

Polarized high-frequency wave propagation beyond the nonlinear Schrödinger approxi- mation

Julian Baumstark, Tobias Jahnke, Christian Lubich

CRC Preprint 2022/28, June 2022

KARLSRUHE INSTITUTE OF TECHNOLOGY

CRC 1173



Wave
phenomena

Participating universities



Universität Stuttgart

EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN



Funded by

DFG

Polarized high-frequency wave propagation beyond the nonlinear Schrödinger approximation*

Julian Baumstark[†] Tobias Jahnke[†] Christian Lubich[‡]

June 22, 2022

Abstract

This paper studies highly oscillatory solutions to a class of systems of semilinear hyperbolic equations with a small parameter, in a setting that includes Klein–Gordon equations and the Maxwell–Lorentz system. The interest here is in solutions that are polarized in the sense that up to a small error, the oscillations in the solution depend on only one of the frequencies that satisfy the dispersion relation with a given wave vector appearing in the initial wave packet. The construction and analysis of such polarized solutions is done using modulated Fourier expansions. This approach includes higher harmonics and yields approximations to polarized solutions that are of arbitrary order in the small parameter, going well beyond the known first-order approximation via a nonlinear Schrödinger equation. The given construction of polarized solutions is explicit, uses in addition a linear Schrödinger equation for each further order of approximation, and is accessible to direct numerical approximation.

1 Introduction

We consider semilinear hyperbolic systems of the form

$$\partial_t u + A(\partial_x)u + \frac{1}{\varepsilon}Eu = \varepsilon T(u, u, u), \quad x \in \mathbb{R}^d, t \in [0, \tau_{\text{end}}/\varepsilon] \quad (1)$$

with vector-valued solution $u(t, x) \in \mathbb{R}^n$ ($d, n \in \mathbb{N}$). Here, $A(\partial_x) = \sum_{\ell=1}^d A_\ell \partial_\ell$ is a first-order differential operator with symmetric matrices $A_1, \dots, A_d \in \mathbb{R}^{n \times n}$, $E \in \mathbb{R}^{n \times n}$ is a skew-symmetric matrix, $T : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a

*Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173

[†]Karlsruher Institut für Technologie, Fakultät für Mathematik, Institut für Angewandte und Numerische Mathematik, julian.baumstark@kit.edu, tobias.jahnke@kit.edu

[‡]Universität Tübingen, Mathematisches Institut, lubich@na.uni-tuebingen.de

trilinear nonlinearity, and $0 < \varepsilon \ll 1$ is a small parameter. This problem class includes, e.g., the Maxwell–Lorentz system and Klein–Gordon systems; cf. Section 2.1 in [CL09].

In [CL09, Lan11] such systems have been considered with initial data

$$u(0, x) = e^{i\kappa \cdot x / \varepsilon} p(x) + c.c., \quad (2)$$

where $\kappa \in \mathbb{R}^d \setminus \{0\}$ is a fixed wave vector and $p : \mathbb{R}^d \rightarrow \mathbb{R}^n$ a smooth envelope independent of ε . (Different scalings such as $p(x/\sqrt{\varepsilon})$ instead of $p(x)$ are also of interest, but will not be considered here.) The dot \cdot denotes the Euclidean scalar product in \mathbb{R}^d and “c.c.” means complex conjugation of the preceding term.

Both the PDE (1) and the initial data (2) contain the small parameter $0 < \varepsilon \ll 1$, and as a consequence, solutions of (1)–(2) typically oscillate rapidly in time and space.

Polarization in the linear case. When $T \equiv 0$ in (1), a situation of particular interest appears for special initial data $p(x) = \alpha(x)v$, where α is a smooth real- or complex-valued function, and v is an eigenvector to an eigenvalue¹ ω of the Hermitian matrix $\sum_{\ell=1}^d \kappa_\ell A_\ell - iE$ (in different terminology, the pair of the wave vector κ and the frequency ω satisfies the dispersion relation). Then, the solution is of the form $u(t, x) = e^{i(\kappa \cdot x - \omega t) / \varepsilon} \mu_L(t, x)v + c.c.$, where μ_L is a smooth, ε -independent complex-valued function. This is a *polarized solution* where the other eigenvalues $\tilde{\omega} \neq \omega$ do not appear in highly oscillatory exponentials $e^{i(\kappa \cdot x - \tilde{\omega} t) / \varepsilon}$, as they would for general initial data.

Nonlinear polarization. In the present paper we contribute to extending the concept of polarization to the *nonlinear* situation (1). There is earlier work by Colin & Lannes [CL09] and Lannes [Lan11], which motivated the present work. They considered initial data (2) with $p(x) = \alpha(x)v + \mathcal{O}(\varepsilon)$ and showed that the solution can then be written as $u(t, x) = e^{i(\kappa \cdot x - \omega t) / \varepsilon} \mu(t, x)v + c.c. + \mathcal{O}(\varepsilon)$ with a function μ that, together with its partial derivatives, is bounded independently of ε . (Here and in the following, the $\mathcal{O}(\cdot)$ notation is with respect to the maximum norm.) This approximation result holds true over times $\mathcal{O}(\varepsilon^{-1})$, which is the time scale over which the nonlinearity yields an $O(1)$ contribution to the solution.

Here, we are interested in approximations of higher order in ε . We will show that for given smooth $p(x) = \alpha(x)v$, there exists a solution of (1) of the form (for an arbitrary fixed odd integer $m \geq 1$)

$$u(t, x) = \sum_{\substack{|j| \leq m \\ j \text{ odd}}} u_j(t, x) e^{ij(\kappa \cdot x - \omega t) / \varepsilon} + \mathcal{O}(\varepsilon^{m+1}) \quad (3)$$

¹In the introduction we assume ω to be a simple eigenvalue.

such that $\mathcal{P}u_1(0, x) = p(x)$ with the orthogonal projection $\mathcal{P} = vv^*$ onto the eigenspace, and the modulation functions u_j with $u_{-j} = \bar{u}_j$ are smooth and, together with their partial derivatives of any order, they are bounded by $u_j = \mathcal{O}(\varepsilon^{|j|-1})$. The dominant modulation function $u_1(t, x)$ is aligned with the eigenvector v up to $\mathcal{O}(\varepsilon)$. Moreover, the representation (3) is unique up to $\mathcal{O}(\varepsilon^{m+1})$. This higher-order approximation result again holds true over times $\mathcal{O}(\varepsilon^{-1})$. Note that here no other frequencies $\tilde{\omega} \neq \omega$ are allowed to appear in the oscillatory exponentials in (3). It is not obvious *a priori* that this can be achieved for the semilinear equation.

Nonlinear Schrödinger approximation. In Section 2.3.2 of [CL09] and Section 2.3 of [Lan11], the solution of (1)–(2) with $p(x) = \alpha(x)v$ is approximated up to $\mathcal{O}(\varepsilon)$ by

$$u_{\text{NLS}}(t, x) = e^{i(\kappa \cdot x - \omega t)/\varepsilon} U(\varepsilon t, x - c_g t) + c.c., \quad (4)$$

where $U(\tau, \xi)$ is the solution of an ε -independent nonlinear Schrödinger equation (NLS) on the rescaled time interval $[0, \tau_{\text{end}}]$. In the second argument of U , c_g is the group velocity.

The modulation function u_1 in (3) admits an asymptotic expansion in powers of ε : $u_1 = u_1^0 + \varepsilon u_1^1 + \varepsilon^2 u_1^2 + \dots$. It turns out that $u_1^0(t, x)$ in (3) equals $U(\varepsilon t, x - c_g t)$, though it arises with a different proof.

Higher harmonics. A shortcoming of the nonlinear Schrödinger approximation is the fact that substituting (4) into the nonlinearity yields

$$\begin{aligned} T(u_{\text{NLS}}, u_{\text{NLS}}, u_{\text{NLS}}) &= e^{i(\kappa \cdot x - \omega t)/\varepsilon} \left(T(U, U, \bar{U}) + T(U, \bar{U}, U) + T(\bar{U}, U, U) \right) \\ &\quad + e^{-i(\kappa \cdot x - \omega t)/\varepsilon} \left(T(U, \bar{U}, \bar{U}) + T(\bar{U}, U, \bar{U}) + T(\bar{U}, \bar{U}, U) \right) \\ &\quad + e^{3i(\kappa \cdot x - \omega t)/\varepsilon} T(U, U, U) + e^{-3i(\kappa \cdot x - \omega t)/\varepsilon} T(\bar{U}, \bar{U}, \bar{U}), \end{aligned}$$

whereas substituting (4) into the left-hand side of (1) does not produce any terms with factor $e^{\pm 3i(\kappa \cdot x - \omega t)/\varepsilon}$. Hence, these *higher harmonics* contribute to the residual and thus to the approximation error. In order to improve the accuracy, these higher harmonics have to be taken into account. This is a first motivation to search for approximations to u of the form (3).

Modulated Fourier expansions and geometric optics. Asymptotic expansions such as (3) are known as modulated Fourier expansions (MFEs) in time, a name that was coined in [HLW02, Chapter XIII]. Multiscale expansions of this type were occasionally used before under different names: in a formal way without rigorous bounds in [Kru58] for analysing the motion of charged particles and in [Whi74] for asymptotics of wave propagation, and with rigorous bounds of modulation functions and remainder term in [HL00]

for studying the long-time behaviour of numerical methods for highly oscillatory Hamiltonian systems. Since then, MFEs were used in many works on the analysis of highly oscillatory ordinary and evolutionary partial differential equations and their numerical approximations; see, e.g., the reviews [HL12, GHL18] and numerous references therein.

In the way MFEs are used here, with an expansion of the modulation functions in powers of ε (which is possible for the problem at hand but not in other cases with near-resonances between frequencies; see e.g. [CHL08, FGL13]), the approach of this paper is actually close to the often-used approach in geometric optics of making an ansatz (see e.g. [Rau12, Chapters 7–9])

$$u(t, x) = \sum_j \varepsilon^j a_j(t, \varepsilon t, x, (\kappa \cdot x - \tilde{\omega}t)/\varepsilon),$$

where $\theta \mapsto a_j(t, \tau, x, \theta)$ is 2π -periodic. Expanding this map into a Fourier series, as is often done, and interchanging the appearing double sum yields an expression that is formally very similar to a modulated Fourier expansion (3) with modulation functions $u_j(t, x)$ expanded in powers of ε . The actual construction of the coefficient functions and the derivation of bounds for them uses, however, different perspectives and details.

For systems similar to or more general than (1) asymptotic expansions of solutions have been constructed in [Rau12, DR97, JMR00, JMR93] and many other works. These articles refer to the setting of geometric optics, where the time interval does not depend on ε . This is to be distinguished from diffractive geometric optics, where the length of the time interval is $\mathcal{O}(1/\varepsilon)$ as in (1). In the diffractive regime, solutions of the initial-value problem for semilinear and quasilinear systems with ε -oscillatory initial data (2) have been approximated with arbitrarily high order in ε in [DJMR96], though with εE instead of E/ε in (1). For (1) with initial data (2) a second-order approximation of the form (3), but with oscillating u_j , has been analyzed in [BJ22]. Note that the problem of nonlinear polarization posed in the present paper is different from the standard initial value problem.

Computational aspects. Because of the high-frequency oscillations, a direct discretization of (1)–(2) would have to use very small time steps and spatial mesh widths and is therefore computationally prohibitive, and even more so the computation of polarized solutions. The smooth, non-oscillatory modulation functions u_j in (3) are constructed explicitly in this paper, requiring only the solution of one nonlinear and a few linear Schrödinger equations that are independent of ε , using co-moving coordinates and the slow time scale $\tau = \varepsilon t$, on the ε -independent rescaled time interval $[0, \tau_{\text{end}}]$. With the NLS approximation or the MFE, one can thus achieve an accuracy of $\mathcal{O}(\varepsilon)$ or $\mathcal{O}(\varepsilon^{m+1})$, respectively, without having to solve any oscillatory PDEs.

While computational aspects were the basic motivation for the present paper, they will not be elaborated here.

Outline. In Section 2 we give a precise formulation of the setting and present our main result, Theorem 2.4, which establishes the polarized modulated Fourier expansion (3). For $m = 1$ equations for the computation of the modulation functions are stated explicitly in Corollary 2.6. The remainder of the paper gives the proof of Theorem 2.4. In Section 3 we construct the MFE formally, in Section 4 we derive bounds for the modulation functions, and in Section 5 we bound the approximation error of the truncated MFE.

2 Statement of the main result

For every $k \in \mathbb{R}^d$ we define the real symmetric matrix

$$A(k) = \sum_{\ell=1}^d k_{\ell} A_{\ell} \in \mathbb{R}^{n \times n} \quad (5)$$

with the same matrices A_{ℓ} as in the differential operator $A(\partial_x)$. A pair (ω, k) fulfills the *dispersion relation* if $i\omega/\varepsilon$ is an eigenvalue of $A(ik/\varepsilon) + E/\varepsilon$, or equivalently, ω is an eigenvalue of the Hermitian matrix $A(k) - iE$.

Assumption 2.1 *Let $\kappa \in \mathbb{R}^d \setminus \{0\}$ be a fixed wave vector, and let $\omega = \omega_1(\kappa)$ be a selected eigenvalue of $A(\kappa) - iE$. We assume that $\omega_1(\cdot)$ is twice continuously differentiable at κ .*

We note that a sufficient (though not necessary) condition for the required differentiability is that ω is a simple eigenvalue of $A(\kappa) - iE$.

The Hermitian matrix

$$\mathcal{L}(\omega, \kappa) = -\omega I + A(\kappa) - iE \in \mathbb{C}^{n \times n}$$

thus has a nontrivial kernel, which is the eigenspace to the eigenvalue ω of $A(\kappa) - iE$. Let \mathcal{P} be the corresponding eigenprojection, i.e., the orthogonal projection onto the kernel of $\mathcal{L}(\omega, \kappa)$. We set $\mathcal{P}^{\perp} = I - \mathcal{P}$.

We consider a function p that takes values in the range of \mathcal{P} :

$$p \in \mathcal{S}(\mathbb{R}^d, \mathbb{C}^n) \quad \text{with} \quad \mathcal{P}p(x) = p(x) \quad \text{for all } x \in \mathbb{R}^d. \quad (6)$$

Here, $\mathcal{S}(\mathbb{R}^d, \mathbb{C}^n)$ is the Schwartz space of smooth functions that, together with all their derivatives, decay faster than every negative power of $|x|$. The function p is assumed to be independent of ε .

Our next assumption concerns the NLS approximation of (4), $u_{\text{NLS}}(t, x) = e^{i(\kappa \cdot x - \omega t)/\varepsilon} U(\varepsilon t, x - c_g t) + c.c.$ with initial data $u_{\text{NLS}}(0, x) = p(x) e^{i\kappa \cdot x} + c.c.$, where $c_g = \nabla \omega_1(\kappa)$ is the group velocity. With the Hessian of ω_1 at κ , denoted $H = \nabla^2 \omega_1(\kappa)$, the function $U = U(\tau, \xi)$ is taken as the solution of the NLS

$$\partial_\tau U - \frac{i}{2} \nabla_\xi \cdot H \nabla_\xi U = \mathcal{P}(T(\bar{U}, U, U) + T(U, \bar{U}, U) + T(U, U, \bar{U})) \quad (7)$$

with initial data $U(0, \cdot) = p$ of (6).

Note that ε does not show up in (7). We assume the following.

Assumption 2.2 *The NLS (7) has a bounded solution U on $[0, \tau_{\text{end}}] \times \mathbb{R}^d$ that has arbitrarily many bounded partial derivatives.*

This is satisfied at least for sufficiently small $\tau_{\text{end}} > 0$, as follows from the local existence result in [Seg63].

For the construction of higher-order approximations in powers of ε we need the following assumption.

Assumption 2.3 *The matrix*

$$\mathcal{L}_j = \mathcal{L}(j\omega, j\kappa) \in \mathbb{C}^{n \times n} \quad (8)$$

is invertible for every odd integer j with $3 \leq j \leq m + 2$.

For $j = 3$ this assumption was also made in [CL09, Assumption 3].

Under these assumptions we have the following theorem, which is the main result of this paper.

Theorem 2.4 (Modulated Fourier expansion) *Let $\kappa \in \mathbb{R}^d \setminus \{0\}$ be a fixed wave vector and let the frequency $\omega = \omega_1(\kappa) \in \mathbb{R}$. Let $p \in \mathcal{S}(\mathbb{R}^d, \mathbb{C}^n)$ with $\mathcal{P}p = p$. Fix an odd positive number m . Under Assumptions 2.1–2.3, there exists a solution u of (1) on the interval $[0, \tau_{\text{end}}/\varepsilon]$ that admits a modulated Fourier expansion*

$$u(t, x) = \sum_{\substack{|j| \leq m \\ j \text{ odd}}} u_j(t, x) e^{ij(\kappa \cdot x - \omega t)/\varepsilon} + R_m(t, x) \quad (9)$$

with $u_{-j}(t, x) = \overline{u_j(t, x)}$ and with the following properties:

1. $\mathcal{P}u_1$ satisfies the initial condition

$$\mathcal{P}u_1(0, \cdot) = p.$$

2. The modulation functions u_j are bounded as follows, with a constant C independent of ε (but depending on m):

$$\|\mathcal{P}u_1\|_{L^\infty([0, \tau_{\text{end}}/\varepsilon] \times \mathbb{R}^d)} \leq C, \quad \|\mathcal{P}^\perp u_1\|_{L^\infty([0, \tau_{\text{end}}/\varepsilon] \times \mathbb{R}^d)} \leq C\varepsilon, \quad (10)$$

$$\|u_j\|_{L^\infty([0, \tau_{\text{end}}/\varepsilon] \times \mathbb{R}^d)} \leq C\varepsilon^{j-1} \quad (\text{for odd } 3 \leq j \leq m), \quad (11)$$

and the same bounds hold true for the spatio-temporal partial derivatives of these functions up to any fixed order. Moreover, the modulation functions u_j admit an asymptotic expansion in powers of ε .

3. The remainder term is bounded by

$$\|R_m\|_{L^\infty([0, \tau_{\text{end}}/\varepsilon] \times \mathbb{R}^d)} \leq C\varepsilon^{m+1}. \quad (12)$$

The modulation functions u_j with these properties (and hence also u) are unique up to $O(\varepsilon^{m+1})$ w.r.t. the maximum norm.

Remark 2.5 (Polarization) We note in particular that the initial value $u(0, x) = (e^{i\kappa \cdot x/\varepsilon} p(x) + \text{c.c.}) + O(\varepsilon)$ is determined by p uniquely up to $O(\varepsilon^{m+1})$. So Theorem 2.4 holds true for special initial values $u(0, \cdot)$ of (1), which we call polarized initial values. As will be seen from the explicit construction in the next section, the polarization map $p \mapsto u(0, \cdot)$ is a nonlinear map. The construction shows that polarized initial values correspond to a slow manifold of a system of highly oscillatory partial differential equations.

For polarized initial values, the other frequencies $\omega_2(\kappa), \dots, \omega_\mu(\kappa)$ (i.e. the other eigenvalues of $A(\kappa) - iE$) do not enter the oscillatory exponentials of the modulated Fourier expansion. For general initial values of type (2), it would be possible to obtain a modulated Fourier expansion that includes also those frequencies, under appropriate resonance or non-resonance conditions among the frequencies. However, this is not the point of the above result, where the interest lies precisely in the fact that the other frequencies do not occur.

Corollary 2.6 Choosing $m = 1$ in (9) gives an $\mathcal{O}(\varepsilon^2)$ approximation on the time interval $[0, \tau_{\text{end}}/\varepsilon]$ that is constructed as follows: We obtain $u_1 = p_1 + q_1$ (with $\mathcal{P}u_1 = p_1$ and $\mathcal{P}^\perp u_1 = q_1$) by setting

$$p_1 = p_1^0 + \varepsilon p_1^1, \quad q_1 = \varepsilon q_1^1,$$

where

- $p_1^0(t, x) = U(\varepsilon t, x - c_g t)$, where $U(\tau, \xi)$ is the solution to the nonlinear Schrödinger equation (7) with initial value $U(0, \cdot) = p$.
- $q_1^1(t, x) = z_1^1(\varepsilon t, x - c_g t)$, where $z_1^1(\tau, \xi)$ is obtained from $\partial_\xi U(\tau, \xi)$ via matrix operations detailed in (33).

- $p_1^1(t, x) = y_1^1(\varepsilon t, x - c_g t)$, where $y_1^1(\tau, \xi)$ is the solution to the linear Schrödinger equation (38) with zero initial value.

Remark 2.7 (Computation of polarized solutions) *Since the proof of Theorem 2.4 gives an explicit construction of the modulation functions, the result can be used to devise numerical schemes to compute polarized solutions of (1), using time steps that are much larger than ε . This will be presented elsewhere.*

3 Construction of the modulated Fourier expansion

3.1 Rescaling of time and co-moving coordinate system

For our analysis it is convenient to change to the new variables

$$\tau = \varepsilon t, \quad \xi = x - c_g t, \quad v(\tau, \xi) = v(\varepsilon t, x - c_g t) = u(t, x), \quad (13)$$

where again $c_g = \nabla \omega_1(\kappa)$ is the group velocity. Since

$$\begin{aligned} \partial_t u(t, x) &= \partial_t v(\varepsilon t, x - c_g t) = \varepsilon \partial_\tau v(\tau, \xi) - c_g \cdot \nabla_\xi v(\tau, \xi), \\ A(\partial_x) u(t, x) &= A(\partial_\xi) v(\tau, \xi), \end{aligned}$$

the change of variables turns the original problem (1) into

$$\partial_\tau v + \frac{1}{\varepsilon} B(\partial_\xi) v + \frac{1}{\varepsilon^2} E v = T(v, v, v), \quad \xi \in \mathbb{R}^d, \tau \in [0, \tau_{\text{end}}] \quad (14)$$

with

$$B(\partial_\xi) = \sum_{\ell=1}^d B_\ell \frac{\partial}{\partial \xi_\ell}, \quad B_\ell = A_\ell - (c_g)_\ell I. \quad (15)$$

We choose a fixed odd number $m \geq 1$ and define the index sets

$$\mathcal{J} = \{\pm 1, \pm 3, \dots, \pm m\} \quad \text{and} \quad \mathcal{J}_+ = \mathcal{J} \cap \mathbb{N}. \quad (16)$$

We aim to construct modulation functions $v_j(\tau, \xi) = u_j(t, x)$ such that in the new variables (τ, ξ) ,

$$v(\tau, \xi) \approx \sum_{j \in \mathcal{J}} e^{ij\kappa \cdot \xi / \varepsilon} e^{ij(\kappa \cdot c_g - \omega)\tau / \varepsilon^2} v_j(\tau, \xi), \quad v_{-j} = \overline{v_j} \quad (17)$$

with the bounds in the maximum norm

$$\mathcal{P} v_1 = \mathcal{O}(1), \quad (18a)$$

$$\mathcal{P}^\perp v_1 = \mathcal{O}(\varepsilon), \quad (18b)$$

$$v_j = \mathcal{O}(\varepsilon^{j-1}) \quad \text{for odd } 3 \leq j \leq m, \quad (18c)$$

and the same bounds for the space and time derivatives up to some fixed order, and with an

approximation error in (17) of size $\mathcal{O}(\varepsilon^{m+1})$ w.r.t. maximum norm. (19)

When this is achieved, this yields (9) in the original variables (t, x) together with the bounds of Theorem 2.4.

3.2 PDEs for v_j

Substituting (17) into (14), collecting terms with the same exponential factor and discarding terms with factors

$$e^{ij\kappa \cdot \xi / \varepsilon} e^{ij(\kappa \cdot c_g - \omega)\tau / \varepsilon^2}, \quad |j| > m,$$

yields the system

$$\begin{aligned} \partial_\tau v_j + \frac{i}{\varepsilon^2} \mathcal{L}_j v_j + \frac{1}{\varepsilon} B(\partial_\xi) v_j &= \sum_{\#J=j} T(v_J), \\ j \in \mathcal{J}_+, \quad t \in [0, \tau_{\text{end}}], \quad x \in \mathbb{R}^d \end{aligned} \quad (20)$$

with \mathcal{L}_j defined in (8) and the notation $J = (j_1, j_2, j_3)$, $\#J := j_1 + j_2 + j_3$, and $T(v_J) = T(v_{j_1}, v_{j_2}, v_{j_3})$. The sum on the right-hand side is taken over the set

$$\left\{ J = (j_1, j_2, j_3) \in \mathcal{J}^3 : \#J = j \right\}.$$

The system of PDEs (20) is compatible with the condition that $v_{-j} = \overline{v_j}$. Since \mathcal{L}_j is invertible for $3 \leq j \in \mathcal{J}_+$ according to Assumption 2.3, (20) can be reformulated as

$$v_j = i\varepsilon \mathcal{L}_j^{-1} \left(\varepsilon \partial_\tau v_j + B(\partial_\xi) v_j - \varepsilon \sum_{\#J=j} T(v_J) \right), \quad 3 \leq j \in \mathcal{J}_+. \quad (21)$$

The case $j = 1$ is special. We distinguish

$$y_1 = \mathcal{P}v_1, \quad y_{-1} = \overline{y_1}, \quad (22a)$$

$$z_1 = \mathcal{P}^\perp v_1, \quad z_{-1} = \overline{z_1}. \quad (22b)$$

In order to derive equations for y_1 and z_1 , we use that $\mathcal{P}\mathcal{L}_1 = \mathcal{L}_1\mathcal{P} = 0$ by definition, and that $\mathcal{P}A(\partial_\xi)\mathcal{P} = \mathcal{P}(c_g \cdot \nabla)$; cf. [Lan11, Lemma 2.9]. It follows that $\mathcal{P}B(\partial_\xi)y_1 = 0$, $\mathcal{P}B(\partial_\xi)z_1 = \mathcal{P}A(\partial_\xi)z_1$ and hence

$$\partial_\tau y_1 + \frac{1}{\varepsilon} \mathcal{P}A(\partial_\xi)z_1 = \sum_{\#J=1} \mathcal{P}T(v_J), \quad (23)$$

$$\partial_\tau z_1 + \frac{i}{\varepsilon^2} \mathcal{L}_1 z_1 + \frac{1}{\varepsilon} \mathcal{P}^\perp B(\partial_\xi)(y_1 + z_1) = \sum_{\#J=1} \mathcal{P}^\perp T(v_J). \quad (24)$$

\mathcal{L}_1 is not invertible, but the restricted mapping $\mathcal{L}_\perp := \mathcal{L}_1|_{\mathcal{P}^\perp \mathbb{C}^n} : \mathcal{P}^\perp \mathbb{C}^n \rightarrow \mathcal{P}^\perp \mathbb{C}^n$ has an inverse $\mathcal{L}_\perp^{-1} = \mathcal{P}^\perp \mathcal{L}_1^{-1} \mathcal{P}^\perp : \mathcal{P}^\perp \mathbb{C}^n \rightarrow \mathcal{P}^\perp \mathbb{C}^n$. Therefore, (24) is equivalent to

$$z_1 = i\varepsilon \mathcal{L}_\perp^{-1} \mathcal{P}^\perp \left(\varepsilon \partial_\tau z_1 + B(\partial_\xi)(y_1 + z_1) - \varepsilon \sum_{\#J=1} T(v_J) \right). \quad (25)$$

3.3 Asymptotic expansion in powers of ε

Our next goal is to approximate a solution of the PDE system (20) with functions \tilde{v}_j that are appropriately bounded together with their partial derivatives and that satisfy the initial condition $\mathcal{P}\tilde{v}_1(0, \cdot) = p$. The condition (18) suggests an ansatz as a truncated power series in ε of the form

$$\tilde{v}_j(\tau, \xi) = \sum_{\ell=j-1}^m \varepsilon^\ell v_j^\ell(\tau, \xi), \quad j \in \mathcal{J}_+, \quad (26)$$

where m is the same number as in the definition of \mathcal{J} in (16). Note that ℓ is a power on ε and is a superscript on v_j . As before we let $v_{-j}^\ell = \overline{v_j^\ell}$. For example, for $m = 5$ we have $\mathcal{J}_+ = \{1, 3, 5\}$ and (26) reads

$$\begin{aligned} v_1 &\approx v_1^0 + \varepsilon v_1^1 + \varepsilon^2 v_1^2 + \varepsilon^3 v_1^3 + \varepsilon^4 v_1^4 + \varepsilon^5 v_1^5, \\ v_3 &\approx \varepsilon^2 v_3^2 + \varepsilon^3 v_3^3 + \varepsilon^4 v_3^4 + \varepsilon^5 v_3^5, \\ v_5 &\approx \varepsilon^4 v_5^4 + \varepsilon^5 v_5^5. \end{aligned} \quad (27)$$

As in (22) we set

$$y_1^\ell = \mathcal{P}v_1^\ell, \quad y_{-1}^\ell = \overline{y_1^\ell}, \quad z_1^\ell = \mathcal{P}^\perp v_1^\ell, \quad z_{-1}^\ell = \overline{z_1^\ell} \quad (28)$$

and in accordance with (18)

$$\tilde{y}_1 = \mathcal{P}\tilde{v}_1 = y_1^0 + \varepsilon y_1^1 + \varepsilon^2 y_1^2 + \dots + \varepsilon^m y_1^m \quad (29)$$

$$\tilde{z}_1 = \mathcal{P}^\perp \tilde{v}_1 = 0 + \varepsilon z_1^1 + \varepsilon^2 z_1^2 + \dots + \varepsilon^m z_1^m. \quad (30)$$

Negative subscripts mean complex conjugation as before. All other terms are set to zero, i.e.

$$z_{\pm 1}^0 = 0, \quad v_j^\ell = 0 \quad \text{for } \ell < |j| - 1. \quad (31)$$

In particular, this implies $v_{\pm 1}^0 = y_{\pm 1}^0$. For multi-indices $J = (j_1, j_2, j_3) \in \mathcal{J}^3$ and $L = (\ell_1, \ell_2, \ell_3) \in \mathbb{N}_0^3$ we let

$$v_J^L = (v_{j_1}^{\ell_1}, v_{j_2}^{\ell_2}, v_{j_3}^{\ell_3}).$$

We will now derive equations for the coefficient functions y_1^ℓ for $0 \leq \ell \leq m$, for z_1^ℓ for $1 \leq \ell \leq m$, and for v_j^ℓ with odd $3 \leq j \leq m$ for $j-1 \leq \ell \leq m$. In order to fulfill the condition $\mathcal{P}u_1(0, \cdot) = p$ (cf. Property 1 in Theorem 2.4) we choose the initial data $y_1^0(0, \cdot) = p$ and $y_1^\ell(0, \cdot) = 0$ for $\ell \geq 1$.

3.4 Equations for y_1^0 and z_1^1

Substituting the expansions (26), (29), and (30) into (23) and (25) and equating like powers of ε yields

$$\partial_\tau y_1^0 + \mathcal{P}A(\partial_\xi)z_1^1 = \sum_{\substack{\#J=1 \\ |L|_1=0}} \mathcal{P}T(v_J^L), \quad (32)$$

$$z_1^1 = i\mathcal{L}_\perp^{-1}\mathcal{P}^\perp B(\partial_\xi)y_1^0, \quad (33)$$

cf. [Lan11, Sect. 2.2]. Substituting (33) into (32) yields

$$\partial_\tau y_1^0 + i\mathcal{P}A(\partial_\xi)\mathcal{L}_\perp^{-1}\mathcal{P}^\perp B(\partial_\xi)y_1^0 = \sum_{\substack{\#J=1 \\ |L|_1=0}} \mathcal{P}T(v_J^L). \quad (34)$$

With (15) it follows that

$$\mathcal{P}^\perp B(\partial_\xi)y_1^0 = \mathcal{P}^\perp A(\partial_\xi)y_1^0 - (c_g)_\ell \mathcal{P}^\perp y_1^0 = \mathcal{P}^\perp A(\partial_\xi)y_1^0,$$

because $\mathcal{P}^\perp y_1^0 = 0$ according to (28). Lemma 2.12 in [Lan11] yields for every smooth $v : \mathbb{R}^d \rightarrow \mathbb{C}^n$

$$\mathcal{P}A(\partial_\xi)\mathcal{L}_\perp^{-1}\mathcal{P}^\perp A(\partial_\xi)\mathcal{P}v(\xi) = -\frac{1}{2}(\nabla_\xi \cdot H\nabla_\xi)\mathcal{P}v(\xi)$$

where $H = \nabla^2\omega_1(\kappa)$ is the Hessian of $\omega_1(\cdot)$ at κ . Substituting these equations into (34) leads to the nonlinear Schrödinger equation

$$\partial_\tau y_1^0 - \frac{i}{2}\nabla_\xi \cdot H\nabla_\xi y_1^0 = \sum_{\substack{\#J=1 \\ |L|_1=0}} \mathcal{P}T(v_J^L). \quad (35)$$

Since

$$\sum_{\substack{\#J=1 \\ |L|_1=0}} T(v_J^L) = T(y_1^0, y_1^0, y_{-1}^0) + T(y_1^0, y_{-1}^0, y_1^0) + T(y_{-1}^0, y_1^0, y_1^0)$$

depends only on y_1^0 and $y_{-1}^0 = \overline{y_1^0}$, (35) is independent of z_1^1 . Thus, one can first solve (35) with initial value $y_1^0(0, \cdot) = p$ to determine y_1^0 . This is precisely the initial value problem for the NLS approximation considered in Assumption 2.2. Therefore, by this assumption a smooth solution $y_1^0 = U$ exists over the interval $[0, \tau_{\text{end}}]$. Once y_1^0 is obtained, z_1^1 is determined by (33).

3.5 Equations for y_1^1 and z_1^2

Proceeding in the same way yields

$$\partial_\tau y_1^1 + \mathcal{P}A(\partial_\xi)z_1^2 = \sum_{\substack{\#J=1 \\ |L|_1=1}} \mathcal{P}T(v_J^L), \quad (36)$$

$$z_1^2 = i\mathcal{L}_\perp^{-1}\mathcal{P}^\perp \left(B(\partial_\xi)(y_1^1 + z_1^1) - \sum_{\substack{\#J=1 \\ |L|_1=0}} T(v_J^L) \right). \quad (37)$$

Inserting (37) into (36) and proceeding as before gives

$$\begin{aligned} \partial_\tau y_1^1 - \frac{i}{2}\nabla_\xi \cdot H\nabla_\xi y_1^1 + i\mathcal{P}A(\partial_\xi)\mathcal{L}_\perp^{-1}\mathcal{P}^\perp \left(B(\partial_\xi)z_1^1 - \sum_{\substack{\#J=1 \\ |L|_1=0}} T(v_J^L) \right) \\ = \sum_{\substack{\#J=1 \\ |L|_1=1}} \mathcal{P}T(v_J^L). \end{aligned} \quad (38)$$

The sum

$$\begin{aligned} \sum_{\substack{\#J=1 \\ |L|_1=1}} T(v_J^L) &= T(v_1^1, v_1^0, v_{-1}^0) + T(v_1^1, v_{-1}^0, v_1^0) + T(v_{-1}^0, v_1^1, v_1^0) \\ &\quad + T(v_1^0, v_1^1, v_{-1}^0) + T(v_1^0, v_{-1}^0, v_1^1) + T(v_{-1}^0, v_1^0, v_1^1) \\ &\quad + T(v_1^0, v_1^0, v_{-1}^1) + T(v_1^0, v_{-1}^1, v_1^0) + T(v_{-1}^1, v_1^0, v_1^0) \end{aligned}$$

depends only on y_1^0 and $v_1^1 = y_1^1 + z_1^1$. Since v_1^1 appears exactly once in each evaluation of the trilinearity and since z_1^1 does not depend on y_1^1 , the PDE (38) is a *linear* inhomogeneous Schrödinger equation for y_1^1 , to be solved with zero initial value: $y_1^1(0, \cdot) = 0$. With the solution y_1^1 we then compute z_1^2 from (37).

3.6 Equations for y_1^2 , z_1^3 , and v_3^2

As in the first two steps one obtains

$$\partial_\tau y_1^2 + \mathcal{P}A(\partial_\xi)z_1^3 = \sum_{\substack{\#J=1 \\ |L|_1=2}} \mathcal{P}T(v_J^L), \quad (39)$$

$$z_1^3 = i\mathcal{L}_\perp^{-1}\mathcal{P}^\perp \left(\partial_\tau z_1^1 + B(\partial_\xi)(y_1^2 + z_1^2) - \sum_{\substack{\#J=1 \\ |L|_1=1}} T(v_J^L) \right). \quad (40)$$

For the first time there is now a contribution from the time-derivative $\partial_\tau z_1$ on the right-hand side of (25). Substituting (40) into (39) leads to

$$\begin{aligned} \partial_\tau y_1^2 - \frac{i}{2} \nabla \cdot H \nabla y_1^2 + i \mathcal{P} A(\partial_\xi) \mathcal{L}_\perp^{-1} \mathcal{P}^\perp \left(\partial_\tau z_1^1 + B(\partial_\xi) z_1^2 - \sum_{\substack{\#J=1 \\ |L|_1=1}} T(v_J^L) \right) \\ = \sum_{\substack{\#J=1 \\ |L|_1=2}} \mathcal{P} T(v_J^L), \end{aligned} \quad (41)$$

which is to be solved with zero initial value. In contrast to the first two steps the sum

$$\sum_{\substack{\#J=1 \\ |L|_1=2}} \mathcal{P} T(v_J^L)$$

on the right-hand side involves not only terms $v_{\pm 1}^\ell$ with subscript ± 1 , but also $\mathcal{P} T(v_J^L) = \mathcal{P} T(v_3^2, v_{-1}^0, v_{-1}^0)$ and two other terms with permuted arguments. Here, v_3^2 has to be computed from (21). Since by construction $\partial_\tau v_3^0 = 0$ and $v_3^1 = 0$ we obtain

$$v_3^2 = -i \mathcal{L}_3^{-1} \sum_{\substack{\#J=3 \\ |L|_1=0}} T(v_J^L). \quad (42)$$

The sum

$$\sum_{\substack{\#J=3 \\ |L|_1=0}} T(v_J^L) = T(v_1^0, v_1^0, v_1^0) = T(y_1^0, y_1^0, y_1^0)$$

depends only on y_1^0 , which is already available.

3.7 Equations for y_1^ℓ , $z_1^{\ell+1}$, and v_j^ℓ

For $\ell = 3, \dots, m$ we obtain y_1^ℓ by solving the PDE

$$\begin{aligned} \partial_\tau y_1^\ell - \frac{i}{2} \nabla \cdot H \nabla y_1^\ell + \mathcal{P} A(\partial_\xi) \left(i \mathcal{L}_\perp^{-1} \mathcal{P}^\perp \left(\partial_\tau z_1^{\ell-1} + B(\partial_\xi) z_1^\ell - \sum_{\substack{\#J=1 \\ |L|_1=\ell-1}} T(v_J^L) \right) \right) \\ = \sum_{\substack{\#J=1 \\ |L|_1=\ell}} \mathcal{P} T(v_J^L) \end{aligned} \quad (43)$$

with zero initial data. Then, we set

$$z_1^{\ell+1} = i \mathcal{L}_\perp^{-1} \mathcal{P}^\perp \left(\partial_\tau z_1^{\ell-1} + B(\partial_\xi) (y_1^\ell + z_1^\ell) - \sum_{\substack{\#J=1 \\ |L|_1=\ell-1}} T(v_J^L) \right) \quad (44)$$

and for $j \in \mathcal{J}_+$ with $3 \leq j \leq \ell + 1$

$$v_j^\ell = i\mathcal{L}_j^{-1} \left(\partial_\tau v_j^{\ell-2} + B(\partial_\xi) v_j^{\ell-1} - \sum_{\substack{\#J=j \\ |L|_1=\ell-2}} T(v_j^L) \right). \quad (45)$$

If $\ell = m$, then (44) need not be computed, because z_1^{m+1} appears neither in the ansatz (26) nor in the construction of the other quantities.

Substituting (26) into (17) yields a modulated Fourier expansion

$$\tilde{v}(\tau, \xi) := \sum_{j \in \mathcal{J}} e^{ij\kappa \cdot \xi / \varepsilon} e^{ij(\kappa \cdot c_g - \omega)\tau / \varepsilon^2} \tilde{v}_j(\tau, \xi) \quad (46)$$

$$= \sum_{j \in \mathcal{J}} e^{ij\kappa \cdot \xi / \varepsilon} e^{ij(\kappa \cdot c_g - \omega)\tau / \varepsilon^2} \sum_{\ell=|j|-1}^m \varepsilon^\ell v_j^\ell(\tau, \xi), \quad (47)$$

with $v_{-j}^\ell = \overline{v_j^\ell}$, as an approximation to the solution of (14).

4 Bounds for the modulation functions

In this section we derive the bounds (18) for the modulation functions \tilde{v}_j constructed in the previous section. A suitable function space for the analysis is the Wiener algebra, which we recall now.

4.1 Wiener algebra

The Wiener algebra is defined by

$$W = \{f \in \mathcal{S}'(\mathbb{R}^d) : \hat{f} \in L^1(\mathbb{R}^d)\}, \quad \|f\|_W = \|\hat{f}\|_{L^1} = \int_{\mathbb{R}^d} |\hat{f}(k)|_2 dk,$$

where $\mathcal{F}f = \hat{f}$ is the Fourier transform of f , i.e.

$$(\mathcal{F}f)(k) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-ik \cdot x} dx.$$

$W(\mathbb{R}^d)$ is a Banach algebra and continuously embedded in $L^\infty(\mathbb{R}^d)$. There is a constant C_T such that the trilinear estimate

$$\|T(f_1, f_2, f_3)\|_W \leq C_T \|f_1\|_W \|f_2\|_W \|f_3\|_W \quad (48)$$

holds for all $f_1, f_2, f_3 \in W$; cf. Section 3.2 in [Lan11] and p. 715 in [CL09]. If, in addition, $g_1, g_2, g_3 \in W$, then it follows from (48) that

$$\|T(f_1, f_2, f_3) - T(g_1, g_2, g_3)\|_W \leq C_T M^2 \sum_{\ell=1}^3 \|f_\ell - g_\ell\|_W \quad (49)$$

with $M := \max\{\|f_\ell\|_W, \|g_\ell\|_W, \ell = 1, 2, 3\}$. For $s \in \mathbb{N}$ we define

$$W^s = \{f \in W(\mathbb{R}^d) : \partial^\alpha f \in W(\mathbb{R}^d) \text{ for all } \alpha \in \mathbb{N}_0^d, |\alpha|_1 \leq s\},$$

$$\|f\|_{W^s} = \sum_{|\alpha|_1 \leq s} \|\partial^\alpha f\|_W.$$

In these spaces (48) and the product rule of differentiation imply the trilinear estimates

$$\|T(f_1, f_2, f_3)\|_{W^s} \leq C_T^{(s)} \sum_{s_1+s_2+s_3=s} \|f_1\|_{W^{s_1}} \|f_2\|_{W^{s_2}} \|f_3\|_{W^{s_3}} \quad (50)$$

for all $f_1, f_2, f_3 \in W^s$.

4.2 Boundedness of the coefficient functions

The right-hand side of (47) can provide a reasonable approximation only if all coefficient functions $y_1^\ell, z_1^\ell, v_j^\ell$ remain bounded on $[0, \tau_{\text{end}}]$. This issue is studied now. We will work with the regularity

$$y_1^0 \in X^s := \bigcap_{i=0}^{\lfloor s/2 \rfloor} C^i([0, \tau_{\text{end}}], W^{s-2i}) \quad (51)$$

for a sufficiently large integer s , where $\lfloor s/2 \rfloor$ denotes the largest integer which is not larger than $s/2$. Since y_1^0 solves the nonlinear Schrödinger equation (35), the regularity (51) with an arbitrary s is implied by Assumption 2.2.

Lemma 4.1 *For every integer $q \geq 0$ there is a constant C independent of ε such that*

$$\|v_j^\ell\|_{X^q} \leq C$$

for all $j \in \mathcal{J}$ and all $\ell = 0, \dots, m$.

In particular, this implies the weaker bounds

$$\|v_j^\ell\|_{L^\infty([0, \tau_{\text{end}}] \times \mathbb{R}^d)} \leq \|v_j^\ell\|_{L^\infty([0, \tau_{\text{end}}], W)} \leq \|v_j^\ell\|_{L^\infty([0, \tau_{\text{end}}], W^q)} \leq C.$$

Proof. The first step in the construction of Section 3.3 is to compute y_1^0 by solving the nonlinear Schrödinger equation (35). All other coefficient functions can be traced back to y_1^0 , and thus their regularity depends on the regularity of y_1^0 , as we will explain now. Since z_1^1 is given by (33), it follows from (51) that $z_1^1 \in X^{s-1}$. In Section 3.5 the next step was to construct y_1^1 by solving the linear inhomogeneous Schrödinger equation (38) with zero initial data. The inhomogeneity includes the term

$$-i\mathcal{P}A(\partial_\xi)\mathcal{L}_\perp^{-1}\mathcal{P}^\perp B(\partial_\xi)z_1^1 \in X^{s-3} \quad (52)$$

and other terms with higher regularity. By standard regularity theory (e.g. based on Section 4.2 in [Paz83]), this yields $y_1^1 \in X^{s-3}$. Since z_1^2 defined by (37) involves $B(\partial_\xi)y_1^1$ and other terms of higher regularity, we obtain that $z_1^2 \in X^{s-4}$.

The coefficient function y_1^2 is the solution of the linear Schrödinger equation (41) with zero initial data. Here, the critical term in the inhomogeneity is

$$-i\mathcal{P}A(\partial_\xi)\mathcal{L}_\perp^{-1}\mathcal{P}^\perp B(\partial_\xi)z_1^2 \in X^{s-6},$$

such that $y_1^2 \in X^{s-6}$. An immediate consequence of (40) is that $z_1^3 \in X^{s-7}$. The function v_3^2 defined by (42) depends only on $y_1^0 \in X^s$, such that $v_3^2 \in W^s$. By continuing this procedure, it is found that all coefficient functions involved in the approximation (47) are bounded if s is sufficiently large. Roughly speaking, three orders of regularity are required if the upper index of y_1^ℓ is increased by one, i.e. $y_1^\ell \in X^{s-3\ell}$. ■

The bounds for the coefficient functions v_j^ℓ in (26) yield the bounds (18) when the truncated expansion \tilde{v}_j is taken in the role of v_j , as we always do in the following.

5 Error bound

We will prove that the modulated Fourier expansion (46) approximates a solution of (14) up to $\mathcal{O}(\varepsilon^{m+1})$. Let r be the residual of \tilde{v} , i.e.

$$r = \partial_\tau \tilde{v} + \frac{1}{\varepsilon} B(\partial_\xi) \tilde{v} + \frac{1}{\varepsilon^2} E \tilde{v} - T(\tilde{v}, \tilde{v}, \tilde{v}). \quad (53)$$

Comparing

$$\begin{aligned} & \partial_\tau \tilde{v} + \frac{1}{\varepsilon} B(\partial_\xi) \tilde{v} + \frac{1}{\varepsilon^2} E \tilde{v} \\ &= \sum_{j \in \mathcal{J}} e^{ij\kappa \cdot \xi / \varepsilon} e^{ij(\kappa \cdot c_g - \omega)\tau / \varepsilon^2} \left(\partial_\tau \tilde{v}_j + \frac{i}{\varepsilon^2} \mathcal{L}_j \tilde{v}_j + \frac{1}{\varepsilon} B(\partial_\xi) \tilde{v}_j \right) \end{aligned}$$

with

$$\begin{aligned} T(\tilde{v}, \tilde{v}, \tilde{v}) &= \sum_{J \in \mathcal{J}^3} e^{i\#J(\kappa \cdot \xi) / \varepsilon} e^{i\#J(\kappa \cdot c_g - \omega)\tau / \varepsilon^2} T(\tilde{v}_{j_1}, \tilde{v}_{j_2}, \tilde{v}_{j_3}) \\ &= \sum_{\substack{j \text{ odd} \\ |j| \leq 3m}} e^{ij(\kappa \cdot \xi) / \varepsilon} e^{ij(\kappa \cdot c_g - \omega)\tau / \varepsilon^2} \sum_{\#J=j} T(\tilde{v}_{j_1}, \tilde{v}_{j_2}, \tilde{v}_{j_3}) \end{aligned}$$

shows that r has the representation

$$r(\tau, \xi) = \sum_{\substack{j \text{ odd} \\ |j| \leq 3m}} e^{ij(\kappa \cdot \xi) / \varepsilon} e^{ij(\kappa \cdot c_g - \omega)\tau / \varepsilon^2} r_j(\tau, \xi), \quad (54)$$

with $r_{-j} = \overline{r_j}$ and

$$r_j(\tau, \xi) = \begin{cases} \partial_\tau \tilde{v}_j + \frac{i}{\varepsilon^2} \mathcal{L}_j \tilde{v}_j + \frac{1}{\varepsilon} B(\partial_\xi) \tilde{v}_j - \sum_{\#J=j} T(\tilde{v}_{j_1}, \tilde{v}_{j_2}, \tilde{v}_{j_3}) & \text{if } 1 \leq j \leq m, \\ - \sum_{\#J=j} T(\tilde{v}_{j_1}, \tilde{v}_{j_2}, \tilde{v}_{j_3}) & \text{if } m < j \leq 3m. \end{cases}$$

The defects r_j with $j \leq m$ are caused by solving the PDEs (20), (23), (24) only *approximately*. For $j > m$ the defects r_j originate from the fact that only the terms with index $j \in \mathcal{J} = \{\pm 1, \pm 3, \dots, \pm m\}$ are used in the ansatz (17).

Lemma 5.1 *Let $q \geq 0$ be arbitrary. There is a constant C independent of ε (but dependent on q) such that*

$$\sup_{\tau \in [0, \tau_{\text{end}}]} \|r(\tau, \cdot)\|_{W^q} \leq C\varepsilon^{m-1}.$$

Proof. We will show that

$$\sup_{\tau \in [0, \tau_{\text{end}}]} \|r_j(\tau, \cdot)\|_{W^q} \leq C\varepsilon^{m-1}$$

for all odd $j = 1, \dots, 3m$ for some constant C . Then, the assertion follows directly from (54).

Lemma 4.1 and (26) imply that $\|\tilde{v}_j\|_{W^q} = \mathcal{O}(\varepsilon^{j-1})$ and, via the trilinear estimate (50), that

$$\|T(\tilde{v}_{j_1}, \tilde{v}_{j_2}, \tilde{v}_{j_3})\|_{W^q} \leq c\varepsilon^{j_1-1}\varepsilon^{j_2-1}\varepsilon^{j_3-1} = c\varepsilon^{\#J-3} \quad (55)$$

for some constant c which depends on $C_T^{(q)}$ from (50). If $j > m$, then $j \geq m + 2$ because j and m are both odd numbers, and we obtain

$$\|r_j(t, \cdot)\|_{W^q} \leq \sum_{\#J=j} \|T(\tilde{v}_{j_1}, \tilde{v}_{j_2}, \tilde{v}_{j_3})\|_{W^q} \leq C\varepsilon^{j-3} \leq C\varepsilon^{m-1},$$

where C depends on c from (55) and on the number of multi-indices $J \in \mathcal{J}^3$ with $\#J = j$.

Now let $j \leq m$. In this case, r_j is given by

$$r_j(\tau, \xi) = \sum_{\ell=j-1}^m \varepsilon^\ell r_j^\ell(\tau, \xi) \quad (56)$$

with

$$r_j^\ell(\tau, \xi) = \partial_\tau v_j^\ell + i\mathcal{L}_j v_j^{\ell+2} + B(\partial_\xi) v_j^{\ell+1} - \sum_{\substack{\#J=j \\ |L|_1=\ell}} T(v_J^\ell).$$

In Subsection 3.3 the coefficients $y_1^\ell, z_1^\ell, v_j^\ell$ have been constructed in such a way that $r_j^\ell = 0$ for $\ell \leq m - 2$. Hence, only

$$r_j^{m-1} = \partial_\tau v_j^{m-1} + B(\partial_\xi) v_j^m - \sum_{\substack{\#J=j \\ |L|_1=m-1}} T(v_J^L), \quad (57a)$$

$$r_j^m = \partial_\tau v_j^m - \sum_{\substack{\#J=j \\ |L|_1=m}} T(v_J^L) \quad (57b)$$

do not vanish. Since $\|r_j^{m-1}\|_{W^q}$ and $\|r_j^m\|_{W^q}$ remain uniformly bounded by Lemma 4.1, it follows from (56) that

$$\sup_{\tau \in [0, \tau_{\text{end}}]} \|r_j(\tau, \cdot)\|_{W^q} \leq C\varepsilon^{m-1}$$

for all $m < j \leq 3m$. This proves the assertion. \blacksquare

Lemma 5.2 *There exists a unique mild solution v of (14) with initial data $v(0, \cdot) = \tilde{v}(0, \cdot)$ in the space $C(I, W^q)$ (with arbitrary $q \geq 0$) on the whole time interval $I = [0, \tau_{\text{end}}]$ of Assumption 2.2, and*

$$\|v - \tilde{v}\|_{L^\infty([0, \tau_{\text{end}}], W^q)} \leq C\varepsilon^{m+1}, \quad (58)$$

where C is independent of ε (but depends on m and q).

In particular, this implies

$$\|v - \tilde{v}\|_{L^\infty([0, \tau_{\text{end}}] \times \mathbb{R}^d)} \leq C\varepsilon^{m+1}.$$

Proof. We split the proof into three parts (a)–(c).

(a) For every $\varepsilon > 0$ the operator

$$\mathcal{A}_\varepsilon = -\frac{1}{\varepsilon}B(\partial_\xi) - \frac{1}{\varepsilon^2}E \quad \text{with domain} \quad D(\mathcal{A}_\varepsilon) = W^{q+1}$$

generates a strongly continuous group $(\exp(\tau\mathcal{A}_\varepsilon))_{\tau \in \mathbb{R}}$ on W^q . The group operators are explicitly given by

$$\mathcal{F}(\exp(\tau\mathcal{A}_\varepsilon)f)(k) = \exp\left(-\frac{\tau}{\varepsilon^2}(i\varepsilon B(k) + E)\right) \widehat{f}(k)$$

for every $f \in W^q$ (with arbitrary $q \geq 0$) and all $\tau \in \mathbb{R}$. Since $E \in \mathbb{R}^{n \times n}$ is skew-symmetric and $iB(k)$ is skew-Hermitian, it follows that

$$\begin{aligned} \|\exp(\tau \mathcal{A}_\varepsilon) f\|_{W^q} &= \sum_{|\alpha|_1 \leq q} \|\mathcal{F}(\exp(\tau \mathcal{A}_\varepsilon) \partial_\xi^\alpha f)\|_{L^1} \\ &= \sum_{|\alpha|_1 \leq q} \int_{\mathbb{R}^d} \left| \exp\left(-\frac{\tau}{\varepsilon^2} (i\varepsilon B(k) + E)\right) k^\alpha \widehat{f}(k) \right|_2 dk \\ &= \sum_{|\alpha|_1 \leq q} \int_{\mathbb{R}^d} |k^\alpha \widehat{f}(k)|_2 dk \\ &= \sum_{|\alpha|_1 \leq q} \|\partial_\xi^\alpha f\|_{W^q} = \|f\|_{W^q} \end{aligned}$$

so that $\exp(t \mathcal{A}_\varepsilon) : W^q \rightarrow W^q$ is an isometry. Together with the trilinear estimate (50), this yields that a unique mild solution v of (14) with initial data $v(0, \cdot) = \tilde{v}(0, \cdot) \in W^q$ exists in the space $C([0, \bar{\tau}], W^q)$ for every $\bar{\tau} < \tau_{\max}$, for some maximal time τ_{\max} where the solution becomes unbounded. We will find later that $\tau_{\max} > \tau_{\text{end}}$.

(b) By Lemma 4.1, $\|\tilde{v}\|_{L^\infty([0, \tau_{\text{end}}], W^q)} \leq C$. Comparing (53) with (14) shows that the error $e = v - \tilde{v}$ solves the evolution equation

$$\partial_\tau e = -\frac{1}{\varepsilon} B(\partial_\xi) e - \frac{1}{\varepsilon^2} E e + [T(v, v, v) - T(\tilde{v}, \tilde{v}, \tilde{v})] - r. \quad (59)$$

Applying Duhamel's principle (variation of constants formula) to (59) and using the trilinear estimate (49) yields

$$\|e(\tau)\|_{W^q} \leq \|e(0)\|_{W^q} + \tilde{C} \int_0^\tau \|e(\sigma)\|_{W^q} d\sigma + \int_0^\tau \|r(\sigma)\|_{W^q} d\sigma.$$

Since $e(0) = 0$ by assumption and since $\|r(\sigma)\|_{W^q} \leq C\varepsilon^{m-1}$ by Lemma 5.1, we obtain

$$\|e(\tau)\|_{W^q} \leq \tilde{C} \int_0^\tau \|e(\sigma)\|_{W^q} d\sigma + C\tau_{\text{end}} \varepsilon^{m-1},$$

such that the error bound (58) follows from Gronwall's lemma. This holds true on subintervals of $[0, \tau_{\text{end}}]$ as long as the solution $v \in C([0, \bar{\tau}], W^q)$ exists, i.e. for $\bar{\tau} < \tau_{\max}$. Since $\|v(\tau)\|_{W^q} \leq \|v - \tilde{v}\|_{W^q} + \|\tilde{v}\|_{W^q}$, the solution remains bounded and hence can be continued up to and beyond $\bar{\tau} = \tau_{\text{end}}$, so that $\tau_{\max} > \tau_{\text{end}}$.

(c) So far, we have only obtained an $\mathcal{O}(\varepsilon^{m-1})$ error bound instead of the stated $\mathcal{O}(\varepsilon^{m+1})$ bound. The improvement of the order of approximation

comes about as follows: We can repeat the same argument with $m + 2$ instead of m . The resulting error, denoted e_{m+2} , is then of size $\mathcal{O}(\varepsilon^{m+1})$ and differs from the above error $e_m = e$ (for the expansion up to terms of order $\mathcal{O}(\varepsilon^m)$) only by the terms $\varepsilon^\ell v_j^\ell$ with $\ell = m + 1, m + 2$, i.e.

$$e_m(\tau, \xi) = e_{m+2}(\tau, \xi) + \sum_{|j| \leq m+2} e^{ij\kappa \cdot \xi / \varepsilon} e^{ij(\kappa \cdot c_g - \omega)\tau / \varepsilon^2} \sum_{\ell \in \{m+1, m+2\}} \varepsilon^\ell v_j^\ell(\tau, \xi).$$

By Lemma 4.1 and the bound for e_{m+2} , each of these terms is $\mathcal{O}(\varepsilon^{m+1})$, and so we obtain the stated error bound. ■

Finally we have proved (17)–(19). By returning to the original coordinates (t, x) , we thus obtain items 1.–4. of Theorem 2.4. The uniqueness of the modulation functions u_j up to $\mathcal{O}(\varepsilon^{m+1})$ follows from the uniqueness of the functions v_j^ℓ in the construction of Section 3. This completes the proof of Theorem 2.4.

References

- [BJ22] J. Baumstark and T. Jahnke. Approximation of high-frequency wave propagation in dispersive media. Technical Report 2022/9, SFB 1173, Karlsruher Institut für Technologie, Karlsruhe, Germany, February 2022.
- [CHL08] D. Cohen, E. Hairer, and C. Lubich. Long-time analysis of nonlinearly perturbed wave equations via modulated Fourier expansions. *Arch. Ration. Mech. Anal.*, 187:341–368, 2008.
- [CL09] M. Colin and D. Lannes. Short pulses approximations in dispersive media. *SIAM J. Math. Anal.*, 41(2):708–732, 2009.
- [DJMR96] Ph. Donnat, J.-L. Joly, G. Metivier, and J. Rauch. Diffractive nonlinear geometric optics. In *Séminaire sur les Équations aux Dérivées Partielles, 1995–1996*, Sémin. Équ. Dériv. Partielles, pages Exp. No. XVII, 25. École Polytech., Palaiseau, 1996.
- [DR97] Ph. Donnat and J. Rauch. Dispersive nonlinear geometric optics. *J. Math. Phys.*, 38(3):1484–1523, 1997.
- [FGL13] E. Faou, L. Gauckler, and C. Lubich. Sobolev stability of plane wave solutions to the cubic nonlinear schrödinger equation on a torus. *Comm. Partial Differential Equations*, 38(7):1123–1140, 2013.

- [GHL18] L. Gauckler, E. Hairer, and C. Lubich. Dynamics, numerical analysis, and some geometry. *Proceedings of the International Congress of Mathematicians – Rio de Janeiro 2018. Vol. I*, pages 453–485, 2018.
- [HL00] E. Hairer and C. Lubich. Long-time energy conservation of numerical methods for oscillatory differential equations. *SIAM J. Numer. Anal.*, 38:414–441, 2000.
- [HL12] E. Hairer and C. Lubich. Modulated Fourier expansions for continuous and discrete oscillatory systems. *Foundations of Computational Mathematics, Budapest 2011*, pages 113–128, 2012.
- [HLW02] E. Hairer, C. Lubich, and G. Wanner. *Geometric Numerical Integration. Structure-Preserving Algorithms for Ordinary Differential Equations*. Springer Series in Computational Mathematics 31. Springer-Verlag, Berlin, 2002.
- [JMR93] J.-L. Joly, G. Métivier, and J. Rauch. Generic rigorous asymptotic expansions for weakly nonlinear multidimensional oscillatory waves. *Duke Math. J.*, 70(2):373–404, 1993.
- [JMR00] J.-L. Joly, G. Metivier, and J. Rauch. Transparent nonlinear geometric optics and Maxwell-Bloch equations. *J. Differential Equations*, 166(1):175–250, 2000.
- [Kru58] M. Kruskal. The gyration of a charged particle. *Rept. PM-S-33 (NYO-7903)*, Princeton University, Project Matterhorn, 1958.
- [Lan11] D. Lannes. High-frequency nonlinear optics: from the nonlinear Schrödinger approximation to ultrashort-pulses equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 141(2):253–286, 2011.
- [Paz83] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.
- [Rau12] J. Rauch. *Hyperbolic partial differential equations and geometric optics*, volume 133 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.
- [Seg63] I. Segal. Non-linear semi-groups. *Ann. of Math.*, 78:339–364, 1963.
- [Whi74] G. B. Whitham. *Linear and nonlinear waves*. Wiley-Interscience, New York, 1974.