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On free fall of quantum matter

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Abstract We propose an approach that allows to systematically take into account gravity in quantum particle physics. It is based on quantum field theory and the general principle of relativity. These are used to build a model for quantum particles in curved spacetime. We compute by its means a deviation from a classical geodesic in the Earth's gravitational field. This shows that free fall depends on quantum-matter properties. Specifically, we find that the free-fall universality and the wave-packet spreading are mutually exclusive phenomena. We then estimate the Eötvös parameter for a pair of atoms freely falling near the Earth's surface, provided that the wave-packet spreading is more fundamental than the weak equivalence principle.

1 Introduction

The theory of quantum fields is well known by now to be extremely successful in describing scattering processes in elementary particle physics. The very notion of an elementary particle is based on the Poincaré group which is in turn the isometry group of Minkowski spacetime. According to the general theory of relativity, Minkowski spacetime describes a universe with no matter and no cosmological constant. The observable Universe is therefore described by a non-Minkowski spacetime. The questions arise then how to model a quantum particle in the absence of the global Poincaré symmetry and how to experimentally test this construction in the presence of a gravitational field.

From an experimental point of view, these questions need to be studied in the background of the Earth's gravitational field. This may be described approximately by the Schwarzschild line element (the Earth's rotation neglected) which, in spherical coordinates, reads

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2\big(d\theta^2 + \sin^2\theta d\phi^2\big)$$

$$\text{with} \quad f(r) \equiv 1 - \frac{r_S}{r}, \tag{1}$$

where r_S is the Schwarzschild radius, which is $r_{S,\oplus} \approx 8.87\times10^{-3}$ m in the case of Earth, while the Earth's radius is $r_\oplus \approx 6.37\times10^6$ m. The ratio $r_{S,\oplus}/r_\oplus \approx 1.39\times10^{-9}$ is negligibly small, yet its gradient is responsible for the gravitational force, whose gradient is in its turn for the tidal effects. These manifestations of the Earth's gravitational field are (for good reason) irrelevant in particle colliders. In light of this, the Schwarzschild line element turns in the zeroth-order approximation into the Minkowski one to underlie the special theory of relativity.

One of the basic postulates of this theory is the special principle of relativity, which states that the laws of physics are the same in all inertial frames [1]. Even though it was formulated prior to quantum theory, this principle is implementable in quantum physics. In particular, it gives rise to Wigner's classification of elementary particles [2], meaning that these are related to unitary and irreducible representations of the Poincaré group, which are distinguished by mass and spin. To put it differently, the existence of an elementary particle is independent on an inertial frame considered. This mathematical construction is in agreement with up-to-date observations in high-energy physics.

Collider-physics experiments are performed in the background of the Earth's gravitational field. For example, non-relativistic neutrons have been shown to fall down in accordance with Newton's gravitational law [3–5]. Theoretical particle physics is obviously incomplete, as it is based on the Minkowski-spacetime approximation and, therefore, the Wigner classification is an approximation as well. The fundamental problem is to find a way how to go beyond of this approximation and yet to stay consistent with collider physics.

From a logical point of view, this should be done by implementing the general principle of relativity in quantum theory, which asserts that physical laws are invariant under general coordinate transformations [6]. Leaving

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aside the group-theoretical aspect of this construction, we have recently shown that the general principle of relativity is implementable in de-Sitter spacetime [7]. It was gained through deriving a non-perturbative (in curvature) wave-packet solution which, first, is invariant under the diffeomorphism group and, second, locally reduces to the superposition of Minkowski plane waves. The latter property implies that this solution may be associated locally with one of the Wigner classes and, therefore, is in agreement with the Einstein equivalence principle – locally and at any point of the spacetime, the Minkowski-spacetime (quantum) physics holds [8].

This article aims to treat this approach further. Specifically, we wish to study the question if free fall is universal in quantum theory. In other words, if the weak equivalence principle [8] holds in quantum physics. Furthermore, we shall show in passing that all gravity corrections to quantum-mechanical phase, which were previously obtained under certain approximations or heuristically, can be systematically derived by building the principles of general relativity in quantum particle physics. We shall find higher-order gravitational corrections to the phase, which, to our knowledge, have not been reported before, and establish that these corrections may not be unambiguously determined without experimental data.

Throughout, we use natural units $c = G = \hbar = 1$, unless otherwise stated.

2 Quantum particles in curved spacetime

2.1 Covariant wave packet

In order to study quantum corrections to free fall, one needs first to introduce a model for quantum particles, since these are elementary "sensors" by means of which one observes free fall in practice. Quantum field theory is by now a fundamental framework which allows us to successfully describe microscopic processes. Its application in particle physics relies, however, on the Minkowski-spacetime approximation. The outstanding problem remains, namely that it is unclear how to consistently take into account gravity in microscopic physics.

We have recently proposed in [7] that an operator, which creates a quantum scalar particle out of vacuum, should be related to a quantum-field operator, $\hat{\Phi}(x)$, as follows:

$$\begin{aligned}\hat{a}^\dagger(\varphi_{X,P}) &\equiv \big(\varphi_{X,P},\, \hat{\Phi}\big)_{\text{Klein–Gordon}} \\ &\equiv -i\int_\Sigma d\Sigma^\mu(x) \\ &\quad\times \Big(\varphi_{X,P}(x)\nabla_\mu\hat{\Phi}^\dagger(x) - \hat{\Phi}^\dagger(x)\nabla_\mu\varphi_{X,P}(x)\Big)\,, \end{aligned} \tag{2}$$

where Σ is a space-like Cauchy surface, $\varphi_{X,P}(x)$ stands for a wave packet whose centre of mass is initially localised in the semi-classical limit at $X = (T, \mathbf{X})$ in coordinate space and at $P = (P^T, \mathbf{P})$ in momentum space, where the on-shell momentum P belongs to the cotangent space at X. This definition arises from a covariant generalisation of asymptotic creation and annihilation operators to underlie S-matrix in collider physics [9].

Accordingly, we obtain from the definition (2) that

$$\big[\hat{a}(\varphi_{X,P}),\, \hat{a}^\dagger(\varphi_{X,P})\big] = \big(\varphi_{X,P},\, \varphi_{X,P}\big)_{\text{Klein–Gordon}}, \tag{3}$$

where we have made use of the quantum-field algebra (e.g. see [10]). This commutator defines the normalisation condition for the wave packet:

$$\big(\varphi_{X,P},\, \varphi_{X,P}\big)_{\text{Klein–Gordon}} = 1, \tag{4}$$

which reduces to the standard normalisation condition known in quantum mechanics if one considers the non-relativistic approximation in the weak-field limit, after having rescaled the wave packet by $1/\sqrt{2M}$, where M is the scalar-field mass. Thus, a single-particle state is

$$|\varphi_{X,P}\rangle \equiv \hat{a}^\dagger(\varphi_{X,P})|\Omega\rangle, \tag{5}$$

where $|\Omega\rangle$ is the quantum vacuum, namely $\hat{a}(\varphi_{X,P})|\Omega\rangle = 0$. The primary task is, therefore, to determine $\varphi_{X,P}(x)$ on physical grounds. It should be emphasised that it is by now a standard approach in quantum theory to search instead for a *global* quantum-field-mode expansion in a given curved space [11]. This approach has been put forward as a generalisation of the global plane-wave expansion in Minkowski spacetime, where quantum field theory serves primarily for the description of scattering processes [12]. Still, the Universe is not globally flat [13]. This means that elementary particle physics is based on the Minkowski-spacetime approximation. It then follows from the Einstein equivalence principle that the global plane-wave expansion of quantum-field operators in particle physics corresponds to a local quantum-field expansion in the non-flat Universe. If we now consider, for instance, the Sunyaev–Zeldovich effect, which is responsible for the CMB-spectrum distortion due to the inverse Compton scattering of the CMB photons by high-energy galaxy-cluster electrons, then we conclude that the plane-wave expansion of quantum-field operators has to locally emerge at any given small-enough space-time region. The equivalence principle suggests further that one needs to deal with an object which depends on a relative distance, rather than on an absolute position as that is the case for quantum-field operators, e.g. $\hat{\Phi}(x)$. The wave function $\varphi_{X,P}(x)$, which describes a particle in quantum theory, seems thereby to be a natural object for that purpose.

The Minkowski-spacetime approximation in elementary particle physics is good enough to describe high-energy processes to take place in colliders. Primary observables here are related to S-matrix elements, each of which provides a probability amplitude for a given initial multi-particle state to evolve into a particular final multi-particle state. Both initial and final states are usually associated with wave functions to have definite momenta – plane waves. However, the LSZ reduction formula [9] (which basically links the mathematical formalism of quantum field theory to physics) uses normalisable wave functions to describe particles having neither definite momentum nor position – superposition of plane waves. Note, S-matrix has also to depend on initial and final center-of-mass positions of quantum particles, otherwise one could not implement the cluster decomposition principle, basically stating that distant experiments give uncorrelated results (see Ch. 4 in [12]). This requires localised-in-space quantum states which correspond to wave packets. We conclude from these trivial observations that $\varphi_{X,P}(x)$ must be representable via superposition of plane waves for x sufficiently close to X, if treated in a local inertial frame.

The general principle of relativity requires that physical laws are the same in all coordinate frames [6]. In particular, the Einstein field equations are invariant under general coordinate transformations. In the semi-classical approximation, the single-particle state $|\varphi_{X,P}\rangle$ enters the Einstein equations through the expectation value of the stress-tensor operator $\hat{T}_{\mu\nu}(x)$ of the scalar field $\hat{\Phi}(x)$ in the quantum state $|\varphi_{X,P}\rangle$. For this expectation value to be tensorial, one must impose a condition that $\varphi_{X,P}(x)$ is a covariant wave packet or scalar in the problem under consideration. Therefore, quantum particle physics must be formulated in an observer-independent manner: All observers, independent on their state of motion or their rest frame, agree on the existence of the single-particle state $|\varphi_{X,P}\rangle$, assuming that a quantum particle, which is described by this state, moves through their detectors (e.g. Wilson's cloud chambers in case of an electrically charged particle). Still, this particle affects their detectors differently and this depends on their state of motion (e.g. curvature of a charged-particle track depends on how a given Wilson chamber moves). This idea logically follows from the general principle of relativity, but remains unexplored in quantum particle physics.

By analogy with the Minkowski-spacetime case, we furthermore suppose that $\varphi_{X,P}(x)$ is a solution of the scalar-field equation, which has the following Fourier-integral representation:

$$
\begin{aligned}
\varphi_{X,P}(x) &\equiv \left(-g(X)\right)^{-\frac{1}{2}} \\
&\times \int \frac{d^4K}{(2\pi)^3}\,\theta(K^T)\,\delta(K^2-M^2)\,F_P(K)\,\phi_{X,K}(x), \qquad (6)
\end{aligned}
$$

$\phi_{X,K}(x)$ satisfies the scalar-field equation on mass shell $g^{AB}(X)K_AK_B \equiv K^2 = M^2$:

$$
\Big(\Box_x + M^2 - \frac{1}{6}\,R(x)\Big)\phi_{X,K}(x) = 0, \qquad (7)
$$

and $F_P(K)$ has a peak at $K = P$ and provides for the normalisation condition (4). It should be remarked at this point that the Klein–Gordon product, which has been defined in (2), is not conserved if the scalar field interacts with itself or other (non-gravitational) fields. In this case, $|\varphi_{X,P}\rangle$ is a dynamical state that may, for example, be unstable. The normalisation condition cannot then be fulfilled for all times (e.g. free neutrons have a mean lifetime of about 15 min). However, this condition holds for all times in curved spacetime, even if it is dynamical (e.g. gravitational waves, collapse etc.) if $(\Box_x + s(x))\hat{\Phi}(x) = 0$ is fulfilled, where $s(x)$ is real.

However, a word of caution is needed on this point. The gravitational interaction between a pair of particles can be thought as virtual-graviton exchange within the effective theory of quantum gravity [14]. This quantum-gravity approach presumes a background gravitational field whose fluctuations are promoted to quantum-field operators. Hence, in quantum gravity, $\hat{\Phi}(x)$ satisfies the scalar-field equation (7) with a non-trivial right-hand side. This results in a dynamical evolution of $\hat{a}(\varphi_{X,P})$. We treat in this article a test-particle approximation – $|\varphi_{X,P}\rangle$ does not source gravity or, in other words, is independent on the graviton field $\hat{h}_{\mu\nu}(x)$.

It proves useful to consider Riemann normal coordinates at X, such those x corresponds to y, while X to $Y \equiv (0, 0, 0, 0)$. In the Riemann frame, i.e. y, the first derivative of the metric tensor vanishes at X. Thereby, geodesics passing through the point X turn into straight lines in the Riemann frame [15]. Note, this particular choice of coordinates does not affect physics as we deal with the covariant wave packet. In the Riemann frame, however, the metric tensor is given through the curvature tensor, its covariant derivatives and their tensorial products in a covariant form:

$$
\begin{aligned}
g_{ab}(y) = \eta_{ab} &- \frac{1}{3}\,R_{acbd}\,y^c y^d - \frac{1}{6}\,R_{acbd;e}\,y^c y^d y^e \\
&- \left(\frac{1}{20}\,R_{acbd;ef} - \frac{2}{45}\,R_{acgd}\,R_{be}{}^{g}{}_{f}\right) y^c y^d y^e y^f \\
&+ \mathrm{O}(y^5), \qquad (8)
\end{aligned}
$$

where it is implicitly understood that the curvature tensor and its derivatives are computed at X (see Sec. 7 in [15]). The Latin indices running over $\{0, 1, 2, 3\}$ are coupled to the capital Greek indices by means of the vierbein $e^a_A(X)$. We shall study in what follows how these three curvature corrections to the Minkowski metric η_{ab} affect a quantum particle.

We shall mainly focus here on the wave-packet propagation in the Schwarzschild geometry in order to find out whether free fall is universal in quantum theory. We have proposed in [7] to characterise the wave-packet propagation by its centre-of-mass position:

$$
\begin{aligned}
\langle y^i \rangle \equiv -i \int_{y^0} &d^3\mathbf{y}\sqrt{-g(y)}\;y^i\,g^{0b}(y) \\
&\times \big(\varphi_{Y,P}(y)\partial_b\bar{\varphi}_{Y,P}(y) - \bar{\varphi}_{Y,P}(y)\partial_b\varphi_{Y,P}(y)\big). \qquad (9)
\end{aligned}
$$

In Minkowski spacetime and in the non-relativistic limit, it reduces to the standard definition of the position expectation value $\langle y^i \rangle$, which is known in quantum mechanics, while $\langle y^0 \rangle = y^0$, i.e. y^0 is generically a c-number. Furthermore, the wave-packet propagation rate then reads

$$\begin{aligned}\partial_0 \langle y^i \rangle &= -i \int_{y^0} d^3\mathbf{y} \sqrt{-g(y)}\, g^{ib}(y) \\ &\quad \times \big(\varphi_{Y,P}(y)\partial_b\bar{\varphi}_{Y,P}(y) - \bar{\varphi}_{Y,P}(y)\partial_b\varphi_{Y,P}(y)\big), \qquad (10)\end{aligned}$$

where we have used the normalisation condition (4), the scalar-field equation (7) re-written in Riemann normal coordinates and also the fact that $\varphi_{Y,P}(y)$ vanishes at spatial infinity due to its localisation in space. It is straightforward to show that this rate reduces to the momentum expectation value divided by M in the quantum-mechanics regime. For all these reasons, $\langle y^i \rangle$ defined in (9) might be a proper starting point to study free fall of quantum matter.

2.2 Non-inertial effects

A physically relevant solution $\phi_{X,K}(x)$ of the scalar-field equation (7) must be determined from observations in particle physics. To our knowledge, there has been made up to now only one observation which reveals the role of the curvature tensor in quantum particle physics and which we shall utilise below. We intend first to neglect the space-time curvature to obtain the zeroth-order term (in curvature) of the wave packet $\varphi_{X,P}(x)$ in curved spacetime.

In the absence of the space-time curvature, we obtain from (7) that

$$\phi^{(0)}_{Y,K}(y) = e^{-iK\cdot y}, \qquad (11)$$

where

$$K\cdot y \equiv K_a y^a, \qquad (12)$$

as this directly follows from quantum field theory in Minkowski spacetime and its application to elementary particle physics [12]. Next, we obtain a covariant Gaussian wave packet [16,17], with the momentum variance $D > 0$, namely

$$\varphi^{(0)}_{Y,P}(y) = N^{(0)}\, \frac{K_1\big(\frac{M^2}{D^2}\Sigma_{Y,P}(y)\big)}{\Sigma_{Y,P}(y)}, \qquad (13)$$

where $K_\nu(z)$ is the modified Bessel function of the second kind,

$$\Sigma_{Y,P}(y) \equiv \left(\frac{1}{4} + i\,\frac{D^2}{M^2}\,P\cdot y - \frac{D^4}{M^2}\,y\cdot y\right)^{\frac{1}{2}} \qquad (14a)$$

and

$$N^{(0)} \equiv \frac{D}{2\pi\sqrt{K_1\big(\frac{M^2}{D^2}\big)}}. \qquad (14b)$$

We refer to Sec. B in [7], where we have computed the center-of-mass propagation of this wave packet and its energy-momentum vector, and found that this packet behaves kinematically as a point-like particle of the same mass if and only if $Mc \gg D$. This may thereby be interpreted as a classical limit in which particles can be treated as effectively having no extent.

It is worth mentioning that the right-hand side of (11) and, consequently, the right-hand side of (13) can be expressed in terms of general coordinates, namely

$$\phi^{(0)}_{X,K}(x) = e^{+iK\cdot\sigma}, \qquad (15)$$

where

$$K\cdot\sigma \equiv K_M\, g^{MN}(X)\, \nabla_N \sigma(x, X) \qquad (16)$$

and $\sigma \equiv \sigma(x, X)$ is the geodetic distance [18]. The capital indices refer to the tangent space at X. The zeroth-order solution of the scalar-field equation in the form (15) was our starting point in [7] to obtain a covariant wave-packet solution in de-Sitter spacetime, which, first, is non-perturbative in curvature and, second, reduces locally to the plane-wave superposition at any point X. The latter property implies that this solution is consistent with the application of quantum field theory to collider physics.

The approximate solution (11) [or, alternatively, (15)] can be used to make "predictions" which can then be compared with observations in quantum particle physics by making use of the relation between the Schwarzschild and Riemann normal coordinates.

2.2.1 Wave-packet propagation

In Riemann normal coordinates, all geodesics which pass through the point X are given by straight lines [15]. For instance, we consider the following (classical) geodesic:

$$y^a(\tau) = (P^a/M)\,\tau, \qquad (17)$$

where τ is the proper time. This geodesic can be re-written in terms of general coordinates, namely

$$x^a(\tau) = x^a(y)\big|_{y=(P/M)\tau}. \qquad (18)$$

This is the result of classical theory. In quantum theory, one should instead consider

$$\begin{aligned}\langle x^a \rangle &\equiv -i \int_{y^0} d^3\mathbf{y} \sqrt{-g(y)}\, x^a(y)\, g^{0b}(y) \\ &\quad \times \big(\varphi_{Y,P}(y)\partial_b\bar{\varphi}_{Y,P}(y) - \bar{\varphi}_{Y,P}(y)\partial_b\varphi_{Y,P}(y)\big). \qquad (19)\end{aligned}$$

This integral cannot in general be evaluated exactly. We shall do this perturbatively in terms of number of metric-tensor derivatives.

Having expanded $g_{ab}(y)$ (and $\varphi_{Y,P}(y)$) in (19) over the curvature tensor, its derivatives and products, and then collected terms having the same number of the metric derivatives, we get

$$\langle x^a \rangle = \langle x^a \rangle_{(0)} + \sum_{s=2}^{\infty} \langle x^a \rangle_{(s)}, \tag{20}$$

where the term with $s = 1$ is absent, since we work in Riemann normal coordinates in which the first derivative of the metric tensor vanishes at X. We have, by definition, for $s = 0$ that

$$\begin{aligned} \langle x^a \rangle_{(0)} &\equiv -i \int_{y^0} d^3\mathbf{y}\, x^a(y)\, \varphi^{(0)}_{Y,P}(y) \partial_0 \bar{\varphi}^{(0)}_{Y,P}(y) + \text{c.c.} \\ &= x^a + \sum_{n=0}^{\infty} \delta^{(n)}_{(0)} x^a. \end{aligned} \tag{21}$$

Next, in the non-relativistic approximation, i.e. the group velocity $V^i \equiv P^i/P^0$ is negligible with respect to the speed of light c, we find at $X = (0,\, 0,\, 0,\, r_\oplus)$ that

$$\delta^{(0)}_{(0)} x^a \xrightarrow[V \to 0]{} 0, \tag{22a}$$

$$\delta^{(1)}_{(0)} x^a \xrightarrow[V \to 0]{} -\frac{g_\oplus \hbar^2}{8D^2c^2}\, \delta^a_3 \left(\Delta^{(1)}_{(0),\,0} + \frac{4D^4}{M^2\hbar^2}\, \Delta^{(1)}_{(0),\,2}\, \tau^2 \right), \tag{22b}$$

$$\delta^{(2)}_{(0)} x^a \xrightarrow[V \to 0]{} + \frac{(g_\oplus)^2\hbar^2}{4D^2c^4}\, \delta^a_0 \left(\Delta^{(2)}_{(0),\,0} + \frac{4D^4}{M^2\hbar^2}\, \Delta^{(2)}_{(0),\,2}\, \tau^2 \right) c\tau, \tag{22c}$$

$$\begin{aligned} \delta^{(3)}_{(0)} x^a \xrightarrow[V \to 0]{} & + \frac{(g_\oplus)^2\hbar^4}{24D^4c^4 r_\oplus}\, \delta^a_3 \Big(\Delta^{(3)}_{(0),\,0} \\ & + \frac{D^2c^2}{\hbar^2}\, \Delta^{(3)}_{(0),\,2}\, \tau^2 + \frac{4D^6c^2}{M^2\hbar^4}\, \Delta^{(3)}_{(0),\,4}\, \tau^4 \Big), \end{aligned} \tag{22d}$$

$$\begin{aligned} \delta^{(4)}_{(0)} x^a \xrightarrow[V \to 0]{} & + \frac{(g_\oplus)^2\hbar^4}{10D^4c^4 (r_\oplus)^2}\, \delta^a_0 \Big(\Delta^{(4)}_{(0),\,0} \\ & + \frac{D^2c^2}{\hbar^2}\, \Delta^{(4)}_{(0),\,2}\, \tau^2 + \frac{4D^6c^2}{M^2\hbar^4}\, \Delta^{(4)}_{(0),\,4}\, \tau^4 \Big) c\tau, \end{aligned} \tag{22e}$$

where we have replaced y^0 by τ in accordance with (17) and the non-relativistic limit,

$$\Delta^{(n)}_{(s),\,k} \equiv 1 + \sum_{l=1}^{\infty} C^{(n)}_{(s),\,k,\,l} \left(\frac{D}{Mc} \right)^{2l}, \tag{23}$$

where $C^{(n)}_{(s),\,k,\,l}$ are numerical coefficients, and

$$g_\oplus \equiv \frac{c^2 r_{S,\oplus}}{2(r_\oplus)^2} \approx 9.81\ \text{m/s}^2 \tag{24}$$

is the free-fall acceleration. If we suppose for the moment that $\tau \ll M\hbar/2D^2$, then all these corrections to the classical geodesic (18) are owing to $D < \infty$. This can be readily understood by looking at the argument of the modified Bessel function in (13). Specifically, we obtain in the limit $M/D \to \infty$ that

$$\frac{M^2}{D^2}\, \Sigma_{Y,P}(y) \approx \frac{M^2}{2D^2} + i\, P\cdot y + \Big((P\cdot y)^2 - M^2 y\cdot y \Big) \frac{D^2}{M^2}, \tag{25}$$

where the first (divergent) term is canceled in (13) by an analogous term in (14b), the second term corresponds to the quantum-mechanical phase, and the last term ensures that the wave packet is suppressed away from the classical geodesic (17). The strength of this suppression is characterised by the momentum variance D. The bigger its value, the smaller the wave-packet localisation region and, therefore, the deviation from the classical geodesic must disappear in the limit $D \to \infty$, but $D \ll Mc < \infty$ holds in practice. However, the wave-packet spreading starts to play a role if $\tau \gtrsim M\hbar/2D^2$ (see Sec. 4 of Ch. 2 in [19]). Hence, we find that the wave-packet spreading enhances the deviation from the geodesic over a long-enough time. Does this lead to a measurable effect?

Up to the first order in derivatives of the metric tensor, we get in the non-relativistic and weak-field limit near the Earth's surface that

$$\langle x^0(\tau) \rangle \approx c\tau, \tag{26a}$$

$$\begin{aligned} \langle x^i(\tau) \rangle \approx\; & x^i(0) + V^i\tau - \frac{g_\oplus \delta^i_3}{2} \\ & \times \left(\left(1 + \frac{D^2}{M^2c^2}\, \Delta^{(1)}_{(0),\,2} \right) \tau^2 + \frac{\hbar^2}{4D^2c^2}\, \Delta^{(1)}_{(0),\,0} \right), \end{aligned} \tag{26b}$$

where the wave-packet spreading yields a contribution which is in fact negligible in the semi-classical approximation $Mc/D \ggg 1$. Thus, this trajectory agrees with experimental tests of free fall of non-relativistic neutrons nearby the Earth's surface [3–5], where the spin degree of freedom does not play any role here, if the semi-classical approximation applies. If not, then the centre of mass of the packet propagates towards Earth with a constant acceleration that differs from $g_\oplus$ by the factor of $1 + (D/Mc)^2$.

Higher-order (in metric derivatives) corrections to the centre-of-mass trajectory may start to play a role over long-enough time intervals, as, for instance, it follows from (22d). However, we need to take into account the curvature tensor, because the space-time curvature may give a non-zero contribution at that order. We shall study this later below.

2.2.2 Wave-packet phase

Thermal neutrons have been shown to acquire a phase shift as a consequence of the Earth's gravitational field [20]. The quantum interference induced by gravity was originally predicted from the Schrödinger equation with the Newtonian gravitational potential [21] and this result immediately follows in the non-relativistic limit from the approximate solution (13):

$$\varphi_{X,P}(x) \propto \exp\left(-iMc^2 t\left(1 + v^{(1)}\right)\right), \tag{27a}$$

where by definition

$$\delta^{(1)} v \equiv \frac{g_\oplus z}{c^2}, \tag{27b}$$

where z is the vertical height to vanish at the Earth's surface.

A few remarks are in order. First, this result has been computed by considering isotropic coordinates, (ct, x, y, z), in Schwarzschild spacetime, such that $X = (0, 0, 0, r_\oplus)$. We get from the coordinate transformation $y = y(x)$ treated at X that

$$y^a = x^a + \sum_{n=1}^{\infty} \delta^{(n)} y^a, \tag{28}$$

where x vanishes from now on at X and, to the first order in derivatives of the metric tensor,

$$\delta^{(1)} y^0 \approx +\frac{g_\oplus z}{c^2}\, t, \tag{29}$$

where the approximate equality means that we have omitted polynomials of the ratio $r_{S,\oplus}/r_\oplus$ in the prefactor of the right-hand side. We shall tacitly do the same below in the higher-order terms in the expansion (28). Second, the on-mass-shell condition $P\cdot P = M^2c^4$ gives

$$P_T \xrightarrow[V\to 0]{} \sqrt{g_{TT}(X)}\, Mc^2 \;\approx\; \left(1 - \frac{r_{S,\oplus}}{2r_\oplus}\right) Mc^2. \tag{30}$$

This result can be derived by taking into account that $P\cdot y = Mc^2\tau$ on the classical geodesic, where, in the non-relativistic approximation, the proper time τ reads

$$\tau = \sqrt{g_{AB}(X) y^A y^B} \;\approx\; \sqrt{g_{TT}(X)}\, y^T. \tag{31}$$

The ratio $r_{S,\oplus}/r_\oplus$ in $g_{TT}(X)$ has been neglected in (27a) as well.

Another experiment which probes the wave-packet-phase structure has been performed by making use of an accelerated interferometer [22]. Namely, the acceleration-induced quantum interference of non-relativistic neutrons has been observed, that agrees with the expectations from the Einstein principle [23]. Expressing Riemann normal coordinates y through Rindler coordinates (t_R, x_R, y_R, z_R), i.e.

$$y^0 = \left(c/a + z_R/c\right)\sinh(at_R/c), \tag{32a}$$

$$y^1 = x_R, \tag{32b}$$

$$y^2 = y_R, \tag{32c}$$

$$y^3 = \left(c^2/a + z_R\right)\cosh(at_R/c), \tag{32d}$$

we obtain

$$P\cdot y \approx Mc^2 t_R\left(1 + \frac{az_R}{c^2} + \frac{(at_R)^2}{6c^2}\right), \tag{33}$$

which coincides in the non-relativistic approximation with the gravity-induced phase shift if we take $a = g_\oplus$. The acceleration-squared term cannot be compared yet with that in gravity, as we first need to take into account higher-order gravity corrections to (27a). We shall study these corrections shortly.

The Colella–Overhauser–Werner experiment [20] shows that the wave function of a freely-falling particle is a superposition of the scalar-field modes which are not eigenfunctions of the Schwarzschild-time translation operator ∂_t, even though it is a Killing vector. For this reason, quantum-field modes, which are eigenfunctions of a time-like Killing vector, do not necessary correspond to modes whose superposition can be related to a physical quantum particle. The Bonse-Wroblewski experiment [22] gives another example for this observation, now in case of Rindler spacetime. However, this circumstance might change if one considers interacting field models. For instance, stationary wave functions above a reflecting plate describe bound states of particles in the Earth's gravitational field [24], which were observed in Nature [25].

The gravity-induced quantum interference has been observed so far in the non-relativistic regime. The wave-packet phase $P\cdot y$ turns on shell into $M\tau$, where τ is the proper time over a geodesic defined by the initial centre-of-mass position X and the initial momentum P. Hence, a general relativistic result for the phase difference in the Colella–Overhauser setup [21] (with $M > 0$ and the curvature tensor neglected) reads

$$\delta\varphi(h) = \left(Mc^2/\hbar\right)\left(\tau_{r_\oplus} - \tau_{r_\oplus + h}\right), \tag{34}$$

where h is a vertical height of the upper horizontal path above the Earth's surface. Moreover, we obtain from $P\cdot y$ up to the first order in $g_\oplus$ that

$$\delta\varphi(h) \approx -\frac{g_\oplus hl}{\hbar c^2}\,\frac{(Mc)^2 + 2P^2}{P}, \tag{35}$$

where l is the length of the horizontal path. The gravity-induced phase shift $\delta\varphi(h)$ reduces to the Colella–Overhauser result [21] in the non-relativistic regime, $Mc \gg P$. In the relativistic limit, $\delta\varphi(h)$ is by a factor of two bigger than that previously reported in [26–28] for photons. Note, we take in (35) into account both the gravitational time dilation and length contraction in the horizontal direction, which nonnegligibly contributes in the relativistic limit. This was also taken into consideration earlier in [29].

2.2.3 Preliminary conclusion

There are several gravity-attributed effects which were experimentally confirmed in non-relativistic quantum physics. In full agreement with [3–5], we have found that the relativistic wave packet is consistent with Ehrenfest's theorem for free fall:

$$\frac{d^2}{d\tau^2}\,\langle x^i(\tau)\rangle \approx -g_\oplus\left(1+\frac{D^2}{M^2c^2}\right)\delta_3^i, \tag{36}$$

iff $Mc \ggg D$ is fulfilled. Besides, thermal neutrons have been shown to non-trivially interfere with each other due to the Earth's gravitational field [20]. This was predicted earlier in [21] by relying on the Schrödinger equation with Newton's potential. Here the covariant wave packet gives the phase shift (35) which also agrees with the observations. Note that both effects are essentially due to the gravitational time dilation (29) (see also [30]).

For these reasons, the mathematical model for an elementary particle in curved spacetime, that has been presented in Sect. 2.1, deserves further scrutiny. The next step we intend now to make is to perturbatively take into account the space-time curvature.

2.3 Curvature effects

The curvature-dependent terms neglected in the previous section may start to play a role over long-enough time intervals even if the wave-packet localisation size is much smaller than the local curvature length. In order to study this issue, we look for the wave-packet solution in the following form:

$$\phi_{Y,K}(y) = \phi_{Y,K}^{(0)}(y) + \sum_{n=2}^{\infty}\phi_{Y,K}^{(n)}(y). \tag{37}$$

Substituting $\phi_{Y,K}(y)$ into the scalar-field equation (7), we obtain

$$\left(\eta^{ab}\partial_a\partial_b + M^2\right)\phi_{Y,K}^{(n)}(y) = j^{(n)}(y), \tag{38}$$

where we have for $n \in \{2,\,3,\,4\}$ that

$$j^{(2)}(y) = -\mathcal{D}^{(2)}\phi_{Y,K}^{(0)}, \tag{39a}$$

$$j^{(3)}(y) = -\mathcal{D}^{(3)}\phi_{Y,K}^{(0)}, \tag{39b}$$

$$j^{(4)}(y) = -\mathcal{D}^{(4)}\phi_{Y,K}^{(0)} - \mathcal{D}^{(2)}\phi_{Y,K}^{(2)}, \tag{39c}$$

and, in vacuum ($R_{ab} = 0$ and, consequently, $\eta^{ab}R_{ab} = 0$),

$$\mathcal{D}^{(2)} = \frac{1}{3}\,R^a{}_c{}^b{}_d\,y^c y^d\partial_a\partial_b, \tag{40a}$$

$$\mathcal{D}^{(3)} = \frac{1}{6}\,R^a{}_c{}^b{}_{d;e}\,y^c y^d y^e\partial_a\partial_b, \tag{40b}$$

$$\mathcal{D}^{(4)} = \left(\frac{1}{20}\,R^a{}_c{}^b{}_{d;ef} + \frac{1}{15}\,R^a{}_{cgd}R^b{}_e{}^g{}_f\right)y^c y^d y^e y^f\partial_a\partial_b - \frac{4}{45}\,R^{fde}{}_a R_{fbec}\,y^a y^b y^c\partial_d. \tag{40c}$$

Note that there is no correction with $n = 1$, since we work here in local inertial coordinates. We now intend to solve this system of differential equations in order.

2.3.1 LO curvature correction

We obtain one curvature-dependent term at the leading order (LO) of perturbation theory, which does not vanish in vacuum, namely

$$\phi_{Y,K}^{(2)}(y) = e^{-iK\cdot y}\,R_{acbd}K^aK^by^cy^d\,v_1, \tag{41}$$

where v_1 is a covariant function of y and K. In other words, v_1 depends on $y\cdot y \equiv y^2$ and $K\cdot y$ only. We obtain then from (38) with $n = 2$ that

$$\partial\cdot\partial v_1 - 2iK\cdot\partial v_1 + 8\dot{v}_1 = \frac{1}{3}, \tag{42a}$$

where the dot stands for the differentiation with respect to y^2, and the first two terms on the left-hand side of this equation can be re-written in terms of y^2 and $K\cdot y$ and derivatives of v_1 with respect to its variables. This equation has infinitely many solutions. However, bearing in mind $j^{(2)}(y) = \mathrm{O}(y^2)$, one should have $v_1 \propto K\cdot y$. This gives from (42a) that

$$v_1 = \frac{i}{6M^2}\,K\cdot y. \tag{42b}$$

It is worth emphasising that this solution is non-unique. For example, v_1+const is another solution of (42a). This constant represents a free parameter. It cannot be determined even in de-Sitter spacetime even if one considers the de-Sitter quantum state. It seems, thereby, that we need experimental data to deal with this mathematical ambiguity.

There has been recently observed a wave-packet phase shift due to the curvature [31]. This effect was predicted earlier by assuming the non-relativistic approximation [32–36]. In order to show how this result follows from the covariant wave packet, we first establish the leading-order correction to (13):

$$\varphi_{Y,P}^{(2)}(y) = \frac{1}{48M^2}\,N^{(0)}R_{acbd}\,P^aP^by^cy^d \times\left(iP\cdot y - 2D^2y^2\right)\frac{K_4\left(\frac{M^2}{D^2}\Sigma_{Y,P}(y)\right)}{(\Sigma_{Y,P}(y))^4}. \tag{43}$$

Second, choosing isotropic coordinates in Schwarzschild spacetime and assuming $Mc \gg D$, we find in the non-relativistic approximation up to the second order in the metric derivatives that

$$\varphi_{X,P}(x) \propto \exp\left(-iMc^2t\left(1 + v^{(1)} + v^{(2)}\right)\right), \tag{44a}$$

where $\delta^{(1)}v$ has been defined in (27b) and

$$v^{(2)} \equiv -\frac{1}{2}\,R_{0i0j}\,x^i x^j + \frac{(g_\oplus t)^2}{6c^2}, \tag{44b}$$

where we have also used the expansion of Riemann coordinates via isotropic coordinates up to the second order in derivatives of the metric, appearing in (28). The numerical factor in front of the curvature-dependent term in the phase correction (44b) follows from the combination of the proper-time expansion over the non-inertial coordinates x at X and the leading-order correction (43) to (13). This coefficient can also be obtained by working in Fermi coordinates and expressing the proper time τ through the Fermi time coordinate (see (45) in [37]). This result agrees with the observation [31] and, therefore, the imaginary part of v_1 given in (42b) is physically correct.

The acceleration-squared term in (44b) remains experimentally unobserved, although, as a matter of principle, it may be testable, as recently argued in [38]. This term manifests itself in the non-inertial frame and is, accordingly, present in (33) as well (see also [23]).

The curvature-dependent correction (43) is covariant. Besides, it vanishes on the classical geodesic (17). Still, the wave packet has a non-vanishing support in space. This correction is, hence, generically non-zero iff $D < \infty$ (note that the packet is proportional to the Wightman function in the limit $D \to \infty$, which, in vacuum, is oblivious to the leading-order curvature correction, see (2.21) in [39]). Specifically, $\varphi_{X,P}(x)$ becomes deformed due to (43). Moreover, this deformation is typical for the gravitational tidal effect, namely the wave packet becomes squeezed in the horizontal direction, while stretched in the vertical one.

For this reason, one might expect that (43) influences the wave-packet propagation solely due to $D < \infty$. Indeed, we have from (20) that

$$\langle x^a \rangle_{(2)} \equiv \sum_{n=0}^{\infty} \delta^{(n)}_{(2)} x^a, \tag{45}$$

where, up to the fourth order in derivatives of the metric tensor, we find in the non-relativistic limit in Schwarzschild spacetime that

$$\delta^{(0)}_{(2)} x^a \xrightarrow[V \to 0]{} 0, \tag{46a}$$

$$\delta^{(1)}_{(2)} x^a \xrightarrow[V \to 0]{} +\frac{(g_\oplus)^2\hbar^4}{3D^4c^4 r_\oplus}\,\delta^a_3\left(\Delta^{(1)}_{(2),\,2} + \frac{D^2c^2}{\hbar^2}\,\Delta^{(1)}_{(2),\,2}\,\tau^2\right), \tag{46b}$$

$$\delta^{(2)}_{(2)} x^a \xrightarrow[V \to 0]{} 0. \tag{46c}$$

It should be noted that there is no contribution to the normalisation factor (14b) at this order of perturbation theory. In fact, both the wave packet and its normalisaition condition (4) do not depend on a coordinate frame. This means that the leading-order correction to (14b) has to be proportional to the Riemann tensor R_{abcd} contracted with the metric tensor η_{ab} and the initial four-momentum P_a, as there are no other tensors in the problem under consideration. Yet, any scalar obtained by contracting R_{abcd} with η_{ab} and P_a is zero in vacuum.

Up to the second order in derivatives of the metric tensor, we obtain in the non-relativistic limit at the Earth's surface for $\tau \ll c/g_\oplus \approx 3.06\times 10^7$ s, i.e. $\tau \approx t$, that

$$\langle x^0(\tau)\rangle \approx c\tau\left(1 + \frac{(g_\oplus)^2\hbar^2}{4D^2c^4}\,\Delta^{(2)}_{(0),\,0}\right), \tag{47a}$$

$$\langle x^i(\tau)\rangle \approx x^i(0) + V^i\tau - \frac{g_\oplus\delta^i_3}{2} \times\left(\left(1 + \frac{D^2}{M^2c^2}\,\Delta^{(1)}_{(0),\,2}\right)\tau^2 + \frac{\hbar^2}{4D^2c^2}\,\Delta^{(1)}_{(0),\,0}\right). \tag{47b}$$

The novel contribution entering the centre-of-mass trajectory of the wave packet at this order influences time duration of free fall (cf. [40]). This result shows that the position expectation values defined with respect to $\tau = \text{const}$ and $\langle x^0(\tau)\rangle = \text{const}$ Cauchy surfaces approximately agree if $\hbar/D \ll r_\oplus(r_\oplus/r_{S,\oplus}) \sim 100\times$solar-system size.

2.3.2 *NLO curvature correction*

There are two curvature-dependent terms in the next-to-leading order (NLO):

$$\phi^{(3)}_{Y,K}(y) = e^{-iK\cdot y} R_{acbd;e} K^a K^b y^c y^d \left(w_1 y^e + w_2 K^e\right), \tag{48}$$

where w_1 and w_2 are covariant functions of y and K, which do not vanish in vacuum. First, we obtain from (38) with $n = 3$ that

$$\partial\cdot\partial w_1 - 2iK\cdot\partial w_1 + 12\dot{w}_1 = \frac{1}{6}. \tag{49a}$$

According to the minimal Ansatz, we need to look for a solution of the inhomogeneous part of this equation only. By analogy with our procedure in the previous section, we obtain

$$w_1 = \frac{i}{12M^2}\,K\cdot y. \tag{49b}$$

Second, we have

$$\partial\cdot\partial w_2 - 2iK\cdot\partial w_2 + 8\dot{w}_2 = 2iw_1 - 2w_1', \tag{50a}$$

where the prime stands for the differentiation with respect to $K\cdot y$. The inhomogeneity of this equation originates from $w_1 \propto K\cdot y$. Therefore, we assume that w_2 depends on $K\cdot y$ only. This results in

$$w_2 = -\frac{i}{24M^4}\,(K\cdot y)^2 + \frac{1}{24M^4}\,K\cdot y, \tag{50b}$$

which has no free parameters.

Even though there are no experimental data, to our knowledge, which need the curvature derivative for their explanation, we can make a prediction by using this particular solution in the non-relativistic regime. This is due to $R_{0c0d;0} = 0$ in Schwarzschild spacetime. Therefore, the wave-packet phase up to the third order in derivatives of the metric tensor reads

$$\varphi_{X,P}(x) \propto \exp\left(-iMc^2 t\left(1 + v^{(1)} + v^{(2)} + v^{(3)}\right)\right), \tag{51a}$$

where $v^{(1)}$ and $v^{(2)}$ have, respectively, been defined in (27b) and (44b), and

$$v^{(3)} \equiv -\frac{1}{6} R_{0i0j;k}\, x^i x^j x^k - \frac{2(g_\oplus t)^2}{3c^2}\frac{z}{r_\oplus}, \tag{51b}$$

and we have also made use of (28) up to the order with $n = 3$. The numerical factor in front of the curvature-dependent term coincides with that to appear in the phase if one works in the Fermi frame (see (45) in [37]). But, there is an extra contribution in the non-inertial frame, that may be interpreted as a time-dependent correction to (27b). It becomes non-negligible with respect to (27b) if $t \gtrsim 16.4$ min, whereas a characteristic curvature time at the surface of Earth is roughly given by $(r_\oplus/c)\sqrt{r_\oplus/r_{S,\oplus}} \approx 9.5$ min. It is unclear if it is feasible to design an experiment that could test this non-inertial-frame contribution to the phase.

The next-to-leading-order curvature correction to the locally Minkowski wave packet (13) vanishes in vacuum on the classical geodesic. Hence, this provides a sub-leading correction to the gravitational tidal effect we have considered above. Nevertheless, we find no contribution to the classical geodesic in the non-relativistic limit in Schwarzschild spacetime, that depends on the first (covariant) derivative of the Riemann tensor, i.e.

$$\langle x^a\rangle_{(3)} \equiv \sum_{n=0}^{\infty}\delta_{(3)}^{(n)} x^a, \tag{52}$$

where, up to the fourth order in the metric derivatives,

$$\delta_{(3)}^{(0)} x^a \xrightarrow[V\to 0]{} 0, \tag{53a}$$

$$\delta_{(3)}^{(1)} x^a \xrightarrow[V\to 0]{} 0. \tag{53b}$$

Note, for the same reason as in the case of the leading-order correction, there is no curvature contribution to the normalisation factor (14b) at NLO.

It is straightforward to compute higher-order corrections to (47). These corrections yield

$$\frac{d^2}{d\tau^2}\langle x^i(\tau)\rangle \approx -g_\oplus\left(1 + \frac{D^2}{M^2c^2} - \frac{r_{S,\oplus}\hbar^2}{48(r_\oplus)^3}\left(\frac{16}{D^2} - \frac{91}{M^2c^2}\right)\right)\delta_3^i, \tag{54}$$

where we only consider τ^2-dependent terms in $\langle x^i(\tau)\rangle$, because higher-order polynomials in τ contribute to higher-order terms in the classical trajectory $x^i(\tau)$. The wave-packet centre of mass falls down with a constant acceleration which depends on the Lagrangian mass M, the momentum variance D and higher-order metric derivatives. This is our main result.

2.3.3 NNLO curvature correction

We find eighteen curvature-dependent terms in the next-to-next-to-leading order (NNLO):

$$\begin{aligned}\phi_{Y,K}^{(4)}(y) = e^{-iK\cdot y}\big(&R_{acbd;ef}K^aK^bI^{cdef}\\ &+R_{acge}R_{bdhf}K^aJ^{bcdefgh}\big),\end{aligned} \tag{55}$$

where by definition

$$\begin{aligned} I^{cdef} \equiv\; & f_1 y^c y^d y^e y^f\\ &+ f_2 y^c y^d K^e y^f + f_3 y^c y^d y^e K^f\\ &+ f_4 y^c y^d K^e K^f + f_5 y^c y^d \eta^{ef},\end{aligned} \tag{56a}$$

$$\begin{aligned} J^{bcdefgh} \equiv\; & f_6 K^b y^c y^d y^e y^f K^g K^h + K^b\big(f_7 y^c y^d y^e y^f\\ &+ f_8 y^c y^d K^e y^f + f_9 y^c y^d K^e K^f\big)\eta^{gh}\\ &+\big(f_{10} y^b y^e y^f\\ &+ f_{11} K^b y^e y^f\\ &+ f_{12} y^b K^e y^f + f_{13} y^b y^e K^f + f_{14} K^b K^e y^f\\ &+ f_{15} y^b K^e K^f + f_{16} y^b \eta^{ef}\\ &+ f_{17} K^b \eta^{ef} + f_{18} K^b K^e K^f\big)\eta^{gh}\eta^{cd}.\end{aligned} \tag{56b}$$

Substituting (55) with (56) into (38) with $n = 4$, we obtain

$$\partial\cdot\partial f_i - 2iK\cdot\partial f_i = F_i, \tag{57}$$

where the first batch of the F-functions reads

$$F_1 = \frac{1}{20} - 16\dot{f}_1, \tag{58a}$$

$$F_2 = 2if_1 - 2f_1' - 12\dot{f}_2, \tag{58b}$$

$$F_3 = 2if_1 - 2f_1' - 12\dot{f}_3, \tag{58c}$$

$$F_4 = 2if_2 - 2f_2' + 2if_3 - 2f_3' - 8\dot{f}_4, \tag{58d}$$

$$F_5 = -2f_1 - 8\dot{f}_5, \tag{58e}$$

$$F_6 = -\frac{1}{3}\left(v_1'' - 2iv_1' - v_1\right) - 16\dot{f}_6, \tag{58f}$$

where we remind that the dot and prime stand for the derivative with respect to y^2 and $K\cdot y$, respectively. Taking into account, first, dimensions of the f-functions and, second, assuming that the curvature-dependent factor in (55) is a polynomial of maximal degree six both in y and in K, as this is the case in de-Sitter spacetime for the solution derived in

[7], we obtain a solution for the first batch of the f-functions, which generically depends on twelve constant parameters, assuming that v_1 is given by (42b). Next, the second batch of the F-functions can be treated, namely

$$F_7 = \frac{1}{15} - 16\dot{f}_7, \tag{59a}$$

$$F_8 = 4if_7 - 4f_7' + \frac{4}{3}(v_1' - iv_1) - 12\dot{f}_8, \tag{59b}$$

$$F_9 = 2if_8 - 2f_8' - 8f_6 - 8\dot{f}_9, \tag{59c}$$

$$F_{10} = -\frac{4i}{45} - 12\dot{f}_{10}, \tag{59d}$$

$$F_{11} = 2if_{10} - 2f_{10}' + 4f_1 - 10f_7 - 8\dot{f}_{11}, \tag{59e}$$

$$F_{12} = 2if_{10} - 2f_{10}' + 4f_1 - \frac{2}{3}v_1 - 8\dot{f}_{12}, \tag{59f}$$

giving the second batch of the f-functions to depend on extra eleven constants. And, finally, we have

$$F_{13} = 2if_{10} - 2f_{10}' - 8f_1 + 8f_7 - 8\dot{f}_{13}, \tag{60a}$$

$$F_{14} = 4if_{11} - 4f_{11}' + 2if_{12} - 2f_{12}' + 2if_{13} - 2f_{13}' - 4f_8 - 4\dot{f}_{14}, \tag{60b}$$

$$F_{15} = 2if_{12} - 2f_{12}' + 2if_{13} - 2f_{13}' + 2f_8 - 4\dot{f}_{15}, \tag{60c}$$

$$F_{16} = -3f_{10} - 4\dot{f}_{16}, \tag{60d}$$

$$F_{17} = -2f_{11} - f_{13} + 2if_{16} - 2f_{16}', \tag{60e}$$

$$F_{18} = 2if_{14} - 2f_{14}' + 2if_{15} - 2f_{15}' - 2f_9. \tag{60f}$$

The f-functions that solve these equations depend on additional twelve constant parameters. In total, (55) depends on 35 dimensionless parameters which need to be determined. It is not clear to us how these can be unambiguously achieved. For this reason, it is impossible at this stage to make any predictions for the phase shift at NNLO.

At this order, the curvature-dependent correction does not vanish on the classical geodesic. This is the main reason why we consider this order. One of the consequences of this property is that the Wightman two-point function, defined as

$$W(x, X) \equiv \lim_{D\to\infty} \frac{D^2}{4\pi^2 N}\,\varphi_{X,P}(x), \tag{61}$$

acquires a non-trivial contribution which can be brought to the form obtained in [39,41] by fixing two (at least in Schwarzschild spacetime) of all free parameters. Another consequence is the fact that the wave-packet normalisation factor depends now on the Riemann tensor at $x = X$. Specifically, we have from (20) that

$$\langle x^a\rangle_{(4)} \equiv \sum_{n=0}^{\infty} \delta_{(4)}^{(n)} x^a, \tag{62}$$

where, up to the fourth order in the metric derivatives, we obtain in Schwarzschild spacetime that

$$\delta_{(4)}^{(0)} x^a \xrightarrow[V\to 0]{} + \frac{(g_\oplus)^2\hbar^4}{D^4c^4(r_\oplus)^2} \sum_{n=-1}^{\infty} C_n\left(\frac{D}{Mc}\right)^{2n} \delta_0^a\, c\tau, \tag{63}$$

where C_n are real and depend on the free parameters. Therefore, the normalisation factor

$$N \xrightarrow[V\to 0]{} N^{(0)}\left(1 - \frac{(r_{S,\oplus})^2\hbar^4}{8D^4(r_\oplus)^6} \sum_{n=-1}^{\infty} C_n\left(\frac{D}{Mc}\right)^{2n}\right) \tag{64}$$

depends on the curvature tensor at the point X.

3 Concluding remarks

The main goal of this article was to study free fall in quantum physics. In order to address this question, however, a quantum-particle model in curved space is required. The conceptual idea we have put forward in this regard is to covariantly generalise asymptotic particle states to underlie S-matrix in elementary particle physics. First, this approach respects the general principle of relativity in the sense that the quantum-particle state $|\varphi_{X,P}\rangle$ unitarily transforms under the diffeomorphism group by construction, i.e.

$$|\varphi_{X,P}\rangle \to |\varphi_{\tilde{X},\tilde{P}}\rangle = \hat{U}\,|\varphi_{X,P}\rangle, \tag{65}$$

where $\hat{U}$ is a unitary operator that is related to the coordinate transformation $x \to \tilde{x} = \tilde{x}(x)$, and, thereby, $\tilde{X} = \tilde{x}(X)$ and $\tilde{P}(\tilde{X})$ is a covariant vector obtained from $P(X)$ by means of the tensorial transformation rule. Second, locally and at any point X, the wave packet $\varphi_{X,P}(x)$ to be associated with $|\varphi_{X,P}\rangle$ is represented in a local inertial frame through the superposition of Minkowski plane waves, implying that $|\varphi_{X,P}\rangle$ is related to one of the unitary and irreducible representations of the Poincaré group. This implements the Einstein equivalence principle at quantum level. In other words, quantum-field-theory techniques applied for the description of microscopic processes in collider physics are reliable in any space-time region of the Universe, assuming that a local curvature length there is much bigger than a characteristic length scale to describe a given quantum system (see [42] for an application of this approach).

It is worth emphasising that we have focused here on the construction of diffeomorphism-invariant quantum states, $|\varphi_{X,P}\rangle$. In the semi-classical limit (see below), these states describe particles (approximately) moving along geodesics. Each of these geodesics is determined by its tangent vector P/M at X. In accordance with Heisenberg's uncertainty principle, $|\varphi_{X,P}\rangle$ are (essentially) localised in a certain space volume at a given moment of time (see Sec. II.3.1 in [43] for further details). Such localised states can thus be employed to obtain pseudo-local diffeomorphism-invariant observables in gravity, proposed in [44].

A few experiments have been performed up to now, that require for their explanation both quantum theory and gravity. The elementary-particle model we have proposed is, first, in full agreement with the quantum interference induced by the Earth's gravitational field [20]. This is a non-inertial-frame effect which is accordingly absent in local inertial frames. It was later experimentally proved that this effect also exists in accelerated frames [22]. Loosely speaking, these experiments show that the quantum interference cannot be used to distinguish between uniform gravity and acceleration [23]. The wave function $\varphi_{X,P}(x)$ is naturally consistent with these observations, as shown in Sect. 2.2.2. Second, there has been recently observed a phase shift due to the space-time curvature [31]. A theoretical basis for this experimental result was established in [33], assuming a stationary spacetime. The wave-function solution $\varphi_{X,P}(x)$ we have obtained in this article agrees with [33] and, consequently, with the observation [31], yet this provides a generic result which is applicable in vacuum in non-stationary spacetimes as well (see also [7] for a non-perturbative (in curvature) result in de-Sitter spacetime).

The covariant wave packet $\varphi_{X,P}(x)$ is therefore suitable for studying free fall of quantum matter. The main result of this article is the observation that the wave-packet centre of mass falls down with the acceleration that depends on both the Lagrangian mass M and the wave-packet momentum variance D, which is given in (54). It implies that free fall is not universal at quantum level. However, it appears hard to predict how accurate an experimental test of free fall must be to observer this effect. In fact, this prediction depends on D which essentially defines the initial quantum-particle state. To pass experimental constraints on the (possible) violation of the universality of free fall, the momentum variance D must necessarily be much bigger than $(\hbar/r_\oplus)\sqrt{r_{S,\oplus}/r_\oplus} \sim 10^{-18}$ eV/c and much smaller than Mc. To put it differently, the characteristic (initial) extent of the wave packet must be much bigger than the Compton wavelength $\hbar/Mc$ and much smaller than the local curvature length $r_\oplus\sqrt{r_\oplus/r_{S,\oplus}} \sim 10^{11}$ m. This qualitatively agrees with the results of [7] (see also appendices in [45]).

The non-universality of free fall implies that the weak equivalence principle cannot hold in quantum theory, if conceived as the free-fall universality [8]. There are many other arguments supporting this assertion, see [46]. For this reason, this reference proposed a quantum version of the weak equivalence principle. If adapted to the position expectation value, this quantum principle basically states that this expectation value cannot depend on M. The lower bound on $\hbar/D$ arises from $-\frac{1}{2}\Gamma^a_{ij}(X)\langle y^i y^j\rangle_{(0)}$ to give a quantum correction to $-\frac{1}{2}\Gamma^a_{00}(X)(y^0)^2$. The expectation value $\langle y^i y^j\rangle_{(0)}$ characterises the wave-packet extent which is subject to spreading. It is a universal quantum phenomenon, see [19], which is described by $(D/M)^2(y^0)^2$. From an experimental point of view, this is the most relevant source of the deviation from the classical law of free fall, as the upper bound on $\hbar/D$ is violated for a wave packet of size comparable to or bigger than the Earth-orbit radius.

For an experiment aiming at testing the weak equivalence principle by making use of a pair of atoms, one may estimate $\hbar/D$ by an atomic diameter. Defining the Eötvös parameter as

$$\eta(A, B) \equiv 2\,\frac{g_A - g_B}{g_A + g_B}, \tag{66}$$

where g_A and g_B stand for free-fall accelerations of two atoms A and B, respectively, we get

$$\eta(\text{Au}, \text{Al}) \approx -8.88\times 10^{-16}. \tag{67}$$

This cannot be compared to the experimental upper bound of the order of 10^{-11} found in [47], as aluminum and gold cylinders used in the experiment give the Eötvös parameter $\eta \sim 10^{-79}$. The theoretical estimate (67) is close to the precision of the MICROSCOPE experiment [48], where, however, a relative acceleration of alloy masses, rather than atoms, has been probed on board of a satellite. Next, if we consider ^{87}Rb and ^{39}K atoms, then the Eötvös parameter is dominated in this pair by potassium atoms, as these are lighter and smaller. In this case, we obtain the following estimate:

$$\eta(\text{Rb}, \text{K}) \approx -1.42\times 10^{-16}. \tag{68}$$

This is by 9 orders of magnitude smaller than the sensitivity of a matter-wave interferometer used to test the free-fall universality with rubidium and potassium atoms [49,50], whereas by 4 orders of magnitude smaller than the atom-interferometer sensitivity (employing ^{85}Rb and ^{87}Rb isotopes) achieved in [51]. A novel measurement technique [52] allowed herein to reduce systematic errors [53], making atom interferometers promising for quantum tests of the weak equivalence principle with even higher sensitivity.

The main result (54) relies on an assumption that the position expectation value is given in the Schwarzschild frame by the integral (19). This definition tacitly uses the constant-proper-time Cauchy surfaces, as $P^i = 0$ has been set in the integral computations. Still, the position expectation value depends on a Cauchy-surface choice [7]. Up to the first order in derivatives of the metric tensor, we find

$$\frac{d^2}{dt^2}\langle x^i(t)\rangle \approx -g_\oplus\left(1 - \frac{D^2}{M^2c^2}\right)\delta^i_3, \tag{69}$$

where Cauchy surfaces of constant t have been considered. Note that the Schwarzschild time t and proper time τ (over the geodesic (17) with $P^i = 0$) roughly equal if $t \ll c/g_\oplus \sim 10^7$ s. At this order of the approximation, the difference between (69) and (36) effectively stems from replacing ∂_0 by $\partial_0 - g_\oplus y^0 \partial_3$. Therefore, quantum tests of the universality of free fall might also show if quantum particles measure proper time. It would then provide another witness for the proper-time role in quantum theory, see [54]. A theoretical argument in favor of that could be based on the circumstance that, in the Riemann normal frame, $\langle y^a \rangle$ exactly equals $(P^a/M)\,\tau$, independent on the ratio D/Mc (at least up to the order studied), when the Cauchy surfaces of constant proper time are treated. Indeed, this choice ensures that the curvature-dependent normalisation factor N does not modify that equality. In this case and in the Riemann normal frame, the wave-function centre of mass propagates over the classical geodesic (17) as if the wave packet were structureless.

The expectation value $\langle x^a \rangle$ makes no use of position operator. It is not even clear how to self-consistently define quantum-mechanics operators with corresponding expectation values in curved spacetime. Still, such quantities as energy and three-momentum of the wave packet can be established without introducing the quantum-mechanics operator formalism by solely employing the quantum-field algebra:

$$\langle p_a \rangle \equiv \int_\Sigma d\Sigma^b(y) \left(\langle \varphi_{Y,P} | \hat{T}_{ab}(y) | \varphi_{Y,P} \rangle - \langle \Omega | \hat{T}_{ab}(y) | \Omega \rangle \right), \tag{70}$$

where the scalar-field stress tensor reads

$$\begin{aligned} \hat{T}_{ab}(y) \equiv\; & \nabla_a \hat{\Phi}(y) \nabla_b \hat{\Phi}(y) \\ & - \frac{1}{2}\, g_{ab}(y) \Big(\nabla_c \hat{\Phi}(y) \nabla^c \hat{\Phi}(y) - M^2 \hat{\Phi}^2(y) \Big) \\ & + \frac{1}{6} \Big(G_{ab}(y) - \nabla_a \nabla_b + g_{ab}(y) \nabla_c \nabla^c \Big) \hat{\Phi}^2(y) \end{aligned} \tag{71}$$

for the scalar-field model studied, where $G_{ab}(y)$ is the Einstein tensor. The integrand in (70) can be expressed through $\varphi_{Y,P}(y)$ and its complex conjugate by means of the quantum-field algebra, wherein $-\langle \Omega | \hat{T}_{ab}(y) | \Omega \rangle$ serves to set the quantum-vacuum four-momentum to zero. If the Riemann tensor is now neglected, $\langle p^i \rangle$ approaches $M \partial_0 \langle y^i \rangle$ in the non-relativistic limit, while $\langle p^0 \rangle$ naturally contains that what would be the quantum-mechanical expectation value of the three-momentum-squared operator. It remains to understand how to obtain $\langle y^a p_b \rangle$ and $\langle p_b y^a \rangle$, which might give a curvature-dependent commutation relation.

The quantum state $|\varphi_{X,P}\rangle$ cannot yet be treated as a satisfactory model for an elementary particle. Indeed, this state does not source gravity, i.e. $\langle \varphi_{X,P} | \hat{h}_{\mu\nu}(x) | \varphi_{X,P} \rangle = 0$, where $\hat{h}_{\mu\nu}(x)$ is the graviton field, although $\langle \varphi_{X,P} | \hat{T}_{\mu\nu}(x) | \varphi_{X,P} \rangle \neq \langle \Omega | \hat{T}_{\mu\nu}(x) | \Omega \rangle$. A proper generalisation of the operator $\hat{a}^\dagger(\varphi_{X,P})$ given in (2) is required to go beyond the test-particle approximation. For this reason, it is unclear whether (54) implies that the gravitational mass differs from the inertial mass and how these two masses are related to the Lagrangian mass M in gravity.

Another aspect of the problem is how to determine the free dimensionless parameters that enter the wave function $\varphi_{X,P}(x)$. It may require study of this problem in other spacetimes, by searching for a unique form of $\varphi_{X,P}(x)$ reducing to preferred wave functions in corresponding curved spacetimes. Besides, it may require quantum gravity for dealing with this question. In particular, it might be feasible at least to re-produce our results by using the effective-theory approach to quantum gravity, by setting up no background gravitational field and replacing the Earth by a heavy quantum particle. This approach has already provided quantum-gravity corrections to hyperbolic-like trajectories [55–57]: In quantum gravity, such trajectories cannot be solutions of the geodesic equation.

Finally, the quantum-mechanical phase is given in the semi-classical limit by $+iS(x,X)/\hbar$, where $S(x,X) = -Mc^2\tau(x,X)$ is the single-particle (on-shell) action. Curvature-dependent corrections in $\varphi_{X,P}(x)$ may in general contribute to this phase on a classical geodesic. In fact, the leading correction in vacuum is proportional to the curvature tensor squared, as shown in Sect. 2.3.3. This means that the classical action for a single particle in curved spacetime has a limited application in quantum theory.

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References

1. A. Einstein, Ann. Phys. **322**, 891 (1905)
2. E. Wigner, Ann. Math. **40**, 149 (1939)
3. A.W. McReynolds, Phys. Rev. **83**, 172 (1951)
4. J.W.T. Dabbs, J.A. Harvey, D. Paya, H. Horstmann, Phys. Rev. **139**, B756 (1965)
5. L. Koester, Phys. Rev. D **14**, 907 (1976)
6. A. Einstein, Ann. Phys. **354**, 769 (1916)

7. V.A. Emelyanov, Eur. Phys. J. C **81**, 189 (2021)
8. E. Di Casola, S. Liberati, S. Sonego, Am. J. Phys. **83**, 39 (2015)
9. H. Lehmann, K. Symanzik, W. Zimmermann, Nuovo Cimento **1**, 205 (1955)
10. B.S. DeWitt, *The Global Approach to Quantum Field Theory* (Oxford University Press, Oxford, 2003)
11. B.S. DeWitt, Phys. Rep. **19**, 295 (1975)
12. S. Weinberg, *Quantum Theory of Fields* (Cambridge University Press, Cambridge, 1995)
13. V.F. Mukhanov, *Physical Foundations of Cosmology* (Cambridge University Press, Cambridge, 2005)
14. J.F. Donoghue, Phys. Rev. D **50**, 3874 (1994)
15. A.Z. Petrov, *Einstein Spaces* (Pergamon Press Ltd., Oxford, 1969)
16. D.V. Naumov, V.A. Naumov, J. Phys. G Nucl. Part. Phys. **37**, 105014 (2010)
17. D.V. Naumov, Phys. Part. Nuclei Lett. **10**, 642 (2013)
18. B.S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, London, 1965)
19. E. Merzbacher, *Quantum Mechanics*, 3rd edn. (Wiley, New York, 1998)
20. R. Colella, A.W. Overhauser, S.A. Werner, Phys. Rev. Lett. **34**, 1472 (1975)
21. R. Colella, A.W. Overhauser, Phys. Rev. Lett. **33**, 1237 (1974)
22. U. Bonse, T. Wroblewski, Phys. Rev. Lett. **51**, 1401 (1983)
23. M. Nauenberg, Am. J. Phys. **84**, 879 (2016)
24. V.I. Luschikov, A.I. Frank, JETP Lett. **28**, 559 (1978)
25. V.V. Nesvizhevsky et al., Nature **415**, 297 (2002)
26. P.D. Mannheim, Phys. Rev. A **57**, 1260 (1998)
27. M. Zych et al., Class. Quantum Gravity **29**, 224010 (2012)
28. D. Rideout et al., Class. Quantum Gravity **29**, 224011 (2012)
29. A. Brodutch et al., Phys. Rev. D **91**, 064041 (2015)
30. A.P. Czarnecka, A. Czarnecki, Am. J. Phys. **89**, 634 (2021)
31. P. Asenbaum et al., Phys. Rev. Lett. **118**, 183602 (2017)
32. J. Anandan, Phys. Rev. D **30**, 1615 (1984)
33. J. Audretsch, K.-P. Marzlin, Phys. Rev. A **50**, 2080 (1994)
34. J. Audretsch, K.-P. Marzlin, J. Phys. II Fr. **4**, 2073 (1994)
35. J. Audretsch, K.-P. Marzlin, Phys. Rev. A **53**, 312 (1996)
36. K. Bongs et al., Appl. Phys. B **84**, 599 (2006)
37. W.-Q. Li, W.-T. Ni, J. Math. Phys. **20**, 1925 (1979)
38. C. Marletto, V. Vedral, Front. Phys. **8**, 176 (2020)
39. T.S. Bunch, L. Parker, Phys. Rev. D **20**, 2499 (1979)
40. L. Viola, R. Onofrio, Phys. Rev. D **55**, 455 (1997)
41. S.M. Christensen, Phys. Rev. D **14**, 2490 (1976)
42. V.A. Emelyanov, F.R. Klinkhamer, Acta Phys. Polon. B **52**, 805 (2021)
43. R. Haag, *Local Quantum Physics. Fields, Particles, Algebras* (Springer, Berlin, 1996)
44. S.B. Giddings, D. Marolf, J.B. Hartle, Phys. Rev. D **74**, 064018 (2006)
45. A. Roura, Phys. Rev. X **10**, 021014 (2020)
46. C. Lämmerzahl, Gen. Relativ. Gravit. **28**, 1043 (1996)
47. P.G. Roll, R. Krotkov, R.H. Dicke, Ann. Phys. **26**, 442 (1964)
48. P. Touboul et al., Phys. Rev. Lett. **119**, 231101 (2017)
49. D. Schlippert et al., Phys. Rev. Lett. **112**, 203002 (2014)
50. H. Albers et al., Eur. Phys. J. D **74**, 145 (2020)
51. P. Asenbaum et al., Phys. Rev. Lett. **125**, 191101 (2020)
52. A. Roura, Phys. Rev. Lett. **118**, 160401 (2017)
53. G. D'Amico et al., Phys. Rev. Lett. **119**, 253201 (2017)
54. M. Zych et al., Nat. Commun. **8**, 505 (2011)
55. N.E.J. Bjerrum-Bohr et al., Phys. Rev. Lett. **114**, 061301 (2015)
56. J.F. Donoghue, B.K. El-Menoufi, J. High Energy Phys. **05**, 118 (2015)
57. N.E.J. Bjerrum-Bohr et al., Int. J. Mod. Phys. D **24**, 1544013 (2015)