# An Improved Planar Graph Product Structure Theorem

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**Abstract**

Dujmović, Joret, Micek, Morin, Ueckerdt and Wood [J. ACM 2020] proved that for every planar graph $G$ there is a graph $H$ with treewidth at most 8 and a path $P$ such that $G \subseteq H \boxtimes P$. We improve this result by replacing "treewidth at most 8" by "simple treewidth at most 6".

**Mathematics Subject Classifications:** 05C10, 05C76

## 1 Introduction

This paper is motivated by the following question: what is the global structure of planar graphs? Recently, Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood [13] gave an answer to this question that describes planar graphs in terms of products of simpler graphs, in particular, graphs of bounded treewidth. In this note, we improve this result in two respects. To describe the result from [13] and our improvement, we need the following definitions.

A *tree-decomposition* of a graph $G$ is a collection $(B_x \subseteq V(G) : x \in V(T))$ of subsets of $V(G)$ (called *bags*) indexed by the nodes of a tree $T$, such that:

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(a) for every edge $uv \in E(G)$, some bag $B_x$ contains both $u$ and $v$, and
(b) for every vertex $v \in V(G)$, the set $\{x \in V(T) : v \in B_x\}$ induces a non-empty (connected) subtree of $T$.

The *width* of a tree decomposition is the size of the largest bag minus 1. The *treewidth* of a graph $G$, denoted by $\text{tw}(G)$, is the minimum width of a tree decomposition of $G$. These definitions are due to Robertson and Seymour [23]. Treewidth is recognised as the most important measure of how similar a given graph is to a tree. Indeed, a connected graph with at least two vertices has treewidth 1 if and only if it is a tree. See [3, 16, 22] for surveys on treewidth.

A tree-decomposition $(B_x : x \in V(T))$ of a graph $G$ is *$k$-simple*, for some $k \in \mathbb{N}$, if it has width at most $k$, and for every set $S$ of $k$ vertices in $G$, we have $|\{x \in V(T) : S \subseteq B_x\}| \leqslant 2$. The *simple treewidth* of a graph $G$, denoted by $\text{stw}(G)$, is the minimum $k \in \mathbb{N}$ such that $G$ has a $k$-simple tree-decomposition. Simple treewidth appears in several places in the literature under various guises [17–19, 24]. The following facts are well-known: A graph has simple treewidth 1 if and only if it is a linear forest. A graph has simple treewidth at most 2 if and only if it is outerplanar. A graph has simple treewidth at most 3 if and only if it has treewidth at most 3 and is planar [18]. The edge-maximal graphs with simple treewidth 3 are ubiquitous objects, called *planar 3-trees* or *stacked triangulations* in structural graph theory and graph drawing [2, 18], called *stacked polytopes* in polytope theory [8], and called *Apollonian networks* in enumerative and random graph theory [15]. It is also known and easily proved that $\text{tw}(G) \leqslant \text{stw}(G) \leqslant \text{tw}(G) + 1$ for every graph $G$ (see [17, 24]).

The *strong product* of graphs $A$ and $B$, denoted by $A \boxtimes B$, is the graph with vertex set $V(A) \times V(B)$, where distinct vertices $(v, x), (w, y) \in V(A \boxtimes B)$ are adjacent if (1) $v = w$ and $xy \in E(B)$, or (2) $x = y$ and $vw \in E(A)$, or (3) $vw \in E(A)$ and $xy \in E(B)$.

Dujmović et al. [13] proved the following theorem describing the global structure of planar graphs.

**Theorem 1** ([13])**.** *Every planar graph $G$ is isomorphic to a subgraph of $H \boxtimes P$, for some planar graph $H$ with treewidth at most 8 and some path $P$.*

Theorem 1 has been used to solve several open problems regarding queue layouts [13], non-repetitive colourings [11], centered colourings [9], clustered colourings [12], adjacency labellings [4, 10, 14], vertex rankings [6], and twin-width [5].

We modify the proof of Theorem 1 to establish the following.

**Theorem 2.** *Every planar graph $G$ is isomorphic to a subgraph of $H \boxtimes P$, for some planar graph $H$ with simple treewidth at most 6 and some path $P$.*

Theorem 2 improves upon Theorem 1 in two respects. First it is for simple treewidth (although it should be said that the proof of Theorem 1 gives the analogous result for simple treewidth 8). The main improvement is to replace 8 by 6, which does require new ideas. The proof of Theorem 2 builds heavily on the proof of Theorem 1, which in turn builds on a result of Pilipczuk and Siebertz [21], who showed that every planar graph has a partition into geodesic paths whose contraction gives a graph with treewidth at most 8.

Dujmović et al. [13] also proved a variant of Theorem 1 in which they lowered the bound on the treewidth of $H$ to 3 at the expense of adding $K_3$ (the complete graph on three vertices) as a third factor to the product.

**Theorem 3** ([13]). *Every planar graph $G$ is isomorphic to a subgraph of $H \boxtimes P \boxtimes K_3$, for some planar graph $H$ with simple treewidth at most 3 and some path $P$.*

Theorems 1 and 2 can be thought of as having an extra $K_1$-factor, since $H \boxtimes K_1 \cong H$. We show another variant of this trade-off between the (simple) treewidth of $H$ and the size of a third clique factor.

**Theorem 4.** *Every planar graph $G$ is isomorphic to a subgraph of $H \boxtimes P \boxtimes K_2$, for some planar graph $H$ with simple treewidth at most 4 and some path $P$.*

Dujmović et al. [13] generalised Theorem 1 for graphs on surfaces. The *Euler genus* of the surface with $h$ handles and $c$ cross-caps is $2h + c$. The *Euler genus* of a graph $G$ is the minimum Euler genus of a surface in which $G$ embeds; see [20] for more about graph embeddings in surfaces. Dujmović et al. [13] showed that every graph with Euler genus $g$ is isomorphic to a subgraph of $H \boxtimes P$, for some graph $H$ with treewidth at most $2g + 8$ and some path $P$. Their proof in conjunction with Theorem 2 instead of Theorem 1 shows the following result. Here $A + B$ is the *complete join* of graphs $A$ and $B$ (obtained from the disjoint union of $A$ and $B$ by adding all edges between $A$ and $B$).

**Theorem 5.** *Every graph with Euler genus $g$ is isomorphic to a subgraph of $(H + K_{2g}) \boxtimes P$, for some planar graph $H$ with simple treewidth at most 6 and some path $P$.*

## 2 Proof of Theorem 2

Our goal is to find a given planar graph $G$ as a subgraph of $H \boxtimes P$ for some graph $H$ of small treewidth and path $P$. Dujmović et al. [13] showed this can be done by partitioning the vertices of $G$ into so-called vertical paths in a BFS spanning tree so that contracting each path into a single vertex gives the graph $H$ (see Theorem 6 and Figure 1 below).

To formalise this idea, we need the following terminology and notation. A *partition* $\mathcal{P}$ of a graph $G$ is a set of connected subgraphs of $G$, such that each vertex of $G$ is in exactly one subgraph in $\mathcal{P}$. The *quotient* of $\mathcal{P}$, denoted $G/\mathcal{P}$, is the graph with vertex set $\mathcal{P}$, where distinct elements $A, B \in \mathcal{P}$ are adjacent in $G/\mathcal{P}$ if there is an edge of $G$ with endpoints in $A$ and $B$. Note that $G/\mathcal{P}$ is a minor of $G$, so if $G$ is planar then $G/\mathcal{P}$ is planar.

If $T$ is a tree rooted at a vertex $r$, then a non-empty path $(x_0, \dots, x_p)$ in $T$ is *vertical* if $\mathrm{dist}_T(x_i, r) = \mathrm{dist}_T(x_0, r) + i$ for all $i \in [0, p]$.

**Lemma 6** ([13]). *Let $T$ be a BFS spanning tree in a connected graph $G$. Let $\mathcal{P}$ be a partition of $G$ into vertical paths in $T$. Then $G$ is isomorphic to a subgraph of $(G/\mathcal{P}) \boxtimes P$, for some path $P$.*

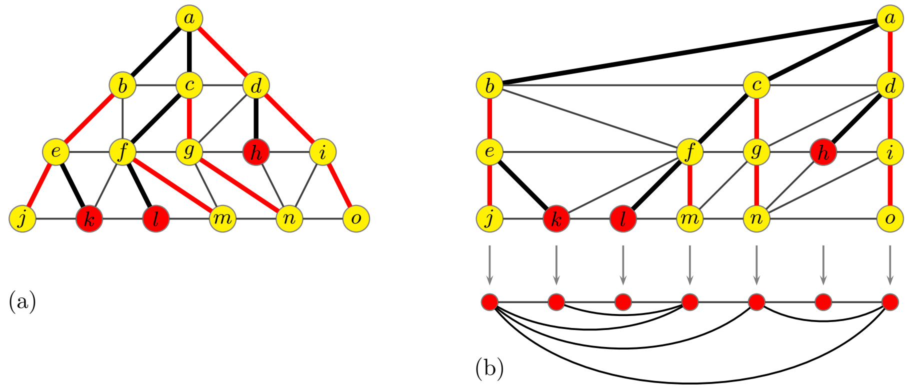


Figure 1: (a) A partition $\mathcal{P}$ of a planar graph $G$ into red vertical paths in a BFS spanning tree. (b) Illustration of $G$ as a subgraph of $(G/\mathcal{P}) \boxtimes P$.

The heart of this paper is Theorem 8 below, which is an improved version of the key lemma from [13]. The statement of Theorem 8 is identical to Lemma 13 from [13], except that we require $F$ to be partitioned into at most 5 instead of 6 paths and that the tree-decomposition of $H$ is 6-simple.

For a cycle $C$, we write $C = [P_1, \dots, P_k]$ if $P_1, \dots, P_k$ are pairwise disjoint non-empty paths in $C$, and the endpoints of each path $P_i$ can be labelled $x_i$ and $y_i$ so that $y_i x_{i+1} \in E(C)$ for $i \in [k]$, where $x_{k+1}$ means $x_1$. This implies that $V(C) = \bigcup_{i=1}^{k} V(P_i)$.

The proof of Theorem 8 employs the following well-known variation of Sperner's Lemma (see [1]). A *near-triangulation* is a 2-connected plane graph in which every internal face is a triangle.

**Lemma 7** (Sperner's Lemma)**.** *Let $G$ be a near-triangulation whose vertices are coloured* $1, 2, 3$*, with the outerface $F = [P_1, P_2, P_3]$ where each vertex in $P_i$ is coloured $i$. Then $G$ contains an internal face whose vertices are coloured* $1, 2, 3$.

**Lemma 8.** *Let $G^+$ be a plane triangulation, let $T$ be a spanning tree of $G^+$ rooted at some vertex $r$ on the outerface of $G^+$, and let $P_1, \dots, P_k$ for some $k \in [5]$, be pairwise disjoint vertical paths in $T$ such that $F = [P_1, \dots, P_k]$ is a cycle in $G^+$. Let $G$ be the near-triangulation consisting of all the edges and vertices of $G^+$ contained in $F$ and the interior of $F$. Then $G$ has a partition $\mathcal{P}$ into paths in $G$ that are vertical in $T$, such that $P_1, \dots, P_k \in \mathcal{P}$ and the quotient $H := G/\mathcal{P}$ has a 6-simple tree-decomposition such that some bag contains all the vertices of $H$ corresponding to $P_1, \dots, P_k$.*

*Proof.* The proof is by induction on $n = |V(G)|$. If $n = 3$, then $G$ is a 3-cycle and $k \leqslant 3$. The partition into vertical paths is $\mathcal{P} = \{P_1, \dots, P_k\}$. The tree-decomposition of $H$ consists of a single bag that contains the $k \leqslant 3$ vertices corresponding to $P_1, \dots, P_k$. Now assume that $n > 3$.

We now set up an application of Sperner's Lemma to the near-triangulation $G$. We begin by colouring the vertices in $k \leqslant 5$ colours. For $i \in \{1, \dots, k\}$, colour each vertex in $P_i$ by $i$. Now, for each remaining vertex $v$ in $G$, consider the path $P_v$ from $v$ to the root

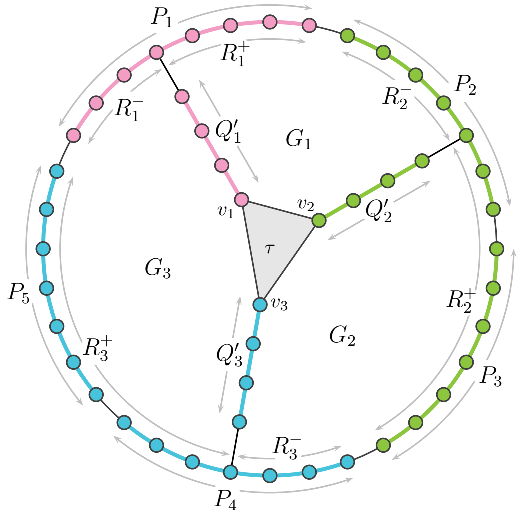


Figure 2: Example of the proof of Theorem 8 with $k = 5$.

of $T$. Since $r$ is on the outerface of $G^+$, $P_v$ contains at least one vertex of $F$. If the first vertex of $P_v$ that belongs to $F$ is in $P_i$, then assign the colour $i$ to $v$. The set $V_i$ of all vertices of colour $i$ induces a connected subgraph of $G$ for each $i \in \{1, \dots, k\}$. Consider the graph $M = G/\{V_1, \dots, V_k\}$ obtained by contracting each colour class $V_i$ into a single vertex $c_i$. Since $G$ is planar, $M$ is planar. (In fact, $M$ is outerplanar, although we will not use this property.) Moreover, if $k \geqslant 3$ then $[c_1, \dots, c_k]$ is a (not necessarily induced) cycle in $M$. Since $M \ncong K_5$, we may assume without loss of generality that either $k \leqslant 4$ or $k = 5$ and $c_2c_5$ is not an edge in $M$; that is, no vertex coloured 2 is adjacent to a vertex coloured 5.

Group consecutive paths from $P_1, \dots, P_k$ as follows:

- If $k = 1$ then, since $F$ is a cycle, $P_1$ has at least three vertices, so $P_1 = [v, P_1', w]$ for two distinct vertices $v$ and $w$. Let $R_1 := v$, $R_2 := P_1'$ and $R_3 := w$.
- If $k = 2$ then, without loss of generality, $P_1$ has at least two vertices, say $P_1 = [v, P_1']$. Let $R_1 := v$, $R_2 := P_1'$ and $R_3 := P_2$.
- If $k = 3$ then let $R_1 := P_1$, $R_2 := P_2$ and $R_3 := P_3$.
- If $k = 4$ then let $R_1 := P_1$, $R_2 := P_2$ and $R_3 := [P_3, P_4]$.
- If $k = 5$ then let $R_1 := P_1$, $R_2 := [P_2, P_3]$ and $R_3 := [P_4, P_5]$.

We now derive a 3-colouring from the $k$-colouring above. For $i \in \{1, 2, 3\}$, colour each vertex in $R_i$ by $i$. Now, for each remaining vertex $v$ in $G$, consider again the path $P_v$ from $v$ to the root of $T$ and if the first vertex of $P_v$ that belongs to $F$ is in $R_i$, then assign the colour $i$ to $v$. Hence, for $k = 3$ we obtain exactly the same 3-colouring as above, while for $k \in \{4, 5\}$ some pairs of colour classes from the $k$-colouring are merged into one colour class in the 3-colouring. In each case, we obtain a 3-colouring of $V(G)$ that satisfies the conditions of Theorem 7. Therefore there exists a triangular face $\tau = v_1v_2v_3$ of $G$ whose vertices are coloured $1, 2, 3$ respectively; see Figure 2.

For each $i \in \{1,2,3\}$, let $Q_i$ be the path in $T$ from $v_i$ to the first ancestor $v_i'$ of $v_i$ in $T$ that is in $F$. Observe that $Q_1$, $Q_2$, and $Q_3$ are disjoint since $Q_i$ consists only of vertices coloured $i$. Note that $Q_i$ may consist of the single vertex $v_i = v_i'$. Let $Q_i'$ be $Q_i$ minus its final vertex $v_i'$. Imagine for a moment that the cycle $F$ is oriented clockwise, which defines an orientation of $R_1$, $R_2$ and $R_3$. Let $R_i^-$ be the subpath of $R_i$ that contains $v_i'$ and all vertices that precede it, and let $R_i^+$ be the subpath of $R_i$ that contains $v_i'$ and all vertices that succeed it.

Consider the subgraph of $G$ that consists of the edges and vertices of $F$, the edges and vertices of $\tau$, and the edges and vertices of $Q_1 \cup Q_2 \cup Q_3$. This graph has an outerface, an inner face $\tau$, and up to three more inner faces $F_1, F_2, F_3$ where $F_i = [Q_i', R_i^+, R_{i+1}^-, Q_{i+1}']$, where we use the convention that $Q_4 = Q_1$ and $R_4 = R_1$. Note that $F_i$ may be *degenerate* in the sense that $[Q_i', R_i^+, R_{i+1}^-, Q_{i+1}']$ may consist only of a single edge $v_i v_{i+1}$.

Consider any non-degenerate $F_i = [Q_i', R_i^+, R_{i+1}^-, Q_{i+1}']$. Note that these four paths are pairwise disjoint, and thus $F_i$ is a cycle. If $Q_i'$ and $Q_{i+1}'$ are non-empty, then each is a vertical path in $T$. Furthermore, each of $R_i^+$ and $R_{i+1}^-$ consists of at most two vertical paths in $T$. Thus, $F_i$ is the concatenation of at most six vertical paths in $T$. Let $k_i$ be the actual number of (non-empty) vertical paths whose concatenation gives $F_i$. Then $k_1 \leqslant 5$ and $k_3 \leqslant 5$ since $R_1^-$ and $R_1^+$ consist of only one vertical path in $T$. Also, if $k \leqslant 4$ then $R_2^+$ consists of only one vertical path in $T$, implying $k_2 \leqslant 5$. If $k = 5$, then in our preliminary $k$-colouring no vertex coloured 2 is adjacent to a vertex coloured 5. Since $v_2 v_3$ is an edge, this means that either $v_2'$ lies on $P_3$ or $v_3'$ lies on $P_4$ or both. In any case, at least one of $R_2^+$ and $R_3^-$ consists of only one vertical path in $T$, which again gives $k_2 \leqslant 5$.

So $F_i$ is the concatenation of $k_i \leqslant 5$ vertical paths in $T$ for each $i \in \{1,2,3\}$. Let $G_i$ be the near-triangulation consisting of all the edges and vertices of $G^+$ contained in $F_i$ and the interior of $F_i$. Observe that $G_i$ contains $v_i$ and $v_{i+1}$ but not the third vertex of $\tau$. Therefore $G_i$ satisfies the conditions of the lemma and has fewer than $n$ vertices. By induction, $G_i$ has a partition $\mathcal{P}_i$ into vertical paths in $T$, such that $H_i := G_i/\mathcal{P}_i$ has a 6-simple tree-decomposition $(B_x^i : x \in V(J_i))$ in which some bag $B_{u_i}^i$ contains the vertices of $H_i$ corresponding to the at most five vertical paths that form $F_i$. Do this for each non-degenerate $F_i$.

We now construct the desired partition $\mathcal{P}$ of $G$. Initialise $\mathcal{P} := \{P_1, \dots, P_k\}$. Then add each non-empty $Q_i'$ to $\mathcal{P}$. Now for each non-degenerate $F_i$, classify each path in $\mathcal{P}_i$ as either *external* (that is, fully contained in $F_i$) or *internal* (with no vertex in $F_i$). Add all the internal paths of $\mathcal{P}_i$ to $\mathcal{P}$. By construction, $\mathcal{P}$ partitions $V(G)$ into vertical paths in $T$ and $\mathcal{P}$ contains $P_1, \dots, P_k$.

Let $H := G/\mathcal{P}$. Next we construct a tree-decomposition of $H$. Let $J$ be the tree obtained from the disjoint union of $J_i$, taken over the $i \in \{1,2,3\}$ such that $F_i$ is non-degenerate, by adding one new node $u$ adjacent to each $u_i$. (Recall that $u_i$ is the node of $J_i$ for which the bag $B_{u_i}^i$ contains the vertices of $H_i$ corresponding to the paths that form $F_i$.) Let the bag $B_u$ contain all the vertices of $H$ corresponding to $P_1, \dots, P_k, Q_1', Q_2', Q_3'$. For each non-degenerate $F_i$, and for each node $x \in V(J_i)$, initialise $B_x := B_x^i$. Recall that vertices of $H_i$ correspond to contracted paths in $\mathcal{P}_i$. Each internal path in $\mathcal{P}_i$ is in $\mathcal{P}$. Each external path $P$ in $\mathcal{P}_i$ is a subpath of $P_j$ for some $j \in [k]$ or is one of $Q_1', Q_2', Q_3'$.

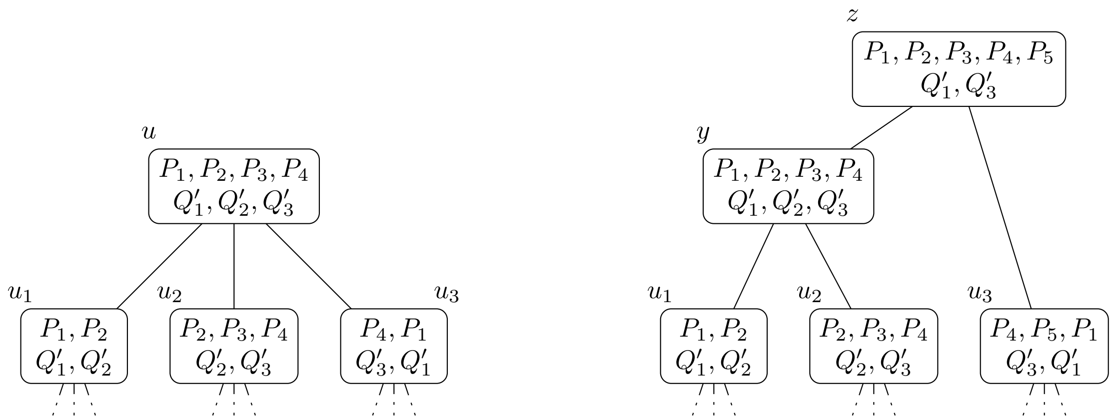


Figure 3: Illustration of 6-simple tree-decomposition for a possible scenario with $k = 4$ (left) and $k = 5$ (right).

For each such path $P$, for every $x \in V(J)$, in bag $B_x$, replace each instance of the vertex of $H_i$ corresponding to $P$ by the vertex of $H$ corresponding to the path among $P_1, \ldots, P_k, Q'_1, Q'_2, Q'_3$ that contains $P$. This completes the description of $(B_x : x \in V(J))$. By construction, $|B_x| \leqslant k + 3 \leqslant 8$ for every $x \in V(J)$.

First we show that for each vertex $a$ in $H$, the set $X := \{x \in V(J) : a \in B_x\}$ forms a subtree of $J$. If $a$ corresponds to a path distinct from $P_1, \ldots, P_k, Q'_1, Q'_2, Q'_3$ then $X$ is fully contained in $J_i$ for some $i \in \{1, 2, 3\}$. Thus, by induction $X$ is non-empty and connected in $J_i$, so it is in $J$. If $a$ corresponds to $P$ which is one of the paths among $P_1, \ldots, P_k, Q'_1, Q'_2, Q'_3$ then $u \in X$ and whenever $X$ contains a vertex of $J_i$ it is because some external path of $\mathcal{P}_i$ was replaced by $P$. In particular, we would have $u_i \in X$ in that case. Again by induction each $X \cap J_i$ is connected and since $uu_i \in E(T)$, we conclude that $X$ induces a (connected) subtree of $J$.

Now we show that, for every edge $ab$ of $H$, there is a bag $B_x$ that contains $a$ and $b$. If $a$ and $b$ are both obtained by contracting any of $P_1, \ldots, P_k, Q'_1, Q'_2, Q'_3$, then $a$ and $b$ both appear in $B_u$. If $a$ and $b$ are both in $H_i$ for some $i \in \{1, 2, 3\}$, then some bag $B^i_x$ contains both $a$ and $b$. Finally, when $a$ is obtained by contracting a path $P_a$ in $G_i - V(F_i)$ and $b$ is obtained by contracting a path $P_b$ not in $G_i$, then the cycle $F_i$ separates $P_a$ from $P_b$ so the edge $ab$ is not present in $H$. This concludes the proof that $(B_x : x \in V(J))$ is a tree-decomposition of $H$. Note that $B_u$ contains the vertices of $H$ corresponding to $P_1, \ldots, P_k$.

By assumption the tree-decomposition $(B^i_x : x \in V(J_i))$ of $H_i$ is 6-simple for $i \in \{1, 2, 3\}$. Since $|B_u \cap B_{u_i}| \leqslant 5$ for each $i \in \{1, 2, 3\}$, the tree-decomposition $(B_x : x \in V(J))$ of $H$ is 6-simple, unless $|B_u| = 8$, which only occurs if $k = 5$ (since $|B_u| \leqslant k + 3$). Now assume that $k = 5$. Recall again that either $v'_2$ lies on $P_3$ or $v'_3$ lies on $P_4$ or both. Without loss of generality, $v'_3$ lies on $P_4$, and thus there is no edge between $Q'_2$ and $P_5$.

We now modify the above tree-decomposition of $H$ in the $k = 5$ case. See Figure 3 for an illustration. First delete node $u$ from $J$ and the corresponding bag $B_u$. Add a new node $y$ to $J$ adjacent to $u_1$ and $u_2$, where $B_y$ consists of the vertices of $H$ corresponding to $P_1, \dots, P_4, Q'_1, Q'_2, Q'_3$. Thus $|B_y| = 7$. Add a node $z$ to $J$ adjacent to $y$ and $u_3$, where $B_z$ consists of the vertices of $H$ corresponding to $P_1, \dots, P_5, Q'_1, Q'_3$. Thus $|B_z| = 7$ and $(B_x : x \in V(J))$ is a tree-decomposition of $H$ with width 6. Since $P_5$ has no vertex in $G_1 \cup G_2$, the vertex of $H$ corresponding to $P_5$ is not in $B_{u_1} \cup B_{u_2}$, and thus the nodes of $J$ whose bags contain this vertex form a connected subtree of $J$. Similarly, the vertex of $H$ corresponding to $Q'_2$ is not in $B_{u_3}$ and thus the nodes of $J$ whose bags contain this vertex form a connected subtree of $J$. The argument for the other vertices of $H$ is identical to that above. This completes the proof that $(B_x : x \in V(J))$ is a tree-decomposition of $H$ with width at most 6. It is 6-simple since the tree-decompositions of $G_1$, $G_2$ and $G_3$ are 6-simple, and $|B_y \cap B_{u_1}| \leqslant 5$ and $|B_y \cap B_{u_2}| \leqslant 5$ and $|B_z \cap B_{u_3}| \leqslant 5$. Moreover, $B_z$ contains the vertices of $H$ corresponding to $P_1, \dots, P_5$ as desired. □

The following corollary of Lemma 8 is a direct analogue of the corresponding result in [13, Theorem 12].

**Corollary 9.** *Let $T$ be a rooted spanning tree in a connected planar graph $G$. Then $G$ has a partition $\mathcal{P}$ into vertical paths in $T$ such that* $\operatorname{stw}(G/\mathcal{P}) \leqslant 6$.

*Proof.* The result is trivial if $|V(G)| < 3$. Now assume $|V(G)| \geqslant 3$. Let $r$ be the root of $T$. Let $G^+$ be a plane triangulation containing $G$ as a spanning subgraph with $r$ on the outerface of $G^+$. The three vertices on the outerface of $G^+$ are vertical (singleton) paths in $T$. Thus, $G^+$ satisfies the assumptions of Theorem 8 with $k = 3$ and $F$ being the outerface, which implies that $G^+$ has a partition $\mathcal{P}$ into vertical paths in $T$ such that $\operatorname{stw}(G^+/\mathcal{P}) \leqslant 6$. Note that $G/\mathcal{P}$ is a subgraph of $G^+/\mathcal{P}$. Hence $\operatorname{stw}(G/\mathcal{P}) \leqslant 6$. □

Theorems 6 and 9 imply Theorem 2 (since we may assume that $G$ is connected).

## 3 Proof of Theorem 4

Here we show that every planar graph is a subgraph of $H \boxtimes P \boxtimes K_2$ for some planar graph $H$ with simple treewidth at most 4 and some path $P$. The proof follows the same approach as before. We consider a connected planar graph $G$ and a rooted spanning tree $T$ of $G$. We then show that $G$ has a partition $\mathcal{P}$, each part being the union of up to two vertical paths whose lower endpoints are adjacent in $G$. Call such a subgraph of $G$ a *bipod*. The same idea except with up to three vertical paths with pairwise adjacent lower endpoints (so called *tripods*) is used by Dujmović et al. [13] to show Theorem 3. The following result is analogous to the key lemma from [13].

**Lemma 10.** *Let $G^+$ be a plane triangulation, let $T$ be a spanning tree of $G^+$ rooted at some vertex $r$ on the boundary of the outerface of $G^+$, and for some $k \in [4]$, let $P_1, \dots, P_k$ be pairwise disjoint bipods such that $F = [P_1, \dots, P_k]$ is a cycle in $G^+$. Let $G$ be the near-triangulation consisting of all the edges and vertices of $G^+$ contained in $F$ and the interior of $F$. Then $G$ has a partition $\mathcal{P}$ into bipods such that $P_1, \dots, P_k \in \mathcal{P}$, and the quotient $H := G/\mathcal{P}$ has a 4-simple tree-decomposition such that exactly one bag contains all the vertices of $H$ corresponding to $P_1, \dots, P_k$.*

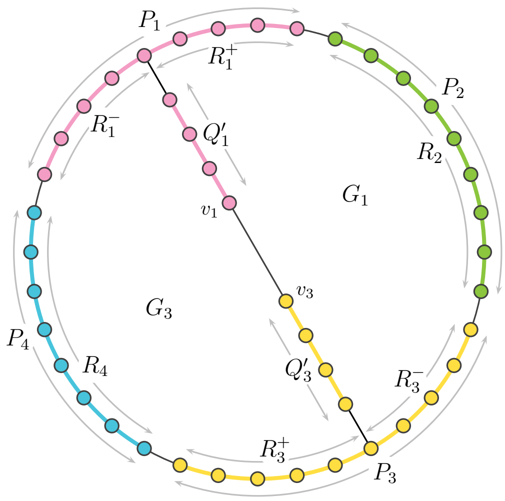


Figure 4: Example of the proof of Theorem 10 with $k = 4$.

*Proof.* The argument is in large part analogous to the proof of Theorem 8. We proceed by induction on the number $n$ of vertices in $G$; the base case $n \leqslant 4$ being trivial.

As before, assign each vertex $v \in V(G)$ colour $i \in [k]$ if the first vertex in path $P_v$ from $v$ to the root of $T$ that belongs to $F$ lies in $P_i$ (this first vertex might be $v$ itself). The set $V_i$ of all vertices coloured $i$ induces a connected subgraph in $G$, and the graph $M = G/\{V_1, \ldots, V_k\}$ obtained from $G$ by contracting each $V_i$ into a single vertex $c_i$ is outerplanar and $|V(M)| = k$. Since $K_4$ is not outerplanar, $M \ncong K_4$. Thus $k \leqslant 3$, or $k = 4$ and we may assume that $c_2c_4$ is not an edge in $M$. In the latter case observe that $M$ is inner triangulated and hence $c_1c_3$ is an edge in $M$.

If $k = 1$ then $|P_1| \geqslant 3$ and we recolour one endpoint of $P_1$ in colour 2 and the other endpoint in colour 3. If $k = 2$ then, without loss of generality, $|P_1| \geqslant 2$ and we recolour one endpoint of $P_1$ in colour 3. In every case, we now have a vertex colouring of $G$ with $\max(3, k)$ colours such that at least one internal edge $e$ of $G$ has its two endpoints of different colours. By potentially renaming colours in the $k \leqslant 3$ case, and the assumption that $c_1c_3$ is an edge of $M$ in the $k = 4$ case, we may assume that $e = v_1v_3$ with $v_1$ of colour 1 and $v_3$ of colour 3.

For $i \in \{1, 3\}$, let $Q_i$ be the path in $T$ from $v_i$ to the first ancestor of $v_i$ in $T$ that belongs to $F$, and $Q_i'$ be its (possibly empty) subpath minus the final vertex. The graph consisting of $e = v_1v_3$ and all vertices and edges of $F$, $Q_1$ and $Q_3$ has two inner faces $F_1, F_3$. Let $R_i$ be the part of $F$ in colour $i$. Let $R_i^+, R_i^-$ be the sub-paths of $R_i$ defined in Theorem 8. Then $F_1 = [Q_1', R_1^+, R_2, R_3^-, Q_3']$ and $F_3 = [Q_3', R_3^+, R_4, R_1^-, Q_1']$; see Figure 4 for an illustration. Note that $R_4 = \varnothing$ if $k \leqslant 3$. Since $Q_1' \cup Q_3'$ is a bipod, each of $F_1$ and $F_3$ is composed of at most four bipods. As before, for $i \in \{1, 3\}$, let $G_i$ be the subgraph of $G$ on all vertices and edges of $F_i$ and the interior of $F_i$. By planarity, no vertex in the interior of $F_1$ is adjacent to any vertex in the interior of $F_3$. For $k = 4$ this relates to the fact that $c_2c_4$ is not an edge in $M$.

By induction, there is a partition $\mathcal{P}_i$ of $G_i$ such that $H_i := G_i/\mathcal{P}_i$ has a 4-simple tree-decomposition $(B^i_x : x \in V(J_i))$ in which exactly one bag $B^i_{u_i}$ contains the vertices of $H_i$ corresponding to the bipods that form $F_i$. We construct the partition $\mathcal{P}$ for $G$ as before by initializing $\mathcal{P} := \{P_1, \ldots, P_k\}$ and adding the bipod $Q'_1 \cup Q'_3$ to $\mathcal{P}$. For each $F_i$, add all internal bipods (those with no vertex in $F_i$) of $\mathcal{P}_i$ to $\mathcal{P}$. This concludes the definition of $\mathcal{P}$.

For a 4-simple tree-decomposition of $H := G/\mathcal{P}$, let $J$ be the tree obtained from $J_1$ and $J_3$ by adding one new node $u$ adjacent to $u_1$ and $u_3$. Let the bag $B_u$ contain all vertices of $H$ corresponding to the bipods that form $F_1$ and $F_3$. These are $P_1, \ldots, P_k$ and $Q'_1 \cup Q'_3$; that is, exactly $k + 1 \leqslant 5$ bipods. For each external bipod $P$ (those with some vertex in $F_i$) of $\mathcal{P}_i$, replace each instance of the vertex of $H_i$ corresponding to $P$ by the vertex of $H$ corresponding to the bipod among $P_1, \ldots, P_k, (Q'_1 \cup Q'_3)$ that contains $P$. By construction, $|B_x| \leqslant 5$ for each node $x \in V(J)$.

The proof that $(B_x : x \in V(J))$ is a tree-decomposition of $H$ is analogous to the proof in Theorem 8. To see that it is 4-simple, i.e., any set of 4 nodes appears in at most two bags, first note that the tree-decompositions of $G_1$ and $G_3$ are 4-simple, and $B_u \cap B_{u_1}$ is a subset of the vertices of $H$ corresponding to $Q'_1 \cup Q'_3, P_1, P_2, P_3$, while $B_u \cap B_{u_3}$ corresponds to a subset of the four bipods $Q'_1 \cup Q'_3, P_2, P_3, P_4$. As (by induction hypothesis) no other bag of $G_1$ contains the vertices corresponding to $Q'_1 \cup Q'_3, P_1, P_2, P_3$, and $P_1$ is disjoint from $G_3$, this 4-tuple of nodes appears only in the two bags $B_u$ and $B_{u_1}$. Similarly, $Q'_1 \cup Q'_3, P_2, P_3, P_4$ appears only in $B_u$ and $B_{u_3}$, as this 4-tuple appears only once in the tree-decomposition of $G_3$ and not at all in the tree-decomposition of $G_1$, since $P_2$ is disjoint from $G_3$. Hence $(B_x : x \in V(J))$ is a 4-simple tree-decomposition of $H$. Moreover, $B_u$ contains the vertices of $H$ corresponding to $P_1, \ldots, P_k$ and is the unique bag with that property, as desired. □

Finally, Theorem 4 follows as a corollary from Theorem 10 in the same way as Theorem 8 gives Theorem 2.

## 4 Discussion

We conclude with an open problem. Bose, Dujmović, Javarsineh, Morin, and Wood [7] defined the *row treewidth* of a graph $G$ to be the minimum integer $k$ such that $G$ is isomorphic to a subgraph of $H \boxtimes P$ for some graph $H$ with treewidth $k$ and for some path $P$. Theorem 1 by Dujmović et al. [13] says that planar graphs have row treewidth at most 8. Our Theorem 2 improves this upper bound to 6. Dujmović et al. [13] proved a lower bound of 3. In fact, they showed that for every integer $\ell$ there is a planar graph $G$ such that for every graph $H$ and path $P$, if $G$ is isomorphic to a subgraph of $H \boxtimes P \boxtimes K_\ell$, then $H$ contains $K_4$ and thus has treewidth at least 3. Determining the maximum row treewidth of a planar graph is a tantalising open problem.

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