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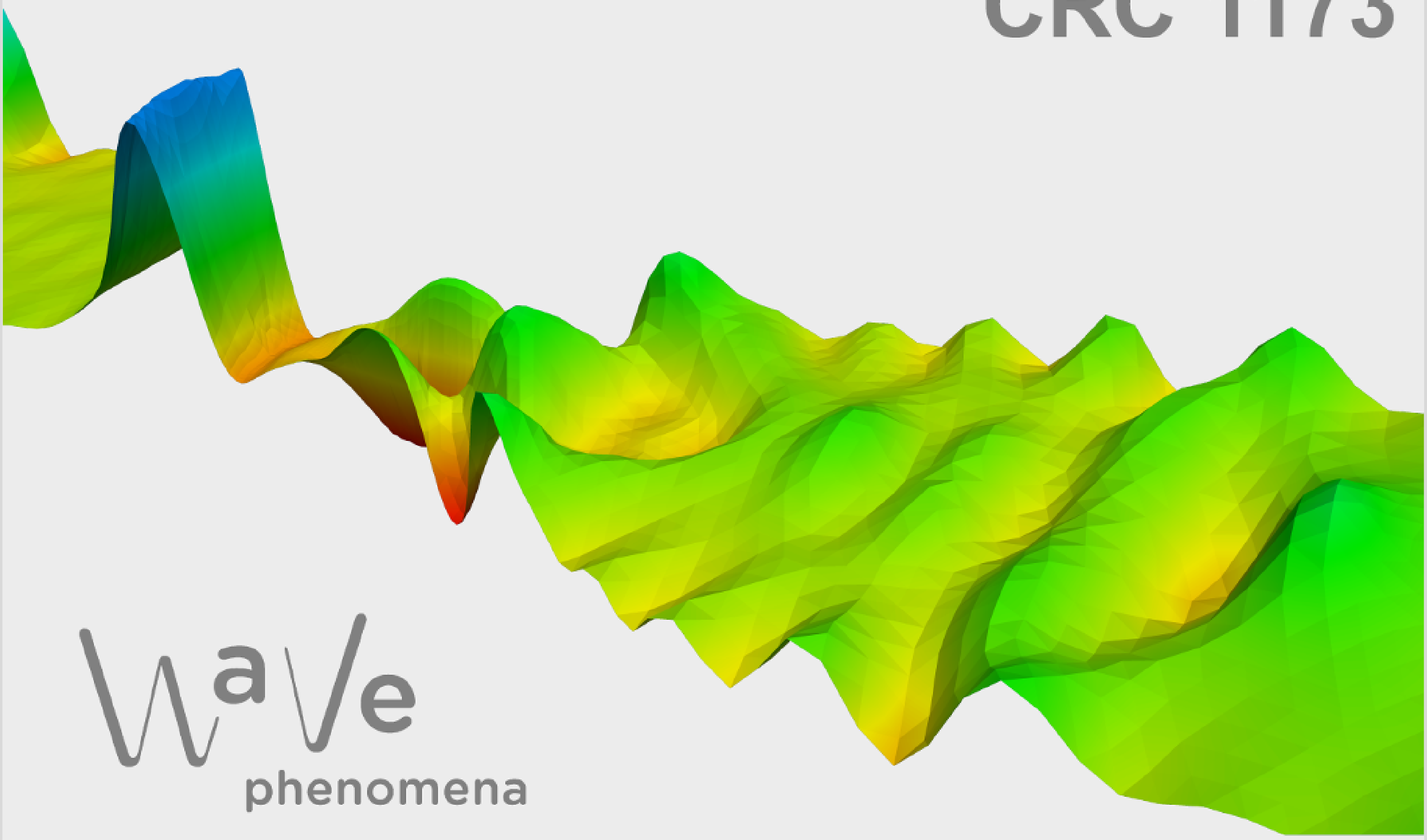

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EIGENVALUE ANALYSIS OF THE LAX OPERATOR FOR THE ONE-DIMENSIONAL CUBIC NONLINEAR DEFOCUSING SCHRÖDINGER EQUATION

XIAN LIAO AND MICHAEL PLUM

Abstract. We analyze the eigenvalues of the Lax operator associated to the one-dimensional cubic nonlinear defocusing Schrödinger equation. With the help of a newly discovered unitary matrix, it reduces to the study of a unitarily equivalent operator, which involves only the amplitude and the phase velocity of the potential. For a specific kind of potentials which satisfy nonzero boundary conditions, the eigenvalues of the Lax operator are characterized via a family of compact operators.

Keywords: Cubic nonlinear defocusing Schrödinger equation, nonzero boundary condition, Lax operator, one-dimensional Dirac operator

AMS Subject Classification (2020): 35Q55, 37K10

1. Introduction

We consider the following one-dimensional cubic nonlinear defocusing Schrödinger equation (NLS)

$$i\partial_t q + \partial_{xx} q = 2|q|^2 q, \tag{1}$$

where $q = q(t,x) : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{C}$ denotes the unknown wave function. By the seminal paper by Zakharov-Shabat [13], the (NLS) can be (formally) formulated in the Lax pair form

$$\partial_t L = PL - LP, \tag{2}$$

where L is the self-adjoint Lax operator

$$L = \begin{pmatrix} i\partial_x & -iq \\ i\bar{q} & -i\partial_x \end{pmatrix}, \tag{3}$$

and P is the following skewadjoint differential operator

$$P = i\begin{pmatrix} 2\partial_x^2 - |q|^2 & -q\partial_x - \partial_x q \\ \bar{q}\partial_x + \partial_x\bar{q} & -2\partial_x^2 + |q|^2 \end{pmatrix}.$$

Here the application of the operator $\partial_x\bar{q}$ on a function f is understood as $\partial_x(\bar{q}f)$. Let $U(t',t)$ be the unitary family generated by the skewadjoint operator P, then by virtue of (2), one can relate the operators $L(t)$ and $L(t')$ at different times by

$$L(t) = U^*(t',t)L(t')U(t',t),$$

such that the spectra of the Lax operator L is formally invariant under the evolutionary NLS-flow (1).

In the (classical) setting of decaying potentials:

$$q(x) \to 0 \text{ as } |x| \to \infty,$$

the spectral problem of the Lax operator L and the associated direct/inverse scattering transform have been extensively studied in the literature, see e.g. the book [2]. The case with the nonzero boundary condition for q at infinity:

$$|q(x)| \to 1 \text{ as } |x| \to \infty \tag{4}$$

has also attracted much attention, see e.g. [1, 4, 5, 6, 7, 8, 9, 14]. In the classical framework where

$$q - 1 \in \mathcal{S}$$

is a Schwartz function, Faddeev-Takhtajan [8] studied the self-adjoint operator L, and showed that its essential spectrum is $(-\infty, -1] \cup [1, \infty)$ and there are at most countably many *simple real* eigenvalues $\{\lambda_m\}$ in $(-1, 1)$. More recently, Demontis et al. [7] studied rigorously the inverse scattering transform in the framework that $q(x)$ tends to $e^{i\theta_\pm} \in \mathbb{S}^1$ as $x \to \pm\infty$ sufficiently fast in the sense that $(1+x^2)(q - e^{i\theta_\pm}) \in L^1(\mathbb{R}^\pm)$. In particular, under some stronger decay assumption

$$(1 + x^4)(q - e^{i\theta_\pm}) \in L^1(\mathbb{R}^\pm),$$

they showed that there are only finitely many discrete eigenvalues which belong to the spectral gap $(-1, 1)$. It was shown recently in [11] that in the low-regularity finite-energy setting

$$q \in L^2_{\mathrm{loc}}(\mathbb{R}) \text{ with } |q|^2 - 1, \partial_x q \in H^{-1}(\mathbb{R}),$$

the essential spectrum of the Lax operator L is $(-\infty, -1] \cup [1, \infty)$, and the spectrum outside the essential spectrum consists of isolated simple eigenvalues in $(-1, 1)$. However, under this weak assumption, there might be eigenvalues embedded in the essential spectrum. We mention also a recent work [3] devoted to the case of the nonzero asymmetric boundary condition

$$q(x) \to q_\pm, \text{ as } |x| \to \infty, \text{ with } |q_+| \neq |q_-|.$$

In this paper we will focus on the study of the eigenvalues of the Lax operator $L = \begin{pmatrix} i\partial_x & -iq \\ i\bar{q} & -i\partial_x \end{pmatrix}$, which is a one-dimensional Dirac operator. A specific kind of piecewise constant potentials

$$q = \begin{cases} e^{i\theta_-} & x < -R, \\ Ae^{i\varphi} & -R < x < R, \\ e^{i\theta_+} & x > R, \end{cases}$$

has been considered in [4], and the authors there estimated the location of the discrete eigenvalues inside the spectral gap $(-1, 1)$ by considering the relation between A and $\cos(\varphi)$. In particular, if $A < 1$, then there is at least one discrete eigenvalue.

Here we propose a new idea to study the operator L, where we take non-vanishing bounded potentials q with finite phase velocity as follows

$$q = |q|e^{i\varphi} \in L^\infty(\mathbb{R}; \mathbb{C}), \quad \partial_x \varphi \in L^\infty(\mathbb{R}; \mathbb{R}). \tag{5}$$

By the following unitary matrix which we believe to be new

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-\frac{1}{2}i(\varphi - \frac{\pi}{2})} & e^{\frac{1}{2}i(\varphi - \frac{\pi}{2})} \\ e^{-\frac{1}{2}i(\varphi - \frac{\pi}{2})} & -e^{\frac{1}{2}i(\varphi - \frac{\pi}{2})} \end{pmatrix} : H^s(\mathbb{R}; \mathbb{C}^2) \to H^s(\mathbb{R}; \mathbb{C}^2), \quad s = 0, 1,$$

we can transform the Lax operator $L : H^1(\mathbb{R};\mathbb{C}^2) \mapsto L^2(\mathbb{R};\mathbb{C}^2)$ to the operator

$$\mathcal{L} = MLM^* = \begin{pmatrix} -u_- & i\partial_x \\ i\partial_x & -u_+ \end{pmatrix} : H^1(\mathbb{R};\mathbb{C}^2) \mapsto L^2(\mathbb{R};\mathbb{C}^2).$$

where $u_\pm = \frac{1}{2}\partial_x\varphi \pm |q|$ are the two Riemannian invariants of the corresponding compressible Euler equations to the hydrodynamic nonlinear Schrödinger equation (see (8) below). In particular, the operators L and $\mathcal{L}$ share the same real eigenvalues λ, and the corresponding eigenvectors ψ and Ψ are related by M as follows:

$$L\psi = \lambda\psi \overset{\Psi=M\psi}{\Longleftrightarrow} \mathcal{L}\Psi = \lambda\Psi.$$

It suffices to study the eigenvalues of the operator $\mathcal{L}$. We can reformulate the eigenvalue problem of $\mathcal{L}$ into a single λ-nonlinear eigenvalue problem if $\lambda + u_- \neq 0$ on $\mathbb{R}$:

$$-\partial_x\Big(\frac{1}{\lambda+u_-}\partial_x\phi\Big) - (\lambda+u_+)\phi = 0,$$

where $\Psi = \begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = \begin{pmatrix}\frac{1}{\lambda+u_-}i\partial_x\phi \\ \phi\end{pmatrix}$, or into the following λ-nonlinear eigenvalue problem if $\lambda + u_+ \neq 0$ on $\mathbb{R}$:

$$-\partial_x\Big(\frac{1}{\lambda+u_+}\partial_x\phi\Big) - (\lambda+u_-)\phi = 0,$$

where $\Psi = \begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = \begin{pmatrix}\phi \\ \frac{1}{\lambda+u_+}i\partial_x\phi\end{pmatrix}$. By integration by parts, one can easily show that for any $c \in \mathbb{R}$, $u_- + c$ (resp. $u_+ - c$) controls the size $c - \lambda$ (resp. $c + \lambda$) in the following sense (with $f^{(+)}, f^{(-)}$ denoting the positive and negative parts of f respectively):

$$c - \lambda \leq \|(u_- + c)^{(+)}\|_{L^\infty} \text{ or } c + \lambda \leq \|(u_+ - c)^{(-)}\|_{L^\infty}.$$

In particular, if $u_+, -u_- \geq c > 0$ (i.e. $|q| \geq c + \frac{1}{2}|\partial_x\varphi|$), then there are no eigenvalues in $(-c, c)$ for $\mathcal{L}$ and L.

These arguments don't take into account of the special boundary condition (4) at infinity, and hence work for all non-vanishing bounded potentials q with finite phase velocity. We believe that this new formulation $\mathcal{L}$ of L will give new observations to interesting problems related to the cubic nonlinear defocusing NLS equation, such as the semiclassical limit, dispersive shock waves, rarefactive waves, the KdV approximation of NLS in the long wave length regime.

Organization of the paper. We study the general case of non-vanishing bounded potentials with finite phase velocity in Section 2, by encoding the above arguments step by step.

In Section 3 we will focus on the special case of the finite potentials $q = |q|e^{i\varphi} \in L^\infty(\mathbb{R};\mathbb{C})$, $\partial_x\varphi \in L^\infty(\mathbb{R};\mathbb{R})$, with the following nonzero boundary conditions

$$|q(x)| \to 1 \text{ and } \partial_x\varphi(x) \to 0 \text{ as } |x| \to \infty,$$

such that one of the two Riemannian invariants is constant:

$$u_+ = \frac{1}{2}\partial_x\varphi + |q| = 1 \text{ or } u_- = \frac{1}{2}\partial_x\varphi - |q| = -1.$$

We will characterize the eigenvalues located inside the spectral gap $(-1, 1)$.

2. General case of non-vanishing bounded potentials with finite phase velocity

In this section we will study the eigenvalue problem of the Lax operator (i.e. the one-dimensional Dirac operator) $L = \begin{pmatrix} i\partial_x & -iq \\ i\bar{q} & -i\partial_x \end{pmatrix}$, in the general case of non-vanishing bounded potentials with finite phase velocity.

After transforming the eigenvalue problem of L to the eigenvalue problem of $\mathcal{L}$ by the unitary matrix M in Subsection 2.1, we reformulate the eigenvalue problem of $\mathcal{L}$ into λ-nonlinear eigenvalue problems in Subsection 2.2. We analyze these λ-nonlinear eigenvalue problems to derive the estimates for the eigenvalues of L and $\mathcal{L}$ in Subsection 2.3.

2.1. Unitary equivalence between L and $\mathcal{L}$.

It is well-known that if $|q| \neq 0$ never vanishes, then by use of the Madelung transform

$$q(t,x) = \sqrt{\rho(t,x)}e^{i\varphi(t,x)},$$

one can write (at least formally) the hydrodynamic formulation of (1) as follows

$$\begin{cases} \partial_t \rho + 2\partial_x(\rho v) = 0, \\ \partial_t v + \partial_x(v^2) + 2\partial_x \rho = \partial_x\big(\partial_x(\frac{1}{2}\frac{\partial_x \rho}{\rho}) + (\frac{1}{2}\frac{\partial_x \rho}{\rho})^2\big), \end{cases} \tag{6}$$

where $\rho = \rho(t,x) : \mathbb{R}\times\mathbb{R} \mapsto (0,\infty)$ denotes the unknown density function and $v = v(t,x) = \partial_x\varphi(t,x) : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}$ denotes the unknown velocity function.

We introduce two real-valued functions

$$u_\pm = \frac{1}{2}v \pm \sqrt{\rho}, \tag{7}$$

which are the two Riemannian invariants for the compressible Euler equations

$$\begin{cases} \partial_t \rho + 2\partial_x(\rho v) = 0, \\ \partial_t v + \partial_x(v^2) + 2\partial_x \rho = 0. \end{cases} \tag{8}$$

The functions $u_\pm$ played an important role in the study of the semiclassical limit from (6) (with the Planck constant $\hbar$ appearing on the right-hand side of the v-equation) to (8) in [10], and in the study of hydrodynamic optical soliton tunneling in [12].

By straightforward calculations, the newly found unitary matrix

$$M = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-\frac{1}{2}i(\varphi-\frac{\pi}{2})} & e^{\frac{1}{2}i(\varphi-\frac{\pi}{2})} \\ e^{-\frac{1}{2}i(\varphi-\frac{\pi}{2})} & -e^{\frac{1}{2}i(\varphi-\frac{\pi}{2})} \end{pmatrix}, \tag{9}$$

transforms the Lax operator L to the following self-adjoint operator

$$\mathcal{L} := \begin{pmatrix} -u_- & i\partial_x \\ i\partial_x & -u_+ \end{pmatrix}. \tag{10}$$

More precisely, we have

Lemma 2.1 (Unitary equivalence between L and $\mathcal{L}$)**.** *For non-vanishing bounded potentials q with finite phase velocity such that $u_\pm \in L^\infty(\mathbb{R};\mathbb{R})$, the operator $L : H^1(\mathbb{R};\mathbb{C}^2) \mapsto L^2(\mathbb{R};\mathbb{C}^2)$ and the operator $\mathcal{L} : H^1(\mathbb{R};\mathbb{C}^2) \mapsto L^2(\mathbb{R};\mathbb{C}^2)$ are unitarily equivalent:*

$$L = M^*\mathcal{L}M.$$

Proof. It follows from the straightforward calculations and the following estimates for $\Psi = M\psi$:

$$\|\Psi\|_{L^2} = \|\psi\|_{L^2}, \quad \|\Psi\|_{H^1} \leq (1+\|\partial_x\varphi\|_{L^\infty})\|\psi\|_{H^1}, \quad \|\psi\|_{H^1} \leq (1+\|\partial_x\varphi\|_{L^\infty})\|\Psi\|_{H^1}.$$

$\square$

2.2. **Eigenvalue problem of $\mathcal{L}$.** We consider the following eigenvalue problem

$$\mathcal{L}\Psi = \begin{pmatrix} -u_- & i\partial_x \\ i\partial_x & -u_+ \end{pmatrix} \Psi = \lambda\Psi, \text{ with } \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}. \tag{11}$$

We are going to reformulate it in different interesting cases. We notice the following symmetry

$$(u_+, u_-, \lambda, \Psi_1, \Psi_2) \mapsto (-u_-, -u_+, -\lambda, -\Psi_2, \Psi_1) \tag{12}$$

in this spectral problem $\mathcal{L}\Psi = \lambda\Psi$, which corresponds to the symmetry

$$(q, \lambda, \psi_1, \psi_2) \mapsto (\bar{q}, -\lambda, \psi_2, \psi_1)$$

in the spectral problem of the Lax operator $L\psi = \lambda\psi$.

2.2.1. *Case* $u_\pm = \pm 1$. If $\lambda \neq 1$, then the above spectral problem reads simply as

$$\begin{cases} (\lambda - 1)\Psi_1 = i\partial_x\Psi_2, \\ -\partial_{xx}\Psi_2 - (\lambda^2 - 1)\Psi_2 = 0, \end{cases}$$

That is, the second component Ψ_2 solves the spectral problem for the free Schrödinger operator $-\partial_{xx}$ with the spectral parameter $\lambda^2 - 1$, and the first component Ψ_1 is given by $\frac{1}{\lambda-1} i\partial_x\Psi_2$.

By a similar argument or by the symmetry property (12), if $\lambda \neq -1$, then the above spectral problem reads as

$$\begin{cases} -\partial_{xx}\Psi_1 - (\lambda^2 - 1)\Psi_1 = 0, \\ (\lambda + 1)\Psi_2 = i\partial_x\Psi_1, \end{cases}$$

where the first component Ψ_1 solves the spectral problem for the free Schrödinger operator $-\partial_{xx}$ with the spectral parameter $\lambda^2 - 1$, and the second component Ψ_2 is given by $\frac{1}{\lambda+1} i\partial_x\Psi_1$.

2.2.2. *Case* $u_- = -1$. If $\lambda \neq 1$, then the spectral problem (11) reads as

$$\begin{cases} (\lambda - 1)\Psi_1 = i\partial_x\Psi_2, \\ -\partial_{xx}\Psi_2 - (\lambda - 1)(\lambda + u_+)\Psi_2 = 0, \end{cases} \tag{13}$$

and it suffices to consider the λ-nonlinear eigenvalue problem for Ψ_2;

$$-\partial_{xx}\phi - (\lambda - 1)(\lambda + u_+)\phi = 0,$$

with the first component Ψ_1 given by $\frac{1}{\lambda-1} i\partial_x\phi$.

Obviously $\lambda = 1$ is not an eigenvalue of $\mathcal{L}$ in this case $u_- = -1$.

2.2.3. *Case* $u_+ = 1$. Similarly as above or by the symmetry property (12), if $\lambda \neq -1$, then the spectral problem (11) reads as

$$\begin{cases} -\partial_{xx}\Psi_1 - (\lambda + 1)(\lambda + u_-)\Psi_1 = 0, \\ (\lambda + 1)\Psi_2 = i\partial_x\Psi_1, \end{cases} \tag{14}$$

and $\lambda = -1$ is not an eigenvalue of $\mathcal{L}$ in this case $u_+ = 1$.

2.2.4. *General case of $u_\pm \in L^\infty(\mathbb{R};\mathbb{R})$ and λ such that $\lambda + u_- \neq 0$ on $\mathbb{R}$.* By straightforward calculations, the spectral problem in (11) reads as

$$\begin{cases} (\lambda+u_-)\Psi_1 = i\partial_x\Psi_2, \\ -\partial_x\Big(\frac{1}{\lambda+u_-}\partial_x\Psi_2\Big) - (\lambda+u_+)\Psi_2 = 0. \end{cases} \tag{15}$$

2.2.5. *General case of $u_\pm \in L^\infty(\mathbb{R};\mathbb{R})$ and λ such that $\lambda + u_+ \neq 0$ on $\mathbb{R}$.* As above, (11) becomes

$$\begin{cases} -\partial_x\Big(\frac{1}{\lambda+u_+}\partial_x\Psi_1\Big) - (\lambda+u_-)\Psi_1 = 0, \\ (\lambda+u_+)\Psi_2 = i\partial_x\Psi_1. \end{cases} \tag{16}$$

To conclude, we have

Lemma 2.2 (Reformulation of the eigenvalue problem of $\mathcal{L}$)**.** *The eigenvalue problem $\mathcal{L}\Psi = \lambda\Psi$ reads,*

(1) if $\lambda + u_- \neq 0$ on $\mathbb{R}$, as

$$-\partial_x\Big(\frac{1}{\lambda+u_-}\partial_x\phi\Big) - (\lambda+u_+)\phi = 0, \tag{17}$$

together with $\Psi = \begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = \begin{pmatrix}\frac{1}{\lambda+u_-}i\partial_x\phi \\ \phi\end{pmatrix}$.

(2) if $\lambda + u_+ \neq 0$ on $\mathbb{R}$, as

$$-\partial_x\Big(\frac{1}{\lambda+u_+}\partial_x\phi\Big) - (\lambda+u_-)\phi = 0, \tag{18}$$

together with $\Psi = \begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = \begin{pmatrix}\phi \\ \frac{1}{\lambda+u_+}i\partial_x\phi\end{pmatrix}$.

2.3. **Analysis of eigenvalues for $\mathcal{L}$.** We consider first the eigenvalues close to ± 1. For notational simplicity we introduce the two real-valued functions

$$V_\pm = u_\pm \mp 1 = \frac{1}{2}v \pm (\sqrt{\rho} - 1), \tag{19}$$

and we decompose $V_\pm$ into their positive and negative parts respectively

$$V_\pm = V_\pm^{(+)} - V_\pm^{(-)}, \text{ with } V_\pm^{(+)} = \max\{V_\pm, 0\},\ V_\pm^{(-)} = \max\{-V_\pm, 0\}.$$

From now on we assume that $V_+^{(-)}, V_-^{(+)} \in L^\infty(\mathbb{R};\mathbb{R})$, and we are going to analyze the eigenvalues of the operator $\mathcal{L}$ in the two cases listed in Lemma 2.2 respectively.

2.3.1. *Case $1-\lambda > \|V_-^{(+)}\|_{L^\infty}$.* In this case,

$$1-\lambda - V_- = 1-\lambda - V_-^{(+)} + V_-^{(-)} > 0 \text{ on } \mathbb{R},$$

and hence the eigenvalue problem $\mathcal{L}\Psi = \lambda\Psi$ reads as (17).

We test the eigenvalue problem (17) by $\bar\phi$ to derive

$$\int_{\mathbb{R}}\Big(\frac{1}{1-\lambda-V_-}|\partial_x\phi|^2 + (1+\lambda+V_+)|\phi|^2\Big)\,\mathrm{d}x = 0.$$

This yields

$$0 \geq \int_{\mathbb{R}}(1+\lambda+V_+)|\phi|^2\,\mathrm{d}x \geq (1+\lambda - \|V_+^{(-)}\|_{L^\infty})\|\phi\|_{L^2}^2,$$

which immediately implies

$$1 + \lambda \leq \|V_+^{(-)}\|_{L^\infty}.$$

2.3.2. *Case* $1 + \lambda > \|V_+^{(-)}\|_{L^\infty}$. Similarly as above, if $1 + \lambda > \|V_+^{(-)}\|_{L^\infty}$, such that $1 + \lambda + V_+ > 0$ on $\mathbb{R}$, then we use (18) and derive

$$1 - \lambda \leq \|V_-^{(+)}\|_{L^\infty}.$$

We conclude that if λ is an eigenvalue of $\mathcal{L}$, then

$$\begin{aligned} 1 - \lambda > \|(u_- + 1)^{(+)}\|_{L^\infty} &\Rightarrow 1 + \lambda \leq \|(u_+ - 1)^{(-)}\|_{L^\infty}, \\ 1 + \lambda > \|(u_+ - 1)^{(-)}\|_{L^\infty} &\Rightarrow 1 - \lambda \leq \|(u_- + 1)^{(+)}\|_{L^\infty}. \end{aligned}$$

Notice that the above two statements are equivalent to each other, and also to the following unconditional statement for the eigenvalues λ of $\mathcal{L}$:

$$1 - \lambda \leq \|(u_- + 1)^{(+)}\|_{L^\infty} \text{ or } 1 + \lambda \leq \|(u_+ - 1)^{(-)}\|_{L^\infty}$$

More generally, $\forall c \in \mathbb{R}$, we can replace ± 1 and $u_\pm \mp 1$ by $\pm c$ and $u_\pm \mp c$ respectively in the above arguments, to derive the following result.

Theorem 2.1 (Eigenvalues of the operator $\mathcal{L}$)**.** *Let* $u_\pm \in L^\infty(\mathbb{R};\mathbb{R})$. *If* $\lambda \in \mathbb{R}$ *is an eigenvalue of the operator* $\mathcal{L} = \begin{pmatrix} -u_- & i\partial_x \\ i\partial_x & -u_+ \end{pmatrix} : H^1(\mathbb{R};\mathbb{C}^2) \mapsto L^2(\mathbb{R};\mathbb{C}^2)$, *then* λ *satisfies for all* $c \in \mathbb{R}$,

$$c - \lambda \leq \|(u_- + c)^{(+)}\|_{L^\infty} \text{ or } c + \lambda \leq \|(u_+ - c)^{(-)}\|_{L^\infty}.$$

In particular if $u_+, -u_- \geq c > 0$, *there are no eigenvalues in* $(-c, c)$ *for* $\mathcal{L}$.

As the Lax operator L and the operator $\mathcal{L}$ are shown to be unitarily equivalent in Lemma 2.1, we have the following result immediately.

Corollary 2.1 (Eigenvalues of the operator L)**.** *Suppose that the non-vanishing bounded function* $q = |q|e^{i\varphi}$ *has finite phase velocity:* $\partial_x\varphi \in L^\infty(\mathbb{R};\mathbb{R})$. *Then the eigenvalues* $\lambda \in \mathbb{R}$ *of the operator* $L : H^1(\mathbb{R};\mathbb{C}^2) \mapsto L^2(\mathbb{R};\mathbb{C}^2)$ *satisfy for all* $c \in \mathbb{R}$,

$$c - \lambda \leq \left\| \left(\frac{1}{2}\partial_x\varphi - |q| + c \right)^{(+)} \right\|_{L^\infty} \text{ or } c + \lambda \leq \left\| \left(\frac{1}{2}\partial_x\varphi + |q| - c \right)^{(-)} \right\|_{L^\infty}.$$

In particular, if $|q| \geq c + \frac{1}{2}|\partial_x\varphi|$ *pointwise for some* $c > 0$, *then there are no eigenvalues of the operator* L *inside* $(-c, c)$.

Remark 2.1 (Examples of potentials)**.** *For the potential*

$$q = |q|e^{i\varphi}, \quad |q| - 1 \geq 0, \quad \varphi = \int_{x_0}^{x} 2(|q| - 1)\,\mathrm{d}y\,,$$

such that $|q| = 1 + \frac{1}{2}|\partial_x\varphi|$, *then there are no eigenvalues of the Lax operator* L *inside* $(-1, 1)$. *If we assume further decay at infinity:* $(1+x^4)(q - e^{i\theta_\pm}) \in L^1(\mathbb{R}^\pm)$ *for some* $\theta_\pm \in \mathbb{R}$, *the spectrum of* L *consists only of essential spectrum* $(-\infty, -1] \cup [1, \infty)$.

It is well-known that for the potential given by the initial data of the dark soliton solution $\varepsilon \tanh(\varepsilon(x - 2\sqrt{1-\varepsilon^2}t)) + i\sqrt{1-\varepsilon^2}$ *of the one-dimensional cubic nonlinear defocusing Schrödinger equation satisfying nonzero boundary condition at infinity:*

$$q_\varepsilon(x) := \varepsilon \tanh(\varepsilon x) + i\sqrt{1-\varepsilon^2}, \quad \varepsilon \in (0, 1),$$

the spectrum of L consists of the essential spectrum $(-\infty,-1]\cup[1,\infty)$ together with a single simple discrete eigenvalue at $-\sqrt{1-\varepsilon^2}=-1+\frac{1}{2}\varepsilon^2+O(\varepsilon^4)$. We calculate the corresponding $u_\pm=\frac{1}{2}\partial_x\varphi\pm|q|$ as

$$u_+=1-\varepsilon^2\operatorname{sech}^2(\varepsilon x)+O(\varepsilon^4),\quad u_-=-1+O(\varepsilon^4),\qquad \textit{as } \varepsilon\to 0.$$

If we take $c=1$, then Corollary 2.1 implies, without any a priori knowledge about the eigenvalue, that

$$1-\lambda\le|O(\varepsilon^4)| \textit{ or } 1+\lambda\le\varepsilon^2+O(\varepsilon^4).$$

In next section we are going to give more characterization of the eigenvalues when $|q(x)|\to 1$ as $|x|\to\infty$.

3. Special case of potentials under nonzero boundary condition

In this section we will consider the Lax operator $L=\begin{pmatrix} i\partial_x & -iq\\ i\bar{q} & -i\partial_x\end{pmatrix}$ with non-vanishing bounded potentials satisfying nonzero boundary conditions at infinity:

$$\begin{aligned} &q=|q|e^{i\varphi}\in L^\infty(\mathbb{R};\mathbb{C}),\quad \partial_x\varphi\in L^\infty(\mathbb{R};\mathbb{R}),\\ &|q(x)|\to 1 \text{ and } \partial_x\varphi(x)\to 0 \text{ as } |x|\to\infty.\end{aligned}\tag{20}$$

This implies the following assumptions on the potentials $u_\pm=\frac{1}{2}\partial_x\varphi\pm|q|$ of the unitarily equivalent operator $\mathcal{L}=\begin{pmatrix} -u_- & i\partial_x\\ i\partial_x & -u_+\end{pmatrix}$:

$$u_\pm\in L^\infty(\mathbb{R};\mathbb{R}),\quad u_\pm(x)\to\pm1 \text{ as } |x|\to\infty.$$

In this section we will study the eigenvalues of the operators L and $\mathcal{L}$ inside the open interval $\lambda\in(-1,1)$, and we consider the following two special cases:

(1) $u_-=-1$ such that $\lambda+u_-\neq 0$, and the eigenvalue problem $\mathcal{L}\Psi=\lambda\Psi$ reads as

$$-\partial_{xx}\phi-(\lambda-1)(\lambda+u_+)\phi=0,\ \text{with } \Psi=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\begin{pmatrix}\frac{1}{\lambda-1}i\partial_x\phi\\ \phi\end{pmatrix};\tag{21}$$

(2) $u_+=1$ such that $\lambda+u_+\neq 0$, and the eigenvalue problem $\mathcal{L}\Psi=\lambda\Psi$ reads as

$$-\partial_{xx}\phi-(\lambda+1)(\lambda+u_-)\phi=0,\ \text{with } \Psi=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\begin{pmatrix}\phi\\ \frac{1}{\lambda+1}i\partial_x\phi\end{pmatrix}.\tag{22}$$

By virtue of the symmetry between (21) and (22):

$$(u_+,u_-,\lambda,\phi)\mapsto(-u_-,-u_+,-\lambda,\phi),\tag{23}$$

it suffices to study the λ-nonlinear eigenvalue problem (21):

$$-\phi''+(1-\lambda^2)\phi=-(1-\lambda)V\phi,\tag{24}$$

where

$$V:=u_+-1\in L^\infty(\mathbb{R};\mathbb{R}),\ \text{with } |V(x)|\to 0 \text{ as } |x|\to\infty.\tag{25}$$

3.1. **Study of a compact operator.** For analyzing the eigenvalues of the problem (24), we first observe that the operator

$$K : H^1(\mathbb{R}) \ni u \mapsto V \cdot u \in L^2(\mathbb{R}) \tag{26}$$

is compact. For any fixed $\beta > 0$, we define the operator

$$K_\beta = (-\partial_x^2 + \beta)^{-1} K : H^1(\mathbb{R}) \mapsto H^1(\mathbb{R}). \tag{27}$$

It is a symmetric and compact operator, when we endow $H^1(\mathbb{R})$ with the inner product

$$\langle u, v\rangle_\beta := \langle u', v'\rangle_{L^2(\mathbb{R})} + \beta \langle u, v\rangle_{L^2(\mathbb{R})}. \tag{28}$$

Thus K_β has an ONB of eigenfunctions and an associated eigenvalue sequence converging to 0. We take all the negative eigenvalues

$$\{\mu_j(\beta)\}_{j\in N_-} \subset (-\infty, 0), \tag{29}$$

and order them non-decreasingly

$$|\mu_1(\beta)| \geq |\mu_2(\beta)| \geq \cdots > 0.$$

Here the set N_- can be $\mathbb{N}$ or a finite set $\{1, \cdots, n\}$ or the empty set $\emptyset$.

Lemma 3.1 (Properties of the set N_- and the functions μ_j)**.** *The following holds true:*

(i) The set $N_- = N_-^{(\beta)}$ is independent of β.

(ii) For each $j \in N_-$, the function

$$\mu_j : (0, \infty) \mapsto (-\infty, 0)$$

is strictly increasing and continuous.

Proof. For $u \in H^1(\mathbb{R})\backslash\{0\}$ and $\beta > 0$, we derive from (27) and (28) that

$$\frac{\langle K_\beta u, u\rangle_\beta}{\langle u, u\rangle_\beta} = \frac{\langle Vu, u\rangle_{L^2}}{\|u'\|_{L^2}^2 + \beta \|u\|_{L^2}^2}.$$

Poincaré's min-max principle implies for any $j \in N_-^{(\beta)}$,

$$\mu_j(\beta) = \min_{U\subset H^1(\mathbb{R}) \text{ subspace}, \dim(U)=j} \ \max_{u\in U\backslash\{0\}} \frac{\langle Vu, u\rangle_{L^2}}{\|u'\|_{L^2}^2 + \beta\|u\|_{L^2}^2}. \tag{30}$$

The minimum is attained at

$$U = U_j^{(\beta)} := \operatorname{Span}\{\psi_1(\beta), \cdots, \psi_j(\beta)\},$$

where $\psi_l(\beta)$ denotes an eigenfunction of K_β associated with $\mu_l(\beta)$, and $\psi_1(\beta), \cdots, \psi_j(\beta)$ are chosen $\langle\cdot,\cdot\rangle_\beta$-orthonormal.

Let $\beta_1 \in (0, \infty)$ and $j \in N_-^{(\beta_1)}$, such that $\mu_j(\beta_1) < 0$. Then (30) implies

$$\frac{\langle Vu, u\rangle_{L^2}}{\|u'\|_{L^2}^2 + \beta_1\|u\|_{L^2}^2} < 0, \quad \forall u \in U_j^{(\beta_1)}\backslash\{0\},$$

and thus $\langle Vu, u\rangle_{L^2} < 0$, and hence

$$\frac{\langle Vu, u\rangle_{L^2}}{\|u'\|_{L^2}^2 + \beta\|u\|_{L^2}^2} < 0, \quad \forall \beta \in (0, \infty), \quad \forall u \in U_j^{(\beta_1)}\backslash\{0\}. \tag{31}$$

By the L^2-compactness of the unit sphere in the finite-dimensional space $U_j^{(\beta_1)}$, we conclude that

$$\max_{u\in U_j^{(\beta_1)}\setminus\{0\}} \frac{\langle Vu,u\rangle_{L^2}}{\|u'\|_{L^2}^2+\beta\|u\|_{L^2}^2} = \max_{u\in U_j^{(\beta_1)},\, \|u\|_{L^2}=1} \frac{\langle Vu,u\rangle_{L^2}}{\|u'\|_{L^2}^2+\beta} < 0,$$

and hence

$$\min_{U\subset H^1(\mathbb{R}) \text{ subspace}, \dim(U)=j} \max_{u\in U\setminus\{0\}} \frac{\langle Vu,u\rangle_{L^2}}{\|u'\|_{L^2}^2+\beta\|u\|_{L^2}^2} < 0, \quad \forall \beta\in(0,\infty).$$

This implies $j\in N_-^{(\beta)}$ for all $\beta\in(0,\infty)$, and we have proved $N_-^{(\beta_1)}\subset N_-^{(\beta)}$ for all $\beta_1,\beta\in(0,\infty)$. The assertion (i) follows.

Let $j\in N_-$ and $0<\beta<\beta_1<\infty$, such that

$$\frac{\langle Vu,u\rangle_{L^2}}{\|u'\|_{L^2}^2+\beta\|u\|_{L^2}^2} < \frac{\langle Vu,u\rangle_{L^2}}{\|u'\|_{L^2}^2+\beta_1\|u\|_{L^2}^2} < 0, \quad \forall u\in U_j^{(\beta_1)}\setminus\{0\}.$$

By the above compactness argument again we deduce

$$\max_{u\in U_j^{(\beta_1)}\setminus\{0\}} \frac{\langle Vu,u\rangle_{L^2}}{\|u'\|_{L^2}^2+\beta\|u\|_{L^2}^2} < \max_{u\in U_j^{(\beta_1)}\setminus\{0\}} \frac{\langle Vu,u\rangle_{L^2}}{\|u'\|_{L^2}^2+\beta_1\|u\|_{L^2}^2} = \mu_j(\beta_1).$$

This implies $\mu_j(\beta)<\mu_j(\beta_1)$, and hence $\mu_j:(0,\infty)\mapsto(-\infty,0)$ is strictly increasing.

Let $\beta_1\in(0,\infty)$ and we are going to show the continuity of μ_j at β_1. We calculate for all $\beta\in[\frac{1}{2}\beta_1, 2\beta_1]$ and all $u\in H^1(\mathbb{R})\setminus\{0\}$ that

$$\begin{aligned}\Big|\frac{\langle Vu,u\rangle_{L^2}}{\|u'\|_{L^2}^2+\beta\|u\|_{L^2}^2} - \frac{\langle Vu,u\rangle_{L^2}}{\|u'\|_{L^2}^2+\beta_1\|u\|_{L^2}^2}\Big| &= \frac{|\langle Vu,u\rangle_{L^2}|\cdot\|u\|_{L^2}^2|\beta-\beta_1|}{(\|u'\|_{L^2}^2+\beta\|u\|_{L^2}^2)\cdot(\|u'\|_{L^2}^2+\beta_1\|u\|_{L^2}^2)}\\ &\le \frac{\|V\|_{L^\infty}|\beta-\beta_1|}{\beta\cdot\beta_1} \le \Big(\frac{2\|V\|_{L^\infty}}{\beta_1}\Big)|\beta-\beta_1|,\end{aligned}$$

which immediately implies

$$\begin{aligned}\frac{\langle Vu,u\rangle_{L^2}}{\|u'\|_{L^2}^2+\beta_1\|u\|_{L^2}^2} - \Big(\frac{2\|V\|_{L^\infty}}{\beta_1}\Big)|\beta-\beta_1| &\le \frac{\langle Vu,u\rangle_{L^2}}{\|u'\|_{L^2}^2+\beta\|u\|_{L^2}^2}\\ &\le \frac{\langle Vu,u\rangle_{L^2}}{\|u'\|_{L^2}^2+\beta_1\|u\|_{L^2}^2} + \Big(\frac{2\|V\|_{L^\infty}}{\beta_1}\Big)|\beta-\beta_1|.\end{aligned}$$

We apply the min-max principle in (30) to arrive at

$$\begin{aligned}\mu_j(\beta_1) &\le \mu_j(\beta) + \Big(\frac{2\|V\|_{L^\infty}}{\beta_1}\Big)|\beta-\beta_1|,\\ \text{and } \mu_j(\beta) &\le \mu_j(\beta_1) + \Big(\frac{2\|V\|_{L^\infty}}{\beta_1}\Big)|\beta-\beta_1|.\end{aligned}$$

This gives

$$|\mu_j(\beta)-\mu_j(\beta_1)| \le \Big(\frac{2\|V\|_{L^\infty}}{\beta_1}\Big)|\beta-\beta_1|,$$

which implies the continuity of μ_j at β_1. □

3.2. **Eigenvalues inside** $(-1,1)$**.** When we reformulate (24) as

$$K_\beta \phi = -\frac{1}{1-\lambda}\phi, \text{ with } \beta = 1-\lambda^2 > 0,$$

By virtue of Lemma 3.1, we define the subset of N_-:

$$N_-^0 := \{j \in N_- \,|\, \mu_j(0_+) < -\frac{1}{2}\}. \tag{32}$$

We have the following results.

Theorem 3.1 (Eigenvalues in $(-1,1)$ when $u_- = -1$ and $V = u_+ - 1$ vanishes at infinity)**.** *Let the negative part of* V*:* $V^{(-)}(x) = \max\{0, -V(x)\}$ *satisfy the following smallness assumption*

$$\|V^{(-)}\|_{L^\infty(\mathbb{R})} \leq 1. \tag{33}$$

Then the following statements hold true:

(a) For each $j \in N_-^0$*, there exists a unique* $\beta_j \in (0,1]$ *such that*

$$\beta_j = -\frac{1}{\mu_j(\beta_j)}(2 + \frac{1}{\mu_j(\beta_j)}). \tag{34}$$

(b) The set of eigenvalues of the eigenvalue problem (24) *in* $(-1,1)$ *is given by*

$$\Big\{\lambda_j \,|\, \lambda_j := 1 + \frac{1}{\mu_j(\beta_j)}, \quad j \in N_-^0\Big\} \subset (-1,1), \tag{35}$$

and the eigenspace of λ_j *coincides with the eigenspace associated with the eigenvalue* $\mu_j(\beta_j)$ *of the operator* K_{β_j}*.*

(c) $(\lambda_j)_{j\in N_-^0}$ *is non-increasing and*

$$0 < \lambda_j + 1 \leq \|V^{(-)}\|_{L^\infty}, \quad \forall j \in N_-^0. \tag{36}$$

Proof. **Proof of (a).** We abbreviate

$$\alpha_j(\beta) = -\frac{1}{\mu_j(\beta)} \in (0,\infty), \quad \forall j \in N_-, \, \beta \in (0,\infty).$$

By Lemma 3.1, for any $j \in N_-$, $\alpha_j(\beta)$ is strictly increasing and continuous. By the definition of N_-^0, we have

$$\alpha_j(0_+) = -\frac{1}{\mu_j(0_+)} < 2, \quad \forall j \in N_-^0. \tag{37}$$

We then search for $\beta_j \in (0,1]$ which satisfies (34):

$$\beta_j = \alpha_j(\beta_j)(2 - \alpha_j(\beta_j)), \quad j \in N_-^0. \tag{38}$$

As in the proof of Lemma 3.1, let $(\psi_j(\beta))_{j\in N_-}$ denote an $\langle\cdot,\cdot\rangle_\beta$-orthonormal system of eigenfunctions of K_β associated with $(\mu_j(\beta))_{j\in N_-}$, such that

$$K_\beta(\psi_j(\beta)) = \mu_j(\beta)\psi_j(\beta),$$

or equivalently,

$$-\psi_j(\beta)'' + \beta\psi_j(\beta) + \alpha_j(\beta)V\psi_j(\beta) = 0 \text{ on } \mathbb{R}. \tag{39}$$

Testing (39) with $\bar{\psi}_j(\beta)$ gives

$$\begin{aligned}0 &= \|\psi_j(\beta)'\|_{L^2}^2 + \beta\|\psi_j(\beta)\|_{L^2}^2 + \alpha_j(\beta)\langle V\psi_j(\beta), \psi_j(\beta)\rangle_{L^2} \\ &\geq \beta\|\psi_j(\beta)\|_{L^2}^2 - \alpha_j(\beta)\|V^{(-)}\|_{L^\infty}\|\psi_j(\beta)\|_{L^2}^2,\end{aligned}$$

and hence

$$\beta \le \alpha_j(\beta)\|V^{(-)}\|_{L^\infty}, \quad \forall \beta \in (0,\infty), \quad \forall j \in N_-. \tag{40}$$

Under the assumption (33), we have

$$\beta \le \alpha_j(\beta), \quad \forall \beta \in (0,\infty), \quad \forall j \in N_-, \tag{41}$$

and in particular

$$1 \le \alpha_j(1), \quad \forall j \in N_-. \tag{42}$$

We define for each $j \in N_-^0$ the strictly increasing and continuous function

$$f_j : (0,1] \mapsto \mathbb{R}, \quad f_j(\beta) = \alpha_j(\beta) - 1 - \sqrt{1-\beta},$$

such that, using (37) and (42),

$$f_j(0_+) < 0, \quad f_j(1) \ge 0.$$

Thus there exists a unique zero $\beta_j \in (0,1]$ of f_j such that

$$\alpha_j(\beta_j) = 1 + \sqrt{1-\beta_j}. \tag{43}$$

Hence β_j satisfies (38) and thus (34).

If $\tilde\beta_j \in (0,1]$ is another solution of (34) and hence of (38), then

$$\alpha_j(\tilde\beta_j) = 1 + \sqrt{1-\tilde\beta_j}, \quad \text{or } \alpha_j(\tilde\beta_j) = 1 - \sqrt{1-\tilde\beta_j}.$$

The second case $\alpha_j(\tilde\beta_j) = 1 - \sqrt{1-\tilde\beta_j}$ contradicts (41). Thus $\beta_j \in (0,1]$ is the unique solution of (34).

Proof of (b). Let β_j, $j \in N_-^0$ be given by (a), and $\lambda_j = 1 + \frac{1}{\mu_j(\beta_j)} \in (-1,1)$ such that

$$1 - \lambda_j = \alpha_j(\beta_j), \quad 1 - \lambda_j^2 = \beta_j \in (0,1].$$

Then the equation (39) with $\beta = \beta_j$ shows that (24) holds for the eigenpairs $(\lambda_j, \psi_j(\beta_j))_{j\in N_-^0}$.

Vice versa, let $(\lambda,\psi) \in (-1,1)\times(H^1(\mathbb{R})\backslash\{0\})$ denote any eigenpair of the eigenvalue problem (24). Then ψ satisfies

$$K_\beta\psi = \mu\psi, \text{ with } \beta := 1-\lambda^2 \in (0,1], \quad \mu := -\frac{1}{1-\lambda} \in (-\infty, -\frac{1}{2}).$$

Since $\mu < 0$, there exists $j \in N_-$ such that

$$\mu = \mu_j(\beta), \quad \psi = \psi_j(\beta).$$

Since $\mu_j : (0,\infty) \mapsto (-\infty, 0)$ is strictly increasing, we have

$$\mu_j(0_+) \le \mu_j(\beta) = \mu < -\frac{1}{2},$$

and thus $j \in N_-^0$. Furthermore, we have from the definitions of β, μ and $\mu = \mu_j(\beta)$ that

$$\beta = -\frac{1}{\mu}(2 + \frac{1}{\mu}) = -\frac{1}{\mu_j(\beta)}(2 + \frac{1}{\mu_j(\beta)}).$$

By (a), $\beta = \beta_j \in (0,1]$ is the unique solution of the above equation. Consequently,

$$\lambda = 1 + \frac{1}{\mu_j(\beta_j)} = \lambda_j$$

for some $j \in N_-^0$.

Proof of (c). We now show $\lambda_j \geq \lambda_{j+1}$ for $j \in N_-^0$ such that $j+1 \in N_-^0$. This is equivalent to showing $\mu_j(\beta_j) \leq \mu_{j+1}(\beta_{j+1})$, and hence $\alpha_j(\beta_j) \leq \alpha_{j+1}(\beta_{j+1})$. If $\beta_j < \beta_{j+1}$, then $\alpha_j(\beta_j) \leq \alpha_{j+1}(\beta_{j+1})$, since $\alpha_j(\beta)$ is non-decreasing with respect to j and strictly increasing with respect to β. This means, by virtue of (43), $\beta_j \geq \beta_{j+1}$, which contradicts the assumption $\beta_j < \beta_{j+1}$. Thus $\beta_j \geq \beta_{j+1}$ and hence $\alpha_j(\beta_j) \leq \alpha_{j+1}(\beta_{j+1})$ holds.

Finally, the estimate (36) comes from (40) straightforward since

$$\beta_j = 1 - \lambda_j^2, \quad \alpha_j(\beta_j) = 1 - \lambda_j, \quad \lambda_j \in (-1,1).$$

□

By virtue of the symmetry (23), we have

Corollary 3.1 (Eigenvalues in $(-1,1)$ in the case of non-vanishing finite potentials q satisfying nonzero boundary conditions)**.** *Let $u_- = -1$ or $u_+ = 1$, and $u_\pm \in L^\infty(\mathbb{R};\mathbb{R})$ satisfy*

$$u_\pm(x) \to \pm 1 \text{ as } |x| \to \infty, \text{ and } \|(u_\pm \mp 1)^{(\pm)}\|_{L^\infty} \leq 1.$$

Then the eigenvalues $\{\lambda_j\} \subset (-1,1)$ of the operators L and $\mathcal{L}$ are given by (a)-(c) in Theorem 3.1 correspondingly.

3.3. **Examples of the existence of eigenvalues of** (24)**.**

Lemma 3.2. *Let $R > 0$. Let $V \in L^\infty(\mathbb{R};\mathbb{R})$ such that*

$$\int_{-R}^{R} V(x)\,\mathrm{d}x < 0 \text{ and } V \leq 0 \text{ outside } [-R,R]. \tag{44}$$

For any $\beta > 0$, let $\{\mu_1(\beta), \mu_2(\beta), \cdots\}_{j\in N_-} \subset (-\infty, 0)$ denote the non-increasing negative eigenvalues of the compact operator K_β given in (27)*.*

Then

$$\mu_1(0_+) = -\infty. \tag{45}$$

Proof. Take $\phi \in C_c^\infty(\mathbb{R})$ such that $\phi = 1$ on $(-1,1)$ and $\phi = 0$ outside $(-2,2)$. For $n \in \mathbb{N}$, we denote $\phi_n(x) = \phi(\frac{x}{n})$, $x \in \mathbb{R}$, such that

$$\|\phi_n\|_{L^2}^2 = n\|\phi\|_{L^2}^2,$$
$$\|\phi_n'\|_{L^2}^2 = \frac{1}{n}\|\phi'\|_{L^2}^2.$$

Hence for $n \geq R$, we derive from the assumption (44) that

$$\langle V\phi_n, \phi_n\rangle_{L^2(\mathbb{R})} = \int_{-R}^{R} V\,\mathrm{d}x + \int_{\mathbb{R}\setminus[-R,R]} V(x)|\phi(\frac{x}{n})|^2\,\mathrm{d}x \leq \int_{-R}^{R} V\,\mathrm{d}x < 0.$$

Therefore for $\beta_{(n)} := \frac{1}{n^2}$, we have

$$\frac{\langle V\phi_n, \phi_n\rangle_{L^2}}{\|\phi_n'\|_{L^2}^2 + \beta_{(n)}\|\phi_n\|_{L^2}^2} \leq -Cn,$$

for some constant $C > 0$ independent of n. Consequently, for all $n \geq R$,

$$\mu_1(\beta_{(n)}) = \min_{u\in H^1(\mathbb{R})\setminus\{0\}} \frac{\langle Vu, u\rangle_{L^2}}{\|u'\|_{L^2}^2 + \beta_{(n)}\|u\|_{L^2}^2} \leq -Cn.$$

This implies (45). □

We conclude from Theorem 3.1 and Lemma 3.2 the existence of eigenvalues of the eigenvalue problem (24).

Corollary 3.2. *If* $V \in L^\infty(\mathbb{R};\mathbb{R})$ *satisfies* (33) *and* (44), *then there exists at least one eigenvalue* $\lambda_1 \in (-1,1)$ *of the eigenvalue problem* (24).

The following example shows the optimality (in some sense) of the existence result above: If $-V = V^{(-)}$ is small on an interval while vanishes outside the interval, then there exists exactly one eigenvalue λ_1 in $(-1,1)$ of the eigenvalue problem (24).

Example. Let $V(x) = \begin{cases} -\varepsilon & \text{on } [0,1], \\ 0 & \text{otherwise,} \end{cases}$ for some $\varepsilon \in (0,1]$.

For $\beta > 0$, we look for negative eigenvalues $\{\mu_j(\beta)\}_{j\in N_-}$ of the operator $K_\beta : H^1(\mathbb{R}) \mapsto H^1(\mathbb{R})$ given in (27):

$$K_\beta u = \mu(\beta)u, \text{ i.e. } -u'' + \beta u = -\alpha(\beta)Vu, \text{ with } \alpha(\beta) = -\frac{1}{\mu(\beta)}.$$

We denote $\kappa = \sqrt{\alpha\varepsilon - \beta}$ with $\operatorname{Im}[\kappa] \geq 0$ and $k = \sqrt{\beta} > 0$, such that the above eigenvalue problem becomes

$$\begin{aligned} -u'' &= \kappa^2 u \text{ on } [0,1], \\ u'' &= k^2 u \text{ outside } [0,1]. \end{aligned}$$

We then search for the non-trivial solution of the following form which are continuously differentiable at 0 and 1:

$$u = \begin{cases} (a+b)e^{kx} & \text{for } x \leq 0, \\ ae^{i\kappa x} + be^{-i\kappa x} & \text{for } x \in [0,1], \\ \frac{ae^{i\kappa}+be^{-i\kappa}}{e^{-k}}e^{-kx} & \text{for } x \geq 1. \end{cases}$$

The C^1-matching conditions at 0 and 1 read then as

$$\begin{pmatrix} i\kappa - k & -i\kappa - k \\ (i\kappa + k)e^{i\kappa} & (-i\kappa + k)e^{-i\kappa} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In order to have a non-trivial solution, the determinant of the matrix on the lefthand side should vanish:

$$\tan(\kappa) = \frac{2k\kappa}{\kappa^2 - k^2}.$$

This equation for κ has countably many positive solutions $\kappa_j(k)$, $j \in \mathbb{N}$ with $\kappa_j(k) \overset{k\to 0_+}{\to} (j-1)\pi$.

Thus

$$\mu_j(\beta) = -\frac{1}{\alpha_j(\beta)} = -\frac{\varepsilon}{\beta + \kappa_j(k)^2} \overset{\beta\to 0_+}{\to} \begin{cases} -\infty & \text{if } j = 1, \\ -\frac{\varepsilon}{(j-1)^2\pi^2} & \text{if } j \geq 2. \end{cases}$$

This together with $\varepsilon \in [0,1]$ implies

$$\mu_j(0_+) \geq -\frac{\varepsilon}{\pi^2} \geq -\frac{1}{\pi^2} > -\frac{1}{2}, \quad \forall j \geq 2.$$

To conclude, $N_-^0 = \{1\}$.

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