = \{\mathfrak{b} \in H^s(\mathbb{R}) \,|\, \langle \eta(x/L-n)\mathfrak{b}, \eta(x/L-n)q\rangle_{H^s} \in \mathbb{R}, \quad \forall n\in\mathbb{Z}\}.$$

We have shown in [14]: Given $q\in X^s$, there exist $L>0$ and $r,R>0$ depending only on $E^s(q)$, so that we can define a map

$$\Psi : l^2_d \times \tilde H^s \to X^s$$

by

$$\Psi((a_n),\mathfrak{b}) = e^{i\theta}(q+\mathfrak{b}),$$
$$\text{with } \theta(x) = \sum_{n\in\mathbb{Z}} a_n \rho(x/L-n).$$

Here the map Ψ restricted to $B_r^{l^2_d\times \tilde H^s}((0)_n,0)$ is bilipschitz to its range, which contains the ball $B_R^{X^s}(q)$ in X^s, and the related coordinate changes and their Fréchet derivatives are bounded by some constants depending only on $E^s(q)$ and s.

The topology of the metric space (X^s,d^s) is nontrivial. It is described by the following theorem which we will prove in Section 2.

Theorem 1.2 (Topology of the metric space)**.** *Let* $s\ge 0$. *Let*

$$Q = \{q_c : -1 \le c \le 1\} \subset X^s$$

where q_c *denotes the 'profile' of the soliton solutions given in* (1.2)

$$q_c = q_c(x) = \sqrt{1-c^2}\tanh(\sqrt{1-c^2}x) + ic, \quad c\in[-1,1],$$

and we recall that $q_1 = q_{-1} = 1$ *in* X^s *due to the identification of functions differing only by phase.*

Then Q *is a strong deformation retract of* (X^s,d^s), *which means that there is a continuous map (called deformation)*

$$\Xi : [0,1]\times X^s \to X^s$$

so that

[2]The coefficient before $d^s(p,q)$ is corrected from $c(1+\|p'\|^{\frac12}_{H^{s-1}} + \||p|^2-1\|^{\frac12}_{H^{s-1}})$ (in [14]) to $c(1+\|p'\|_{H^{s-1}} + \||p|^2-1\|^{\frac12}_{H^{s-1}})$ below.

(1) $\Xi(0,q)=q, \quad \forall q\in X^s$,
(2) $\Xi(t,q_c)=q_c, \quad \forall t\in[0,1], \forall c\in[-1,1]$,
(3) $\Xi(1,q)\subset Q, \quad \forall q\in X^s$.

This corrects a statement in [14] where we stated incorrectly that all balls in X^s are contractible. In particular (X^s,d^s) is homotopy equivalent to the circle. The following argument shows that the topology of the phase space is nontrivial. If $q=|q|e^{i\varphi}\in X^1$ never vanishes and $\partial_x q\in L^1$, we can define the asymptotic change of the phase on the real line as

$$\begin{aligned}\Theta(q) &:= \operatorname{Im}\int_{\mathbb{R}} \frac{q}{|q|}\partial_x\frac{\bar q}{|\bar q|}\,\mathrm{d}x\\ &= \operatorname{Im}\int_{\mathbb{R}}\frac{\partial_x \bar q}{\bar q}\,\mathrm{d}x\\ &= -\lim_{x\to+\infty}(\varphi(x)-\varphi(-x))\in\mathbb{R}/(2\pi\,\mathbb{Z}),\end{aligned}\tag{1.11}$$

which is also conserved by the Gross-Pitaevskii flow (on the time interval where the solution never vanishes). It is a Casimir function which means that it Poisson commutes with every Hamiltonian: $\frac{\delta\Theta}{\delta q}=0$, and hence the Hamiltonian vector field of Θ vanishes.

If we consider the finite-energy solution $q\in X^1$ (even with no zeros), then the above phase change function may not be well-defined (keeping in mind of the example $e^{i\ln(1+|x|)}\in X^1$). We are going to extend the definition of Θ to general $q\in X^0$ with $\partial_x q\in L^1$ (see Theorem 1.3 below for more details), and similar as the mass $\mathcal{M}$, this quantity Θ will play an important role later in the analysis of low regularity case.

Notice that the (soliton) solutions defined (1.2) have the following asymptotic behaviors at infinity (modulo $\mathbb{S}^1$):

$$\lim_{x\to\infty} q_c(t,x)=\sqrt{1-c^2}+ic, \quad \lim_{x\to-\infty} q_c(t,x)=-\sqrt{1-c^2}+ic, \quad c\in(-1,1),$$

and in particular under the identification modulo $\mathbb{S}^1$ in X^s,

$$q_{-1}=q_1=1 \text{ in } X^s.$$

If we consider modulo $2\pi\,\mathbb{Z}$, then

$$\Theta(q_c)=2\arccos(c),$$

and $c\to\Theta(q_c)\in\mathbb{R}/(2\pi\,\mathbb{Z})$ is continuous. However it is not possible to lift $\Theta(q_c):[-1,1]\mapsto\mathbb{R}$ continuously. Notice that $c\to 2\arccos(c)$ is a monotonically decreasing function

$$[-1,1]\ni c\mapsto 2\arccos(c)\in[0,2\pi] \text{ with } -1,0,1\mapsto 2\pi,\pi,0 \text{ respectively.}$$

Thus

$$\{q\in X^s:\partial_x q\in\mathcal{S}\}\tag{1.12}$$

is not simply connected (indeed Theorem 1.2 implies that X^s is homotopy equivalent to the circle). Nevertheless the asymptotic phase change Θ is closely related to the momentum $\mathcal{P}$ by view of the Hamiltonian H_1 defined below.

Theorem 1.3 (Asymptotic phase change Θ and renormalized momentum H_1)**.** *The asymptotic change of phase Θ has a unique continuous and smooth extension to $\{q \in X^0 : \partial_x q \in L^1\}$ modulo $2\pi\,\mathbb{Z}$*

$$\Theta : \{q \in X^0 : \partial_x q \in L^1\} \to \mathbb{R}\,/(2\pi\,\mathbb{Z}).$$

The Hamiltonian

$$H_1(q) = -\mathrm{Im} \int_{\mathbb{R}} \big((|q|^2 - 1)\partial_x \log q\big)\, dx \in \mathbb{R}/(2\pi\,\mathbb{Z}) \tag{1.13}$$

which is defined on

$$\{q \in X^{\frac{1}{2}+\varepsilon} : q \neq 0\}, \quad \varepsilon > 0$$

has a unique continuous and smooth extension to $X^{\frac{1}{2}}$ modulo $2\pi\,\mathbb{Z}$. It is conserved under the flow of the Gross-Pitaevskii equation (1.1)*.*

We sketch the proof of Theorem 1.3 and leave the details for Section 2. For the extensions of both Θ and H_1, given $q_0 \in X^s$, we define a continuous map

$$B_\delta^{X^s}(q_0) \ni q \to \tilde{q}$$

so that $\tilde{q} \in X^{s+2}$ does not vanish. For the extension of Θ on $\{q \in X^0 : \partial_x q \in L^1\}$, we can take furthermore $\tilde{q}$ (see Lemma 2.4) such that

$$q - \tilde{q} \in L^2, \partial_x \tilde{q} \in L^1,$$

and set (with an abuse of notations)

$$\Theta = \Theta(\tilde{q}),$$

where $\Theta(\tilde{q})$ is the winding number of $\tilde{q}$ given in (1.11). Different choices of $\tilde{q}$ may lead to different values of $\Theta(\tilde{q})$, which is nevertheless unique up to a multiple of 2π, and $\Theta : \{q \in B_\delta^{X^s}(q_0) : \partial_x q \in L^1\} \mapsto \mathbb{R}\,/(2\pi\,\mathbb{Z})$ is continuous.

For the extension of H_1 on $X^{\frac{1}{2}}$, we proceed in a similar fashion, so that

$$q - \tilde{q} \in H^{\frac{1}{2}}.$$

If $q \in X^{\frac{1}{2}+\varepsilon}$ with $\varepsilon > 0$ and $\partial_x q, \partial_x \tilde{q} \in L^1$, we define

$$H_1(q) = -\mathrm{Im} \int_{\mathbb{R}} \Big(\bar{q}\partial_x q - \partial_x \tilde{q}/\tilde{q}\Big)\, \mathrm{d}x \, \in \mathbb{R}\,/(2\pi\,\mathbb{Z}), \tag{1.14}$$

which coincides with the definition (1.13) if q never vanishes. The first term is the momentum $\mathcal{P}$ (see (1.5)) and the second term is again the winding number of $\tilde{q}$ if $\partial_x \tilde{q} \in L^1$:

$$H_1(q) = \mathcal{P}(q) - \Theta(q). \tag{1.15}$$

On the other hand, we may rewrite (1.14) as

$$H_1(q) = -\mathrm{Im} \int_{\mathbb{R}} \Big((\overline{q - \tilde{q}})\partial_x q - \partial_x \bar{\tilde{q}}(q - \tilde{q}) + \frac{1}{\tilde{q}}(|\tilde{q}|^2 - 1)\partial_x \tilde{q}\Big)\, \mathrm{d}x\,, \tag{1.16}$$

which in this form is easily seen to be well defined on $X^{\frac{1}{2}}$, and, with a continuous choice of $\tilde{q}$, it is continuous resp. smooth modulo $2\pi\,\mathbb{Z}$. Note that $H_1(q)$ cannot be defined on X^s if $0 \leq s < \frac{1}{2}$, since the momentum $\mathcal{P}(q) = \mathrm{Im}\int_{\mathbb{R}} q\partial_x \bar{q} dx$ cannot be defined even on $\{q \in X^s : q(x) = 1 \text{ if } |x| \geq 1\}$.

To see the conservation of the Hamiltonian $H_1(q(t, x))$ under the Gross-Pitaevskii flow, we recall the construction of the solution in [14, Section 2]. Let $q_0 \in X^s$, $s \geq 0$ (more precisely a representative of the equivalence class) and let $\tilde{q} \in X^\infty$ be its regularisation (depending only on the initial data q_0, but not on t, which may be different from the regularisation above). We make the Ansatz $q = \tilde{q} + b$ for the solution of the Gross-Pitaevskii equation (1.1) and search for b as a solution to

$$i\partial_t b + b_{xx} = 2|b|^2 b + 4|b|^2\tilde{q} + 2\bar{\tilde{q}}b^2 + (4|\tilde{q}|^2 - 2)b + 2(\tilde{q})^2 b + 2(|\tilde{q}|^2 - 1)\tilde{q} - \tilde{q}_{xx}$$

with initial data $b(0) = q_0 - \tilde{q} \in H^s$. Using Strichartz estimates we obtain a unique solution $b \in C([0,T], H^s)$ for some time $T > 0$. If $q_0 \in 1 + \mathcal{S}$, then $q(t) \in 1 + \mathcal{S}$ and with $\tilde{q} = 1$ we have $\Theta(\tilde{q}) = 0$. It is clearly independent of time. The momentum is conserved for $q_0 \in 1 + \mathcal{S}$ and hence by virtue of the relation (1.15), H_1 is conserved on $1 + \mathcal{S}$. By the density of the set $1 + \mathcal{S}$ in (X^s, d^s) and the local continuity of H_1 in $X^{\frac{1}{2}}$, H_1 is also conserved for initial data in $X^{\frac{1}{2}}$.

1.2. **Energies and transmission coefficient.** Recently continuous families of conserved energies have been constructed for the Korteweg-de Vries equation (KdV), the modified KdV and NLS equations independently by Killip, Visan and Zhang [13], and the first author and Tataru [15]. We recall the local-in-time well-posedness result of the Gross-Pitaevskii equation in (X^s, d^s), $s \geq 0$, as well as the global-in-time well-posedness result for the case $s > \frac{1}{2}$ (by constructing a family of conserved energy functionals) in our previous work [14] below.

Theorem 1.4 (LWP in X^s, $s \geq 0$ & GWP in X^s, $s > \frac{1}{2}$, [14])**.** *Let $s \geq 0$. The Gross-Pitaevskii equation* (1.1) *is locally-in-time well-posed in the metric space (X^s, d^s) in the following sense: For any initial data $q_0 \in X^s$, there exists a positive time $\bar{t} \in (0, \infty)$ and a unique local-in-time solution $q \in \mathcal{C}((-\bar{t}, \bar{t}); X^s)$ of* (1.1) *and for any $t \in (0, \bar{t})$, the Gross-Pitaevskii flow map $X^s \ni q_0 \mapsto q \in \mathcal{C}([-t, t]; X^s)$ is continuous.* [3]

Let $s > \frac{1}{2}$, then the above holds for all $\bar{t} \in \mathbb{R}^+$ and hence the Gross-Pitaevskii equation (1.1) *is globally-in-time well-posed in the metric space (X^s, d^s). Furthermore, for any initial data $q_0 \in X^s$, there exists τ_0 depending only on $E^s(q_0)$ such that the unique solution $q \in \mathcal{C}(\mathbb{R}; X^s)$ of the Gross-Pitaevskii equation* (1.1) *satisfies the following uniform bound*

$$E^s_{\tau_0}(q(t)) \leq 2E^s_{\tau_0}(q_0), \quad \forall t \in \mathbb{R}\,. \tag{1.17}$$

In the above, we have used the rescaled energy norm in the metric space (X^s, d^s):

$$E^s_\tau(q) = \Big(\||q|^2 - 1\|^2_{H^{s-1}_\tau} + \|\partial_x q\|^2_{H^{s-1}_\tau} \Big)^{\frac{1}{2}}, \quad s \geq 0, \tag{1.18}$$

where the rescaled Sobolev norm is defined as

$$\|f\|_{H^s_\tau} = \|D^s_\tau f\|_{L^2}, \tag{1.19}$$

and the rescaled operator

$$D_\tau = (-\partial_x^2 + \tau^2)^{\frac{1}{2}}$$

[3] Here the solution $q \in \mathcal{C}((-\bar{t}, \bar{t}); X^s)$ is defined in terms of the representatives in (X^s, d^s) as follows. There is $\tilde{q} : (-\bar{t}, \bar{t}) \to H^s_{loc}(\mathbb{R})$ which satisfies that

$$(-\bar{t}, \bar{t}) \ni t \to \tilde{q}(t) - \tilde{q}(0) \in L^2(\mathbb{R}),$$

is weakly continuous and

$$\|\tilde{q}(\cdot) - \tilde{q}_{0,\varepsilon}\|_{L^4([a,b]\times\mathbb{R})} < \infty,$$

for some regularized initial data $\tilde{q}_{0,\varepsilon}$ of $\tilde{q}(0)$ and for all time intervals $[a, b] \subset (-\bar{t}, \bar{t})$ with $0 \in [a, b]$, such that the equation (1.1) holds in the distributional sense on $(-\bar{t}, \bar{t}) \times \mathbb{R}$ and $\tilde{q}(t)$ projects to $q(t)$.

reads in terms of Fourier transform as

$$\widehat{D_\tau f}(\xi) = (\xi^2+\tau^2)^{\frac{1}{2}}\hat{f}(\xi).$$

Notice that when $\tau = 2$, $D_2 = D = \langle \partial_x \rangle$ is defined in (1.6) and $E^s_2(q) = E^s(q)$ is the energy norm defined in (1.8).

We aim to show the almost conservation law (1.17) for all $s \geq 0$, and in particular we are going to construct a family of conserved energies $\mathcal{E}^s_\tau(q)$ which is equivalent to $(E^s_\tau(q))^2$ provided $\tau^{-\frac{1}{2}}E^0_\tau(q)$ is small. Our main result reads as follows.

Theorem 1.5 (GWP and conserved energies in X^s, $s \geq 0$)**.** *Let $s \geq 0$. Then the Gross-Pitaevskii equation* (1.1) *is globally-in-time well-posed in the metric space (X^s, d^s) in the sense in Theorem 1.4. Furthermore, there exist a constant $C \geq 2$ and a family of analytic (in both variables τ and q) energy functionals $(\mathcal{E}^s_\tau)_{\tau\geq 2} : X^s \mapsto [0,\infty)$, such that*

- *$\mathcal{E}^s_\tau(q)$ is equivalent to $(E^s_\tau(q))^2$ in the following sense: If $q \in X^s$ with $\frac{1}{\sqrt{\tau}}E^0_\tau(q) < \frac{1}{2C}$, then*

$$|\mathcal{E}^s_\tau(q) - (E^s_\tau(q))^2| \leq C\frac{E^0_\tau(q)}{\sqrt{\tau}}(E^s_\tau(q))^2, \quad s \geq 0. \tag{1.20}$$

- *$\mathcal{E}^s_\tau(\cdot)$, $\tau \geq 2$ is conserved by Gross-Pitaevskii flow* (1.1)*.*

Correspondingly, for any initial data $q_0 \in X^s$, there exists $\tau_0 \geq C$ depending only on $E^0(q_0)$ such that the unique solution $q \in \mathcal{C}(\mathbb{R}; X^s)$ of the Gross-Pitaevskii equation (1.1) *satisfies the following energy bound:*

$$E^s_{\tau_0}(q(t)) \leq 2E^s_{\tau_0}(q_0), \quad \forall t \in \mathbb{R}\,. \tag{1.21}$$

The construction of the conserved energy functionals $(\mathcal{E}^s_\tau)$ is related to the complete integrability fact (by means of the inverse scattering method) of the Gross-Pitaevskii equation (1.1), which can be viewed as the compatibility condition for the following two ODE systems (see Zakharov-Shabat [17])

$$\begin{aligned} u_x &= \begin{pmatrix} -i\lambda & q \\ \bar{q} & i\lambda \end{pmatrix} u, \\ u_t &= i\begin{pmatrix} -2\lambda^2 - (|q|^2-1) & -2i\lambda q + \partial_x q \\ -2i\lambda\bar{q} - \partial_x\bar{q} & 2\lambda^2 + (|q|^2-1) \end{pmatrix} u, \end{aligned} \tag{1.22}$$

where $u = u(t,x) : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{C}^2$ is the unknown vector-valued solutions. The first ODE system can be formulated as the spectral problem $Lu = \lambda u$ of the Lax operator

$$L = \begin{pmatrix} i\partial_x & -iq \\ i\bar{q} & -i\partial_x \end{pmatrix}. \tag{1.23}$$

If the potential satisfies $q \in 1 + \mathcal{S}$, the Lax operator is self adjoint with essential spectrum $(-\infty,-1]\cup[1,\infty)$ and finitely many discrete eigenvalues in $(-1,1)$ (see [6] and Theorem 1.6 below). For

$$\lambda \in \mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$$

we denote by z the square root of λ^2-1 with positive imaginary part. The map $\mathbb{C}\setminus((-\infty,-1]\cup[1,\infty)) \ni \lambda \to z$ is holomorphic.

The construction of the energies $(\mathcal{E}^s_\tau)$ depends on the transmission coefficient $T^{-1}(\lambda;q)$ of the Lax operator (1.23) (see e.g. (1.37) below for the definition of $\mathcal{E}^s_\tau$

and see Section 3 below for the definition of T^{-1}), and by an abuse of notation we call T^{-1}, while not T, the transmission coefficient, and we will use the same abuse for the renomarlized transmission coefficient T_c^{-1} below. In the classical framework $q \in 1+\mathcal{S}$, the logarithm of this transmission coefficient is shown in [7] to have an asymptotic expansion as follows

$$\begin{aligned} \ln T^{-1}(\lambda) = i\sum_{l=0}^{k-1} \mathcal{H}_l (2z)^{-l-1} + (\ln T^{-1}(\lambda))_{\geq k+1}, \quad \operatorname{Im}\lambda > 0, \\ \text{with } |(\ln T^{-1}(\lambda))_{\geq k+1}| = O(|\lambda|^{-k-1}) \text{ as } |\lambda| \to \infty. \end{aligned} \tag{1.24}$$

Here $\mathcal{H}_0$ is the mass $\mathcal{M}$, $\mathcal{H}_1$ is the momentum $\mathcal{P}$, $\mathcal{H}_2$ is the energy $\mathcal{E}$, and

$$\mathcal{H}_3 = \operatorname{Im} \int_{\mathbb{R}} \big(\partial_x q \partial_{xx}\bar{q} + 3(|q|^2-1) q\partial_x \bar{q}\big)\, \mathrm{d}x \,-\mathcal{P} \tag{1.25}$$

which cannot be defined on X^s for any $s \geq 0$ (since $\mathcal{P}$ cannot be defined on X^s for any $s \geq 0$).

We proved in [14] that,

$$X^s \ni q \to \underline{T_c}^{-1}(\lambda) := \exp\Big(\ln T^{-1}(\lambda) - i\mathcal{M}(2z)^{-1} - i\mathcal{P}(2z(\lambda+z))^{-1}\Big) \tag{1.26}$$

defines a continuous map to holomorphic functions if $s > \frac{1}{2}$. We expand its logarithm as

$$-i \ln \underline{T_c}^{-1}(\lambda) \sim \sum_{l\geq 2} \underline{H}_l (2z)^{-l-1},$$

where $\underline{H}_{2n} = \mathcal{H}_{2n}$ and

$$\underline{H}_3 = \operatorname{Im} \int_{\mathbb{R}} \big(\partial_x q \partial_{xx}\bar{q} + 3(|q|^2-1) q\partial_x \bar{q}\big)\, \mathrm{d}x\,,$$

which is well-defined on $X^{\frac{3}{2}}$.

We prove in this paper that there is a unique extension of a renormalized transmission coefficient

$$q \to T_c^{-1}(\lambda) := \exp\Big(\ln T^{-1}(\lambda) - i\mathcal{M}(2z)^{-1} - i\Theta(2z(\lambda+z))^{-1}\Big) \tag{1.27}$$

to X^0 modulo $\exp(2\pi i (2z(\lambda+z))^{-1}\,\mathbb{Z})$, or, equivalently, if we fix an element $\tilde{1}$ in the fiber of 1 and $\Theta(\tilde{1}) = 0$, there is a unique continuous extension of (1.27) to the universal covering space of X^0. One obtains an asymptotic expansion

$$-i \ln T_c^{-1}(\lambda) \sim \sum_{l\geq 1} H_l (2z)^{-l-1}$$

on the covering space, where H_1 is defined in (1.16), $H_2 = \mathcal{E}$ and $H_{2n} = \underline{H}_{2n} = \mathcal{H}_{2n}$. Justification of the asymptotic expansion follows from a combination of the techniques in [14] and in this paper. We do not pursue this here to avoid a distraction by sidelines.

Theorem 1.6 (Renormalized transmission coefficient)**.** *Suppose that $q \in X^0$. Then the renormalized transmission coefficient of the Lax operator (1.23): $\lambda \to T_c^{-1}(\lambda)$ is a holomorphic function on $\mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$. The zeroes are simple and contained in $(-1,1)$, which coincide with the eigenvalues of the Lax operator.*

There exists a constant $C \geq 2$ (as in Theorem 1.5), such that

- *if $q \in X^s$ satisfies the smallness condition $\frac{1}{\sqrt{\tau}} E^0_\tau(q) \le \frac{1}{2C}$ for $\tau := 2\mathrm{Im}\, z \ge 2$, then*

$$\begin{aligned}\Big|\frac{1}{2}\Big(\ln T_c^{-1}(\lambda) + \ln \overline{T_c^{-1}(-\bar\lambda)}\Big)\\ + \frac{i}{2z}\int_{\mathbb{R}} \frac{1}{\xi^2 - 4z^2}\Big(|\widehat{\partial_x q}|^2(\xi) + |(\widehat{|q|^2 - 1})|^2(\xi)\Big) d\xi\Big| \le C\Big(\frac{1}{\sqrt{\tau}} E^0_\tau(q)\Big)^3,\end{aligned} \tag{1.28}$$

- *if $q \in X^s$, $s \in (\frac{1}{2}, \frac{3}{2})$ never vanishes and satisfies $\tau_0^{-1-\varepsilon} E_{\tau_0}^{\frac{1}{2}+\varepsilon}(q) \le \frac{1}{2C}$ for some $\varepsilon \in (0, \frac{1}{2})$ and $\tau_0 \ge 2$, then for all $\tau \ge \tau_0$,*

$$\begin{aligned}\Big|\frac{1}{2i}\Big(\ln T_c^{-1}(\lambda) - \ln \overline{T_c^{-1}(-\bar\lambda)}\Big) - \frac{\lambda}{4z^3} H_1 - \frac{\lambda}{4z^3}\int_{\mathbb{R}} \frac{\xi}{\xi^2 - 4z^2} |\widehat{\partial_x q}|^2 d\xi\Big|\\ \le C\tau^{-1-2s}(\tau_0^{-1-\varepsilon} E_{\tau_0}^{\frac{1}{2}+\varepsilon}(q))(E^s_{\tau_0}(q))^2,\end{aligned} \tag{1.29}$$

where H_1 is given in (1.13) in Theorem 1.3.

Remark 1.7. *We will also obtain a more detailed expression for the odd part: We take the function $r = \tau^2 D_\tau^{-2} q \in X^2$ and a non-vanishing function $\tilde q \ne 0$ with $\tilde q - q \in L^2$, such that if $\frac{1}{\sqrt{\tau}} E^0_\tau(q) \le \frac{1}{2C}$ for $\tau = 2\mathrm{Im}\, z \ge 2$ and $\mathrm{Im}\,\lambda > 0$, then*

$$\begin{aligned}&\Big|\frac{1}{2i}\Big(\ln T_c^{-1}(\lambda) - \ln \overline{T_c^{-1}(-\bar\lambda)}\Big) - \frac{1}{4z^2}\int_{\mathbb{R}} \frac{\xi(\tau^4 + 4z^2(2\tau^2 + \xi^2))}{(\tau^2 + \xi^2)^2(\xi^2 - 4z^2)} |\widehat{\partial_x q}|^2 d\xi\\ &+ \frac{\lambda - z}{2z}\mathrm{Im}\int_{\mathbb{R}} \big(\bar r \partial_x r - \partial_x \log \tilde q\big) dx + \frac{(\lambda - z)^2}{4z^2}\mathrm{Im}\int_{\mathbb{R}} (|r|^2 - 1)(\bar r \partial_x r)\, \mathrm{d}x\\ &- \frac{\lambda - z}{4z^2}\int_{x<y} e^{2iz(y-x)}\Big((|q|^2 - 1)(x)\mathrm{Im}\,[r\partial_x \bar r](y)\\ &\qquad + \mathrm{Im}\,[r\partial_x \bar r](x)(|q|^2 - 1)(y)\Big) dxdy\\ &- \frac{\lambda - z}{2z}\int_{x<y} e^{2iz(y-x)}\Big((|q(x)|^2 - 1)\mathrm{Im}\,[\bar r(q - r)](y)\\ &\qquad + \mathrm{Im}\,[r(\overline{q - r})](x)(|q(y)|^2 - 1)\Big) dxdy\Big| \le C\Big(\frac{1}{\sqrt{\tau}} E^0_\tau(q)\Big)^3.\end{aligned} \tag{1.30}$$

If $s \in [0, \frac{7}{4}]$ and the smallness condition $\frac{1}{\sqrt{\tau_0}} E^0_{\tau_0}(q) \le \frac{1}{2C}$ holds for some fixed $\tau_0 \ge 2$, then for all $\tau \ge \tau_0$,

$$\text{Lefthandsides of (1.28) and (1.30)} \le C\tau^{-\frac{3}{2}-2s} E^0_{\tau_0}(q)(E^s_{\tau_0}(q))^2. \tag{1.31}$$

Corollary 1.8 (Conserved energy and momentum sequences)**.** *There exist a sequence of conserved energies H_{2n}, $n \ge 1$ with*

$$\Big|H_{2n} - (E^n(q))^2\Big| \le c(E^0(q))\, E^0(q)(E^n(q))^2, \tag{1.32}$$

and a sequence of conserved Hamiltonians $\tilde H_{2n+1}$, $n \ge 1$ with

$$\Big|\tilde H_{2n+1} - \mathrm{Im}\int_{\mathbb{R}} q^{(n)} \bar q^{(n+1)}\, \mathrm{d}x\Big| \le c(\varepsilon, E^{\frac{1}{2}+\varepsilon}(q))\, E^{\frac{1}{2}+\varepsilon}(q)(E^{n+1/2}(q))^2. \tag{1.33}$$

Remark 1.9. *The conserved energy $\mathcal{E}^s_\tau(q)$ given in Theorem 1.5 here is indeed the same as the conserved energy given in [14], which is defined in terms of the real part of the renormalized transmission coefficient located on the imaginary axis (see e.g. (1.37) below): Indeed, if we take the purely imaginary points $(\lambda, z) =$*

$(i\sqrt{\tau^2/4-1}, i\tau/2)$, $\tau \geq 2$, *then the real parts of the renormalized transmission coefficients in [14] (see* (1.26)*) and here (see* (1.27)*) coincide.*

1.3. **Ideas of the proofs.** In order to consider the low regularity case $q \in X^0$, we first perform a regularisation procedure: We define the regularisation $r = \tau^2(\tau^2 - \partial_x^2)^{-1} q = \tau^2 D_\tau^{-2} q \in X^2$ for $q \in X^0$ and $\tau \geq 2$. If $E_\tau^0(q) \leq \tau^{1/2}$ then (see Lemma 2.2 below)

$$\|r\|_{L^\infty} \leq c\tau, \qquad \|q-r\|_{L^2} + \tau^{-1}\||r|^2-1\|_{L^2} + \tau^{-1}\|\partial_x r\|_{L^2} \leq cE_\tau^0(q). \tag{1.34}$$

This regularisation r will play an important role in the proofs.

We will prove Theorem 1.5 by use of Theorem 1.6, and mention the proof of Corollary 1.8 below. We will also sketch below the proof ideas of Theorem 1.2 and Theorem 1.3 (whose detailed proofs can be found in Section 2), and of Theorem 1.6 (whose detailed proofs can be found in Section 3).

1.3.1. *Idea of the proof of Theorem 1.2.* The construction for the deformation in Theorem 1.2 is as follows: First we define a deformation to the set of functions which do not have zeros outside an interval $[-R_0, R_0]$. This is followed by a second essentially linear deformation to the set Q, basically by unwinding the rotation near infinity. Implementing these ideas is more delicate than this (simple) description: Because of the low regularity assumption $q \in X^0$, we have to resort to its regularisation (1.34).

1.3.2. *Idea of the proof of Theorem 1.3.* The crucial point in the proof of Theorem 1.3 is a further regularisation map $q \mapsto \tilde{q}$, with the regularisation $\tilde{q}$ having on zeros. More precisely, given $q_0 \in X^s$, $s \geq 0$, we will construct (see Lemma 2.4 below) a continuous map defined on a small ball $B_\delta^{X^0}(q_0)$ in X^0:

$$B_\delta^{X^0}(q_0) \ni q \to \tilde{q} \in X^{s+2}$$

so that $|\tilde{q}| = 1$, $q - \tilde{q} \in H^s$. Furthermore, with the assumption $\partial_x q \in L^1$, we have $\partial_x \tilde{q} \in L^1$, and we can then define

$$\Theta(q) = -\mathrm{Im} \int_{\mathbb{R}} \partial_x \ln \tilde{q} \, \mathrm{d}x \in \mathbb{R}/(2\pi\mathbb{Z}).$$

Since X^s is homotopy equivalent to $\mathbb{S}^1$, Θ can be lifted to a map from the universal covering space to $\mathbb{R}$.

Direct estimates show then that

$$H_1(q) = \mathrm{Im} \int_{\mathbb{R}} \Big((q-\tilde{q})\partial_x \bar{q} - \overline{(q-\tilde{q})}\partial_x \tilde{q} \Big) \, \mathrm{d}x$$

can be defined on the universal covering space of $X^{\frac{1}{2}}$, and it is unique up to the addition of multiplies of 2π.

1.3.3. *Proof of Theorem 1.5 by virtue of Theorem 1.4 and Theorem 1.6.* Thanks to the local-in-time well-posedness result in Theorem 1.4, the equivalence relation (1.20) between the conserved energy $\mathcal{E}_\tau^s(q)$ and the square of the energy norm $(E_\tau^s(q))^2$ implies immediately the (almost) conservation of the energy (1.21) and then the global-in-time well-posedness in Theorem 1.5, by a standard continuation argument (as e.g. the end of [14, Section 1]), which is omitted here.

It remains to prove the equivalent relation (1.20) and the conservation of $\mathcal{E}_\tau^s(q)$. We are going to construct the energy functionals $(\mathcal{E}_\tau^s(q))_{\tau \geq 2}$ in terms of the time-independent renormalized transmission coefficient, and then show that (1.20) is a consequence of the more precise statements in Theorem 1.6, in the low regularity regime $s \in [0,2)$. In this paper we focus on the case $s < 2$ in order to avoid a more technical presentation.

Case $s \in [0,1)$**.** We first consider the following integral for $\tilde{s} \in (-1,0)$ and $\tau_0 > 0$,

$$-\frac{2}{\pi}\sin(\pi\tilde{s})\int_{\tau_0}^{\infty}(\tau^2-\tau_0^2)^{\tilde{s}}\tau(\tau^2+\xi^2)^{-1}d\tau,$$

which reads as, by using the standard branches of the logarithm and fractional powers in the cut domain $\mathbb{C}\setminus(-\infty,0]$,

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\Big[\frac{1}{\pi i}\int_{\tau_0}^{\infty}((i\tau-\varepsilon)^2+\tau_0^2)^{\tilde{s}}i\tau(\xi^2+\tau^2)^{-1}id\tau \\ +\frac{1}{\pi i}\int_{\tau_0}^{\infty}((i\tau+\varepsilon)^2+\tau^2)^{\tilde{s}}i\tau(\xi^2+\tau^2)^{-1}id\tau\Big].\end{aligned}$$

Let γ be the path from $i\infty-0$ to $i\tau_0$ and then to $i\infty+0$, then we derive

$$\begin{aligned}&-\frac{2}{\pi}\sin(\pi\tilde{s})\int_{\tau_0}^{\infty}(\tau^2-\tau_0^2)^{\tilde{s}}\tau(\tau^2+\xi^2)^{-1}d\tau\\ &\quad=\frac{1}{\pi i}\int_{\gamma}(z^2+\tau_0^2)^{\tilde{s}}z(\xi^2-z^2)^{-1}dz=(\tau_0^2+\xi^2)^{\tilde{s}},\quad \tilde{s}\in(-1,0),\end{aligned}\tag{1.35}$$

where we moved the contour of integration to the real axis for the last equality where only residues contribute since the integrand is odd. Thus, for $-1<\tilde{s}<0$,

$$\begin{aligned}\|f\|^2_{H^{\tilde{s}}_{\tau_0}}&=\int_{\mathbb{R}}|\hat{f}(\xi)|^2(\tau_0^2+\xi^2)^{\tilde{s}}d\xi\\ &=-\frac{2}{\pi}\sin(\pi\tilde{s})\int_{\tau_0}^{\infty}\int_{\mathbb{R}}(\tau^2-\tau_0^2)^{\tilde{s}}\tau(\tau^2+\xi^2)^{-1}|\hat{f}(\xi)|^2d\xi d\tau\\ &=-\frac{2}{\pi}\sin(\pi\tilde{s})\int_{\tau_0}^{\infty}(\tau^2-\tau_0^2)^{\tilde{s}}\tau\|f\|^2_{H^{-1}_{\tau}}d\tau.\end{aligned}\tag{1.36}$$

Observe that the limit of the right hand side is $\|f\|^2_{H^{-1}_{\tau_0}}$ as $\tilde{s}\to -1$.

For $s\in(0,1)$ and $\tau_0\geq 2$, we define our conserved energies in terms of the real part of the renormalized transmission coefficient given by Theorem 1.6 on the imaginary axis as

$$\mathcal{E}^s_{\tau_0}(q)=\frac{2}{\pi}\sin(\pi(s-1))\int_{\tau_0}^{\infty}(\tau^2-{\tau_0}^2)^{s-1}\tau^2\mathrm{Re}\,\ln T_{\mathrm{c}}^{-1}\Big(i\sqrt{\frac{\tau^2}{4}-1}\Big)d\tau,\tag{1.37}$$

and for $s=0$, we define

$$\mathcal{E}^0_{\tau}(q)=-\tau\mathrm{Re}\,\ln T_{\mathrm{c}}^{-1}\Big(i\sqrt{\frac{\tau^2}{4}-1}\Big).\tag{1.38}$$

We derive (1.20) for $s=0$ straightforward from the estimate (1.28) with $(\lambda,z)=(i\sqrt{\tau^2/4-1},i\tau/2)$ and $\tau\geq\max\{2,2C(E^0_\tau(q))^2\}$ in Theorem 1.6:

$$\big|-\mathcal{E}^0_\tau(q)+(E^0_\tau(q))^2\big|\leq C\frac{E^0_\tau(q)}{\sqrt{\tau}}\big(E^0_\tau(q)\big)^2.\tag{1.39}$$

Similarly, for $0<s<1$, and $\tau_0\geq\max\{2,2C(E^0_{\tau_0}(q))^2\}$ (such that $\tau\geq\max\{2,2C(E^0_\tau(q))^2\}$ holds for all $\tau\geq\tau_0$ trivially), we derive from (1.28), (1.36) and the trivial estimates

$E^0_\tau(q) \le \tau^{-s} E^s_\tau(q)$, $E^0_\tau(q) \le E^0_{\tau_0}(q)$ and $E^s_\tau(q) \le E^s_{\tau_0}(q)$, that

$$\begin{aligned}
&\Big| -\mathcal{E}^s_{\tau_0}(q) + (E^s_{\tau_0}(q))^2 \Big| \\
&\le -\frac{2}{\pi}\sin(\pi(s-1)) \int_{\tau_0}^\infty (\tau^2-{\tau_0}^2)^{s-1} \Big| \tau^2 \mathrm{Re}\, \ln T_c^{-1}\Big(i\sqrt{\frac{\tau^2}{4}-1}\Big) + \tau (E^0_\tau(q))^2 \Big| d\tau \\
&\lesssim -\frac{2}{\pi}\sin(\pi(s-1)) \int_{\tau_0}^\infty (\tau^2-{\tau_0}^2)^{s-1} \tau^{\frac12 - 2s} d\tau E^0_{\tau_0}(q) (E^s_{\tau_0}(q))^2 \\
&\lesssim \frac{E^0_{\tau_0}(q)}{\sqrt{\tau_0}} (E^s_{\tau_0}(q))^2 .
\end{aligned}$$

Case $s \in [1,2)$**.** If $s=1$, then $\mathcal{E}^1_\tau(q)$ is simply the conserved energy (1.3):

$$\mathcal{E}^1_\tau(q) = (E^1_\tau(q))^2 = (E^1(q))^2 = \mathcal{E} = \|(|q|^2-1, \partial_x q)\|^2_{L^2} = H_2(q).$$

The same calculation as for (1.35) gives for $0 \le \tilde{s} < 1$

$$-\frac{2}{\pi}\sin(\pi\tilde{s}) \int_{\tau_0}^\infty (\tau^2-\tau_0^2)^{\tilde{s}} \tau \big[(\tau^2+\xi^2)^{-1} - \tau^{-2}\big] d\tau + {\tau_0}^{2\tilde{s}} = (\tau_0^2+\xi^2)^{\tilde{s}}, \tag{1.40}$$

and hence for $s \in (1,2)$,

$$\begin{aligned}
(E^s_{\tau_0}(q))^2 = -\frac{2}{\pi}\sin(\pi(s-1)) \int_{\tau_0}^\infty (\tau^2-{\tau_0}^2)^{s-1} \big[\tau (E^0_\tau(q))^2 \\
- \tau^{-1}(E^1(q))^2 \big] d\tau + {\tau_0}^{2(s-1)} (E^1(q))^2 .
\end{aligned} \tag{1.41}$$

Therefore, for $s \in (1,2)$, we define

$$\begin{aligned}
\mathcal{E}^s_{\tau_0}(q) = \frac{2}{\pi}\sin(\pi(s-1)) \int_{\tau_0}^\infty (\tau^2-{\tau_0}^2)^{s-1} \\
\times \left(\tau^2 \mathrm{Re}\, \ln T_c^{-1}\Big(i\sqrt{\frac{\tau^2}{4}-1}\Big) + \tau^{-1}(E^1(q))^2 \right) d\tau + {\tau_0}^{2(s-1)} (E^1(q))^2 .
\end{aligned} \tag{1.42}$$

The difference $\mathcal{E}^s_{\tau_0}(q) - (E^s_{\tau_0}(q))^2$ is given by

$$\frac{2}{\pi}\sin(\pi(s-1)) \int_{\tau_0}^\infty (\tau^2-\tau_0^2)^{s-1} \Big(\tau^2 \mathrm{Re}\, \ln T_c^{-1}\Big(i\sqrt{\frac{\tau^2}{4}-1}\Big) + \tau (E^0_\tau(q))^2 \Big) d\tau,$$

and we estimate the integrand using (1.28) and indeed (1.31) by

$$(\tau^2-\tau_0^2)^{s-1} \tau^{\frac12 - 2\sigma} E^0_{\tau_0}(q) (E^\sigma_{\tau_0}(q))^2 \text{ for some } \sigma \in [1, \frac{7}{4}].$$

Integration yields the searched estimate (1.20) for $s = \sigma \in (1, \frac74]$. If $\frac74 \le s < 2$ we take $\sigma = \frac74$ above

$$(\tau^2-\tau_0^2)^{s-1} \tau^{-3} E^0_{\tau_0}(q) (E^{\frac74}_{\tau_0}(q))^2,$$

which decays at the rate τ^{2s-5} with $2s-5 < -1$ as $\tau \to \infty$, and the integration with respect to τ gives (1.20):

$$\Big| \mathcal{E}^s_{\tau_0}(q) - (E^s_{\tau_0})^2 \Big| \lesssim \tau_0^{2(s-2)} E^0_{\tau_0}(q) (E^{\frac74}_{\tau_0}(q))^2 \lesssim \frac{E^0_{\tau_0}(q)}{\tau_0^{1/2}} (E^s_{\tau_0}(q))^2,$$

where we used the trivial estimate $E^{\frac74}_{\tau_0}(q) \le {\tau_0}^{-s+\frac74} E^s_{\tau_0}(q)$ for the second inequality.

In this paper we only consider the case $s < 2$, and for $s > 1$ we have expanded above the rescaled energy norm $E^0_\tau(q)$ with respect to τ in the formulation (1.41) of $(E^s_\tau(q))^2$, and similarly we expanded the real part $\operatorname{Re}\ln T_c^{-1}$ on the imaginary axis $(\lambda, z) = (i\sqrt{\tau^2/4-1}, i\tau/2)$ in the definition (1.42) of $\mathcal{E}^s_\tau(q)$. For $s \geq 2$ one has to expand these terms up to higher-order terms exactly as in [14]. Then one may replace the norm $\||q|^2-1\|_{l^2DU^2} + \|\partial_x q\|_{l^2DU^2}$ in [14] by $E^0(q)$ here, which is (1.20) for general $s \geq 0$. This yields immediately the estimates (1.32) in Corollary 1.8 as a special case. We do not go into details to avoid a more technical presentation.

1.3.4. *Ideas of the proof of Theorem 1.6.* Let $q \in 1 + \mathcal{S}$. Let $T^{-1}(\lambda; q)$ be the time independent transmission coefficient of the associated Lax operator to the one dimensional Gross-Pitaevskii equation (1.1), and $T_c^{-1}(\lambda; q)$ be the renormalized transmission coefficient modulo the mass and the asymptotic phase change as in (1.27):

$$T_c^{-1}(\lambda) = T^{-1}(\lambda) e^{-i\mathcal{M}(2z)^{-1} - i\Theta(2z(\lambda+z))^{-1}}. \tag{1.43}$$

This renormalized transmission coefficient T_c^{-1} is defined on $\{q \in X^0 : |q|^2-1, \partial_x q \in L^1,\, q \neq 0\}$, and has a unique continuous extension to X^0 modulo $\exp(2\pi i(2z(\lambda + z))^{-1}\,\mathbb{Z})$, or equivalently, on the universal covering space of X^0 with values in $\mathbb{C}$.

We approximately diagonalize the Lax operator using the regularisation r (see (1.34) above) and $\zeta = \lambda + z$ with $\operatorname{Im}\zeta > 0$ (see (3.12) below), and obtain an expansion

$$T_c^{-1}(\lambda) = e^{\Phi(\lambda)} \sum_{n=0}^{\infty} T_{2n}(\lambda),$$

where the phase correction function Φ reads explicitly as

$$\Phi(\lambda) = -\int_{\mathbb{R}} \frac{i\zeta\big[|r|^2 - 1 + 2\operatorname{Re}(\bar r(q-r))\big] - \bar r\partial_x r}{|r|^2 - \zeta^2}\,\mathrm{d}x - \frac{i}{2z}\mathcal{M} - \frac{i}{2z(\lambda+z)}\Theta, \tag{1.44}$$

and T_{2n} are $2n$ dimensional integrals essentially from a Picard iteration from $x = -\infty$ which can be found in (3.24) below. A direct estimate gives, e.g. on the imaginary axis $(\lambda, z) = (i\sqrt{\tau^2/4-1}, i\tau/2)$ and $\tau \geq \max\{2, 2C(E^0_\tau(q))^2\}$,

$$\Big|\sum_{n=2}^{\infty} T_{2n}(\lambda)\Big| \leq C\Big(\frac{E^0_\tau(q)}{\sqrt{\tau}}\Big)^4.$$

such that if we take the logarithm:

$$\Big|\ln T_c^{-1} - (\Phi + T_2)\Big| \leq C\Big(\frac{E^0_\tau(q)}{\sqrt{\tau}}\Big)^4.$$

Careful estimates give

$$\Big|\operatorname{Re}(\Phi + T_2)\Big|_{\lambda = i\sqrt{\tau^2/4-1}} + \frac{1}{\tau}(E^0_\tau(q))^2\Big| \leq C\Big(\frac{E^0_\tau(q)}{\sqrt{\tau}}\Big)^3,$$

which implies the energy estimate (1.28) of Theorem 1.6 on the imaginary axis.

The odd conserved Hamiltonians $\tilde H_{2n+1}$ in (1.33) in Corollary 1.8 are defined by the asymptotic series

$$\frac{1}{2i}\Big(\ln T_c^{-1}(\lambda) - \ln \overline{T_c^{-1}(-\bar\lambda)}\Big) \sim \frac{\lambda}{z}\sum_{n=0}^{\infty} \tilde H_{2n+1}(2z)^{-2(n+1)}.$$

The claim follows by combining the estimates of [14] with (1.29).

Organisation of the paper. We will discuss the topology of the metric space (X^s, d^s) in Section 2 where we will prove Theorem 1.2 and Theorem 1.3. Theorem 1.6 will be proven in Section 3.

2. The topology of X^s

In this section we prove Theorem 1.2 that Q is a strong deformation retract and Theorem 1.3 about the asymptotic phase shift Θ and the renormalized momentum H_1. The first step consists in estimates for an approximation, which is nontrivial since X^s is not a linear function space.

2.1. **The regularization: Estimates for r.**

One of the difficulties of dealing with X^s with $s \le \frac{1}{2}$ is that $H^s(\mathbb{R})$ is not an algebra. We regularize q by $r = \tau^2 D_\tau^{-2} q$ and derive estimates for the crucial functions $|r|^2-1$, r' and $q-r$ in terms of $E_\tau^0(q)$ in Lemma 2.2 below.

Definition 2.1. *Let $\tau \ge 2$, $s \in \mathbb{R}$ and $1 < p < \infty$. We define*

$$\|f\|_{W_\tau^{s,p}(\mathbb{R})} = \|D_\tau^s f\|_{L^p(\mathbb{R})}$$

where $D_\tau = (-\partial_x^2 + \tau^2)^{\frac{1}{2}}$ is defined by the Fourier multiplier as in (1.19).

The case $s = -1$ is of particular interest. Let $j_\tau = \chi_{\{x<0\}} e^{\tau x} = \begin{cases} e^{\tau x} & \text{if } x < 0 \\ 0 & \text{if } x \ge 0 \end{cases}$. Then for $1 < p < \infty$, we have by the above definition

$$\|f\|_{W_\tau^{-1,p}(\mathbb{R})} = \|D_\tau^{-1} f\|_{L^p(\mathbb{R})} \sim \|j_\tau * f\|_{L^p(\mathbb{R})}. \tag{2.45}$$

Lemma 2.2. *Let $\tau \ge 2$ and $q \in X^0$. There exists r so that $r \in X^2$ and*

$$\begin{aligned} C_p^{-1}\|q'\|_{W_\tau^{-1,p}} \le {} & \|q-r\|_{L^p} + \tau^{-1}\|r'\|_{L^p} \le C_p \|q'\|_{W_\tau^{-1,p}}, \quad \forall p \in (1,\infty), \\ \|r\|_{L^\infty} & \le C\tau\big(1+\tau^{-1/2}E_\tau^0(q)\big), \\ \||r|^2-1\|_{L^p} & \le C_p \tau\big(1+\tau^{-1/2}E_\tau^0(q)\big)\|\big(|q|^2-1, q'\big)\|_{W_\tau^{-1,p}}, \quad \forall p \in [2,\infty), \end{aligned} \tag{2.46}$$

where $C_p > 0$ is some universal constant depending only on p.

In particular if $q \in X^0$, then $r \in X^2$, with the energy

$$E^1(r) = \big\|(|r|^2-1, r')\big\|_{L^2} \le C\big(1+E^0(q)\big)E^0(q).$$

Proof. We choose

$$r = \tau^2 D_\tau^{-2} q = \tau^2(-\partial_{xx} + \tau^2)^{-1} q,$$

and obtain by a multiple use of Hörmander's multiplier theorem the first line of (2.46) for $p \in (1,\infty)$ by

$$\begin{aligned} \|r'\|_{W_\tau^{-1,p}} &= \|\tau^2 D_\tau^{-3} q'\|_{L^p} \le C_p \|D_\tau^{-1} q'\|_{L^p}, \\ \|r'\|_{L^p} &= \|\tau^2 D_\tau^{-2} q'\|_{L^p} \le C_p \tau \|D_\tau^{-1} q'\|_{L^p}, \\ \|r-q\|_{L^p} &= \|D_\tau^{-2}\partial_x q'\|_{L^p} \le C_p \|D_\tau^{-1} q'\|_{L^p}, \\ \|D_\tau^{-1} q'\|_{L^p} &\le \|D_\tau^{-1} r'\|_{L^p} + \|D_\tau^{-1}(q-r)'\|_{L^p} \le C_p \tau^{-1}\|r'\|_{L^p} + C_p\|q-r\|_{L^p}. \end{aligned}$$

We turn to the proof of the second line in (2.46). Let I be an interval with the center $x_0 \in \mathbb{R}$ and the length τ^{-1}. Let $J \supset I$ have the same center $x_0 \in \mathbb{R}$ and the length $3\tau^{-1}$. Let $\eta \in C_c^\infty(\mathbb{R};[0,1])$ be supported in $(-1,1)$, identically 1 on $[-1/2,1/2]$, and $\eta_\tau = \tau\eta(\tau x)$, then

$$\begin{aligned}
\int_I |q|^2 dx &= \int_I \eta(\tau(x_0-x))|q|^2(x)\,\mathrm{d}x \\
&\le \int_J \eta(\tau(x_0-x))(|q|^2-1)(x)\,\mathrm{d}x + \int_J \eta(\tau(x_0-x))\,\mathrm{d}x \\
&\lesssim \||q|^2-1\|_{H_\tau^{-1}}\tau^{-1}\|\eta_\tau\|_{H_\tau^1} + \tau^{-1} \\
&\lesssim \tau^{1/2}\||q|^2-1\|_{H_\tau^{-1}} + \tau^{-1}.
\end{aligned}$$

Hence, by use of $(2.46)_1$: $\|q-r\|_{L^2} \lesssim \|q'\|_{H_\tau^{-1}}$,

$$\begin{aligned}
\|q\|_{L^2(I)} + \|r\|_{L^2(I)} &\le \|r-q\|_{L^2(I)} + \|q\|_{L^2(I)} \\
&\lesssim \tau^{1/4}\||q|^2-1\|_{H_\tau^{-1}}^{1/2} + \|q'\|_{H_\tau^{-1}} + \tau^{-1/2}.
\end{aligned} \tag{2.47}$$

For the second estimate $(2.46)_2$ we observe that

$$\tau^2 r - r_{xx} = \tau^2 q,$$

and hence by Sobolev's inequality

$$\begin{aligned}
\|r\|_{L^\infty(I)} &\lesssim \tau^{1/2}\|r\|_{L^2(I)} + \tau^{-3/2}\|r''\|_{L^2(I)} \\
&\lesssim \tau^{1/2}\big(\|r\|_{L^2(I)} + \|q\|_{L^2(I)}\big) \\
&\lesssim \tau + \tau^{1/2}E_\tau^0(q).
\end{aligned}$$

Before we address the third line of (2.46) we claim the Sobolev inequality (recalling $j_\tau = \chi_{\{x<0\}}e^{\tau x}$ and $\|j_\tau * f\|_{L^1}$ can be used to define the norm $\|f\|_{W_\tau^{-1,1}}$)

$$\|f\|_{L^p(I)} \lesssim \tau^{-1/p}\|f'\|_{L^1(I)} + \tau^{2-1/p}\|(1+\tau^2|x-x_0|^2)^{-1}(j_\tau * f)(x)\|_{L^1(\mathbb{R})}. \tag{2.48}$$

By scaling and translation it suffices to prove inequality (2.48) for $\tau = 1$ and $x_0 = 0$. This simplifies the notation. By the fundamental theorem of calculus, if $I = [-1/2,1/2]$ is an interval of length 1 and $\eta \in C_b^1(I)$ with integral 1 (we can choose η so that $\|\eta + \eta_x\|_{L^\infty} \le 6$), then

$$\begin{aligned}
\|g\|_{L^p(I)} &\le \|g\|_{L^\infty(I)} \le \|g'\|_{L^1(I)} + \Big|\int_I g\eta\,\mathrm{d}x\Big| \\
&= \|g'\|_{L^1(I)} + \Big|\int_{\mathbb{R}} (j_1 * g)(\partial_x+1)\eta\,\mathrm{d}x\Big| \\
&\le \|g'\|_{L^1(I)} + 6\|j_1 * g\|_{L^1(\mathbb{R})}.
\end{aligned} \tag{2.49}$$

This is almost what we need, up to the nonlocality of the second term on the right hand side of (2.48). Notice that by the exponential decay property of the kernel $j_1(x) = \chi_{\{x<0\}}e^x$, we have

$$\|j_1 * (\rho f)\|_{L^1(\mathbb{R})} \sim \|\rho(j_1 * f)\|_{L^1(\mathbb{R})}, \tag{2.50}$$

where ρ is chosen to be continuous and positive so that $\rho = 1$ on I and

$$\rho(x) = (1+|x-x_0|^2)^{-1} \qquad \text{for } |x-x_0| \ge 3/2.$$

Indeed, in order to show $\|\rho(j_1 * f)\|_{L^1} \lesssim \|j_1 * (\rho f)\|_{L^1}$, we take $g = j_1 * (\rho f)$ resp.

$$f = \rho^{-1}(\partial_x - 1)g,$$

and it suffices to show

$$\|\rho\big(j_1 * (\rho^{-1}(\partial_x - 1)g)\big)\|_{L^1} \lesssim \|g\|_{L^1}.$$

Similarly, in order to show $\|j_1 * (\rho f)\|_{L^1} \lesssim \|\rho(j_1 * f)\|_{L^1}$, it suffices to show

$$\|j_1 * \big(\rho(\partial_x - 1)\rho^{-1}g\big)\|_{L^1} \lesssim \|g\|_{L^1}.$$

If we exchange the orders of ρ^{-1} and the derivative $(\partial_x - 1)$, then one arrives at the identity, and hence it suffices to bound the commutators $\rho j_1 * [\rho^{-1}, \partial_x]g$ and $j_1 * (\rho[\partial_x, \rho^{-1}]g)$ as follows

$$\|\rho j_1 * (\rho^{-1}\frac{\rho'}{\rho}g)\|_{L^1} \lesssim \|g\|_{L^1}, \quad \|j_1 * (\frac{\rho'}{\rho}g)\|_{L^1} \lesssim \|g\|_{L^1}.$$

The second inequality holds obviously by Young's inequality and it suffices to show

$$\|\rho j_1 * (\rho^{-1}h)\|_{L^1} \lesssim \|h\|_{L^1}.$$

This is ensured by the boundedness of the integral operator $h \mapsto \int_{\mathbb{R}} K(x,y)h(y)\,\mathrm{d}y$ in L^1 with the integral kernel

$$K(x,y) = \left(\frac{\rho(x)}{\rho(y)}\right) 1_{\{y<x\}} e^{-(x-y)}$$

satisfying Schur's criterium

$$\sup{}_x \|K(x,.)\|_{L^1(\mathbb{R})} + \sup{}_y \|K(.,y)\|_{L^1(\mathbb{R})} < \infty.$$

Thus (2.50) follows. Finally we apply (2.49) with $g = \rho f$, (2.50) and rescaling to derive (2.48).

We turn to the proof of the last line of (2.46), by use of (2.47), (2.48) and (2.50). Let $p \geq 2$, then

$$\begin{aligned}
\||r|^2 - 1\|_{L^p(I)} &\lesssim \tau^{-\frac{1}{p}}\|\bar{r}r'\|_{L^1(I)} + \tau^{2-\frac{1}{p}}\|(1+\tau^2|x-x_0|^2)^{-1}\big(j_\tau * (|r|^2-1)\big)(x)\|_{L^1(\mathbb{R})} \\
&\lesssim \tau^{-\frac{1}{2}}\|r\|_{L^2(I)}\|r'\|_{L^p(I)} + \tau^{1-\frac{1}{p}}\|(1+\tau^2|x-x_0|^2)^{-1}(|q|^2-|r|^2)\|_{L^1(\mathbb{R})} \\
&\quad + \tau^{2-\frac{1}{p}}\|(1+\tau^2|x-x_0|^2)^{-1}\big(j_\tau * (|q|^2-1)\big)(x)\|_{L^1(\mathbb{R})} \\
&\lesssim \tau^{-\frac{1}{2}}\|r\|_{L^2(I)}\|r'\|_{L^p(I)} \\
&\quad + \tau^{\frac{1}{2}} \sup_{I',|I'|=\tau^{-1}} \|(q,r)\|_{L^2(I')}\|(1+\tau^2|x-x_0|^2)^{-\frac{1}{p}}(q-r)\|_{L^p(\mathbb{R})} \\
&\quad + \tau\|(1+\tau^2|x-x_0|^2)^{-1/p}\big(j_\tau * (|q|^2-1)\big)(x)\|_{L^p(\mathbb{R})}.
\end{aligned}$$

We have used (the rescaled version of) (2.50) for the second inequality with $\rho_\tau(x) := (1+\tau^2|x-x_0|^2)^{-1}$:

$$\begin{aligned}
\tau^{2-\frac{1}{p}}\|\rho_\tau(j_\tau * (|q|^2-|r|^2))\|_{L^1(\mathbb{R})} &\lesssim \tau^{2-\frac{1}{p}}\|j_\tau * (\rho_\tau(|q|^2-|r|^2))\|_{L^1(\mathbb{R})} \\
&\leq \tau^{1-\frac{1}{p}}\|\rho_\tau(|q|^2-|r|^2)\|_{L^1(\mathbb{R})},
\end{aligned}$$

which has been further bounded in the last inequality by Hölder's inequality and change of variables $\tau x \mapsto x$ as

$$\begin{aligned}
&\tau^{\frac{1}{p'}} \|\rho_\tau^{\frac{1}{p'}} |(q,r)|\|_{L^{p'}(\mathbb{R})} \|\rho_\tau^{\frac{1}{p}}(q-r)\|_{L^p(\mathbb{R})} = \tau^{\frac{1}{p'}} \Big(\int_{\mathbb{R}} \rho_\tau |(q,r)|^{p'} \,\mathrm{d}x \Big)^{\frac{1}{p'}} \|\rho_\tau^{\frac{1}{p}}(q-r)\|_{L^p(\mathbb{R})} \\
&\lesssim \tau^{\frac{1}{p'}} \Big(\sup_{I', |I'|=\tau^{-1}} \|(q,r)\|_{L^2(I')} \Big) \tau^{-(\frac{1}{2}-\frac{1}{p})} \Big(\sum_{n\in\mathbb{Z}} \|\rho\|_{L^{\frac{1-\frac{1}{p}}{\frac{1}{2}-\frac{1}{p}}}([n,n+1])} \Big)^{\frac{1}{p'}} \|\rho_\tau^{\frac{1}{p}}(q-r)\|_{L^p(\mathbb{R})} \\
&\lesssim \tau^{\frac{1}{2}} \sup_{I', |I'|=\tau^{-1}} \|(q,r)\|_{L^2(I')} \|\rho_\tau^{\frac{1}{p}}(q-r)\|_{L^p(\mathbb{R})}.
\end{aligned}$$

We raise the inequality to the power p and sum over intervals I to obtain

$$\||r|^2-1\|_{L^p} \le C\tau\big(1+\tau^{-1/2}E^0_\tau(q)\big)\Big(\|q'\|_{W^{-1,p}_\tau} + \||q|^2-1\|_{W^{-1,p}_\tau}\Big).$$

The proof for the estimates (2.46) is finished. □

2.2. **Proof of Theorem 1.2.**

Proof of Theorem 1.2. We will assume $s=0$ for notational simplicity and clarity, but we will give an argument which immediately applies to $s\in[0,\infty)$. We define $r=\tau^2(\tau^2-\partial^2)^{-1}q = h*q$, $h := \frac{\tau}{2}e^{-\tau|x|}$ as in Lemma 2.2. We fix τ large enough such that the regularised solitons $r_c := h * q_c$ satisfy

$$|r_c(R)| \in [\frac{3}{4},1], \quad \forall |R|\ge 1, \quad \forall c\in[-1,1]. \tag{2.51}$$

The map $X^0 \ni q \mapsto r \in X^2$ is Lipschitz continuous (see Lemma 2.1 [14]).

We are going to construct the deformation from X^0 to Q essentially in two steps: We first construct a deformation $\Xi_1(t,q)$ defined on $(t,q)\in[0,1]\times X^0$, such that some regularised function $\tilde r$ of $q\in\{\Xi_1(1,p) | p=\Xi_1(0,p)\in X^0$ and away from Q : $d^0(p,Q)>2\delta\}$ has no zeros outside the fixed space interval $[-1,1]$. At the same time we keep Q fixed under Ξ_1: $Q=\Xi_1(t,Q)$.

We then investigate the asymptotic behaviours of the regularised dark solitons r_c of $q_c\in Q$, such that the regularised function $\tilde r$ of $q\in\{\Xi_1(1,p)|p\in X^0$ and close to Q : $d^0(p,Q)<2\delta\}$ has no zeros outside some large space interval $[-R,R]\supset[-1,1]$. We construct the second deformation Ξ_2 by unwinding the rotations at infinity.

Step 1: Construction of Ξ_1. We first have the following observation. Given an interval I we define

$$E^1(r;I) = \Big(\int_I \big(|r'|^2 + (|r|^2-1)^2\big)\,\mathrm{d}x\Big)^{\frac{1}{2}}.$$

We assume that ε is sufficiently small so that $|r(x)|\in(\frac{1}{2},2)$ if $E^1(r;(x-1,x+1)))<2\varepsilon$. For $q\in X^0$ with $E^1(r)>\varepsilon$, we can define a continuous map

$$q \to (a_-,a_+)\in\mathbb{R}^2$$

where

$$\begin{aligned}
&E^1(r;(-\infty,a_-+1)) + \frac{\varepsilon}{2}(1+\tanh(a_-+1)) \\
&\qquad = E^1(r;(a_+-1,\infty)) + \frac{\varepsilon}{2}(1-\tanh(a_+-1)) = \varepsilon.
\end{aligned}$$

Then $a_\pm$ are uniquely defined since the functions are strictly monotone, and $a_-+1 < a_+ - 1$ since otherwise

$$\begin{aligned}\varepsilon < E^1(r;(-\infty,a_-+1)) + E^1(r;(a_+-1,\infty))\\ = \varepsilon + \frac{\varepsilon}{2}(\tanh(a_+-1) - \tanh(a_-+1)) \le \varepsilon,\end{aligned}$$

which is impossible. We want to extend the definition of $a_\pm$ as continuous functions $x_\pm$ on the whole of X^0. Let

$$\rho_0(t) = \begin{cases} 0 & \text{if } t \le 1 \\ t-1 & \text{if } 1 < t \le 2 \\ 1 & \text{if } t > 2. \end{cases}$$

We define

$$x_+(q) = 1 + \rho_0(E^1(r)/\varepsilon)(a_+(q)-1), \qquad x_-(q) = -1 + \rho_0(E^1(r)/\varepsilon)(a_-(q)+1)$$

which is now continuous on X^0. Then $x_\pm = \pm 1$ if $E^1(r) \le \varepsilon$, and

$$|r(x)| \in (\frac{1}{2}, 2), \quad \forall x \in (-\infty, x_-) \cup (x_+, \infty),$$

and hence the function r does not vanish outside (x_-, x_+) with

$$x_+ - x_- \ge 2.$$

Given $\delta > 0$ we define a continuous function $\rho : X^0 \mapsto [0,1]$ as follows:

$$\rho(q) = \rho_0(d^0(q,Q)/\delta), \text{ where } Q = \{q_c \,|\, c \in [-1,1]\}.$$

Notice that $\rho(q_c) = 0$, $\forall q_c \in Q$, and $\rho(q) = 1$ for q away from Q: $d^0(q,Q) > 2\delta$. We define the first deformation via

$$\big(\Xi_1(t,q)\big)(y) = q\Big((1+t\rho(q)(x_+-x_-)/2)y + t\rho(q)(x_++x_-)/2\Big), \quad t \in [0,1]. \tag{2.52}$$

It is the identity if $t = 0$. It is clearly continuous and the identity on Q. Furthermore, since $t\rho(q)$ take values in $[0,1]$ and $x_+ - x_- \ge 2 > 0$, the function $x = x(t,y) = (1+t\rho(q)(x_+-x_-)/2)y + t\rho(q)(x_++x_-)/2$ as the argument for the q-function above in (2.52) satisfies

$$\begin{aligned} x \ge 1 + t\rho(q)x_+ \text{ if } y \ge 1,\\ x \le -1 + t\rho(q)x_- \text{ if } y \le -1.\end{aligned}$$

We denote the composition of $r(x)$ with this transformation of coordinates $y \mapsto x(t,y)$ by $\tilde r$,

$$\tilde r(t,y) = r\Big((1+t\rho(q)(x_+-x_-)/2)y + t\rho(q)(x_++x_-)/2\Big). \tag{2.53}$$

We observe that for $q \in X^0$ away from Q with $d^0(q,Q) > 2\delta$,

$$\begin{aligned} \tilde r(1,y) = r\big((1+(x_+-x_-)/2)y + (x_++x_-)/2\big)\\ \text{with } |\tilde r(1,y)| \in (\frac{1}{2}, 2) \text{ for } |y| \ge 1.\end{aligned} \tag{2.54}$$

For $q \in X^0$ close to Q: $d^0(q,Q) < \delta$,

$$\tilde r(t,y) = r(y). \tag{2.55}$$

We are going to choose $\delta > 0$ small enough in next step such that the above (2.54) holds also for q close to Q: $d^0(q,Q) < 2\delta$ if $|y| \ge R_0$ for $R_0 > 0$ large enough.

Step 2. Construction of Ξ_2. We first investigate the regularised dark solitons $r_c(R) := (h * q_c)(R)$, $h = \frac{\tau}{2}e^{-\tau|x|}$, for large $|R|$ in the following lemma.

Lemma 2.3. *There exists $R_0 > 0$ so that for all $R \geq R_0$, the map*

$$(-1,1] \ni c \to \phi(c) := r_c(R)/r_c(-R)$$

is injective and

$$\frac{3}{4} \leq |r_c(R)|, |r_c(-R)| \leq 1, \quad \forall c \in (-1,1]. \tag{2.56}$$

Proof. By virtue of (2.51), the bound (2.56) is clearly achieved by choosing $R_0 \geq 1$. By construction $r_c(x) = -\overline{r_c(-x)}$ and hence

$$r_c(R)/r_c(-R) = -\left(r_c(R)/|r_c(R)|\right)^2.$$

It suffices to prove that

$$\partial_c \arg r_c(R) > 0, \quad \lim_{c\to 1} \arg r_c(R) = \frac{\pi}{2}, \quad \lim_{c\to -1} \arg r_c(R) = -\frac{\pi}{2}. \tag{2.57}$$

The limits $c \to \pm 1$ are obvious and we will prove that the derivative $\partial_c \arg r_c(R)$ never vanishes for $R \geq R_0$. For fixed $R > 0$, wse compute first

$$\begin{aligned}
\partial_c q_c(R) &= \partial_c\Big[ic + \sqrt{1-c^2}\tanh(\sqrt{1-c^2}R)\Big] \\
&= i - cR\operatorname{sech}^2(\sqrt{1-c^2}R) - \frac{c}{\sqrt{1-c^2}}\tanh(\sqrt{1-c^2}R).
\end{aligned}$$

and hence, if $-1 < c < 1$

$$\begin{aligned}
\partial_c \arg q_c(R) &= \operatorname{Im}\frac{\partial_c q_c(R)}{q_c(R)} \\
&= \frac{\frac{1}{\sqrt{1-c^2}}\tanh(\sqrt{1-c^2}R) + c^2 R\operatorname{sech}^2(\sqrt{1-c^2}R)}{1-(1-c^2)\operatorname{sech}^2(\sqrt{1-c^2}R)} \\
&\geq \frac{1}{2}\Big(\tanh(\sqrt{1-c^2}R) + c^2\sqrt{1-c^2}R\operatorname{sech}^2(\sqrt{1-c^2}R)\Big) \\
&\geq \mu > 0
\end{aligned}$$

for some $\mu > 0$ if $R > 1$. This implies the claim for r_c if R is sufficiently large since $\lim_{R\to\infty}|q_c'(R)| = 0$. □

Recall the definition of $\tilde{r}(t,y)$ in (2.53) and the fact (2.54) for q away from Q. We choose $R_0 \geq 1$ large enough and δ small enough, so that (by virtue of the continuity of the map $X^0 \ni q \mapsto r \in X^2$) for all $q \in X^0$ close to Q: $d^0(q,Q) < 2\delta$,

$$\begin{aligned}
&|\tilde{r}(1,y)| = \Big|r\Big(\big(1+\rho_0(\frac{d^0(q,Q)}{\delta})\frac{x_+ - x_-}{2}\big)y + \rho_0(\frac{d^0(q,Q)}{\delta})\frac{x_+ + x_-}{2}\Big)\Big| \in [\frac{1}{2}, 2], \\
&\quad \forall y \notin [-R_0, R_0].
\end{aligned}$$

Thus $\tilde{r}(1,y)$ does not have zeros for all $q \in X^0$ and $|y| \geq R_0$.

In the following we simply take $R = R_0$ and denote (with an abuse of notations) $\tilde{r}(1,y)$ by $\tilde{r}(y)$. Since the map

$$\beta : Q \ni q_c \to \frac{r_c(R)}{r_c(-R)} \in \mathbb{S}^1$$

is a homeomorphism, we define the retraction B by

$$B(q) = \beta^{-1}\left(\frac{\tilde r(R)}{|\tilde r(R)|}\frac{\overline{\tilde r(-R)}}{|\tilde r(-R)|}\right) : X^s \mapsto Q.$$

This is a continuous map from X^s to Q which is the identity on Q.

Recall the first deformation (2.52). We fix the representative in the equivalence class of $q \in X^s$ by requiring

$$\tilde r(-R) \in (\frac{1}{2}, 2).$$

There exists a smooth $\alpha(y) = \alpha(y;\tilde r)$ supported on $(-\infty, -R+1) \cup (R-1, \infty)$ with bounded derivatives so that

$$\tilde r(y) = \begin{cases} \exp(i\alpha(y))b(y) & \text{if } y < -R \\ \exp(i\alpha(y))b(y)\frac{\tilde r(R)}{|\tilde r(R)|} & \text{if } y > R \end{cases}$$

where $\alpha(-R) = \alpha(R) = 0$ and $b - 1 \in L^2$ are real valued with $\frac{1}{2} \le b \le 2$. We define the second deformation

$$\Xi_2(t, \Xi_1(1,q)) = \Xi_1(1,q)\exp(it(\alpha(y;B(q)) - \alpha(y;\tilde r))),$$

which again fixes Q. As a consequence

$$\Xi_2(1, \Xi_1(1,q)) - B(q) \in L^2.$$

We define the final deformation by

$$\Xi_3(t, \Xi_2(1, \Xi_1(1,q))) = (1-t)\Xi_2(1, \Xi_1(1,q)) + tB(q).$$

□

2.3. **Proof of Theorem 1.3.** Theorem 1.3 studies the asymptotic phase shift Θ and the modified momentum H_1. Before we start the construction rigorously, we recall that if $q \in X^0$ never vanishes with $\partial_x q \in L^1$, we define

$$\Theta(q) = -\mathrm{Im}\int_{\mathbb R} \frac{\partial_x q}{q} dx$$

and if $q \in X^s$ with $s > \frac{1}{2}$ never vanishes, then we define

$$H_1(q) = -\mathrm{Im}\int_{\mathbb R} (|q|^2 - 1)\frac{\partial_x q}{q} dx.$$

The issue is to find a continuous resp. smooth extension across q with zeros, on the covering space of X^0.

We first regularise $q \mapsto \tilde q$ as follows, such that $\tilde q$ does not vanish.

Lemma 2.4. *Given $q_0 \in X^s$, $s \ge 0$, there exist $\delta \in (0,1)$ depending only on $E^0(q_0)$ and a continuous map*

$$B_\delta^{X^0}(q_0) \ni q \to \tilde q \in X^{s+2}$$

so that $|\tilde q| = 1$,

$$\|q - \tilde q\|_{H^s} \le c(E^0(q_0))E^s(q), \tag{2.58}$$

and, if $\partial_x q \in L^1$ then $q - \tilde q \in L^1$ and $\partial_x \tilde q \in L^1$.

For $q \in 1 + \mathcal S$ we may choose $\tilde q = 1$ if we give up the estimate (2.58). We postpone the proof of Lemma 2.4 and complete the proof of Theorem 1.3.

Proof of Theorem 1.3. We define for $q \in X^0$ with $\partial_x q \in L^1$,

$$\Theta = -\mathrm{Im} \int_{\mathbb{R}} \partial_x \log \tilde{q} \, \mathrm{d}x \,.$$

Since Θ is the winding number of $x \to \tilde{q}(x)$ if $\partial_x q \in L^1$, it is obvious that different choices lead to the same Θ, up to $2\pi \mathbb{Z}$.

For $q \in X^s$, $s \geq \frac{1}{2}$ and $\tilde{q}$ from Lemma 2.4, we define H_1 as in (1.16):

$$H_1(q) = \mathrm{Im} \int_{\mathbb{R}} (q - \tilde{q}) \partial_x \bar{q} - \overline{(q - \tilde{q})} \partial_x \tilde{q} dx.$$

By Lemma 2.4, with the continuity of the maps

$$B_\delta^{X^0}(q_0) \cap X^s \ni q \to q - \tilde{q} \in H^{1/2}$$

and

$$B_\delta^{X^0}(q_0) \cap X^s \ni q \to \partial_x \tilde{q} \in H^{-1/2},$$

the map $q \to H_1(q)$ is continuous on $B_\delta^{X^0}(q_0) \cap X^s$. Hence we obtain a unique continuous map for $s \geq \frac{1}{2}$

$$H_1(q) : X^s \to \mathbb{R} / (2\pi \mathbb{Z}).$$

Conservation of $H_1(q)$ by the Gross-Pitaevskii flow has been proven at the end of Subsection 1.1. □

Proof of Lemma 2.4. The construction of the map $q \to \tilde{q}$ is elementary but delicate. It suffices to construct $\tilde{q}$ with $|\tilde{q}| \geq \frac{1}{4}$ since then $\frac{\tilde{q}}{|\tilde{q}|}$ has the same properties.

Let $\delta > 0$ and $q_0 \in X^s$, $s \geq 0$. Let $r = h * q \in X^{s+2}$, $h = \frac{\tau}{2} e^{-\tau|x|}$ (as in Lemma 2.2) for some $\tau \geq 2$ large enough, so that q in the unit ball $B_1^{X^0}(q_0)$ centered at q_0 are uniformly close to its regularisation r:

$$\|q - r\|_{L^2} < \delta, \quad \forall q \in B_1^{X^0}(q_0). \tag{2.59}$$

This is the only relevance of τ (which depends on δ and $E^0(q_0)$). We are going to construct $q \mapsto \tilde{q}$ for a single $q \in X^s$ first, and then to extend this construction $q \mapsto \tilde{q}$ continuously to the small ball $B_\delta^{X^0}(q_0)$ for some $\delta < 1$ small enough.

Step 1. Construction of $q \mapsto \tilde{q}$. Let $q \in X^s$, $s \geq 0$, and $r = h * q \in X^{s+2}$, $h = \frac{\tau}{2} e^{-\tau|x|}$ for some $\tau \geq 2$. The set $\{x : |r(x)| < \frac{1}{2}\}$ is an at most countable union of open intervals. In the complement we would like to define $\tilde{q} = r$. If I is one of the intervals and $|r| \geq \frac{1}{4}$ on I, we also want to set $\tilde{q} = r$ on I. There are at most *finitely many* other intervals [4]. The actual construction is more involved.

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[4]Indeed, the energy on the interval $J_k = (a_k, b_k)$ reads roughly as $\int_{J_k} (|r|^2 - 1)^2 + |r'|^2 \gtrsim |J_k| + \frac{1}{|J_k|}$, since $||r| - 1||_{J_k} > \frac{1}{2}$, and the derivative of $|r|$ on some subinterval $\tilde{J}_k = (a_k, c_k) \subset (a_k, b_k)$ with $|r(c_k)| = \frac{1}{4}$ is comparable with $\frac{1}{|\tilde{J}_k|}$ and hence $\int_{\tilde{J}_k} (|r|')^2 \gtrsim \frac{1}{|\tilde{J}_k|} \geq \frac{1}{|J_k|}$. Finite energy assumption implies the finiteness of the number of such intervals J_k, and the upper and lower bounds for its length $|J_k|$ in terms of the energy of r.

There is a finite number K (possibly 0) bounded by a constant depending only on the $E^0(q)$ of disjoint intervals

$$\begin{aligned} & I_k = (a_k, b_k), \quad 0 < k \le K < \infty, \\ & \text{with } |r(x)| \ge \frac{1}{2} \quad \text{if } \frac{b_k - a_k}{6} \le \big|x - \frac{a_k + b_k}{2}\big| \le \frac{b_k - a_k}{2} \\ & \text{and } |r(x)| < \frac{1}{4} \text{ for at least one } x \in I_k, \end{aligned} \tag{2.60}$$

with a length $|I_k|$ bounded from below and above by a fixed constant depending only on $E^0(q)$, so that $|r(x)| \ge \frac{1}{2}$ if $x \notin \cup_{k=1}^K I_k$. Notice that that $|r| \ge \frac{1}{2}$ also in the outer thirds of the interval I_k: $[a_k, a_k + \frac{b_k - a_k}{3}] \cup [b_k - \frac{b_k - a_k}{3}, b_k]$, while $|r| < \frac{1}{4}$ on some subinterval in the inner third of the interval I_k: $[a_k + \frac{b_k - a_k}{3}, b_k - \frac{b_k - a_k}{3}]$.

We define $\tilde q = r$ on the complement of these intervals. It remains to define $\tilde q$ on the intervals $I_k = (a_k, b_k)$. To do that we need some preparations. We fix a smooth monotone function ϕ identically 0 on $(-\infty, 1/3]$ and identically 1 for $x \ge 2/3$ and a smooth nonnegative function η supported in $(0,1)$, identically 1 on $[1/3, 2/3]$. We define

$$\phi_k(x) = \phi\big(\frac{x - a_k}{b_k - a_k}\big), \quad \eta_k(x) = \eta\big(\frac{x - a_k}{b_k - a_k}\big).$$

We write r in polar coordinates in the outer thirds of the interval I_k,

$$r(x) = \rho(x) e^{i\theta(x)} \qquad \text{on } \left\{ \frac{b_k - a_k}{6} \le \Big|x - \frac{b_k + a_k}{2}\Big| \le \frac{b_k - a_k}{2} \right\}$$

with $0 \le \theta(b_k) - \theta(a_k) < 2\pi$. We define on I_k

$$\begin{aligned} \tilde q(x) = & \Big((1 - \eta_k(x))\rho(x) + \eta_k(x)\big(\rho(a_k) + \phi_k(x)(\rho(b_k) - \rho(a_k))\big)\Big) \\ & \times \exp\Big(i\big((1 - \eta_k(x))\theta(x) + \eta_k(x)\big(\theta(a_k) + \phi_k(x)(\theta(b_k) - \theta(a_k))\big)\big)\Big). \end{aligned} \tag{2.61}$$

Then $\tilde q \in X^{s+2}$ satisfies $|\tilde q| \ge \frac{1}{4}$, and

$$\|q - \tilde q\|_{H^s} \le \|q - r\|_{H^s} + \|r - \tilde q\|_{H^s} \le c(E^0(q), \tau) E^s(q). \tag{2.62}$$

Step 2. Construction of $q \mapsto \tilde q$ on $B_\delta^{X^0}(q_0)$. Now we want to fix a continuous choice of $\tilde q$ *locally* in a neighborhood of q_0. We do the previous construction for q_0 and fix the chosen intervals $I_k = (a_k, b_k)$. Let $q \in B_\delta^{X^0}(q_0)$ with $\delta < 1$ sufficiently small (to be determined later), that is,

$$d^0(q, q_0) = \left(\int_{\mathbb{R}} \inf_{|\lambda|=1} \|\operatorname{sech}(x - y)(q(x) - \lambda q_0(x))\|_{L^2_x(\mathbb{R})}^2 dy \right)^{\frac{1}{2}} < \delta.$$

Let $r = h * q$, $h = \frac{\tau}{2} e^{-\tau|x|}$ with τ large enough such that (2.59) holds.

In the construction of $\tilde q$ below we are going to use some facts on the complete metric space (X^s, d^s) in Theorem 1.1, as well as some regularity results, which can be found in [14, Section 6]. Indeed, let $\lambda(y)$ in the definition of $d^0(q, q_0)$ above be the constant at y. Then we have the following regularity for the function λ (see also [14, Lemma 6.4])

$$|\lambda(x) - \lambda(y)| \le C\delta|x - y|, \tag{2.63}$$

for some constant C depending on $E^0(q_0)$. We write λ as $\lambda(y) = \exp(i\alpha(y))$ for some real smooth function α with derivatives bounded by $C\delta$. We fix a point x_0

such that $|r(x)| \geq \frac{1}{2}$ on $x \in (x_0 - 1, x_0 + 1)$ and add a multiple of 2π so that

$$|\alpha(x_0)| < \frac{\pi}{2}.$$

With this choice the function α is uniquely determined. Then, with a possibly different constant, we derive from (2.59) that

$$\int_{\mathbb{R}} \| \operatorname{sech}(x-y)(r(x) - e^{i\alpha(x)} r_0(x)) \|_{L^2_x}^2 dy \leq C\delta^2.$$

We take a small enough δ, such that (recalling by the construction in Step 1, $|r_0| \geq \frac{1}{2}$ on the outer thirds of the interval I_k)

$$|r(x)| \geq \frac{1}{4} \qquad \text{for } x \in \mathbb{R} \setminus \bigcup_k \Big(a_k + \frac{1}{3}(b_k - a_k), b_k - \frac{1}{3}(b_k - a_k) \Big).$$

We define

$$\tilde{q}(x) = r(x) \quad \text{ in } \mathbb{R} \setminus \bigcup_k (a_k, b_k)$$

and write r again in polar coordinates in outer thirds of the intervals I_k

$$r(x) = \rho(x) \exp(i\theta(x)).$$

On each side θ is uniquely defined up to the addition of a multiple of 2π. We choose the multiples of 2π so that

$$|\theta(a_k) - \theta_0(a_k)|, |\theta(b_k) - \theta_0(b_k)| < \pi/2$$

which we can do by choosing δ sufficiently small.

With this choice we define $\tilde{q}$ by (2.61) inside the intervals (a_k, b_k). Then $|\tilde{q}| \geq \frac{1}{4}$ and $\tilde{q} \in X^{s+2}$. Since the parameter τ depends only on $E^0(q_0)$, the estimate (2.58) follows as in (2.62). By the construction, $\tilde{q}' \in L^1$ if $q' \in L^1$. □

3. The transmission coefficient

In this section we will introduce the (renormalized) transmission coefficient associated to the Gross-Pitaevskii equation, give and analyze its asymptotic expansion, and finally prove Theorem 1.6.

We will first explain the Lax-pair formulation (L, P) of the Gross-Pitaevskii equation in Subsection 3.1. In the classical framework $q \in 1 + \mathcal{S}$, we will introduce the transmission coefficient $T^{-1}(\lambda)$ associated to the Lax operator

$$L = \begin{pmatrix} i\partial_x & -iq \\ i\overline{q} & -i\partial_x \end{pmatrix},$$

where the spectral parameters $(\lambda, z) \in \mathcal{R}$ will be defined in (3.5) below.

Recall the regularization r of q in Lemma 2.2, such that

$$\| |r|^2 - 1 \|_{L^2}, \, \|\partial_x r\|_{L^2}, \, \|r\|_{L^\infty}$$

can be bounded in terms of the (rescaled) energy norm $E^0(q)$. Using this regularization r we will in Subsection 3.2 approximately diagonalize the spectral equation of the Lax operator $Lu = \lambda u$ as

$$\tilde{L}v = z\, v, \text{ with } \tilde{L} = \begin{pmatrix} i\partial_x + iq_1 & -iq_2 \\ iq_3 & -i\partial_x + i(q_4 - q_1) \end{pmatrix}$$

where the elements q_j, $j = 1, 2, 3, 4$ are formulated in terms of q, r (and λ) explicitly, such that

$$\|q_j\|_{L^2}, \quad j = 1, 2, 3, 4$$

can also be bounded in terms of the (rescaled) energy norm $E^0(q)$.

We will then in Subsection 3.3 solve the renormalized Lax equation

$$\tilde{\tilde{L}}w = \begin{pmatrix} 0 & 0 \\ 0 & 2z \end{pmatrix} w, \text{ with } \tilde{\tilde{L}} = \begin{pmatrix} i\partial_x & -iq_2 \\ iq_3 & -i\partial_x + iq_4 \end{pmatrix}$$

$$\text{provided with the initial condition } \lim_{x \to -\infty} w(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

by Picard iteration. Observe in the classical setting $q \in 1 + \mathcal{S}$ that the limit of the first component of the solution w^1 at infinity is related to the transmission coefficient $T^{-1}(\lambda)$ as $\lim_{x\to\infty} w^1(x) = e^{\int_{\mathbb{R}} q_1 \,\mathrm{d}x} T^{-1}(\lambda)$, such that we have the asymptotic expansion of the transmission coefficient (up to the correction in terms of q_1) as follows

$$e^{\int_{\mathbb{R}} q_1 \,\mathrm{d}x} T^{-1}(\lambda) = \sum_{n=0}^{\infty} T_{2n},$$

where T_{2n} are $2n$ dimensional integrals essentially from the Picard iteration for fixed λ. They are holomorphic functions in λ for $\lambda \in \mathbb{C} \setminus \big((-\infty, -1] \cup [1, \infty)\big)$ and they depend on the choice of r which we have fixed. We will also estimate the terms T_{2n}, $n \geq 2$ and their sum in Subsection 3.3, which yields an estimate for the following difference for $q \in X^0$:

$$\left| \ln\Big(\sum_{n=0}^{\infty} T_{2n}\Big) - T_2 \right| = \left| \ln T^{-1}(\lambda) + \int_{\mathbb{R}} q_1 \,\mathrm{d}x \, - T_2 \right|.$$

We will simplify T_2 up to tolerable cubic error terms (denoted by $O_\tau(\mathcal{E}^3)$) in Subsection 3.4, which is the main result in this section. Correspondingly we will rewrite the corrected term $T_2 + \Phi$ in Subsection 3.5 as

$$T_2 + \Phi = A + B + O_\tau(\mathcal{E}^3),$$

where the correction function Φ is given in (1.44) (with $\mathcal{M}, \Theta$ denoting the mass and the asymptotic phase change respectively):

$$\Phi = -\int_{\mathbb{R}} q_1 \,\mathrm{d}x \, - \frac{i}{2z}\mathcal{M} - \frac{i}{2z(\lambda + z)}\Theta. \tag{3.1}$$

Both the terms A, B are well-defined for $q \in X^0$, and we will precise them. We recall that Θ depends on the choice of $\tilde{q}$ and can be considered as analytic function in q for $q \in X^0$ with $\partial_x q \in L^1$ modulo $2\pi\mathbb{Z}$, or as an analytic function on the universal covering space.

To conclude, for $q \in X^0$, we will in Subsection 3.6 define our renormalized transmission coefficient

$$\begin{aligned} T_{\mathrm{c}}^{-1}(\lambda) &= e^{\Phi(\lambda)}\Big(\sum_{n=0}^{\infty} T_{2n}(\lambda)\Big) \\ \text{that is, } \ln T_{\mathrm{c}}^{-1}(\lambda) &= \big(\Phi + T_2\big) + \Big(\ln\big(\sum_{n=0}^{\infty} T_{2n}\big) - T_2\Big). \end{aligned} \tag{3.2}$$

It is indeed $T_c^{-1} := T^{-1}(\lambda)\exp\left(-\frac{i\mathcal{M}}{2z} - \frac{i\Theta}{2z(\lambda+z)}\right)$ if $q \in 1+\mathcal{S}$. We will then complete the proof of Theorem 1.6. Recall that we always take $(\lambda, z) \in \mathcal{R}$ (see (3.5) below, and in particular $z = \sqrt{\lambda^2 - 1}$ has positive imaginary part). We restrict ourselves to the case $\operatorname{Im}\lambda \geq 0$ and

$$\tau := 2\operatorname{Im} z \geq 2$$

from Subsection 3.3 to Subsection 3.5, to simplify the presentation. We will consider other cases for $\lambda \in \mathbb{C} \setminus (-\infty, -1] \cup [1, \infty)$ and make the conclusions in Subsection 3.6.

3.1. **The Lax-Pair and the transmission coefficient.** The Gross-Pitaevskii equation (1.1) is completely integrable by means of the inverse scattering method. It can be viewed as the compatibility condition for the following two ODE systems (see Zakharov-Shabat [17])

$$\begin{aligned} u_x &= \begin{pmatrix} -i\lambda & q \\ \bar{q} & i\lambda \end{pmatrix} u, \\ u_t &= i\begin{pmatrix} -2\lambda^2 - (|q|^2 - 1) & -2i\lambda q + \partial_x q \\ -2i\lambda\bar{q} - \partial_x\bar{q} & 2\lambda^2 + (|q|^2 - 1) \end{pmatrix} u, \end{aligned} \tag{3.3}$$

where $u = u(t, x) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{C}^2$ is the unknown vector-valued solutions. Equivalently, the Gross-Pitaevskii equation can formally be reformulated in the Lax-pair (L, P) form

$$L_t = PL - LP,$$

where L is the self-adjoint Lax operator

$$L = \begin{pmatrix} i\partial_x & -iq \\ i\bar{q} & -i\partial_x \end{pmatrix}, \tag{3.4}$$

and P is the following skewadjoint differential operator

$$P = i\begin{pmatrix} 2\partial_x^2 - (|q|^2 - 1) & -q\partial_x - \partial_x q \\ \bar{q}\partial_x + \partial_x\bar{q} & -2\partial_x^2 + (|q|^2 - 1) \end{pmatrix}.$$

In the above, the first system in (3.3) reads as the spectral problem $Lu = \lambda u$ and the righthand side of the second system of (3.3) reads as Pu. The operators $L(t)$ and $L(t')$ at different times are related by the unitary family $U(t', t)$ generated by the skewadjoint operator P as

$$L(t) = U^*(t', t)L(t')U(t', t),$$

and the spectra of the Lax operator L is formally invariant by time evolution. This inverse scattering transform relates the evolution of the Gross-Pitaevskii flow to the spectral property of the Lax operator L. See [1, 4, 5, 6, 7, 10, 17] for the study between the potential q and the spectral information of L.

If $q - 1$ is Schwartz function, then by [6, 7], the self-adjoint operator L has essential spectrum $\mathcal{I} = (-\infty, -1] \cup [1, \infty)$ and finitely many *simple real* eigenvalues $\{\lambda_m\}$ in $(-1, 1)$. We are going to define its transmission coefficient

$$T^{-1}(\lambda) \text{ such that } |T^{-1}(\lambda)|\big|_{\lambda\in\mathcal{I}} \geq 1, \quad T^{-1}(\lambda_m) = 0,$$

by solving the spectral problem of the Lax operator $Lu = \lambda u$, i.e. the ordinary differential equation $u_x = \begin{pmatrix} -i\lambda & q \\ \bar{q} & i\lambda \end{pmatrix} u$ in (3.3).

We consider the spectral problem on the Riemann surface $\{(\lambda,z)\,|\,z^2=\lambda^2-1\}$. More precisely, we first notice that if $q=1$ then the matrix $\begin{pmatrix}-i\lambda & 1\\ 1 & i\lambda\end{pmatrix}$ has eigenvalues

$$iz \text{ and } -iz, \quad \text{with } z=z(\lambda)=\sqrt{\lambda^2-1},$$

together with the corresponding eigenvectors

$$\begin{pmatrix}1\\ i(\lambda+z)\end{pmatrix} \text{ and } \begin{pmatrix}1\\ i(\lambda-z)\end{pmatrix}.$$

In the following of this paper we will always take (λ,z) on the upper sheet of the Riemann surface

$$\begin{aligned}&\mathcal{R}=\big\{(\lambda,z)\,|\,\lambda\in\mathcal{V},\quad z=z(\lambda)=\sqrt{\lambda^2-1}\in\mathcal{U}\big\},\\ &\text{where } \mathcal{V}:=\mathbb{C}\setminus\mathcal{I},\quad \mathcal{I}=(-\infty,-1]\cup[1,\infty),\\ &\text{and } \mathcal{U}:=\{z\in\mathbb{C}\,|\,\operatorname{Im}z>0\} \text{ is the open upper complex plane.}\end{aligned} \tag{3.5}$$

The conformal mapping

$$\zeta=\zeta(\lambda)=\lambda+z \tag{3.6}$$

maps from $(\lambda,z)\in\mathcal{R}$ to $\zeta\in\mathcal{U}$ and has an inverse mapping $\zeta\mapsto\lambda=\lambda(\zeta)=\frac{1}{2}(\zeta+\frac{1}{\zeta})$.

In the classical framework $q-1\in\mathcal{S}$, as $q\to1$ at infinity, we solve indeed the boundary value problem of the above ODE with respect to the space variable $x\in\mathbb{R}$ in (3.3):

$$u_x=\begin{pmatrix}-i\lambda & q\\ \bar q & i\lambda\end{pmatrix}u,\quad u=e^{-izx}\begin{pmatrix}1\\ i(\lambda-z)\end{pmatrix}+o(1)e^{(\operatorname{Im}z)x} \text{ as } x\to-\infty. \tag{3.7}$$

We define the transmission coefficient $T^{-1}(\lambda)$ on $\mathcal{R}$ by the asymptotic behavior of the (Jost) solution u of (3.7) at infinity

$$u=e^{-izx}T^{-1}(\lambda)\begin{pmatrix}1\\ i(\lambda-z)\end{pmatrix}+o(1)e^{(\operatorname{Im}z)x} \text{ as } x\to+\infty. \tag{3.8}$$

Then $T^{-1}(\lambda)$ is a holomorphic function on $\mathcal{R}$, $\lim_{|\lambda|\to\infty}T^{-1}(\lambda)=1$ and

$$\overline{T}^{-1}(\lambda)=T^{-1}(\overline{\lambda}),\quad \text{for } (\lambda,z),\,(\overline{\lambda},-\overline{z})\in\mathcal{R}. \tag{3.9}$$

We multiply the above solution u by $e^{-2iz\lambda t}$ such that it solves also the time evolutionary equation in (3.3) and $\partial_t T^{-1}(\lambda)=0$. Hence the transmission coefficient $T^{-1}(\lambda)$ is conserved by the Gross-Pitaevskii flow.

The logarithm of the transmission coefficient has the asymptotic expansion (1.24) if $q\in 1+\mathcal{S}$.

3.2. Approximate diagonalization of the Lax equation: Estimates for q_j.

Let $(\lambda,z)\in\mathcal{R}$ (defined in (3.5)), such that

$$\operatorname{Im}\lambda\ge0,\quad \tau:=2\operatorname{Im}z\ge2. \tag{3.10}$$

In this situation, $\zeta=\lambda+z\in\mathcal{U}$ satisfies $\operatorname{Im}\zeta=\operatorname{Im}\lambda+\operatorname{Im}z\in[\operatorname{Im}z,2\operatorname{Im}z)\subset\mathbb{R}^+$.

Let $q\in X^0$, and let $r\in X^2$ be given in Lemma 2.2. We will diagonalize the Lax equation (3.7) for the Jost solution u into the ordinary differential equations (3.12) for v below, where the elements q_j, $j=1,2,3,4$ will be estimated in Lemma 3.1 afterwards.

A straightforward calculation shows that

$$\begin{aligned}
&\frac{1}{|r|^2-\zeta^2}\Big[\begin{pmatrix}-i\zeta & r\\ \bar r & i\zeta\end{pmatrix}\begin{pmatrix}-i\lambda & q\\ \bar q & i\lambda\end{pmatrix}-\begin{pmatrix}|r|^2-1 & 0\\ 0 & |r|^2-1\end{pmatrix}\Big]\begin{pmatrix}-i\zeta & r\\ \bar r & i\zeta\end{pmatrix}\\
&\quad=\begin{pmatrix}-iz & 0\\ 0 & iz\end{pmatrix}+\frac{1}{|r|^2-\zeta^2}\begin{pmatrix}-i\zeta & r\\ \bar r & i\zeta\end{pmatrix}\begin{pmatrix}0 & q-r\\ \bar q-\bar r & 0\end{pmatrix}\begin{pmatrix}-i\zeta & r\\ \bar r & i\zeta\end{pmatrix}\\
&\quad=\begin{pmatrix}-iz & 0\\ 0 & iz\end{pmatrix}+\frac{1}{|r|^2-\zeta^2}\begin{pmatrix}-i\zeta\big[\bar r(q-r)+r(\bar q-\bar r)\big] & r^2(\bar q-\bar r)+\zeta^2(q-r)\\ \zeta^2(\bar q-\bar r)+\bar r^2(q-r) & i\zeta\big[r(\bar q-\bar r)+\bar r(q-r)\big]\end{pmatrix}.
\end{aligned}$$

If u satisfies the Lax equation (the first equation in (3.3)) then we take

$$v=\begin{pmatrix}-i\zeta & r\\ \bar r & i\zeta\end{pmatrix}u,\ \text{ or equivalently, } u=\frac{1}{|r|^2-\zeta^2}\begin{pmatrix}-i\zeta & r\\ \bar r & i\zeta\end{pmatrix}v, \tag{3.11}$$

such that v solves

$$\begin{aligned}
v_x=&\frac{1}{|r|^2-\zeta^2}\Big[\begin{pmatrix}-i\zeta & r\\ \bar r & i\zeta\end{pmatrix}\begin{pmatrix}-i\lambda & q\\ \bar q & i\lambda\end{pmatrix}\begin{pmatrix}-i\zeta & r\\ \bar r & i\zeta\end{pmatrix}+\begin{pmatrix}0 & r'\\ \bar r' & 0\end{pmatrix}\begin{pmatrix}-i\zeta & r\\ \bar r & i\zeta\end{pmatrix}\Big]v\\
=&\begin{pmatrix}-iz & 0\\ 0 & iz\end{pmatrix}v+\begin{pmatrix}-q_1 & q_2\\ q_3 & q_4-q_1\end{pmatrix}v,
\end{aligned}\tag{3.12}$$

where q_j, $j=1,2,3,4$ are given by

$$\begin{aligned}
q_1&=\frac{i\zeta\big[|r|^2-1+2\mathrm{Re}\,(\bar r(q-r))\big]-\bar r r'}{|r|^2-\zeta^2},\\
q_2&=\frac{r(|r|^2-1)+i\zeta r'+r^2(\bar q-\bar r)+\zeta^2(q-r)}{|r|^2-\zeta^2}\\
&=\frac{r\big[|r|^2-1+2\mathrm{Re}\,(\bar r(q-r))\big]+i\zeta r'}{|r|^2-\zeta^2}-(q-r),\\
q_3&=\frac{\bar r(|r|^2-1)-i\zeta\bar r'+\bar r^2(q-r)+\zeta^2(\bar q-\bar r)}{|r|^2-\zeta^2}\\
&=\frac{\bar r\big[|r|^2-1+2\mathrm{Re}\,(\bar r(q-r))\big]-i\zeta\bar r'}{|r|^2-\zeta^2}-(\overline{q-r})\\
q_4&=\frac{2i\zeta\big[|r|^2-1+2\mathrm{Re}\,(\bar r(q-r))\big]+2i\mathrm{Im}\,(r\bar r')}{|r|^2-\zeta^2}.
\end{aligned}\tag{3.13}$$

Lemma 3.1. *The map $(2,\infty)\ni\tau\to E^s_\tau(q)$, $s\in[0,1]$ is monotonically decreasing.*

Let $(\lambda,z)\in\mathcal{R}$ with $\mathrm{Im}\,\lambda\ge 0$ and $\mathrm{Im}\,z=\frac{\tau}{2}$, $2\le\tau_0\le\tau$. Let $q\in X^0$ with $E^0_\tau(q)\le\tau^{1/2}$. Let r and q_j's be given in Lemma 2.2 and (3.13) respectively. Then we have the following estimates for q_j, $j=1,2,3,4$:

$$\|q_j\|_{L^p}\lesssim\frac{1}{\tau}\|(|r|^2-1,r')\|_{L^p}+\|q-r\|_{L^p},\quad\forall p\in[1,\infty],\tag{3.14}$$

and

$$\|q_j\|_{L^p}\le C_p\|(|q|^2-1,q')\|_{W^{-1,p}_\tau}\le C_p\tau^{-1}E_{\tau_0}^{1+\frac12-\frac1p}(q),\quad\forall p\in[2,\infty),\tag{3.15}$$

where in particular

$$\|q_j\|_{L^2}\le cE^0_\tau(q)\le cE^0_{\tau_0}(q).\tag{3.16}$$

Proof. Monotonicity of $\tau \to \|f\|_{H^{s-1}_\tau}$, $s \in [0,1]$ is obvious after a Fourier transform.

We recall that $|r| \lesssim \tau$ if $E^0_\tau(q) \le \tau^{1/2}$ by Lemma 2.2 and observe that $\operatorname{Im}\zeta \in [\frac{\tau}{2}, \tau)$. Thus

$$||r|^2 - \zeta^2| = ||r|^2 - (\operatorname{Re}\zeta)^2 + (\operatorname{Im}\zeta)^2 - 2i\operatorname{Im}\zeta\operatorname{Re}\zeta| \gtrsim |\zeta|^2 + |r|^2. \tag{3.17}$$

The inequality (3.14) is an immediate consequence of the structure of the q_j.

We use the estimates (2.46) in Lemma 2.2 to derive from (3.14) that

$$\|q_j\|_{L^p} \le C_p(1 + \tau^{-1/2}E^0_\tau(q))\|(|q|^2 - 1, q')\|_{W^{-1,p}_\tau}, \quad \forall p \in [2,\infty),$$

and hence (3.16) holds. We use further the estimate

$$\|f\|_{W^{s,p}_\tau} \lesssim \tau^{s-\sigma}\|f\|_{W^{\sigma,p}_\tau} \tag{3.18}$$

whenever $s, \sigma \in \mathbb{R}$, $s \le \sigma$ and the Sobolev inequality $\|f\|_{L^p} \le C_p\|f\|_{H^{\frac12 - \frac1p}}$, $p \in [2,\infty)$, to derive (3.15) by the following inequality and the trivial bound $E^{1+\frac12-\frac1p}(q) \le E^{1+\frac12-\frac1p}_{\tau_0}(q)$, $\tau_0 \ge 2$,

$$\|q_j\|_{L^p} \le C_p\|(|q|^2 - 1, q')\|_{W^{-1,p}_\tau} \le C_p\tau^{-1}\|(|q|^2-1, q')\|_{L^p} \le C_p\tau^{-1}E^{1+\frac12-\frac1p}(q).$$

The proof of Lemma 3.1 is complete. □

3.3. **The renormalized Lax equation: Estimates for T_{2n}, $n \ge 2$.** Recall the (approximately) diagonalized Lax equation (3.12)-(3.13) for the unknown vector-valued function v, which is related to the original Jost solution u by the transformation (3.11). The Jost solution u satisfies the original Lax equation (3.7) and has the asymptotic behaviour (3.8).

Let

$$w = -\frac{1}{2iz}e^{izx + \int_0^x q_1\,\mathrm{d}m} v = -\frac{1}{2iz}e^{izx+\int_0^x q_1\,\mathrm{d}m}\begin{pmatrix} -i\zeta & r \\ \bar r & i\zeta \end{pmatrix} u, \tag{3.19}$$

then it satisfies the renormalized ODE (of the original ODE (3.7) for u)

$$w_x = \begin{pmatrix} 0 & 0 \\ 0 & 2iz \end{pmatrix} w + \begin{pmatrix} 0 & q_2 \\ q_3 & q_4 \end{pmatrix} w, \quad \lim_{x\to-\infty} w(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.20}$$

We call w the renormalized Jost solution. It satisfies the following integral equation

$$\begin{aligned} w(x) = {} & \lim_{y\to-\infty} w(y) \\ & + \int_{-\infty}^x \begin{pmatrix} 0 & q_2(x_1) \\ e^{2iz(x-x_1)+\int_{x_1}^x q_4\,\mathrm{d}m} q_3(x_1) & 0 \end{pmatrix} w(x_1)\,dx_1, \end{aligned}$$

with the following asymptotics as $x \to \pm\infty$ (recalling u's asymptotics (3.8)):

$$\begin{aligned} w(\lambda, x) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + o(1) \text{ as } x \to -\infty, \\ w(\lambda, x) &= \begin{pmatrix} e^{\int_{-\infty}^\infty q_1\,\mathrm{d}m} T^{-1}(\lambda) \\ 0 \end{pmatrix} + o(1) \text{ as } x \to +\infty. \end{aligned} \tag{3.21}$$

Hence we use the following Picard type iterative procedure to derive the first component of the vector-valued Jost solution w, taking two steps at one time

$$\begin{aligned} w^1 &= \sum_{n=0}^{\infty} w_n, \quad w_0 = 1, \\ w_n(x) &= \int_{-\infty}^{x} q_2(y_1) \int_{-\infty}^{y_1} e^{2iz(y_1-y_2)+\int_{y_2}^{y_1} q_4 dm} q_3(y_2) w_{n-1}(y_2) dy_2 dy_1. \end{aligned} \tag{3.22}$$

Lemma 3.2. *Let w_n be defined in (3.22) iteratively. Suppose that*

$$\|q_4\|_{L^2}^2 \leq 4C \operatorname{Im} z, \tag{3.23}$$

for some positive constant $C > 0$. Then the following estimate holds

$$\|w_n\|_{C_b} + \|\partial_x w_n\|_{L^1} \leq e^C (\operatorname{Im} z)^{-1} \|q_2\|_{L^2} \|q_3\|_{L^2} \|w_{n-1}\|_{L^\infty}.$$

Proof. Let $\tau = 2\operatorname{Im} z$. We observe for $y_1 < y_2$,

$$\begin{aligned} -2\operatorname{Im} z(y_1-y_2) + \int_{y_1}^{y_2} \operatorname{Re} q_4 dm &\leq -\tau(y_1-y_2) + |y_1-y_2|^{\frac{1}{2}} \|q_4\|_{L^2} \\ &\leq -\frac{1}{2}\tau(y_1-y_2) + \frac{1}{2}\tau^{-1}\|q_4\|_{L^2}^2 \\ &\leq -\operatorname{Im} z(y_1-y_2) + C. \end{aligned}$$

Hence, by Young's inequality for convolutions

$$\begin{aligned} \|w_n\|_{L^\infty} &\leq \int_{y_2<y_1} e^{C-\operatorname{Im} z(y_1-y_2)} |q_2(y_1)||q_3(y_2)| dy_2 dy_1 \|w_{n-1}\|_{L^\infty} \\ &\leq e^C (\operatorname{Im} z)^{-1} \|q_2\|_{L^2} \|q_3\|_{L^2} \|w_{n-1}\|_{L^\infty}. \end{aligned}$$

Finally, since

$$|\partial_x w_n(x)| \leq |q_2(x)| \cdot \int_{-\infty}^{x} e^{C-\operatorname{Im} z(x-y)} |q_3(y)| \, \mathrm{d}y \cdot \|w_{n-1}\|_{L^\infty},$$

we have

$$\begin{aligned} \|\partial_x w_n\|_{L^1} &\leq \int_{y<x} |q_2(x)| e^{C-\operatorname{Im} z(x-y)} |q_3(y)| \, \mathrm{d}x \, \mathrm{d}y \, \|w_{n-1}\|_{L^\infty} \\ &\leq e^C (\operatorname{Im} z)^{-1} \|q_2\|_{L^2} \|q_3\|_{L^2} \|w_{n-1}\|_{L^\infty}. \end{aligned}$$

□

If (3.23) holds, then we define the limit of w_n at infinity as

$$\begin{aligned} T_{2n}(\lambda) &:= \lim_{x\to\infty} w_n(x) \\ &= \int_{x_1<y_1<\cdots<x_n<y_n} \prod_{j=1}^{n} e^{\left(2iz(y_j-x_j)+\int_{x_j}^{y_j} q_4(m)dm\right)} q_3(x_j) q_2(y_j) dx_j dy_j, \quad n \geq 0, \end{aligned} \tag{3.24}$$

(with $T_0 = 1$), such that by Lemma 3.2 it satisfies

$$|T_{2n}(\lambda)| \leq \left(e^C (\operatorname{Im} z)^{-1} \|q_2\|_{L^2} \|q_3\|_{L^2}\right)^n. \tag{3.25}$$

If $q \in 1+\mathcal{S}$, recalling the limit of the first component w^1 at infinity in (3.21): $\lim_{x\to\infty} w^1(x) = e^{\int_{\mathbb{R}} q_1\,\mathrm{d}x} T^{-1}(\lambda)$, it holds

$$\sum_{n=0}^{\infty} T_{2n} = e^{\int_{\mathbb{R}} q_1\,\mathrm{d}x} T^{-1}(\lambda). \tag{3.26}$$

Lemma 3.3 (Estimates for T_{2n}, $n \ge 2$)**.** *Let $(\lambda, z) \in \mathcal{R}$ with $\operatorname{Im}\lambda \ge 0$ and $\operatorname{Im} z = \frac{\tau}{2}$, $\tau \ge 2$, and $q \in X^0$ with $E^0_\tau(q) \le \tau^{1/2}$. Let r, q_j's, T_{2n}'s be given in Lemma 2.2, (3.13) and (3.24) respectively.*

There exist c, δ (independent of λ, z, q) such that

$$|T_{2n}(\lambda)| \le c^n \tau^{-n} (E^0_\tau(q))^{2n}, \quad n \ge 1, \tag{3.27}$$

and hence for $q \in X^s$, $s \in [0,2]$, and some fixed $\tau_0 \ge 2$,

$$|T_{2n}(\lambda)| \lesssim c^n \tau^{-n-2s} (E^0_{\tau_0}(q))^{2n-2} (E^s_{\tau_0}(q))^2, \quad n \ge 2, \quad \forall \tau \ge \tau_0,$$

which implies for $q \in X^s$, $s \in [0,2]$ with smallness condition $E^0_{\tau_0}(q) \le \delta {\tau_0}^{1/2}$ for some $\tau_0 \ge 2$,

$$\sum_{n=2}^{\infty} |T_{2n}(\lambda)| \le c\tau^{-2-2s} (E^0_{\tau_0}(q))^2 (E^s_{\tau_0}(q))^2, \quad \forall \tau \ge \tau_0.$$

Proof. We derive (3.27) straightforward from $E^0_\tau(q) \le \tau^{\frac{1}{2}}$, (3.16) and (3.25):

$$|T_{2n}(\lambda)| \le \left(\frac{c}{\tau} \|q_2\|_{L^2} \|q_3\|_{L^2}\right)^n \le \left(\frac{c}{\tau}\right)^n (E^0_\tau(q))^{2n}, \quad \forall n \ge 1.$$

If $s \in [0,1]$, then by virtue of the trivial estimates $E^0_\tau(q) \le \tau^{-s} E^s_\tau(q)$ which follows from $\|f\|_{H^{-1}_\tau} \le \tau^{-s} \|f\|_{H^{-1+s}_\tau}$ if $s \ge 0$ and $E^s_\tau(q) \le E^s_{\tau_0}(q)$ if $0 \le s \le 1$ and $\tau \ge \tau_0$, we derive

$$|T_{2n}(\lambda)| \le c^n \tau^{-n-2s} (E^0_\tau(q))^{2n-2} (E^s_\tau(q))^2 \le c^n \tau^{-n-2s} (E^0_{\tau_0}(q))^{2n-2} (E^s_{\tau_0}(q))^2.$$

If $s \in (1,2]$ and $n \ge 2$, then we take $\sigma = s/2 \in (0,1]$ such that

$$\begin{aligned} |T_{2n}(\lambda)| &\le c^n \tau^{-n-2s} (E^0_\tau(q))^{2n-4} (E^\sigma_\tau(q))^4 \\ &\le c^n \tau^{-n-2s} (E^0_{\tau_0}(q))^{2n-4} (E^\sigma_{\tau_0}(q))^4, \end{aligned}$$

which, together with interpolation

$$E^\sigma_{\tau_0}(q) \le (E^0_{\tau_0}(q))^{1-\sigma} (E^1_{\tau_0}(q))^\sigma \le (E^0_{\tau_0}(q) E^s_{\tau_0}(q))^{1/2},$$

implies

$$|T_{2n}(\lambda)| \le c^n \tau^{-n-2s} (E^0_{\tau_0}(q))^{2n-2} (E^s_{\tau_0}(q))^2, \quad \forall \tau \ge \tau_0.$$

The bound for

$$\sum_{n=2}^{\infty} |T_{2n}| \le \sum_{n=2}^{\infty} c^n (\tau^{-\frac{1}{2}} E^0_\tau(q))^{2n-4} (\tau^{-\frac{1}{2}} E^0_\tau(q))^4$$

follows by a geometric sum and the trivial bound

$$\tau^{-\frac{1}{2}} E^0_\tau(q) \le {\tau_0}^{-\frac{1}{2}} E^0_{\tau_0}(q), \quad \forall \tau \ge \tau_0.$$

The proof is complete. $\square$

Corollary 3.4. *Assume the same assumptions as in Lemma 3.3. There exists a small constant δ_1, such that for all $q \in X^s$, $s \in [0,2]$ with smallness condition $E^0_{\tau_0}(q) \leq \delta_1 {\tau_0}^{1/2}$ for some $\tau_0 \geq 2$, we have*

(3.28)
$$\left| \ln \Big(\sum_{n=0}^{\infty} T_{2n}(\lambda) \Big) - T_2(\lambda) \right| \leq c\tau^{-2}(E^0_\tau(q))^4 \leq c\tau^{-2-2s}(E^0_{\tau_0}(q))^2(E^s_{\tau_0}(q))^2, \quad \forall \tau \geq \tau_0.$$

Proof. By the Lipschitz continuity of the logarithm on $\{\zeta \in \mathbb{C} : |\zeta - 1| \leq \frac{1}{2}\}$ and the triangle inequality

$$\begin{aligned} \left| \ln \Big(\sum_{n=0}^{\infty} T_{2n}(\lambda) \Big) - T_2(\lambda) \right| &\lesssim \left| \Big(\sum_{n=0}^{\infty} T_{2n}(\lambda) \Big) - (1 + T_2) \right| \\ &\quad + \left| \Big(\sum_{n=0}^{\infty} T_{2n}(\lambda) \Big) \right| \Big| \exp(T_2) - (1+T_2) \Big| \\ &\lesssim \sum_{n=2}^{\infty} |T_{2n}(\lambda)| + |T_2(\lambda)|^2. \end{aligned}$$

Under the smallness condition $E^0_{\tau_0}(q) \leq \delta_1 \tau_0^{1/2}$ and $s \in [0,2]$, the estimate (3.28) follows from Lemma 3.3:

$$\Big| \sum_{n \geq 2} T_{2n} \Big| \leq c\tau^{-2-2s}(E^0_{\tau_0}(q))^2(E^s_{\tau_0}(q))^2$$

and

$$|T_2(\lambda)|^2 \leq c\tau^{-2}(E^0_\tau(q))^4 \leq c\tau^{-2-2s}(E^0_{\tau_0}(q))^2(E^s_{\tau_0}(q))^2.$$

□

3.4. **The term T_2.** Let $(\lambda, z) \in \mathcal{R}$ with $\operatorname{Im} \lambda \geq 0$ and $\operatorname{Im} z = \frac{\tau}{2}$, $\tau \geq 2$. Let $q \in X^0$, and let r be given in Lemma 2.2. We recall that in (3.24) we have defined

$$T_2 = \int_{x<y} \exp\Big(-2iz(x-y) + \int_x^y q_4(m)dm \Big) q_3(x) q_2(y) dxdy,$$

where

$$\begin{aligned} q_2 &= \frac{r\big[|r|^2 - 1 + 2\mathrm{Re}\,(\bar{r}(q-r))\big] + i\zeta r'}{|r|^2 - \zeta^2} - (q-r), \\ q_3 &= \frac{\bar{r}\big[|r|^2 - 1 + 2\mathrm{Re}\,(\bar{r}(q-r))\big] - i\zeta \bar{r}'}{|r|^2 - \zeta^2} - (\overline{q-r}), \\ q_4 &= \frac{2i\zeta\big[|r|^2 - 1 + 2\mathrm{Re}\,(\bar{r}(q-r))\big] + 2i\mathrm{Im}\,(r\bar{r}')}{|r|^2 - \zeta^2}. \end{aligned}$$

Our task in this subsection is to extract the leading term in T_2 with respect to large τ. We begin with bounds for multi-linear terms.

Lemma 3.5. *Let $a > 0$. Suppose that for $y > x$*

$$\operatorname{Re} \phi(x,y) \geq a(y-x)$$

then for $1 \leq p_1, p_2 \leq \infty$ with $\frac{1}{p_1} + \frac{1}{p_2} \geq 1$,

$$\left| \int_{x<y} e^{-\phi(x,y)} f(x) g(y) dxdy \right| \leq a^{-2+\frac{1}{p_1}+\frac{1}{p_1}} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}. \tag{3.29}$$

Similarly, if for $x_1 \leq x_2 \leq x_3$

$$\operatorname{Re}\phi(x_1,x_2,x_3) \geq a(x_3-x_1),$$

then, for $1 \leq p_1, p_2, p_3 \leq \infty$ with $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3} \geq 1$

$$\begin{aligned}&\Big|\int_{x_1<x_2<x_3} e^{-\phi(x_1,x_2,x_3)} f(x_1)g(x_2)h(x_3)dx_1dx_2dx_3\Big| \\ &\leq a^{-3+\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}}\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}\|h\|_{L^{p_3}}.\end{aligned} \tag{3.30}$$

Proof. We estimate by Young's inequality for convolutions, with $\frac{1}{r}+\frac{1}{p_1}+\frac{1}{p_2}=2$

$$\begin{aligned}\left|\int_{x<y} e^{-\phi(x,y)}f(x)g(y)dxdy\right| &\leq \int_{\mathbb{R}}\big((e^{ax}\chi_{x<0})*|f|\big)(y)|g(y)|\,\mathrm{d}y \\ &\leq \|f\|_{L^{p_1}}\|g\|_{L^{p_2}}\|e^{ax}\chi_{x<0}\|_{L^r},\end{aligned}$$

which implies (3.29). We apply a dual of this estimate with respect to x_3

$$\begin{aligned}&\Big|\int_{x_1<x_2<x_3} e^{-\phi(x_1,x_2,x_3)} f(x_1)g(x_2)h(x_3)dx_1dx_2dx_3\Big| \\ &\quad\leq \int_{x_1<x_2<x_3} e^{-a(x_3-x_1)}|f(x_1)||g(x_2)|\,|h(x_3)|dx_1dx_2dx_3 \\ &\quad= \int_{x_1<x_2} e^{-a(x_3-x_2)}|f(x_1)||g(x_2)|\,\big(e^{ax}\chi_{x<0}*|h|\big)(x_2)dx_1dx_2 \\ &\quad\leq a^{-2+\frac{1}{p_1}+\frac{1}{r}}\|f\|_{L^{p_1}}\||g|\,(e^{ax}\chi_{x<0}*|h|)\|_{L^r} \\ &\quad\leq a^{-2+\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}}\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}\|e^{ax}\chi_{x<0}*h\|_{L^{p_3}}\end{aligned}$$

where we chose $r \geq 1$ so that $\frac{1}{r}=\frac{1}{p_3}+\frac{1}{p_2}$. Then (3.30) follows since

$$\|e^{ax}\chi_{x<0}*h\|_{L^{p_3}} \leq a^{-1}\|h\|_{L^{p_3}}.$$

□

To simplify the presentation we formalize the notion of tolerable cubic errors.

Definition 3.6 (Tolerable cubic error)**.** *Let $\tau \geq 2$. We call a term F a tolerable cubic error if there exists a positive constant c such that for all $p \in [2,3]$*

$$|F| \leq c\tau^{-3+\frac{3}{p}}\|(|q|^2-1,\partial_x q)\|^3_{W^{-1,p}_\tau}.$$

We write

$$F = O_\tau(\mathcal{E}^3).$$

In this subsection we simplify T_2 up to tolerable cubic errors. Their relevance is described in the following lemma.

Lemma 3.7 (Estimates for tolerable cubic errors)**.** *Suppose $2 \leq \tau_0 \leq \tau$ and $F = O_\tau(\mathcal{E}^3)$. Then for $0 \leq s \leq \frac{7}{4}$*

$$|F| \leq c\tau^{-\frac{3}{2}-2s}E^0_{\tau_0}(q)(E^s_{\tau_0}(q))^2.$$

Proof. The ideas of proof are exactly as in the proof of Lemma 3.3. If $0 \leq s \leq \frac{3}{2}$ we take $p=2$ to bound

$$|F| \leq c\tau^{-\frac{3}{2}}(E^0_\tau(q))^3.$$

If $s \in [0,1]$, then we are done by virtue of $E^0_\tau(q) \le \tau^{-s} E^s_\tau(q)$ and $E^s_\tau(q) \le E^s_{\tau_0}(q)$. If $1 < s \le \frac32$ we obtain by interpolation

$$\begin{aligned} |F| &\le c\tau^{-\frac32 - 2s}(E_\tau^{\frac23 s}(q))^3 \\ &\le c\tau^{-\frac32 - 2s}(E_{\tau_0}^{\frac23 s}(q))^3 \\ &\le c\tau^{-\frac32 - 2s} E^0_{\tau_0}(q)(E^s_{\tau_0}(q))^2. \end{aligned}$$

If $s = \frac74$ we take $p = 3$ to bound by Sobolev embedding $H^{\frac16}(\mathbb{R}) \hookrightarrow L^3(\mathbb{R})$ and interpolation

$$|F| \le c\tau^{-5}\|(|q|^2-1, q')\|^3_{L^3} \le c\tau^{-5}(E^{\frac76}(q))^3 \le c\tau^{-5}(E^{\frac76}_{\tau_0}(q))^3 \le c\tau^{-5}E^0_{\tau_0}(q)(E^{\frac74}_{\tau_0}(q))^2.$$

If $\frac32 < s < \frac74$ we proceed in the same fashion but with an exponent $p = (\frac32 - \frac{2s}{3})^{-1} \in (2,3)$:

$$\begin{aligned} |F| \le c\tau^{-3+\frac3p}\|(|q|^2-1, q')\|^3_{W_\tau^{-1,p}} &\le c\tau^{-6+\frac3p}(E^{\frac32 - \frac1p}(q))^3 \\ &\le c\tau^{-6+\frac3p}(E^{\frac32-\frac1p}_{\tau_0}(q))^3 \le c\tau^{-\frac32-2s}E^0_{\tau_0}(q)(E^s_{\tau_0}(q))^2. \end{aligned}$$

□

Remark 3.8. *We observe that $\frac32 + 2\frac74 = 5$, which gives a decay τ^{-5} and corresponds to the Hamiltonian H_3.*

The main result of this section is:

Proposition 3.9 (Leading term in T_2)**.** *Assume the same assumptions as in Lemma 3.3. Then*

$$\begin{aligned} T_2(\lambda) = \int_{x<y} & e^{2iz(y-x)}\Big\{ \frac{1}{4z^2}\big[\bar q'(x) q'(y) + (|q(x)|^2-1)(|q(y)|^2-1)\big] \\ &+ \frac{1}{4z^2\zeta}(|q|^2-1)(x)\mathrm{Im}\,[r\bar r'](y) + \mathrm{Im}\,[r\bar r'](x)(|q|^2-1)(y) \\ &+ \frac{i}{2z\zeta}\big[(|q|^2-1)(x)\mathrm{Im}\,[\bar r(q-r)](y) + \mathrm{Im}\,[r(\overline{q-r})](x)(|q|^2-1)(y)\big]\Big\} dxdy \\ &+ \frac{1}{4z^2}\int_{\mathbb{R}} (q\bar q' - r\bar r')\,\mathrm{d}x \\ &- \frac{1}{2iz}\int_{\mathbb{R}}|q-r|^2\,\mathrm{d}x \; - \frac{i}{4z^2\zeta}\int_{\mathbb{R}}(|q|^2-1)(|r|^2-1)\,\mathrm{d}x \; + O_\tau(\mathcal{E}^3). \end{aligned}$$

Proof. We are going to simplify T_2 up to tolerable cubic errors in four steps.
Step 1. We can remove the integral of q_4 from the exponential:

$$T_2 = \int_{x<y} e^{-2iz(x-y)} q_2(x) q_3(y)\,\mathrm{d}x\;\mathrm{d}y \; + O_\tau(\mathcal{E}^3). \tag{3.31}$$

Indeed, since by Lemma 3.1

$$\left|\int_x^y q_4 dm\right| \le |x-y|^{\frac12}\|q_4\|_{L^2} \le c\tau^{1/2}|x-y|^{1/2}\tau^{-1/2}E^0_\tau(q),$$

we have if $\tau^{-1/2}E^0_\tau(q) \le 1$

$$\left|1 - \exp\Big(\int_x^y q_4 dm\Big)\right| \le \int_x^y |q_4(m)| dm \exp(c\tau^{1/2}|x-y|^{1/2}).$$

Since

$$\exp(c\tau^{1/2}|x-y|^{1/2}) \lesssim \exp(\tau|x-y|/2),$$

we have

$$\begin{aligned}&\Big|\int_{x<y} e^{-\tau(y-x)}\Big|1-\exp\Big(\int_x^y q_4 dm\Big)\Big||q_2(x)||q_3(y)|dxdy\Big|\\&\qquad\lesssim \int_{x<m<y} e^{-\tau(y-x)/2}|q_2(x)||q_4(m)||q_3(y)|dxdmdy.\end{aligned}$$

The simplication (3.31) follows now from Lemma 3.1 and Lemma 3.5.
Step 2. We can replace the denominator $|r|^2-\zeta^2$ by $1-\zeta^2$:

$$\begin{aligned}T_2 = \int_{x<y} & e^{2iz(y-x)}\Big[\frac{\bar r(|r|^2-1+2\mathrm{Re}\,(\bar r(q-r)))-i\zeta\bar r'}{1-\zeta^2}-(\overline{q-r})\Big](x)\\&\times\Big[\frac{r(|r|^2-1+2\mathrm{Re}\,(\bar r(q-r)))+i\zeta r'}{1-\zeta^2}-(q-r)\Big](y)\,\mathrm{d}x\ \mathrm{d}y\\&+O_\tau(\mathcal{E}^3).\end{aligned}\tag{3.32}$$

Noticing $\frac{1}{|r|^2-\zeta^2}=\frac{1}{1-\zeta^2}-\frac{|r|^2-1}{(|r|^2-\zeta^2)(1-\zeta^2)}$, and by (3.17) $||r|^2-\zeta^2|\gtrsim|\zeta^2|+|r|^2$ and by Lemma 2.2 $\|r\|_{L^\infty}\lesssim\tau$, we apply the bilinear estimate in Lemma 3.5 followed by Hölder's inequality with $p_1=p_2=p_3=p\in[2,3]$

$$\begin{aligned}&\Big|\int_{x<y} e^{2iz(y-x)}q_3(x)\frac{(|r|^2-1)(r\big(|r|^2-1+2\mathrm{Re}\,[\bar r(q-r)]\big)+i\zeta r')}{(|r|^2-\zeta^2)(1-\zeta^2)}dxdy\Big|\\&\quad\lesssim\tau^{-3}\int_{x<y}e^{-\tau(y-x)}|q_3(x)|\Big[||r|^2-1|\cdot\big|\big(|r|^2-1,\tau(q-r),r'\big)\big|(y)\Big]\,\mathrm{d}x\ \mathrm{d}y\\&\quad\le\tau^{-3-2+\frac{3}{p}}\|q_3\|_{L^p}\||r|^2-1\|_{L^p}\|\big(|r|^2-1,\tau(q-r),r'\big)\|_{L^p},\end{aligned}$$

which is $O_\tau(\mathcal{E}^3)$ by Lemma 2.2 and Lemma 3.1. In the same fashion we replace the denominator $|r|^2-\zeta^2$ by $1-\zeta^2$ in q_3 and we obtain (3.32).
Step 3. We exchange $r(x)$ resp. $\bar r(x)$ and $r(y)$ resp. $\bar r(x)$ and replace $|r|^2$ by 1 with tolerable cubic errors:

$$\begin{aligned}T_2=\int_{x<y}&e^{2iz(y-x)}\Big\{\frac{\zeta^2}{(1-\zeta^2)^2}\bar r'(x)r'(y)+\frac{1}{(1-\zeta^2)^2}(|q(x)|^2-1)(|q(y)|^2-1)\\&+\frac{i\zeta}{(1-\zeta^2)^2}\Big[(|q|^2-1)(x)\bar r r'(y)-r\bar r'(x)(|q|^2-1)(y)\Big]\\&-\frac{1}{1-\zeta^2}\Big[(\bar r(|q|^2-1)-i\zeta\bar r')(x)(q-r)(y)+(\overline{q-r})(x)(r(|q|^2-1)+i\zeta r')(y)\Big]\\&+(\overline{q-r})(x)(q-r)(y)\Big\}dxdy+O_\tau(\mathcal{E}^3).\end{aligned}\tag{3.33}$$

We first notice that replacing $r(x)$ by $r(y)$ leads to a tolerable cubic error via (3.30)

$$\begin{aligned}&\Big|\int_{x<y}e^{2iz(y-x)}(r(x)-r(y))f_1(x)f_2(y)dxdy\Big|\\&\qquad=\Big|\int_{x<m<y}e^{2iz(y-x)}r'(y)f_1(x)f_2(y)dxdmdy\Big|\\&\qquad\le c\tau^{-3+\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}}\|r'\|_{L^{p_1}}\|f_1\|_{L^{p_2}}\|f_2\|_{L^{p_3}}\end{aligned}$$

for $1 \le p_1, p_2, p_3$ with $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \ge 1$. We thus derive from (3.32) in Step 2 that

$$\begin{aligned} T_2 = \int_{x<y} e^{2iz(y-x)} \Big[& \frac{1}{(1-\zeta^2)^2} f(x) \big[|r|^2 f\big](y) + \frac{\zeta^2}{(1-\zeta^2)^2} \bar{r}'(x) r'(y) \\ & + \frac{i\zeta}{(1-\zeta^2)^2} \Big(f(x) \big[\bar{r} r'\big](y) - \big[r \bar{r}'\big](x) f(y) \Big) \\ & - \frac{1}{1-\zeta^2} \Big((\bar{r} f)(x)(q-r)(y) + (\overline{q-r})(x)(rf)(y) \Big) \\ & + \frac{i\zeta}{1-\zeta^2} \Big(-(\overline{q-r})(x) r'(y) + \bar{r}'(x)(q-r)(y) \Big) \\ & + (\overline{q-r})(x)(q-r)(y) \Big] dxdy + O_\tau(\mathcal{E}^3), \end{aligned}$$

where we denote

$$f := |r|^2 - 1 + 2\mathrm{Re}\,[\bar{r}(q-r)] = |r|^2 - 1 + r(\overline{q-r}) + \bar{r}(q-r).$$

We can then harmlessly replace $|r|^2$ by 1 in the first summand, since by (3.29) combined with Hölder's inequality

$$\begin{aligned} & \tau^{-4} \int_{x<y} e^{-\tau(y-x)} |f(x)||(|r(y)|^2 - 1) f(y)| dxdy \\ & \quad \le \tau^{-3 + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}} \big(\tau^{-1} \|f\|_{L^{p_1}} \big) \big(\tau^{-1} \|f\|_{L^{p_2}} \big) \big(\tau^{-1} \||r|^2 - 1\|_{L^{p_3}} \big). \end{aligned}$$

Noticing that

$$|q|^2 - 1 = |r|^2 - 1 + r(\overline{q-r}) + \bar{r}(q-r) + |q-r|^2 = f + (q-r)^2,$$

we replace f by $|q|^2 - 1$ to arrive at (3.33),

$$\begin{aligned} & \left| \frac{1}{(1-\zeta^2)^2} \int_{x<y} e^{2iz(y-x)} f(x)(q-r)^2(y) dxdy \right| \\ & \qquad \le \tau^{-5 + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}} \Big(\tau^{-1} \|f\|_{L^{p_1}} \Big) \|q-r\|_{L^{p_2}} \|q-r\|_{L^{p_3}}. \end{aligned}$$

We argue similarly with the remaining terms.

Step 4. We derive the claim of Proposition 3.9 by integration by parts. We first work with the $\bar{r}'(x) r'(y)$ term in (3.33). Recall that $\frac{\zeta^2}{(1-\zeta^2)^2} = (2z)^{-2}$, and integrate by parts,

$$\begin{aligned} & \frac{1}{4z^2} \int_{x<y} e^{2iz(y-x)} (\bar{q}'(x) q'(y) - \bar{r}'(x) r'(y)) dxdy \\ & \quad = \frac{1}{4z^2} \int_{x<y} e^{2iz(y-x)} \Big[(\overline{q-r})'(x) r'(y) + \bar{r}'(x)(q-r)'(y) \\ & \qquad + ((\overline{q-r})'(x)(q-r)'(y) \Big] dxdy \\ & \quad = \int_{x<y} e^{2iz(y-x)} \Big[-\frac{1}{2iz} \Big((\overline{q-r})(x) r'(y) - \bar{r}'(x)(q-r)(y) \Big) \\ & \qquad + (\overline{q-r})(x)(q-r)(y) \Big] dxdy \\ & \qquad - \frac{1}{4z^2} \int_{\mathbb{R}} (q\bar{q}' - r\bar{r}')\, \mathrm{d}x + \frac{1}{2iz} \int_{\mathbb{R}} |q-r|^2\, \mathrm{d}x\,. \end{aligned}$$

This gives

$$\begin{aligned}T_2(z) = \int_{x<y} & e^{2iz(y-x)}\Big\{\frac{1}{4z^2}\bar{q}'(x)q'(y) + \frac{1}{(2z\zeta)^2}(|q(x)|^2-1)(|q(y)|^2-1)\\ &+\frac{i}{4z^2\zeta}\Big[(|q|^2-1)(x)\bar{r}r'(y) - r\bar{r}'(x)(|q|^2-1)(y)\Big]\\ &+\frac{1}{2z\zeta}\Big[(|q|^2-1)(x)(\bar{r}(q-r))(y) + (r(\overline{q-r}))(x)(|q|^2-1)(y)\Big]\Big\}dxdy\\ &+\frac{1}{4z^2}\int_{\mathbb{R}}(q\bar{q}'-r\bar{r}')\,\mathrm{d}x - \frac{1}{2iz}\int_{\mathbb{R}}|q-r|^2dx + O_\tau(\mathcal{E}^3).\end{aligned}$$

We rewrite the third term as

$$\begin{aligned}&\frac{i}{8z^2\zeta}\int_{x<y} e^{2iz(y-x)}(|q|^2-1)(x)(\bar{r}r'-r\bar{r}')(y) + (\bar{r}r'-r\bar{r}')(x)(|q|^2-1)(y)dxdy\\ &\quad+\frac{i}{8z^2\zeta}\int_{x<y} e^{2iz(y-x)}(|q|^2-1)(x)(|r|^2-1)'(y) - (|r|^2-1)'(x)(|q|^2-1)(y)dxdy,\end{aligned}$$

where by integration by parts the second integral reads as

$$\begin{aligned}\frac{1}{4z\zeta}\int_{x<y} & e^{2iz(y-x)}\Big[(|q|^2-1)(x)(|r|^2-1)(y) + (|r|^2-1)(x)(|q|^2-1)(y)\Big]\,\mathrm{d}x\ \mathrm{d}y\\ &-\frac{i}{4z^2\zeta}\int_{\mathbb{R}}(|q|^2-1)(|r|^2-1)\,\mathrm{d}x\,.\end{aligned}$$

Hence using $\frac{1}{4z^2\zeta^2}+2\cdot\frac{1}{4z\zeta}=\frac{1}{4z^2}$, we derive Proposition 3.9. □

3.5. **The term $T_2+\Phi$.** Recall the corrected function Φ given in (1.44) if $q\in X^0$ with $|q|^2-1, q'\in L^1$:

$$\Phi = -\int_{\mathbb{R}} q_1\,\mathrm{d}x - \frac{i}{2z}\mathcal{M} - \frac{i}{2z\zeta}\Theta, \tag{3.34}$$

where q_1 is given in (3.13):

$$q_1 = \frac{i\zeta\big[|r|^2-1+2\mathrm{Re}\,(\bar{r}(q-r))\big]-\bar{r}r'}{|r|^2-\zeta^2},$$

$\mathcal{M}=\int_{\mathbb{R}}(|q|^2-1)\,\mathrm{d}x$ denotes the mass and Θ denotes the asymptotic change of the phase given in Theorem 1.3.

We correct the term T_2 by Φ as follows.

Proposition 3.10 (Corrected term $T_2+\Phi$). *Assume the same assumptions as in Lemma 3.3.*

The corrected function Φ has a unique continuous and smooth extension to X^0 modulo $\pi i(z\zeta)^{-1}\mathbb{Z}$, and more precisely, with a choice of $\tilde{q}$ such that $\tilde{q}\in X^2$, $q-\tilde{q}\in L^2$ and $|\tilde{q}|\geq\frac{1}{4}$,

$$\begin{aligned}\Phi = &-\frac{i}{2z}\int_{\mathbb{R}}\frac{(|q|^2-1)(|r|^2-1)}{|r|^2-\zeta^2}\,\mathrm{d}x + i\zeta\int_{\mathbb{R}}\frac{|q-r|^2}{|r|^2-\zeta^2}\,\mathrm{d}x\\ &-\frac{i}{2z\zeta}\mathrm{Im}\int_{\mathbb{R}}\Big(\bar{r}r'-\frac{\tilde{q}'}{\tilde{q}}\Big)\,\mathrm{d}x + \frac{1}{2z\zeta}\int_{\mathbb{R}}\frac{(|r|^2-1)\bar{r}r'}{|r|^2-\zeta^2}\,\mathrm{d}x \mod \pi i(z\zeta)^{-1}\mathbb{Z},\end{aligned} \tag{3.35}$$

where the integral $\mathrm{Im}\int_{\mathbb{R}}\Big(\bar r r' - \frac{\tilde q'}{\tilde q}\Big)\,\mathrm{d}x$ *is understood as*

$$\mathrm{Im}\int_{\mathbb{R}}\Big((\bar r-\bar{\tilde q})r' - \bar{\tilde q}'(r-\tilde q) + \frac{\tilde q'}{\tilde q}(|\tilde q|^2-1)\Big)\,\mathrm{d}x\,.$$

Then

$$T_2+\Phi = A+B+O_\tau(\mathcal{E}^3), \quad \bmod \pi i(z\zeta)^{-1}\,\mathbb{Z}, \tag{3.36}$$

where

$$\begin{aligned} A(\lambda) =& \frac{i}{4z^2}\int_{\mathbb{R}}\mathrm{Im}\,(q\bar q' - r\bar r')\,\mathrm{d}x \\ &+\frac{1}{4z^2}\int_{x<y} e^{2iz(y-x)}\Big(\bar q'(x)q'(y) + (|q|^2-1)(x)(|q|^2-1)(y)\Big)dxdy, \end{aligned} \tag{3.37}$$

and

$$\begin{aligned} B(\lambda) = &-\frac{i}{2z\zeta}\mathrm{Im}\int_{\mathbb{R}}(\bar r r' - \frac{\tilde q'}{\tilde q})\,\mathrm{d}x\, - \frac{i}{4z^2\zeta^2}\int_{\mathbb{R}}(|r|^2-1)\mathrm{Im}\,(\bar r r')\,\mathrm{d}x \\ &+\frac{1}{4z^2\zeta}\int_{x<y}e^{2iz(y-x)}\Big[(|q|^2-1)(x)\mathrm{Im}\,[r\bar r'](y) \\ &\qquad + \mathrm{Im}\,[r\bar r'](x)(|q|^2-1)(y)\Big]\,\mathrm{d}x\ \mathrm{d}y \\ &+\frac{i}{2z\zeta}\int_{x<y}e^{2iz(y-x)}\Big[(|q|^2-1)(x)\mathrm{Im}\,[\bar r(q-r)](y) \\ &\qquad + \mathrm{Im}\,[r(\overline{q-r})](x)(|q|^2-1)(y)\Big]\,\mathrm{d}x\ \mathrm{d}y\,. \end{aligned} \tag{3.38}$$

In particular, we have that

(1) $B(\lambda)+\overline{B(-\bar\lambda)}=0$ *and*

$$\frac12(A(\lambda)+\overline{A(-\bar\lambda)}) = -\frac{i}{2z}\int_{\mathbb{R}}\frac{1}{\xi^2-4z^2}\Big(|\widehat{q'}(\xi)|^2 + |(\widehat{|q|^2-1})(\xi)|^2\Big)\,\mathrm{d}\xi\,. \tag{3.39}$$

(2) $\frac{1}{2i}\big(A(\lambda)-\overline{A(-\bar\lambda)}\big)$ *reads as*

$$\frac{1}{4z^2}\int_{\mathbb{R}}\Big(-\frac{1}{\xi}\big(|\widehat{q'}(\xi)|^2 - |\widehat{r'}(\xi)|^2\big) + \frac{\xi}{\xi^2-4z^2}|\widehat{q'}(\xi)|^2\Big)\,\mathrm{d}x\,. \tag{3.40}$$

In particular with $r=\tau^2D_\tau^{-2}q$, *it reads as*

$$\frac{1}{4z^2}\int_{\mathbb{R}}\frac{\xi(\tau^4+4z^2(2\tau^2+\xi^2))}{(\tau^2+\xi^2)^2(\xi^2-4z^2)}|\widehat{q'}(\xi)|^2\,\mathrm{d}x\,, \tag{3.41}$$

which can be bounded by $c\tau^{-1}(E^0_\tau(q))^2$.

(3) *There exists* $\delta_2>0$ *and* $\tau_0\ge 2$ *such that if* $q\in X^s$, $s\in(\frac12,\frac32)$ *does not have zeros and satisfies the smallness condition*

$$\tau_0^{-1-\varepsilon}E_{\tau_0}^{\frac12+\varepsilon}(q)\le\delta_2, \tag{3.42}$$

for some $0<\varepsilon<\min\{s-\frac12,\frac12\}$, *then we can take* $r=q=\tilde q$, *such that*

(3.43)
$$\begin{aligned} &\Big|\frac{1}{2i}\Big((T_2+\Phi)(\lambda)-\overline{(T_2+\Phi)(-\bar\lambda)}\Big) - \frac{\lambda}{4z^3}H_1 - \frac{\lambda}{4z^3}\int_{x<y}e^{2iz(y-x)}\mathrm{Im}\,[\bar q'(x)q'(y)]dxdy\Big| \\ &\le c\tau^{-1-2s}(\tau_0^{-1-\varepsilon}E_{\tau_0}^{\frac12+\varepsilon}(q))(E^s_{\tau_0}(q))^2, \quad \forall\tau\ge\tau_0, \end{aligned}$$

where H_1 is defined in Theorem 1.3. Correspondingly (1.29) *holds.*

Proof. We do more detailed calculations for Φ, A and the case $s > \frac{1}{2}$ respectively as follows.

Calculation of Φ. We first recall the definition of q_1 in (3.13)

$$\int_{\mathbb{R}} q_1 dx = \int_{\mathbb{R}} \frac{i\zeta(|r|^2-1) + i\zeta\big[r(\bar{q}-\bar{r}) + \bar{r}(q-r)\big] - \bar{r}r'}{|r|^2-\zeta^2}\,\mathrm{d}x$$
$$= \int_{\mathbb{R}} \Big(\frac{i\zeta}{|r|^2-\zeta^2}(|q|^2-1-|q-r|^2) - \frac{\bar{r}r'}{|r|^2-\zeta^2}\Big)\,\mathrm{d}x\,.$$

Noticing $\frac{1}{|r|^2-\zeta^2} = \frac{1}{1-\zeta^2} - \frac{|r|^2-1}{(1-\zeta^2)(|r|^2-\zeta^2)} = \frac{1}{1-\zeta^2} - \frac{|r|^2-1}{(1-\zeta^2)^2} + \frac{(|r|^2-1)^2}{(1-\zeta^2)^2(|r|^2-\zeta^2)}$ and $\frac{i\zeta}{1-\zeta^2} = -\frac{i}{2z}$, we derive

$$\int_{\mathbb{R}} q_1 dx = -\frac{i}{2z}\int_{\mathbb{R}} (|q|^2-1) - \frac{(|r|^2-1)}{1-\zeta^2}(|q|^2-1) + \frac{(|r|^2-1)^2}{(1-\zeta^2)(|r|^2-\zeta^2)}(|q|^2-1)\,\mathrm{d}x$$
$$+ \frac{i}{2z}\int_{\mathbb{R}} \Big[|q-r|^2 - \frac{|r|^2-1}{|r|^2-\zeta^2}|q-r|^2\Big]\,\mathrm{d}x$$
$$+ \frac{1}{2z\zeta}\int_{\mathbb{R}} \Big[\bar{r}r' + \frac{|r|^2-1}{2z\zeta}\bar{r}r' - \frac{(|r|^2-1)^2}{2z\zeta(|r|^2-\zeta^2)}\bar{r}r'\Big]\,\mathrm{d}x\,.$$

Thus the correction function $\Phi = -\int_{\mathbb{R}} q_1\,\mathrm{d}x - \frac{i}{2z}\mathcal{M} - \frac{i}{2z\zeta}\Theta$ reads as in (3.35), and

$$\Phi = \frac{i}{4z^2\zeta}\int_{\mathbb{R}} (|r|^2-1)(|q|^2-1)\,\mathrm{d}x - \frac{i}{2z}\int_{\mathbb{R}} |q-r|^2\,\mathrm{d}x$$
$$- \frac{1}{4z^2\zeta^2}\int_{\mathbb{R}} (|r|^2-1)\bar{r}r'\,\mathrm{d}x - \frac{i}{2z\zeta}\mathrm{Im}\int_{\mathbb{R}} \Big(\bar{r}r' - \frac{\tilde{q}'}{\tilde{q}}\Big)\,\mathrm{d}x + O_\tau(\mathcal{E}^3). \tag{3.44}$$

Hence (3.36) follows from Proposition 3.9.

Calculation of A. To calculate more precisely A, we first notice the following fact by use of the definition of Fourier transformations

$$\int_{x<y} e^{2iz(y-x)}\bar{f}(x)g(y)dxdy = \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{x<y} e^{2iz(y-x)-i\xi x+i\eta y}\overline{\hat{f}(\xi)}\hat{g}(\eta)d\xi d\eta dydx$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{-i(\xi+2z)}\int_{\mathbb{R}^2} e^{-i(\xi-\eta)y}\overline{\hat{f}(\xi)}\hat{g}(\eta)dyd\eta d\xi = \int_{\mathbb{R}} \frac{i}{(\xi+2z)}\overline{\hat{f}(\xi)}\hat{g}(\xi)d\xi.$$

Similarly, we derive

$$\int_{x<y} e^{2iz(y-x)}f(x)\bar{g}(y)dxdy = \int_{\mathbb{R}} \frac{-i}{(\xi-2z)}\hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi.$$

Since if $(\lambda, z) \in \mathcal{R}$, then $(-\bar{\lambda}, -\bar{z}) \in \mathcal{R}$ with $\mathrm{Im}\,(-\bar{\lambda}) = \mathrm{Im}\,\lambda$ and $\mathrm{Im}\,(-\bar{z}) = \mathrm{Im}\,z$, we have $B(\lambda) + \overline{B(-\bar{\lambda})} = 0$ and

$$\frac{1}{2}\big(A(\lambda) + \overline{A(-\bar{\lambda})}\big)$$
$$= \frac{1}{4z^2}\int_{x<y} e^{2iz(y-x)}\Big(\frac{1}{2}(\bar{q}'(x)q'(y) + q'(x)\bar{q}'(y)) + (|q|^2-1)(x)(|q|^2-1)(y)\Big)\,\mathrm{d}x\,\mathrm{d}y\,,$$

which reads in terms of their Fourier transforms as

$$\frac{1}{4z^2}\int_{\mathbb{R}} \frac{1}{2}\Big(\frac{i}{\xi+2z} - \frac{i}{\xi-2z}\Big)\Big(|\widehat{q'}(\xi)|^2 + |\widehat{(|q|^2-1)}(\xi)|^2\Big)\,\mathrm{d}x\,\mathrm{d}y\,.$$

This is (3.39).

Similarly we arrive at (3.40) for $\frac{1}{2i}(A(\lambda)-\overline{A(-\bar\lambda)})$ by use of Plancherel's identity, which is (3.41) if $r=\tau^2 D_\tau^{-2} q$. It can be bounded by $c\tau^{-1}(E^0_\tau(q))^2$ by virtue of $\tau = 2\mathrm{Im}\, z\geq 2$.

Special case $s\in(\frac12,\frac32)$**.** If $q\in X^s$, $s>\frac12$, then we can take $q=r$ (i.e. we do not need the regularisation procedure) in the definitions of q_j's in (3.13). This is the setting of [14], and we are going to show below similar estimates as in Lemma 3.3 (for terms T_{2n}), Lemma 3.7 (for cubic errors) and (3.36)-(3.39)-(3.40) (for A,B) in this setting.

We proved in [14, Proposition 5.1] that

$$\Big|T_{2n}(\lambda)|_{\lambda=i\sqrt{\tau^2/4-1}}\Big| \leq \big(C(1+\tau^{-1}\|q'\|_{l^2_\tau DU^2})\tau^{-1}\|(|q|^2-1,q')\|_{l^2_\tau DU^2}\big)^{2n},$$

which can be compared with (3.27) above. We refer to [14] for the definition of the space $l^2_\tau DU^2$, and we have $\frac{1}{\sqrt\tau}E^0_\tau(q)\lesssim \frac1\tau\|(|q|^2-1,q')\|_{l^2_\tau DU^2}$ and the crucial property that

$$\|f\|_{l^2_\tau DU^2}\leq c_\sigma \tau^{-\frac12-\sigma}\|f\|_{H^\sigma_\tau},\quad \forall\sigma>-\frac12,$$

that is,

$$\tau^{-1}\|(|q|^2-1,q')\|_{l^2_\tau DU^2}\leq c_\sigma\tau^{-\frac12-\sigma}E^\sigma_\tau(q),\quad\forall\sigma>\frac12.$$

Then, provided $E^{\frac12+\varepsilon}_{\tau_0}(q)\leq \delta_2{\tau_0}^{1+\varepsilon}$ for some $\varepsilon>0$ such that $\sigma=\frac12+\varepsilon<\min\{s,1\}$ and for some small enough δ_2, and some $\tau_0\geq 2$,

$$\Big|\sum_{n=2}^\infty T_{2n}(\lambda)|_{\lambda=i\sqrt{\tau^2/4-1}}\Big|\leq c\tau^{-1-2s}({\tau_0}^{-1-\varepsilon}E^{\frac12+\varepsilon}_{\tau_0}(q))^2(E^s_{\tau_0}(q))^2,\quad\forall\tau\geq\tau_0, \tag{3.45}$$

and we continue with a variant of the argument above.

We call F a cubic error if

$$|F|\leq c\tau^{-3}\|(|q|^2-1,q')\|^3_{l^3_\tau DU^2}.$$

Using

$$\|f\|_{l^3_\tau DU^2}\leq\|f\|_{l^2_\tau DU^2}\qquad\text{and}\qquad \|f\|_{l^3_\tau DU^2}\leq c\tau^{-\frac23}\|f\|_{L^3},$$

we have the following estimate for cubic errors for $s\in(\frac12,\frac32)$:

$$|F|\leq c\tau^{-1-2s}({\tau_0}^{-1-\varepsilon}E^{\frac12+\varepsilon}_{\tau_0}(q))(E^s_{\tau_0}(q))^2. \tag{3.46}$$

If $q=r$, then we have from (3.37)-(3.38) that

$$\begin{aligned}A+B={}&-\frac{i}{2z\zeta}\mathrm{Im}\int_{\mathbb R}(\bar q q'-\frac{\tilde q'}{\tilde q})\,\mathrm{d}x-\frac{i}{4z^2\zeta^2}\int_{\mathbb R}(|q|^2-1)\mathrm{Im}\,(\bar q q')\,\mathrm{d}x\\ &+\frac{1}{4z^2}\int_{x<y}e^{2iz(y-x)}\Big(\bar q'(x)q'(y)+(|q|^2-1)(x)(|q|^2-1)(y)\Big)dxdy\\ &+\frac{1}{4z^2\zeta}\int_{x<y}e^{2iz(y-x)}\Big[(|q|^2-1)(x)\mathrm{Im}\,[q\bar q'](y)+\mathrm{Im}\,[q\bar q'](x)(|q|^2-1)(y)dxdy.\end{aligned}$$

We can integrate by parts to rewrite the last integral above as

$$\begin{aligned}&\frac{i}{4z^3\zeta}\int_{\mathbb{R}}(|q|^2-1)\mathrm{Im}\,[q\bar{q}']\,\mathrm{d}x\\&+\frac{1}{4iz^3\zeta}\int_{x<y}e^{2iz(y-x)}\Big(\mathrm{Re}\,[q\bar{q}'](x)\mathrm{Im}\,[q\bar{q}'](y)-\mathrm{Im}\,[q\bar{q}'](x)\mathrm{Re}\,[q\bar{q}'](y)\Big)dxdy,\end{aligned}$$

where the double integral reads further as

$$\begin{aligned}&-\frac{1}{4iz^3\zeta}\int_{x<y}e^{2iz(y-x)}\mathrm{Im}\,[(q\bar{q}')(x)(\bar{q}q')(y)]dxdy\\&=\frac{i}{4z^3\zeta}\int_{x<y}e^{2iz(y-x)}\mathrm{Im}\,[\bar{q}'(x)q'(y)]dxdy\\&\quad-\frac{i}{4z^3\zeta}\int_{x<y}e^{2iz(y-x)}\mathrm{Im}\,\Big[\bar{q}'(x)\int_x^y q'(m)\,\mathrm{d}m\,(\bar{q}q')(y)\Big]dxdy\\&\quad+\frac{i}{4z^3\zeta}\int_{x<y}e^{2iz(y-x)}\mathrm{Im}\,[\bar{q}'(x)((|q|^2-1)q')(y)]dxdy.\end{aligned}\tag{3.47}$$

We have showed in [14, Appendix A] that [5]

$(T_2+\Phi)-(A+B)$,

$$\begin{aligned}&\text{with } A+B=-\frac{i}{2z\zeta}\mathrm{Im}\int_{\mathbb{R}}(\bar{q}q'-\frac{\tilde{q}'}{\tilde{q}})\,\mathrm{d}x-\Big(\frac{i}{4z^2\zeta^2}+\frac{i}{4z^3\zeta}\Big)\int_{\mathbb{R}}(|q|^2-1)\mathrm{Im}\,(\bar{q}q')\,\mathrm{d}x\\&+\frac{1}{4z^2}\int_{x<y}e^{2iz(y-x)}\Big(\bar{q}'(x)q'(y)+(|q|^2-1)(x)(|q|^2-1)(y)\Big)dxdy\\&+\frac{i}{4z^3\zeta}\int_{x<y}e^{2iz(y-x)}\mathrm{Im}\,[\bar{q}'(x)q'(y)]dxdy\end{aligned}$$

is a cubic error, which can be estimated as in (3.46) under the smallness assumption (3.42).

Finally if $q\neq 0$ and we take $\tilde{q}=q$, then by Theorem 1.3 we have

$$H_1=\int_{\mathbb{R}}(|q|^2-1)\mathrm{Im}\,[\frac{\bar{q}'}{\bar{q}}]\,\mathrm{d}x\quad \mathrm{mod}\ 2\pi\,\mathbb{Z},$$

and the integral difference

$$\int_{\mathbb{R}}(|q|^2-1)\mathrm{Im}\,[q\bar{q}']\,\mathrm{d}x\ -H_1=\int_{\mathbb{R}}(|q|^2-1)\frac{|q|^2-1}{|q|}\mathrm{Im}\,[\frac{q\bar{q}'}{|q|}]\,\mathrm{d}x\,,$$

can be controlled as in (3.46). Thus we have (3.43) by view of $\frac{1}{2z\zeta}+\frac{1}{4z^2\zeta^2}=\frac{1}{4z^2}$ and $\frac{1}{4z^2}+\frac{1}{4z^3\zeta}=\frac{\lambda}{4z^3}$.

Correspondingly (1.29) follows by virtue of (3.45).

□

3.6. **Conclusions.**

Finally we deduce the claims made in Theorem 1.6 about the holomorophy of the renormalized transmission coefficient.

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[5] Instead of Φ given by (3.34) if $q\in X^0$ with $|q|^2-1, q'\in L^1$ here, the correction function $\underline{\Phi}$ in [14] was given by $\underline{\Phi}=-\int_{\mathbb{R}}q_1\,\mathrm{d}x-\frac{i}{2z}\mathcal{M}-\frac{i}{2z\zeta}\mathcal{P}$, if $q\in X^{(\frac{1}{2})+}$ with $|q|^2-1, q'\in L^1$. Thus with a choice of $\tilde{q}$, $T_2+\Phi=T_2+\underline{\Phi}-\frac{i}{2z\zeta}\mathrm{Im}\,\int_{\mathbb{R}}(\bar{q}q'-\frac{\tilde{q}'}{\tilde{q}})\,\mathrm{d}x$.

3.6.1. *Case* $(\lambda, z) \in \mathcal{R}$ *with* $\operatorname{Im}\lambda \geq 0$, $2\operatorname{Im} z = \tau > 2$ *and* $q \in X^0$ *with* $E^0_\tau(q) \leq \delta_1 \tau^{\frac{1}{2}}$. This is the case considered from Subsection 3.2 to Subsection 3.5. Let r be given in Lemma 2.2. Let Φ be given in Proposition 3.10.

By the obtained results in Subsections 3.2-3.5, we can define for $q \in X^0$ under the smallness condition $E^0_\tau(q) \leq \delta_1 \tau^{\frac{1}{2}}$,

$$\ln T_{\mathrm{c}}^{-1}(\lambda) = (T_2 + \Phi) + \Big(\ln \sum_{n=0}^{\infty} T_{2n} - T_2\Big) \in \mathbb{C}\,/(\pi i (z\zeta)^{-1}\,\mathbb{Z}),$$

where $T_2+\Phi$ is characterized in Proposition 3.10, and the estimates for $\ln \sum_{n=0}^{\infty} T_{2n} - T_2$ can be found in Corollary 3.4. Hence (1.28) and (1.30) in Theorem 1.6 follow from Corollary 3.4, Lemma 3.7 and Proposition 3.10.

Correspondingly the renormalized transmission coefficient T_{c}^{-1} reads as

$$T_{\mathrm{c}}^{-1}(\lambda) = e^{\Phi(\lambda)} \sum_{n=0}^{\infty} T_{2n}(\lambda) \in \mathbb{C}\,/(e^{\pi i (z\zeta)^{-1}\,\mathbb{Z}}), \tag{3.48}$$

where Φ is defined in Proposition 3.10, and T_{2n} are defined in (3.24).

If $q \in 1 + \mathcal{S}$, then the (original) transmission coefficient T^{-1} reads as $T^{-1}(\lambda) = e^{-\int_{\mathbb{R}} q_1 \,\mathrm{d}x} \sum_{n=0}^{\infty} T_{2n}$ (see (3.26)), and we have the relation

$$T_{\mathrm{c}}^{-1}(\lambda) = T^{-1}(\lambda) e^{-\frac{i}{2z}\mathcal{M} - \frac{i}{2z\zeta}\Theta}.$$

We recall that Θ is uniquely defined modulo $2\pi\,\mathbb{Z}$. This implies that the definition of the renormalized transmission coefficient is independent of the choice of r. We hence can typically choose $r = \tau^2 D_\tau^{-2} q$.

By the proof of Theorem 1.3 and more precisely Lemma 2.4, we can choose $\tilde{q}$ continuously in small balls and hence we find a continuous branch of $\Phi(\lambda; q)$ and hence $T_c^{-1}(\lambda; q)$ on the covering space of $X^0 \ni q$. For $q = 1$ we may choose $\tilde{q} = 1$ so that $T_c^{-1}(\lambda; 1) = 1$ (we identify 1 with $\tilde{1}$ in the covering space).

3.6.2. *Case* $(\lambda, z) \in \mathcal{R}$ *with* $\operatorname{Im}\lambda \geq 0$, $2\operatorname{Im} z = \tau > 2$ *and* $q \in X^0$. We are now in the position to remove the smallness condition on the energy.

Given any small enough δ_0, there exists R so that

$$\||r|^2 - 1\|_{L^2(\mathbb{R}\setminus(-R,R))} + \tau\|r - q\|_{L^2(\mathbb{R}\setminus(-R,R))} + \|r'\|_{L^2(\mathbb{R}\setminus(-R,R))} < \delta_0 \tau.$$

As the first step, following exactly the ideas in Subsection 3.2 and Subsection 3.3, we can solve the w-ODE (3.20) on $(-\infty, -R]$. Then we solve the original spectral ODE (3.7) for u on the finite interval $[-R, R]$ with initial data at $x = -R$ given by $w(-R)$ under the transformation (3.19) obtained in the first step. On $[R, \infty)$ we solve the w-ODE again by iteration with initial data at $x = R$ given by $u(R)$ and the transformation (3.19). The constant δ_0 exists such that these arguments work and we can still define the renormalized transmission coefficient in terms of the limit of the first component of the solution w as

$$T_{\mathrm{c}}^{-1}(\lambda) = e^{\Phi} \lim_{x\to\infty} w^1(x).$$

To prove holomorphy of $T_{\mathrm{c}}^{-1}(\lambda; q)$ in λ, for any z_0 with $\operatorname{Im} z_0 > 1$, we take a small open disk around z_0 and fix $\tau = 2\operatorname{Im} z_0$ (instead of $\tau = 2\operatorname{Im} z$) in this small disk, to define $r = \tau^2 D_\tau^{-2} q$. Holomorphy of $T_c^{-1}(\lambda)$ is then a consequence of holomorphy of the q_j with respect to λ and of the Picard iteration-mapping (3.22). Similarly the analyticity of $T_{\mathrm{c}}^{-1}(\lambda; q)$ in q is obvious from the construction.

3.6.3. *Case* $(\lambda, z) \in \mathcal{R}$ *with* $\mathrm{Im}\,\lambda < 0$, $\tau = 2\mathrm{Im}\, z > 2$ *and* $q \in X^0$. In the case $(\lambda, z) \in \mathcal{R}$ with $\mathrm{Im}\,\lambda < 0$ (we recall that always $\mathrm{Im}\, z > 0$), we have $\zeta^{-1} = (\lambda + z)^{-1}) = \lambda - z$ satisfying $-\mathrm{Im}\,\zeta^{-1} = -\mathrm{Im}\,\lambda + \mathrm{Im}\, z \in (\mathrm{Im}\, z, 2\,\mathrm{Im}\, z) \in \mathbb{R}^+$. We define

$$v^- = \begin{pmatrix} -i\zeta^{-1} & r \\ \bar{r} & i\zeta^{-1} \end{pmatrix} u, \text{ or equivalently, } u = \frac{1}{|r|^2 - \zeta^{-2}} \begin{pmatrix} -i\zeta^{-1} & r \\ \bar{r} & i\zeta^{-1} \end{pmatrix} v^-,$$

such that v solves

$$(v^-)_x = \begin{pmatrix} iz & 0 \\ 0 & -iz \end{pmatrix} v^- + \begin{pmatrix} q_4^- - q_1^- & q_2^- \\ q_3^- & -q_1^- \end{pmatrix} v^-,$$

where q_k, $k = 1, 2, 3, 4$ are given by

$$\begin{aligned} q_1^- &= \frac{-i\zeta^{-1}\big[|r|^2 - 1 + 2\mathrm{Re}\,(\bar{r}(q-r))\big] - r\bar{r}'}{|r|^2 - \zeta^{-2}}, \\ q_2^- &= \frac{r\big[|r|^2 - 1 + 2\mathrm{Re}\,(\bar{r}(q-r))\big] + i\zeta^{-1} r'}{|r|^2 - \zeta^{-2}} - (q - r), \\ q_3^- &= \frac{\bar{r}\big[|r|^2 - 1 + 2\mathrm{Re}\,(\bar{r}(q-r))\big] - i\zeta^{-1} \bar{r}'}{|r|^2 - \zeta^{-2}} - (\overline{q - r}), \\ q_4^- &= \frac{-2i\zeta^{-1}\big[|r|^2 - 1 + 2\mathrm{Re}\,(\bar{r}(q-r))\big] - 2i\mathrm{Im}\,(r\bar{r}')}{|r|^2 - \zeta^{-2}}. \end{aligned} \tag{3.49}$$

There are similar estimates for q_j^- as in Lemma 3.1 if $\mathrm{Im}\,\lambda < 0$, and all the results in Subsection 3.2-Subection 3.5 hold correspondingly. The arguments above in Paragraph 3.6.2 work also correspondingly.

3.6.4. *Case* $(\lambda, z) \in \mathcal{R}$ *with* $2\mathrm{Im}\, z = \tau \leq 4$ *and* $q \in X^0$. We consider now the case when $\tau = 2\mathrm{Im}\, z \in (0, 4]$ is bounded. For $\delta_0 > 0$ sufficient small (to be determined later), we can take $\tau_0 \geq 2$ such that $E^0_{\tau_0}(q) \leq \delta_0$. We take $r = \tau_0^2 D_{\tau_0}^{-2} q$, such that by Lemma 2.2,

$$\begin{aligned} &\|r\|_{L^\infty} \leq c\tau_0, \text{ and} \\ &\|(\frac{1}{\tau_0} r', \frac{1}{\tau_0}(|r|^2 - 1), q - r)\|_{L^p} \leq C_p \|(|q|^2 - 1, q')\|_{W^{-1,p}_{\tau_0}}, \quad \forall p \in [2, \infty). \end{aligned}$$

If $(\lambda, z) \in \mathcal{R}$ with $\mathrm{Im}\,\lambda \geq 0$, then $\zeta \in \mathcal{U}$ with $\mathrm{Im}\,\zeta > 0$ and $|\zeta| \geq 1$, and hence $\zeta^2 \in \mathbb{C} \setminus \mathbb{R}^+$ with $|\zeta^2| \geq 1$. Thus similarly as in Lemma 3.1 we have also

$$\|q_j\|_{L^p} \leq C_p(\zeta) \|(|q|^2 - 1, q')\|_{W^{-1,p}_{\tau_0}}, \quad \forall p \in [2, \infty).$$

We then have by the argument in Subsection 3.3 that

$$|T_{2n}(\lambda)| \leq (C_\lambda)^n (E^0_{\tau_0}(q))^{2n} \leq (C_\lambda {\delta_0}^2)^n, \quad \forall n \geq 1,$$

where C_λ is some positive constant depending on λ, z, ζ. We then take $\delta_0 > 0$ sufficiently small (depending on λ) such that $\sum_{n=0}^\infty T_{2n}$ converges, and we define the renormalized transmission coefficient T_c^{-1} as before (see e.g. (3.48)).

Together with Paragraph 3.6.2 and Paragraph 3.6.3, we have the holomorphy of the renormalized transmission coefficient across the real interval $\lambda \in (-1, 1)$.

3.6.5. *The renormalized transmission coefficient and the Lax operator.* We conclude from Paragraphs 3.6.1-3.6.2-3.6.3-3.6.4 above that the renormalized transmission coefficient $T_c^{-1}(\lambda; q)$ is well-defined for $\lambda \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$ and $q \in X^0$. It is understood as a continuous map from the universal covering space of X^0 to the holomorphic functions on $\mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$. By construction it is independent of time.

We are now in a position to study the spectrum of the Lax operator (3.4), if $q \in X^0$. By use of Lemma 2.4, we can take its regularisation $\tilde{q}$ which satisfies $\tilde{q} - q \in L^2$, $|\tilde{q}| = 1$ and $\tilde{q}' \in H^1$. Then the spectrum of L is the same as the spectrum of conjugated Lax operator:

$$\begin{aligned} \begin{pmatrix} \overline{\sqrt{\tilde q}} & 0 \\ 0 & \sqrt{\tilde q} \end{pmatrix} \begin{pmatrix} i\partial_x & -iq \\ i\bar q & -i\partial_x \end{pmatrix} \begin{pmatrix} \sqrt{\tilde q} & 0 \\ 0 & \overline{\sqrt{\tilde q}} \end{pmatrix} &= \begin{pmatrix} i\partial_x + \frac{i}{2}\bar{\tilde q}\tilde q' & -i\bar{\tilde q} q \\ i\tilde q \bar q & -i\partial_x - \frac{i}{2}\tilde q \bar{\tilde q}' \end{pmatrix} \\ &= \begin{pmatrix} i\partial_x & -i \\ i & -i\partial_x \end{pmatrix} + \begin{pmatrix} \frac{i}{2}\bar{\tilde q}\tilde q' & -i(\bar{\tilde q} q - 1) \\ i(\tilde q \bar q - 1) & -\frac{i}{2}\tilde q \bar{\tilde q}' \end{pmatrix} \end{aligned}$$

where the entries of the second matric are in L^2. We obtain a compact perturbation of the Lax operator as an operator from $H^{1/2} \to H^{-1/2}$. By a Fourier transform the left operator becomes the multiplication operator by

$$\begin{pmatrix} -\xi & -i \\ i & \xi \end{pmatrix}, \quad \xi \in \mathbb{R},$$

whose eigenvalues are $\pm\sqrt{\xi^2+1}$, and hence the spectrum is the continuous spectrum, which is $(-\infty, -1] \cup [1, +\infty)$. When we consider the Lax operator as an operator from $H^{1/2}$ to $H^{-1/2}$ then the multiplication by the second operator (whose entries are in L^2) is a compact perturbation. Hence the essential spectrum of the Lax operator is $(-\infty, -1] \cup [1, \infty)$ and the spectrum outside the essential spectrum consists of isolated eigenvalues (in $(-1, 1)$ since the Lax operator is self adjoint). If λ is outside the continuous spectrum then the space of solutions in $L^2((-\infty, 0])$ of

$$Lu - \lambda u = 0$$

is spanned by the left Jost function and thus the geometric multiplicity of eigenvalues in $(-1, 1)$ is 1. The algebraic multiplicity equals the geometric multiplicity since the operator is selfadjoint. Any eigenfunction (with eigenvalue outside the essential spectrum) is a multiple of the left and the right Jost function and hence the transmission coefficient T_c^{-1} vanishes at eigenvalues. The algebraic multiplicity is the order of the zero. Since the algebraic multiplicity is 1 the zeroes are simple. Thus the renormalized transmission coefficient has simple zeros in $(-1, 1)$.

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