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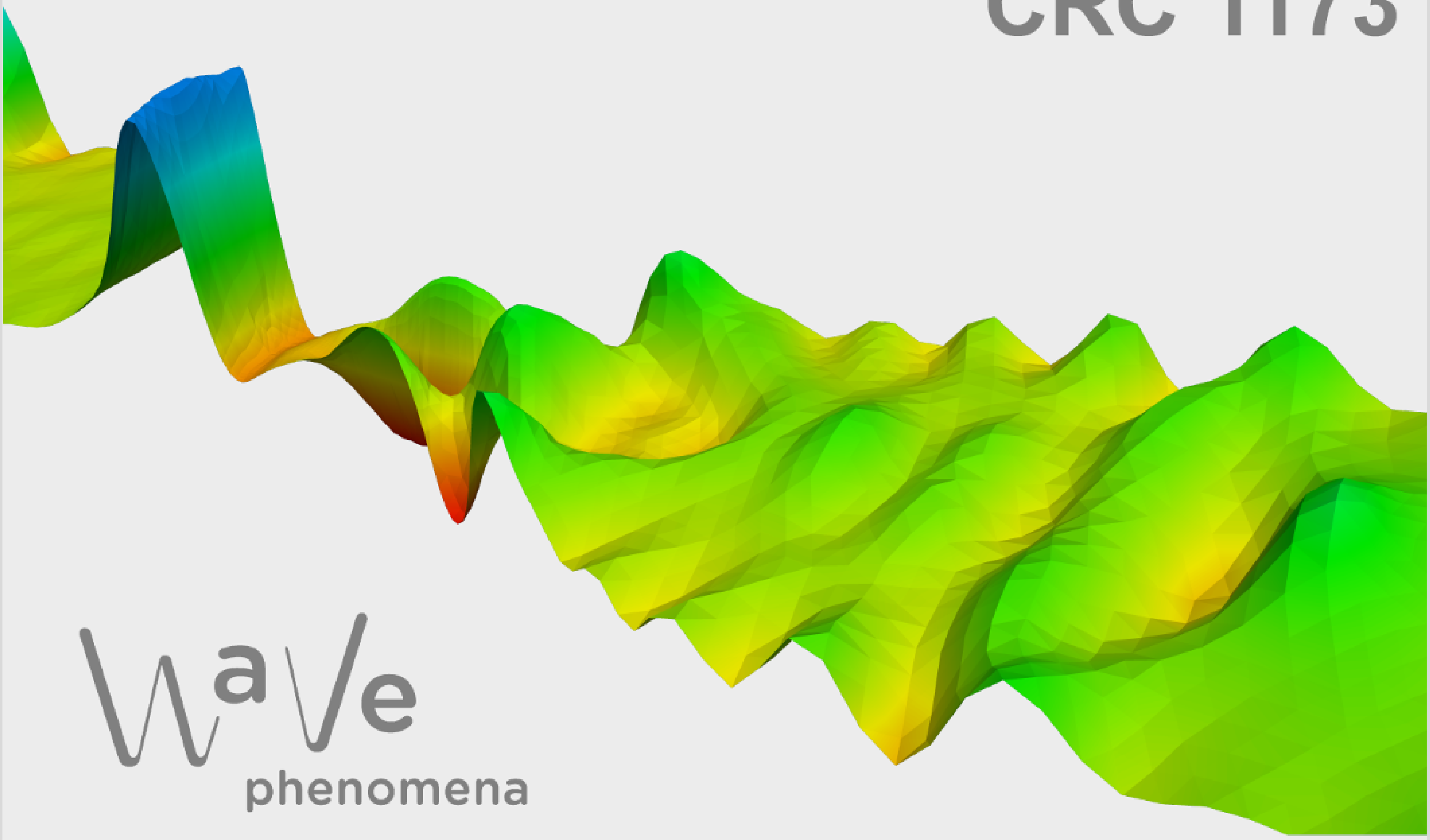

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Approximate mechanical impedance of a thin linear elastic slab

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Abstract

An elastic deformable body covered with a thin elastic layer is considered. In the literature, it is well known that the effect of thin layers can be described by impedance boundary conditions. In this paper, we consider a planar geometry, and compute the mechanical impedance defined on the exterior surface of the elastic slab. It is given through a relation between the traction tensor and the displacement. In Fourier space it is a pseudodifferential operator of order one. After computing the exact mechanical impedance, we give an approximate impedance problem at order three with respect to the thickness of the thin elastic layer.

Keywords: Linear elasticity, thin layer, mechanical impedance, Lamé system, abstract differential system, approximation.

Mathematics Subject Classifications: 35Q74, 35J25, 34B05, 34E05, 35S15.

1 Introduction

Studying physical phenomena defined on objects comprising thin parts required always a particular treatment. A thin part of a domain can be described through a geometry having a length which is small enough in one or two dimensions compared to the others [18]. A typical example of such structures are thin shells, which often give rise to numerical instabilities in the numerical approximation of the solution [3]. Notably, the small thickness of the thin shell requires a discretization at the same length scale. This leads to a huge number of meshes, costing computations, and not precise results. To avoid these numerical instabilities, an analytical solution was offered. It consists in replacing the initial problem by an equivalent one which doesn't take into account any more the thin part. Rather, it is written through a boundary condition called the *impedance boundary condition*.

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The notion of impedance boundary condition is widely used in the field of the scattering of time-harmonic electromagnetic or acoustic waves, where the obstacles are in general coated with thin shells. For the first time, an impedance boundary condition was formulated by Leontovich [14]; it was presented as the boundary condition modeling the penetration of a wave in an imperfectly conducting metal barrier. In the case of the electromagnetic waves system, the impedance boundary condition links the tangential components of the electric field to the magnetic field, and it is written on the exterior surface of the obstacle.

For some geometries, it is not possible to write the exact impedance condition. Therefore, approximations of different orders with respect to the thickness of the coating have been derived (Enquist and Nédélec [8], Senior and Volakis [19], Bartoli and Bendali [4], Bendali and Lemrabet [5], Haddar and Joly [12], Poignard [17], Yuferev and Ida [20]).

In [4], N. Bartoli and A. Bendali overcame the numerical instabilities through the development of an efficient Padé-like approximation for the case of the Helmholtz equation and developed numerical procedures to efficiently solve the related unusual boundary value problems. In [6], A. Bendali, F. Mbarek, K. Lemrabet, and P. Sébastien gave a Padé-like approximation of order three for the diffraction of a time-harmonic electromagnetic wave by an obstacle coated by a thin shell of a dielectric material, and proved the efficiency of the approximation. In [10], F. Z. Goffi, K. Lemrabet and T. Arens gave a Padé approximation at order three in the case of a multi-layered contrasted thin coating.

For the case of elastic deformable bodies, to which we are interested in this research, the state of the stress and displacement at a point is described through the linear elasticity theory [11]. Here, the use of the impedance boundary conditions was also initiated for modeling the effect of a thin coating of an elastic body (see for instance A. Abdallaoui [1]). Namely, the mechanical impedance [2] links the traction and the displacement on the surface separating the elastic body from the thin shell. Hence, it transforms the transmission boundary problem into an impedance boundary problem set on the fixed domain defining the elastic body.

In the present work, in section 2, we define the transmission problem for an elastic body Ω_- coated by a thin elastic shell Ω^δ, of small thickness $\delta > 0$. We define also the mechanical impedance as a boundary condition defined on the interface Γ separating Ω_- and Ω^δ, which depends strongly on the parameter δ. In section 3, we use the differential operators ∇ (gradient), $\nabla\cdot$ (divergence), and $\nabla\times$ (curl) to write the Lamé system in the thin shell and recall an existence and regularity result for a mixed boundary value problem used to define the impedance operator of the shell. In section 4, we consider the case of a thin slab $\Omega^\delta = \Gamma \times (0, \delta)$. We introduce the tangential gradient (∇_Γ), the tangential divergence (div_Γ), the surface vector curl ($\overrightarrow{\mathrm{curl}}_\Gamma$), and the scalar surface curl (curl_Γ) (for more details see [16]). These operators will be used for rewriting the Lamé system as a first order differential system for the normal variable, whose coefficients are differential operators in the tangential variable. The new formulation is called the abstract Lamé system. Notably, we set the vectors

$$\mathbf{X}\left(z\right) = \left(u_T, u_n\right)^t(\cdot, z)\,; \quad \mathbf{Y}\left(z\right) = \left(\mathbf{n} \times \left(\nabla \times \mathbf{u}\right), \nabla \cdot \mathbf{u}\right)^t(\cdot, z)\,,$$

where the notation ^{t} refers to the transpose of a vector field, u_T and u_n are the tangential and the normal components of the displacement field $\mathbf{u}$. We write the Lamé system as an abstract differential system of order one in the normal variable z

$$\frac{d}{dz}\begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix}(z) = \mathcal{M}\begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix}(z) + \mathbf{H}(z). \tag{1}$$

The entries of the matrix $\mathcal{M}$ are differential operators in the tangential directions, and $\mathbf{H}$ is a given vector function.

To any field W defined on Ω^δ, we associate a field $\widehat{W}$ through the partial Fourier transform

$$\widehat{W}(\xi, z) = \int_{\mathbb{R}^2} e^{ix\cdot\xi} W(x, z)\, \mathrm{d}x. \tag{2}$$

Hence, the system (1) can be written

$$\frac{d}{dz}\begin{pmatrix} \widehat{\mathbf{X}} \\ \widehat{\mathbf{Y}} \end{pmatrix}(z) = \mathcal{S}(\mathcal{M})\begin{pmatrix} \widehat{\mathbf{X}} \\ \widehat{\mathbf{Y}} \end{pmatrix}(z) + \widehat{\mathbf{H}}(z),$$

where $\mathcal{S}(\mathcal{M})$ (the symbol of $\mathcal{M}$, see for example [7]) is a function of ξ. Then, we compute the propagator $\exp(z\mathcal{S}(\mathcal{M}))$ to get

$$\begin{pmatrix} \widehat{\mathbf{X}} \\ \widehat{\mathbf{Y}} \end{pmatrix}(z) = \exp(z\mathcal{S}(\mathcal{M}))\begin{pmatrix} \widehat{\mathbf{X}} \\ \widehat{\mathbf{Y}} \end{pmatrix}(0) + \int_0^z (\exp((z-t)\,\mathcal{S}(\mathcal{M})))\,\widehat{\mathbf{H}}(t)\, dt.$$

In section 5, we link the traction vector $\mathbf{T}(z) = \left((\sigma\mathbf{n})_T, (\sigma\mathbf{n})_n\right)^t(\cdot, z)$ to the displacement vector $\mathbf{X}(z) = (u_T, u_n)^t(\cdot, z)$ and the shear and compression vector $\mathbf{Y}(z) = (\mathbf{n}\times(\nabla\times\mathbf{u}), \nabla\cdot\mathbf{u})^t(\cdot, z)$. Here, σ represents the stress tensor.

Using $\exp(z\mathcal{S}(M))$, we compute $\left(\widehat{\mathbf{X}}, \widehat{\mathbf{T}}\right)^t(\xi, z)$ as a function of $\left(\widehat{\mathbf{X}}, \widehat{\mathbf{T}}\right)^t(\xi, 0)$, such that

$$\begin{pmatrix} \widehat{\mathbf{X}} \\ \widehat{\mathbf{T}} \end{pmatrix}(\xi, z) = \begin{pmatrix} \widehat{N}_{11} & \widehat{N}_{12} \\ \widehat{N}_{21} & \widehat{N}_{22} \end{pmatrix}(\xi, z)\begin{pmatrix} \widehat{\mathbf{X}} \\ \widehat{\mathbf{T}} \end{pmatrix}(\xi, 0) + \left(\widehat{R}\widehat{\mathbf{H}}\right)(\xi, z).$$

For the homogeneous case, the impedance operator is given by its symbol $\widehat{Z_\delta}(\cdot) = -\left(\left(\widehat{N}_{22}\right)^{-1}\widehat{N}_{21}\right)(\cdot, \delta)$.

In the last section 6, we take an expansion of $\widehat{Z_\delta}$ with respect to the small parameter δ to get a Padé-like approximation $\widehat{Z_\delta^*}$ at order three. We prove the existence and uniqueness of the solution to the approximate impedance value problem.

2 Impedance problem

2.1 Notations

For Ω an open domain in $\mathbb{R}^3$, we introduce the following general notations that will be used throughout the paper: a spatial location $\mathbf{x} \in \mathbb{R}^3$ is written $\mathbf{x} = (x, z)$, such that $x = (x_1, x_2)$. The normal vector outwardly directed to the open set is denoted $\mathbf{n}$. Vector fields that are defined on Ω are written through their tangential and normal components, such as $\mathbf{u} = (u_T, u_n)^t$, for $u_T = \mathbf{n}\times(\mathbf{u}\times\mathbf{n})$ and $u_n = \mathbf{n}\cdot\mathbf{u}$. We highlight that for the sake of simplicity, we denote with bold letters only $3D$ vectors, while their tangential components are denoted with non-bold letters. The identity matrices of sizes 2 and 3 are denoted I_3 and I_2, respectively.

In this section, we define the transmission problem and rewrite it in the form of an impedance boundary problem.

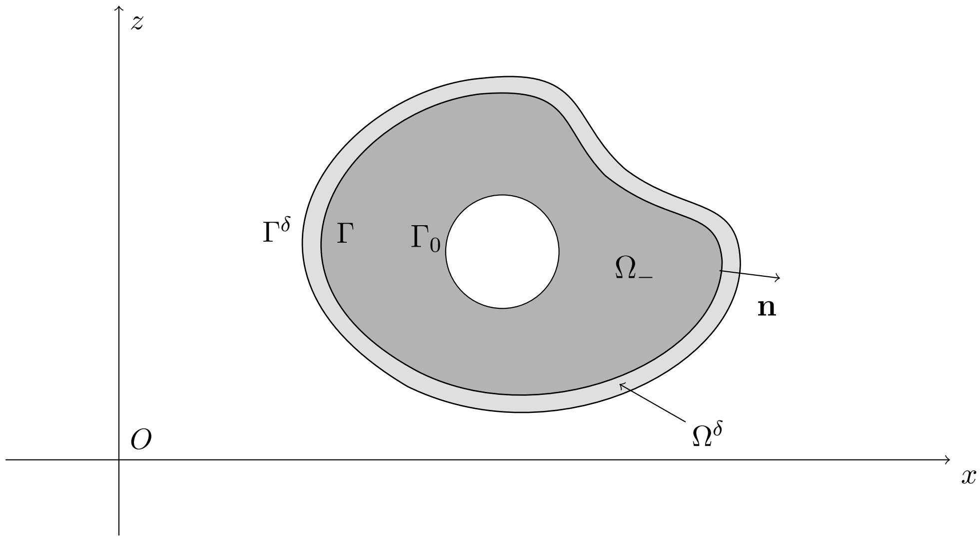

Figure 1: Bounded elastic body Ω_- of boundary $\partial\Omega_- = \Gamma \cup \Gamma_0$ and coated with an elastic thin layer Ω^δ of thickness δ and exterior boundary Γ^δ. $\mathbf{n}$ is the outward directed normal.

2.2 The transmission problem

Let Ω_- be a bounded and regular open set in $\mathbb{R}^3$ and $\partial\Omega_- = \Gamma \cup \Gamma_0$ its boundary, such that $\Gamma \cap \Gamma_0 = \emptyset$ (see Fig. 1). We set

$$\Omega^\delta = \{\mathbf{x} \notin \Omega_-;\ \mathrm{dist}\,(\mathbf{x}, \Gamma) < \delta\}\,, \quad \Gamma^\delta = \{\mathbf{x} \notin \Omega_-;\ \mathrm{dist}\,(\mathbf{x}, \Gamma) = \delta\}\,,$$

where $\delta > 0$ is a small parameter representing the thickness of the thin shell Ω^δ. We suppose that Ω_- and Ω^δ are elastic bodies and consider the following transmission boundary value problem:

1. Equilibrium equations
$$\begin{cases} \nabla \cdot \sigma_- (\mathbf{u}_-) + \mathbf{f}_- = 0 & \text{in } \Omega_- \\ \nabla \cdot \sigma (\mathbf{u}) + \mathbf{f} = 0 & \text{in } \Omega^\delta. \end{cases} \tag{3}$$
In the elastic thin shell Ω^δ, we refer by σ to the stress tensor, $\mathbf{u}$ to the displacement vector filed, and $\mathbf{f}$ is a given external volume force. Respectively, on the elastic body Ω_- the same ingredients are denoted through the index "$_-$".

2. Transmission conditions at the interface Γ
$$\begin{cases} \mathbf{u}_- = \mathbf{u} & \text{on } \Gamma \\ \sigma_- \mathbf{n} = \sigma \mathbf{n} & \text{on } \Gamma, \end{cases} \tag{4}$$
where $\mathbf{u}_-$ (resp. $\mathbf{u}$) is the trace of the displacement vector, defined on Ω_- (resp. Ω^δ), on the surface Γ, and $\mathbf{n}$ is the outward normal vector to Ω_- on Γ. These transmission conditions mean that the displacement and the traction are continuous through the interface Γ.

3. Dirichlet boundary condition on Γ_0
$$\mathbf{u}_- = 0 \quad \text{on } \Gamma_0. \tag{5}$$
It means that the body Ω_- is clamped on the part Γ_0 of its boundary.

4. Neumann boundary condition on Γ^{δ}

$$\sigma \mathbf{n} = \mathbf{g} \quad \text{on } \Gamma^{\delta}, \tag{6}$$

for $\mathbf{g}$ being a given external surface force applied on Γ^{δ}.

2.3 Impedance operator

It is well known that if the thickness δ of the elastic shell Ω^{δ} is small enough, solving the transmission problem via classical numerical methods is not convenient because of numerical instabilities.

We introduce the mechanical impedance of the thin shell Ω^{δ}, which allows to reduce the transmission problem set on $\Omega_{-} \cup \Omega^{\delta}$ (and depending on the small parameter δ) to a boundary value problem set on the fixed domain Ω_{-}. The equilibrium equation on Ω^{δ}, the transmission condition on Γ, and the boundary conditions on Γ^{δ} are embodied in the form a boundary condition depending on δ and set on Γ.

The mechanical impedance operator of the thin shell Ω^{δ} is given by

$$Z_{\delta}\left(\mathbf{u}_{|\Gamma}, \mathbf{f}, \mathbf{g}\right) = \sigma\left(\mathbf{u}\right)\mathbf{n}, \quad \text{on} \quad \Gamma, \tag{7}$$

such that

$$\mathbf{u} = \left(u_{1,} u_{2,} u_{n}\right)^{t} : \Omega^{\delta} \to \mathbb{R}^{3}$$

is the unique solution to the boundary value problem

$$\begin{cases} -\nabla \cdot \sigma\left(\mathbf{u}\right) = \mathbf{f} & \text{in } \Omega^{\delta} \\ \mathbf{u} = \varphi & \text{on } \Gamma \\ \sigma\left(\mathbf{u}\right)\mathbf{n} = \mathbf{g} & \text{on } \Gamma^{\delta}. \end{cases}$$

After defining the impedance operator, now we can set the impedance boundary problem defined through this operator.

2.4 Impedance boundary problem

Thanks to the transmission condition (4) written on the surface Γ, and the definition of the impedance operator (7) given in 2.3, we can write

$$\sigma\left(\mathbf{u}_{-}\right)\mathbf{n}_{|\Gamma} = \sigma\left(\mathbf{u}\right)\mathbf{n}_{|\Gamma} = Z_{\delta}\left(\mathbf{u}_{|\Gamma}, \mathbf{f}, \mathbf{g}\right) = Z_{\delta}\left(\mathbf{u}_{-|\Gamma}, \mathbf{f}, \mathbf{g}\right).$$

Solving the transmission problem (3)-(4)-(5)-(6) is then equivalent to solving the following impedance problem set on the fixed domain Ω_{-}:

$$\begin{cases} -\nabla \cdot \sigma\left(\mathbf{u}_{-}\right) = \mathbf{f}_{-} & \text{in } \Omega_{-} \\ \mathbf{u}_{-} = 0 & \text{on } \Gamma_{0} \\ \sigma\left(\mathbf{u}_{-}\right)\mathbf{n} = Z_{\delta}\left(\mathbf{u}_{-|\Gamma}, \mathbf{f}; \mathbf{g}\right) & \text{on } \Gamma. \end{cases}$$

The small parameter appears only in the impedance operator Z_{δ}. Here, we emphasize that we cannot reach the exact expression of this operator in all geometries. One can only compute its approximation with respect to the parameter δ.

3 The Lamé system on the thin shell

In the case of an isotropic linear elastic shell Ω^δ, the stress tensor $\sigma(\mathbf{u})$ and the deformation tensor $\varepsilon(\mathbf{u}) := \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^t\right)$ are linked by the Hooke's law [11]

$$\sigma(\mathbf{u}) = \lambda \operatorname{tr}_3(\varepsilon(\mathbf{u})) I_3 + 2\mu\varepsilon(\mathbf{u}), \tag{8}$$

where the Lamé coefficients λ and μ characterize the elastic body Ω^δ ($\mu > 0, \lambda \geq 0$). Here, tr_3 is the trace of a matrix of size 3.

3.1 Well known results for the Lamé system

Theorem 1. *For a given* $\mathbf{f}$ *in* $\mathbb{L}^2\left(\Omega^\delta\right)$, $\boldsymbol{\varphi}$ *in* $\mathbb{H}^{\frac{1}{2}}(\Gamma)$, *and* $\mathbf{g}$ *in* $\mathbb{H}^{-\frac{1}{2}}\left(\Gamma^\delta\right)$, *the boundary value problem*

$$\begin{cases} -\nabla\cdot\sigma(\mathbf{u}) = \mathbf{f} & \text{in } \Omega^\delta \\ \mathbf{u} = \boldsymbol{\varphi} & \text{on } \Gamma \\ \sigma(\mathbf{u})\,\mathbf{n} = \mathbf{g} & \text{on } \Gamma^\delta, \end{cases}$$

has a unique variational solution $\mathbf{u}$ *in* $\mathbb{H}^1\left(\Omega^\delta\right)$. *Moreover, for* $\mathbf{f}$ *in* $\mathbb{H}^s\left(\Omega^\delta\right)$, $\boldsymbol{\varphi}$ *in* $\mathbb{H}^{s+\frac{3}{2}}(\Gamma)$, *and* $\mathbf{g}$ *in* $\mathbb{H}^{s+\frac{1}{2}}\left(\Gamma^\delta\right)$ *the solution* $\mathbf{u}$ *is in* $\mathbb{H}^{s+2}\left(\Omega^\delta\right)$ *(See for instance [15]).*

We emphasize that the problem defining the impedance operator is linear, we have

$$Z_\delta(\boldsymbol{\varphi}, \mathbf{f}, \mathbf{g}) = Z_\delta(\boldsymbol{\varphi}, 0, 0) + Z_\delta(0, \mathbf{f}, 0) + Z_\delta(0, 0, \mathbf{g}).$$

Our aim in what follows is to give explicit formula for the impedance operator by expressing each of its three terms.

3.2 Other expression for the Lamé system

From the definition of the deformation tensor ε, we have $\operatorname{tr}_3\varepsilon(\mathbf{u}) = \nabla\cdot\mathbf{u}$. Further, we have the following identities $\nabla\cdot((\nabla\cdot\mathbf{u})\,I_3) = \nabla(\nabla\cdot\mathbf{u})$ and $\nabla\cdot\left((\nabla\mathbf{u})^T\right) = \nabla(\nabla\cdot\mathbf{u})$. We recall also the identity $\overrightarrow{\Delta}\mathbf{u} = \nabla(\nabla\cdot\mathbf{u}) - \nabla\times(\nabla\times\mathbf{u})$, such that $\overrightarrow{\Delta}$ is the vector laplacian. Using this fact, we can write the Lamé system $\nabla\cdot\sigma(\mathbf{u}) = -\mathbf{f}$ as

$$(\lambda + 2\mu)\,\nabla(\nabla\cdot\mathbf{u}) - \mu\nabla\times(\nabla\times\mathbf{u}) = -\mathbf{f}. \tag{9}$$

4 The Lamé system in the slab

As the exact expression of the impedance operator is not reachable in all geometries, we are going to consider in the sequel only the case of a thin slab $\Omega^\delta = \mathbb{R}^2\times(0,\delta)$ of thickness $\delta > 0$, with the boundaries $\Gamma = \mathbb{R}^2\times\{0\}$ and $\Gamma^\delta = \mathbb{R}^2\times\{\delta\}$, we set also $\Gamma_0 = \mathbb{R}^2\times\{-1\}$ (see Fig. 2). We write an exact impedance boundary condition using a partial Fourier transform and give stable approximations with respect to the small parameter δ.

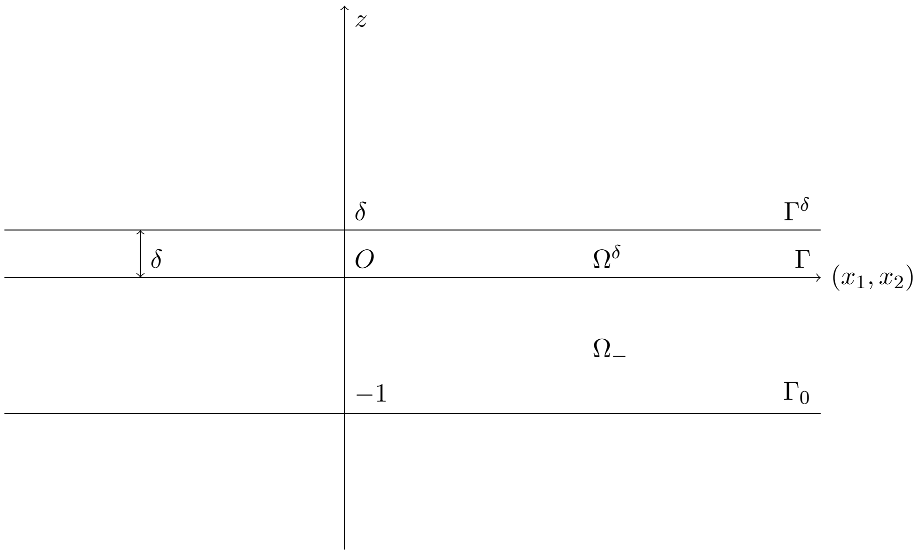

Figure 2: Thin elastic slab Ω_- coated with an elastic thin layer Ω^δ of thickness δ.

4.1 Surface differential operators

On the surface Γ, we recall the definitions of the surface differential operators.

For a scalar function $\phi : \Omega^\delta \to \mathbb{R}$, which is well defined on the neighborhood of the surface Γ, we define the surface gradient applied on ϕ as $\nabla_\Gamma \phi := \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}\right)^t$ and the surface vector curl as $\overrightarrow{\operatorname{curl}}_\Gamma \phi := \left(\frac{\partial \phi}{\partial x_2}, -\frac{\partial \phi}{\partial x_1}\right)^t = -\left(\mathbf{n} \times \nabla_\Gamma \phi\right)^t$.

For a tangent vector field $v_T : \Omega^\delta \to \mathbb{R}^2$, written $v_T = (v_1, v_2)^t$, we define the surface divergence of v_T as $\operatorname{div}_\Gamma v_T := \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}$ and the scalar surface curl as $\operatorname{curl}_\Gamma v_T := -\operatorname{div}_\Gamma \left(\mathbf{n} \times v_T\right)$.

4.2 Expression of the traction

For describing the stress intensity at a given point, it is sufficient to write the traction vector $\mathbf{T}$, representing the variation of the resultant forces with respect to an infinitesimal oriented area. By definition, the traction vector is written

$$\mathbf{T} := \sigma \mathbf{n},$$

which can be decomposed through its tangential and normal components

$$\mathbf{T} = \left((\sigma \mathbf{n})_T, (\sigma \mathbf{n})_n\right)^t.$$

From the Hooke's law (8), we have

$$\begin{aligned} \sigma(\mathbf{u})\,\mathbf{n} &= \big(\lambda \operatorname{tr}_3 \varepsilon(\mathbf{u})\, I_3 + 2\mu \varepsilon(\mathbf{u})\big)\mathbf{n} \\ &= \lambda (\nabla \cdot \mathbf{u})\,\mathbf{n} + 2\mu (\nabla \mathbf{u})\,\mathbf{n} - \mu\big(\nabla \mathbf{u} - (\nabla \mathbf{u})^t\big)\mathbf{n}. \end{aligned}$$

A simple algebraic computation gives

$$(\nabla \mathbf{u})\,\mathbf{n} = \begin{pmatrix} \nabla_\Gamma u_n \\ \frac{\partial u_n}{\partial z} \end{pmatrix},$$

and

$$\left(\nabla \mathbf{u} - (\nabla \mathbf{u})^t\right)\mathbf{n} = \mathbf{n} \times (\nabla \times \mathbf{u}) \, .$$

Finally, we obtain

$$\sigma(\mathbf{u})\,\mathbf{n} = \begin{pmatrix} 2\mu \nabla_\Gamma u_n - \mu\, \mathbf{n} \times (\nabla \times \mathbf{u}) \\ 2\mu \dfrac{\partial u_n}{\partial z} + \lambda (\nabla \cdot \mathbf{u}) \end{pmatrix} .$$

In the elasticity theory, the term $(\nabla \cdot \mathbf{u})(x,z)$ represents the compression at (x,z) and $\mathbf{n} \times (\nabla \times \mathbf{u})(x,z)$ is the shear stress at (x,z).

4.3 The Lamé system as an abstract differential system

A crucial step for writing the mechanical impedance is to rewrite the Lamé system as a differential system whose coefficients are differential operators. Thus the terminology *abstract differential Lamé system*. Similar strategy was previously used for the case of the scattering theory of the electromagnetic waves (see for e.g. [9, 10]). The approach is based mainly on decomposing the differential operators in the PDEs according to the derivatives in the tangential and normal directions. At this regard, from the definition of the surface differential operators, we have

$$\begin{aligned} \nabla \times \mathbf{u} &= \begin{pmatrix} \overrightarrow{\operatorname{curl}}_\Gamma u_n - \dfrac{\partial}{\partial z}(\mathbf{u} \times \mathbf{n}) \\ \operatorname{curl}_\Gamma u_T \end{pmatrix} \\ \nabla \cdot \mathbf{u} &= \operatorname{div}_\Gamma u_T + \frac{\partial u_n}{\partial z}, \end{aligned} \tag{10}$$

which give

$$\begin{aligned} \frac{\partial}{\partial z} u_T &= -\mathbf{n} \times (\nabla \times \mathbf{u}) + \nabla_\Gamma u_n \\ \frac{\partial}{\partial z} u_n &= (\nabla \cdot \mathbf{u}) - \operatorname{div}_\Gamma u_T. \end{aligned}$$

From (10), we get

$$\nabla \times (\nabla \times \mathbf{u}) = \begin{pmatrix} \overrightarrow{\operatorname{curl}}_\Gamma (\nabla \times \mathbf{u})_n - \dfrac{\partial}{\partial z}\left[(\nabla \times \mathbf{u}) \times \mathbf{n}\right] \\ \operatorname{curl}_\Gamma (\nabla \times \mathbf{u})_T \end{pmatrix},$$

which leads to

$$\begin{aligned} \left[\nabla \times (\nabla \times \mathbf{u})\right]_T &= \overrightarrow{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma u_T + \frac{\partial}{\partial z}\left[\mathbf{n} \times (\nabla \times \mathbf{u})\right] \\ \left[\nabla \times (\nabla \times \mathbf{u})\right]_n &= -\operatorname{div}_\Gamma \left[\mathbf{n} \times (\nabla \times \mathbf{u})\right]. \end{aligned}$$

Hence, we can write the Lamé system (9) through its tangential and normal components as

$$\begin{aligned} (\lambda + 2\mu)\, \nabla_\Gamma (\nabla \cdot \mathbf{u}) - \mu \left[\nabla \times (\nabla \times \mathbf{u})\right]_T &= -f_T \\ (\lambda + 2\mu)\, \frac{\partial}{\partial z} (\nabla \cdot \mathbf{u}) - \mu \left[\nabla \times (\nabla \times \mathbf{u})\right]_n &= -f_n, \end{aligned}$$

and then

$$\mu \frac{\partial}{\partial z} \left[\mathbf{n} \times (\nabla \times \mathbf{u}) \right] = (\lambda + 2\mu) \nabla_\Gamma (\nabla \cdot \mathbf{u}) - \mu \overrightarrow{\mathrm{curl}}_\Gamma \, \mathrm{curl}_\Gamma \, u_T + f_T$$
$$(\lambda + 2\mu) \frac{\partial}{\partial z} (\nabla \cdot \mathbf{u}) = -\mu \, \mathrm{div}_\Gamma \left[\mathbf{n} \times (\nabla \times \mathbf{u}) \right] - f_n.$$

Next, we will be able to write the abstract differential Lamé system whose coefficients are differential operators in the tangent directions. We denote by $\mathcal{M}$ the coefficient matrix, written

$$\mathcal{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

for,

$$M_{11} = \begin{pmatrix} 0_2 & \nabla_\Gamma \\ -\,\mathrm{div}_\Gamma & 0 \end{pmatrix}, \qquad M_{12} = \begin{pmatrix} -I_2 & 0_{21} \\ 0_{12} & 1 \end{pmatrix},$$
$$M_{21} = \begin{pmatrix} -\overrightarrow{\mathrm{curl}}_\Gamma \, \mathrm{curl}_\Gamma & 0_{21} \\ 0_{12} & 0 \end{pmatrix}, \quad M_{22} = \begin{pmatrix} 0_2 & \frac{(\lambda+2\mu)}{\mu} \nabla_\Gamma \\ -\frac{\mu}{(\lambda+2\mu)} \, \mathrm{div}_\Gamma & 0 \end{pmatrix},$$

we refer by I_2 and 0_2 to the identity and zero matrices of order 2, respectively. While, $0_{21} = (0,0)^t$ and $0_{12} = (0,0)$. Furthermore, we write

$$\mathcal{L} = \begin{pmatrix} \frac{1}{\mu} I_2 & 0_{21} \\ 0_{12} & -\frac{1}{\lambda+2\mu} \end{pmatrix}.$$

Recall that

$$\mathbf{X}(z) = (u_T, u_n)^t (\cdot, z), \quad \mathbf{Y}(z) = (\mathbf{n} \times (\nabla \times \mathbf{u}), \nabla \cdot \mathbf{u})^t (\cdot, z).$$

Now, we can rewrite the abstract formula for the Lamé system as a differential system of order one in the normal variable z as

$$\frac{d}{dz} \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} (z) = \mathcal{M} \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} (z) + \begin{pmatrix} 0 \\ \mathcal{L}\mathbf{f} \end{pmatrix}. \tag{11}$$

We emphasize again that the entries of the coefficient matrix $\mathcal{M}$ are differential operators for the tangential variable x.

4.4 Linking the traction to the compression and the shear stress

The relation between the traction vector and the compression and the shear stress can be written explicitly from their definitions. Recall

$$\mathbf{T}(z) = \big(\sigma(\mathbf{u})\,\mathbf{n}\big)(z), \quad \text{such that} \quad \sigma(\mathbf{u})\,\mathbf{n} = \begin{pmatrix} 2\mu \nabla_\Gamma u_n - \mu \, \mathbf{n} \times (\nabla \times \mathbf{u}) \\ 2\mu \frac{\partial u_n}{\partial z} + \lambda (\nabla \cdot \mathbf{u}) \end{pmatrix}.$$

Then, we can write

$$\mathbf{T}(z) = P_1 \mathbf{X}(z) + P_2 \mathbf{Y}(z), \tag{12}$$

with

$$P_1 = \begin{pmatrix} 0_2 & 2\mu \nabla_\Gamma \\ -2\mu \, \mathrm{div}_\Gamma & 0 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} -\mu I_2 & 0_{21} \\ 0_{12} & (\lambda + 2\mu) \end{pmatrix}.$$

We can also write

$$\mathbf{Y}(z) = Q_1 \mathbf{X}(z) + Q_2 \mathbf{T}(z), \tag{13}$$

with

$$Q_1 = \begin{pmatrix} 0_2 & 2\nabla_\Gamma \\ \frac{2\mu}{(\lambda+2\mu)} \mathrm{div}_\Gamma & 0 \end{pmatrix} \text{ and } Q_2 = \begin{pmatrix} -\frac{1}{\mu} I_2 & 0_{21} \\ 0_{12} & \frac{1}{(\lambda+2\mu)} \end{pmatrix}.$$

From the definition of the operator $\mathcal{L}$, we can see that $Q_2 = -\mathcal{L}$. These formulas will be used in the next section for writing the mechanical impedance for a thin shell.

4.5 The differential system via Fourier transform

The solution $(\mathbf{X}, \mathbf{Y})^t(z)$ to the homogeneous Lamé system (11) for a given initial data $(\mathbf{X}, \mathbf{Y})^t(0)$ is written

$$\begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix}(z) = \exp(z\mathcal{M}) \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix}(0),$$

where $\exp(z\mathcal{M})$ can be obtained explicitly through the partial Fourier transform (2) by solving

$$\frac{d}{dz} \begin{pmatrix} \widehat{\mathbf{X}} \\ \widehat{\mathbf{Y}} \end{pmatrix}(z) = \mathcal{S}(\mathcal{M}) \begin{pmatrix} \widehat{\mathbf{X}} \\ \widehat{\mathbf{Y}} \end{pmatrix}(z). \tag{14}$$

Here, $\mathcal{S}(\mathcal{M})$ refers to the symbol of coefficient matrix $\mathcal{M}$ for the Lamé system (11). We write

$$\mathcal{S}(\mathcal{M}) = \begin{pmatrix} \mathcal{S}(M)_{11} & \mathcal{S}(M)_{12} \\ \mathcal{S}(M)_{21} & \mathcal{S}(M)_{22} \end{pmatrix}.$$

Direct computation of the Fourier Transform for each matrix M_{ij}, for $i, j = 1, 2$, gives

$$\mathcal{S}(M)_{11}(\xi) = \begin{pmatrix} 0_2 & i\xi \\ -i\xi^t & 0 \end{pmatrix}, \qquad \mathcal{S}(M)_{12}(\xi) = \begin{pmatrix} -I_2 & 0_{21} \\ 0_{12} & 1 \end{pmatrix},$$

$$\mathcal{S}(M)_{21}(\xi) = \begin{pmatrix} (i\xi \times \mathbf{n})(i\xi \times \mathbf{n})^t & 0_{21} \\ 0_{12} & 0 \end{pmatrix}, \quad \mathcal{S}(M)_{22}(\xi) = \begin{pmatrix} 0_2 & \frac{(\lambda+2\mu)}{\mu} i\xi \\ -\frac{\mu}{(\lambda+2\mu)} i\xi^t & 0 \end{pmatrix},$$

such that

$$i\xi = \begin{pmatrix} i\xi_1 \\ i\xi_2 \end{pmatrix}, \qquad (i\xi \times \mathbf{n}) = \begin{pmatrix} i\xi_2 \\ -i\xi_1 \end{pmatrix},$$

and

$$(i\xi)(i\xi)^t = -\begin{pmatrix} \xi_1^2 & \xi_1\xi_2 \\ \xi_1\xi_2 & \xi_2^2 \end{pmatrix}, \quad (i\xi \times \mathbf{n})(i\xi \times \mathbf{n})^t = -\begin{pmatrix} \xi_2^2 & -\xi_1\xi_2 \\ -\xi_1\xi_2 & \xi_1^2 \end{pmatrix}.$$

Finally, we write the solution for the non homogeneous Lamé system (11). For a given initial data $(\mathbf{X}, \mathbf{Y})^t(0)$, we write

$$\begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix}(z) = \exp(z\mathcal{M}) \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix}(0) + \int_0^z \exp((z-t)\mathcal{M}) \begin{pmatrix} 0 \\ \mathcal{L}\mathbf{f} \end{pmatrix}(t)\, dt.$$

More precisely, we have

$$\mathbf{X}(z) = (\exp(z\mathcal{M}))_{11} \mathbf{X}(0) + (\exp(z\mathcal{M}))_{12} \mathbf{Y}(0) + \int_0^z (\exp((z-t)\mathcal{M}))_{12} \mathcal{L}\mathbf{f}(t)\, dt \tag{15}$$

$$\mathbf{Y}(z) = (\exp(z\mathcal{M}))_{21} \mathbf{X}(0) + (\exp(z\mathcal{M}))_{22} \mathbf{Y}(0) + \int_0^z (\exp((z-t)\mathcal{M}))_{22} \mathcal{L}\mathbf{f}(t)\, dt \tag{16}$$

In the next paragraph, we compute the exponential of the symbol of $\mathcal{M}$ explicitly.

4.6 Computation of $\exp\left(z\mathcal{S}\left(\mathcal{M}\right)\right)$

We recall that for $\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}\right)^t(0)$ being an initial data, the solution to the homogeneous Lamé system (14) is written

$$\begin{pmatrix} \widehat{\mathbf{X}} \\ \widehat{\mathbf{Y}} \end{pmatrix}(z) = \exp\left(z\mathcal{S}\left(\mathcal{M}\right)\right)\begin{pmatrix} \widehat{\mathbf{X}} \\ \widehat{\mathbf{Y}} \end{pmatrix}(0).$$

For writing the solution explicitly, we have to compute the exponential of the matrix given through the symbol of $\mathcal{M}$, which is not an evident task. In this case, we proceed in a non direct way. We start by solving the second equation for the system (14) by means of the Helmholtz decomposition for the displacement field. After that, we solve the first equation.

At first, we use the Helmholtz decomposition of $\mathbf{u}$, written $\mathbf{u} = \Phi + \Psi$, with $\nabla \times \Phi = 0$ and $\nabla \cdot \Psi = 0$. The homogeneous Lamé system (9) is then written

$$(\lambda + 2\mu)\nabla\left(\nabla \cdot \Phi\right) - \mu\nabla \times \left(\nabla \times \Psi\right) = 0.$$

By applying independently the $\nabla\cdot$ operator, at first, on the last formula for the Lamé system and then the $\nabla\times$ operator, we obtain

$$\begin{aligned} \Delta\left(\nabla \cdot \mathbf{u}\right) &= \Delta\left(\nabla \cdot \Phi\right) = 0 \\ \overrightarrow{\Delta}\left(\nabla \times \mathbf{u}\right) &= \overrightarrow{\Delta}\left(\nabla \times \Psi\right) = 0. \end{aligned}$$

After applying the partial Fourier transform on the system, the solution to the second order ODE system for the $z-$variable is written

$$\widehat{\mathbf{Y}}\left(\xi, z\right) = \mathbf{E}\left(\xi\right)\exp(-(\delta - z)|\xi|) + \mathbf{F}\left(\xi\right)\exp\left(-z|\xi|\right).$$

For a fixed ξ and $z = 0$, we have the following relation

$$e^{-\delta|\xi|}\mathbf{E} + \mathbf{F} = \widehat{\mathbf{Y}}\left(0\right). \tag{17}$$

From the system (14), we have

$$\frac{d}{dz}\widehat{\mathbf{Y}} = \mathcal{S}\left(M\right)_{21}\widehat{\mathbf{X}} + \mathcal{S}\left(M\right)_{22}\widehat{\mathbf{Y}}.$$

Thus

$$|\xi|\left(e^{-\delta|\xi|}\mathbf{E} - \mathbf{F}\right) = \mathcal{S}\left(M\right)_{21}\widehat{\mathbf{X}}\left(0\right) + \mathcal{S}\left(M\right)_{22}\widehat{\mathbf{Y}}\left(0\right). \tag{18}$$

From (17) and (18), we obtain the following two equations linking $\mathbf{E}$ and $\mathbf{F}$ to $\widehat{\mathbf{X}}\left(0\right)$ and $\widehat{\mathbf{Y}}\left(0\right)$, written

$$\begin{aligned} 2e^{-\delta|\xi|}\mathbf{E} &= \left(I + \frac{1}{|\xi|}\mathcal{S}\left(M\right)_{22}\right)\widehat{\mathbf{Y}}\left(0\right) + \frac{1}{|\xi|}\mathcal{S}\left(M\right)_{21}\widehat{\mathbf{X}}\left(0\right) \\ 2\mathbf{F} &= \left(I - \frac{1}{|\xi|}\mathcal{S}\left(M\right)_{22}\right)\widehat{\mathbf{Y}}\left(0\right) - \frac{1}{|\xi|}\mathcal{S}\left(M\right)_{21}\widehat{\mathbf{X}}\left(0\right), \end{aligned}$$

and

$$
\begin{aligned}
e^{(-\delta|\xi|)}\widehat{\mathbf{Y}}(z) =& \frac{e^{-(\delta-z)|\xi|}}{2}\left(\frac{1}{|\xi|}\mathcal{S}(M)_{21}\widehat{\mathbf{X}}(0)+\left(I+\frac{1}{|\xi|}\mathcal{S}(M)_{22}\right)\widehat{\mathbf{Y}}(0)\right) \\
&+\frac{e^{-(\delta+z)|\xi|}}{2}\left(-\frac{1}{|\xi|}\mathcal{S}(M)_{21}\widehat{\mathbf{X}}(0)+\left(I-\frac{1}{|\xi|}\mathcal{S}(M)_{22}\right)\widehat{\mathbf{Y}}(0)\right) \\
=& \frac{\left(e^{-(\delta-z)|\xi|}-e^{-(\delta+z)|\xi|}\right)}{2|\xi|}\mathcal{S}(M)_{21}\widehat{\mathbf{X}}(0) \\
&+\left(\frac{\left(e^{-(\delta-z)|\xi|}+e^{-(\delta+z)|\xi|}\right)}{2}+\frac{\left(e^{-(\delta-z)|\xi|}-e^{-(\delta+z)|\xi|}\right)}{2|\xi|}\mathcal{S}(M)_{22}\right)\widehat{\mathbf{Y}}(0).
\end{aligned}
\tag{19}
$$

We emphasize that through the Helmholtz decomposition for the displacement field, we could explicitly write the field $\mathbf{Y}$ expressing the shear stress and compression. This last, will be used for writing the solution to

$$
\frac{d}{dz}\widehat{\mathbf{X}}=\mathcal{S}(M)_{11}\widehat{\mathbf{X}}+\mathcal{S}(M)_{12}\widehat{\mathbf{Y}}.
$$

From the ODEs theory, the solution $\mathbf{X}$ to the non homogeneous system is written in the following integral form

$$
\widehat{\mathbf{X}}(z)=\exp\left(z\mathcal{S}(M)_{11}\right)\widehat{\mathbf{X}}(0)+\int_0^z\exp\Big((z-t)\,\mathcal{S}(M)_{11}\Big)\mathcal{S}(M)_{12}\widehat{\mathbf{Y}}(t)\,dt,
$$

which is expressed at the end in terms of $\widehat{\mathbf{X}}(0)$ and $\widehat{\mathbf{Y}}(0)$. For writing explicitly the solution, we make use of computer algebra tools. At first, we have

$$
\exp\Big(z\mathcal{S}(M)_{11}\Big)=\begin{pmatrix}
-\frac{1}{|\xi|^2}(i\xi\times\mathbf{n})(i\xi\times\mathbf{n})^t-\frac{\left(e^{z|\xi|}+e^{-z|\xi|}\right)}{2|\xi|^2}(i\xi)(i\xi)^t & \frac{\left(e^{z|\xi|}-e^{-z|\xi|}\right)}{2|\xi|}i\xi \\
-\frac{\left(e^{z|\xi|}-e^{-z|\xi|}\right)}{2|\xi|}i\xi^t & \frac{\left(e^{z|\xi|}+e^{-z|\xi|}\right)}{2}
\end{pmatrix}.
$$

Setting

$$
\begin{aligned}
2A(z)&=\int_0^z\Big(\exp(z-t)\,\mathcal{S}(M)_{11}\Big)\exp(t|\xi|)\,dt \\
2B(z)&=\int_0^z\Big(\exp(z-t)\,\mathcal{S}(M)_{11}\Big)\exp(-t|\xi|)\,dt,
\end{aligned}
$$

we can write

$$
\begin{aligned}
\widehat{\mathbf{X}}(z)=&\left(e^{z\mathcal{S}(M)_{11}}+\Big(A(z)-B(z)\Big)\frac{1}{|\xi|}\mathcal{S}(M)_{21}\right)\widehat{\mathbf{X}}(0) \\
&+\left(\Big(A(z)+B(z)\Big)\mathcal{S}(M)_{12}+\Big(A(z)-B(z)\Big)\frac{1}{|\xi|}\mathcal{S}(M)_{12}\mathcal{S}(M)_{22}\right)\widehat{\mathbf{Y}}(0).
\end{aligned}
\tag{20}
$$

Straightforward calculations give

$$
A(z)=\begin{pmatrix}
-\frac{\left(e^{z|\xi|}-1\right)}{|\xi|^3}(i\xi\times\mathbf{n})(i\xi\times\mathbf{n})^t-\frac{\left(e^{z|\xi|}-e^{-z|\xi|}\right)+2z|\xi|e^{z|\xi|}}{4|\xi|^3}(i\xi)(i\xi)^t & -\frac{\left(e^{z|\xi|}-e^{-z|\xi|}\right)-2z|\xi|e^{z|\xi|}}{4|\xi|^2}i\xi \\
\frac{\left(e^{z|\xi|}-e^{-z|\xi|}\right)-2z|\xi|e^{z|\xi|}}{4|\xi|^2}(i\xi)^t & \frac{\left(e^{z|\xi|}-e^{-z|\xi|}\right)+2z|\xi|e^{z|\xi|}}{4|\xi|}
\end{pmatrix}
$$

$$B(z) = \begin{pmatrix} \frac{\left(e^{-z|\xi|}-1\right)}{|\xi|^3} (i\xi \times \mathbf{n}) (i\xi \times \mathbf{n})^t - \frac{\left(e^{z|\xi|}-e^{-z|\xi|}\right)+2z|\xi|e^{-z|\xi|}}{4|\xi|^3} (i\xi) (i\xi)^t & \frac{\left(e^{z|\xi|}-e^{-z|\xi|}\right)-2z|\xi|e^{-z|\xi|}}{4|\xi|^2} i\xi \\ -\frac{\left(e^{z|\xi|}-e^{-z|\xi|}\right)-2z|\xi|e^{-z|\xi|}}{4|\xi|^2} (i\xi)^t & \frac{\left(e^{z|\xi|}-e^{-z|\xi|}\right)+2z|\xi|e^{-z|\xi|}}{4|\xi|} \end{pmatrix}.$$

Finally, after writing the solutions $\mathbf{X}$ and $\mathbf{Y}$, given by (20) and (19), we can deduce the formula of $\exp(z\mathcal{S}(\mathcal{M}))$, written

$$\exp(z\mathcal{S}(\mathcal{M})) = \begin{pmatrix} \left(e^{z\mathcal{S}(\mathcal{M})}\right)_{11} & \left(e^{z\mathcal{S}(\mathcal{M})}\right)_{12} \\ \left(e^{z\mathcal{S}(\mathcal{M})}\right)_{21} & \left(e^{z\mathcal{S}(\mathcal{M})}\right)_{22} \end{pmatrix}.$$

Setting $\alpha = \frac{\mu}{\lambda+2\mu}$, we get

- $\left(e^{z\mathcal{S}(\mathcal{M})}\right)_{11}(\xi, z) = \cosh z|\xi| \begin{pmatrix} I_2 & \frac{1}{|\xi|}(\tanh z|\xi|)\, i\xi \\ -\frac{1}{|\xi|}(\tanh z|\xi|)\, i\xi^t & 1 \end{pmatrix}$
- $\left(e^{z\mathcal{S}(\mathcal{M})}\right)_{12}(\xi, z) =$

$$\cosh z|\xi| \begin{pmatrix} \begin{array}{l} -z(1-\alpha)\dfrac{i\xi\,(i\xi)^t}{2|\xi|^2} + \\ \tanh z|\xi| \left(-(1+\alpha)\dfrac{i\xi\,(i\xi)^t}{2|\xi|^3} + 2\dfrac{(i\xi \times \mathbf{n})(i\xi \times \mathbf{n})^t}{2|\xi|^3}\right) \end{array} & -z(\tanh z|\xi|)\left(\frac{1-\alpha}{\alpha}\right)\frac{i\xi}{2|\xi|} \\ z(\tanh z|\xi|)(1-\alpha)\frac{i\xi^t}{2|\xi|} & -z\frac{1}{2}\left(\frac{1-\alpha}{\alpha}\right) - \frac{(\tanh z|\xi|)}{8|\xi|}\left(\frac{1-2\alpha}{\alpha}\right) \end{pmatrix}$$

- $\left(e^{z\mathcal{S}(\mathcal{M})}\right)_{21}(\xi, z) = \cosh z|\xi| \begin{pmatrix} (\tanh z|\xi|)\frac{1}{|\xi|}(i\xi \times \mathbf{n})(i\xi \times \mathbf{n})^t & 0 \\ 0 & 0 \end{pmatrix}$
- $\left(e^{z\mathcal{S}(\mathcal{M})}\right)_{22}(\xi, z) = \cosh z|\xi| \begin{pmatrix} I_2 & (\tanh z|\xi|)\frac{1}{\alpha}\frac{i\xi}{|\xi|} \\ -(\tanh z|\xi|)\,\alpha\frac{i\xi^t}{|\xi|} & 1 \end{pmatrix}.$

These formulas will be useful for the next section, in which we compute the mechanical impedance for a slab elastic body.

5 Mechanical impedance boundary condition

A mechanical impedance condition is a relation linking the displacement vector $\mathbf{X}(z) = (u_T, u_n)^t(\cdot, z)$ to the shear and compression vector $\mathbf{Y}(z) = (\mathbf{n} \times (\nabla \times \mathbf{u}), \nabla \cdot \mathbf{u})^t(\cdot, z)$. It can also be seen as a relation linking the displacement vector $\mathbf{X}(z) = (u_T, u_n)^t(\cdot, z)$ to the traction field vector $\mathbf{T}(z) = ((\sigma\mathbf{n})_T, (\sigma\mathbf{n})_n)^t(\cdot, z)$ (see [19]). In the following section we show how to transfer the mechanical impedance from the exterior surface of the elastic thin body to the surface of the fixed part.

5.1 Transfer of the impedance condition through the slab

From the Neumann boundary condition (6), we have

$$\mathbf{g} = \mathbf{T}(\delta).$$

Further, from the relation (12) linking the traction to the displacement, shear, and compression, the impedance condition defined on the surface Γ^δ is written

$$\mathbf{g} = P_1\mathbf{X}(\delta) + P_2\mathbf{Y}(\delta).$$

When replacing the fields $\mathbf{X}(\delta)$ and $\mathbf{Y}(\delta)$ by their explicit formulas (15)-(16), the impedance condition can be transferred from the surface Γ^δ to the impedance condition on the boundary Γ as follows:

$$\begin{aligned}\mathbf{g} =& \left[P_1 (\exp\delta\mathcal{M})_{11} + P_2 (\exp\delta\mathcal{M})_{21}\right]\mathbf{X}(0) + \left[P_1 (\exp\delta\mathcal{M})_{12} + P_2 (\exp\delta\mathcal{M})_{22}\right]\mathbf{Y}(0) \\ &+ \int_0^z \Big(P_1 \big(\exp\big((\delta - t)\mathcal{M}\big)\big)_{12} + P_2 \big(\exp\big((\delta - t)\mathcal{M}\big)\big)_{22}\Big)\mathcal{L}\mathbf{f}(t)\,dt.\end{aligned}$$

From (13), we have

$$\mathbf{Y}(0) = Q_1\mathbf{X}(0) + Q_2\mathbf{T}(0).$$

Thus, the transferred impedance condition on the surface Γ is written

$$\begin{aligned}\mathbf{g} =& \begin{bmatrix} P_1\left[(\exp\delta\mathcal{M})_{11} + (\exp\delta\mathcal{M})_{12} Q_1\right] \\ +P_2\left[(\exp\delta\mathcal{M})_{21} + (\exp\delta\mathcal{M})_{22} Q_1\right]\end{bmatrix}\mathbf{X}(0) \\ &+ \left[P_1 (\exp\delta\mathcal{M})_{12} + P_2 (\exp\delta\mathcal{M})_{22}\right] Q_2\mathbf{T}(0) \\ &+ \int_0^z \Big(P_1 \big(\exp\big((\delta - t)\mathcal{M}\big)\big)_{12} + P_2 \big(\exp\big((\delta - t)\mathcal{M}\big)\big)_{22}\Big)\mathcal{L}\mathbf{f}(t)\,dt.\end{aligned}$$

The last formula represents another way for writing the impedance condition on the surface Γ. This formula will be given explicitly in the next sub-section by writing the exact formula for the impedance operator.

However, we can consider the following two special cases. If we consider the boundary condition $\mathbf{Y}(\delta) = \mathbf{g}$, it gives

$$\mathbf{Y}(0) = \big((\exp\delta\mathcal{M})_{22}\big)^{-1}\left(\mathbf{g} - (\exp\delta\mathcal{M})_{21}\mathbf{X}(0) - \int_0^\delta \big(\exp((\delta - t)\mathcal{M})\big)_{22}\mathcal{L}\mathbf{f}(t)\,dt\right),$$

or

$$\begin{aligned}\mathbf{g} =& \big((\exp\delta\mathcal{M})_{21} + (\exp\delta\mathcal{M})_{22} Q_1\big)\mathbf{X}(0) + (\exp\delta\mathcal{M})_{22} Q_2\mathbf{T}(0) \\ &+ \int_0^\delta \big(\exp((\delta - t)\mathcal{M})\big)_{22}\mathcal{L}\mathbf{f}(t)\,dt.\end{aligned}$$

Else, the boundary condition $\mathbf{X}(\delta) = \mathbf{g}$ can be transferred into

$$\mathbf{X}(0) = \big((\exp\delta\mathcal{M})_{11}\big)^{-1}\left(\mathbf{g} - (\exp\delta\mathcal{M})_{12}\mathbf{Y}(0) - \int_0^\delta \big(\exp((\delta - t)\mathcal{M})\big)_{12}\mathcal{L}\mathbf{f}(t)\,dt\right).$$

From these formulations, we need to set some properties of the operators $(\exp\delta\mathcal{M})_{22}$ and $(\exp\delta\mathcal{M})_{11}$, that we show in the following proposition.

Proposition 2. *1. Let*

$$\frac{1}{\cosh z\,|\xi|}\left(e^{z\mathcal{S}(\mathcal{M})}\right)_{22}(\xi,z)=\begin{pmatrix} I_2 & (\tanh z\,|\xi|)\,\frac{1}{\alpha}\frac{i\xi}{|\xi|} \\ -(\tanh z\,|\xi|)\,\alpha\frac{i\xi^t}{|\xi|} & 1\end{pmatrix}.$$

For fixed $z\geq 0$, the operator

$$\begin{array}{rccc} \frac{1}{\cosh z|\xi|}\left(e^{z\mathcal{S}(\mathcal{M})}\right)_{22}: & \mathbb{H}^s\left(\mathbb{R}^2;\mathbb{C}^3\right) & \to & \mathbb{H}^s\left(\mathbb{R}^2;\mathbb{C}^3\right) \\ & \phi & \to & \frac{1}{\cosh z|\xi|}\left(e^{z\mathcal{S}(\mathcal{M})}\right)_{22}\phi \end{array}$$

is linear and continuous for every s in $\mathbb{R}$.

2. The matrix $\frac{1}{\cosh z|\xi|}\left(e^{z\mathcal{S}(\mathcal{M})}\right)_{22}$ is invertible with

$$\left(\frac{1}{\cosh z\,|\xi|}\left(e^{z\mathcal{S}(\mathcal{M})}\right)_{22}\right)^{-1}$$
$$=\begin{pmatrix} -\frac{1}{|\xi|^2}(i\xi\times\mathbf{n})(i\xi\times\mathbf{n})^t-\frac{1}{\left(1-\tanh^2 z|\xi|\right)}\frac{1}{|\xi|^2}(i\xi)(i\xi)^t & -\frac{1}{\left(1-\tanh^2 z|\xi|\right)}\frac{1}{\alpha}\frac{(\tanh z|\xi|)}{|\xi|}i\xi \\ \frac{1}{\left(1-\tanh^2 z|\xi|\right)}\alpha\frac{(\tanh z|\xi|)}{|\xi|}i\xi^t & \frac{1}{\left(1-\tanh^2 z|\xi|\right)}\end{pmatrix}.$$

3. Let

$$\mathcal{H}^s_{\cosh}\left(\mathbb{R}^2;\mathbb{C}^3\right)=\left\{\phi\in\mathbb{H}^s\left(\mathbb{R}^2;\mathbb{C}^3\right):\phi\cosh z\,|\xi|\in\mathbb{H}^s\left(\mathbb{R}^2;\mathbb{C}^3\right)\right\}.$$

The operator

$$\begin{array}{rccc} \left(\left(e^{z\mathcal{S}(\mathcal{M})}\right)_{22}\right)^{-1}: & \mathcal{H}^s_{\cosh}\left(\mathbb{R}^2;\mathbb{C}^3\right) & \to & \mathcal{H}^s_{\cosh}\left(\mathbb{R}^2;\mathbb{C}^3\right) \\ & \phi & \to & \left(\left(e^{z\mathcal{S}(\mathcal{M})}\right)_{22}\right)^{-1}\phi \end{array}$$

is linear and continuous for $z\geq 0$.

Proof. 1. The coefficients of the matrix

$$\frac{1}{\cosh z\,|\xi|}\left(e^{z\mathcal{S}(\mathcal{M})}\right)_{22}(\xi,z)=\begin{pmatrix} 1 & 0 & (\tanh z\,|\xi|)\,\frac{1}{\alpha}\frac{i\xi_1}{|\xi|} \\ 0 & 1 & (\tanh z\,|\xi|)\,\frac{1}{\alpha}\frac{i\xi_2}{|\xi|} \\ -(\tanh z\,|\xi|)\,\alpha\frac{i\xi_1}{|\xi|} & -(\tanh z\,|\xi|)\,\alpha\frac{i\xi_2}{|\xi|} & 1\end{pmatrix}$$

are bounded.

2. With the help of an algebraic computation software, we can get

$$\left(\frac{1}{\cosh z\,|\xi|}\left(e^{z\mathcal{S}(\mathcal{M})}\right)_{22}(\xi,z)\right)^{-1}$$
$$=\begin{pmatrix} \frac{\xi_2^2}{|\xi|^2}+\frac{1}{\left(1-\tanh^2 z|\xi|\right)}\frac{\xi_1^2}{|\xi|^2} & -\frac{\xi_1\xi_2}{|\xi|^2}+\frac{1}{\left(1-\tanh^2 z|\xi|\right)}\frac{\xi_1\xi_2}{|\xi|^2} & -\frac{1}{\left(1-\tanh^2 z|\xi|\right)}\frac{1}{\alpha}\frac{(\tanh z|\xi|)}{|\xi|}i\xi_1 \\ -\frac{\xi_1\xi_2}{|\xi|^2}+\frac{1}{\left(1-\tanh^2 z|\xi|\right)}\frac{\xi_1\xi_2}{|\xi|^2} & \frac{\xi_1^2}{|\xi|^2}+\frac{1}{\left(1-\tanh^2 z|\xi|\right)}\frac{\xi_2^2}{|\xi|^2} & -\frac{1}{\left(1-\tanh^2 z|\xi|\right)}\frac{1}{\alpha}\frac{(\tanh z|\xi|)}{|\xi|}i\xi_2 \\ \frac{1}{\left(1-\tanh^2 z|\xi|\right)}\alpha\frac{(\tanh z|\xi|)}{|\xi|}i\xi_1 & \frac{1}{\left(1-\tanh^2 z|\xi|\right)}\alpha\frac{(\tanh z|\xi|)}{|\xi|}i\xi_2 & \frac{1}{\left(1-\tanh^2 z|\xi|\right)}\end{pmatrix}$$
$$=\begin{pmatrix} -\frac{1}{|\xi|^2}(i\xi\times\mathbf{n})(i\xi\times\mathbf{n})^t-\frac{1}{\left(1-\tanh^2 z|\xi|\right)}\frac{1}{|\xi|^2}(i\xi)(i\xi)^t & -\frac{1}{\left(1-\tanh^2 z|\xi|\right)}\frac{1}{\alpha}\frac{(\tanh z|\xi|)}{|\xi|}i\xi \\ \frac{1}{\left(1-\tanh^2 z|\xi|\right)}\alpha\frac{(\tanh z|\xi|)}{|\xi|}i\xi^t & \frac{1}{\left(1-\tanh^2 z|\xi|\right)}\end{pmatrix}.$$

3. We have
$$\frac{1}{\left(1-\tanh^2 z\,|\xi|\right)} = -\cosh^2 z\,|\xi|\,.$$
Then, we can see that the coefficients of $\left(\frac{1}{\cosh z|\xi|}\left(e^{z\mathcal{S}(M)}\right)_{22}(\xi,z)\right)^{-1}$ are bounded by $\cosh z\,|\xi|$.

□

Remark 3. *Regarding the operator* $(\exp\delta\mathcal{M})_{11}$*, we have*
$$\left(\frac{1}{\cosh z\,|\xi|}\left(e^{z\mathcal{S}(\mathcal{M})}\right)_{11}(\xi,z)\right)^{-1}$$
$$= \begin{pmatrix} -\frac{1}{|\xi|^2}(i\xi\times\mathbf{n})(i\xi\times\mathbf{n})^t - \frac{1}{\left(1-\tanh^2 z|\xi|\right)}\frac{1}{|\xi|^2}(i\xi)(i\xi)^t & -\frac{1}{\left(1-\tanh^2 z|\xi|\right)}\frac{(\tanh z|\xi|)}{|\xi|}i\xi \\ \frac{1}{\left(1-\tanh^2 z|\xi|\right)}\frac{(\tanh z|\xi|)}{|\xi|}i\xi^t & \frac{1}{\left(1-\tanh^2 z|\xi|\right)} \end{pmatrix}.$$

In the next sub-section, we will give formulations for the mechanical impedance, which are more developed.

5.2 Expressing the solution $(\mathbf{X},\mathbf{T})^t(z)$ for given $(\mathbf{X},\mathbf{T})^t(0)$

The solution $(\mathbf{X},\mathbf{T})^t(z)$ of the Lamé system for given $(\mathbf{X},\mathbf{T})^t(0)$ can be obtained by the Fourier transform applied on the relations (12)-(13) linking $(\mathbf{X},\mathbf{Y})^t$ and $(\mathbf{X},\mathbf{T})^t$, as well as the solution (15)-(16).

We have
$$\widehat{\mathbf{T}}(z) = \mathcal{S}(P_1)\,\widehat{\mathbf{X}}(z) + \mathcal{S}(P_2)\,\widehat{\mathbf{Y}}(z)$$
$$\widehat{\mathbf{Y}}(z) = \mathcal{S}(Q_1)\,\widehat{\mathbf{X}}(z) + \mathcal{S}(Q_2)\,\widehat{\mathbf{T}}(z)\,,$$
with
$$\mathcal{S}(P_1) = \begin{pmatrix} 0_2 & 2\mu i\xi \\ -2\mu i\xi^t & 0 \end{pmatrix}, \qquad \mathcal{S}(P_2) = \begin{pmatrix} -\mu I_2 & 0_{21} \\ 0_{12} & (\lambda+2\mu) \end{pmatrix}$$
$$\mathcal{S}(Q_1) = \begin{pmatrix} 0_2 & 2i\xi \\ \frac{2\mu}{(\lambda+2\mu)}i\xi^t & 0 \end{pmatrix}, \qquad \mathcal{S}(Q_2) = \begin{pmatrix} -\frac{1}{\mu}I_2 & 0_{21} \\ 0_{12} & \frac{1}{(\lambda+2\mu)} \end{pmatrix}.$$
Then we use
$$\widehat{\mathbf{X}}(z) = \left(e^{z\mathcal{S}(\mathcal{M})}\right)_{11}\widehat{\mathbf{X}}(0) + \left(e^{z\mathcal{S}(\mathcal{M})}\right)_{12}\widehat{\mathbf{Y}}(0) + \int_0^z \left(e^{(z-t)\mathcal{S}(\mathcal{M})}\right)_{12}\mathcal{L}\mathbf{f}(t)\,dt$$
$$\widehat{\mathbf{Y}}(z) = \left(e^{z\mathcal{S}(\mathcal{M})}\right)_{21}\widehat{\mathbf{X}}(0) + \left(e^{z\mathcal{S}(\mathcal{M})}\right)_{22}\widehat{\mathbf{Y}}(0) + \int_0^z \left(e^{(z-t)\mathcal{S}(\mathcal{M})}\right)_{22}\mathcal{L}\mathbf{f}(t)\,dt.$$

We get

$$\widehat{\mathbf{X}}(z) = \Big(\left(e^{z\mathcal{S}(\mathcal{M})}\right)_{11} + \left(e^{z\mathcal{S}(\mathcal{M})}\right)_{12} \mathcal{S}(Q_1) \Big) \widehat{\mathbf{X}}(0) + \left(e^{z\mathcal{S}(\mathcal{M})}\right)_{12} \mathcal{S}(Q_2)\, \widehat{\mathbf{T}}(0) + \int_0^z \left(e^{(z-t)\mathcal{S}(\mathcal{M})}\right)_{12} \widehat{\mathcal{L}\mathbf{f}}(t)\, dt$$

$$\begin{aligned}\widehat{\mathbf{T}}(z) =& \Big(\mathcal{S}(P_1) \left(e^{z\mathcal{S}(\mathcal{M})}\right)_{11} + \mathcal{S}(P_2) \left(e^{z\mathcal{S}(\mathcal{M})}\right)_{21} + \left(\mathcal{S}(P_1) \left(e^{z\mathcal{S}(\mathcal{M})}\right)_{12} + \mathcal{S}(P_2) \left(e^{z\mathcal{S}(\mathcal{M})}\right)_{22} \right) \mathcal{S}(Q_1) \Big) \widehat{\mathbf{X}}(0) \\ &+ \Big(\mathcal{S}(P_1) \left(e^{z\mathcal{S}(\mathcal{M})}\right)_{12} + \mathcal{S}(P_2) \left(e^{z\mathcal{S}(\mathcal{M})}\right)_{22} \Big) \mathcal{S}(Q_2)\, \widehat{\mathbf{T}}(0) \\ &+ \int_0^z \Big(\mathcal{S}(P_1) \left(e^{(z-t)\mathcal{S}(\mathcal{M})}\right)_{12} + \mathcal{S}(P_2) \left(e^{(z-t)\mathcal{S}(\mathcal{M})}\right)_{22} \Big) \widehat{\mathcal{L}\mathbf{f}}(t)\, dt. \end{aligned} \tag{21}$$

Hence, we write

$$\begin{pmatrix} \widehat{\mathbf{X}} \\ \widehat{\mathbf{T}} \end{pmatrix}(z) = \widehat{\mathbb{N}}(z) \begin{pmatrix} \widehat{\mathbf{X}} \\ \widehat{\mathbf{T}} \end{pmatrix}(0) + \begin{pmatrix} \widehat{R}_1 \widehat{\mathbf{f}} \\ \widehat{R}_2 \widehat{\mathbf{f}} \end{pmatrix}(z) \tag{22}$$

$$= \begin{pmatrix} \widehat{N}_{11} & \widehat{N}_{12} \\ \widehat{N}_{21} & \widehat{N}_{22} \end{pmatrix}(z) \begin{pmatrix} \widehat{\mathbf{X}} \\ \widehat{\mathbf{T}} \end{pmatrix}(0) + \begin{pmatrix} \widehat{R}_1 \widehat{\mathbf{f}} \\ \widehat{R}_2 \widehat{\mathbf{f}} \end{pmatrix}(z). \tag{23}$$

Straightforward, but cumbersome, computations give the following explicit formulas for the coefficients' matrix $\mathbb{N}$.

- $$\frac{1}{\cosh z|\xi|}\widehat{N}_{11} = \begin{pmatrix} I_2 - z\frac{1}{|\xi|}\frac{(\lambda+\mu)}{(\lambda+2\mu)}(i\xi)(i\xi)^t & -\left(\frac{1}{|\xi|}(\tanh z|\xi|)\frac{\mu}{(\lambda+2\mu)} + z\frac{(\lambda+\mu)}{(\lambda+2\mu)}\right) i\xi \\ \left(-z\frac{(\lambda+\mu)}{(\lambda+2\mu)} + \frac{1}{|\xi|}(\tanh z|\xi|)\frac{\mu}{(\lambda+2\mu)}\right) i\xi^t & 1 - z|\xi|(\tanh z|\xi|)\frac{(\lambda+\mu)}{(\lambda+2\mu)} \end{pmatrix}$$

- $$\frac{1}{\cosh z|\xi|}\widehat{N}_{12} = \begin{pmatrix} \begin{matrix} -(\tanh z|\xi|)\frac{(\lambda+3\mu)}{2\mu(\lambda+2\mu)}\frac{1}{|\xi|^3}(i\xi)(i\xi^t) \\ -z\frac{(\lambda+\mu)}{2\mu(\lambda+2\mu)}\frac{1}{|\xi|^2}(i\xi)(i\xi^t) \\ -(\tanh z|\xi|)\frac{1}{\mu|\xi|^3}(i\xi\times\mathbf{n})(i\xi\times\mathbf{n})^t \end{matrix} & -z\frac{1}{|\xi|}(\tanh z|\xi|)\frac{(\lambda+\mu)}{2\mu(\lambda+2\mu)} i\xi \\ -z\frac{1}{|\xi|}(\tanh z|\xi|)\frac{(\lambda+\mu)}{2\mu(\lambda+2\mu)} i\xi^t & -z\frac{(\lambda+\mu)}{2\mu(\lambda+2\mu)}(\tanh z|\xi|)\frac{(\lambda+3\mu)}{2\mu(\lambda+2\mu)} \end{pmatrix}$$

- $$\frac{1}{\cosh z|\xi|}\widehat{N}_{21} = \begin{pmatrix} 2\mu \begin{pmatrix} -\frac{1}{2}\frac{1}{|\xi|}\tanh z|\xi|\,(i\xi\times\mathbf{n})(i\xi\times\mathbf{n})^t \\ +(\alpha-1)\frac{1}{|\xi|}(\tanh z|\xi| + z|\xi|)\, i\xi i\xi^t \end{pmatrix} & 2\mu z|\xi|(\alpha-1)\tanh z|\xi|\ \ i\xi \\ 2\mu z|\xi|(\alpha-1)\tanh z|\xi|\ \ i\xi^t & -2\mu|\xi|(\alpha-1)(\tanh z|\xi| - z|\xi|) \end{pmatrix}$$

- $$\frac{1}{\cosh z|\xi|}\widehat{N}_{22} = \begin{pmatrix} I_2 + \frac{z}{|\xi|}(\alpha-1)\tanh z|\xi|\, i\xi(i\xi)^t & \left(\alpha\frac{1}{|\xi|}\tanh z|\xi| + z(\alpha-1)\right)(i\xi) \\ -\left(\alpha\frac{1}{|\xi|}\tanh z|\xi| + z(1-\alpha)\right)(i\xi)^t & (1 - z|\xi|(1-\alpha)(\tanh z|\xi|)) \end{pmatrix}.$$

For writing the impedance condition, we need to analyze the matrix coefficient $\widehat{N}_{22}$.

Proposition 4. *The matrix $\widehat{N}_{22}(z,\xi)$ is invertible.*

Proof. The determinant of $\widehat{N}_{22}(z,\xi)$ is given by

$$\det N_{22}(z,\xi) = \cosh z\,|\xi|\,\Phi\left(z\,|\xi|\right),$$

where $\Phi : [0,+\infty) \to [0,+\infty)$ is defined by

$$\begin{aligned}\Phi(t) &= \left((\alpha-1)^2 t^2 \left(1-\tanh^2 t\right) + \left(1-\alpha^2\tanh^2 t\right)\right)\\ &= \frac{1}{2\cosh^2 t}\left(\left(1-\alpha^2\right)\cosh 2t + 2t^2(1-\alpha)^2 + \alpha^2 + 1\right.\end{aligned}$$

The function Φ is bounded and satisfies

$$\begin{aligned}&\Phi(t) > 0 : \forall t \in [0,+\infty),\\ &\Phi(0) = 1, \lim_{t\to\infty}\Phi(t) = 1-\alpha^2.\end{aligned}$$

□

The inverse matrix of $\widehat{N}_{22}(z,\xi)$ is given by

$$\left(\widehat{N}_{22}\right)^{-1}(z,\xi) = \frac{1}{\cosh z\,|\xi|}\begin{pmatrix} -\frac{1}{|\xi|^2}(i\xi\times\mathbf{n})(i\xi\times\mathbf{n})^t - \frac{(1+z|\xi|(\alpha-1)(\tanh z|\xi|))}{|\xi|^2\Phi} i\xi i\xi^t & -\frac{(\alpha(\tanh z|\xi|)+z|\xi|(\alpha-1))}{|\xi|\Phi} i\xi \\ \frac{(\alpha(\tanh z|\xi|)+(1-\alpha)z|\xi|)}{|\xi|\Phi} i\xi^t & \frac{(1+z|\xi|(1-\alpha)\tanh z|\xi|)}{\Phi}\end{pmatrix} \quad (24)$$

Proposition 5. *The entries of the principal part of the symbol* $\left(\widehat{N}_{22}\right)^{-1}$ *are bounded by* $C(\alpha,z)$ *(*$C(\alpha,z)>0$ *depending only on* α *and* z*). For fixed* $z\in(0,\delta)$*, the matrix* $\left(\widehat{N}_{22}\right)^{-1}(\cdot,z)$ *is the symbol of a pseudodifferential operator of order zero.*

Proof. The functions

$$\begin{aligned} t &\to \frac{(1\pm(\tanh t)\,t)}{\Phi(t)\cosh t}\\ t &\to \frac{(\tanh t)\pm t}{\Phi(t)\cosh t}\end{aligned}$$

are bounded for $t\geq 0$. □

5.3 Impedance operator

Before going deeper for expressing and analyzing the explicit formula of the impedance operator, we use the following notation. For the sake of simplicity, we write the components of the operator $\widehat{\mathbb{N}}$ given in (22) as

$$\widehat{N}_{ij}(\cdot,z) = \widehat{N}_{ij}(z), \qquad i,j=1,2;$$

this notation is also used for their inverses, i.e. $(\widehat{N}_{ij}(\cdot,z))^{-1} = (\widehat{N}_{ij}(z))^{-1}$ for $i,j=1,2$.

From the boundary condition (6) and the solution (23) of the Lamé system given through $\mathbf{X}$ and $\mathbf{T}$, we have

$$\widehat{\mathbf{g}} = \widehat{\mathbf{T}}(\delta) = \widehat{N}_{21}(\delta)\,\widehat{\boldsymbol{\varphi}} + \widehat{N}_{22}(\delta)\,\widehat{\mathbf{T}}(0) + \widehat{R}_2(\delta)\,\widehat{\mathbf{f}}.$$

Then, the impedance operator Z_δ is given through

$$\widehat{\mathbf{T}}(0) = \widehat{Z}_\delta\left(\widehat{\boldsymbol{\varphi}}, \widehat{\mathbf{f}}, \widehat{\mathbf{g}}\right) = \left(\widehat{N}_{22}(\delta)\right)^{-1}\left(\widehat{\mathbf{g}} - \widehat{R}_2(\delta)\,\widehat{\mathbf{f}} - \widehat{N}_{21}(\delta)\,\widehat{\boldsymbol{\varphi}}\right).$$

The linearity of the impedance operator allows to write

$$\begin{aligned}\widehat{Z}_\delta\left(\widehat{\boldsymbol{\varphi}}, \widehat{\mathbf{f}}, \widehat{\mathbf{g}}\right) &= \widehat{Z}_\delta(\widehat{\boldsymbol{\varphi}}, 0, 0) + \widehat{Z}_\delta(0, 0, \widehat{\mathbf{g}}) + \widehat{Z}_\delta\left(0, \widehat{\mathbf{f}}, 0\right)\\ &= \widehat{Z_\delta^1}(\widehat{\boldsymbol{\varphi}}) + \widehat{Z_\delta^2}(\widehat{\mathbf{g}}) + \widehat{Z_\delta^3}\left(\widehat{\mathbf{f}}\right).\end{aligned}$$

In this way, we can write the three terms of the impedance operator as follows:

$$\begin{aligned}
\widehat{Z_\delta^1}(\xi) &:= -\left(\left(\widehat{N}_{22}\right)^{-1}\widehat{N}_{21}(\delta)\right)(\xi) = \begin{pmatrix} \left(\widehat{Z}_\delta^1\right)_{11} & \left(\widehat{Z}_\delta^1\right)_{12} \\ \left(\widehat{Z}_\delta^1\right)_{21} & \left(\widehat{Z}_\delta^1\right)_{22} \end{pmatrix}(\xi)\\
\widehat{Z_\delta^2}(\xi) &:= \left(\widehat{N}_{22}(\delta)\right)^{-1}(\xi) = \begin{pmatrix} \left(\widehat{Z_\delta^2}\right)_{11} & \left(\widehat{Z_\delta^2}\right)_{12} \\ \left(\widehat{Z_\delta^2}\right)_{21} & \left(\widehat{Z_\delta^2}\right)_{22} \end{pmatrix}(\xi)\\
\widehat{Z_\delta^3}(\xi) &:= -\left(\widehat{N}_{22}(\delta)\right)^{-1}(\xi)\,\widehat{R}_2(\delta) = \begin{pmatrix} \left(\widehat{Z_\delta^3}\right)_{11} & \left(\widehat{Z_\delta^3}\right)_{12} \\ \left(\widehat{Z_\delta^3}\right)_{21} & \left(\widehat{Z_\delta^3}\right)_{22} \end{pmatrix}(\xi).
\end{aligned}$$

The operator $\widehat{Z_\delta^2}$ was written explicitly in (24). Therefore, the focus will be made on more developing and analyzing the operators $\widehat{Z_\delta^1}$ as well $\widehat{Z_\delta^3}$.

5.3.1 Analysis of the operator Z_δ^1

Straightforward computation of $\widehat{Z_\delta^1}(\xi)$ leads to:

- $$\left(\widehat{Z}_\delta^1\right)_{11}(\xi) = \begin{pmatrix} \mu\frac{1}{|\xi|}\tanh\delta|\xi|\,(i\xi\times\mathbf{n})\,(i\xi\times\mathbf{n})^t \\ -2\mu\frac{1}{\Phi}\begin{pmatrix} \frac{1}{|\xi|}(1+\delta|\xi|(\alpha-1)(\tanh\delta|\xi|))(\alpha-1)(\tanh\delta|\xi|+\delta|\xi|) \\ -\frac{1}{|\xi|}(\alpha(\tanh\delta|\xi|)+\delta|\xi|(\alpha-1))\,\delta|\xi|(\alpha-1)\tanh\delta|\xi| \end{pmatrix} i\xi\,(i\xi)^t \end{pmatrix}$$

- $$\left(\widehat{Z}_\delta^1\right)_{12}(\xi) = -2\mu\frac{1}{\Phi}\begin{pmatrix} (1+\delta|\xi|(\alpha-1)(\tanh z|\xi|))\,\delta|\xi|(\alpha-1)\tanh\delta|\xi| \\ +(\alpha(\tanh\delta|\xi|)+\delta|\xi|(\alpha-1))(\alpha-1)(\tanh\delta|\xi|-\delta|\xi|) \end{pmatrix} i\xi$$

- $$\left(\widehat{Z}_\delta^1\right)_{21}(\xi) = -2\mu\frac{1}{\Phi}\begin{pmatrix} -(\alpha(\tanh\delta|\xi|)+(1-\alpha)\delta|\xi|)(\alpha-1)(\tanh\delta|\xi|+\delta|\xi|) \\ +(1+\delta|\xi|(1-\alpha)\tanh\delta|\xi|)\,z|\xi|(\alpha-1)\tanh\delta|\xi| \end{pmatrix}(i\xi)^t$$

- $$\left(\widehat{Z}_\delta^1\right)_{22}(\xi) = -2\mu\frac{1}{\Phi}\begin{pmatrix} -(\alpha(\tanh\delta|\xi|)+(1-\alpha)\delta|\xi|)\,\delta|\xi|^2(\alpha-1)\tanh\delta|\xi| \\ -(1+\delta|\xi|(1-\alpha)\tanh\delta|\xi|)\,|\xi|(\alpha-1)(\tanh\delta|\xi|-z|\xi|) \end{pmatrix}.$$

Which can be simplified as

- $$\left(\widehat{Z}_\delta^1\right)_{11}(\xi) = \mu \frac{1}{|\xi|} \tanh \delta |\xi| \, (i\xi \times \mathbf{n}) \, (i\xi \times \mathbf{n})^t - 2\mu \frac{1}{\Phi(\delta|\xi|)} \frac{1}{|\xi|} (\alpha - 1) \left(\delta |\xi| \left(1 - \tanh^2 \delta |\xi|\right) + \tanh \delta |\xi|\right) i\xi \, (i\xi)^t$$

- $\left(\widehat{Z}_\delta^1\right)_{12}(\xi) = -2\mu \frac{1}{\Phi(\delta|\xi|)} (\alpha - 1) \left((1 - \alpha) \delta^2 |\xi|^2 \left(1 - \tanh^2 \delta |\xi|\right) + \alpha \tanh^2 \delta |\xi|\right) i\xi$

- $\left(\widehat{Z}_\delta^1\right)_{21}(\xi) = 2\mu \frac{1}{\Phi(\delta|\xi|)} (\alpha - 1) \left((1 - \alpha) \delta^2 |\xi|^2 \left(1 - \tanh^2 \delta |\xi|\right) + \alpha \tanh^2 \delta |\xi|\right) (i\xi)^t$

- $\left(\widehat{Z}_\delta^1\right)_{22}(\xi) = -2\mu \frac{1}{\Phi(\delta|\xi|)} (\alpha - 1) \left(\delta |\xi| \left(1 - \tanh^2 \delta |\xi|\right) - \tanh \delta |\xi|\right) |\xi|.$

Proposition 6. *The entries of the principal part of the symbol $\widehat{Z_\delta^1}(\xi)$ are bounded by $C(\alpha, \delta)|\xi|$ ($C(\alpha, \delta) > 0$ depending only on α and δ).*

We draw the attention of the reader to the fact that the matrix $\widehat{Z_\delta^1}$ is the symbol of a pseudodifferential operator of order one.

Proof. The functions $t \to t^2 \left(1 - \tanh^2 t\right)$ and $t \to \Phi(t)$ are bounded for $t \geq 0$. □

The operator $\widehat{Z_\delta^1}(\xi) = -\left(\widehat{N}_{22}(\delta)\right)^{-1} \widehat{N}_{21}(\delta)(\xi)$ is the symbol of the impedance operator Z_δ^1 and

$$\begin{array}{rccc} Z_\delta^1 : & \mathbb{H}^s(\Gamma) & \to & \mathbb{H}^{s-1}(\Gamma) \\ & \boldsymbol{\varphi} = \mathbf{u}_{|\Gamma} & \to & Z_\delta^1(\boldsymbol{\varphi}) = (\sigma \mathbf{n})_{|\Gamma} \end{array}$$

is bounded.

5.3.2 Analysis of the operator Z_δ^3

Now, we move to analyzing the third part of the impedance operator, which represents the more technical part.

Proposition 7. *The operator*

$$\begin{array}{rccc} Z_\delta^3 : & \mathbb{L}^2\left((0, \delta) \times \mathbb{R}^2, \mathbb{R}^3\right) & \to & \mathbb{H}^{-\frac{1}{2}}\left(\mathbb{R}^2, \mathbb{R}^3\right) \\ & \mathbf{f} & \to & Z_\delta^3(\mathbf{f}), \end{array}$$

defined by its symbol

$$\widehat{Z_\delta^3(\mathbf{f})}(\xi) = \widehat{Z_\delta^3}\left(\widehat{\mathbf{f}}\right)(\xi) = -\left(\left(\widehat{N}_{22}(\delta)\right)^{-1}(\xi) \left(\widehat{R}_2(\delta) \widehat{\mathbf{f}}\right)(\delta, \xi)\right),$$

is a linear bounded pseudodifferential operator.

Proof. First, remember that $Q_2 = -\mathcal{L}$. Thus, from (21) we can see that

$$(\widehat{R}_2\widehat{\mathbf{f}})(\delta,\xi) = -\int_0^\delta \widehat{N}_{22}\left(\delta - t, \xi\right)\widehat{\mathbf{f}}(t)dt.$$

It can be developed as follows:

$$\left(\widehat{R}_2\widehat{\mathbf{f}}\right)(\delta,\xi) =$$

$$-\int_0^\delta \begin{pmatrix} \cosh\left(\delta-t\right)|\xi| - \left(\delta-t\right)|\xi|\left(\sinh\left(\delta-t\right)|\xi|\right)\left(1-\alpha\right)\frac{i\xi i\xi^t}{|\xi|^2} & \left(\left(\sinh\left(\delta-t\right)|\xi|\right)\alpha - \left(1-\alpha\right)\left(\delta-t\right)|\xi|\cosh\left(\delta-t\right)|\xi|\right)\frac{1}{|\xi|}i\xi \\ -\begin{pmatrix} \left(1-\alpha\right)\left(\delta-t\right)|\xi|\left(\cosh\left(\delta-t\right)|\xi|\right) + \\ \left(\sinh\left(\delta-t\right)|\xi|\right)\alpha \end{pmatrix}\frac{1}{|\xi|}i\xi^t & \cosh\left(\delta-t\right)|\xi| - \left(\delta-t\right)|\xi|\left(\sinh\left(\delta-t\right)|\xi|\right)\left(1-\alpha\right) \end{pmatrix}\widehat{\mathbf{f}}(t,\xi)dt.$$

Then the operators under the integral are written:

- $\star$ $\left(\left(\widehat{N}_{22}\right)^{-1}\left(\delta,\xi\right)\widehat{N}_{22}\left(\delta-t,\xi\right)\right)_{11} =$

$$\frac{1}{\cosh\delta|\xi|}\begin{pmatrix} -\frac{\cosh(\delta-t)|\xi|}{|\xi|^2}\left(i\xi\times\mathbf{n}\right)\left(i\xi\times\mathbf{n}\right)^t \\ +\begin{pmatrix} \left(1-\left(\tanh\delta|\xi|\right)\left(1-a\right)\delta|\xi|\right)\left(\cosh\left(\delta-t\right)|\xi| - \left(\delta-t\right)|\xi|\left(\sinh\left(\delta-t\right)|\xi|\right)\left(1-\alpha\right)\right) \\ +\left(a\left(\tanh\delta|\xi|\right) - \delta|\xi|\left(1-a\right)\right)\left(\left(1-\alpha\right)\left(\delta-t\right)|\xi|\left(\cosh\left(\delta-t\right)|\xi|\right) + \left(\sinh\left(\delta-t\right)|\xi|\right)\alpha\right) \end{pmatrix}\frac{1}{|\xi|^2\Phi}i\xi i\xi^t \end{pmatrix}$$

- $\star$ $\left(\left(\widehat{N}_{22}\right)^{-1}\left(\delta,\xi\right)\widehat{N}_{22}\left(\delta-t,\xi\right)\right)_{21} =$

$$\frac{1}{\cosh\delta|\xi|}\begin{pmatrix} \left(\left(a\left(\tanh\delta|\xi|\right) + \delta|\xi|\left(1-a\right)\right)\left(\cosh\left(\delta-t\right)|\xi| + \left(z-t\right)|\xi|\left(\sinh\left(\delta-t\right)|\xi|\right)\left(1-\alpha\right)\right)\right) \\ -\left(1+\left(1-a\right)\delta|\xi|\left(\tanh\delta|\xi|\right)\right)\left(\left(\left(1-\alpha\right)\left(\delta-t\right)|\xi|\left(\cosh\left(\delta-t\right)|\xi|\right) + \left(\sinh\left(\delta-t\right)|\xi|\right)\alpha\right)\right) \end{pmatrix}\frac{i\xi^t}{\Phi|\xi|}$$

- $\star$ $\left(\left(\widehat{N}_{22}\right)^{-1}\left(\delta,\xi\right)\widehat{N}_{22}\left(\delta-t,\xi\right)\right)_{12} =$

$$\frac{1}{\cosh\delta|\xi|}\begin{pmatrix} \left(1+\left(\tanh\delta|\xi|\right)\left(1-a\right)\delta|\xi|\right)\left(\left(\sinh\left(\delta-t\right)|\xi|\right)\alpha - \left(1-\alpha\right)\left(\delta-t\right)|\xi|\cosh\left(\delta-t\right)|\xi|\right) \\ -\left(a\left(\tanh\delta|\xi|\right) - \delta|\xi|\left(1-a\right)\right)\left(\cosh\left(\delta-t\right)|\xi| - \left(\delta-t\right)|\xi|\left(\sinh\left(\delta-t\right)|\xi|\right)\left(1-\alpha\right)\right) \end{pmatrix}\frac{i\xi}{|\xi|\Phi}$$

- $\star$ $\left(\left(\widehat{N}_{22}\right)^{-1}\left(\delta,\xi\right)\widehat{N}_{22}\left(\delta-t,\xi\right)\right)_{22} =$

$$-\frac{1}{\cosh\delta|\xi|}\frac{1}{\Phi}\begin{pmatrix} \left(a\left(\tanh\delta|\xi|\right) + \delta|\xi|\left(1-a\right)\right)\left(\left(\sinh\left(\delta-t\right)|\xi|\right)\alpha - \left(1-\alpha\right)\left(\delta-t\right)|\xi|\cosh\left(\delta-t\right)|\xi|\right) \\ -\left(1+\left(1-a\right)\delta|\xi|\left(\tanh\delta|\xi|\right)\right)\left(\cosh\left(\delta-t\right)|\xi| - \left(\delta-t\right)|\xi|\left(\sinh\left(\delta-t\right)|\xi|\right)\left(1-\alpha\right)\right) \end{pmatrix}.$$

Now, we prove that

$$\int_0^\delta \left(\left(\widehat{N}_{22}\right)^{-1}\left(\delta,\xi\right)\widehat{N}_{22}\left(\delta-t,\xi\right)\right)_{11}\widehat{f_T}\left(t,\xi\right)dt \in \mathbb{H}^{-\frac{1}{2}}\left(\mathbb{R}^2,\mathbb{R}^2\right)$$
$$\int_0^\delta \left(\left(\widehat{N}_{22}\right)^{-1}\left(\delta,\xi\right)\widehat{N}_{22}\left(\delta-t,\xi\right)\right)_{21}\widehat{f_T}\left(t,\xi\right)dt \in \mathbb{H}^{-\frac{1}{2}}\left(\mathbb{R}^2,\mathbb{R}^2\right)$$
$$\int_0^\delta \left(\left(\widehat{N}_{22}\right)^{-1}\left(\delta,\xi\right)\widehat{N}_{22}\left(\delta-t,\xi\right)\right)_{12}\widehat{f}_n\left(t,\xi\right)dt \in H^{-\frac{1}{2}}\left(\mathbb{R}^2,\mathbb{R}\right)$$
$$\int_0^\delta \left(\left(\widehat{N}_{22}\right)^{-1}\left(\delta,\xi\right)\widehat{N}_{22}\left(\delta-t,\xi\right)\right)_{22}\widehat{f}_n\left(t,\xi\right)dt \in H^{-\frac{1}{2}}\left(\mathbb{R}^2,\mathbb{R}\right).$$

We use the hypothesis $\widehat{\mathbf{f}}$ in $\mathbb{L}^2 \in ((0,\delta) \times \mathbb{R}^2, \mathbb{R}^3)$ and Cauchy-Schwarz inequality. We highlight that it is sufficient to prove that

$$\int_0^\delta \left| \left(\widehat{N}_{22}^{-1}(\delta,\xi)\, \widehat{N}_{22}(\delta - t, \xi) \right)_{ij} \right|^2 dt \leq C(\alpha,\delta)(1 + |\xi|), \quad 1 \leq i,j \leq 2,$$

with $C(\alpha,\delta) > 0$ (not depending on ξ). We give the sketch of the proof for $\left(\widehat{N}_{22}^{-1}(\delta,\xi)\, \widehat{N}_{22}(\delta - t, \xi) \right)_{11}$. It is clear that this estimate is satisfied for $|\xi|$ bounded. It remains then to prove it for $|\xi|$ large enough.

The most difficult term to handle is the one having the factor $|\xi|^2$. Expanding the products and using the equality

$$(\cosh \delta |\xi|)(\cosh(\delta - t)|\xi|) - (\sinh \delta |\xi|)(\sinh(\delta - t)|\xi|) = \cosh t |\xi|,$$

we get

$$\begin{aligned}
&\left(\left(\widehat{N}_{22} \right)^{-1} (\delta,\xi)\, \widehat{N}_{22}(\delta - t, \xi) \right)_{11} = \\
&- \frac{\cosh(\delta - t)|\xi|}{\cosh \delta |\xi|} \frac{(i\xi \times \mathbf{n})(i\xi \times \mathbf{n})^t}{|\xi|^2} - (1-\alpha)^2 \delta |\xi|^2 (\delta - t) \frac{\cosh t|\xi|}{\cosh \delta |\xi|} \frac{i\xi i\xi^t}{|\xi|^2 \Phi} \\
&+ \frac{1}{\cosh \delta |\xi|} (\cosh(\delta - t)|\xi| - (1-\alpha)|\xi|(\delta - t)(\sinh(\delta - t)|\xi|)) \frac{i\xi i\xi^t}{|\xi|^2 \Phi} \\
&- \frac{1}{\cosh \delta |\xi|} ((\tanh \delta |\xi|)(1-\alpha)\delta |\xi|)(\cosh(\delta - t)|\xi|) \frac{i\xi i\xi^t}{|\xi|^2 \Phi} \\
&+ \frac{1}{\cosh \delta |\xi|} (\alpha \tanh \delta |\xi|)((1-\alpha)(\delta - t)|\xi|(\cosh(\delta - t)|\xi|) + (\sinh(\delta - t)|\xi|)\alpha) \frac{i\xi i\xi^t}{|\xi|^2 \Phi} \\
&- \frac{1}{\cosh \delta |\xi|} (\delta |\xi|(1-\alpha)(\sinh(\delta - t)|\xi|)\alpha) \frac{i\xi i\xi^t}{|\xi|^2 \Phi}.
\end{aligned}$$

Then, we use the estimates

$$\begin{gathered}
|\tanh \delta |\xi|| \leq 1, \forall \xi \in \mathbb{R}^2 \\
\exists C_N; \frac{|\xi|^N}{\cosh \delta |\xi|} \leq C_N, \forall N \in \mathbb{N}.
\end{gathered}$$

We recall the properties of the function Φ:

$$\begin{aligned}
&\Phi(t) > 0 \ \forall t \in [0, +\infty[, \\
&\Phi(0) = 1, \ \lim_{t \to \infty} \Phi(t) = 1 - a^2,
\end{aligned}$$

and the relations

$$\begin{aligned}
2\cosh^2 s|\xi| &= 1 + \cosh 2s|\xi| \\
2\sinh^2 s|\xi| &= -1 + \cosh 2s|\xi|.
\end{aligned}$$

Hence, we get

$$\int_0^\delta \left|\left(\widehat{N}_{22}^{-1}(\delta,\xi)\,\widehat{N}_{22}(\delta-t,\xi)\right)_{11}\right|^2 dt$$
$$\leq C_1(\alpha,\delta)\,\frac{1}{\cosh^2\delta|\xi|}\left(|\xi|^4\int_0^\delta(\delta-t)^2\cosh(2t|\xi|)\,dt+|\xi|^2\int_0^\delta\cosh(2(\delta-t)|\xi|)\,dt+|\xi|^2\int_0^\delta(\delta-t)^2\cosh(2(\delta-t)|\xi|)\,dt\right).$$

Integrating by parts, we obtain

$$\int_0^\delta(\delta-t)^2\cosh(2t|\xi|)\,dt=\frac{\delta}{2|\xi|^2}-\frac{1}{4|\xi|^3}+\frac{\cosh(2\delta|\xi|)}{4|\xi|^3}$$
$$\int_0^\delta\cosh(2(\delta-t)|\xi|)\,dt=\frac{1}{2|\xi|}\sinh 2\delta|\xi|$$
$$\int_0^\delta(\delta-t)^2\cosh(2(\delta-t)|\xi|)\,dt=\delta^2\frac{1}{2|\xi|}\sinh 2\delta|\xi|-\delta\frac{1}{2|\xi|^2}\cosh 2\delta|\xi|+\frac{1}{4|\xi|^3}\sinh 2\delta|\xi|\,.$$

And we use the fact that

$$\begin{aligned}|\tanh\delta|\xi||&\leq 1\\ \sinh 2\delta|\xi|&=2\sinh\delta|\xi|\cosh\delta|\xi|\\ \cosh 2\delta|\xi|&=\cosh^2\delta|\xi|\left(1+\tanh^2\delta|\xi|\right).\end{aligned}$$

Hence, we conclude the existence of a constant $C_2(\alpha,\delta)$ (not depending on ξ), such that

$$\int_0^\delta\left|\left(\widehat{N}_{22}^{-1}(\delta,\xi)\,\widehat{N}_{22}(\delta-t,\xi)\right)_{11}\right|^2 dt\leq C_2(\alpha,\delta)\,|\xi|\,.$$

The other terms of the operator $\widehat{Z_\delta^3}$ can be handled similarly.

□

In the next section, we give the approximate mechanical impedance for the homogeneous case.

6 Approximate mechanical impedance at order three

6.1 The approximate impedance operator

We consider the homogeneous case when $\mathbf{f}=0$ and $\mathbf{g}=0$, then the impedance operator is only given by Z_δ^1, such that

$$\widehat{Z_\delta^1}(\xi)=-\left(\left(\widehat{N}_{22}(\delta)\right)^{-1}\widehat{N}_{21}(\delta)\right)(\xi)\,.$$

In order to obtain an approximation at order three with respect to the thickness δ for the impedance operator, we expand the coefficients of the matrix $\widehat{Z_\delta^1}(\xi)$ in powers of δ at order three.

From the numerical point of view, Padé-like approximations are more stable. Therefore, in this paper we give an approximation of fractional type, and we expand the numerator and the denominator separately.

We give the following approximations at order three:

- $\dfrac{1}{|\xi|}\tanh\delta|\xi| = \delta\left(1-\dfrac{1}{3}\delta^2|\xi|^2\right)+o\left(\delta^3\right)$
$$= \delta\frac{1}{\left(1+\frac{1}{3}\delta^2|\xi|^2\right)}+o\left(\delta^3\right)$$
- $\dfrac{1}{|\xi|}(\alpha-1)\left(\delta|\xi|\left(1-\tanh^2\delta|\xi|\right)+\tanh\delta|\xi|\right) = 2\delta(\alpha-1)\left(1-\dfrac{2}{3}\delta^2|\xi|^2\right)+o\left(\delta^3\right)$
$$= 2\delta(\alpha-1)\frac{1}{\left(1+\frac{2}{3}\delta^2|\xi|^2\right)}+o\left(\delta^3\right)$$
- $(\alpha-1)\left((1-\alpha)\delta^2|\xi|^2\left(1-\tanh^2\delta|\xi|\right)+\alpha\tanh^2\delta|\xi|\right) = \delta^2|\xi|^2(\alpha-1)+o\left(\delta^3\right)$
- $-(\alpha-1)\left(\delta|\xi|\left(1-\tanh^2\delta|\xi|\right)-\tanh\delta|\xi|\right)|\xi| = \dfrac{2}{3}\delta^3|\xi|^4(\alpha-1)+o\left(\delta^3\right).$

Regarding the function Φ, we have

$$\begin{aligned}\Phi(\delta|\xi|) &= \frac{1}{2\cosh^2 z|\xi|}\left(\left(1-\alpha^2\right)\cosh 2\delta|\xi|+2\delta^2|\xi|^2(1-\alpha)^2+\alpha^2+1\right)\\ &= 1+\delta^2\left(|\xi|^2(1-2\alpha)\right)+o\left(\delta^3\right).\end{aligned}$$

Moreover,

$$\left(1+\delta^2|\xi|^2(1-2\alpha)\right)\left(1+\frac{2}{3}\delta^2|\xi|^2\right) = 1+\delta^2\frac{1}{3}|\xi|^2(5-6\alpha)+o\left(\delta^3\right).$$

Thus, we can choose the approximations

$$\begin{aligned}\left(\widehat{Z_\delta^{1*}}\right)_{11}(\xi) &= \mu\delta\frac{1}{\left(1+\frac{1}{3}\delta^2|\xi|^2\right)}(i\xi\times\mathbf{n})(i\xi\times\mathbf{n})^t-2\mu(\alpha-1)2\delta\frac{i\xi(i\xi)^t}{1+\delta^2\frac{1}{3}|\xi|^2(5-6\alpha)}\\ \left(\widehat{Z_\delta^{1*}}\right)_{21}(\xi) &= -2\mu(\alpha-1)\delta^2|\xi|^2\frac{i\xi^t}{1+\delta^2\frac{1}{3}|\xi|^2(5-6\alpha)}\\ \left(\widehat{Z_\delta^{1*}}\right)_{12}(\xi) &= 2\mu(\alpha-1)\delta^2|\xi|^2\frac{i\xi}{1+\delta^2\frac{1}{3}|\xi|^2(5-6\alpha)}\\ \left(\widehat{Z_\delta^{1*}}\right)_{22}(\xi) &= 2\mu(\alpha-1)\frac{2}{3}\delta^3\frac{|\xi|^4}{1+\delta^2\frac{1}{3}|\xi|^2(5-6\alpha)}.\end{aligned}$$

For the term $\left(\widehat{Z_\delta^{1*}}\right)_{11}(\xi)$, since $\alpha=\frac{\mu}{\lambda+2\mu}$, then we have $5-6\alpha = 5-6\frac{\mu}{\lambda+2\mu} = \frac{5\lambda+4\mu}{\lambda+2\mu}>0$. Thus, the term $1+\delta^2\frac{1}{3}|\xi|^2(5-6\alpha)$ is positive for all $\xi\in\mathbb{R}^2$. Consequently, the operator

$$1-\delta^2\frac{(5-6\alpha)}{3}\Delta_\Gamma : H^s(\Gamma)\to H^{s-2}(\Gamma)$$

is invertible .

Finally, the approximate mechanical impedance in the form of a differential operator is then given by

$$Z_\delta^{1*}\boldsymbol{\varphi} = \begin{pmatrix}(Z_\delta^{1*})_{11} & (Z_\delta^{1*})_{12}\\ (Z_\delta^{1*})_{21} & (Z_\delta^{1*})_{22}\end{pmatrix}\begin{pmatrix}\varphi_T\\ \varphi_n\end{pmatrix},$$

with

$$\left(Z_\delta^{1*}\right)_{11}\varphi_T = \delta\mu\overrightarrow{\operatorname{curl}}_\Gamma\left(1-\frac{1}{3}\delta^2\Delta_\Gamma\right)^{-1}\operatorname{curl}_\Gamma\varphi_T - 2\mu\left(1-a\right)2\delta\nabla_\Gamma\left(1-\delta^2\frac{1}{3}\left(5-6\alpha\right)\Delta_\Gamma\right)^{-1}\operatorname{div}_\Gamma\varphi_T$$

$$\left(Z_\delta^{1*}\right)_{12}\varphi_n = 2\mu\left(\alpha-1\right)\delta^2\nabla_\Gamma\left(1-\delta^2\frac{1}{3}\left(5-6\alpha\right)\Delta_\Gamma\right)^{-1}\Delta_\Gamma\varphi_n$$

$$\left(Z_\delta^{1*}\right)_{21}\varphi_T = -2\mu\left(\alpha-1\right)\delta^2\Delta_\Gamma\left(1-\delta^2\frac{1}{3}\left(5-6\alpha\right)\Delta_\Gamma\right)^{-1}\operatorname{div}_\Gamma\varphi_T$$

$$\left(Z_\delta^{1*}\right)_{22}\varphi_n = 2\mu\left(\alpha-1\right)\frac{2}{3}\delta^3\Delta_\Gamma\left(1-\delta^2\frac{1}{3}\left(5-6\alpha\right)\Delta_\Gamma\right)^{-1}\Delta_\Gamma\varphi_n.$$

Remark 8. *The bilinear form for the approximate mechanical operator is written*

$$\begin{aligned}\left(Z_\delta^{1*}\boldsymbol{\varphi},\boldsymbol{\varphi}\right) = {} & \delta\mu\int_\Gamma\left(1-\frac{1}{3}\delta^2\Delta\right)^{-1}\operatorname{curl}_\Gamma\varphi_T\operatorname{curl}_\Gamma\varphi_T d\Gamma \\ & + \delta2\mu\left(\alpha-1\right)\int_\Gamma\left(1-\delta^2\frac{1}{3}\left(5-6\alpha\right)\Delta_\Gamma\right)^{-1}\operatorname{div}_\Gamma\varphi_T\operatorname{div}_\Gamma\varphi_T d\Gamma \\ & - \delta^2 2\mu\left(\alpha-1\right)\int_\Gamma\left(1-\delta^2\frac{1}{3}\left(5-6\alpha\right)\Delta_\Gamma\right)^{-1}\Delta_\Gamma\varphi_n\operatorname{div}_\Gamma\varphi_T d\Gamma \\ & - \delta^2 2\mu\left(\alpha-1\right)\int_\Gamma\left(1-\delta^2\frac{1}{3}\left(5-6\alpha\right)\Delta_\Gamma\right)^{-1}\operatorname{div}_\Gamma\varphi_T\Delta_\Gamma\varphi_n d\Gamma \\ & + 2\mu\left(\alpha-1\right)\frac{2}{3}\delta^3\int_\Gamma\left(1-\delta^2\frac{1}{3}\left(5-6\alpha\right)\Delta_\Gamma\right)^{-1}\Delta_\Gamma\varphi_n\Delta_\Gamma\varphi_n d\Gamma.\end{aligned}$$

Using Young inequality $2AB\leq A^2\gamma+\frac{1}{\gamma}B^2$ with $\frac{3}{2}<\gamma<2$, we get

$$\left(Z_\delta^{1*}\boldsymbol{\varphi},\boldsymbol{\varphi}\right)\geq C\delta\left[\begin{array}{c}\int_\Gamma\left|\left(1-\frac{1}{3}\delta^2\Delta\right)^{-\frac{1}{2}}\operatorname{curl}_\Gamma\varphi_T\right|^2 d\Gamma \\ +\int_\Gamma\left|\left(1-\delta^2\frac{1}{3}\left(5-6\alpha\right)\Delta_\Gamma\right)^{-\frac{1}{2}}\operatorname{div}_\Gamma\varphi_T\right|^2 d\Gamma \\ +\int_\Gamma\left|\left(1-\delta^2\frac{1}{3}\left(5-6\alpha\right)\Delta_\Gamma\right)^{-\frac{1}{2}}\Delta\varphi_n\right|^2 d\Gamma\end{array}\right].$$

6.2 Variational formulation of the approximate impedance problem

We assume that the linear elastic body Ω_- is isotropic with Lamé's coefficients λ_- and μ_-. We consider the space

$$\mathbb{V}\left(\Omega_-\right) = \left\{\left(v_T,v_n\right)=\mathbf{v}\in\mathbb{H}^1\left(\Omega_-\right);\mathbf{v}_{|\Gamma_0}=0,v_{n|\Gamma}\in H^1\left(\Gamma\right)\right\},$$

equipped with the inner product

$$\langle\mathbf{u},\mathbf{v}\rangle = \langle\mathbf{u},\mathbf{v}\rangle_{\mathbb{H}^1(\Omega_-)}+\left\langle\mathbf{u}_{|\Gamma},\mathbf{v}_{|\Gamma}\right\rangle_{H^1(\Gamma)},$$

and for $\mathbf{u}, \mathbf{v}$ in $\mathbb{V}(\Omega_-)$, we set

$$a_{\Omega_-}(\mathbf{u}, \mathbf{v}) = \int_{\Omega_-} \sigma(\mathbf{u}_-)\, \varepsilon(\mathbf{v}_-)\, d\Omega_-$$

$$a_\Gamma^1(\mathbf{u}, \mathbf{v}) = \int_\Gamma \begin{bmatrix} \mu\left(1 - \frac{1}{3}\delta^2 \Delta\right)^{-1} \operatorname{curl}_\Gamma u_T \operatorname{curl}_\Gamma v_T \\ +4\mu(1-a)\left(1 - \delta^2 \frac{1}{3}(5 - 6\alpha)\Delta_\Gamma\right)^{-1} \operatorname{div}_\Gamma u_T \operatorname{div}_\Gamma v \end{bmatrix} d\Gamma$$

$$a_\Gamma^2(\mathbf{u}, \mathbf{v}) = \int_\Gamma \begin{bmatrix} -2\mu(1-a)\left(1 - \delta^2 \frac{1}{3}(5 - 6\alpha)\Delta_\Gamma\right)^{-1} \Delta_\Gamma u_n . \operatorname{div}_\Gamma v_T \\ -2\mu(1-a) \operatorname{div}_\Gamma u_T \left(1 - \delta^2 \frac{1}{3}(5 - 6\alpha)\Delta_\Gamma\right)^{-1} \Delta_\Gamma v_n \end{bmatrix} d\Gamma$$

$$a_\Gamma^3(\mathbf{u}, \mathbf{v}) = \int_\Gamma \frac{1}{3}\mu(1-a)\Delta_\Gamma u_n \left(1 - \delta^2 \frac{1}{3}(5 - 6\alpha)\Delta_\Gamma\right)^{-1} \Delta_\Gamma v_n d\Gamma.$$

Remark 9. *1. The surface bilinear form $a_\Gamma^1(\mathbf{u}, \mathbf{v})$ is defined for $\mathbf{u}, \mathbf{v}$ in $\mathbb{H}^1(\Omega_-)$. It involves only the trace of the extension (tangential component) and $\delta a_\Gamma^1(\mathbf{u}, \mathbf{v})$ represents the effect of the thin slab at order one.*

2. *If we drop the stabilizing terms $\left(1 - \frac{1}{3}\delta^2\Delta\right)^{-1}$ and $\left(1 - \delta^2\frac{1}{3}(5 - 6\alpha)\Delta_\Gamma\right)^{-1}$ we recover the approximation of order one given in [13]. The slab behaves as a linear elastic surface Γ whose Lamé's coefficients are μ and $\lambda^* = 2\mu(1-a) = \frac{2\lambda\mu}{\lambda+2\mu}$.*

3. *The surface bilinear form $a_\Gamma^2(\mathbf{u}, \mathbf{v})$ is defined for $\mathbf{u}, \mathbf{v}$ in $\mathbb{H}^1(\Omega_-)$. It involves only the trace of the extension and the trace of the flexion (normal component). Further, the forms $\delta a_\Gamma^1(\mathbf{u}, \mathbf{v}) + \delta^2 a_\Gamma^2(\mathbf{u}, \mathbf{v})$ represent the effect of the slab at order two.*

4. *The definition of $a_\Gamma^3(\mathbf{u}, \mathbf{v})$ requires more regularrity on the trace of the flexion. It is well defined when the traces of u_n and v_n are in $H^1(\Gamma)$. The effect of the slab at order three is given by $\delta a_\Gamma^1(\mathbf{u}, \mathbf{v}) + \delta^2 a_\Gamma^2(\mathbf{u}, \mathbf{v}) + \delta^3 a_\Gamma^3(\mathbf{u}, \mathbf{v})$.*

Theorem 10. *For a given $\mathbf{f}_-$ in $\mathbb{L}^2(\Omega_-)$, there exists a unique solution $\mathbf{u}_-^*$ in $\mathbb{V}(\Omega_-)$ solution to the approximate impedance problem*

$$\begin{cases} -\nabla \cdot \sigma\left(\mathbf{u}_-^*\right) = \mathbf{f}_- & \text{in } \Omega_- \\ \mathbf{u}_-^* = 0 & \text{on } \Gamma_0 \\ \sigma\left(\mathbf{u}_-^*\right)\mathbf{n} = Z_\delta^*\left(\mathbf{u}_-^*\right) & \text{on } \Gamma. \end{cases}$$

Its variational formulation is given by

$$a\left(\mathbf{u}_-^*, \mathbf{v}_-\right) = l(\mathbf{v}_-), \forall \mathbf{v}_- \in \mathbb{V}(\Omega_-),$$

with

$$a = a_{\Omega_-} + \delta a_\Gamma^1 + \delta^2 a_\Gamma^2 + \delta^3 a_\Gamma^3$$
$$l(\mathbf{v}_-) = \int_{\Omega_-} \mathbf{f}_- \mathbf{v}_- d\Omega_-.$$

Moreover, $\mathbf{u}_-^* \in \mathbb{H}^2(\Omega_-)$, $\left(\sigma\left(\mathbf{u}_-^*\right)\mathbf{n}\right)_{T|\Gamma} \in \left(H^{3/2}(\Gamma)\right)^2$, and $(u_n^*)_{|\Gamma} \in H^{5/2}(\Gamma)$.

Proof. 1. The existence of the solutions follows form the coercivity estimate

$$a_{\Omega_-}(\mathbf{u},\mathbf{u}) + \delta a^1_\Gamma(\mathbf{u},\mathbf{u}) + \delta^2 a^2_\Gamma(\mathbf{u},\mathbf{u}) + \delta^3 a^3_\Gamma(\mathbf{u},\mathbf{u}) \geq C\delta^3 \|\mathbf{u}\|^2_{\mathbb{V}(\Omega_-)}.$$

2. The regularity results are obtained gradually in different steps combining well known smoothness results for boundary value problems for the Lamé system, the definition of the approximate impedance boundary condition, and the ellipticity of the operator $(Z^*_\delta)_{22}$.

First, since $(u^*_T)_{|\Gamma}$ in $\left(H^{1/2}(\Gamma)\right)^2$, then $(Z^*_\delta)_{21}(u^*_T) \in H^{-1/2}(\Gamma)$. We also have $\left(\sigma\left(\mathbf{u}^*_-\right)\mathbf{n}\right)_n$ in $H^{-1/2}(\Gamma)$, defined by

$$\left(\sigma\left(u^*_-\right)\mathbf{n}\right)_n = (Z^*_\delta)_{21}(u^*_T) + (Z^*_\delta)_{22}(u^*_n);$$

from which we deduce that

$$(Z^*_\delta)_{22}\, u^*_{-n} = \frac{1}{3}\delta^3\mu 2(1-a)\Delta_\Gamma\left(1-\delta^2\frac{1}{3}(5-6\alpha)\Delta_\Gamma\right)^{-1}\Delta_\Gamma u^*_{-n} \in H^{-1/2}(\Gamma).$$

This in turn implies by the regularity of the elliptic operator $(Z^*_\delta)_{22}$ that $u^*_n \in H^{3/2}(\Gamma)$.

Second, the fact that $u^*_n \in H^{3/2}(\Gamma)$ leads to

$$(Z^*_\delta)_{12}(u^*_n) = -\delta^2 2\mu(1-a)\nabla_\Gamma\left(1-\delta^2\frac{1}{3}(5-6\alpha)\Delta_\Gamma\right)^{-1}\Delta_\Gamma u^*_{-n} \in H^{1/2}(\Gamma),$$

and u^*_{-T} in $\left(H^{1/2}(\Gamma)\right)^2$ gives

$$\begin{aligned}(Z^*_\delta)_{11}(u^*_T) &= \delta\mu\overrightarrow{\operatorname{curl}}_\Gamma\left(1-\frac{1}{3}\delta^2\Delta\right)^{-1}\operatorname{curl}_\Gamma u^*_{-T}\\ &\quad - \delta 4\mu(1-a)\nabla_\Gamma\left(1-\delta^2\frac{1}{3}(5-6\alpha)\Delta_\Gamma\right)^{-1}\operatorname{div}_\Gamma u^*_{-T} \quad \in \left(H^{1/2}(\Gamma)\right)^2.\end{aligned}$$

Then, the boundary condition

$$\left(\sigma\left(\mathbf{u}^*_-\right)\mathbf{n}\right)_T = (Z^*_\delta)_{11}(u^*_T) + (Z^*_\delta)_{12}(u^*_n)$$

implies that $\left(\sigma\left(\mathbf{u}^*_-\right)\mathbf{n}\right)_T \in \left(H^{1/2}(\Gamma)\right)^2$. Thus, we deduce that

$$\begin{cases} -\nabla\cdot\sigma\left(\mathbf{u}^*_-\right) = \mathbf{f}_- & \text{in } \mathbb{L}^2(\Omega_-)\\ \mathbf{u}^*_- = 0 & \text{on } \Gamma_0\\ \left(\sigma\left(\mathbf{u}^*_-\right)\mathbf{n}\right)_T & \text{in } \left(H^{1/2}(\Gamma)\right)^2\\ \mathbf{u}^*_n & \text{in } H^{3/2}(\Gamma).\end{cases}$$

This boundary value problem for the Lamé system implies that $\mathbf{u}^*_- \in \mathbb{H}^2(\Omega_-)$, $u^*_T \in \left(H^{3/2}(\Gamma)\right)^2$, and $\left(\sigma\left(\mathbf{u}^*_-\right)\mathbf{n}\right) \in \mathbb{H}^{1/2}(\Gamma)$.

Third, from the last result, $u^*_T \in \left(H^{3/2}(\Gamma)\right)^2$, we get $(Z^*_\delta)_{21}(u^*_T) \in H^{1/2}(\Gamma)$ and

$$(Z^*_\delta)_{22}\left(u^*_{-n}\right) = \left(\sigma\left(u^*_-\right)\mathbf{n}\right)_n - (Z^*_\delta)_{21}\left(u^*_{-T}\right) \in H^{1/2}(\Gamma),$$

which gives by the regularity of the elliptic operators $(Z^*_\delta)_{22}$, $u^*_n \in H^{5/2}(\Gamma)$.

Finally, from $u_n^* \in H^{5/2}(\Gamma)$, we get $(Z_\delta^*)_{12}(u_n^*) \in \left(H^{3/2}(\Gamma)\right)^2$, and from $u_{-T}^* \in \left(H^{3/2}(\Gamma)\right)^2$ we get $(Z_\delta^*)_{11}(u_T^*) \in \left(H^{3/2}(\Gamma)\right)^2$. Then

$$\left(\sigma\left(\mathbf{u}_-^*\right)\mathbf{n}\right)_T = (Z_\delta^*)_{11}(u_T^*) + (Z_\delta^*)_{12}(u_n^*) \in \left(H^{3/2}(\Gamma)\right)^2.$$

$\square$

Remark 11. *If* $\mathbf{f}_-$ *is in* $\mathbb{H}^s(\Omega_-)$ $(s \geq 0)$*, we get by applying recursively the same arguments* $\mathbf{u}_-^* \in \mathbb{H}^{s+3/2}(\Omega_-)$*,* $u_n^* \in H^{s+5/2}(\Gamma)$ *, and* $\left(\sigma\left(\mathbf{u}_-^*\right)\mathbf{n}\right)_T \in \left(H^{s+3/2}(\Gamma)\right)^2$.

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