Master Thesis

Ramsey numbers for set-colorings

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Abstract

For $s, t, n \in \mathbb{N}$ with $s \geq t$, an (s,t)-coloring of K_n is an edge coloring of K_n in which each edge is assigned a set of t colors from $\{1, \ldots, s\}$. For $k \in \mathbb{N}$, a monochromatic K_k is a set of k vertices S such that for some color $i \in [s]$, $i \in c(uv)$ for all distinct $u, v \in S$. As in the case of the classical Ramsey number, we are interested in the least positive integer $n = R_{s,t}(k)$ such that for any (s,t)-coloring of K_n, there exists a monochromatic K_k. We estimate upper and lower bounds for general cases and calculate close bounds for some small cases of $R_{s,t}(k)$.

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1 Introduction

Given a positive integer k, the Ramsey number $R(k)$ is the least positive integer n such that every 2-edge coloring of a complete graph K_n contains a monochromatic K_k. Since the first theorem by Frank Plumpton Ramsey in 1930 which proved the existence of $R(k)$, Ramsey theory has been widely and deeply studied. The overall results say that, a sufficiently large object must contain a specific structure. There are various kinds of Ramsey numbers and associated results. However, still very few exact values of Ramsey numbers are known. In this thesis, we study (s,t)-colorings of K_n, where each edge of K_n is assigned a set of t colors from $\{1,\dots,s\}$. By $R_{s,t}(k)$ we denote the least positive integer n such that any (s,t)-coloring of K_n contains a monochromatic K_k. We call $R_{s,t}(k)$ the *Ramsey number for set-coloring*, or the (s,t) Ramsey number. As for the classical Ramsey number, we will prove that $R_{s,t}(k)$ is well-defined and estimate its bounds. In addition, we show that with the existence of some specific designs and Hadamard matrices, we can gain information about lower bounds of some (s,t) Ramsey numbers. In case of specific resolvable designs, we show that those lower bounds are tight.

1.1 The basics

We assume that the readers are familiar with basic terms in graph theory and combinatorics. We introduce here some definitions that are used throughout this thesis. For undefined terms we refer to books by Diestel [7] for graph theory and Stinson [24], Beth, Jungnickel and Lenz [4] for combinatorics. We consider graphs up to isomorphism and label the vertices explicitly as needed. If two graphs G and H are isomorphic, we write $G \cong H$.

We only deal with finite graphs without loops or multiple edges. For a graph G, its vertex set is $V(G)$, its order $|V(G)|$ is also denoted by $|G|$, its size is the number of edges and is denoted by $||G|| := e(G) := |E(G)|$.

For positive integers s,t, we denote by $[s]$ the set $\{1,2,\dots,s\}$, and call any t-element set a *t-set*. For any set S, the *power set* of S is the set containing all subsets of S and is denoted by 2^S.

For $n \in \mathbb{N}$. we denote by $\mathbb{Z}_n$ the set $\mathbb{Z}/n\mathbb{Z}$ of integers modulo n.

For sets A and B, the *symmetric difference* of A and B is denoted by $A\Delta B$, is the set containing elements that are either in A or in B (but not in both A and B). That is $A\Delta B = (A \setminus B) \cup (B \setminus A)$.

For graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_1 \cap V_2 = \emptyset$, we denote by $G_1 + G_2$ the graph $G_1 \cup G_2$ together with all the edges connecting vertices of G_1 and G_2. That is $G_1 + G_2 = (V, E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup \{uv : \ u \in V_1, \ v \in V_2\}$. If G_2

consists of a single vertex x, we write $G_1 + x$ for $G_1 + \{x\}$.

For $n \in \mathbb{N}$, a *complete graph* on n vertices is denoted by K_n, and K_3 is called a *triangle*. For sets A, B, C, we denote by $K_{A,B}$ the complete bipartite graph with parts A and B, and by $K_{A,B,C}$ the complete tripartite graphs with parts A, B, C.

For any graph H and any integer $n \geq |H|$, the *extremal number* $ex(n, H)$ denotes the maximum size of a graph on n vertices that does not contain H as a subgraph. Additionally, $EX(n, H)$ is the set of H-free graphs on n vertices with size $ex(n, H)$.

For integers $1 \leq r \leq n$, the Turán graph $T_r(n)$ is the unique r-partite graph on n vertices whose partition sets (often called just "parts") differ in size by at most 1. We denote $||T_r(n)||$ by $t_r(n)$. If $n = r \cdot s$ for some $s \in \mathbb{N}$, we also write $T_r(n) = K_r^s$.

Among all r-partite graphs of order n, the graph $T_r(n)$ has the largest size. Turán's Theorem states that $T_r(n)$ is also has the largest size among all graphs of order n that does not contain K_{r+1} as a subgraph, i.e. $EX(n, K_{r+1}) = \{T_r(n)\}$.

For any positive integers s, n, an *s-coloring* of $E(K_n)$ or an s-edge-coloring of K_n is a function

$$c : E(K_n) \to [s],$$

where each edge of K_n is assigned a color from $[s]$. A set of edges E is called *strong monochromatic* if all edges in E have the same color. The classical *multicolor Ramsey number* $R_s(k)$ is the least positive integer n such that any s-coloring of $E(K_n)$ contains a strong monochromatic K_k. The classical *Ramsey number* is $R_2(k)$ and we write $R(k) := R_2(k)$. Ramsey's Theorem proves that $R(k)$ exists for any $k \in \mathbb{N}$. Moreover, it is known that $\sqrt{2}^k \leq R(k) \leq 2^{2k-3}$ for any $k \in \mathbb{N}$ (see e.g. Diestel [7]). Known results of small Ramsey numbers can be found in a dynamic survey by Radziszowski [20]. We also refer to Graham, Rothschild and Spencer [11] for an overview of Ramsey theory. For related Ramsey theory, we refer to some other variants of Ramsey numbers such as the *list Ramsey numbers* by Alon et al. [2], and the *fractional Ramsey numbers* by Jacobson, Levin and Scheinerman [14], as well as by Scheinerman and Ullman [22].

1.2 Outline

In Section 2, we define formally the Ramsey number for set-coloring $R_{s,t}(k)$ as well as the related objects in our study and state the main results of the thesis. We then estimate general bounds for $R_{s,t}(k)$ in Section 3 and compute bounds for some concrete small cases in Section 4. In Sections 5 and 6, we study the connection between (s, t) Ramsey numbers and specific block designs and Hadamard matrices. Section 7 investigates the upper bounds of $R_{s,t}(3)$. In Section 8, we introduce the off-diagonal version of Ramsey numbers for set-colorings and estimate its bounds. Section 9 summarizes the results of the previous sections and poses some open problems for further study.

2 Definitions and main results

We first define formally the Ramsey number for set-colorings and related terms.

2.1 Definitions

In the following, let s, t, n, k be positive integers such that $s \geq t$, $n \geq k$ and let G be any non-empty graph. We call c an (s,t)-*coloring of* G if c is an edge coloring of G where each edge is colored with t distinct colors from $[s]$. More formally, c is a function

$$c : E(G) \to \binom{[s]}{t},$$

where $\binom{X}{k} := \{S \subseteq X : |S| = k\}$. We also call an (s,t)-coloring a *set-coloring.*

Let c be an (s,t)-coloring of G. A set of edges $E \subseteq E(G)$ is *monochromatic* if there is some color $i \in [s]$ such that $i \in c(e)$ for all edges $e \in E$. A set of vertices $S \subseteq V(G)$ is *monochromatic* if the set of edges $\{uv : u, v \in S, u \neq v\}$ is monochromatic. A graph G is called *monochromatic* if $V(G)$ is monochromatic. We denote by $R_{s,t}(k)$ the least $n \in \mathbb{N}$ such that for any (s,t)-coloring of K_n, there exists a monochromatic K_k. We call $R_{s,t}(k)$ the *Ramsey number for set-coloring*, or for short the (s,t) Ramsey number. We will prove in Proposition 3.2 that $R_{s,t}(k)$ is well-defined.

If c is an (s,t)-coloring of K_n that has no monochromatic K_k, we say that c *witnesses a lower bound on* $R_{s,t}(k)$ or c is a *witness coloring to the lower bound* $R_{s,t}(k) > n$. To prove a lower bound $n < R_{s,t}(k)$, it is sometimes useful to construct a coloring of K_n where each edge does not necessarily have exactly t colors but at least t of them. We now define c as an (s,t^+)-*coloring* of G if c is an edge coloring of G, where each edge has at least t colors from $[s]$, i.e. c is a function

$$c : E(G) \to \binom{[s]}{t^+},$$

where $\binom{X}{k^+} := \{S \subseteq X : |S| \geq k\}$. Clearly if c is an (s,t) coloring of K_n, then c is also an (s,t^+)-coloring of K_n.

Let c be an (s,t^+)-coloring of G. For any edge $e \in E(G)$, we call the set $c(e)$ the *color set* or *color combination* of e, and define $c(G) := \{c(e) : e \in E(G)\}$. For $i \in [s]$ and any vertex $u \in V(G)$, $N_i(u)$ denotes the set of neighbors of u which are connected to u by an edge with color i, i.e.

$$N_i(u) := \{v \in N(u) : i \in c(uv)\}.$$

Let c be an (s,t^+)-coloring of G. For $i \in [s]$, let G_i be a subgraph of G with vertex set $V(G)$ and edges with color i, that means for vertices $u, v \in V(G)$, $uv \in E(G_i)$ if and only if $i \in c(uv)$. We call G_i the *color graph* of c (*in color* i). Observe that for all colors $i \in [s]$, c has a monochromatic K_k in color i if and only if G_i contains K_k as a subgraph.

In this thesis, we identify colors $1, 2, 3, 4$ with *red, blue, green* and *orange*, respectively.

Remark 2.1. $R_{2,1}(k)$ is the classical Ramsey number and $R_{s,1}(k)$ with $s \geq 2$ is the classical multicolor Ramsey number.

Example 2.2. Figure 1 shows different ways to demonstrate a $(4, 2)$-coloring of K_4. Two distinct colors are chosen from $\{1, 2, 3, 4\}$ to color each edge. In this coloring there is no monochromatic triangle. In particular, $R_{4,2}(3) > 4$. For vertex u_1, the red neighborhood is $N_1(u_1) = \{u_2, u_4\}$, the blue neighborhood is $N_2(u_1) = \{u_2\}$, further the green neighborhood is $N_3(u_1) = \{u_3, u_4\}$ and the orange neighborhood is $N_4(u_1) = \{u_3\}$.

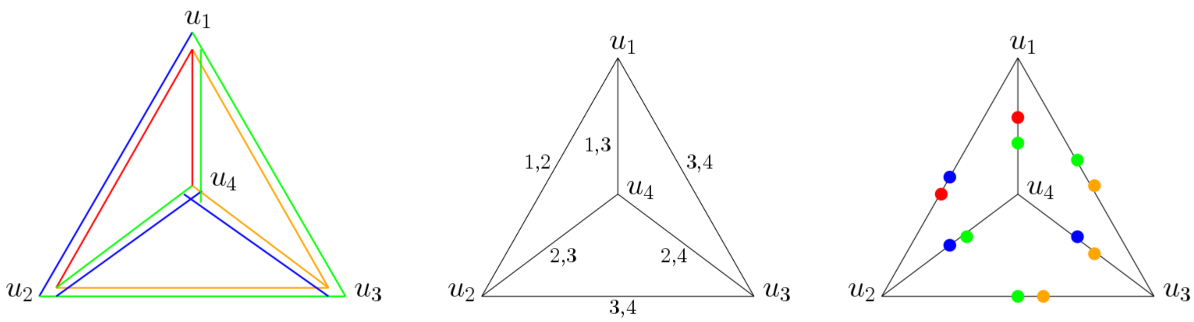


Figure 1: A $(4, 2)$-coloring with no monochromatic K_3

Figure 2 shows another $(4, 2)$-coloring of K_4, which contains a monochromatic K_3: $4 \in c(u_1u_2) \cap c(u_1u_3) \cap c(u_2u_3)$. The color graph G_4 has a triangle $u_1u_2u_3$.

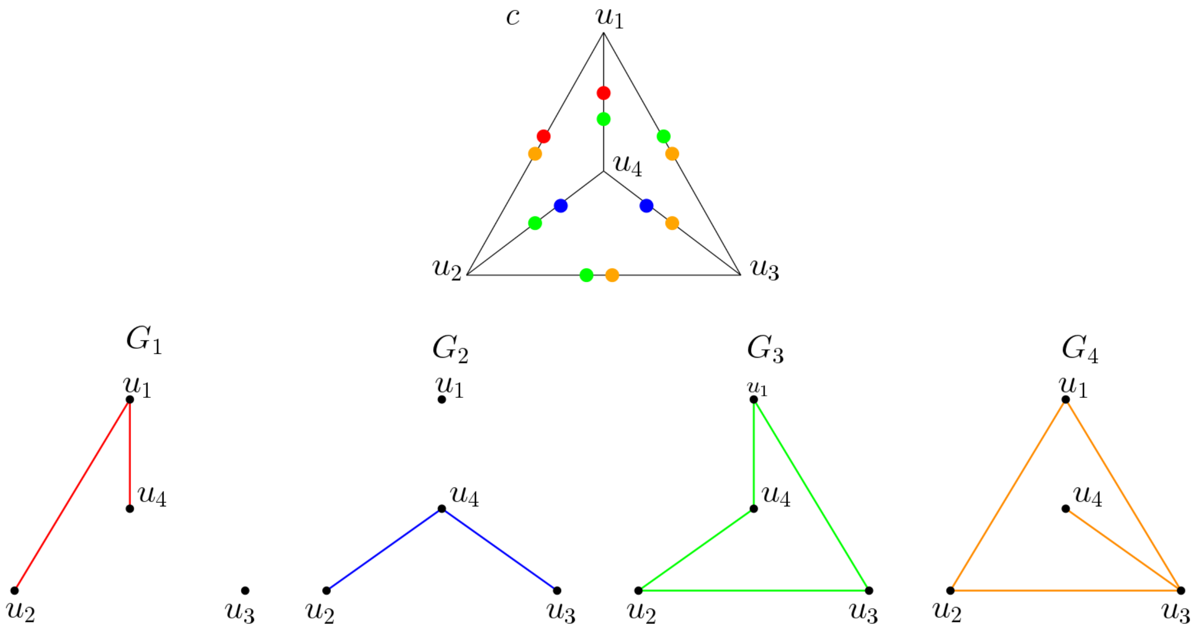


Figure 2: A $(4,2)$-coloring with color graphs G_1, G_2, G_3, G_4.

2.2 Main results

For all $s, t, k \in \mathbb{N}$ with $s > t, k \geq 3$, we have bounds for $R_{s,t}(k)$:

$$\left\lceil \frac{k}{e} \cdot s^{-\frac{1}{k}} \cdot \left(\frac{s}{t}\right)^{\frac{k-1}{2}} \right\rceil \leq R_{s,t}(k) \leq 2 - \frac{s}{s-t} + C_{s,t} \cdot \left(\frac{s}{t}\right)^{sk},$$

where $C_{s,t} = \frac{s}{s-t}\left(\frac{t}{s}\right)^{2s}$.

For all $n \in \mathbb{N}$, if there exists a Hadamard matrix of order $2n$, then we have a better lower bound for $R_{2n,n}(3)$:

$$R_{2n,n}(3) > 4n.$$

For all $v, k, \lambda \in \mathbb{N}$ such that $v > k \geq 2$, if there exists a resolvable (v, k, λ) design, then for $s = \frac{\lambda(v-1)}{k-1}$,

$$R_{s,s-\lambda}\left(\frac{v}{k} + 1\right) = v + 1.$$

For definitions of (resolvable) designs and Hadamard matrices, see Sections 5 and 6. The existence of a resolvable design gives the exact values of some $R_{s,t}(k)$. However, as for Hadamard matrices, this method is only applicable to some specific parameters s, t, k.

3 General bounds

In this section we compute general bounds for $R_{s,t}(k)$.

3.1 Basic bounds

We first consider $(s, 1)$-colorings, or s-colorings. We remind the reader that $R_{s,1}(k) = R_s(k)$ is the classical multicolor Ramsey number and again refer to Graham, Rothschild and Spencer [11] for further information.

Proposition 3.1. *For all positive integers s, k such that $s \geq 2$,*

$$R_{s,1}(k) = R_s(k) \leq s^{sk}.$$

Proof. Let $n := s^{sk}$ and c any s-coloring of K_n. We show that there exists a monochromatic K_k in c.

Let $V := V(K_n)$ and fix a $v_1 \in V$. By the pigeonhole principle, there are $\lceil \frac{n-1}{s} \rceil = \left\lceil \frac{s^{sk}-1}{s} \right\rceil = s^{sk-1}$ edges at v with a common color, say $i_1 \in [s]$. Let $V_1 \subseteq N_{i_1}(v_1), |V_1| = s^{sk-1}$. Similarly, let $v_2 \in V_1$, then there is some color $i_2 \in [s]$ and set $V_2 \subset V_1$ with $|V_2| = \frac{|V_1|}{s} = s^{sk-2}$ such that all edges between v_2 and V_2 have color i_2. In the same manner, we define a sequence of vertices v_i and sets V_i, $i \in [sk]$, $i \geq 2$, as follows: $v_i \in V_{i-1}, V_i \subset V_{i-1}, |V_i| = s^{sk-i}$ such that all edges between v_i and V_i have the same color. Then for any $i, j, l \in [sk]$ with $j, l > i$ we have that $c(v_i v_j) = c(v_i v_l)$. By the pigeonhole principle, there are at least k vertices from $\{v_i : i \in [sk]\}$ that send the same color to the vertices in $\{v_i : i \in [sk]\}$ with higher index. By construction, these k vertices form a monochromatic set in c, which completes the proof of the proposition. $\square$

Proposition 3.2. *The Ramsey number $R_{s,t}(k)$ is well defined, i.e. for any positive integers $s, t, k, s \geq t$, there exists a positive integer n such that any (s,t)-coloring of K_n contains a monochromatic K_k.*

Proof. In an (s, t)-coloring of any complete graph, there are $\binom{s}{t}$ possible ways to color each edge. We can consider each possible color combination as a single color and apply Proposition 3.1. We have

$$R_{s,t}(k) \leq R_{\binom{s}{t}}(k) \leq \binom{s}{t}^{k\binom{s}{t}},$$

which proves the existence of $R_{s,t}(k)$. $\square$

Remark 3.3. For this upper bound $n = \binom{s}{t}^{k\binom{s}{t}}$ and an (s, t)-coloring of K_n, we not only find a monochromatic K_k in some color $i \in [s]$, but a clique of size k whose all edges have the same color combination, i.e. a K_k that is monochromatic in t colors.

It is clear that $R_{s,t}(k) \leq R_{s,t}(l)$ if $k \leq l$. We now consider how the number $R_{s,t}(k)$ behaves if we change the parameters s and t.

Proposition 3.4. *For all positive integers $s, t, k, s \geq t$, we have*

(i) $R_{s,t}(k) \leq R_{s+1,t}(k)$.

(ii) If $t \geq 2$ then $R_{s,t}(k) \leq R_{s-1,t-1}(k)$.

(iii) If $t \geq 2$ then $R_{s,t}(k) \leq R_{s,t-1}(k)$.

Proof. Let n be a lower bound of $R_{s,t}(k)$ for some $n \in \mathbb{N}$, and c be an (s,t)-coloring of K_n without monochromatic K_k.

Since an (s,t)-coloring of K_n is also an $(s+1,t)$-coloring of K_n, c is an $(s+1,t)$-coloring of K_n avoiding monochromatic K_k. Thus $R_{s+1,t}(k) > n$, or n is also a lower bound on $R_{s+1,t}(k)$, which proves Item (i).

For Item (ii), we define a new coloring c' of K_n from c by removing one color on each edge as follows. For any edge $uv \in E(K_n)$ such that $s \in c(uv)$, let $c'(uv) = c(uv) \setminus \{s\}$. For any other edge, i.e. for all $uv \in E(K_n)$ such that $s \notin c(uv)$, let $c'(uv) = c(uv) \setminus \{\min\{c(uv)\}\}$. Now the new coloring c' still has no monochromatic K_k and is an $(s-1, t-1)$-coloring of K_n. Thus $R_{s-1,t-1}(k) > n$, proving (ii).

Similarly, for Item (iii), we define a new coloring c' from c as follows: for any edge $uv \in E(K_n), c'(uv) = c(uv) \setminus \{\min\{c(uv)\}\}$. Then c' still has no monochromatic K_k and is an $(s, t-1)$-coloring of K_n. Hence we have $R_{s,t-1}(k) > n$ and $R_{s,t}(k) \leq R_{s,t-1}(k)$.

Alternatively we can prove Item (iii) by applying Items (i) and (ii). Item (i) implies that $R_{s,t}(k) \leq R_{s+1,t}(k)$, and by Item (ii), $R_{s+1,t}(k) \leq R_{s,t-1}(k)$, hence $R_{s,t}(k) \leq R_{s,t-1}(k)$. □

Corollary 3.5. *For all positive integers $s, t, k, s \geq t$,*

$$R_{s,t}(k) \leq R_{s,1}(k) = R_s(k).$$

Proof. We apply Proposition 3.4 (iii) multiple times and obtain:

$$R_{s,t}(k) \leq R_{s,t-1}(k) \leq R_{s,t-2}(k) \leq \cdots \leq R_{s,t-1}(k).$$

□

Another way to prove Corollary 3.5 is to replace the color set of each edge with only one color, in order to create an $(s,1)$-coloring. Let $n \in \mathbb{N}$ and c be an (s,t)-coloring of K_n without monochromatic K_k. We define c' as an s-coloring of K_n as follows: For any edge $uv \in E(K_n)$, let $c'(uv) = \max\{c(uv)\}$. Then c' also has no monochromatic K_k, hence $R_{s,1}(k) > n$ and $R_{s,1}(k) \geq R_{s,t}(k)$.

Corollary 3.6. *For all positive integers s, t, k with $s \geq t$,*

$$R_{s,t}(k) \leq s^{sk}.$$

Proof. By applying Corollary 3.5 and 3.1 we have $R_{s,t}(k) \leq R_{s,1}(k) = R_s(k) \leq s^{sk}$. □

Corollary 3.7. *For all positive integers s, t, k with $s \geq t$,*

$$R_{s,t}(k) \leq (s-t+1)^{k(s-t+1)}.$$

If $s = 2t$ then

$$R_{2t,t}(k) \leq (t+1)^{k(t+1)}.$$

We note that this upper bound on $R_{s,t}(k)$ is an improvement over the bounds in Propositions 3.2 and 3.6.

Proof. We apply Proposition 3.4 (ii) multiple times and obtain

$$R_{s,t}(k) \leq R_{s-1,t-1}(k) \leq \cdots \leq R_{s-t+1,1}(k).$$

The statement now follows from Proposition 3.1. □

Starting from an (s,t)-coloring of K_n without monochromatic K_k, we can construct an (ms, mt)-coloring of K_n, for a factor $m \in \mathbb{N}$, with the same property. The following proposition shows that a lower bound for $R_{s,t}(k)$ is also a lower bound for $R_{ms,mt}(k)$.

Proposition 3.8. *For all positive integers s, t, m with $s \geq t$,*

$$R_{s,t}(k) \leq R_{ms,mt}(k).$$

Proof. Suppose c is an (s,t)-coloring of K_n without monochromatic K_k, then $R_{s,t}(k) > n$. We construct from c an (ms, mt)-coloring c' of K_n without monochromatic K_k in order to prove $R_{ms,mt}(k) > n$, which then implies $R_{s,t}(k) \leq R_{ms,mt}(k)$.

Let $e \in E(K_n)$ be an arbitrary edge and write $c(e) = \{c_1, c_2, \ldots, c_t\}$ where $c_i \in [s]$ and pairwise distinct for all $i \in [t]$. Now let $c'(e)$ be defined from $c(e)$ as

$$\begin{aligned} c'(e) =& \{m(c_1-1)+1, \ldots, m(c_1-1)+m,\ m(c_2-1)+1, \ldots, m(c_2-1)+m, \\ & \ldots, m(c_t-1)+1, \ldots, m(c_t-1)+m\} \\ =& \{m(c_i-1)+j :\ i \in [t],\ j \in [m]\}. \end{aligned}$$

We prove that c' is an (ms, mt)-coloring of K_n that contains no monochromatic K_k.

First note that any color of $c'(e)$ has the form $m(c_i-1)+j$ for some $i \in [t]$ and $j \in [m]$. Since $c_i \in [s]$ for all $i \in [t]$, we have

$$m(1-1)+1 \leq m(c_i-1)+j \leq m(s-1)+m,$$

hence $1 \leq m(c_i - 1) + j \leq ms$. Therefore there are ms possible colors for c'. Next, we show that any two colors of $c'(e)$ are distinct. Consider for $c'(e)$ any two colors $m(c_i - 1) + k$ and $m(c_j - 1) + l$ where $i, j \in [t]$ are distinct and $1 \leq k, l \leq m$. By the definition of c, $c_i \neq c_j$. Without loss of generality, suppose that $c_j = c_i + p$ for some $p \in [s-1]$. Then

$$\begin{aligned} m(c_j - 1) + l &= m(c_i + p - 1) + l \\ &= m(c_i - 1) + mp + l \\ &\geq m(c_i - 1) + m + 1 \\ &> m(c_i - 1) + k. \end{aligned}$$

In particular, $m(c_j - 1) + 1 \neq m(c_i - 1) + k$, i.e any two colors of $c'(e)$ are distinct. Thus $|c'(e)| = t$ for any edge $e \in E(K_n)$ and c' is an (ms, mt)-coloring of K_n.

Note that for an edge $e \in E(K_n)$, if $r \in c'(e)$ for some color $r \in [ms]$, then there exist positive integers $i \in [s]$, $j \in [m]$ such that $r = m(i-1) + j$ and $i \in c(e)$.

Now assume for the sake of contradiction that c' has a monochromatic $G \cong K_k$ in color r for some $r \in [ms]$, that is $r \in c'(e)$ for all edges $e \in E(G)$. Then there exist positive integers $i \in [s]$, $j \in [m]$ such that $r = m(i-1) + j$ and $i \in c(e)$ for all edges $e \in E(G)$. It follows that c has a monochromatic K_k in color i, which is a contradiction to the assumption of c. Therefore, c' has no monochromatic K_k, which proves $R_{ms,mt}(k) > n$. $\square$

As a result we have the classical Ramsey number as a lower bound for $R_{2m,m}(k)$ by applying Proposition 3.8 with $s = 2, t = 1$.

Corollary 3.9. *For all $m, k \in \mathbb{N}$,*

$$R(k) = R_{2,1}(k) \leq R_{2m,m}(k).$$

For classical Ramsey numbers we have the following comparison.

Corollary 3.10. *For all $k \in \mathbb{N}$,*

$$R_{3,2}(k) \leq R_{2,1}(k) = R(k) \leq R_{4,2}(k).$$

Proof. Corollary 3.9 with $m = 2$ implies $R_{2,1}(k) \leq R_{4,2}(k)$. The inequality $R_{3,2}(k) \leq R_{2,1}(k)$ follows from Proposition 3.4 (ii). $\square$

Sometimes it is easier to construct an (s, t^+)-coloring of K_n without monochromatic K_k than an exact (s,t)-coloring. We show that such an (s, t^+)-coloring suffices to prove a lower bound $R_{s,t}(k) > n$.

Proposition 3.11. *For all positive integers s, t, n, k with $s \geq t$, suppose c is an (s, t^+)-coloring of K_n without monochromatic K_k. Then $R_{s,t}(k) > n$.*

Proof. For any edge uv of K_n that has more than t colors, we delete colors on uv such that uv has t colors left. Formally, for any $e \in E(K_n)$ with $c(e) = \{i_1, \ldots, i_{t+r}\} \subseteq [s]$ for $1 \leq r \leq s - t$, let $c'(e) := \{i_1, \ldots, i_t\}$. For all edges $e \in E(K_n)$ with $|c(e)| = t$, let $c'(e) := c(e)$. Then c' is an (s,t)-coloring of K_n without monochromatic K_k. Therefore $R_{s,t}(k) > n$. $\square$

3.2 Lower bounds by probabilistic method

In 1947, Erdős [8] introduced the probabilistic method in proving a lower bound for $R(k)$:

$$R(k) \geq 2^{k/2}.$$

Spencer [23] used Lováz Local Theorem with the probabilistic method to attain a slightly better lower bound:

$$R(k) \geq k \cdot 2^{k/2}[(\sqrt{2}/e) + o(1)].$$

For an overview of the probabilistic method, see Alon [1]. Now we use the probabilistic method to determine a lower bound for $R_{s,t}(k)$.

Theorem 3.12. *For all $s, t, k, n \in \mathbb{N}$ such that $s \geq t, k \geq 3$ and $n = \left\lfloor \frac{k}{e} \cdot s^{-\frac{1}{k}} \cdot \binom{s}{t}^{\frac{k-1}{2}} \right\rfloor$,*

$$R_{s,t}(k) > n,$$

i.e. there exists an (s,t)-coloring of K_n without monochromatic K_k.

Note that $R_{s,t}(k) > n$ also means $R_{s,t}(k) \geq \left\lceil \frac{k}{e} \cdot s^{-\frac{1}{k}} \cdot \binom{s}{t}^{\frac{k-1}{2}} \right\rceil$.

Proof. We modify the proof of Graham, Rothschild and Spencer in [11] for (s,t)-colorings. We color edges of K_n uniformly and independently at random, each edge gets a t-set from $[s]$ with probability $1/\binom{s}{t}$. Consider a set U of k vertices. If U is monochromatic in some color $i \in [s]$, then each edge in U has $(t-1)$ remaining colors that can be chosen from $[s] \setminus \{i\}$. Therefore, for any edge e induced by U, we have

$$\mathbb{P}[i \in c(e)] = \frac{\binom{s-1}{t-1}}{\binom{s}{t}} = \frac{t}{s}.$$

Thus the probability that U induces a clique whose each edge contains color i is $\left(\frac{t}{s}\right)^{\binom{k}{2}}$, since U has $\binom{k}{2}$ edges. Moreover, there are s possibilities to choose a color that all edges in U have, and $\binom{n}{k}$ possible cliques of order k in K_n that can be monochromatic. Let A

denote the event "There is some monochromatic K_k", then the probability of the event A is

$$\begin{aligned}\mathbb{P}[A] &= \mathbb{P}[\text{There is some monochromatic } K_k] \\ &= \mathbb{P}[\bigcup_{\substack{U\subseteq V(K_n)\\|U|=k}} U \text{ is monochromatic}] \\ &\leq \sum_{\substack{U\subseteq V(K_n)\\|U|=k}} \mathbb{P}[U \text{ is monochromatic}] \\ &= s\binom{n}{k}\left(\frac{t}{s}\right)^{\binom{k}{2}}.\end{aligned}$$

By the assumption, $n < \frac{k}{e} \cdot s^{-\frac{1}{k}} \cdot \left(\frac{s}{t}\right)^{\frac{k-1}{2}}$, which implies

$$s^{\frac{1}{k}} \cdot \frac{en}{k}\left(\frac{t}{s}\right)^{\frac{k-1}{2}} < 1.$$

It follows that

$$s\left(\frac{en}{k}\right)^{k}\left(\frac{t}{s}\right)^{\frac{k(k-1)}{2}} < 1.$$

On the other hand, $\binom{n}{k} \leq \frac{n^k}{k!} \leq \left(\frac{en}{k}\right)^k$, thus

$$\mathbb{P}[A] \leq s\binom{n}{k}\left(\frac{t}{s}\right)^{\binom{k}{2}} \leq s\left(\frac{en}{k}\right)^{k}\left(\frac{t}{s}\right)^{\frac{k(k-1)}{2}} < 1.$$

Therefore $\mathbb{P}[A^C] > 0$, and there must be some (s,t)-coloring of K_n that has no monochromatic K_k. □

We apply Theorem 3.12 for $s = mt$.

Corollary 3.13. *For all $m, t, k \in \mathbb{N}$,*

$$R_{mt,t}(k) > \left\lfloor \frac{k}{e} \cdot (mt)^{-1/k} \cdot m^{\frac{k-1}{2}} \right\rfloor.$$

For $s = m, t = 1$, we obtain the following well-known lower bound on $R_m(k)$.

Corollary 3.14. *For all $m, k \in \mathbb{N}$,*

$$R_{m,1}(k) = R_m(k) > \left\lfloor \frac{k}{e} \cdot m^{-\frac{1}{k}} \cdot m^{\frac{k-1}{2}} \right\rfloor.$$

Remark 3.15. By Proposition 3.8, for all $m, t, k \in \mathbb{N}$, we have $R_{mt,t}(k) > R_{m,1}(k)$ and hence

$$R_{mt,t}(k) > \left\lfloor \frac{k}{e} \cdot m^{-\frac{1}{k}} \cdot m^{\frac{k-1}{2}} \right\rfloor.$$

This is an improvement over the bound in Corollary 3.13.

4 Concrete cases

A general method to prove a lower bound $n < R_{s,t}(k)$ is to construct an (s,t) or (s,t^+)-coloring of K_n whose color graphs are the Tuán graphs $T_{k-1}(n)$. We will develop this concept in the next sections. First we consider $R_{s,t}(k)$ for some small parameters s,t,k.

Let $s,t,k \in \mathbb{N}$, some trivial cases are: If $s = t$, then $R_{t,t}(k) = k$, and for $k = 2$, $R_{s,t}(2) = 2$. For $t = 1$, we get the classical (multicolor) Ramsey numbers, so we only consider cases where $t > 1$ here. The next nontrivial case is then $R_{3,2}(3)$.

4.1 Bounds for $R_{3,2}(3)$ and $R_{4,2}(3)$

In a $(3,2)$-coloring of a triangle, there are three possibilities to color an edge, with $\{1,2\}$, $\{1,3\}$, or $\{2,3\}$. We show that if any two edges in a triangle get the same color set, then the triangle is monochromatic.

Proposition 4.1. *The only $(3,2)$-coloring of K_3 without monochromatic K_3 has all 3 possible color combinations on 3 edges, $c(K_3) = \{\{1,2\},\{1,3\},\{2,3\}\}$ (see Figure 3).*

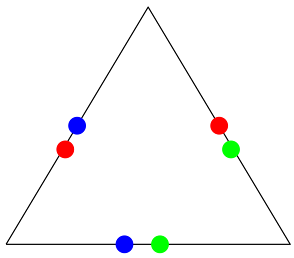


Figure 3: $(3,2)$-coloring avoiding monochromatic K_3

Proof. Let $i,j,k \in \{1,2,3\}$ pairwise distinct. Let c be a $(3,2)$-coloring of a triangle $u_1u_2u_3$ and suppose that two edges have the same color combination. Without loss of generality, $c(u_1u_2) = c(u_1u_3) = \{i,j\}$. If $c(u_2u_3) \in \{\{i,j\},\{i,k\}\}$, then $u_1u_2u_3$ is monochromatic in color i. If $c(u_2u_3) = \{j,k\}$, then the triangle is monochromatic in color j, which proves the proposition. $\square$

Proposition 4.2. *We have that $R_{3,2}(3) = 5$. Moreover, the witness coloring to the lower bound $R_{3,2}(3) > 4$ is unique up to permutation of colors and is given in Figure 4.*

We remark that in this unique $(3,2)$-coloring (up to permutation of colors) of K_4 without monochromatic K_3, any two non-adjacent edges have the same color set.

Proof. Figure 4 illustrates a $(3,2)$-coloring of K_4 avoiding monochromatic triangles, thus shows the lower bound $R_{3,2}(3) > 4$. In this coloring, any triangle in K_4 has all three possible color sets: $\{\{1,2\},\{1,3\},\{2,3\}\}$. By Proposition 4.1, this is the only $(3,2)$-coloring of K_4 that has no monochromatic K_3.

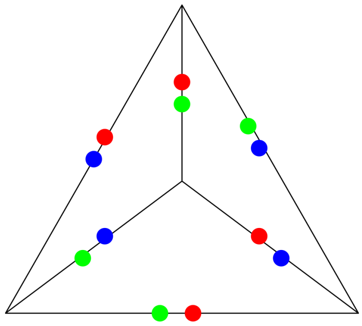


Figure 4: Witness coloring to the lower bound on $R_{3,2}(3)$ of K_4

For the upper bound, fix some vertex $u \in V(K_5)$. By Proposition 4.1, to avoid a monochromatic K_3, any two edges at u must have different color combinations. There are 4 edges incident to u, but only 3 possible color combinations: $\{1,2\},\{1,3\},\{2,3\}$. Thus, there must be one color set which appears at least twice. By Proposition 4.1, there is a monochromatic K_3. $\square$

Now we can use Proposition 4.2 to prove an upper bound for $R_{4,2}(3)$. For the lower bound, we construct a $(4,2^+)$-coloring without monochromatic K_3 using Turán graphs.

Theorem 4.3. *We have that* $8 < R_{4,2}(3) \leq 10$.

Proof. To prove the upper bound, let c be a $(4,2)$-coloring of $G \cong K_{10}$. We show that c has a monochromatic triangle. Fix some vertex $u \in V(G)$. Then u is incident to 9 edges, each is colored with 2 colors, so we have 18 colors in total. Without loss of generality, we assume that at least $\lceil \frac{18}{4} \rceil = 5$ edges at u have color 1. Note that for this proof, we just ignore the second color on those edges. Let $V := \{v_1, v_2, v_3, v_4, v_5\} \subset V(G)$, such that $V \subseteq N_1(u)$, i.e. for all $v \in V, 1 \in c(uv)$. Let $H = G[V]$ be the induced subgraph on V, note that $H \cong K_5$. If any edge in H has color 1, i.e. there exist $i, j \in [5]$ such that $1 \in c(v_i v_j)$, then u, v_i, v_j will form a monochromatic triangle in color 1. Otherwise, if color 1 is not used on any edge of H, then we have a $(3,2)$-coloring of H. By Proposition 4.2, there is one monochromatic triangle in H, and so in G.

G_1		G_2		G_3		G_4	
L_1	R_1	L_2	R_2	L_3	R_3	L_4	R_4
u_1	v_1	u_1	v_1	u_1	v_1	u_1	v_1
u_2	v_2	v_2	u_2	v_2	u_2	u_2	v_2
u_3	v_3	v_3	u_3	u_3	v_3	v_3	u_3
u_4	v_4	u_4	v_4	v_4	u_4	v_4	u_4

Table 1: Construction of $(4,2)$-coloring of K_8 without monochromatic K_3

For the lower bound, we construct a $(4,2^+)$-coloring c of K_8 where each edge of K_8 gets at least 2 colors. Label the vertices of K_8 as $V(K_8) = \{u_1, v_1, \ldots, u_4, v_4\}$. We first define

$G_k = K_{L_k,R_k}$, $k \in [4]$ as the complete bipartite graphs with bipartitions L_k, R_k. The elements of L_k, R_k are shown in Table 1.

Now let $c : E(K_8) \to 2^{[4]}$ be the coloring with color graphs G_k, $k \in [4]$. That means, for any edge $e \in E(K_8)$,

$$c(e) = \{k \in [4] : e \in E(G_k)\}.$$

We will show that any edge in K_8 has at least 2 colors, or any edge of K_8 is edge in at least two of the graphs G_k.

In the following let $i, j \in [4]$ be distinct. It is clear that any edge of the form $u_i v_i$ is an edge of all graphs G_k, $k \in [4]$. Other edges have the form $u_i v_j, u_i u_j$ or $v_i v_j$. We claim that these edges are in exactly two of the graphs G_k. In order to prove that, we first define a 4×4 matrix $H = (h_{i,j})_{i,j=1}^4$ as follows: for any $i, k \in [4]$, $h_{i,k} = 1$ if and only if $u_i \in L_k$ and $v_i \in R_k$, otherwise $h_{i,k} = -1$ if and only if u_i, v_i are swapped in G_k, that is if $u_i \in R_k$ and $v_i \in L_k$. Then the matrix H is determined as follows

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}.$$

We observe that $u_i v_j$ is an edge of G_k if and only if u_i and v_j lie in different partitions of G_k, or $h_{i,k} = h_{j,k}$. On the other hand, $u_i u_j \in E(G_k)$ is equivalent to $v_i v_j \in E(G_k)$, both if and only if $h_{i,k} \cdot h_{j,k} = -1$, i.e. u_i and u_j lie in different partitions of G_k.

Let $r_i, i \in [4]$ be the rows of H. Notice that the inner product of any two rows of H is 0, i.e for distinct $i, j \in [4]$,

$$0 = \langle r_i, r_j \rangle = \sum_{\substack{k=1 \\ h_{i,k}=h_{j,k}}}^{4} 1 + \sum_{\substack{k=1 \\ h_{i,k}=h_{j,k}}}^{4} -1,$$

which implies

$$|\{k \in [4] : h_{i,k} = h_{j,k}\}| = |\{k \in [4] : h_{i,k} \cdot h_{j,k} = -1\}| = 2.$$

This means every edge of the form $u_i u_j, v_i v_j, u_i v_j$ where $i, j \in [4]$ distinct, is contained in exactly two of the graphs G_1, G_2, G_3, G_4.

Since all color graphs c are bipartite, c contains no monochromatic K_3. By Proposition 3.11, $R_{4,2}(3) > 8$, which completes the proof of the theorem. □

The following improves Theorem 4.3.

Theorem 4.4. *We have that $R_{4,2}(3) = 9$.*

Proof. The lower bound is given by Theorem 4.3. To prove that $R_{4,2}(3) \leq 9$, let c be a $(4,2)$-coloring of K_9. We assume for the sake of contradiction that c has no monochromatic K_3. For the proof, we will use a result from Proposition 4.2 that in a $(3,2)$-coloring of K_4 without monochromatic K_3, any two non-adjacent edges have the same color set.

First we prove the following statements:

(i) $|N_i(u)| = 4$ for all $u \in V(K_9)$ and $i \in [4]$.

(ii) $|N_i(u) \cap N_j(u)| \leq 2$ for all $u \in V(K_9)$ and $i, j \in [4], i \neq j$.

Let u be a vertex in K_9. There are 8 edges at u which need 16 colors. By the pigeonhole principle, there is one color $i \in [4]$ that appears at least 4 times on those edges, i.e. $|N_i(u)| \geq 4$. On the other hand, since there is no monochromatic triangle in color i, $K_9[N_1(u)]$ has no edge in color i, thus we have a $(3,2)$-coloring of $K_9[N_1(u)]$. If $|N_i(u)| \geq 5$, then Proposition 4.2 implies that $K_9[N_1(u)]$ contains a monochromatic triangle in some color $j \in [4] \setminus \{i\}$, a contradiction. Therefore, $|N_i(u)| \leq 4$, which implies $|N_i(u)| = 4$. Now the remaining colors $[4] \setminus \{i\}$ have to appear 12 times in total on the edges from u to $N(u)$. By the same argument, $|N_j(u)| = 4$ for all $j \in [4]$, which proves Item (i).

To prove (ii), assume that $|N_i(u) \cap N_j(u)| \geq 3$ for some $i, j \in [4], i \neq j$, i.e. there are 3 distinct vertices $x, y, z \in V(K_9)$, such that $c(ux) = c(uy) = c(uz) = \{i, j\}$. The edges xy, yz, xz are not colored with i or j, otherwise there would be a monochromatic triangle in color i or j. Hence, $c(xy) = c(yz) = c(xz) = \{k, l\}$ with $k, l \in [4] \setminus \{i, j\}$. The triangle xyz is then monochromatic in color k and l, a contradiction, which proves Item (ii).

Now let $u_i, i \in [9]$, be the vertices of K_9. We first consider u_1 and claim that without loss of generality the following hold:

(iii) $N_1(u_1) = \{u_2, u_3, u_4, u_5\}$ and $N_1(u_1) \cap N_2(u_1) = \{u_2, u_3\}$.

(iv) $c(u_2u_3) = c(u_4u_5) = \{3, 4\}$.

(v) $c(u_1u_4) = \{1, 3\}$ and $c(u_1u_5) = \{1, 4\}$.

By Item (i), without loss of generality, suppose that $1 \in c(u_1u_i)$ for $i \in \{2, 3, 4, 5\}$. It follows that there is no edge in color 1 in $K_9[\{u_2, u_3, u_4, u_5\}]$. The second color on each edge $\{u_1u_2, u_1u_3, u_1u_4, u_1u_5\}$ is chosen from colors $\{2, 3, 4\}$. There is some color $j \in \{2, 3, 4\}$ which appears at least twice on those edges. Without loss of generality, $j = 2$ and $c(u_1u_2) = c(u_1u_3) = \{1, j\} = \{1, 2\}$. In addition, by Item (ii), $|N_1(u_1) \cap N_2(u_1)| \leq 2$. Then $|N_1(u_1) \cap N_2(u_1)| = 2$ and Item (iii) follows.

Next, we prove Item (iv). Triangle $u_1u_2u_3$ is not monochromatic, hence $c(u_2u_3) = \{3, 4\}$. Since we have a $(3,2)$-coloring in $K_9[\{u_2, u_3, u_4, u_5\}]$ and by Proposition 4.2, edge u_4u_5

has the same color set as u_2u_3, thus $c(u_4u_5) = \{3, 4\}$, proving Item (iv).

Since $|N_i(u) \cap N_j(u)| \leq 2$ by Item (ii), and by Item (iii) we have $N_1(u_1) \cap N_2(u_1) = \{u_2, u_3\}$, it follows that $2 \notin c(u_1u_4) \cup c(u_1u_5)$. We consider triangle $u_1u_4u_5$, since u_1u_4 and u_1u_5 are not colored with 2, $c(u_1u_4), c(u_1u_5) \in \{\{1, 3\}, \{1, 4\}\}$. If $c(u_1u_4) = c(u_1u_5)$, then $u_1u_4u_5$ is monochromatic in color 3 or 4, a contradiction. Hence $c(u_1u_4) \neq c(u_1u_5)$. Without loss of generality, let $c(u_1u_4) = \{1, 3\}$ and $c(u_1u_5) = \{1, 4\}$, and Item (v) follows.

Next we consider $N_2(u_1)$ and prove the following:

(vi) Without loss of generality, $N_2(u_1) = \{u_2, u_3, u_6, u_7\}$.

(vii) $c(u_2u_3) = c(u_6u_7) = \{3, 4\}$

(viii) $c(u_1u_6) = \{2, 3\}, c(u_1u_7) = \{2, 4\}$.

By Item (i), $|N_2(u_1)| = 4$ and since $2 \notin c(u_1u_4) \cup c(u_1u_5)$, $|N_2(u_1) \cap \{u_6, u_7, u_8, u_9\}| = 2$. Without loss of generality, assume $2 \in c(u_1u_6) \cap c(u_1u_7)$. In total we have $N_2(u_1) = \{u_2, u_3, u_6, u_7\}$, which is Item (vi).

Consider $K_9[\{u_2, u_3, u_6, u_7\}]$, by Item (vi) we must have a $(3, 2)$-coloring of K_4 without color 2, Proposition 4.2 implies that $c(u_6u_7) = c(u_2u_3) = \{3, 4\}$, showing Item (vii).

By Items (i) and (ii), we may write $c(u_1u_6) = \{2, i\}$ and $c(u_1u_7) = \{2, j\}$ for some $i, j \in \{3, 4\}$. If $i = j$, then by Item (vii), vertices u_1, u_6, u_7 would form a monochromatic triangle in color i, thus $i \neq j$ or $c(u_1u_6) \neq c(u_1u_7)$. Without loss of generality, $c(u_1u_6) = \{2, 3\}, c(u_1u_7) = \{2, 4\}$, and Item (viii) follows. We now have the situation in Figure 5.

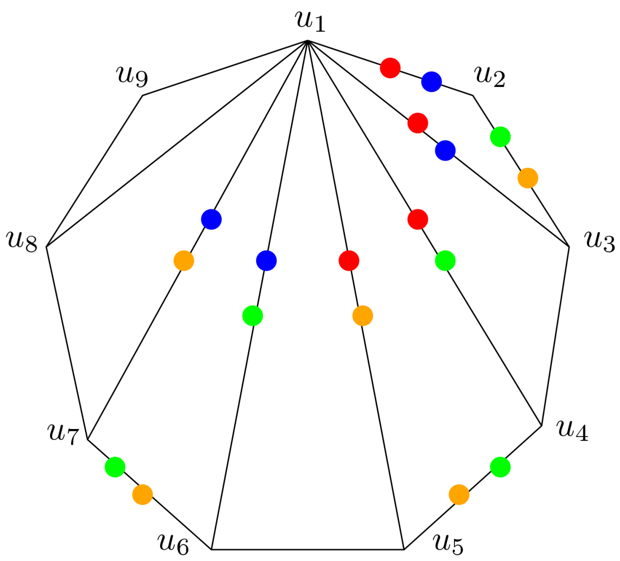


Figure 5: Assumption: $(4, 2)$-coloring of K_9 without monochromatic K_3.

Next we consider u_8, u_9 and prove the following:

(ix) $c(u_1u_8) = c(u_1u_9) = \{3, 4\}$.

(x) $c(u_8u_9) = c(u_4u_6) = c(u_5u_7) = \{1, 2\}$.

By Item (i), we have that $|N_3(u_1)| = |N_4(u_1)| = 4$. On the other hand, by Items (iii) and (vi), $N_1(u_1) \cup N_2(u_1) = \{u_2, \dots, u_7\}$. It follows that edges u_1u_8 and u_1u_9 are not colored with 1 and 2, proving Item (ix) that $c(u_1u_8) = c(u_1u_9) = \{3, 4\}$.

For triangle $u_1u_8u_9$ it follows that $c(u_8u_9) = \{1, 2\}$. Consider $N_3(u_1)$ which is $\{u_4, u_6, u_8, u_9\}$ by (v) and (viii), in $K_9[\{u_4, u_6, u_8, u_9\}]$, c is a $(3, 2)$-coloring with colors $\{1, 2, 4\}$. By Proposition 4.2, $c(u_4u_6) = c(u_8u_9) = \{1, 2\}$. Similarly, for $N_4(u_1) = \{u_5, u_7, u_8, u_9\}$, c is in $K_9[\{u_5, u_7, u_8, u_9\}]$ a $(3, 2)$-coloring with colors $\{1, 2, 3\}$, hence $c(u_5u_7) = c(u_8u_9) = \{1, 2\}$, which shows Item (x).

Finally, we consider the edges at u_4 and prove that

(xi) $c(u_4u_7) = \{1, 3\}$.

We have that $u_1, u_6 \in N_1(u_4)$, and by Item (i), $|N_1(u_4) \cap V(K_9) \setminus \{u_1, u_6\}| = 2$. Since c has no monochromatic triangle in color 1, $u_2, u_3, u_5 \notin N_1(u_4)$. Therefore, $N_1(u_4) \cap \{u_7, u_8, u_9\} = 2$. On the other side, we have $c(u_8u_9) = \{1, 2\}$ by Item (x), the edges u_4u_8 and u_4u_9 cannot both have color 1, which implies that $u_7 \in N_1(u_4)$.

Similarly, $|N_3(u_4) \cap V(K_9) \setminus \{u_1, u_5\}| = 2$. We have $3 \notin c(u_4u_6) \cup c(u_4u_8) \cup c(u_4u_9)$ to avoid monochromatic triangle in color 3. Moreover, by Item (vii), $c(u_2u_3) = \{2, 3\}$, thus $3 \notin c(u_4u_2) \cap c(u_4u_3)$. Therefore, $3 \in c(u_4u_7)$. Altogether we have $c(u_4u_7) = \{1, 3\}$, which is Item (xi).

Now consider u_7, we have $|N_4(u_7) \cap V(K_9) \setminus \{u_1, u_6\}| = 2$. Then $4 \notin c(u_7, u_5) \cup c(u_7u_8) \cup c(u_7u_9)$ to avoid monochromatic triangle in color 4. The color set of edge u_7u_4 is already determined by Item (xi). In particular, $4 \notin c(u_7u_4)$, hence $4 \in c(u_7u_2) \cap c(u_7u_3)$. This is a contradiction, since $u_2u_3u_7$ forms a monochromatic triangle in color 4. This completes the proof of the theorem. $\square$

Following personal communication with Dr. Torsten Ueckerdt [25], we have a shorter proof of Theorem 4.4 using a result of Meringer [18]. Here, the author listed all 4-regular graphs of order 9 (see Figure 6).

Alternative proof of Theorem 4.4. Let c be a $(4, 2)$-coloring of K_9 without monochromatic K_3. By Item (i) above, $|N_i(u)| = 4$ for all vertices $u \in V(K_9)$ and colors $i \in [4]$. Therefore, each color graph G_i, $i \in [4]$, is a 4-regular graphs that contain no triangles. However, all 4-regular graphs on 9 vertices do contain a triangle. Hence, any $(4, 2)$-coloring of K_9 must contain a monochromatic triangle. $\square$

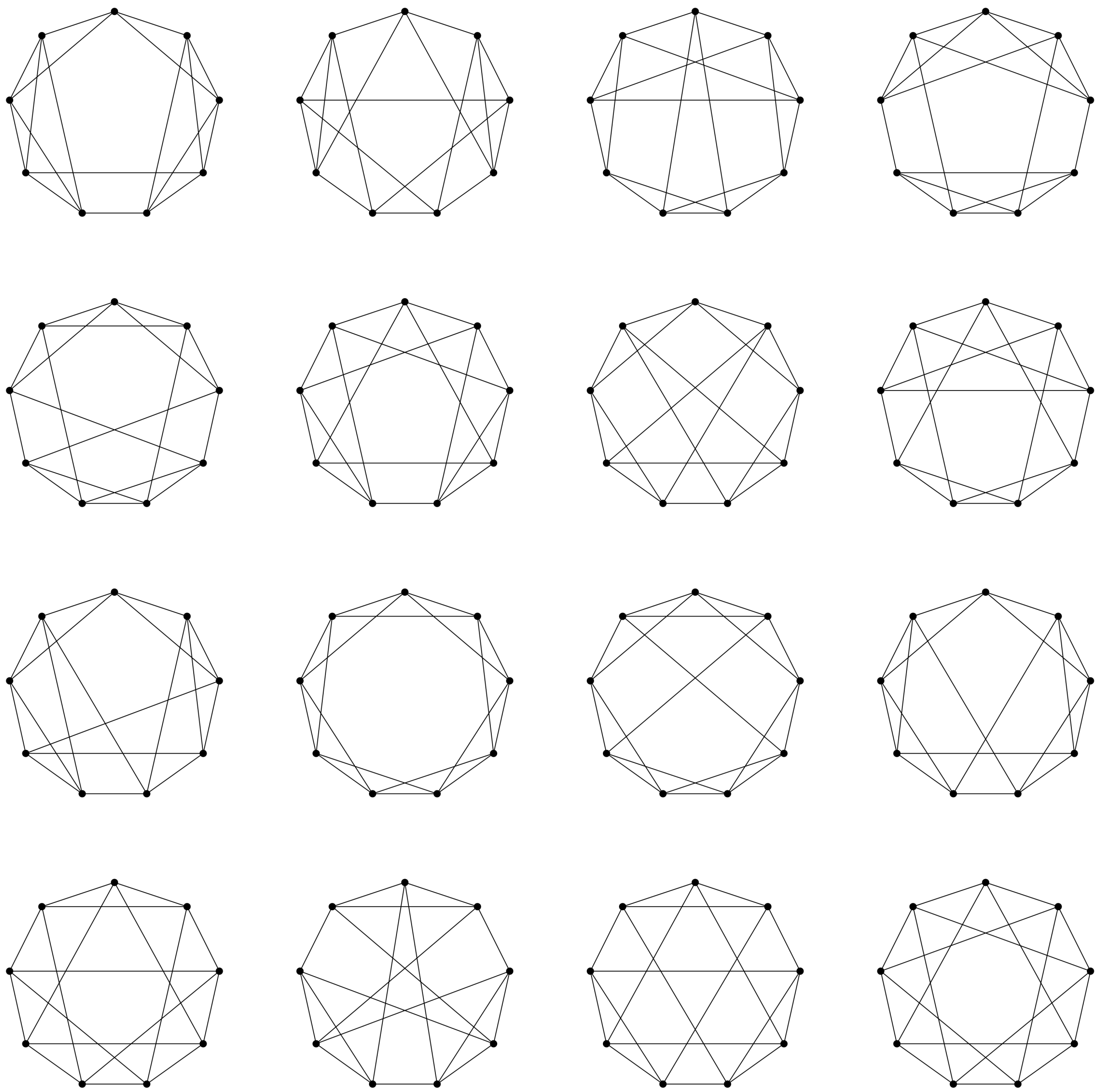


Figure 6: [18] All 4-regular graphs on 9 vertices

At this point, we recall Corollary 3.10 for $k = 3$:

$$R_{3,2}(3) = 5 \leq R(3) = 6 \leq R_{4,2}(3) = 9.$$

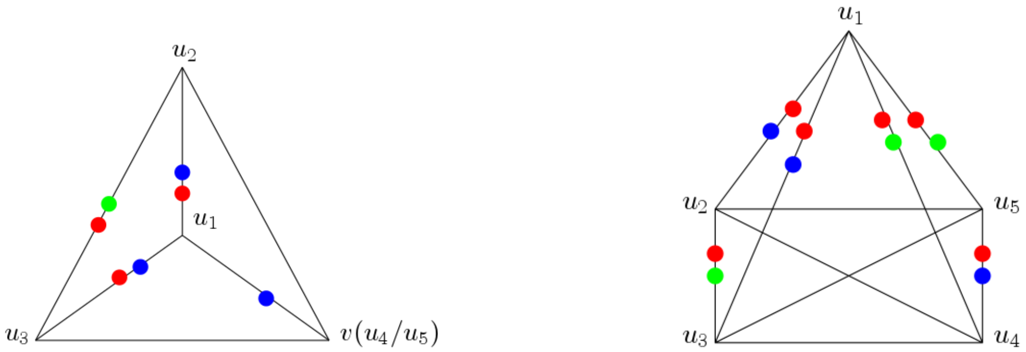
4.2 Bounds for $R_{3,2}(4)$ and $R_{4,3}(4)$

We can find an upper bound for $R_{3,2}(4)$ by using information about $R_{3,2}(3)$. Proposition 4.2 says that any $(3,2)$-coloring of K_5 contains a monochromatic triangle. The next proposition shows that a $(3,2)$-coloring of K_5 either contains a monochromatic K_4, or two monochromatic K_3, each of a distinct color.

Proposition 4.5. *Suppose c is a $(3,2)$-coloring of K_5 that avoids monochromatic K_4. Then there exist distinct $i, j \in [3]$ such that c has a monochromatic K_3 in color i and a monochromatic K_3 in color j.*

Proof. We proceed with a proof by contradiction. Let $u_1, \dots, u_5$ be vertices of K_5. By Proposition 4.2, without loss of generality, the triangle $u_1u_2u_3$ is monochromatic in color 1. By assumption, c has no monochromatic triangle in color 2 or 3. In $u_1u_2u_3$, the color of each edge has the form $\{1, i\}$ for $i \in \{2, 3\}$. Hence, color 2 or 3 must appear at least twice in $u_1u_2u_3$. On the other hand, since c has no monochromatic K_3 of color 2 or 3, color 2 or 3 can appear at most twice in $u_1u_2u_3$. Therefore, $u_1u_2u_3$ has exactly two edges that are colored the same. Without loss of generality, $c(u_1u_2) = c(u_1u_3) = \{1, 2\}$ and $c(u_2u_3) = \{1, 3\}$. Note that u_1 is then incident to 2 edges with color 2. In the following we show that the other two edges at u_1 avoid color 2.

Claim A: $c(u_1u_4) = c(u_1u_5) = \{1, 3\}$ and $c(u_4u_5) = \{1, 2\}$.



(a) Assumption for contradiction of Claim A: $2 \in c(u_1v)$.

(b) Claim A: $c(u_1u_4) = c(u_1u_5) = \{1, 3\}, c(u_4u_5) = \{1, 2\}$.

Figure 7: Claim A: Assumption and result

Let $v \in \{u_4, u_5\}$. For the sake of contradiction, suppose that $2 \in c(u_1v)$ (see Figure 7 (a)). Since u_1u_2v is not monochromatic in color 2, $c(u_2v) = \{1, 3\}$. Similarly, since u_1u_3v is not monochromatic in color 2, $c(u_3v) = \{1, 3\}$. Now we have $c(u_2u_3) = c(u_2v) = c(u_3v) = \{1, 3\}$, a contradiction. Thus, $c(u_1u_4) = c(u_1u_5) = \{1, 3\}$. Since $u_1u_4u_5$ is not monochromatic in color 3, we have $c(u_4u_5) = \{1, 2\}$, which proves Claim A.

Now we have the situation in Figure 7 (b). Next we prove the following.

Claim B: $c(u_2u_4) = c(u_3u_4) = \{2, 3\}$.

By the symmetry of u_2, u_3, without loss of generality, we prove $c(u_3u_4) = \{2, 3\}$. For the sake of contradiction, suppose that $1 \in c(u_3u_4)$ (see Figure 8 (a)). Note that u_3u_4 is edge of two K_4: $\{u_1, u_2, u_3, u_4\}$ and $\{u_1, u_3, u_4, u_5\}$. Both of these K_4 now have 5 edges with color 1. Since each of the induced subgraphs on $\{u_1, u_2, u_3, u_4\}$ and $\{u_1, u_3, u_4, u_5\}$ is not monochromatic in color 1, it must follow that $c(u_2u_4) = c(u_3u_5) = \{2, 3\}$ and we have the situation in Figure 8 (b). On the other hand, the triangle $u_2u_3u_4$ is not monochromatic in color 3, thus $c(u_3u_4) = \{1, 2\}$. Since $u_3u_4u_5$ is now monochromatic in color 2, we have a contradiction to the assumption on c, which completes the proof of Claim B.

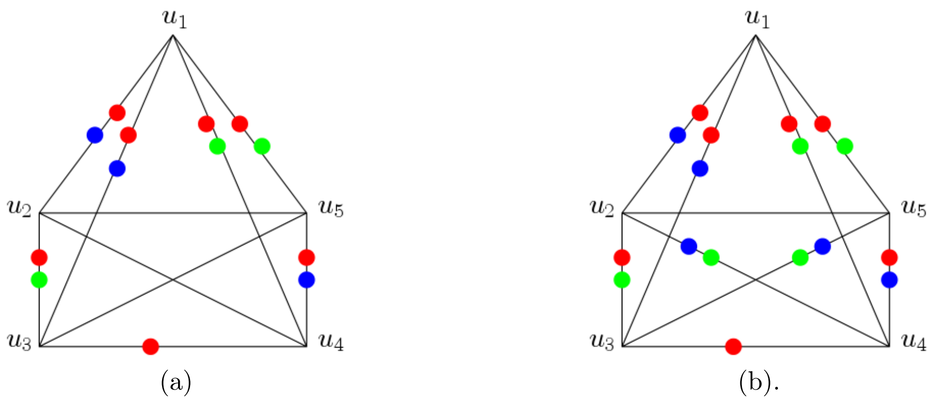


Figure 8: Assumption of Claim B

By Claims A and B, $c(u_2u_3) = \{1, 3\}$ and $c(u_2u_4) = c(u_3u_4) = \{2, 3\}$. Then the triangle $u_2u_3u_4$ has color 3 on all edges, which is a contradiction to our assumption on c. This completes the proof of the proposition. □

Theorem 4.6. *We have that* $9 < R_{3,2}(4) \leq 14$.

Proof. For the upper bound, let $u \in V(K_{14})$, then u is incident to 13 edges. There are 3 possible ways to assign colors to each edge, $\{1, 2\}, \{1, 3\}$ or $\{2, 3\}$. By the pigeonhole principle, there is one color combination that appears at least $\lceil \frac{13}{3} \rceil = 5$ times. Without loss of generality, let $V := \{v_1, v_2, v_3, v_4, v_5\}$ and $c(uv) = \{1, 2\}$ for all $v \in V$. If $K_{14}[V]$ contains a monochromatic K_4, so does K_{14}. If $K_{14}[V]$ has no monochromatic K_4, Proposition 4.5 implies that there are one monochromatic triangle in color i and one in color j, with $i, j \in \{1, 2, 3\}$ and $i \neq j$. The triangle in color i or j forms together with u a monochromatic K_4 in color 1 or 2, proving the upper bound.

For the lower bound, we form a suitable $(3, 2^+)$-coloring c of K_9 by defining complete tripartite graphs G_1, G_2, G_3 as follows. Label the vertices of K_9 as $V(K_9) = V = \{(i, j) : i, j \in \mathbb{Z}_3\}$. For all $i \in \mathbb{Z}_3$, we define the partite sets of G_1, G_2, G_3:

- $A_i := \{(i, j) : j \in \mathbb{Z}_3\}$ and $G_1 := K_{A_o, A_1, A_2}$,
- $B_i := \{(j, i) : j \in \mathbb{Z}_3\}$ and $G_2 := K_{B_0, B_1, B_2}$,
- $C_i := \{(j, k) : j + k \equiv i \pmod 3\}$ and $G_3 := K_{C_0, C_1, C_2}$.

Explicitly, for $i, j, k, l \in \mathbb{Z}_3$ and distinct vertices $(i, j), (k, l) \in V$,

- $\{(i, j), (k, l)\} \in E(G_1)$ if and only if $i \neq k$,
- $\{(i, j), (k, l)\} \in E(G_2)$ if and only if $j \neq l$,
- $\{(i, j), (k, l)\} \in E(G_3)$ if and only if $i + j \not\equiv k + l \pmod 3$.

The structure of the tripartite graphs and their partitions can be seen in Figure 9.

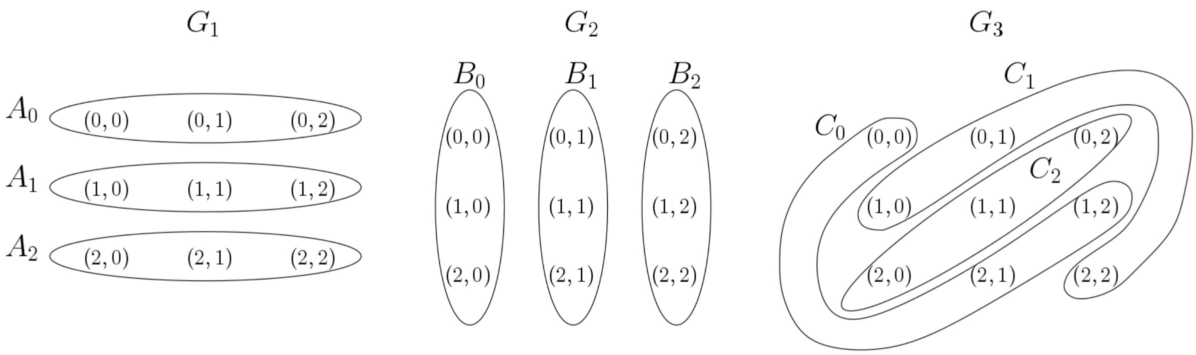


Figure 9: Tripartite graphs G_1, G_2, G_3 with their partitions A_i, B_i, C_i

Let $v := (i, j)$ and $w := (k, l)$ be distinct elements of V, $i, j, k, l \in \mathbb{Z}_3$. We claim that $\{v, w\}$ is an edge of at least two of the graphs G_1, G_2, G_3. Note that for $j, l \in \mathbb{Z}_3$, if $j \neq l$, then $j \not\equiv l \pmod 3$. We consider 4 cases.

Case 1: Suppose $i = k$ and $j \neq l$. By definition, $vw \notin E(G_1)$ and $vw \in E(G_2)$. Moreover, we have $i + j \neq i + l = k + l$, and since $i, j, k, l \in \mathbb{Z}_3$, $i + j \not\equiv k + l \pmod 3$. That means v and w lie in different partite sets of G_3, hence $vw \in E(G_3)$.

Case 2: If $i \neq k$ and $j = l$, then $vw \in E(G_1) \setminus E(G_2)$. Similarly to Case 1, we have $i + j \not\equiv k + l \pmod 3$, thus $vw \in E(G_3)$.

Case 3: Now suppose $i \neq k, j \neq l$ and $i + j \equiv k + l \pmod 3$. Then $vw \in E(G_1) \cap E(G_2)$ and $vw \notin E(G_3)$.

Case 4: Finally, suppose $i \neq k, j \neq l$ and $i + j \not\equiv k + l \pmod 3$. By definition, $vw \in E(G_1) \cap E(G_2) \cap E(G_3)$, which completes the proof of the claim.

We now define c as the coloring with color graphs G_1, G_2, G_3, i.e. for all edges $vw \in E(K_9)$,

$$c(vw) = \{i \in [3] : vw \in E(G_i)\}.$$

Then c is a $(3, 2^+)$-coloring and by construction of G_1, G_2, G_3, c contains no monochromatic K_4. Proposition 3.11 implies that $R_{3,2}(4) > 9$. $\square$

We can construct a $(4,3)$-coloring of K_9 without monochromatic K_4 in a similar way and attain a lower bound for $R_{4,3}(4)$.

Theorem 4.7. *We have that $R_{4,3}(4) > 9$.*

Proof. To build a $(4,3)$-coloring of K_9 without monochromatic K_4, we first define the color graphs G_1, G_2, G_3, G_4 as follows. Let $V := \{(i,j) : i,j \in \mathbb{Z}_3\}$ be the vertex set of K_9 and let G_1, G_2, G_3, G_4 be the complete tripartite graphs with parts A_i, B_i, C_i, D_i respectively for $i \in \mathbb{Z}_3$, where

- $A_i := \{(i,j) : j \in \mathbb{Z}_3\}$,
- $B_i := \{(j,i) : j \in \mathbb{Z}_3\}$,
- $C_i := \{(j,k) : j,k \in \mathbb{Z}_3 \text{ and } j+k \equiv i \pmod 3\}$,
- $D_i := \{(j,k) : j,k \in \mathbb{Z}_3 \text{ and } k-j \equiv i \pmod 3\}$.

Explicitly, for $i,j,k,l \in \mathbb{Z}_3$ and distinct vertices (i,j), $(k,l) \in V$,

- $\{(i,j),(k,l)\} \in E(G_1)$ if and only if $i \neq k$,
- $\{(i,j),(k,l)\} \in E(G_2)$ if and only if $j \neq l$,
- $\{(i,j),(k,l)\} \in E(G_3)$ if and only if $i+j \not\equiv k+l \pmod 3$,
- $\{(i,j),(k,l)\} \in E(G_4)$ if and only if $j-i \not\equiv l-k \pmod 3$.

Figure 10 illustrates the graphs G_1, G_2, G_3, G_4 with their partite sets.

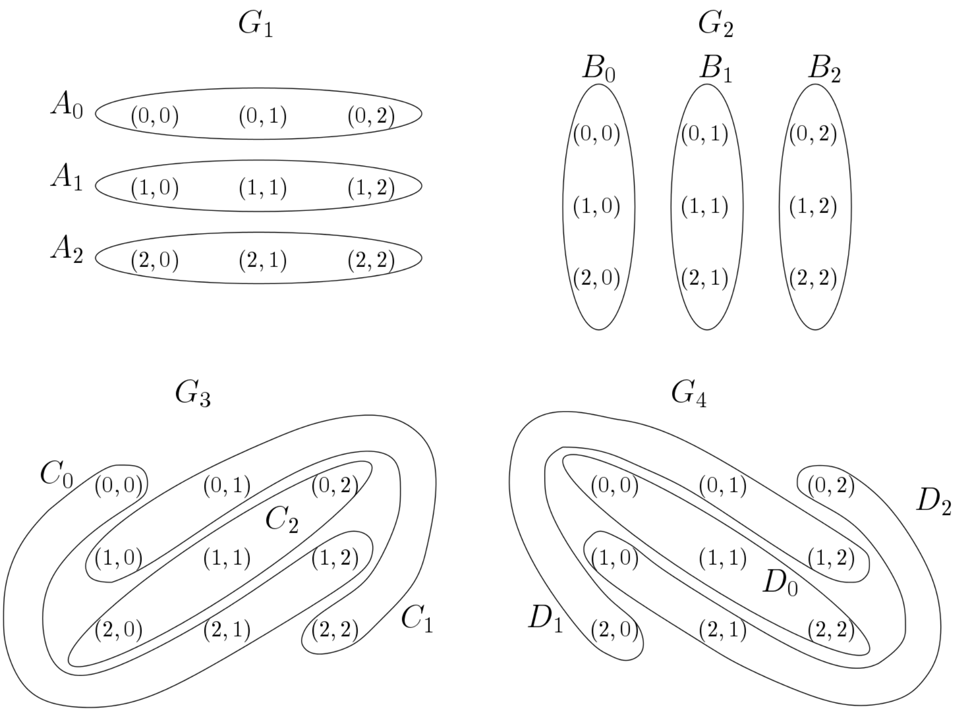


Figure 10: Tripartite graphs G_1, G_2, G_3, G_4 with partitions

Let $i, j, k, l \in \mathbb{Z}_3$ and $v := (i, j)$ and $w := (k, l)$ be distinct elements of V. We claim that $\{v, w\}$ is edge of exactly three of the graphs G_1, G_2, G_3, G_4.

Before analyzing different cases of i, j, k, l as before, we consider vector $d := (i-k, j-l)$. First note that $\mathbb{Z}_3$ is a Galois field and $\mathbb{Z}_3 \times \mathbb{Z}_3$ is a vector space. Then $\mathbb{Z}_3 \times \mathbb{Z}_3$ is an affine plane of order 3, where each point (x, y) lies on 4 lines from 4 parallel classes for $x, y \in \mathbb{Z}_3$ (see Beth et al. [4]). The vector d (as point in this affine plane) can be written as $d = mu$, where $m \in \mathbb{Z}_3$ and $u \in \{(0, 1), (1, 0), (1, 1), (1, 2)\}$. Now we consider 4 cases and write $\equiv$ for $\equiv \pmod 3$.

Case 1: Suppose $i = k$ and $j \neq l$. Then $d = (i - k, j - l)$ must have the form $d = m \cdot (0, 1) = (0, m)$ for some $m \neq 0$, i.e. $j = l + m$ for $m \in \{1, 2\}$. By definition, $vw \notin E(G_1)$ and $vw \in E(G_2)$. Moreover, $i + j = i + l + m \not\equiv i + l = k + l$, thus $vw \in E(G_3)$. For G_4, $j - i = l + m - i \not\equiv l - i = l - k$, thus $vw \in E(G_4)$.

Case 2: If $i \neq k$ and $j = l$, then $d = m(1, 0) = (m, 0), m \neq 0$, that is $i = k + m$ for $m \in \{1, 2\}$. Then $i + j = k + m + j \not\equiv k + j = k + l$, which implies that $vw \in E(G_3)$, and $j - i = j - (k + m) = l - k - m \not\equiv l - k$, thus $vw \in E(G_4)$. Clearly, $vw \in E(G_1)$ and $vw \notin E(G_2)$.

Case 3: Suppose $i \neq k, j \neq l$ and $i + j \equiv k + l$. Then $i - k \equiv l - j = -(j - l)$. It follows that $d = m(1, 2) = (m, 2m)$ for $m \neq 0$, since $2 = -1$ in $\mathbb{Z}_3$. We have $i = k + m$ and $j = l + 2m$, hence $i + j = k + l + 3m \equiv k + l$, which means $vw \notin E(G_3)$. Since $j - i = l + 2m - k - m = l - k + m \not\equiv l - k$, we have $vw \in E(G_4)$. Clearly $vw \in E(G_1) \cap E(G_2)$.

Case 4: Finally, suppose $i \neq k, j \neq l$ and $i + j \not\equiv k + l$. By definition, $vw \in E(G_1) \cap E(G_2)$. There is only one possibility left for vector d, namely $d = m(1, 1)$ for $m \neq 0$. It means $i = k + m$ and $j = l + m$. It follows that $i + j = l + k + 2m \in \{l + k + 2, l + k + 1\}$. In particular, $i + j \not\equiv l + k$, thus $vw \in E(G_3)$. For G_4, we have $j - i = (l + m) - (k + m) \equiv l - k$, thus $vw \notin E(G_4)$. This completes the proof of the claim.

We now define coloring c as follows: For any edge vw of K_9, let

$$c(vw) = \{i \in [4] : vw \in G_i\}.$$

Then c is a $(4, 3)$-coloring with color graphs G_1, G_2, G_3, G_4 and by construction, all color graphs are tripartite and contain no K_4. Therefore, c has no monochromatic K_4, which proves $R_{4,3}(4) > 9$. □

Remark 4.8. Theorems 4.6 and 4.7 were presented in the chronological order how we approached the problems. Alternatively, we can prove the lower bound of Theorem 4.6 as a result of Theorem 4.7. By Proposition 3.4 (ii), $R_{4,3}(4) \leq R_{3,2}(4)$, and Theorem 4.7 implies that $R_{3,2}(4) > 9$.

5 Design construction

Considering the proof of Theorem 4.7, we can label the vertices of K_9 in a different way such that

$$\begin{pmatrix} (0,0) & (0,1) & (0,2) \\ (1,0) & (1,1) & (1,2) \\ (2,0) & (2,1) & (2,2) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix},$$

then the partite sets of the tripartite graphs G_1, G_2, G_3, G_4 are listed in Table 2.

$A_0 = \{1,2,3\}$	$B_0 = \{1,4,7\}$	$C_0 = \{2,4,9\}$	$D_0 = \{1,5,9\}$
$A_1 = \{4,5,6\}$	$B_1 = \{2,5,8\}$	$C_1 = \{3,5,7\}$	$D_1 = \{3,4,8\}$
$A_2 = \{7,8,9\}$	$B_2 = \{3,6,9\}$	$C_2 = \{1,6,8\}$	$D_2 = \{2,6,7\}$

Table 2: Partitions of color graphs in a $(4,3)$-coloring without monochromatic K_4

This is a resolution of a $(9,3,1)$ design that we will define below. We raise the question whether Theorem 4.7 can be generalized, or whether a particular design could yield a lower bound on some Ramsey numbers. We first introduce basic terms and well-known results in design theory. For an overview of the history of design theory, we refer to Wilson [27].

5.1 Basics of design theory

In this section let t, v, k, λ be positive integers such that $v \geq k \geq t$.

A $t-(v,k,\lambda)$ *design* is a pair $(V, \mathcal{B})$ with point set V and a multiset $\mathcal{B}$ of sets (called blocks) of points with

- $|V| = v$,
- $|B| = k$ for any $B \in \mathcal{B}$, and
- each set of t points is a subset of exactly λ blocks.

We often refer to $\mathcal{B}$ as the design without mentioning V. A $t-(v,k,\lambda)$ design is also denoted by $S_\lambda(t,k,v)$. If $\lambda = 1$, we call $S_1(t,k,v) := S(t,k,v)$ a Steiner system of order v.

Two designs $(V_1, \mathcal{B}_1)$ and $(V_2, \mathcal{B}_2)$ are *isomorphic* if there exists a bijection $\phi : V_1 \to V_2$ such that

$$\{\{\phi(x) : x \in B\} : B \in \mathcal{B}_1\} = \mathcal{B}_2.$$

The bijection ϕ is then called an *isomorphism*.

If $\mathcal{B}$ is a $t-(v,k,\lambda)$ design then the number of blocks in $\mathcal{B}$ is $b := |\mathcal{B}| = \lambda \cdot \binom{v}{t} / \binom{k}{t}$. If $b = v$, the design is called *symmetric*. Since the number of blocks plays an important role in our application, we also write for convenience (v, b, k, λ) designs.

Let $\mathcal{B} := \{B_i : i \in [b]\}$ a $t-(v, b, k, \lambda)$ design with point set $V = \{x_1, \ldots, x_v\}$. The incidence matrix of $\mathcal{B}$ is a $v \times b$ matrix $A = (a_{i,j})$ such that

$$a_{i,j} = \begin{cases} 1, & \textit{if } x_i \in B_j \\ 0, & \textit{if } x_i \notin B_j \end{cases}$$

If $I \subseteq [v]$ is a set of size $1 \leq i \leq t$, then the number of blocks containing I is

$$r_i = \lambda \cdot \binom{v-i}{t-i} \Big/ \binom{k-i}{t-i}.$$

If $t = 2$, then each point of V is contained in

$$r := r_1 = \frac{\lambda(v-1)}{k-1}$$

blocks.

In design theory, an important question is when a particular design exists. We see that if a $t-(v, k, \lambda)$ design exists, then b and r_i must be positive integers. As a result, necessary conditions for the existence of a $t-(v, k, \lambda)$ design are

$$\binom{k-i}{t-i} \Bigm| \lambda \cdot \binom{v-i}{t-i} \text{ for all } 0 \leq i \leq t-1. \tag{1}$$

We call (1) the *divisibility conditions*. For $t = 2$, the necessary divisibility conditions (1) reduce to:

$$\begin{cases} k(k-1) \mid \lambda v(v-1), \\ (k-1) \mid \lambda(v-1). \end{cases} \tag{2}$$

An open problem in design theory for a long time was the Existence Conjecture: for any fixed parameters $t, k, \lambda \in \mathbb{N}$, there exists a $t-(v, k, \lambda)$ design for any $v \in \mathbb{N}$ if v is sufficiently large and satisfies the divisibility conditions. Wilson [26] proved the conjecture for $t = 2$. Recently, Keevash [15] proved the conjecture by a method which he called randomized algebraic constructions. An alternative proof was given by Glock et al. [10], based on iterative absorption.

For our purpose, we consider furthermore a particular kind of designs - the resolvable designs.

A *parallel class* of a $t-(v, k, \lambda)$ design $(V, \mathcal{B})$ is a set of pairwise disjoint blocks forming a partition of the point set V. A partition of $\mathcal{B}$ into parallel classes is called *resolution*. If a design has a resolution, then it is called *resolvable*.

A *Steiner triple system* is $S(t,3,v)$. A resolvable Steiner triple system is called a *Kirkman triple system*.

Since each block has k elements, each parallel class has $\frac{v}{k}$ blocks. There are b blocks in $\mathcal{B}$, thus the number of parallel classes is $\frac{bk}{v}$.

In this section, we only consider designs with $v > k \geq t = 2$. We call such a $2-(v,k,\lambda)$ design a *balanced incomplete block design* (BIBD) and write (v,k,λ) or (v,b,k,λ) design. Note that in this case ($t = 2$), the number of blocks is

$$b = \frac{\lambda \cdot v(v-1)}{k(k-1)}.$$

Example 5.1. A $(7,3,1)$ design is unique up to isomorphism. This particular design is referred to as the *Fano plane*. We have $V = [v] = [7], k = 3, \lambda = 1$, and $\mathcal{B}$ contains triples $B_i, i \in [7]$, as blocks.

$$\begin{array}{l} B_1 = \{1,2,3\} \\ B_2 = \{1,4,5\} \\ B_3 = \{1,6,7\} \\ B_4 = \{2,4,6\} \\ B_5 = \{3,5,7\} \\ B_6 = \{3,4,7\} \\ B_7 = \{2,5,6\} \end{array} \qquad A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Each pair of points are contained in exactly 2 blocks. The Fano plane is symmetric, since $v = b = 7$, but not resolvable, since $3 \nmid 7$. Its incidence matrix A is a 7×7 matrix.

Example 5.2. This example shows a resolvable $(4,2,1)$ design, which is unique up to isomorphism. Here we have $V = v = [4], k = 2, \lambda = 1$, then $b = 6$. The number of parallel classes is $s = bk/v = 3$, each parallel class has $v/k = 2$ points. A resolution is $\mathcal{B} = \mathcal{B}_1 \dot\cup \mathcal{B}_2 \dot\cup \mathcal{B}_3$,

$$\begin{array}{ccc} \mathcal{B}_1 & \mathcal{B}_2 & \mathcal{B}_3 \\ B_1 = \{1,2\} & B_3 = \{1,3\} & B_5 = \{1,4\} \\ B_2 = \{3,4\} & B_4 = \{2,4\} & B_6 = \{2,3\}. \end{array}$$

This is an affine plane of order 2 with 4 points and 6 lines. In general, an affine plane of order q is equivalent to a $(q^2,q,1)$ design, the lines in the affine plane correspond to the blocks of the design. Such a design is always resolvable.

Theorem 5.3. *(Stinson [24]) For any prime power q, there exists a resolvable $(q^2,q,1)$ design.*

The $(9, 3, 1)$ design in Table 2 refers to the affine plane $\mathbb{Z}_3 \times \mathbb{Z}_3$, as mentioned in the proof of Theorem 4.7.

Let $\mathcal{B}$ a resolvable (v, b, k, λ) design, then the number of parallel classes of $\mathcal{B}$ is

$$s := \frac{bk}{v} = \frac{k}{v} \cdot \frac{\lambda v(v-1)}{k(k-1)} = \frac{\lambda(v-1)}{k-1} = r.$$

Clearly, $k \mid v$ must be satisfied. Together with the condition (2), it implies that the necessary condition for the existence of a resolvable (v, k, λ) is

$$v \equiv k \pmod{k(k-1)}. \qquad (3)$$

Ray-Chaudhuri and Wilson [21] proved that (3) is also the sufficient condition for $k = 3$ and $\lambda = 1$.

Theorem 5.4. *[21] A Kirkman triple system $S(2, 3, v)$ exists for $v \in \mathbb{N}$ if and only if $v \equiv 3 \pmod 6$.*

In this case, $b = \frac{v(v-1)}{6}$ and $s = \frac{v-1}{2}$.

For a $(v, k = 4, \lambda = 1)$ design, $b = \frac{v(v-1)}{12}$ and $s = \frac{v-1}{3}$. In 1971, Ray-Chaudhuri and Wilson together with Hanani proved in [13] that the neccesary condition (3) is also sufficient for the existence of a resolvable $(v, 4, 1)$ design.

Theorem 5.5. *[13] A resolvable $(v, 4, 1)$ design exists if and only if $v \in \mathbb{N}$ with $v \equiv 4 \pmod{12}$.*

Keevash [16] resolved the Existence Conjecture for general cases. For sufficiently large v, the divisibility condition (1) is also sufficient for the existence of a (resolvable) $t-(v, k, \lambda)$ design.

Theorem 5.6. *[16] For all $k, t, \lambda \in \mathbb{N}$ with $k \geq t$ there exists $n_0 = n_0(k, t, \lambda) \in \mathbb{N}$ such that for all $v \geq n_0$, if $k \mid v$ and $\binom{k-i}{t-i} \mid \lambda\binom{v-i}{t-i}$ for all $0 \leq i \leq t-1$, then there is a resolvable $t - (v, k, \lambda)$ design.*

With $t = 2$ we have the following corollary.

Corollary 5.7. *For all $k, \lambda \in \mathbb{N}$ with $k \geq 2$ there exists $n_0 = n_0(k, \lambda) \in \mathbb{N}$ such that for all $v \geq n_0$, if $k \mid v$ and $(k-1) \mid \lambda(v-1)$, then there is a resolvable (v, k, λ) design.*

Proof. The divisibility condition (1)

$$\binom{k-i}{t-i} \,\Big|\, \lambda \cdot \binom{v-i}{t-i} \text{ for all } 0 \leq i \leq t-1$$

implies for $t = 2$:

$$\begin{cases} k(k-1) \mid \lambda v(v-1), \\ (k-1) \mid \lambda(v-1). \end{cases} \tag{2}$$

With $k \mid v$ and $(k-1) \mid \lambda(v-1)$, the condition $k(k-1) \mid \lambda v(v-1)$ is always satisfied. The corollary then follows from Theorem 5.6. □

We emphasize here that our aim is not to construct (resolvable) designs. Constructions of designs are extensively described in the literature on designs, for example by Lindner and Rodger [17] for Steiner or Kirkman systems, by Beth, Jungnickel, and Lenz [4] for general designs. Our purpose is to build colorings and find bounds for certain Ramsey numbers based on existing designs.

5.2 Design colorings

We assume again for the rest of this section that v, b, k, λ are positive integers with $v > k \geq 2$, unless otherwise stated. We can construct a coloring with desired properties based on an existing resolvable design. Recall that for a resolution of a resolvable (v, b, k, λ) design, there are v/k blocks in each parallel class and the number of parallel classes is

$$s = \frac{bk}{v} = \frac{\lambda(v-1)}{k-1},$$

which implies

$$s - \lambda = \frac{\lambda(v-1) - \lambda(k-1)}{k-1} = \frac{\lambda(v-k)}{k-1}.$$

If a (v, b, k, λ) design is resolvable, then clearly $k \mid v$ and $s, s-\lambda$ are positive integers.

Definition 5.8 (Design coloring)**.** Let $(V, \mathcal{B})$ be a resolvable (v, b, k, λ) design. Then there exists a resolution $\mathcal{B} = \mathcal{B}_1 \dot\cup \ldots \dot\cup \mathcal{B}_s$ with parallel classes $\mathcal{B}_1, \ldots, \mathcal{B}_s$, where $s = bk/v$. For any $i \in [s]$, $\mathcal{B}_i$ is a partition of $[v]$ into v/k disjoint blocks: $\mathcal{B}_i = \{B_{i,1}, \ldots, B_{i,v/k}\}$, where $B_{i,j}$ is a block of the design $\mathcal{B}$ for any $j \in [v/k]$, and $\dot\bigcup_{j=1}^{v/k} B_{i,j} = V$.

Let $V(K_v) = V$ and for $i \in [s]$, let G_i be the complete (v/k)-partite graph, whose parts are the blocks $\{B_{i,j} : j \in [v/k]\}$ of the parallel class $\mathcal{B}_i$.

An edge coloring c of K_v is called a (v, b, k, λ) *design coloring* of K_v if c has color graphs G_i, i.e. for all edges xy of K_v,

$$c(xy) := \{i \in [s] : xy \in E(G_i)\}.$$

Lemma 5.9. *Suppose that $(V, \mathcal{B})$ is a resolvable (v, b, k, λ) design and c is a (v, b, k, λ) design coloring of K_v. Then c is an $(s, s-\lambda)$-coloring of K_v, where $s = bk/v$.*

Proof. Note that s is the number of parallel classes in a resolution of $\mathcal{B}$. Since c is a (v, b, k, λ) design coloring, there is a resolution $\mathcal{B} = \mathcal{B}_1 \dot{\cup} \ldots \dot{\cup} \mathcal{B}_s$ such that for any $i \in [s]$, the color graph G_i of c is the complete (v/k)-partite graph whose parts are v/k blocks of $\mathcal{B}_i$. Clearly, there are s color graphs, hence there are s possible colors for c. We just need to show that each edge of K_v has exactly $(s - \lambda)$ colors, or is edge of $(s - \lambda)$ of the color graphs G_i.

For a parallel class $\mathcal{B}_i, i \in [s]$, we can write $\mathcal{B}_i = \{B_{i,1}, \ldots, B_{i,v/k}\}$, where $B_{i,j}$ are blocks of $\mathcal{B}$, $j \in [v/k]$. Let $x, y \in V = V(K_v)$ be distinct vertices. For $i \in [s]$, $\{x, y\}$ is an edge of G_i if and only if x, y are not in the same partition, thus not in the same block $B_{i,j}$ for all $j \in [v/k]$. By the definition of $\mathcal{B}$, x and y are both contained in λ blocks. In particular, there are exactly λ parallel classes where x and y lie in the same block. Accordingly, in the remaining $(s - \lambda)$ parallel classes, x and y are in different blocks. It follows that $\{x, y\}$ is an edge of $(s - \lambda)$ of the color graphs G_i. This proves that c is an $(s, s - \lambda)$-coloring of K_v. □

Remark 5.10. If a resolvable (v, b, k, λ) design exists, then a (v, b, k, λ) design coloring is an $(s, s - \lambda)$-coloring of K_v for $s = bk/v$, where all color graphs G_i are the Turán graphs $T_{v/k}(v) = K^k_{v/k}$.

We will prove that the opposite direction also holds, i.e. if all color graphs of an $(s, s-\lambda)$-coloring are the Turán graphs $K^k_{v/k}$, then that coloring is a (v, b, k, λ) design coloring.

Proposition 5.11. *Let v, s, t be positive integers with $s > t$. Suppose c is an (s, t)-coloring of K_v such that for all $i \in [s]$, color graphs G_i of c are the Turán graphs $T_n(v)$ for some $n \in \mathbb{N}$ with $n \mid v$. Then there exists a resolvable (v, b, k, λ) design such that c is a (v, b, k, λ) design coloring, where $k = \frac{v}{n}, b = \frac{sv}{k}, \lambda = s - t$.*

Proof. Let $V := V(K_v)$. For $i \in [s]$, since G_i is the Turán graph $T_n(v)$, we can assume that G_i has partite sets $B_{i,j}$, $j \in [n]$. We consider $B_{i,j}$ as blocks, then each block contains $v/n = k$ vertices. Let $\mathcal{B} = \{B_{i,j} : i \in [s], j \in [n]\}$. We show that $(V, \mathcal{B})$ is a resolvable (v, b, k, λ) design.

Since there are s color graphs, we have in total $s \cdot n = \frac{sv}{k} = b$ blocks. Let x, y be distinct vertices of K_v. Since c is an (s, t)-coloring, $\{x, y\}$ is an edge in t color graphs. In other words, $\{x, y\}$ is not an edge in $s - t = \lambda$ color graphs, which implies that x and y lie in the same partite set in λ color graphs. That means x, y are contained in λ of the blocks $B_{i,j}$. Therefore, $\mathcal{B}$ is a (v, b, k, λ) design.

Further, for $i \in [s]$, let $\mathcal{B}_i = \{B_{i,j} : j \in [n]\}$, i.e. $\mathcal{B}_i$ contains all partite sets of color graph G_i. Then $\mathcal{B}_i$ is a disjoint partition of V for all $i \in [s]$ and $\dot{\cup}_{i \in [s]} \mathcal{B}_i = \mathcal{B}$ is a resolution of $\mathcal{B}$ into parallel classes $\mathcal{B}_i$. This proves that $\mathcal{B}$ is resolvable.

Moreover, for any $i \in [s]$, color graph G_i of c is the complete (v/k)-partite graphs whose partite sets are the blocks in the parallel class $\mathcal{B}_i$. Thus c is a (v, b, k, λ) design coloring. □

We prove that a resolvable design coloring implies a lower bound for a Ramsey number.

Proposition 5.12. *If there exists a resolvable (v,b,k,λ) design, then a (v,b,k,λ) design coloring has no monochromatic $K_{\frac{v}{k}+1}$. In particular we have*

$$R_{\frac{bk}{v},\frac{bk}{v}-\lambda}\left(\frac{v}{k}+1\right) > v.$$

Proof. Let c be a (v,b,k,λ) design coloring of K_v. By Lemma 5.9, c is a $\left(\frac{bk}{v},\frac{bk}{v}-\lambda\right)$-coloring of K_v. Let G_i be the color graphs of c, $i \in [bk/v]$, i.e. G_i is the Turán graph $T_{v/k}(v)$. It follows that G_i does not contain $K_{\frac{v}{k}+1}$ as a subgraph for all $i \in [bk/v]$, thus c contains no monochromatic $K_{\frac{v}{k}+1}$. This completes the proof of the proposition. □

It turns out that this lower bound is best possible. We now state our main result for a resolvable (v,b,k,λ) design.

Theorem 5.13. *If there exists a resolvable (v,b,k,λ) design, then we have*

$$R_{\frac{bk}{v},\frac{bk}{v}-\lambda}\left(\frac{v}{k}+1\right) = v+1.$$

We need some steps for the proof. The main idea is follows: given a (v,b,k,λ) coloring of K_v without monochromatic K_n with $n=\frac{v}{k}+1$, one cannot extend this coloring to a larger clique than K_v without creating a monochromatic K_n. On the other hand, every coloring of K_v without monochromatic K_n must have a design construction, which is again not extendable. These two facts then prove the theorem. We now define a coloring extension formally.

Definition 5.14. For positive integers s,t,k,n_1,n_2 with $s \geq t$ and $n_2 \geq n_1+1$, let c_1 be an (s,t)-coloring of K_{n_1} and c_2 be an (s,t)-coloring of K_{n_2}. We say c_2 is an *extension* of c_1 (*to* K_{n_2}) if K_{n_2} has a subgraph $H \cong K_{n_1}$ such that $c_2(e)=c_1(e)$ for all $e \in E(H)$.

If both c_1 and c_2 have no monochromatic K_k, then c_2 is called a K_k*-free extension* of c_1.

If c_1 has a K_k-free extension, then c_1 is called K_k*-free extendable*, otherwise we say c_1 is K_k*-free maximal*.

Lemma 5.15. *Let $\mathcal{B}$ be a resolvable (v,b,k,λ) design and $s := bk/v$. Let c be a (v,b,k,λ) design coloring of K_v. Then c is $K_{\frac{v}{k}+1}$-free maximal.*

Proof. First note that c is an $(s,s-\lambda)$-coloring that has no monochromatic $K_{\frac{v}{k}+1}$ by Propositions 5.9 and 5.12. Let $\mathcal{B}=\mathcal{B}_1\dot\cup\ldots\dot\cup\mathcal{B}_s$ be a resolution with parallel classes $\mathcal{B}_1,\ldots,\mathcal{B}_s$ and blocks $B_{i,j}$, $i \in [s]$, $j \in [\frac{v}{k}]$. Let G_i be the corresponding color graphs of c, i.e. G_i are the Turán graph $T_{v/k}(v)$ for all $i \in [s]$.

It suffices to show that any extension coloring of c to K_{v+1} has a monochromatic $K_{\frac{v}{k}+1}$. For the sake of contradiction, let $H := K_v + x$ for a vertex $x \notin V(K_v)$ and assume c' is an extension of c to $K_v + x$. Note that $H \cong K_{v+1}$ and c' is an $(s, s-\lambda)$-coloring of K_{v+1}. We consider c', there are v edges at x. By the pigeonhole principle, there exists a color $i \in [s]$ that appears at least $\lceil m \rceil$ times on these v edges, where $m = \frac{v(s-\lambda)}{s}$. With $s = \frac{\lambda(v-1)}{k-1}$ and $s - \lambda = \frac{\lambda(v-k)}{k-1}$, we have

$$\begin{aligned} m &= \frac{v(s-\lambda)}{s} = \frac{v\lambda(v-k)}{k-1} \cdot \frac{k-1}{\lambda(v-1)} \\ &= \frac{v(v-k)}{v-1} \\ &= \frac{(v-1)(v-k)}{v-1} + \frac{v-k}{v-1} \\ &= (v-k) + \frac{v-k}{v-1}. \end{aligned}$$

Since $0 < \frac{v-k}{v-1} < 1$, $\lceil m \rceil = v - k + 1$, thus there are at least $(v-k+1)$ edges with color i at vertex x. Moreover, $v - k + 1 = k\left(\frac{v}{k} - 1\right) + 1$. Since each partite set of G_i has k vertices, there is at least one edge with color i from x to all (v/k) partite sets $B_{i,j}$, $j \in [\frac{v}{k}]$, of the complete (v/k)-partite graph G_i. Together with a monochromatic $K_{v/k}$ in G_i, these edges forms a monochromatic $K_{\frac{v}{k}+1}$ of c' in color i. This proves that c is not $K_{\frac{v}{k}+1}$-free extendable. $\square$

Example 5.16. A $(9,3,1)$ design is unique up to isomorphism and resolvable. Table 2 shows a resolution of it with blocks A_i, B_i, C_i, D_i for $i \in \mathbb{Z}_3$ and a corresponding design coloring. There are 4 parallel classes, each of which contains 3 blocks. Each color graph associates with a parallel class of the resolution. We have $R_{4,3}(4) > 9$. Let $G \cong K_9$ and c be a $(9,3,1)$ design coloring of K_9. See figure 11 for illustration.

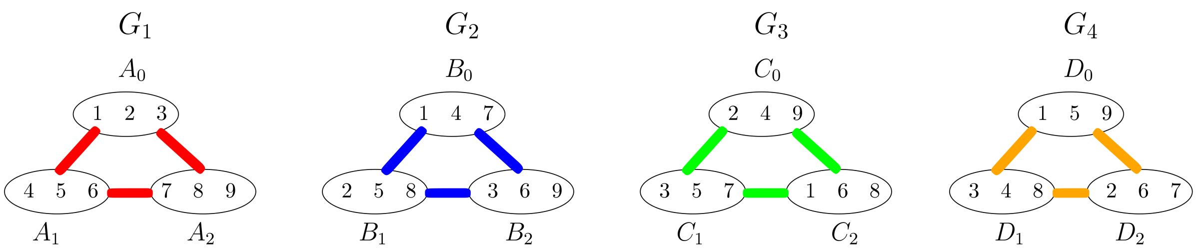


Figure 11: $(9,3,1)$ design coloring

Now add a vertex x to K_9 and let c' be an extension of c to $G+x$. Note that both c and c' are $(4,3)$-colorings. There are 9 edges at x, hence 27 colors are needed. There is some color $i \in [4]$ that appears on at least $\lceil 27/4 \rceil = 7$ of these 9 edges at x, say color 1, i.e. $|N_1(x)| \geq 7$. Then x is connected to all 3 partite sets A_0, A_1, A_2 of G_1. This implies that c' has a monochromatic K_4 in color 1 (see Figure 12). Thus c is not K_4-free extendable.

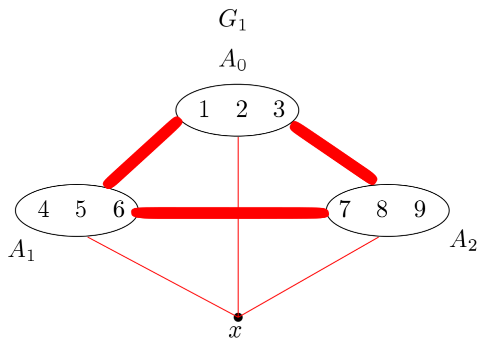


Figure 12: Extension of a $(9, 3, 1)$ design coloring to $K_9 + x$

Lemma 5.17. *Suppose there exists a resolvable (v, b, k, λ) design and*

$$s := \frac{bk}{v},\ n := \frac{v}{k} + 1.$$

Then an $(s, s-\lambda)$ coloring of K_v without monochromatic K_n must be a (v, b, k, λ) design coloring.

Proof. Let c be an $(s, s-\lambda)$ coloring of K_v without monochromatic K_n. By Proposition 5.11, it suffices to show that all color graphs of c are Turán graphs $T_{v/k}(v) = K^k_{v/k}$.

For $i \in [s]$, we consider the size of color graph G_i of c. Since c has no monochromatic K_n, G_i has size at most

$$e(G_i) \leq ex(v, K_n) = t_{\frac{v}{k}}(v) = \binom{v}{2} - \frac{v}{k}\binom{k}{2}.$$

On the other hand, since each edge has $(s - \lambda)$ colors, the total number of edges of all color graphs is

$$\begin{aligned}
\sum_{i=1}^{s} e(G_i) &= (s-\lambda)\binom{v}{2} \\
&= s\binom{v}{2} - \lambda\binom{v}{2} \\
&= s\binom{v}{2} - \frac{bk(k-1)}{v(v-1)} \cdot \frac{v(v-1)}{2} \qquad (\text{since } b = \frac{\lambda v(v-1)}{k(k-1)}) \\
&= s\binom{v}{2} - b\binom{k}{2} \\
&= s\binom{v}{2} - s\frac{v}{k}\binom{k}{2} \\
&= s\left[\binom{v}{2} - \frac{v}{k}\binom{k}{2}\right] \\
&= s \cdot t_{\frac{v}{k}}(v).
\end{aligned}$$

If there is a color graph G_i with $e(G_i) < t_{v/k}(v)$ for some $i \in [s]$, then there must be a color graph G_j with $j \in [s], j \neq i$, such that $E(G_j) > t_{v/k}(v)$, which is a contradiction to the maximum size of G_j. Therefore, all color graphs G_i must have equal size and have exactly $t_{v/k}(v)$ edges. It means all color graphs G_i of c are the Turán graph $T_{v/k}(v)$, thus c is a (v, b, k, λ) design coloring by Proposition 5.11. □

Now we can prove the main theorem.

Proof of Theorem 5.13. The lower bound is by Proposition 5.12. To prove the upper bound, for the sake of contradiction, assume c' is an $(s, s-\lambda)$-coloring of K_{v+1} without monochromatic $K_{\frac{v}{k}+1}$. Let x be a vertex of K_{v+1} and c be the restriction of c' on $E(G)$ with $G := K_{v+1} - x$, i.e. $c = c'|_{E(G)}$. Then c is an $(s, s-\lambda)$-coloring of K_v and has no monochromatic $K_{\frac{v}{k}+1}$. In particular, c' is a $K_{\frac{v}{k}+1}$-free extension of c. By Lemma 5.17, c is a (v, b, k, λ) design coloring. Lemma 5.15 implies that c is not $K_{\frac{v}{k}+1}$-free extendable, which is a contradiction. This proves the upper bound $R_{s,s-\lambda}\left(\frac{v}{k}+1\right) \leq v+1$ and completes the proof of the theorem. □

In the following we present some consequences of Theorem 5.13.

With the existence of a resolvable $(9, 3, 1)$ design we have the exact value of $R_{4,3}(4)$, which extends Theorem 4.7.

Corollary 5.18. *We have that* $R_{4,3}(4) = 10$.

This is a special case of affine planes $(q^2, q, 1)$ with $q = 3$. For general affine planes of order q we have the following result.

Corollary 5.19. *For any prime power q, we have*

$$R_{q+1,q}(q+1) = q^2 + 1.$$

Proof. By Theorem 5.3 there exists a resolvable $(v = q^2, k = q, \lambda = 1)$ design. With

$$s = \frac{\lambda(v-1)}{k-1} = \frac{q^2-1}{q-1} = q+1$$

and $s - \lambda = q$, the statement follows from Theorem 5.13. □

Example 5.20. Example 5.2 shows an affine plane of order $q = 2$, or a resolvable $(4, 2, 1)$ design. A $(4, 2, 1)$ design coloring c is a $(3, 2)$-coloring of K_4 which has no monochromatic K_3. Let $V(K_4) = \{u_1, u_2, u_3, u_4\}$, then

$$\begin{aligned} c(u_1u_2) &= c(u_3u_4) = \{2,3\}, \\ c(u_1u_3) &= c(u_2u_4) = \{1,3\}, \\ c(u_1u_4) &= c(u_2u_3) = \{1,2\}. \end{aligned}$$

and we have $R_{3,2}(3) = 5$. Since the design is unique up to isomorphism, so is the coloring c. We just proved Theorem 4.2 in an alternative way.

Now we consider design colorings with Kirkman triple systems.

Corollary 5.21. *For all $n \in \mathbb{N}$, we have*

$$R_{3n+1,3n}(2n+2) = 6n+4$$

Proof. Let $v = 6n+3$. By Theorem 5.4 there exists a Kirkman triple system of order v, or a resolvable $(v, k = 3, \lambda = 1)$ design. Then the number of parallel classes in a resolution of the design is

$$s = \frac{v-1}{2} = 3n+1,$$

and $s - \lambda = s - 1 = 3n$. The corollary follows by Theorem 5.13 with $\frac{v}{k} = 2n+1$. □

Corollary 5.22. *For any $n \in \mathbb{N}$, we have*

$$R_{4n+1,4n}(3n+2) = 12n+5.$$

Proof. Let $v = 12n+4$. By Theorem 5.5, there exists a resolvable $(v, 4, 1)$ design. The corollary follows by Theorem 5.13 with parameters

$$\begin{aligned} s &= \frac{bk}{v} = \frac{v-1}{3} = 4n+1, \\ s - \lambda &= s - 1 = 4n, \\ \frac{v}{k} &= \frac{12n+4}{4} = 3n+1. \end{aligned}$$

□

Proposition 5.23. *We have that $R_{7,4}(3) = 9$.*

Proof. There exists a resolvable $(8, 14, 4, 3)$ design. A resolution is listed below.

$\mathcal{B}_1$	$\mathcal{B}_2$	$\mathcal{B}_3$	$\mathcal{B}_4$
$B_{1,1} = \{1,2,3,4\}$	$B_{2,1} = \{1,2,5,6\}$	$B_{3,1} = \{1,2,7,8\}$	$B_{4,1} = \{1,3,5,7\}$
$B_{1,2} = \{5,6,7,8\}$	$B_{2,2} = \{3,4,7,8\}$	$B_{3,2} = \{3,4,5,6\}$	$B_{4,2} = \{2,4,6,8\}$

$\mathcal{B}_5$	$\mathcal{B}_6$	$\mathcal{B}_7$
$B_{5,1} = \{1,4,6,7\}$	$B_{6,1} = \{1,3,6,8\}$	$B_{7,1} = \{1,4,5,8\}$
$B_{5,2} = \{2,3,5,8\}$	$B_{6,2} = \{2,4,5,7\}$	$B_{7,2} = \{2,3,6,7\}$

The proposition follows from Theorem 5.13. □

With the result of Keevash's study [16] on the existence of resolvable designs, we get the following.

Theorem 5.24. *For all $k, \lambda \in \mathbb{N}$ with $k \geq 2$, there exists $n_0 = n_0(k, \lambda) \in \mathbb{N}$ such that for all $v \geq n_0$, if $k \mid v$ and $(k-1) \mid \lambda(v-1)$, then for $s = \frac{\lambda(v-1)}{k-1}$,*

$$R_{s,s-\lambda}\left(\frac{v}{k}+1\right) = v+1.$$

Proof. By Corollary 5.7, for all $v \geq n_0$, there exists a resolvable (v, k, λ) design with s parallel classes. The statement follows from Theorem 5.13. □

Remark 5.25. We emphasize the condition that v must be sufficiently large for Theorem 5.24. For example, consider the design $(v = 6, k = 3, \lambda = 2)$. The divisibility conditions are satisfied: $k \mid v$ and $(k-1) \mid \lambda(v-1)$. For $s = \frac{\lambda(v-1)}{k-1} = 5$, it *would* follow that $R_{s,s-\lambda}\left(\frac{v}{k}+1\right) = R_{5,3}(3)$ equals 7. We will show later in Proposition 7.8 that $R_{5,3}(3) = 5$. The reason is that v is not large enough for Theorem 5.24. More precisely, v is not large enough for the existence of a resolvable (v, k, λ) design in this case. There is a unique $(6, 3, 2)$ design (see Colbourn and Dinitz [6]) and it is not resolvable. The blocks of the design are listed below, the number of blocks is $b = 10$.

$$\begin{array}{lllll} B_1 = \{1,2,3\} & B_3 = \{1,3,5\} & B_5 = \{1,5,6\} & B_7 = \{2,4,5\} & B_9 = \{3,4,5\} \\ B_2 = \{1,2,4\} & B_4 = \{1,4,6\} & B_6 = \{2,3,6\} & B_8 = \{2,5,6\} & B_{10} = \{3,4,6\}. \end{array}$$

6 Hadamard construction and lower bounds for $R_{2n,n}(3)$

We recall the proof of Theorem 4.3. To prove the lower bound $R_{4,2}(3) > 8$, we constructed a $(4,2)$-coloring of K_8 without monochromatic triangle. The matrix H describing this coloring is

$$H = H_0 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}.$$

This is a Hadamard matrix which we will define below. In this section we study the connection between Hadamard matrices and certain (s,t)-colorings without monochromatic triangle. We start with definitions and known results for Hadamard matrices.

6.1 Basics of Hadamard matrices

A *Hadamard matrix of order* n is an $n \times n$ matrix H with coefficients $+1$ or -1 such that

$$HH^T = nI$$

If H is a Hadamard matrix, then so is H^T. Any two columns and any two rows of H are orthogonal. If we permute rows or columns of H or multiply some rows or columns by -1, the resulting matrix is again a Hadamard matrix and is called *equivalent* to H. If two Hadamard matrices H and H' are equivalent, we write $H \sim H'$.

A Hadamard matrix H is called *standardized* if the first row and the first column contain only $+1$. Any Hadamard matrix can be transformed to a standardized one. If we delete the first row (or column) of a standardized Hadamard matrix, then the remaining rows (or columns) have as many $+1$ as -1.

For a Hadamard matrix $H = (h_{i,j})_{i,j=1}^n$, we denote the columns of H by h_i and rows of H by r_i, $i \in [n]$.

Example 6.1.

(i) Both (1) and (-1) are Hadamard matrices of order 1.

(ii) A standardized Hadamard matrix of order 2 is

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

(iii) Hadamard matrices of order 4 are for example

$$H_1 = \begin{pmatrix} \mathbf{1} & \mathbf{1} & 1 & 1 \\ \mathbf{-1} & \mathbf{1} & -1 & 1 \\ \mathbf{-1} & \mathbf{1} & 1 & -1 \\ \mathbf{1} & \mathbf{1} & -1 & -1 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{-1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix},$$

$$H_3 = \begin{pmatrix} 1 & \mathbf{-1} & 1 & 1 \\ 1 & \mathbf{1} & -1 & 1 \\ 1 & \mathbf{1} & 1 & -1 \\ 1 & \mathbf{-1} & -1 & -1 \end{pmatrix}, \qquad H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \mathbf{-1} & \mathbf{1} & \mathbf{1} & \mathbf{-1} \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}.$$

All of these are equivalent to H_0 from above. We can transform H_0 to H_1 by permuting the first two columns, to H_2 by permuting the first two rows. We attain H_3 from H_0 by negating the second column and H_4 from H_0 by negating the second row.

A popular interest in the study of Hadamard matrices is the existence of such. Below is a necessary condition.

Theorem 6.2. *(Stinson [24]) If there exists a Hadamard matrix of order $n > 2$, then $n \equiv 0 \pmod 4$.*

The Hadamard Conjecture states that there exists a Hadamard matrix of order n for all $n \in \mathbb{N}$ with $n \equiv 0 \pmod 4$ (see [24]). For study on the existence and construction of specific Hadamard matrices, see for example Paley [19], Baumert, Golomb and Hall [3].

Hadamard matrices have a strong connection to a class of designs.

Theorem 6.3. *[24] Let m be positive integer with $m > 1$. Then there exists a Hadamard matrix of order $4m$ if and only if there exists a $(v = 4m-1, k = 2m-1, \lambda = m-1)$ design.*

Note that for this design, the number of blocks is $b = \frac{\lambda v(v-1)}{k(k-1)} = 4m - 1 = v$, thus it is a symmetric design.

Given a Hadamard matrix H of order n, a $(4m-1, 2m-1, m-1)$ design can be induced as follows. Since every Hadamard matrix is equivalent to a standardized one, we can assume that H is standardized. We delete the first row and column of H, and replace in the remaining matrix every "-1" entry with "0". Then the resulting matrix A is the incidence matrix of a symmetric $(4m-1, 2m-1, m-1)$ design.

Conversely, let A be the incidence matrix fo a $(4m-1, 2m-1, m-1)$ design. Now replace every entry of "0" with "-1", then add a row and column of "1". The resulting matrix is a Hadamard matrix of order $4m$.

Example 6.4. The incidence matrix of the $(7,3,1)$ design in Example 5.1 is equivalent to a Hadamard matrix of order 8.

$$A = \begin{pmatrix} 1&1&1&0&0&0&0\\ 1&0&0&1&1&0&0\\ 1&0&0&0&0&1&1\\ 0&1&0&1&0&1&0\\ 0&1&0&0&1&0&1\\ 0&0&1&1&0&0&1\\ 0&0&1&0&1&1&0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&1&-1&-1&-1&-1\\ 1&1&-1&-1&1&1&-1&-1\\ 1&1&-1&-1&-1&-1&1&1\\ 1&-1&1&-1&1&-1&1&-1\\ 1&-1&1&-1&-1&1&-1&1\\ 1&-1&-1&1&1&-1&-1&1\\ 1&-1&-1&1&-1&1&1&-1 \end{pmatrix}.$$

6.2 Hadamard colorings

We state and prove our main result for Hadamard matrices.

Theorem 6.5. *If there exists a Hadamard matrix H of order $n \in \mathbb{N}$, n even, then*

$$R_{n,\frac{n}{2}}(3) > 2n.$$

Note that theorem 6.5 is not tight for $n = 2$, since $R_{2,1}(3) = R(3) = 6 > 4 + 1 = 5$.

Definition 6.6. For any integer $n \geq 2$ and any Hadamard matrix $H = (h_{i,j})_{i,j=1}^{n}$, we define the set system $\mathcal{S} = \mathcal{S}_H = \{S_1, \ldots, S_n\}$ such that for all $i, k \in [n]$, $S_k \subseteq [n]$ and $i \in S_k$ if and only if $h_{i,k} = -1$.

We then define the edge coloring $c = c_H : E(K_{2n}) \to 2^{[n]}$ of K_{2n} as follows. First, fix some partition of $V(K_n)$ into two n-element sets $L = \{u_1, \ldots, u_n\}$ and $R = \{v_1, \ldots, v_n\}$. Then for all $k \in [n]$, let

$$\begin{aligned} L_k &:= \{u_i : i \in [n] \setminus S_k\} \cup \{v_i : i \in S_k\} \\ R_k &:= V(K_{2n}) \setminus L_k. \end{aligned}$$

Additionally, for all $k \in [n]$, let $G_k := K_{L_k,R_k}$, the balanced complete bipartite graph with parts L_k and R_k. Finally, for every $uv \in E(K_{2n})$, let $c(uv) := \{k \in [n] : uv \in E(G_k)\}$, i.e. c is the coloring with color graphs G_k. We call c_H a *Hadamard coloring* of K_{2n} (*induced by* H).

Note that for $i, k \in [n]$, $u_i \in L_k$ means $v_i \in R_k$, which occurs if and only if $h_{i,k} = 1$ and we say the pair (u_i, v_i) is *fixed* in G_k. Otherwise if $u_i \in R_k$, we say (u_i, v_i) is *swapped*, which is the case if and only if $h_{i,k} = -1$. The set $\mathcal{S}_H$ describes which pairs are swapped and is called the *swapping set* of c_H.

Example 6.7. We consider H_4 from Example 6.1 and the associated Hadamard coloring.

$$H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}.$$

The swapping set $\mathcal{S}_{H_4}$ is determined by the columns of H_4. The first column implies that the pair (u_2, v_2) is the only pair that is swapped in G_1, i.e. $u_2 \in R_1$ and $v_2 \in L_1$. Similarly, the second column implies that in color graph G_2, only the pair (u_3, v_3) is swapped. We have $\mathcal{S}_{H_4} = \{S_1, S_2, S_3, S_4\}$ where $S_1 = \{2\}$, $S_2 = \{3\}$, $S_3 = \{4\}$, $S_4 = \{2, 3, 4\}$. The induced Hadamard coloring is of the following scheme.

G_1		G_2		G_3		G_4	
L_1	R_1	L_2	R_2	L_3	R_3	L_4	R_4
u_1	v_1	u_1	v_1	u_1	v_1	u_1	v_1
$\boldsymbol{v_2}$	$\boldsymbol{u_2}$	u_2	v_2	u_2	v_2	$\boldsymbol{v_2}$	$\boldsymbol{u_2}$
u_3	v_3	$\boldsymbol{v_3}$	$\boldsymbol{u_3}$	u_3	v_3	$\boldsymbol{v_3}$	$\boldsymbol{u_3}$
u_4	v_4	u_4	v_4	$\boldsymbol{v_4}$	$\boldsymbol{u_4}$	$\boldsymbol{v_4}$	$\boldsymbol{u_4}$

The swapped pairs are marked bold. Each color graph is a complete bipartite graph $K_{4,4}$.

We first show a property of a Hadamard coloring that u_i and v_i have complementing roles in its color graphs.

Proposition 6.8. *Let H be a Hadamard matrix of order n with $n \in \mathbb{N}, n \geq 2$ and c_H be the corresponding Hadamard coloring of K_{2n}. Then for all $i \in [n]$ and any vertex w of $V(K_{2n}) \setminus \{u_i, v_i\}$, the following hold:*

(i) $u_i w \in E(G_k)$ *if and only if* $v_i w \notin E(G_k)$,

(ii) $N_k(u_i) = (N_k(v_i))^C = V(K_{2n}) \setminus N_k(v_i)$,

(iii) $c(u_i w) = (c(v_i w))^C = [2n] \setminus c(v_i w)$.

Proof. If $u_i w \in E(G_k)$, then u_i and w lie on two partite sets of G_k. Without loss of generality, let $u_i \in L_k$ and $w \in R_k$. Then $v_i \in R_k$, which implies $v_i w \notin E(G_k)$. The opposite direction follows similarly, which proves Item (i). Item (ii) follows from Item (i) and implies Item (iii) directly. □

Let H be a Hadamard matrix of order n and c_H be the induced Hadamard coloring of K_{2n}. An edge e of K_{2n} has one of the forms $u_i u_j$, $v_i v_j$ or $u_i v_j$, for some $i, j \in [n]$. We examine when e is an edge of color graph $G_k, k \in [n]$.

Proposition 6.9. *Let $H = (h_{i,j})_{i,j=1}^n$ be a Hadamard matrix of order n for a positive integer $n \geq 2$ and c_H be the Hadamard coloring induced by H. Let $i, j, k \in [n], i \neq j$ and w a vertex of K_{2n}. Then the following are equivalent:*

(i) $h_{i,k} \cdot h_{j,k} = 1$ *(or $h_{i,k} = h_{j,k}$),*

(ii) $u_i v_j \in E(G_k)$,

(iii) $u_j v_i \in E(G_k)$,

(iv) $u_i u_j \notin E(G_k)$,

(v) $v_i v_j \notin E(G_k)$.

Proof. We look into the pair $(h_{i,k}, h_{j,k})$ and the relationship between the pairs (u_i, v_i) and $(u_j v_j)$ in G_k. There are 4 cases which are shown in Table 3.

$(h_{i,k}, h_{j,k}) = (1,1)$		$(h_{i,k}, h_{j,k}) = (1,-1)$		$(h_{i,k}, h_{j,k}) = (-1,1)$		$(h_{i,k}, h_{j,k}) = (-1,-1)$	
L_k	R_k	L_k	R_k	L_k	R_k	L_k	R_k
u_i	v_i	u_i	v_i	v_i	u_i	v_i	u_i
u_j	v_j	v_j	u_j	u_j	v_j	v_j	u_j

Table 3: $h_{i,k} \cdot h_{j,k}$ determines edges and non-edges in G_k

It is clear that $u_i v_j \in E(G_k)$ means also $u_j v_i \in E(G_k)$. The edge $u_i v_j$ is an edge in the bipartite graph G_k if and only if the pairs (u_i, v_i) and (u_j, v_j) are both fixed or both swapped in G_k, i.e. $(h_{i,k}, h_{j,k}) \in \{(1,1), (-1,-1)\}$. Thus we have the equivalence of Items (i), (ii) and (iii). Moreover, if (u_i, v_i) and (u_j, v_j) are both fixed or both swapped in G_k, then u_i and u_j are on the same partite set of G_k and so $u_i u_j \notin E(G_k)$. The same holds for $v_i v_j$. Edges $u_i u_j$ and $v_i v_j$ are contained in G_k only in the other two cases $(h_{i,k}, h_{j,k}) \in \{(1,-1), (-1,1)\}$, which shows the equivalence of Items (iv), (v) and (i). $\square$

Now we prove that a Hadamard matrix of order n induces an $(n, \frac{n}{2}^+)$-coloring of K_{2n}, where each edge has at least $n/2$ colors.

Lemma 6.10. *Suppose H is a Hadamard matrix of order n where $n \in \mathbb{N}, n \geq 2$ and let $c = c_H$ be the Hadamard coloring of K_{2n} induced by H. Then the following hold.*

(i) For all $i \in [n]$, $|c(u_i v_i)| = n$.

(ii) For all distinct $i, j \in [n], |c(u_i u_j)| = |c(u_i v_j)| = |c(v_i v_j)| = \frac{n}{2}$.

In particular, a Hadamard coloring of K_{2n} is an $(n, \frac{n}{2}^+)$-coloring of K_{2n}.

Proof. First, we observe that for all $i, k \in [n], u_i v_i$ is always an edge of G_k, thus each edge $u_i v_i$ have n colors, which proves Item (i).

For $i, j, k \in [n], i \neq j$, $u_i v_j \in E(G_k)$ if and only if $h_{i,k} \cdot h_{j,k} = 1$ by Proposition 6.9. The case $u_i u_j \in E(G_k)$ is equivalent to $v_i v_j \in E(G_k)$, both if and only if $h_{i,k} \cdot h_{j,k} = -1$. Let r_i be the rows of H, $i \in [n]$. Since the inner product of every two rows of H is 0, we have

$$0 = \langle r_i, r_j \rangle = \sum_{\substack{k=1, \\ h_{i,k} \cdot h_{j,k} = 1}}^{n} 1 + \sum_{\substack{k=1, \\ h_{i,k} \cdot h_{j,k} = -1}}^{n} -1.$$

Therefore,

$$|\{k \in [n] : h_{i,k} \cdot h_{j,k} = 1\}| = |\{k \in [n] : h_{i,k} \cdot h_{j,k} = -1\}| = \frac{n}{2}.$$

It follows that for all distinct $i, j \in [n]$, each edge of the form $u_i v_j$, $u_i u_j$, $v_i v_j$ is an edge of $\frac{n}{2}$ of the color graphs $\{G_k : k \in [n]\}$. Hence each of these edges get $\frac{n}{2}$ colors, which proves Item (ii). □

Proof of Theorem 6.5. Let c_H be the Hadamard coloring of K_{2n} induced by H and G_i the corresponding color graphs of c_H. By Lemma 6.10, c_H is an $(n, \frac{n}{2}^+)$-coloring of K_{2n}. Since all color graphs of c_H are bipartite graphs, c_H has no monochromatic triangle. By Proposition 3.11, we have $R_{n,\frac{n}{2}}(3) > 2n$. □

The following result arises from the equivalence between Hadamard matrices and certain designs.

Corollary 6.11. *Suppose there exists a* $(4m-1, 2m-1, m-1)$ *design for some integer* $m > 1$. *Then*

$$R_{4m,2m}(3) > 8m.$$

Proof. By Theorem 6.3, there exists a Hadamard matrix of order $4m$. The corollary follows by Theorem 6.5. □

Example 6.12.

- A Hadamard matrix of order 2 implies

$$R_{2,1}(3) = R(3) > 4.$$

It is known that $R(3) = 6$, thus the lower bound given by a Hadamard construction in this case is not tight.

- A Hadamard matrix of order 4 implies

$$R_{4,2}(3) > 8.$$

We know from Theorem 4.4 that $R_{4,2}(3) = 9$, hence the lower bound given by a Hadamard construction in this case is best possible.

- The $(7,3,1)$ design in Example 5.1 is a $(4m-1, 2m-1, m-1)$ design with $m = 2$ and yields

$$R_{8,4}(3) > 16.$$

- There exists a $(11,5,2)$ design (see Charles and Dinitz [6]), which is a $(4m-1, 2m-1, m-1)$ design with $m = 3$. This is equivalent to the existence of a Hadamard matrix of order 12 and implies

$$R_{12,6}(3) > 24.$$

Remark 6.13. We compare the lower bound of $R_{2n,n}(3)$ given by the Hadamard construction and the lower bound by the probabilistic method. Let $n \in \mathbb{N}$, by Proposition 3.12,

$$\begin{aligned} R_{2n,n}(3) &> \left\lfloor \frac{3}{e}(2n)^{-1/3} \cdot 2 \right\rfloor \\ &= \left\lfloor \frac{6 \cdot 2^{-1/3}}{e} n^{-1/3} \right\rfloor . \end{aligned}$$

By Theorem 6.5, if there exists a Hadamard matrix of order $2n$, then

$$R_{2n,n}(3) > 4n.$$

Accordingly, the existence of a Hadamard matrix of order $2n$ gives a better lower bound for $R_{2n,n}(3)$. Moreover, we can construct a witness coloring to the lower bound $R_{2n,n}(3) > 4n$ with a Hadamard coloring, as opposed to the probabilistic method which is not constructive.

A Hadamard matrix H can be transformed to another Hadamard matrix equivalent to H by permuting rows/columns, or by multiplying rows/columns of H by -1. An equivalent matrix, however, does not necessarily induce the same Hadamard coloring. In the following we observe how the four operations of Hadamard equivalence affect c_H when applied to H.

For the rest of this section, we assume that $H = (h_{i,j})_{i,j=1}^{n}$ is a Hadamard matrix of order n with $n \in \mathbb{N}$, n even, $n \geq 2$ and $c = c_H$ is the Hadamard coloring of K_{2n} induced by H with swapping set $\mathcal{S}_H$ and color graphs $G_k = K_{L_k,R_k}$ for $k \in [n]$. Moreover, let $H' = (h'_{i,j})_{i,j=1}^{n}$ be an equivalent Hadamard matrix to H which we will define explicitly in each case. Let $c' = c_{H'}$ be the Hadamard coloring of K_{2n} induced by H' whose color graphs are G'_k with partite sets L'_k and R'_k, $k \in [n]$. Let $\mathcal{S}_{H'} = \{S'_k : k \in [n]\}$ be the swapping set of c'.

First note that any edge of the form u_iv_i with $i \in [n]$ always has all n colors in a Hadamard coloring, and so in c'. For this reason we only need to consider how other edges are colored in c'. We recall the essential fact that by Lemma 6.9, the product $h_{i,k} \cdot h_{j,k}$ determines whether u_iv_j, u_jv_i and u_iu_j, v_iv_j are edges of a color graph G_k. In other words, we can consider all edges of G_k by looking into the value of $h_{i,k} \cdot h_{j,k}$ for all $i, j \in [n]$.

Proposition 6.14. *For distinct $k, l \in [n]$, let H' be the equivalent matrix to H attained by swapping columns k and l of H. Then the color graphs G_k and G_l are swapped, i.e. $G'_k = G_l$ and $G'_l = G_k$, and $G'_i = G_i$ for all $i \in [n] \setminus \{k, l\}$. For any edge $e \in E(K_{2n})$,*

(i) $k \in c'(e)$ if and only if $l \in c(e)$,

(ii) $l \in c'(e)$ if and only if $k \in c(e)$.

Proof. Swapping columns k and l of H means swapping the sets S_k and S_l, i.e for all $i \in [n]$, $i \in S'_k$ if and only if $i \in S_l$ and $i \in S'_l$ if and only if $i \in S_k$. Then $h'_{i,k} = h_{i,l}$ and $h'_{i,l} = h_{i,k}$ for all $i \in [n]$. It follows that for all $i, j \in [n]$,

$$h'_{i,k} \cdot h'_{j,k} = h_{i,l} \cdot h_{j,l},$$

and

$$h'_{i,l} \cdot h'_{j,l} = h_{i,k} \cdot h_{j,k}.$$

By Lemma 6.9 and Table 3, for any edge $e \in E(K_{2n})$, $e \in E(G'_k)$ if and only if $e \in G_l$, which implies Item (i). Similarly, $e \in E(G'_l)$ if and only if $e \in E(G_k)$, which proves Item (ii).

Note that Items (i) and (ii) also hold for any edge of the form $e = u_iv_i$ for $i \in [n]$, since in this case $c'(e) = c(e) = [n]$. Also for any edge e with $k, l \notin c(e)$, we have $c'(e) = c(e)$. $\square$

Proposition 6.15. *Let $i, j \in [n], i \neq j$ and H' be attained from H by swapping rows i and j of H. Then the roles of (u_i, v_i) and (u_j, v_j) are swapped in c'. More precisely, for any vertex $w \in V(K_{2n}) \setminus \{u_i, u_j, v_i, v_j\}$,*

(i) $c'(u_iw) = c(u_jw)$ and $c'(u_jw) = c(u_iw)$,

(ii) $c'(v_iw) = c(v_jw)$ and $c'(v_jw) = c(v_iw)$,

(iii) For any edge $e \in \{u_iu_j, v_iv_j, u_iv_j, u_jv_i\}$, we have $c'(e) = c(e)$.

Proof. By swapping rows i and j of H, we swap i and j in all sets S_i of $\mathcal{S}_H$. It means for $k \in [n]$, $i \in S_k$ if and only if $j \in S'_k$, $j \in S_k$ if and only if $i \in S'_k$. In other words, $h'_{i,k} = h_{j,k}$ and $h'_{j,k} = h_{i,k}$. For $l \in [n] \setminus \{i, j\}$, the entries in row l of H does not change, that means for $k \in [n]$, $h'_{l,k} = h_{l,k}$. Thus for $l \in [n] \setminus \{i, j\}$,

$$h'_{i,k} \cdot h'_{l,k} = h_{j,k} \cdot h_{l,k}$$

and

$$h'_{j,k} \cdot h'_{l,k} = h_{i,k} \cdot h_{l,k}.$$

It follows that the roles of u_i and u_j are swapped, as well as v_i and v_j. Let w be a vertex of K_{2n} such that $w \notin \{u_i, u_j, v_i, v_j\}$, then $w \in \{u_l, v_l\}$ for some $l \in [n] \setminus \{i, j\}$. Then by Lemma 6.9,

- $u_i w \in E(G'_k)$ if and only if $u_j w \in E(G_k)$,
- $u_j w \in E(G'_k)$ if and only if $u_i w \in E(G_k)$,

  and
- $v_i w \in E(G'_k)$ if and only if $v_j w \in E(G_k)$,
- $v_j w \in E(G'_k)$ if and only if $v_i w \in E(G_k)$.

The first two facts imply that for any $k \in [n]$, $k \in c'(u_i w)$ if and only if $k \in c(u_j w)$ and $k \in c'(u_j w)$ if and only if $k \in c(u_i w)$. Thus we have Item (i) that $c'(u_i w) = c(u_j w)$ and $c'(u_j w) = c(u_i w)$. Similarly, Item (ii) follows from the last two facts.

For Item (iii), observe that for all $k \in [n]$,

$$h'_{i,k} \cdot h'_{j,k} = h_{j,k} \cdot h_{i,k},$$

which implies that the relationship between (u_i, v_i) and (u_j, v_j) does not change in G_k. Thus the colors of edges $u_i u_j, v_i v_j, u_i v_j, u_j v_i$ stay the same in c', which completes the proof of the proposition. $\square$

Proposition 6.16. *Let $k \in [n]$ and H' be attained from H by negating column k of H. Then $c' = c$.*

Proof. Note that except for the column k of H which is negated, all other columns are the same in H' as in H. Then for all $l \in [n] \setminus \{k\}$, color graphs G'_l of c' are the same as G_l and we only need to consider G'_k. Column k of H' has entries

$$h'_{i,k} = -h_{i,k}$$

for all $i \in [n]$. For the swapping set $\mathcal{S}_H$ it means that $i \in S'_k$ if and only if $i \notin S_k$. For $i \in [n]$, a pair (u_i, v_i) is fixed in G'_k if and only if it is swapped in G_k. However, this swapping causes no change in edges of G_k, since

$$h'_{i,k} \cdot h'_{j,k} = (-h_{i,k}) \cdot (-h_{j,k}) = h_{i,k} \cdot h_{j,k}$$

for all distinct $i, j \in [n]$. That means, for any edge $e \in E(K_{2n})$, e is an edge of G'_k if and only if e is also an edge of G_k. Accordingly, $G'_k = G_k$ and all color graphs stay the same in c' as in c, thus $c' = c$. □

We can transform Hadamard matrix by negating more than one columns. Applying Proposition 6.16 for each column, we obtain the same Hadamard coloring induced by the resulting equivalent matrix.

Corollary 6.17. *Let H' be attained from H by negating columns of H. Then*

$$c' = c_{H'} = c.$$

We consider the last equivalence operation for H.

Proposition 6.18. *Let $i \in [n]$ and H' be attained from H by negating row i of H. Then the roles of u_i and v_i are swapped. More precisely,*

(i) $c'(u_iv_i) = c(u_iv_i)$,

(ii) $c'(e) = c(e)$ *for all edges* $e \in E(K_{2n})$ *that are not incident to* u_i *or* v_i,

(iii) $c'(u_iw) = c(v_iw)$ *and* $c'(v_iw) = c(u_iw)$ *for any vertex* $w \in V(K_{2n}) \setminus \{u_i, v_i\}$.

Proof. Item (i) is trivial since $c'(u_iv_i) = c(u_iv_i) = [n]$.

For $\mathcal{S}_{H'}$, $i \in S'_k$ if and only if $i \notin S_k$ for all $k \in [n]$. For all $k \in [n]$, $h'_{i,k} = -h_{i,k}$ and for all $j \in [n] \setminus i$, we have $h'_{j,k} = h_{j,k}$. For $j, l \in [n] \setminus \{i\}$, since

$$h'_{j,k} \cdot h'_{l,k} = h_{j,k} \cdot h_{l,k},$$

Lemma 6.9 implies that the property of edges $u_ju_l, u_jv_l, u_lv_j, v_jv_l$ does not change in all G_k, which proves Item (ii).

For Item (iii), notice that

$$h'_{i,k} \cdot h'_{j,k} = -h_{i,k} \cdot h_{j,k}.$$

Let w be a vertex of K_{2n}, $w \notin \{u_i, v_i\}$ and $k \in [n]$, then $w \in \{u_j, v_j :\ j \in [n] \setminus \{j\}\}$. It follows that $u_iw \in E(G'_k)$ if and only if $u_iw \notin E(G_k)$, which by Proposition 6.8 is equivalent to $v_iw \in E(G_k)$. Hence $k \in c'(u_iw)$ if and only if $k \in c(v_iw)$, which implies $c'(u_iw) = c(v_iw)$. Moreover, by Proposition 6.8,

$$c'(v_iw) = (c'(u_iw))^C = (c(v_iw))^C = c(u_iw).$$

This completes the proof of Item (iii) and the proposition. □

Table 4 summarizes the effect of 4 Hadamard equivalence operations on c_H. Let $i, j, k \in [n]$, $i \neq j$.

Hadamard equivalence	$\mathcal{S}_H$	c_H
Swap columns i, j	swap sets S_i and S_j	swap color graphs G_i and G_j
Swap rows i, j	swap i and j in all sets $S \subset \mathcal{S}_H$	relabel (u_i, v_i) and (u_j, v_j)
Negating column i	new set $S_i' = [n] \backslash S_i$	coloring stays the same
Negating row i	new set $S_k' = S_k \Delta \{i\}$ for all $k \in [n]$	swap u_i and v_i

Table 4: Hadamard equivalence applied on H and Hadamard coloring c_H

The symmetric difference $S_k \Delta \{i\}$ means $i \in S_k'$ if and only if $i \notin S_k$.

Example 6.19. We consider again H_0 and the induced Hadamard coloring. Let $H = H_0$ and $c_H = c_{H_0}$ be the Hadamard coloring of K_8 induced by H with color graphs G_k, $k \in [4]$. We assume furthermore that H' is an equivalent matrix to H and $c' = c_{H'}$ is the Hadamard coloring induced by H' with color graphs G_k', $k \in [4]$. The structure of H' is determined explicitly in each individual case below.

$$H_0 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \qquad \begin{array}{cc} \multicolumn{2}{c}{G_1} \\ L_1 & R_1 \\ u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \\ u_4 & v_4 \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{G_2} \\ L_2 & R_2 \\ u_1 & v_1 \\ v_2 & u_2 \\ v_3 & u_3 \\ u_4 & v_4 \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{G_3} \\ L_3 & R_3 \\ u_1 & v_1 \\ v_2 & u_2 \\ u_3 & v_3 \\ v_4 & u_4 \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{G_4} \\ L_4 & R_4 \\ u_1 & v_1 \\ u_2 & v_2 \\ v_3 & u_3 \\ v_4 & u_4 \end{array}$$

(i) Let $H' = H_1$ be attained from H by swapping the first and second columns of H. Then the color graphs G_1 and G_2 are swapped, i.e. $G_1' = G_2$ and $G_2' = G_1$.

$$H_1 = \begin{pmatrix} \mathbf{1} & \mathbf{1} & 1 & 1 \\ \mathbf{-1} & \mathbf{1} & -1 & 1 \\ \mathbf{-1} & \mathbf{1} & 1 & -1 \\ \mathbf{1} & \mathbf{1} & -1 & -1 \end{pmatrix} \qquad \begin{array}{cc} \multicolumn{2}{c}{\boldsymbol{G_1'}} \\ L_1' & R_1' \\ \boldsymbol{u_1} & \boldsymbol{v_1} \\ \boldsymbol{v_2} & \boldsymbol{u_2} \\ \boldsymbol{v_3} & \boldsymbol{u_3} \\ \boldsymbol{u_4} & \boldsymbol{v_4} \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{\boldsymbol{G_2'}} \\ L_2' & R_2' \\ \boldsymbol{u_1} & \boldsymbol{v_1} \\ \boldsymbol{u_2} & \boldsymbol{v_2} \\ \boldsymbol{u_3} & \boldsymbol{v_3} \\ \boldsymbol{u_4} & \boldsymbol{v_4} \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{G_3'} \\ L_3' & R_3' \\ u_1 & v_1 \\ v_2 & u_2 \\ u_3 & v_3 \\ v_4 & u_4 \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{G_4'} \\ L_4' & R_4' \\ u_1 & v_1 \\ u_2 & v_2 \\ v_3 & u_3 \\ v_4 & u_4 \end{array}$$

(ii) Let $H' = H_2$ be obtained from H by swapping rows 1 and 2.

$$H_2 = \begin{pmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{-1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

G_1'		G_2'		G_3'		G_4'	
L_1'	R_1'	L_2'	R_2'	L_3'	R_3'	L_4'	R_4'
$\boldsymbol{u_1}$	$\boldsymbol{v_1}$	$\boldsymbol{v_1}$	$\boldsymbol{u_1}$	$\boldsymbol{v_1}$	$\boldsymbol{u_1}$	$\boldsymbol{u_1}$	$\boldsymbol{v_1}$
$\boldsymbol{u_2}$	$\boldsymbol{v_2}$	$\boldsymbol{u_2}$	$\boldsymbol{v_2}$	$\boldsymbol{u_2}$	$\boldsymbol{v_2}$	$\boldsymbol{u_2}$	$\boldsymbol{v_2}$
u_3	v_3	v_3	u_3	u_3	v_3	v_3	u_3
u_4	v_4	u_4	v_4	v_4	u_4	v_4	u_4

In this particular case, the color graphs G_1 and G_4 do not change in c', since $h_{1,1} = h_{2,1} = 1$ and $h_{1,4} = h_{2,4} = 1$, columns $h_1' = h_1$, $h_4' = h_4$. Figure 13 illustrates the comparison between G_2, G_3 and G_2', G_3'.

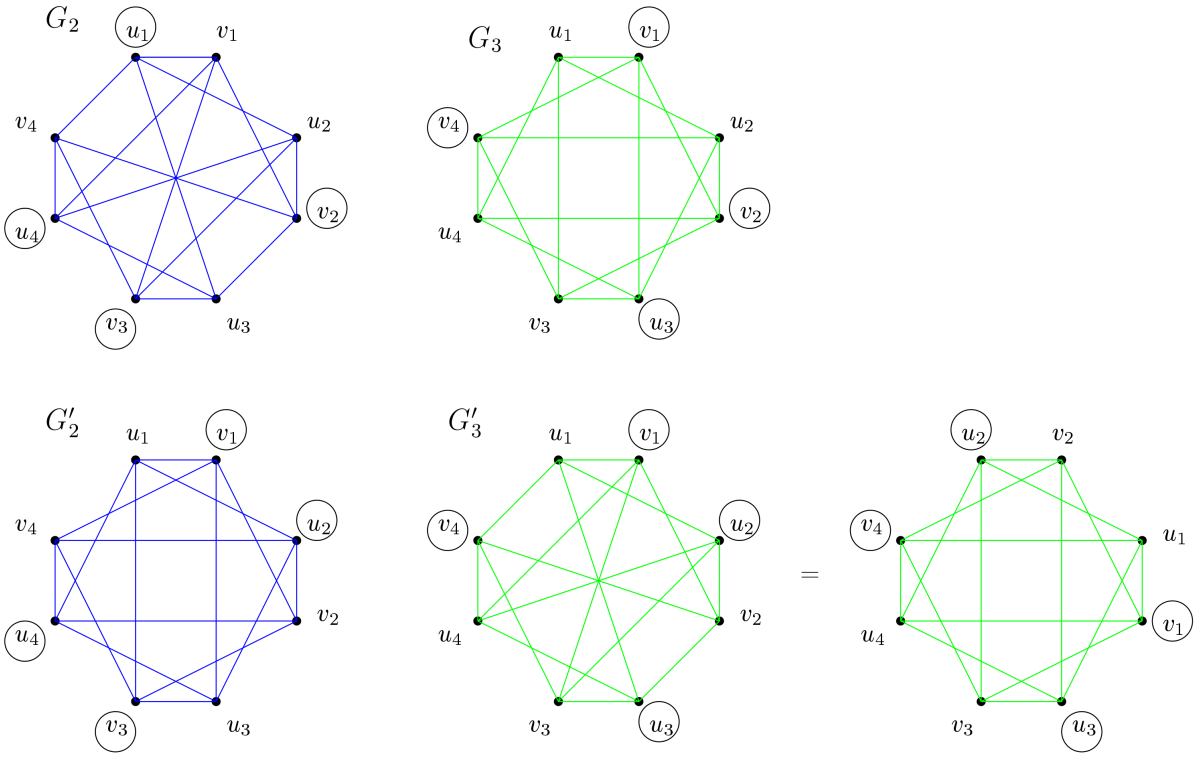


Figure 13: Color graphs of c and c_{H_2} in color 2 and 3

The elements of the partite sets L_k and L_k' are circled for all $k \in [4]$. The second figure of G_3' is created by swapping the positions of u_1 and u_2, as well as v_1 and v_2. The result is the same as G_3, with the swapping roles of (u_1, v_1) and (u_2, v_2).

Moreover, we have in this case

$$H_2 = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} =: H_2'$$

and

$$H_2' = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} = H.$$

We attain H_2' from H_2 by negating columns 2 and 3 of H_2. By Corollary 6.17, $c' = c_{H_2} = c_{H_2'}$. On the other hand, H_2' can be attained from H by swapping the second and third columns of H. By Proposition 6.14, it means the color graphs G_2 and G_3 are swapped, we have $G_2' = G_3$ and $G_3' = G_2$. This effect is also reflected in Figure 13.

(iii) Let $H' = H_3$ be attained from H by negating the second column of H.

$$H_3 = \begin{pmatrix} 1 & \mathbf{-1} & 1 & 1 \\ 1 & \mathbf{1} & -1 & 1 \\ 1 & \mathbf{1} & 1 & -1 \\ 1 & \mathbf{-1} & -1 & -1 \end{pmatrix}$$

G_1'		G_2'		G_3'		G_4'	
L_1'	R_1'	L_2'	R_2'	L_3'	R_3'	L_4'	R_4'
u_1	v_1	$\boldsymbol{v_1}$	$\boldsymbol{u_1}$	u_1	v_1	u_1	v_1
u_2	v_2	$\boldsymbol{u_2}$	$\boldsymbol{v_2}$	v_2	u_2	u_2	v_2
u_3	v_3	$\boldsymbol{u_3}$	$\boldsymbol{v_3}$	u_3	v_3	v_3	u_3
u_4	v_4	$\boldsymbol{v_4}$	$\boldsymbol{u_4}$	v_4	u_4	v_4	u_4

The only change are in G_2, the partite sets L_2 and R_2 are swapped, but the edges stay the same in G_2 and in all other color graphs.

(iv) Let $H' = H_4$ be attained from H by negating the second row of H.

$$H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \mathbf{-1} & \mathbf{1} & \mathbf{1} & \mathbf{-1} \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

G_1'		G_2'		G_3'		G_4'	
L_1'	R_1'	L_2'	R_2'	L_3'	R_3'	L_4'	R_4'
u_1	v_1	u_1	v_1	u_1	v_1	u_1	v_1
$\boldsymbol{v_2}$	$\boldsymbol{u_2}$	$\boldsymbol{u_2}$	$\boldsymbol{v_2}$	$\boldsymbol{u_2}$	$\boldsymbol{v_2}$	$\boldsymbol{v_2}$	$\boldsymbol{u_2}$
u_3	v_3	v_3	u_3	u_3	v_3	v_3	u_3
u_4	v_4	u_4	v_4	v_4	u_4	v_4	u_4

In the coloring c', the roles of u_2 and v_2 are swapped. Figure 14 illustrates the color graphs of c and c', where vertices of L_k and L_k' are circled for all $k \in [4]$. Let w be a vertex of K_8 with $w \neq u_2, v_2$. By Propositions 6.8, for all $k \in [4]$,

- $u_2 w \in E(G_k)$ if and only if $v_2 w \in E(G_k')$,
- $v_2 w \in E(G_k)$ if and only if $u_2 w \in E(G_k')$,
- $c(u_2 w) = c'(v_2 w)$ and $c(v_2 w) = c'(u_2 w)$.

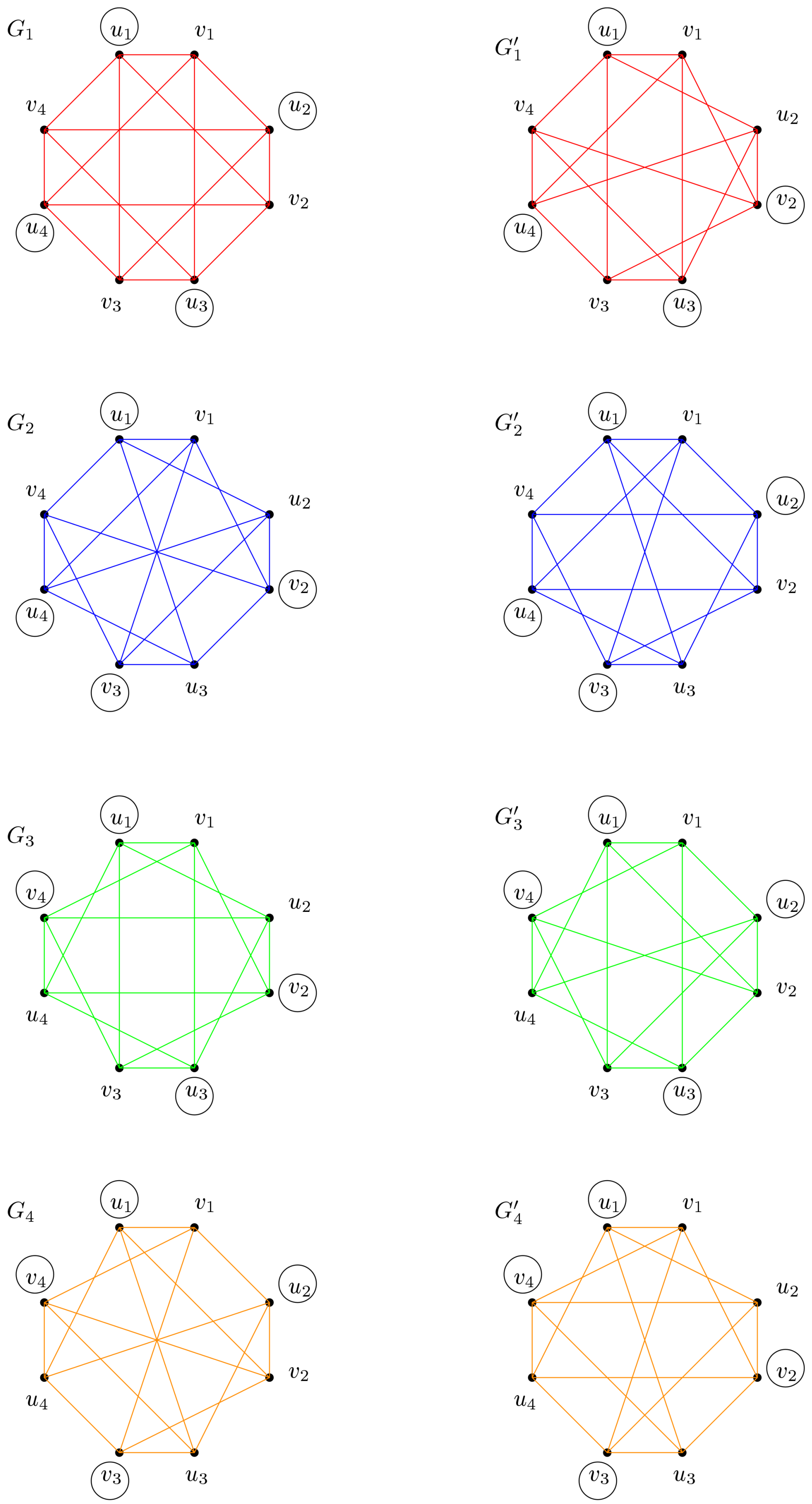


Figure 14: Color graphs of $c = c_H$ and $c' = c_{H_4}$

7 Upper bounds for $R_{s,t}(3)$

In Section 4, we proved that $R_{3,2}(3) = 5$ and $R_{4,2}(3) = 9$. In the previous section, we gained a lower bound for $R_{2n,n}(3) > 4n$ from a Hadamard matrix of order $2n$. In this section, we investigate upper bounds for Ramsey numbers of the form $R_{s,t}(3)$. We first consider upper bounds for general cases, then apply these bounds for some concrete cases.

7.1 General bounds

In Proposition 4.3, we use information about $R_{3,2}(3)$ to find an upper bound for $R_{4,2}(3)$. This idea can be generalized to find the relationship between $R_{s,t}(3)$ and $R_{s-1,t}(3)$.

Proposition 7.1. *For all $s, t \in \mathbb{N}$ with $s \geq t \geq 2$,*

$$R_{s,t}(3) \leq \left\lfloor \frac{s \cdot R_{s-1,t}(3)}{t} - \frac{s}{t} \right\rfloor + 2.$$

Proof. Let $n := R_{s,t}(3) - 1$ and c be an (s,t)-coloring of K_n. Note that c has no monochromatic triangle. Fix some vertex $u \in V(K_n)$. Note that u is incident to $n-1$ edges, each edge has t distinct colors. It follows that, there exist $m := \left\lceil \frac{(n-1)\cdot t}{s} \right\rceil$ distinct vertices $v_1, \dots, v_m \in V(K_n) \backslash \{u\}$ such that $uv_1, \dots, uv_m$ have some color $i \in [s]$. In other words, $|N_i(u)| \geq m$ and $\{v_i : i \in [m]\} \subseteq N_i(u)$. Since c has no monochromatic K_3, it follows that the induced subgraph $K_n[\{v_1, \dots, v_m\}]$ has no edge with color i. In particular, this graph is $(s-1,t)$-colored and avoids monochromatic K_3. By definition of $R_{s-1,t}(3)$ and using the fact that $n = R_{s,t}(3) - 1$, we obtain:

$$\begin{aligned}
& \left\lceil \frac{(R_{s,t}(3) - 2) \cdot t}{s} \right\rceil \leq R_{s-1,t}(3) - 1 \\
\Leftrightarrow \quad & R_{s,t}(3) - 2 \leq \left\lfloor (R_{s-1,t}(3) - 1) \cdot \frac{s}{t} \right\rfloor \\
\Leftrightarrow \quad & R_{s,t}(3) \leq \left\lfloor \frac{s \cdot R_{s-1,t}(3)}{t} - \frac{s}{t} \right\rfloor + 2.
\end{aligned}$$

This completes the proof of the proposition.

□

In case $t = 2, k = 3$ we have

- $R_{2,2}(3) = 3$,
- $R_{3,2}(3) = 5$,
- $R_{4,2}(3) \leq 10$.

Applying Proposition 3.4 inductively, we can find a bound for $R_{s,2}(3)$ with $s \geq 5$.

Theorem 7.2. *For all positive integers $s \geq 5$,*

$$R_{s,2}(3) \leq \frac{5}{6} \cdot \frac{s!}{2^{s-3}} - s! \sum_{i=3}^{s-2} \frac{i-2}{(i+2)!\, 2^{s-i-1}}. \tag{7.1}$$

Proof. We proceed with a proof by induction on s.

For $s = 5$, by Proposition 7.1 we have

$$R_{5,2}(3) \leq \frac{5 \cdot R_{4,2}(3)}{2} - \frac{5}{2} + 2 \leq \frac{5 \cdot 10}{2} - \frac{1}{2} = 25 - \frac{1}{2}.$$

The right side of the inequality (7.1) is equal to

$$\frac{5}{6} \cdot \frac{5!}{2^2} - \frac{5!}{5!\, 2^{5-1-3}} = 25 - \frac{1}{2}.$$

Hence, the induction starts for $s = 5$. For the induction step, we consider

$$\begin{aligned} R_{s+1,2}(3) &\leq \frac{(s+1) \cdot R_{s,2}(3)}{2} - \frac{s+1}{2} + 2 \\ &= \frac{(s+1) \cdot R_{s,2}(3)}{2} - \frac{s-3}{2}. \end{aligned}$$

By the induction hypothesis, we have

$$\begin{aligned} R_{s+1,2}(3) &\leq \frac{s+1}{2} \left(\frac{5}{6} \cdot \frac{s!}{2^{s-3}} - s! \sum_{i=3}^{s-2} \frac{i-2}{(i+2)!\, 2^{s-i-1}} \right) - \frac{s-3}{2} \\ &= \frac{5}{6} \cdot \frac{(s+1)!}{2^{s-2}} - (s+1)! \sum_{i=3}^{s-2} \frac{i-2}{(i+2)!\, 2^{s-i}} - \frac{s-3}{2} \\ &= \frac{5}{6} \cdot \frac{(s+1)!}{2^{s-2}} - (s+1)! \sum_{i=3}^{\boldsymbol{s-1}} \frac{i-2}{(i+2)!\, 2^{s-i}}, \end{aligned}$$

which completes the induction and the proof of the theorem. □

For Ramsey numbers of the form $R_{s,t}(3)$ where $t \geq 3$, we get a similar formula. First we consider the case $s = t + 1$.

Proposition 7.3. *For all integers $t \geq 3$, $R_{t+1,t}(3) = 3$.*

Proof. There are t colors on each edge, hence $3t$ colors are needed to color K_3. From these only $(t + 1)$ colors are distinct. The pigeonhole principle implies that, there exist one color $i \in [t + 1]$ that appears on at least $\lceil \frac{3t}{t+1} \rceil$ edges. We have that

$$\left\lceil \frac{3t}{t+1} \right\rceil = \left\lceil \frac{3t+3-3}{t+1} \right\rceil = \left\lceil 3 - \frac{3}{t+1} \right\rceil = 3,$$

since for all $t \geq 3, 0 < \frac{3}{t+1} < 1$, hence $3 > 3 - \frac{3}{t+1} > 2$. That means, there is a monochromatic triangle in color i, showing $R_{t+1,t}(3) \leq 3$. Clearly $R_{t+1,t}(3) \geq 3$, which proves the proposition. □

Theorem 7.4. *For all integers* $t \geq 3, n \geq 2$,

$$R_{t+n,t}(3) \leq \frac{3(t+n)!}{(t+1)!\, t^{n-1}} + (t+n)! \sum_{i=2}^{n} \frac{t-i}{(t+i)!\, t^{n+1-i}}. \tag{7.2}$$

Let $s := t+n \geq t+2$, *then we can rewrite (7.2):*

$$R_{s,t}(3) \leq \frac{3 \cdot s!}{(t+1)!\, t^{s-t-1}} + s! \sum_{i=2}^{s-t} \frac{t-i}{(t+i)!\, t^{s-t+1-i}}.$$

Proof. We fix t and apply induction on n. For $n = 2$, by Proposition 7.1,

$$\begin{aligned} R_{t+2,t}(3) &\leq \frac{(t+2)\, R_{t+1,t}(3)}{t} - \frac{t+2}{t} + 2 \\ &= \frac{(t+2)\, R_{t+1,t}(3)}{t} + \frac{t-2}{t}. \end{aligned}$$

Since $t \geq 3$, $R_{t+1,t}(3) = 3$ by Proposition 7.3,

$$R_{t+2,t}(3) \leq \frac{3(t+2)}{t} + \frac{t-2}{2}.$$

On the other hand, the right side of (7.2) is equal to

$$\frac{3(t+2)!}{(t+1)!\, t} + \frac{(t+2)!\, (t-2)}{(t+2)!\, t^{2+1-2}} = \frac{3(t+2)}{t} + \frac{t-2}{t}.$$

Therefore, the theorem holds for $n = 2$.

We now consider the induction step. By Proposition 7.1,

$$\begin{aligned} R_{t+n+1,t}(3) &\leq \frac{(t+n+1)\, R_{t+n,t}(3)}{t} - \frac{t+n+1}{t} + 2 \\ &= \frac{(t+n+1)\, R_{t+n,t}(3)}{t} + \frac{t-n-1}{t}. \end{aligned}$$

The induction hypothesis implies that

$$\begin{aligned} R_{t+n+1,t}(3) &\leq \frac{t+n+1}{t}\left[\frac{3(t+n)!}{(t+1)!\,t^{n-1}} + (t+n)!\sum_{i=2}^{n}\frac{t-i}{(t+i)!\,t^{n+1-i}}\right] + \frac{t-n-1}{t} \\ &= \frac{3(t+n+1)!}{(t+1)!\,t^{n}} + (t+n+1)!\sum_{i=2}^{n}\frac{t-i}{(t+i)!\,t^{n+2-i}} + \frac{t-n-1}{t} \\ &= \frac{3(t+n+1)!}{(t+1)!\,t^{n}} + (t+n+1)!\sum_{i=2}^{\boldsymbol{n+1}}\frac{t-i}{(t+i)!\,t^{n+2-i}}, \end{aligned}$$

which completes the proof of the theorem. $\square$

We use Stirling's formula to approximate this upper bound. For $n \in \mathbb{N}$,

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}, \quad n \to \infty.$$

Theorem 7.5. *Let $k \geq 2$ be fixed. Then for all $n \geq 3$,*

$$R_{kn,n}(3) \leq [1+o(1)]\,\sqrt{k}(kn-n+2)\,e^{n}\,\left(\frac{k}{e}\right)^{kn}.$$

Proof. By Theorem 7.4 we have

$$R_{kn,n}(3) \leq \underbrace{\frac{3\cdot(kn)!}{(n+1)!\,n^{kn-n-1}}}_{=:A} + \sum_{i=2}^{kn-n}\underbrace{\frac{(kn)!\,(n-i)}{(n+i)!\,n^{kn-n+1-i}}}_{=:B_i}.$$

We first approximate the term A.

$$\begin{aligned} A &\sim \frac{3\sqrt{2k\pi n}\left(\frac{kn}{e}\right)^{kn}}{\sqrt{2\pi(n+1)}\left(\frac{n+1}{e}\right)^{n+1}\,n^{kn-n-1}} \\ &= \underbrace{\sqrt{\frac{n}{n+1}}}_{\sim 1}\;\frac{3\sqrt{k}\,(kn)^{kn}\,e^{n+1-kn}}{(n+1)^{n+1}\,n^{kn-n-1}} \\ &\sim 3\sqrt{k}\,e^{n+1-kn}\frac{k^{kn}\,n^{kn}}{(n+1)^{n+1}\,n^{kn-n-1}} \\ &= 3\sqrt{k}\,e^{n+1-kn}\,k^{kn}\frac{n^{n+1}\,n^{kn-n-1}}{(n+1)^{n+1}\,n^{kn-n-1}} \\ &= 3\sqrt{k}\,e^{n+1-kn}\,k^{kn}\underbrace{\left(\frac{n}{n+1}\right)^{n+1}}_{\sim e^{-1}}\;\underbrace{\left(\frac{n}{n}\right)^{kn-n-1}}_{=1} \\ &\sim 3\sqrt{k}\,e^{n-kn}\,k^{kn} \\ &= 3\sqrt{k}\,e^{n}\,\left(\frac{k}{e}\right)^{kn}. \end{aligned}$$

For $i \in [2, kn-n]$ we approximate B_i.

$$\begin{aligned}
B_i &\sim \frac{\sqrt{2\pi kn}\,\left(\frac{kn}{e}\right)^{kn}\ (n-i)}{\sqrt{2\pi(n+i)}\,\left(\frac{n+i}{e}\right)^{n+i}\ n^{kn-n+1-i}} \\
&= \underbrace{\sqrt{\frac{n}{n+i}}}_{\sim 1}\ \sqrt{k}\ k^{kn}\ e^{n+i-kn}\ (n-i)\frac{n^{kn}}{(n+i)^{n+i}\ n^{kn-n+1-i}} \\
&\sim \sqrt{k}\ k^{kn}\ e^{n+i-kn}\ (n-i)\frac{n^{n+i}\ n^{kn-n-i}}{(n+i)^{n+i}\ n^{kn-n-i}} \\
&= \sqrt{k}\ k^{kn}\ e^{n+i-kn}\ \underbrace{\frac{n-i}{n}}_{\sim 1}\ \underbrace{\left(\frac{n}{n+i}\right)^{n+i}}_{\sim e^{-i}}\ \underbrace{\left(\frac{n}{n}\right)^{kn-n-i}}_{=1} \\
&\sim \sqrt{k}\ k^{kn}\ e^{n-kn} \\
&= \sqrt{k}\ e^{n}\ \left(\frac{k}{e}\right)^{kn}.
\end{aligned}$$

Altogether we obtain

$$\begin{aligned}
R_{kn,n}(3) &\leq A + \sum_{i=2}^{kn-n} B_i \\
&\sim 3\sqrt{k}\ e^{n}\ \left(\frac{k}{e}\right)^{kn} + \sum_{i=2}^{kn-n} \sqrt{k}\ e^{n}\ \left(\frac{k}{e}\right)^{kn} \\
&\sim \sqrt{k}\ e^{n}\ \left(\frac{k}{e}\right)^{kn}\ [3 + (kn-n-1)] \\
&\sim \sqrt{k}(kn-n+2)\ e^{n}\ \left(\frac{k}{e}\right)^{kn}.
\end{aligned}$$

This completes the proof of the theorem. □

In case $k = 2$, Theorem 7.5 implies the following.

Corollary 7.6. *For all positive integers $n \geq 3$,*

$$R_{2n,n}(3) \leq [1 + o(1)]\sqrt{2}(n+2)\left(\frac{4}{e}\right)^{n}.$$

Remark 7.7. Together with results from the previous section about the Hadamard construction, by Theorem 6.5, if there exists a Hadamard matrix of order $2n$ for $n \in \mathbb{N}$, then

$$4n < R_{2n,n}(3) \leq [1 + o(1)]\sqrt{2}(n+2)\left(\frac{4}{e}\right)^{n}.$$

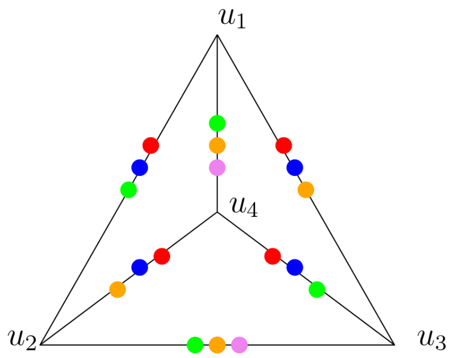
In case $n = 2$, the lower bound resulted from Hadamard colorings is tight. We have $R_{4,2}(3) > 8$ and $R_{4,2}(3) = 9$. However, for general cases, it stays unclear how close the lower bound by a Hadamard coloring $(4n)$ is to the actual value of $R_{2n,n}(3)$. At this point, there is a great gap between the linear bound $4n$ and the exponential bound above.

7.2 Application

We will apply the results above to compute some concrete (s,t) Ramsey numbers of the form $R_{s,t}(3)$.

Proposition 7.8. *We have* $R_{5,3}(3) = 5$.

Proof. Figure 15 shows a $(5,3)$-coloring of K_4 without monochromatic triangle. We identify color 5 with purple. This proves the lower bound $R_{5,3}(4) > 4$.



$$\begin{aligned}
c(u_1u_2) &= \{1,2,3\}\\
c(u_1u_3) &= \{1,2,4\}\\
c(u_1u_4) &= \{3,4,5\}\\
c(u_2u_3) &= \{3,4,5\}\\
c(u_2u_4) &= \{1,2,4\}\\
c(u_3u_4) &= \{1,2,3\}
\end{aligned}$$

Figure 15: Witness coloring to the lower bound $R_{5,3}(3) > 4$

On the other hand, by Proposition 7.3, $R_{4,3}(3) = 3$, and by Proposition 7.1,

$$\begin{aligned}
R_{5,3}(3) &\leq \left\lfloor \frac{5(R_{4,3}(3)-1)}{3} \right\rfloor + 2\\
&= \left\lfloor \frac{5\cdot(3-1)}{3} \right\rfloor + 2\\
&= 5,
\end{aligned}$$

which proves the upper bound and hence $R_{5,3}(3) = 5$. □

Now we consider $R_{6,3}(3)$. First note that by Propositions 7.1 and 7.8,

$$\begin{aligned}
R_{6,3}(3) &\leq \left\lfloor \frac{6\cdot(R_{5,3}(3)-1)}{3} \right\rfloor + 2\\
&= \left\lfloor \frac{6\cdot 4}{3} \right\rfloor + 2 = 10.
\end{aligned}$$

Next we prove $R_{6,3}(9) < 10$.

Proposition 7.9. *We have* $R_{6,3}(3) = 9$.

Proof. For the lower bound, we recall that $R_{7,4}(3) = 9$ by Proposition 5.23. It follows from Proposition 3.4 (ii) that

$$R_{6,3}(3) \geq R_{7,4}(3) = 9.$$

We now prove the upper bound. In order to show that $R_{6,3}(3) \leq 9$, we proceed with a similar proof as the alternative proof of Theorem 4.4 that $R_{4,2}(3) \leq 9$. For the sake of contradiction, we assume that c is a $(6,3)$-coloring of K_9 without monochromatic triangle. Let $V(K_9) = \{u_1, \dots, u_9\}$. Fix some vertex of K_9, say u_1. There are 8 edges at u_1, which have in total $8 \cdot 3 = 24$ colors. By the pigeonhole principle, there are some color $i \in [6]$, say color 1, that appears at least $\lceil \frac{24}{6} \rceil = 4$ times on all edges at u_1, i.e. $|N_1(u_1)| \geq 4$. For any edge uv with $u, v \in N_1(u_1)$, if $1 \in c(uv)$ then uv, u_1u, u_1v will form a monochromatic triangle in color 1. By the assumption that c has no monochromatic triangle, $1 \notin c(uv)$ for all $u, v \in N_1(u_1)$. Hence in $K_9[N_1(u_1)]$ we have a $(5,3)$-coloring. Since $R_{5,3}(3) = 5$ by Proposition 7.8, it must follow that $|N_1(u_1)| \leq 4$. Altogether we have $|N_1(u_1)| = 4$.

There are now 20 colors left on all edges at u_1. Again there is some color $i \in [6] \setminus \{1\}$ that appears at least $\lceil \frac{20}{5} \rceil = 4$ times on those edges. By the same argument as for $N_1(u_1)$, we have $|N_i(u_j)| = 4$ for all colors $i \in [6]$ and all vertices $u_j \in V(K_9)$. Therefore, each color graph G_i of $c, i \in [6]$, is a triangle-free 4-regular graph on 9 vertices. Such a graph does not exists (see Figure 6 and [18]), which proves that there is no $(6,3)$-coloring of K_9 without monochromatic triangle. This completes the proof of the proposition. $\square$

8 Off-diagonal (s,t) Ramsey numbers

In this section we introduce the off-diagonal version of Ramsey numbers for set-colorings.

Definition 8.1. For positive integers $s \geq t$, and $k_1, k_2, \dots, k_s \geq 2$, we define $R_{s,t}(k_1, \dots, k_s)$ to be the least positive integer n such that for any (s,t)-coloring of K_n, there is a monochromatic K_{k_i} in color i for at least one $i \in [s]$.

Remark 8.2. We note some special cases of t, s and k_i, $i \in [s]$.

- By definition $R_{s,t}(k, \dots, k) = R_{s,t}(k)$.
- In case $t = 1$, we have the classical off-diagonal Ramsey number $R_{s,1}(k_1, k_2, \dots, k_s) = R(k_1, k_2, \dots, k_s)$, which denotes the least positive integer n such that any s-coloring of K_n has a monochromatic K_{k_i} in color i for some $i \in [s]$.
- $R_{s,s}(k_1, \dots, k_s) = \min\{k_i : i \in [s]\}$.
- $R_{s,t}(2, k_2, \dots, k_s) = R_{s-1,t}(k_2, \dots, k_s)$.
- $R_{s,t}(k_1, \dots, k_s)$ is invariant under the reordering of $k_1, \dots, k_s$.

For the classical off-diagonal Ramsey numbers, there is an implicit upper bound (see Greenwood and Gleason [12]):

$$R(k_1, \dots, k_s) \leq 2 - s + \sum_{i=1}^{s} R(k_1, \dots, k_{i-1}, k_i - 1, k_{i+1}, \dots, k_s)$$

In case $s = 2$ this bound reduces to

$$R_{2,1}(k_1, k_2) = R(k_1, k_2) \leq R(k_1 - 1, k_2) + R(k_1, k_2 - 1).$$

We prove a similar formulation for $R_{s,t}(k_1, \dots, k_s)$.

Proposition 8.3. *For all positive integers $s, t, s \geq t$ and $k_1, \dots, k_s \geq 2$, we have*

$$R_{s,t}(k_1, k_2, \dots, k_s) \leq 2 - \frac{s}{t} + \frac{1}{t} \sum_{i=1}^{s} R_{s,t}(k_1, \dots, k_{i-1}, k_i - 1, k_{i+1}, \dots, k_s)$$

Proof. Let $n = R_{s,t}(k_1, k_2, \dots, k_s) - 1$ and let c be an (s,t)-coloring of K_n without monochromatic K_{k_i} for all $i \in [s]$. Fix some vertex $u \in V(K_n)$, then u is incident to $(n-1)$ edges. For $i \in [s]$, let $V_i := N_i(u) = \{v \in V(K_n) : i \in c(uv)\}$, i.e. V_i contains all neighbors of u that are connected to u by an edge with color i. Since c is an (s,t)-coloring, we have

$$|V_1| + |V_2| + \cdots + |V_s| = (n-1) \cdot t,$$

which implies

$$n = 1 + \frac{1}{t}\sum_{i=1}^{s} |V_i|.$$

On the other hand, V_{k_i} contains no monochromatic K_{k_i-1} in color i, otherwise together with u, V_i would form a K_{k_i} in color i. Furthermore, V_i contains no monochromatic K_{k_j} in color j for all $j \in [s]$, $j \neq i$. Altogether,

$$|V_i| \leq R_{s,t}(k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_s) - 1$$

for all $i \in [s]$. Since $n = 1 + \frac{1}{t}\sum_{i=1}^{s} |V_i|$ and $n = R_{s,t}(k_1, \ldots, k_s) - 1$, we have

$$\begin{aligned} R_{s,t}(k_1, \ldots, k_s) &= 1 + n \\ &= 2 + \frac{1}{t}\sum_{i=1}^{s} |V_i| \\ &\leq 2 + \frac{1}{t}\left(\sum_{i=1}^{s} R_{s,t}(k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_s) - s\right), \end{aligned}$$

which completes the proof of the proposition. □

We want to find an upper bound on $R_{s,t}(k_1, \ldots, k_s)$. For $s = 2$, Erdős and Szekeres [9] proved in 1935 that

$$R_{2,1}(k_1, k_2) = R(k_1, k_2) \leq \binom{k_1 + k_2 - 2}{k_1 - 1}.$$

We attempt below to find an equivalent formula for general s. We first consider a special case, where $k_i = 2$ in at least $(s - t + 1)$ positions of k_i.

Lemma 8.4. *For all $s, t \in \mathbb{N}$, $s > t$ and $\sum_{i=1}^{s} k_i \leq 2s + t - 1$, we have*

$$R_{s,t}(k_1, \ldots, k_s) = 2.$$

Proof. Since $k_i \geq 2$ for all $i \in [s]$, we have $\sum_{i=1}^{s} k_i \geq 2s$. Then the assumption $\sum_{i=1}^{s} k_i \leq 2s + t - 1$ implies that there are at most $(t - 1)$ positions of k_i such that $k_i > 2$, in other words, there are at least $(s - t + 1)$ positions of k_i with $k_i = 2$. Without loss of generality, let $k_i = k_2 = \cdots = k_{s-t+1} = 2$. It follows that

$$\begin{aligned} R_{s,t}(k_1, \ldots, k_s) &= R_{s,t}(k_1 = 2, k_2 = 2, \ldots, k_{s-t+1} = 2, k_{s-t+2}, \ldots, k_s) \\ &= R_{s-1,t}(k_2 = 2, \ldots, k_{s-t+1} = 2, k_{s-t+2}, \ldots, k_s) \\ &= \ldots \\ &= R_{t,t}(k_{s-t+1} = 2, k_{s-t+2}, \ldots, k_s) \\ &= \min\{2, k_{s-t+2}, \ldots, k_s\} \\ &= 2, \end{aligned}$$

which proves the lemma. □

Example 8.5. For $s = 5, t = 3$ and $\sum_{i=1}^{5} k_i = 2s + t - 1 = 12$, there are only two possible ways to arrange $(k_1, \dots, k_5)$ up to permutation of k_i. In the first case we have

$$\begin{aligned} R_{5,3}(2,3,2,2,3) &= R_{5,3}(2,2,2,3,3) \\ &= R_{4,3}(2,2,3,3) \\ &= R_{3,3}(2,3,3) \\ &= \min\{2,3,3\} = 2. \end{aligned}$$

In the other case we have $R_{5,3}(2,2,2,2,4) = R_{4,3}(2,2,2,4) = 2$.

We now prove an upper bound for $R_{s,t}(k_1, \dots, k_s)$.

Theorem 8.6. *For all $s, t, k_1, \dots, k_s \in \mathbb{N}$ such that $s > t$ and $k_i \geq 2$, $i \in [s]$, let $K = \sum_{i=1}^{s} k_i$. Then*

$$R_{s,t}(k_1, \dots, k_s) \leq 2 - \frac{s}{s-t} + \frac{C}{t^K}\binom{K-s}{k_1 - 1, \dots, k_s - 1}, \tag{8.1}$$

where $C = \frac{s}{s-t} \cdot \frac{t!}{s!} \cdot t^{2s+t-1}$.

Proof. Let M be the right side of the inequality (8.1),

$$M = 2 - \frac{s}{s-t} + \frac{C}{t^K}\binom{K-s}{k_1 - 1, \dots, k_s - 1}.$$

We proceed with a proof by induction on K. First we proof the base case when $K \leq 2s + t - 1$. Then there are at least $(s - t + 1)$ positions of k_i with $k_i = 2$. Without loss of generality, let $k_1 = \dots = k_{s-t+1} = 2$. Then

$$\begin{aligned} \sum_{i=s-t+2}^{s} k_i &\leq 2s + t - 1 - 2(s - t + 1) \\ &= 3t - 3. \end{aligned}$$

By Lemma 8.4, $R_{s,t}(k_1, \dots, k_s) = 2$. Further consider

$$\begin{aligned} \binom{K-s}{k_1 - 1, \dots, k_s - 1} &= \binom{K-s}{1, \dots, 1, k_{s-t+2} - 1, \dots, k_s - 1} \\ &= \frac{(K-s)!}{(k_{s-t+2} - 1)! \dots (k_s - 1)!}. \end{aligned}$$

Since $k_i \geq 2$ for all $i \in [s]$, $K \geq 2s$ and $K - s \geq s$. Moreover, since $\sum_{i=s-t+2}^{s} k_i \leq 3t - 3$,

$$\begin{aligned} (k_{s-t+2} - 1)! \dots (k_s - 1)! &\leq \underbrace{(2-1)! \dots (2-1)!}_{(t-2) \text{ times}} \cdot (3t - 3 - 2(t-2) - 1))! \\ &= t!. \end{aligned}$$

It follows that

$$\binom{K-s}{k_1-1,\dots,k_s-1} \geq \frac{s!}{t!}.$$

Further, since $K \leq 2s+t-1$, we have $t^K \leq t^{2s+t-1}$, thus

$$\frac{1}{t^K} \geq \frac{1}{t^{2s+t-1}}.$$

Altogether we have

$$\begin{aligned}
M &\geq 2 - \frac{s}{s-t} + \frac{s}{s-t} \cdot \frac{t!}{s!} \cdot \frac{t^{2s+t-1}}{t^{2s+t-1}} \cdot \frac{s!}{t!} \\
&= 2 \\
&= R_{s,t}(k_1,\dots,k_s),
\end{aligned}$$

which proves the base case. For the induction step, let $K > 2s+t-1$. By Proposition 8.3 and the induction hypothesis,

$$\begin{aligned}
R_{s,t}(k_1,\dots,k_s) &\leq 2 - \frac{s}{t} + \frac{1}{t}\sum_{i=1}^{s} R_{s,t}(k_1,\dots,k_{i-1},k_i-1,k_{i+1},\dots,k_s) \\
&\leq 2 - \frac{s}{t} + \frac{1}{t}\sum_{i=1}^{s}\left[2 - \frac{s}{s-t} + \frac{C}{t^{K-1}}\binom{\sum_{i=1}^{s} k_i - 1 - s}{k_1-1,\dots,k_1-1,\dots,k_s-1}\right] \\
&= 2 - \frac{s}{t} + \frac{s}{t}\left(2 - \frac{s}{s-t}\right) + \frac{C}{t^K}\sum_{i=1}^{s}\binom{K-s-1}{k_1-1,\dots,k_i-2,\dots,k_s-1} \\
&= 2 + \frac{s}{t}\left(1 - \frac{s}{s-t}\right) + \frac{C}{t^K}\binom{K-s}{k_1-1,\dots,k_s-1} \\
&= 2 + \frac{s}{t}\cdot\frac{s-t-s}{s-t} + \frac{C}{t^K}\binom{K-s}{k_1-1,\dots,k_s-1} \\
&= 2 - \frac{s}{s-t} + \frac{C}{t^K}\binom{K-s}{k_1-1,\dots,k_s-1}.
\end{aligned}$$

This completes the proof of the theorem. ☐

We apply Theorem 8.6 for $s = 2t$.

Corollary 8.7. *For all positive integers t and $k_1,\dots,k_{2t} \geq 2$, let $K = \sum_{i=1}^{2t} k_i$. Then*

$$R_{2t,t}(k_1,\dots,k_s) \leq \frac{2\cdot t!}{s!}\cdot\frac{t^{5t-1}}{t^K}\binom{K-2t}{k_1-1,\dots,k_{2t}-1}.$$

Remark 8.8. For $t=1, s=2$ and positive integers $k_1,k_2 \geq 2$, Theorem 8.6 reduces to the well-known formula:

$$R_{2,1}(k_1,k_2) = R(k_1,k_2) \leq \binom{k_1+k_2-2}{k_1-1,k_2-1} = \binom{k_1+k_2-1}{k_1-1}.$$

For $s = 2t$ and $k_1 = \cdots = k_s = k$, we have $R_{s,t}(k_1, \ldots, k_s) = R_{2t,t}(k)$ and the following result.

Corollary 8.9. *For positive integers t, k such that $k \geq 2$, there exists a positive number C_t such that*

$$R_{2t,t}(k) \leq C_t \cdot 2^{2kt}.$$

Proof. By Corollary 8.7, for $k_1 = \cdots = k_{2t} = k$,

$$R_{2t,t}(k_1, \ldots, k_{2t}) = R_{2t,t}(k) \leq \frac{2 \cdot t!}{s!} \cdot \frac{t^{5t-1}}{t^{2kt}} \binom{2kt - 2t}{k-1, \ldots, k-1}.$$

Note that

$$\binom{2kt - 2t}{k-1, \ldots, k-1} \leq (2t)^{2kt-2t},$$

therefore,

$$\begin{aligned} R_{2t,t}(k) &\leq \frac{2 \cdot t!}{s!} \cdot \frac{t^{5t-1}}{t^{2kt}} \cdot 2^{2kt-2t} \cdot t^{2kt-2t} \\ &= \frac{t!}{s!} \cdot t^{3t-1} \cdot 2^{1-2t} \cdot 2^{2kt}. \end{aligned}$$

The corollary follows from setting $C := \frac{t!}{s!} \cdot t^{3t-1} \cdot 2^{1-2t}$. $\square$

Alternatively we can prove an upper bound on $R_{s,t}(k_1, \ldots, k_s)$ without using multinomial coefficients.

Theorem 8.10. *For all $s, t, k_1, \ldots, k_s \in \mathbb{N}$ such that $s > t$ and $k_i \geq 2$, $i \in [s]$, let $K = \sum_{i=1}^{s} k_i$. Then*

$$R_{s,t}(k_1, \ldots, k_s) \leq 2 - \frac{s}{s-t} + C_{s,t} \left(\frac{s}{t}\right)^K, \tag{8.2}$$

where $C_{s,t} = \frac{s}{s-t} \left(\frac{t}{s}\right)^{2s}$.

Proof. Similarly to proving Theorem 8.6, we apply induction on K.

For the base case, let $K \leq 2s + t - 1$. By Lemma 8.4, we have $R_{s,t}(k_1, \ldots, k_s) = 2$. Since $K \geq 2s$, consider the right side of 8.2:

$$\begin{aligned} 2 - \frac{s}{s-t} + C_{s,t} \left(\frac{s}{t}\right)^{\sum_{i=1}^{s} k_i} &\geq 2 - \frac{s}{s-t} + \frac{s}{s-t} \left(\frac{t}{s}\right)^{2s} \left(\frac{s}{t}\right)^{2s} \\ &= 2 \\ &= R_{s,t}(k_1, \ldots, k_s). \end{aligned}$$

For the induction step, let $\sum_{i=1}^{s} k_i > 2s+t-1$. Proposition 8.3 implies that

$$R_{s,t}(k_1,\dots,k_s) \le 2 - \frac{s}{t} + \frac{1}{t}\sum_{i=1}^{s} R_{s,t}(k_1,\dots,k_{i-1},k_i-1,k_{i+1},\dots,k_s).$$

By the induction hypothesis,

$$\begin{aligned} R_{s,t}(k_1,\dots,k_s) &\le 2 - \frac{s}{t} + \frac{1}{t}\sum_{i=1}^{s}\left[2 - \frac{s}{t} + C_{s,t}\left(\frac{s}{t}\right)^{K-1}\right] \\ &= \underbrace{2 - \frac{s}{t} + s\cdot\frac{1}{t}\left(2 - \frac{s}{s-t}\right)}_{=2-\frac{s}{s-t}} + \underbrace{\frac{s}{t}\cdot C_{s,t}\left(\frac{s}{t}\right)^{K-1}}_{=C_{s,t}\left(\frac{s}{t}\right)^{K}} \\ &= 2 - \frac{s}{s-t} + C_{s,t}\left(\frac{s}{t}\right)^{K}. \end{aligned}$$

This completes the proof of the theorem. ☐

Remark 8.11. The induction step of the proof follows from the induction hypothesis no matter what value $C_{s,t}$ has. The term $C_{s,t}$ was determined merely for the sake of the base case.

We apply Theorem 8.10 for $s = 2t$.

Corollary 8.12. *For all positive integers t and $k_i \ge 2, i \in [2t]$,*

$$R_{2t,t}(k_1,\dots,k_{2t}) \le C\cdot 2^{\sum_{i=1}^{s} k_i},$$

where $C := 2^{1-4t}$. If $k_i = k$ for all $i \in [2t]$ then

$$R_{2t,t}(k) \le C\cdot 2^{2kt}.$$

Remark 8.13. At this point, we have several bounds for general $R_{2t,t}(k)$.

$$\begin{aligned} &R_{2t,t}(k) \le (t+1)^{k(t+1)} && \text{(Corollary 3.7)},\\ &R_{2t,t}(k) \le \frac{t!}{s!}\cdot t^{3t-1}\cdot 2^{1-2t}\cdot 2^{2kt} && \text{(Corollary 8.9)},\\ &R_{2t,t}(k) \le 2^{1-4t}\, 2^{2kt} && \text{(Corollary 8.12)}. \end{aligned}$$

The second and third bounds have the same growth and both are an improvement for the bound in Corollary 3.7.

9 Conclusion

We first summarize the main results from the previous sections.

9.1 Main results

The classical Ramsey number $R(k)$ can be seen as a special case of the (s,t) Ramsey number $R_{s,t}(k)$ with $s=2, t=1$. The off-diagonal Ramsey number $R_{s,t}(k_1,\ldots,k_s)$ is again a generalization of the (s,t)-Ramsey number. Computing upper bounds for $R_{s,t}(k)$ as a special case of $R_{s,t}(k_1,\ldots,k_s)$ where $k_i=k$ for all $i\in[s]$ gives a better upper bound as we have seen in Remark 8.13. In general, for all $s,t,k\in\mathbb{N}$ such that $s>t, k\geq 3$, we have

$$\left\lceil \frac{k}{e}\cdot s^{-\frac{1}{k}}\cdot\left(\frac{s}{t}\right)^{\frac{k-1}{2}}\right\rceil \leq R_{s,t}(k)\leq 2-\frac{s}{s-t}+C_{s,t}\cdot\left(\frac{s}{t}\right)^{sk},$$

where $C_{s,t}:=\frac{s}{s-t}\left(\frac{t}{s}\right)^{2s}$.

Some exact values of Ramsey numbers of the form $R_{s,t}(3)$ were determined and are listed below.

- $R_{3,2}(3)=5$
- $R_{4,2}(3)=9$
- $R_{5,3}(3)=5$
- $R_{7,4}(3)=9$
- $R_{6,3}(3)=9$
- $R_{t+1,t}(3)=3$ for all $t\in\mathbb{N}, t\geq 3$.

For $R_{3,2}(4)$ we have the estimation $10\leq R_{3,2}(4)\leq 14$.

In some cases, the study on small Ramsey numbers $R_{s,t}(k)$ can be generalized. For example, a witness coloring to the lower bound $R_{4,2}(3)>8$ was generalized to Hadamard colorings. In addition, the concept of design colorings arises from the witness coloring to the lower bound $R_{4,3}(4)>9$.

For all $n\in\mathbb{N}, n$, if there exists a Hadamard matrix of order $2n$, then

$$4n<R_{2n,n}(3)\leq[1+o(1)]\sqrt{2}(n+2)\left(\frac{4}{e}\right)^n.$$

The lower bound for $R_{2n,n}(3)$ given by a Hadamard matrix of order $2n$ is better than by the probabilistic method. However, it remains an open question how close the lower bound by the Hadamard construction is to the exact value of $R_{2n,n}(3)$. An advantage is that we can construct a $(2n, n^+)$-coloring of K_{4n} without monochromatic triangle with Hadamard colorings, in contrast to the non-constructive probabilistic method.

A resolvable design gives the exact value of some Ramsey numbers. For all $v, k, \lambda \in \mathbb{N}$ such that $v > k \geq 2$, if there exists a resolvable (v, k, λ) design, then for $s = \frac{\lambda(v-1)}{k-1}$,

$$R_{s,s-\lambda}\left(\frac{v}{k}+1\right) = v+1.$$

A disadvantage of the design construction, similar to the Hadamard construction, is that it is only applicable in specific cases.

9.2 A new definition

To end this thesis, we introduce a new direction for further study. We first recall that for graphs G, H, $R(G, H)$ denotes the least positive integer n such that any 2-coloring of $E(K_n)$ contains a red G or a blue H.

For any positive integers s, t with $s \geq t$ and graph G, we denote by $R_{s,t}(G)$ the least positive integer n such that any (s, t)-coloring of K_n contains a monochromatic copy of G. We also call $R_{s,t}(G)$ the *graph Ramsey number for set-coloring.* A prospective study is to find lower and upper bounds for $R_{s,t}(G)$ with various kinds of graphs G such as cycles, stars, paths, etc.

For motivation, we give a small example by determining $R_{3,2}(C_4)$. Note that by the same arguments as in Proposition 3.4, the following holds. For all graphs G and $s, t \in \mathbb{N}$ with $s \geq t \geq 2$,

$$R_{s,t}(G) \leq R_{s-1,t-1}(G).$$

It was proved by Chvátal and Harary [5] that $R(C_4) = 6$. Hence, $R_{3,2}(C_4) \leq 6$.

Theorem 9.1. *We have that $R_{3,2}(C_4) = 5$. Moreover, up to relabeling of colors and vertices, the only $(3, 2)$-coloring of K_4 without monochromatic C_4 is given in Figure 16.*

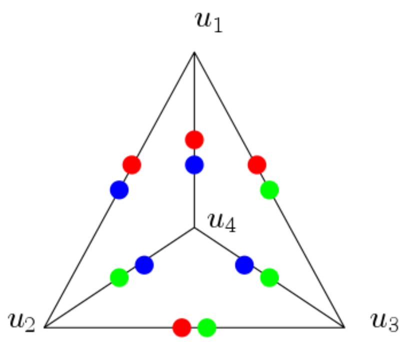


Figure 16: $(3, 2)$-coloring of K_4 without monochromatic C_4

Proof. The coloring in Figure 16 proves the lower bound $R_{3,2}(C_4) > 4$. We will prove that it is unique up to the permutation of vertices and colors. Let $V(K_4) = \{u_1, u_2, u_3, u_4\}$ and c be a $(3,2)$-coloring of K_4 without monochromatic C_4. An edge e of K_4 can have color set $\{1,2\}, \{1,3\}$ or $\{2,3\}$. In the following let i, j, k be arbitrary elements of $\{1,2,3\}$ and pairwise distinct. We first prove the following claims for the coloring c.

(i) For any spanning cycle C of K_4, $c(C) = \{\{1,2\},\{1,3\},\{2,3\}\}$, i.e. all 3 color combinations must appear on edges of C.

(ii) For any vertex $u \in V(K_4)$, there are at most two edges at u with the same color set.

(iii) Any two edges of K_4 that are not adjacent have distinct color sets.

For Item (i), if C has at most two color sets, $c(C) = \{\{i,j\},\{i,k\}\}$ or $c(C) = \{\{i,j\}\}$, then C is monochromatic in color i, which is a contradiction. This proves Item (i). Note that Item (i) also means that there are at most two edges of C have the same color set.

For Item (ii), for the sake of contradiction, assume without loss of generality that $c(u_1u_2) = c(u_1u_3) = c(u_1u_4) = \{i,j\}$. We apply Item (i) for 3 spanning cycles of K_4. For cycle $(u_1, u_2, u_3, u_4, u_1)$, Item (i) implies $\{c(u_2u_3), c(u_3u_4)\} = \{\{i,k\},\{j,k\}\}$. Without loss of generality, let $c(u_2u_3) = \{i,k\}$ and $c(u_3u_4) = \{j,k\}$. For cycle (u_1, u_2, u_4, u_3), we have $c(u_2u_4) = \{i,k\}$. It follows that cycle $(u_1, u_4, u_2, u_3, u_1)$ is monochromatic in color i, which is a contradiction and thus proves Item (ii).

For Item (iii), we now assume without loss of generality that u_1u_2, u_3u_4 are not adjacent and $c(u_1u_2) = c(u_3u_4) = \{i,j\}$. We prove Item (iii) again by applying Item (i) for spanning cycles of K_4. For cycle $(u_1, u_2, u_3, u_4, u_1)$, without loss of generality, $c(u_2u_3) = \{i,k\}$, $c(u_1u_4) = \{j,k\}$. For cycle $(u_1, u_2, u_3, u_4, u_1)$, Item (i) implies that $c(u_2u_4) = \{1,2\}$ or $c(u_1u_3) = \{1,2\}$. On the other hand, Item (i) applied for cycle $(u_1, u_2, u_4, u_3, u_1)$ yields $\{c(u_1,u_3), c(u_2u_4)\} = \{\{1,3\},\{2,3\}\}$. In particular, $\{1,2\} \notin \{c(u_1u_3), c(u_2u_4)\}$, which is a contradiction. This proves Item (iii).

With Items (i), (ii), (iii), we can prove the uniqueness of c. By Items (i), without loss of generality, $c(u_1u_2) = \{1,2\}, c(u_2u_3) = \{1,3\}, c(u_3u_4) = \{2,3\}$. By Item (iii), $c(u_1u_4) \neq c(u_2u_3) = \{1,3\}$, without loss of generality, $c(u_1u_4) = \{1,2\}$. There are now 2 edges at u_1 with the same color set: $c(u_1u_2) = c(u_1u_4) = \{1,2\}$. By Item (ii), $c(u_1u_3) \neq \{1,2\}$, without loss of generality, let $c(u_1u_3) = \{1,3\}$. By Item (i) for cycle $(u_1, u_4, u_2, u_3, u_1)$, $c(u_2u_4) = \{2,3\}$. Therefore, the witness coloring to the lower bound $R_{3,2}(C_4) > 4$ is given in Figure 16.

For the upper bound, let c be any $(3,2)$-coloring of K_5. We prove that c contains a monochromatic C_4. For the sake of contradiction, we assume that c has no monochromatic C_4. Due to the uniqueness of the $(3,2)$-coloring of K_4 without monochromatic C_4

up to relabeling of colors and vertices, we can assume without loss of generality that c is as in Figure 17. We have

$$c(u_1u_2) = c(u_1u_4) = \{1, 2\},$$
$$c(u_2u_3) = c(u_1u_3) = \{1, 3\},$$
$$c(u_2u_4) = c(u_3u_4) = \{2, 3\}.$$

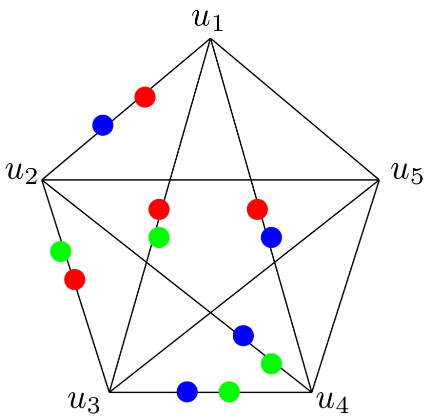


Figure 17: Assumption: $(3, 2)$-coloring of K_5 without monochromatic C_4

Observe that in the cycle $(u_2, u_4, u_3, u_5, u_2)$, there are two edges with the same color set: $c(u_2u_4) = c(u_3u_4) = \{2, 3\}$. Then by Item (i), $c(u_2u_5) \neq \{2, 3\}$. Similarly, for cycle $(u_1, u_3, u_2, u_5, u_1)$, two edges have the same color set: $c(u_1u_3) = c(u_2u_3) = \{1, 3\}$, hence $c(u_2u_5) \neq \{1, 3\}$. Consequently, $c(u_2u_5) = \{1, 2\}$. Then cycle $(u_1, u_2, u_5, u_4, u_1)$ has 3 edges with the same color set: $c(u_4u_1) = c(u_1u_2) = c(u_2u_5) = \{1, 2\}$. Item (i) implies that $K_5[\{u_1, u_2, u_4, u_5\}]$ has a monochromatic C_4, which is a contradiction to the assumption of c. This completes the proof of the theorem. $\square$

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Erklärung

Ich versichere wahrheitsgemäß, die Arbeit selbstständig verfasst, alle benutzten Hilfsmittel vollständig und genau angegeben und alles kenntlich gemacht zu haben, was aus Arbeiten anderer unverändert oder mit Abänderungen entnommen wurde, sowie die Satzung des KIT zur Sicherung guter wissenschaftlicher Praxis in der jeweils gültigen Fassung beachtet zu haben.

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