Spaces with local chronological structure and the cosheaf of fundamental semicategories

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Abstract

In the present work, we examine a structure that generalizes the causal structure of a Lorentzian spacetime. In contrast to similar definitions in the literature, we define the chronological relations locally, that is, on open subsets of a topological space. This has the advantage that we do not need to employ causality conditions for the whole space. The space of timelike homotopy classes of paths in such a space X forms an algebraic structure that we call the fundamental semicategory $\Pi^{\mathrm{t}}(X)$.

We provide a van-Kampen theorem for fundamental semicategories, show that the isomorphism class of $\Pi^{\mathrm{t}}(X)$ determines the topology and isomorphism class of X, and put a topology on the total morphism space of $\Pi^{\mathrm{t}}(X)$ that is locally homeomorphic to $X \times X$.

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Contents

1 Introduction

Lorentzian geometry is the mathematical framework of general relativity, which is, at the moment, the best available physical theory for gravity at the large scale of planets, solar systems, galaxies, and up to the size of the observable universe. It was derived from Riemannian geometry, which was conceived by Bernhard Riemann to describe curved, non-euclidean geometries. The main difference is that a Lorentzian metric consists of an indefinite symmetric bilinear form of signature $(1, n-1)$ on every tangent space, whereas a Riemannian metric consists of scalar products.

Many basic definitions and results are valid in both Riemannian geometry and Lorentzian geometry, for example, the existence and uniqueness of the Levi-Civita connection, local existence of geodesics, and the definition of sectional, Ricci- and scalar curvature, see [1, pp. 22–32]. Nevertheless, the methods in researching Riemannian and Lorentzian geometry differ vastly. A Riemannian metric can be used to define lengths of curves and a distance metric between points, while the analogous Lorentzian distance function does not satisfy the triangle inequality. On the other hand, a Lorentzian metric defines a set of *timelike* or *causal* directions in every tangent space. A Lorentzian manifold is called a *spacetime* if these time directions can consistently be classified into *future*- and *past*-directed components. On a spacetime, there are two relations $\leq$ and $\ll$ that describe which pairs of points are joined by a timelike or causal curve.

One broad goal of global differential geometry is the classification of manifolds by using geometry. An example of an effort toward this goal is the following finiteness theorem:

Theorem 1.0.1 (Perelman, see [2])**.** *Let $n \in \mathbb{N}$, $k \in \mathbb{R}$, $D > 0$, $v > 0$. The class of closed n-dimensional Riemannian manifolds with sectional curvature $\geq k$, diameter $\leq D$, and volume $\geq v$ has only finitely many topological types of manifolds.*

A corollary from this result is that there are only countably many topological types of manifolds. Perelmans' proof of the finiteness theorem relies heavily on *Alexandrov spaces*, which are length spaces that generalize Riemannian manifolds with a lower bound on their sectional curvatures. Discussing the details would go beyond the scope of this work, but the idea is that an Alexandrov space is used to describe a certain kind of limit of a sequence of Riemannian manifolds with curvature bounds.

Lorentzian length spaces were defined by Kunzinger and Sämann in [3] to provide a generalization of Lorentzian manifolds with curvature bounds analogous to Alexandrov spaces.

Several other authors have also defined spaces that generalize either the geometry of spacetime manifolds or only their causal structure. For example, a causal space according to Kronheimer and Penrose [4] is a topological space with two relations that satisfy certain axioms.

To the best of the author's knowledge, all of these generalizations forbid the existence of closed timelike curves. This postulate makes sense from a physics point of view and ensures that the causal relations are not trivial. However, it also severely restricts the possible topologies. For example, every compact spacetime manifold (without boundary) contains a closed timelike curve, see [1, Proposition 3.10].

The goal of this thesis is to provide a generalization of Lorentzian manifolds that does not have these restrictions. We will introduce the notion of a *locally chronological space* that consists of a topological space and a chronological relation for each open subset. These relations satisfy a so-called *cosheaf condition*, which roughly means that the relations on bigger subsets are determined by those on smaller subsets. As our axioms are local conditions, we can ensure that every spacetime manifold is a locally chronological space regardless of causality conditions.

Most of this work is dedicated to the study of the *fundamental semicategory* $\Pi^{\mathrm{t}}(X)$ of such a locally chronological space X, which is defined in analogy to the fundamental groupoid of a topological space.

The definition of locally chronological spaces and fundamental semicategories is inspired by the *directed spaces* in [5]. However, directed spaces in general lack some properties that are important for the investigation of the topology of Lorentzian manifolds, especially openness of timelike future and past sets.

This work is structured as follows: In Chapter 2 we will give an overview of basic concepts in Riemannian and Lorentzian geometry, with emphasis on causality theory of Lorentzian manifolds. Furthermore, we define the necessary notions from category theory and briefly discuss fundamental groups and groupoids.

In Chapter 3 we will define locally chronological spaces and discuss the reasoning behind our choice of axioms. We will also show that the class of all spacetime manifolds is a proper subclass of the locally chronological spaces. Moreover, we define the fundamental semicategory $\Pi^{\mathrm{t}}(X)$ of a locally chronological space X in this chapter.

In the rest of the chapters, we will derive several interesting properties of $\Pi^{\mathrm{t}}(X)$ by using three different approaches. In Chapter 4 we will prove that the isomorphism class of $\Pi^{\mathrm{t}}(X)$ already determines the isomorphism class of X under some mild assumptions. If X is a spacetime manifold, this means that $\Pi^{\mathrm{t}}(X)$ also determines the conformal class of the Lorentzian metric, if the differentiable structure is given.

In Chapter 5 we will view Π^{t} as a functor from the category of locally chronological spaces to the category of semicategories and prove a theorem analogous to the theorem of Seifert-van Kampen.

Finally, in Chapter 6 we will see that there is a natural topology on the total morphism set of $\Pi^{\mathrm{t}}(X)$ that is locally homeomorphic to $X \times X$. This also induces a local chronological structure on the total morphism set of $\Pi^{\mathrm{t}}(X)$.

2 Preliminaries

In this chapter, we will introduce the concepts necessary for understanding and putting the following chapters into perspective. In Section 2.1, we start with a brief review of the definition of a Riemannian manifold and see how it generalizes to so-called length spaces. In Section 2.2, we define Lorentzian manifolds and discuss their length functions and causality theory. In Section 2.3, we introduce semicategories, categories, semifunctors, and functors. In Section 2.4, we introduce two invariants from algebraic topology, namely the fundamental group and the fundamental groupoid.

2.1 Riemannian geometry and length spaces

At the beginning of the 19th century, mathematicians became increasingly interested in non-Euclidean geometries, particularly in spherical and hyperbolic geometry. Similarly to Euclidean geometry, these geometries are characterized by a set of axioms about their points and lines. The geometry of surfaces in $\mathbb{R}^3$ was also a topic of active research: In his *Theorema Egregium*[6] from 1827, Gauss proved that the Gaussian curvature of a surface can be expressed purely in terms of lengths measured on the surface. This means that the curvature describes the internal geometry of the surface and is independent of the chosen embedding into $\mathbb{R}^3$.

In his *Habilitationsschrift*[7] from 1854, Riemann defined a structure that describes all the aforementioned geometries and even generalizes to an arbitrary number of dimensions. To achieve this, he assigned *lengths* to continuously differentiable curves on a manifold by integrating their "line element" that he defined to be "the square root of a differential expression of degree two". In modern terms, this means:

Definition 2.1.1. Let M be a path-connected smooth manifold.

i) A *Riemannian metric* g is an assignment of a scalar product g_p on the tangent space[1] T_pM for every point $p \in M$ that is smooth in the following sense: In any chart with

[1]Readers unfamiliar with differential geometry should keep the example $M = \mathbb{R}^n$ in mind and also identify the tangent spaces T_pM with $\mathbb{R}^n$. The matrix with entries $g_{ij}(p)$ is then the Gram matrix, or fundamental matrix, of the scalar product g_p. In the case of Euclidean space, we have $g_{ij}(p) = \delta_{ij}$.

coordinates $(x_1, \dots, x_n)$ the scalar product is given by

$$g_p = \sum_{i,j=1}^{n} g_{ij}(p)\, \mathrm{d}x_i\, \mathrm{d}x_j,$$

where g_{ij} are smooth functions on the domain of the chart.

ii) The *length of a vector* $v \in T_pM$ is defined to be $\|v\| = \sqrt{g_p(v, v)}$.

iii) The *length of a curve* $c\colon [a, b] \to M$ is

$$L(c) := \int_a^b \|c'(t)\|\, \mathrm{d}t,$$

where a *curve* is understood to be piecewise continuously differentiable.

iv) The *length metric* or *distance* between two points $x, y \in M$ is

$$d(x, y) := \inf\{\, L(c) \mid c \text{ is a curve from } x \text{ to } y \,\}.$$

If we drop the assumption that the space is a smooth manifold, we can still define the length of a path by approximating the path with finitely many points:

Definition 2.1.2. Let (X, d) be a metric space.

i) Let c be a path, that is, a continuous map $c\colon [a, b] \to X$. The *length of* c is defined as

$$L(c) := \sup \left\{ \sum_{i=1}^{k-1} d(c(t_i), c(t_{i+1})) \;\middle|\; t_1, \dots, t_k \in \mathbb{R},\ a = t_1 < t_2 < \dots < t_k = b \right\}.$$

A path c is called *rectifiable* if $L(c)$ is finite.

ii) The *induced length metric* on (X, d) is

$$\lambda(d)(x, y) := \inf \{\, L(c) \mid c \text{ is a path from } x \text{ to } y \}$$

if there is a rectifiable path from x to y, and $\lambda(d)(x, y) = \infty$ otherwise

iii) If $\lambda(d) = d$ holds, then (X, d) is called a *length (metric) space*.

Remark 2.1.3. In both Definitions 2.1.1 and 2.1.2, we have the following properties:

i) The length of a constant path is zero and the length of a non-constant curve or path is positive.

ii) The length is invariant under reparametrization: Let $c\colon [a,b] \to X$ be a path and $f\colon [a',b'] \to [a,b]$ a continuous, strictly monotonous function. In the case of Riemannian geometry, suppose that f and c are piecewise continuously differentiable.
Then, we have $L(c) = L(c \circ f)$.

In Riemannian geometry, this follows from the fact that the integrand is homogeneous, which means that $\|\lambda v\| = |\lambda| \cdot \|v\|$ for all $v \in TM$, $\lambda \in \mathbb{R}$.

iii) The length is additive: Let c and d be curves that have finite lengths $L(c)$ and $L(d)$ such that the endpoint of c is the starting point of d. Then, the length of the concatenated curve is $L(c) + L(d)$.

iv) The length metric on a Riemannian manifold or a metric space is a (possibly infinite) metric. It is positive definite because of the positive length of non-constant paths or curves. The symmetry $d(x,y) = d(y,x)$ follows from the invariance of L under reparametrization, which allows us to reverse the direction of all curves. The triangle inequality $d(x,z) \geq d(x,y) + d(y,z)$ follows from the additivity of L and the fact that d is defined as an infimum of lengths of curves.

The Riemannian distance is indeed a length metric because two notions of length $L(c)$ coincide for any curve c. In this setting, Definition 2.1.2 is more general because it allows us to assign lengths to paths that are not piecewise continuously differentiable.

2.2 Lorentzian geometry

2.2.1 Lorentzian manifolds, general relativity, and causal structure

The definition of a Lorentzian metric is analogous to a Riemannian metric:

Definition 2.2.1. Let M be a smooth n-dimensional manifold. A *Lorentzian metric* g is an assignment of a symmetric bilinear form g_p with signature $(n-1,1)$ on the tangent space T_pM for every point $p \in M$ that is smooth in the same sense as in Definition 2.1.1.

A Lorentzian manifold is called a *spacetime* if it is connected and has a vector field V such that $g(V,V) < 0$. We say that V defines a *time orientation* on M.

The key difference between Riemannian and Lorentzian manifolds is that the bilinear forms g_p are not positive definite but indefinite. The most basic example of a smooth spacetime is the following:

Example 2.2.2 (Minkowski space)**.** Let $n \in \mathbb{N}$ and $M = \mathbb{R}^n$.
With the usual identification $T_pM = \mathbb{R}^n$, we define

$$g_p\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}\right) \coloneqq -x_1\, y_1 + \frac{1}{c^2}\big(x_2\, y_2 + x_3\, y_3 + \dots + x_n\, y_n\big),$$

where $c > 0$ is a constant, called the *speed of light* (usually set to either 1 or the actual speed of light). This Lorentzian manifold is called the n-dimensional *Minkowski space*[2]. It is time-oriented by the constant vector field $V = (1, 0, \dots, 0)$.

Historically, Minkowski space was developed as the mathematical model of a universe in Einstein's theory of special relativity. Einstein later noticed that he needed curvature to describe gravity in his theory of general relativity, so he generalized from flat Minkowski space to arbitrary smooth spacetimes.

A Lorentzian metric allows us to classify tangential vectors into the following classes:

Definition 2.2.3. In a Lorentzian manifold M, a vector $v \in T_pM$ is called

- *timelike* if $g(v, v) < 0$,
- *lightlike* or *null* if $g(v, v) = 0$,
- *spacelike* if $g(v, v) > 0$,
- *causal* or *nonspacelike* if $g(v, v) \leq 0$.

If M is time-oriented by a vector field V, a causal vector $v \in T_pM$ is called

- *future-directed* if $g_p(V_p, v) \leq 0$,
- *past-directed* if $g_p(V_p, v) \geq 0$.

If one of the above adjectives applies to all tangential vectors $c'(t)$ of a piecewise continuously differentiable curve c, the curve also has that adjective (e.g. a *future-directed timelike curve*).

Note that the set of causal vectors is a closed double cone in T_pM and the timelike vectors make up the interior of that double cone. With a time orientation, one of the cones is future-directed and the other cone is past-directed, as shown in Figure 2.1.

In our definition, the zero-vector is both future- and past-directed causal, which is uncommon, but has some advantages. First, the set of future-directed causal vectors is now simply the

[2]Note that there is a multitude of different notations and conventions. In physics, it is customary to set $n = 4$ and denote the time coordinate by x^0 or x^4, and the space coordinates by x^1, x^2, x^3 with an upper index. Signature is often denoted as $-{+}{+}{+}$ for the signs of the coefficients in g_p. The signatures $+{-}{-}{-}$, $-{-}{-}{+}$, and $+{+}{+}{-}$ are also commonly used by different authors depending on the application.

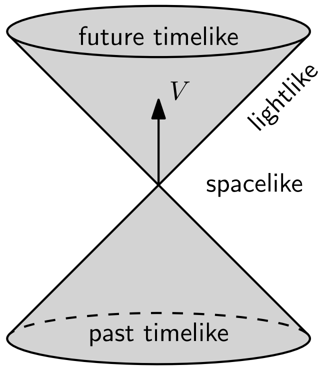


Figure 2.1: Light cones in a Lorentzian manifold. Lightlike vectors form the boundary of the double cone and separate timelike from spacelike vectors.

closure of the future-directed timelike vectors (and analogously for past-directed vectors). Second, it makes sense that the space of causal curves contains constant curves, as this will make Definition 2.2.5 more consistent, and constant curves act as identity morphisms in the fundamental category $\Pi^c(X)$ in [8].

Example 2.2.4. In physics, points of a Lorentzian manifold are called *events*. An event in the n-dimensional Minkowski space is described by one time coordinate x_1 and $n-1$ space coordinates $(x_2, \dots x_n)$.

Imagine that a curve $\gamma\colon [a,b] \to \mathbb{R}^n$, $t \mapsto \big(\gamma_1(t), \dots, \gamma_n(t)\big)$ in Minkowski space describes the motion of a particle through space and time. Let us assume that γ is future-directed, which means that γ_1 increases monotonically. This prevents situations in which the particle travels backward in time, see Subsection 2.2.3 for more discussion on that topic.

The velocity of γ in the coordinate system $(x_1, \dots, x_n)$ is defined as

$$v(t) \coloneqq \frac{\sqrt{\gamma_2'(t)^2 + \cdots + \gamma_n'(t)^2}}{\gamma_1'(t)},$$

that is, how fast the spatial position changes over time. The term $\gamma_1'(t)$ in the denominator is included to make $v(t)$ independent of the parametrization of γ. Without loss of generality, we can reparametrize γ such that $\gamma_1(t) = t$.

Because of $g\big(\gamma'(t), \gamma'(t)\big) = -\gamma_1'(t)^2 \left(1 - \frac{v(t)}{c}\right)^2$, the curve γ is lightlike if it moves at the speed of light, that is, $v(t) = c$ for all $t \in [a,b]$, causal if $v(t) \leq c$ for all t, and timelike if $v(t) < c$ for all t.

There are several theoretical and empirical arguments against particles or waves moving faster than the speed of light. It is therefore generally assumed that an event x cannot *cause* an effect at y unless there is a curve from x to y that does not surpass the speed of

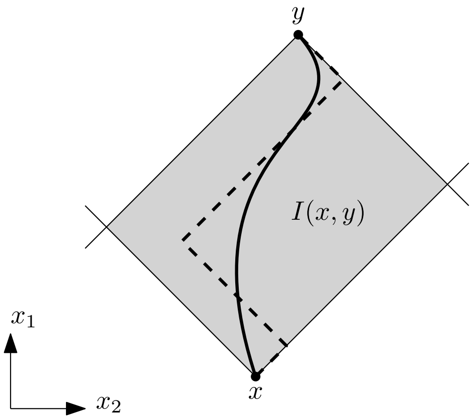


Figure 2.2: Causal and chronological structure in Minkowski space. The chronological diamond $I(x, y)$ (interior of the gray area) contains all future-directed timelike curves from x to y, for example, the thick solid line. The causal diamond $J(x, y)$, in this case the closure of $I(x, y)$, contains all causal curves from x to y. The dashed curve is lightlike and therefore has Lorentzian length 0.

light—hence the name “causal curve”. The name “timelike curve” stems from the fact that the time axis $t \mapsto (t, 0, \ldots, 0)$ of a coordinate system is such a timelike curve.

Definition 2.2.5. Let M be a smooth spacetime, $U \subseteq M$ an open subset, and $x, y \in U$. The *causal relation* $\leq_U$ and *chronological* or *timelike relation* $\ll_U$ are defined by

$$x \leq_U y :\Longleftrightarrow \text{ there is a future-directed causal curve in } U \text{ from } x \text{ to } y,$$
$$x \ll_U y :\Longleftrightarrow \text{ there is a future-directed timelike curve in } U \text{ from } x \text{ to } y.$$

The the *chronological* and *causal future* and *past* of a point $x \in U$ is defined by

$$I_U^+(x) \coloneqq \{ y \in M \mid x \ll_U y \}, \qquad J_U^+(x) \coloneqq \{ y \in M \mid x \leq_U y \},$$
$$I_U^-(x) \coloneqq \{ y \in M \mid y \ll_U x \}, \qquad J_U^-(x) \coloneqq \{ y \in M \mid y \leq_U x \},$$

and the *chronological* and *causal diamonds* are defined by

$$I_U(x, y) \coloneqq I_U^+(x) \cap I_U^-(y),$$
$$J_U(x, y) \coloneqq J_U^+(x) \cap J_U^-(y).$$

For brevity, the subscript is left out if $U = M$.

Remember that, according to our definition, constant curves are causal but not timelike, so $x \leq x$ is true for all points, while $x \ll x$ implies that there is a (non-constant) closed timelike curve from x to x. The following properties are well known in Lorentian geometry:

Lemma 2.2.6 ([1, p. 55]).

i) The sets $I^{\pm}(x)$ are open, but the sets $J^{\pm}(x)$ might be neither open nor closed.

ii) The relation $\ll$ is a subrelation of $\leq$.

iii) The relations $\leq$ and $\ll$ are transitive.

iv) The relations satisfy the so-called push-up property

$$\forall x, y, z \in M : x \ll y \leq z \text{ or } x \leq y \ll z \implies x \ll z.$$

These properties are also true for $I^{\pm}_U$, $J^{\pm}_U$, $\ll_U$, and $\leq_U$, respectively, for every open $U \subseteq M$.

2.2.2 Lengths in Lorentzian manifolds

In Section 2.1, we saw that a Riemannian metric induces a length functional on curves and a length metric on the manifold. Remember that the length metric of a Riemannian manifold is invariant under reparametrization because of the homogeneity of the norm $\|v\| = \sqrt{g(v,v)}$, as discussed in Remark 2.1.3. A Lorentzian metric induces a similar *distance function*, but we will have to work around some issues and it will not be a genuine metric.

In a Lorentzian manifold, g is not positive definite, so we have to use $\sqrt{|g(v,v)|}$ in place of a norm[3]. Consequently, we have to assign length zero to all lightlike curves even if they are not constant curves. As any curve can be approximated with a zig-zag lightlike curve, which is illustrated in Figure 2.2, the definition $d(x,y) := \inf \{ L(c) \mid c \text{ curve from } x \text{ to } y \}$ would assign distance zero to any pair of points on a connected Lorentzian manifold. The supremum of path lengths, on the other hand, would be infinite because we can find arbitrarily long paths by just going forward and backward any number of times. Luckily, in a spacetime, we can forbid this kind of behavior by restricting ourselves to future-directed curves:

Definition 2.2.7 ([9, p. 105]). Let M be a smooth spacetime with Lorentzian metric g.

i) The *length of a future-directed causal curve* $c\colon [a,b] \to M$ is defined as

$$L(c) := \int_a^b \sqrt{|g(c'(t), c'(t))|}\, \mathrm{d}t.$$

ii) The *Lorentzian distance between two points* $x, y \in M$ is defined as

$$d(x,y) := \begin{cases} \sup \{ L(c) \mid c \text{ future-directed causal curve from } x \text{ to } y \} & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

[3]The function $T_pM \to \mathbb{R}$, $v \mapsto \sqrt{|g(v,v)|}$ is, up to scaling, the only non-negative homogeneous function that is invariant under the Lorentz group, which is the group of vector space automorphisms of T_pM that leave g_p invariant.

Note that $d(x, y)$ is positive if and only if $x \ll y$. This is because a future-directed causal curve from x to y has nonzero length if and only if at least some part of it is timelike. The push-up property in Lemma 2.2.6 implies that such a curve exists if and only if $x \ll y$. The convention $d(x, y) = 0$ for $x \not\leq y$ is chosen because it continuously extends d under certain conditions (namely, if the spacetime is globally hyperbolic, see Definition 2.2.9). However, without further assumptions on the causal structure of M, the distance function d might be infinite or non-continuous. We will further discuss this topic at the end of Subsection 2.2.3.

The additivity of L, together with the supremum in the definition of d, implies an *inverse triangle inequality*

$$d(x, z) \geq d(x, y) + d(y, z) \quad \text{for all } x, y, z \in M \text{ with } x \leq y \leq z.$$

Therefore, in contrast to Remark 2.1.3, taking detours makes the Lorentzian length shorter instead of longer. We have already seen this kind of behavior in Figure 2.2. The reason is that $\sqrt{|g(v, v)|}$ is convex for a Riemannian metric g but concave for a Lorentzian metric g. This fact is best demonstrated in Minkowski space:

Example 2.2.8. We calculate the Lorentzian distance function for Minkowski space (M, g), give another proof for the inverse triangle inequality in this special case, and show that a straight line has the maximal length among all future-directed causal curves from x to y:

Let us first examine the integrand of the length functional

$$\begin{aligned} \sigma \colon C &\to \mathbb{R}, \\ v &\mapsto \sqrt{|g(v, v)|} = \sqrt{-g(v, v)} \end{aligned}$$

where $C \subseteq \mathbb{R}^n$ is the convex cone of future-directed causal vectors. It is not hard to see that σ is homogeneous, which means that $\sigma(\lambda\, v) = \lambda\, \sigma(v)$ holds for all $\lambda > 0$ and $v \in C$. Furthermore, it is concave, which means that

$$\sigma\big((1 - \alpha)\, v + \alpha\, w\big) \geq (1 - \alpha)\, \sigma(v) + \alpha\, \sigma(w)$$

holds for all $v, w \in C$ and $\alpha \in [0, 1]$. Setting $\alpha = \frac{1}{2}$ yields the inverse triangle inequality

$$\begin{aligned} & \sigma(v + w) \geq \sigma(v) + \sigma(w) && \text{for } v, w \in C \\ \implies\ & \sigma(z - x) \geq \sigma(z - y) + \sigma(y - x) && \text{for } x \leq y \leq z. \end{aligned}$$

If c is any future-directed causal curve from x to y, we see

$$L(c) = \int_a^b \sigma\left(c'(t)\right) \mathrm{d}t \geq \sigma\left(\int_a^b c'(t)\, \mathrm{d}t\right) = \sigma(y - x)$$

by Jensen's inequality for integrals[4]. If c is a straight line, c' is constant and therefore the equality $L(c) = \sigma(y - x)$ holds. The Lorentzian distance is therefore

$$d(x,y) = \begin{cases} \sqrt{|g(y-x,y-x)|} & \text{if } y - x \text{ is future-directed causal,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that d has finite values and is a continuous function $d\colon M \times M \to \mathbb{R}$.

A straight line c has the property that its tangential vectors $c'(t)$ are parallel along c. In a general Lorentzian manifold such curves are called *geodesics*. It is true that any longest curve between two points in a Lorentzian manifold is a geodesic, but the existence of a longest curve is not guaranteed.

2.2.3 Causality conditions and Alexandrov topology

Minkowski space has the special property that the time-coordinate x_1 increases monotonically along any future-directed causal curve. Therefore, such a curve can never return to its starting point.

For general spacetimes, it might well be possible that a closed timelike loop exists. A person following it would meet themselves at an earlier time and could prevent their journey, so they would never meet their earlier self, which leads to a paradoxon. Discussing the ramifications of time-traveling is beyond the scope of this work, but we will discuss some of the causality conditions that are frequently imposed on spacetimes to prevent this kind of situation.

Definition 2.2.9 ([1, Section 3.2])**.** Let M be a smooth spacetime with causal and chronological relations $\leq$ and $\ll$.

i) M is *totally vicious* if any two points $x, y \in M$ are joined by a future-directed timelike curve, hence $\ll$ is the trivial relation on M.

ii) M is *chronological* if there is no timelike curve that starts and ends in the same point. This is equivalent to $\forall x \in M : x \not\ll x$.

iii) M is *causal* if there is no non-constant[5] future-directed causal curve that starts and ends in the same point. This is the case if and only if $\leq$ is antisymmetric and therefore a partial order.

[4] Note that Jensen's inequality is usually stated for convex functions, see [10]. As σ is concave, we apply the inequality to the convex function $-\sigma$, which is why the direction of the inequality is reversed.

[5] Remember that, in our definition, constant curves are causal but not timelike.

iv) A subset $V \subseteq M$ is called *causally convex in M* if every future-directed causal curve that starts and ends in V lies completely in V.
This is equivalent to $\forall x, y \in V : J_M(x,y) \subseteq V$.

M is *strongly causal* if for any neighborhood $U \subseteq M$ of $x \in M$ there exists a sub-neighborhood $V \subseteq U$ of p that is causally convex in M.

v) M is *globally hyperbolic* if it is strongly causal and $J(x,y)$ is compact for all $x, y \in M$.

These conditions (together with other conditions in between) are sometimes called the causal ladder because of the implications

$$\begin{aligned} & M \text{ is globally hyperbolic} \\ \Longrightarrow\ & M \text{ is strongly causal} \\ \Longrightarrow\ & M \text{ is causal} \\ \Longrightarrow\ & M \text{ is chronological} \\ \Longrightarrow\ & M \text{ is not totally vicious.} \end{aligned}$$

The definition of a strongly causal spacetime formalizes the idea that there are no *almost closed* causal curves, that is, curves that start near x, move away, and then come back arbitrarily close to x, in the sense of topology. Therefore, it makes sense that we can reconstruct the topology from the causal structure in such a spacetime:

Lemma 2.2.10 ([4, p. 34])**.** *Let M be a smooth spacetime. The topology generated by the chronological diamonds $I(x,y)$ for all $x, y \in M$, which is called* Alexandrov topology*, coincides with the manifold topology of M if and only if M is strongly causal.*

Proof. Note that a subset $V \subseteq M$ is causally convex if and only if $J(x,y) \subset V$ for all $x, y \in V$. Because $\ll$ is a subrelation of $\leq$, this also implies $I(x,y) \subset V$ for all $x, y \in V$.

First, assume that M is strongly causal, so for any neighborhood U of $x \in M$, we can find a causally convex neighborhood $V \subseteq U$ and a future-directed timelike curve c with $c\left(\frac{1}{2}\right) = x$. For any small enough $\varepsilon > 0$, we have $c\left(\frac{1}{2} \pm \varepsilon\right) \in V$, which implies $x \in I\Big(c\left(\frac{1}{2} - \varepsilon\right), c\left(\frac{1}{2} + \varepsilon\right)\Big) \subseteq V \subseteq U$ because of the causal convexity of V. As U was an arbitrary neighborhood of x, this shows that the family $\left\{ I\Big(c\left(\frac{1}{2} - \varepsilon\right), c\left(\frac{1}{2} + \varepsilon\right)\Big) \,\Big|\, \varepsilon > 0 \right\}$ is a neighborhood basis of x.

For the reverse implication, let U be any neighborhood of $x \in M$. If the Alexandrov topology coincides with the manifold topology, there are $y, z \in M$ with $x \in I(y,z) \subseteq U$. By the push-up property in Lemma 2.2.6 the set $I(y,z)$ is causally convex. □

Global hyperbolicity is the strongest and one of the most important causality conditions. Its name comes from the fact that certain hyperbolic wave equations, e.g. the Klein-Gordon equation

$$\sum_{i,j} g_{ij} \frac{\partial^2 \psi(x)}{\partial x_i \partial x_j} + \mu^2 \psi(x) = 0$$

in local coordinates, have a unique global solution for suitable initial conditions on a *Cauchy hypersurface*.

Definition 2.2.11. Let M be a smooth spacetime. A curve $c\colon (a,b) \to M$ is called inextendible if it cannot be continuously extended to the domain $[a,b)$ or $(a,b]$.

A *Cauchy hypersurface* is a subset $S \subseteq M$ that every inextendible causal curve intersects exactly once.

Theorem 2.2.12 ([11, 12]). *Let M be a smooth spacetime. The following are equivalent:*

i) M is globally hyperbolic.

ii) M has a Cauchy hypersurface S.

iii) M is diffeomorphic to $\mathbb{R} \times S$ and every hypersurface $\{t\} \times S$ is a Cauchy hypersurface.

The theorem does not imply that a globally hyperbolic spacetime necessarily carries a product metric, it is just diffeomorphic to a product manifold. But if $M = \mathbb{R} \times S$ has indeed a Lorentzian product metric $-\mathrm{d}t^2 + g$, where (S, g) is a Riemannian manifold, then M is globally hyperbolic if and only if (S, g) is a complete metric space, see [1, p. 50]. In this sense, global hyperbolicity has the flavor of a completeness condition. Compactness of M, on the other hand, is not a very useful condition for Lorentzian spacetimes, as a compact spacetime cannot be chronological, see [1, Proposition 3.10].

It is also interesting to look at the properties of the Lorentzian distance function d on spacetimes with different causality conditions. If there is a closed timelike curve that starts and ends in $x \in M$, it is immediately clear that $d(x,x) = \infty$, as we can go around this loop multiple times to get arbitrarily long curves from x to x. By the inverse triangle inequality $d(x,x) = \infty$ also implies $d(y,z) = \infty$ whenever $y \leq x \leq z$. For totally vicious spacetimes we even have $d(x,y) = \infty$ for all $x, y \in M$.

Closed timelike loops are not the only cause for infinite Lorentzian distances. Even on a strongly causal spacetime, there might be points $x, y \in M$ with $d(x,y) = \infty$. This can only happen if $J(x,y)$ is not compact, which intuitively means that causal curves from x to y can go “far out” and spend an arbitrarily long time before coming back to y. Therefore, globally hyperbolic spacetimes have the most well behaved distance functions:

Lemma 2.2.13. *([13, 14]) Let M be a globally hyperbolic spacetime.*

The Lorentzian distance between any two points is finite and $d\colon M \times M \to \mathbb{R}$ is continuous.

For any two points $x, y \in M$, there is future-directed causal curve from x to y with length $d(x, y)$.

2.3 Semicategories, categories, and groupoids

Category theory has been developed to provide a common description of many different fields of mathematics, but has later proven to yield interesting and useful algebraic structures in itself. Semicategories are a generalization of categories in which we do not postulate the existence of identity morphisms. They are therefore less commonly used, but will be very useful in our analysis of chronological structures.

To motivate the following definitions, let us point out some analogies between set theory, group theory and topology:

	set theory	group theory	topology
objects	sets	groups	topological spaces
morphisms	maps	group homomorphisms	continuous maps
isomorphisms	bijective maps	group isomorphisms	homeomorphisms

In these three cases, the objects are sets that might have the additional structure of a group or a topology. Maps that respect these structures are generally called *morphisms* between the objects. The composition of any two morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ is another morphism $gf\colon X \to Z$, and this composition is associative. If a morphism f between sets/groups/topological spaces has an inverse map f^{-1} that is also a morphism, we call f an *isomorphism*. Isomorphic objects are regarded as practically identical: The isomorphism f only renames the elements but does not change the structure. These common structural properties are axiomatized in the definition of a category. We will denote the discussed categories by **Set**, **Grp**, and **Top**.

Definition 2.3.1. A *semicategory* $\mathcal{C}$ consists of

- a class $\mathrm{Obj}(\mathcal{C})$, called the class of *objects* or *points*,
- a class $\mathcal{C}(x, y)$ for any $x, y \in \mathrm{Obj}(\mathcal{C})$, called the class of *morphisms from x to y*. Morphisms are often denoted by $f\colon x \to y$ or $x \xrightarrow{f} y$ for $f \in \mathcal{C}(x, y)$.

- and an *operation*

$$\begin{aligned}\mathcal{C}(y,z)\times\mathcal{C}(x,y) &\to \mathcal{C}(x,z)\\ (a,b) &\mapsto ab\end{aligned}$$

  for all $x, y, z \in \mathrm{Obj}(\mathcal{C})$, such that $(ab)\,c = a\,(bc)$ holds whenever ab and bc are defined.

We will denote the *total morphism class* of a semicategory $\mathcal{C}$ by

$$\mathrm{Mor}(\mathcal{C}) := \bigsqcup_{x,y\in\mathrm{Obj}(\mathcal{C})} \mathcal{C}(x,y),$$

where $\bigsqcup$ denotes a disjoint union of classes.

The *start- and endpoint map* of a semicategory is defined as

$$\begin{aligned}(s,e)\colon\ \mathrm{Mor}(\mathcal{C}) &\to \mathrm{Obj}(\mathcal{C})\times\mathrm{Obj}(\mathcal{C})\\ a &\mapsto (x,y) \quad \text{if } a\in\mathcal{C}(x,y).\end{aligned}$$

A semicategory $\mathcal{C}$ is called *small* if $\mathrm{Obj}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{C})$ are sets.

Readers who are not familiar with the distinction between classes and sets can use these terms synonymously, at least for the moment. The main object of interest in this work, $\Pi^{\mathrm{t}}(X)$, is indeed a small semicategory.

Note that the objects and morphisms of a semicategory are just abstract elements. Despite the notation, morphisms do not need to be actual maps.

Definition 2.3.2. Let $\mathcal{C}$ be a semicategory.

A morphism $\mathrm{id}_x \in \mathcal{C}(x,x)$ is called an *identity* (or *identity morphism*) of x if

$$\forall y\in\mathrm{Obj}(\mathcal{C}), a\in\mathcal{C}(x,y), b\in\mathcal{C}(y,x) : a\,\mathrm{id}_x = a \text{ and } \mathrm{id}_x\, b = b.$$

If an identity of x exists, it is unique.

Two morphisms $a\in\mathcal{C}(x,y)$ and $b\in\mathcal{C}(y,x)$ are called *inverse to each other* if $ab = \mathrm{id}_y$ and $ba = \mathrm{id}_x$. In this case, we write $b = a^{-1}$ and call both a and b an *isomorphism*.

Definition 2.3.3. A semicategory $\mathcal{C}$ is called a *category* if it has an identity morphism id_x for every object $x\in\mathrm{Obj}(\mathcal{C})$.

A small category is called a *groupoid* if every morphism in $\mathcal{C}$ is an isomorphism.

Structure preserving maps between (semi-)categories are called (semi-)functors:

Definition 2.3.4. Let $\mathcal{C}$, $\mathcal{D}$ be semicategories. A *semifunctor*, denoted by $F\colon \mathcal{C} \to \mathcal{D}$, consists of two maps $\mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$ and $\mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$, called *object map* and *morphism map*, that satisfy the following axioms. We will call both maps F, as it will always be clear from the context which map is meant.

i) For all objects $x, y \in \mathrm{Obj}(\mathcal{C})$, we have

$$a \in \mathcal{C}(x, y) \implies F(a) \in \mathcal{D}\big(F(x), F(y)\big).$$

ii) If $a \in \mathcal{C}(y, z)$ and $b \in \mathcal{C}(x, y)$, then $F(ab) = F(a)F(b)$.

If $\mathcal{C}$ and $\mathcal{D}$ are categories, a *functor* $F\colon \mathcal{C} \to \mathcal{D}$ is a semifunctor that additionally satisfies

$$F(\mathrm{id}_x) = \mathrm{id}_{F(x)} \text{ for all } x \in \mathrm{Obj}(\mathcal{C}).$$

Remark 2.3.5. Let $\mathcal{C}$ be a small semicategory, and $x \in \mathrm{Obj}(\mathcal{C})$.

The operation $\mathcal{C}(x, x) \times \mathcal{C}(x, x) \to \mathcal{C}(x, x)$ is associative, so $\mathcal{C}(x, x)$ is a *semigroup.*

If $\mathcal{C}$ is a category, $\mathcal{C}(x, x)$ has an identity element, so it is a *monoid.*

If $\mathcal{C}$ is a groupoid, every morphism is invertible, so $\mathcal{C}(x, x)$ is a *group.*

On the other hand, every semigroup, monoid, or group is (the morphism set of) a semicategory, category, or groupoid, respectively, with one object. A homomorphism (of semigroups, monoids, or groups) is then the same as a (semi-)functor.

Based on these observations, it might be helpful to regard semicategories, categories, and groupoids as a generalization of semigroups, monoids, and groups in that the operation $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is only partially defined. We cannot multiply any two arbitrary elements (morphisms), but only ones that have compatible start- and endpoints. This also makes it necessary to introduce multiple neutral elements or identities.

Lemma 2.3.6. *The class* **SemCat** *of all small semicategories, with semifunctors as morphisms, is a category.*

The class **Cat** *of all small categories, with functors as morphisms, is a category.*

The class **Grpd** *of all small groupoids, with functors as morphisms, is a category.*

An isomorphism of semicategories, categories, or groupoids $\mathcal{C}, \mathcal{D}$ is a (semi-)functor that has an inverse (semi-)functor.

There is also the more general notion of *equivalence* of categories, which often provides a more insightful point of view on categories. However, equivalences can only be defined between categories, not between semicategories, because semicategories do not have identities and isomorphisms.

Definition 2.3.7. Let $\mathcal{C}, \mathcal{D}$ be two semicategories and $F, G \colon \mathcal{C} \to \mathcal{D}$ be two semifunctors. A *natural transformation* η from F to G is an assignment of a morphism $\eta_x \in \mathcal{D}\big(F(x), G(x)\big)$ to every $x \in \mathrm{Obj}(\mathcal{C})$ such that for any $f \in \mathcal{C}(x, y)$ the square

$$\begin{array}{ccc} F(x) & \xrightarrow{\eta_x} & G(x) \\ \Big\downarrow{\scriptstyle F(f)} & & \Big\downarrow{\scriptstyle G(f)} \\ F(y) & \xrightarrow{\eta_x} & G(y) \end{array}$$

commutes.

If $\mathcal{C}$ and $\mathcal{D}$ are categories, $F, G \colon \mathcal{C} \to \mathcal{D}$ are functors, and η_x is an isomorphism for every $x \in \mathrm{Obj}(\mathcal{C})$, then η is called a *natural isomorphism*.

Two categories $\mathcal{C}$ and $\mathcal{D}$ are called *equivalent* if there are functors $F \colon \mathcal{C} \to \mathcal{D}$, $H \colon \mathcal{D} \to \mathcal{C}$ such that there are natural isomorphisms from $H \circ F$ and $F \circ H$ to the identity functor of $\mathcal{C}$ and $\mathcal{D}$, respectively. In this case, G is called a *weak inverse* of F and vice versa.

2.4 Homotopies and algebraic topology

The broad goal of algebraic topology is the description of topological spaces by *algebraic invariants*, which usually are functors from a category of topological spaces to a category of algebraic objects like groups, rings, modules, groupoids, etc. Algebraic invariants encode certain details of the shape of a topological space and therefore can be used to distinguish topological spaces.

If one has enough suitable algebraic invariants of a space X, one can even hope to reconstruct the topology of X from these invariants. In Chapter 4 we will accomplish exactly that for locally chronological spaces.

Algebraic invariants are often invariant under continuous deformations, called *homotopy equivalences*. Let us first define how to continuously deform maps and spaces:

Definition 2.4.1.

i) Let X, Y be topological spaces. A family of maps $f_s \colon X \to Y$, for $s \in [0, 1]$ is called a *homotopy* (of maps) if the map

$$\begin{aligned} H \colon X \times [0,1] &\to Y \\ (x, s) &\mapsto f_s(x) \end{aligned}$$

is continuous. If such a homotopy exists, the maps f_0 and f_1 are called *homotopic to each other*, written as $f_0 \sim f_1$.

ii) Two topological spaces X, Y are *homotopy equivalent* if there are continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$. In this case, the maps f and g are called *inverses up to homotopy*.

2.4.1 Fundamental groups

In this section, we will closely follow the beginning of Chapter 1 in [15] and showcase the fundamental group $\pi_1(X)$, which is one of the simplest and most important algebraic invariants of a topological space. It is constructed from homotopy classes of closed paths:

Definition 2.4.2. Let X be a topological space.

i) A *path* in X is a continuous map $[0,1] \to X$. The set of all paths in X is called the *path space of* X and denoted by $P(X)$. The subset

$$P(X)(x,y) = \{ c \in P(X) \mid c(0) = x,\, c(1) = y \}$$

is the space of *paths from* $x \in X$ *to* $y \in X$.

ii) For two paths $c \in P(X)(x,y)$ and $d \in P(X)(y,z)$, their *concatenation* is given by

$$\begin{aligned} dc\colon [0,1] &\to X \\ t &\mapsto \begin{cases} c(2t) & \text{if } t \in \left[0, \frac{1}{2}\right] \\ d(2t-1) & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases} \end{aligned}$$

iii) For a path $c \in P(X)(x,y)$, its *reverse path* is defined as

$$\begin{aligned} \overline{c}\colon [0,1] &\to X \\ t &\mapsto c(1-t) \end{aligned}$$

iv) A *free homotopy* between paths c_0 and c_1 is a homotopy $(c_s)_{s \in [0,1]}$ of maps.

v) A *homotopy relative to the endpoints* between paths c_0 and c_1 from x to y is a free homotopy with $c_s \in P(X)(x,y)$ for all $s \in [0,1]$.

For brevity, we will often just write "homotopy" instead of "homotopy relative to the endpoints", and only use the notation $c_0 \sim c_1$ in this case.

Note that in this work the notation dc means "first c then d", analogous to the composition of maps or morphisms in a category. Defining it the other way around, as it is done in [15], will give an isomorphic group structure on $\pi_1(X)$.

To get an intuitive understanding of homotopies, imagine paths as (infinitely stretchable) rubber strings on a surface. To indicate the direction, mark the beginning and end of a string differently. A free homotopy may deform a string and move the endpoints around. It is easy to imagine that any two paths in the same path-connected component of X are freely homotopic to each other. For a homotopy (relative to the endpoints), the start and endpoint are pinned down at fixed locations. This gives a more interesting equivalence relation because the string can "get caught" on certain features of the space. If the start and endpoint are equal, we even get a group structure:

Definition 2.4.3. Let X be a topological space, fix a *basepoint* $x \in X$ and let

$$\pi_1(X,x) = P(X)(x,x) \big/ \sim$$

be the set of homotopy classes $[c]$ of paths from x to x. The operation

$$\begin{aligned} \pi_1(X,x) \times \pi_1(X,x) &\to \pi_1(X,x) \\ [d][c] &\mapsto [dc] \end{aligned}$$

is well-defined and turns $\pi_1(X,x)$ into a group, called the *fundamental group of X with basepoint x*. The neutral element is represented by the constant path in x, and inverses are represented by reversed paths, $[c]^{-1} = [\bar{c}]$.

Lemma 2.4.4. *If there is a path from x to y in X, the groups $\pi_1(X,x)$ and $\pi_1(X,y)$ are isomorphic to one another. If X is path-connected, one usually just writes $\pi_1(X)$ instead of $\pi_1(X,x)$*

We will postpone the proof of the lemma until after Definition 2.4.5. For the details of the proof that Definition 2.4.3 actually defines a group, and for more images see e.g. [15]. We will only indicate the necessary steps, using the rubber string analogy from above.

Note that concatenation of paths is not associative[6], but $(ab)c$ and $a(bc)$ are reparametrizations of the same path. We can write down a homotopy that interpolates continuously between these two parametrizations, so the operation in $\pi_1(X)$ is indeed associative. A path or rubber string represents the neutral element if and only if it can be pulled together to a point. Concatenation means tying the end of one string to the beginning of the other; the juncture can now be moved around, as it is no longer the start or endpoint of the concatenated curve. If we concatenate a path with its reverse path, we get a cord with two strands that is only fixed at one point. After letting go of the junction, the cord will pull itself to the basepoint x, which illustrates that the concatenated path represents the neutral element.

It is intuitively clear that on a 2-dimensional sphere every loop can be pulled together to a point, so $\pi_1(S^2)$ is a trivial group. On a torus, the same is not possible if the loop goes through the hole, so $\pi_1(T^1)$ is nontrivial.

[6] In Subsection 2.4.3, we will give an alternative definition of a path space, in which concatenation is actually associative and therefore the path space becomes a category.

2.4.2 Fundamental groupoids

In Remark 2.3.5, we introduced groups as a special case of groupoids with only one object. In this sense, a fundamental group is the special case of a fundamental groupoid, which we are about to define. From a historical perspective, this is of course backward: Fundamental groups were introduced in 1895 by Poincaré [16], while fundamental groupoids were introduced in the 1930s (see [17]) to overcome certain limitations of fundamental groups, which we will discuss in Chapter 5.

Definition 2.4.5. Let X be a topological space and let

$$\Pi(X)(x,y) = P(X)(x,y) \big/ \sim$$

be the set of homotopy classes $[c]$ of paths from x to $y \in X$. For any $x, y, z \in X$, the operation

$$\begin{aligned} \Pi(X)(y,z) \times \Pi(X)(x,y) &\to \Pi(X)(x,z) \\ \big([d],[c]\big) &\mapsto [d][c] := [dc] \end{aligned}$$

is well-defined and turns $\Pi(X)$ into a groupoid, called the *fundamental groupoid of X*. The identities are represented by constant paths, and inverses are given by reversed paths, $[c]^{-1} = [\overline{c}]$.

For any subset $A \subseteq X$, called a *set of basepoints*, we define $\Pi(X,A)$ to be the subcategory of $\Pi(X)$ with $\mathrm{Obj}\big(\Pi(X,A)\big) = A$ and $\Pi(X,A)(x,y) = \Pi(X)(x,y)$ for all $x, y \in A$.

To show that $\Pi(X)$ is a groupoid, we need to take essentially the same steps as we need to show that $\pi_1(X)$ is a group. In fact we have $\pi_1(X,x) = \Pi(X)(x,x)$. In this context, it is now easy to prove Lemma 2.4.4: If there is a path c in X from x to y, the maps

$$\begin{aligned} \Pi(X)(x,x) &\to \Pi(X)(y,y) \\ [g] &\mapsto [c][g][c]^{-1} \end{aligned} \qquad\qquad \begin{aligned} \Pi(X)(y,y) &\to \Pi(X)(x,x) \\ [g] &\mapsto [c]^{-1}[g][c] \end{aligned}$$

are group homomorphisms (called *change-of-basepoint isomorphisms*) that are inverse to each other, so

$$\pi_1(X,x) = \Pi(X)(x,x) \cong \Pi(X)(y,y) = \pi_1(X,y).$$

Spaces with trivial fundamental groups are of special interest in algebraic topology:

Definition 2.4.6. A topological space X is *simply connected* if $\pi_1(X,x)$ is trivial for every $x \in X$.

A semicategory $\mathcal{C}$ is *connected* if every morphism class $\mathcal{C}(x,y)$ contains at least one morphism.

A semicategory $\mathcal{C}$ is *simply connected* or *thin* if every morphism class $\mathcal{C}(x, y)$ contains at most one morphism.

Lemma 2.4.7. *A topological space X is path-connected if and only if its fundamental groupoid $\Pi(X)$ is connected.*

A topological space X is simply connected if and only if its fundamental groupoid $\Pi(X)$ is simply connected.

Proof. The first statement is clear from the definitions.

If $\Pi(X)$ is simply connected, then $\pi_1(X, x) = \Pi(X)(x, x)$ contains id_x but no other morphism, hence $\pi_1(X, x)$ is a trivial group for every x.

If X is simply connected, any two $a, b \in \Pi(X)(x, y)$ satisfy

$$b^{-1}a \in \Pi(X)(x, x) = \pi_1(X, x) = \{\, \mathrm{id}_x \,\}$$

and therefore

$$a = \mathrm{id}_y a = bb^{-1}a = b\,\mathrm{id}_x = b.$$

□

In and of itself, the fundamental groupoid $\Pi(X)$ of a single space X contains little more information than the fundamental groups $\pi_1(X, x)$ for all x. Its power as a tool in algebraic topology mainly comes from the fact that we can relate different topological spaces or subspaces via arbitrary continuous maps:

Lemma 2.4.8. *Any continuous map $f \colon X \to Y$ between topological spaces X, Y induces a functor*

$$\begin{aligned} f_* \colon \Pi(X) &\to \Pi(Y) \\ [c] &\mapsto [f \circ c] \end{aligned}$$

The object map $X \mapsto \Pi(X)$ together with the morphism map $f \mapsto f_*$ forms a functor $\Pi \colon \mathbf{Top} \to \mathbf{Grpd}$, called fundamental groupoid functor*.*

Corollary 2.4.9. *Let $\mathbf{Top}_\bullet$ be the category of* pointed topological spaces*, that is, pairs (X, x) with $x \in X$. A morphism $(X, x) \to (Y, y)$ is a continuous map $f \colon X \to Y$ with $f(x) = y$.*

The object map $(X, x) \mapsto \pi_1(X, x)$ together with the morphism map $f \mapsto f_*$ forms a functor $\pi_1 \colon \mathbf{Top}_\bullet \to \mathbf{Grp}$, called fundamental group functor*.*

In particular, the functoriality of Π and π_1 permits us to calculate the fundamental group or groupoid of a union of topological spaces $X = X_1 \cup X_2$ from the fundamental groups or groupoids of X_1, X_2, and $X_1 \cap X_2$. This is the celebrated Seifert-van Kampen theorem. As we will first need to introduce some more constructions for groups and groupoids, we postpone any further explanation to Chapter 5, in which we will also prove an analogous theorem for fundamental semicategories.

2.4.3 Path spaces as categories

In Definition 2.4.2, we used the same domain $[0,1]$ for all paths, and therefore had to define the composition of paths with an intrinsic reparametrization. As a consequence, the composition of paths was not associative. We can avoid this blemish by making a slight change to the definitions:

Definition 2.4.10. Let X be a topological space.

i) Let

$$\begin{aligned} P'(X)(x,y) &:= \{\, c\colon [0,\ell] \to X \text{ continuous} \mid \ell \geq 0,\ c(0) = x,\ c(\ell) = y \,\}\,, \\ P'(X) &:= \bigsqcup_{x,y \in X} P'(X)(x,y). \end{aligned}$$

ii) For two paths $c \in P'(X)(x,y)$ and $d \in P'(X)(y,z)$ with domains $[0,\ell_c]$ and $[0,\ell_d]$, their *concatenation* is given by

$$\begin{aligned} dc\colon [0,\ell_c+\ell_d] &\to X \\ t &\mapsto \begin{cases} c(t) & \text{if } t \in [0,\ell_c] \\ d(t-\ell_c) & \text{if } t \in [\ell_c, \ell_c+\ell_d]\,. \end{cases} \end{aligned}$$

iii) Two paths $c, d \in P'(X)(x,y)$ with domains $[0,\ell_c]$ and $[0,\ell_d]$ are *equivalent* if $\mathrm{rep}(c)$ and $\mathrm{rep}(d)$ are homotopic (rel. endpoints) to one another, where rep denotes reparametrization to the unit interval:

$$\begin{aligned} \mathrm{rep}\colon P'(X) &\to P(X) \\ (c\colon [0,\ell] \to X) &\mapsto \begin{pmatrix} \mathrm{rep}(c)\colon [0,1] \to X \\ \qquad t \mapsto c\,(t\,\ell) \end{pmatrix}. \end{aligned}$$

It is not hard to see that the composition in $P'(X)$ is associative and paths with domain $\{0\} = [0,0]$ act as identity morphisms, so $P'(X)$ is a category with object set X and morphism sets $P'(X)(x,y)$. Note that this is a slight abuse of notation because we use the symbol $P'(X)$ for both the category and the set of paths, which is, strictly speaking, the total morphism set $\mathrm{Mor}\big(P'(X)\big)$ of that category.

If we replace "$P(X)$" with "$P'(X)$" and "homotopy classes" with "equivalence classes" in Definition 2.4.5, we get the same fundamental groupoid $\Pi(X)$. This is how the fundamental groupoid is defined in [18]. Furthermore, the quotient map $p\colon P'(X) \to \Pi^{\mathrm{t}}(X), c \mapsto [c]$ is a functor.

While $P'(X)$ is certainly more appealing than $P(X)$ from a category theory point of view, we will use $P(X)$ for most of this work. This choice removes the need to keep track of the domain lengths ℓ and also avoids special treatment of the case $\ell = 0$.

3 Locally chronological spaces and timelike homotopies

In Definition 2.2.5 we already introduced the causal and chronological relations $\leq_U$ and $\ll_U$ on subsets U of a smooth spacetime. Each of these relations conveys a lot of information about the geometry and topology of a spacetime, which we will later demonstrate in Theorem 4.4.2. Inspired by this observation, our goal in this chapter is to define a more general structure by imbuing a topological space, which is not necessarily a manifold, with a system of transitive relations $\ll_U$.

In Section 3.1 we will discuss some axioms that relate the chronological structure and the topology with the goal of defining a useful generalization of smooth spacetimes. On our way, we will also define timelike homotopies and the fundamental semicategory $\Pi^{\mathrm{t}}(X)$ of a locally chronological space X, which will be the object of study for the following chapters.

Our *Ansatz* differs from many others in the literature (for example in [4]) in that we consider the chronological relations locally, that is, on all (small) open subsets U, and all of our axioms are local conditions. Therefore, we can show that every smooth spacetime satisfies these axioms, not just chronological or causal spacetimes.

In Section 3.2, we briefly discuss timelike boundaries. In Section 3.3 we will turn the class of locally chronological spaces into a category by defining morphisms between them. In Sections 3.4 and 3.5, we will construct several different examples of locally chronological spaces that are not manifolds.

3.1 Axioms for locally chronological spaces

3.1.1 Local chronological structure and timelike paths

In this work, we will focus on chronological relations and timelike paths rather than causal ones because they will be easier to relate to the topology.

Definition 3.1.1. Let X be a topological space.

A *system of (transitive) relations* $\ll_\bullet$ on X consists of a (transitive) relation $\ll_U$ on every open subset $U \subset X$.

A continuous path $c\colon [0,1] \to X$ is called *timelike*[1] with respect to $\ll_\bullet$ if

$$\forall \text{open } U \subseteq X,\ a, b \in [0,1],\ a < b,\ c\big([a,b]\big) \subseteq U : c(a) \ll_U c(b). \tag{3.1.1}$$

We define $P^{\mathrm{t}}(U)$ to be the set of all timelike paths whose image is contained in U, and $P^{\mathrm{t}}(U)(x,y)$ to be the subset of timelike paths from x to y.

Remember that we only defined piecewise continuously differentiable timelike curves on a smooth spacetime in Definition 2.2.3 and used these to define the relations $\ll_U$. All of these curves are also timelike by the above definition, but there are also timelike paths that are not piecewise continuously differentiable.

Note that on a smooth spacetime, most authors impose the condition in equation (3.1.1) only for so-called *convex normal neighborhoods* U, instead of all open U, see [1, p. 54]. Since convex normal neighborhoods form a basis of the manifold topology, Lemma 3.1.5 will show that this leads to an equivalent characterization of timelike paths.

Definition 3.1.2. Let X be a topological space. We say that a system $\ll_\bullet$ of relations on X is *induced by its timelike paths* if

$$P^{\mathrm{t}}(U)(x,y) \neq \emptyset \iff x \ll_U y$$

holds for all open $U \subseteq X$ and $x, y \in U$.

Smooth spacetimes satisfy this property because $\ll_\bullet$ is defined using timelike curves, see Definition 2.2.5. In general, only the implication " $\implies$ " is true in the situation of Definition 3.1.1.

If equivalence holds, the system of relations has a particular structure:

Definition 3.1.3. Let $\ll_\bullet$ be a system of transitive relations on a topological space X.

i) We call $\ll_\bullet$ a *precosheaf of transitive relations* if $x \ll_U y \implies x \ll_V y$ for all open $U \subseteq V \subseteq X$ and points $x, y \in U$.

ii) We call $\ll_\bullet$ a *cosheaf of transitive relations* if it is a precosheaf and satisfies the following:
For any system $\{U_i\}_{i\in I}$ of open subsets of X with $U = \bigcup_{i\in I} U_i$, the relation $\ll_U$ is the smallest transitive relation that contains each of the relations $\ll_{U_i}$.

    In other words, $\ll_U$ is the transitive hull of $\bigcup_{i\in I} \ll_{U_i}$.

[1] For brevity, we will stop using the qualifier "future-directed" from now on, as all paths in $P^{\mathrm{t}}(X)$ are understood to be future-directed.

Lemma 3.1.4. *If a system $\ll_\bullet$ of transitive relations on X is induced by its timelike paths, it is a cosheaf of transitive relations.*

The reverse implication is not true, as we will see in Example 3.5.3.

Proof. If $U \subseteq V \subseteq X$ are open subsets and $x, y \in U$, the implications

$$x \ll_U y \implies P^{\mathrm{t}}(U)(x,y) \neq \emptyset \implies P^{\mathrm{t}}(V)(x,y) \neq \emptyset \implies x \ll_V y$$

show that $\ll_\bullet$ is a precosheaf.

Let $U = \bigcup_{i\in I} U_i$. We already know that $\ll_U$ is transitive and contains $\ll_{U_i}$ for all $i \in I$; we only need to show that it is the smallest such relation.

For any $x, y \in U$ with $x \ll_U y$, there is a timelike path $c\colon [0,1] \to U$ from x to y. The compact domain $[0,1]$ is covered by the open subsets $c^{-1}(U_i)$, $i \in I$. Using the Lebesgue covering lemma, we can find $0 = t_0 < t_1 < \cdots < t_n = 1$ and $i_1, \ldots, i_n \in I$ such that $c\big([t_{k-1}, t_k]\big) \subseteq U_{i_k}$ for all $k \in 1, \ldots, n$. As the restriction $c|_{[t_{k-1},t_k]}$ is timelike in U_{i_k}, we get a finite chain

$$x = c(t_0) \ll_{U_{i_1}} c(t_1) \ll_{U_{i_2}} \cdots \ll_{U_{i_n}} c(t_n) = y.$$

If $\lll$ is any transitive relation on U that contains all $\ll_{U_i}$, we have just shown

$$x \ll_U y \implies \exists x_0, \ldots x_n \in U : x = x_0 \lll \cdots \lll x_n = y \implies x \lll y,$$

which means that $\ll_U$ is the smallest of all such relations $\lll$. □

If $\ll_\bullet$ satisfies the (pre-)cosheaf condition, certain local conditions are true for all open subsets if they are true for the open sets in a topological basis:

Lemma 3.1.5. *Let $\ll_U$ be a system of transitive relations on X and B be a basis of the topology of X.*

i) If $\ll_\bullet$ is a precosheaf, then a path $c\colon [0,1] \to X$ is timelike if and only if

$$\forall U \in \mathsf{B},\ a, b \in [0,1],\ a < b,\ c\big([a,b]\big) \subseteq U : c(a) \ll_U c(b) \tag{3.1.2}$$

ii) If $\ll_\bullet$ is a cosheaf and the equivalence

$$P^{\mathrm{t}}(U)(x,y) \neq \emptyset \iff x \ll_U y$$

is true for all $x, y \in U \in \mathsf{B}$, then it is also true for all open $U \subseteq X$.

Proof.

i) If c is timelike, equation (3.1.1) implies (3.1.2).

   On the other hand, every open set is a union $U = \bigcup_{i \in I} U_i$ of sets $U_i \in \mathsf{B}$. If equation (3.1.2) is satisfied, and there are $a, b \in [0,1]$ with $a < b$ and $c\big([a,b]\big) \subseteq U$, we can use a similar argument as in the proof of Lemma 3.1.4 to show

$$c(a) \ll_{U_{i_1}} c(t_1) \ll_{U_{i_2}} \cdots \ll_{U_{i_n}} c(b) \implies c(a) \ll_U \cdots \ll_U c(b) \implies c(a) \ll_U c(b).$$

   We only used the fact that $\ll_U$ contains all the $\ll_{U_i}$, which is the precosheaf condition, not the cosheaf condition, because $\ll_U$ does not need to be the smallest such relation.

ii) The implication $P^{\mathrm{t}}(U)(x,y) \neq \emptyset \implies x \ll_U y$ is always true by definition. We only need to show the reverse implication.

   Every open set is a union $U = \bigcup_{i \in I} U_i$ of sets $U_i \in \mathsf{B}$. For any $x, y \in U$ with $x \ll_U y$ there is a chain of points $x_0, \ldots x_n \in X$ with

$$x = x_0 \ll_{U_{i_1}} x_1 \ll_{U_{i_2}} \cdots \ll_{U_{i_n}} x_n = y$$

   since $\ll_\bullet$ is a cosheaf. As the relations $\ll_{U_i}$ are induced by timelike paths, there exists a timelike path $c_k \in P^{\mathrm{t}}(U_{i_k})(x_{i-1}, x_i)$ for each $k \in \{1, \ldots, n\}$. We can concatenate these paths to get a timelike path $c = c_n \cdots c_1 \in P^{\mathrm{t}}(U)(x,y)$. □

3.1.2 Open future and past

When we have a system of relations $\ll_\bullet$ on a topological space, we can define the chronological future, past, and diamond in the same way as in Definition 2.2.5:

$$\begin{aligned} I_U^+(x) &\coloneqq \{ y \in U \mid x \ll_U y \}, \\ I_U^-(x) &\coloneqq \{ y \in U \mid y \ll_U x \}, \\ I_U(x,y) &\coloneqq I_U^+(x) \cap I_U^-(y). \end{aligned}$$

Our second axiom for a locally chronological space is that $I_U^+(x)$ and $I_U^-(x)$ are open for any open $U \subseteq X$ and $x \in U$, just as in a smooth spacetime, see Lemma 2.2.6. This axiom is essential for most of the theorems in this work because it relates the chronological structure to the topology. Together with our first axiom (the system $\ll_\bullet$ is induced by its timelike paths), it tells us that we can slightly change the start- and endpoint of a timelike path in any direction and still get a timelike path between these new points:

Lemma 3.1.6. *Let X be a topological space with a system $\ll_\bullet$ of transitive relations that is induced by its timelike paths. Let $U \subseteq X$ be an open subset. Then the following two statements are equivalent:*

i) For all $x \in U$ the sets $I_U^+(x)$ and $I_U^-(x)$ are open.

ii) The relation $\ll_U$ is an open subset of $U \times U$.

Proof. If ii) is true, the set $I_U^+(x)$ is open because it is the preimage of $\ll_U$ under the continuous inclusion map $U \to U \times U$, $y \mapsto (x, y)$. Analogously, it can be shown that $I_U^-(x)$ is open.

If $x, y \in U$ are two points with $x \ll_U y$, then there is a timelike path c with $c(0) = x$ and $c(1) = y$. The point $z := c\left(\frac{1}{2}\right)$ satisfies $x \ll_U z \ll_U y$. The inclusions

$$\begin{aligned}(x, y) \in\ & I_U^-(z) \times I_U^+(z) \\ =\ & \left\{ (x', y') \,\middle|\, x' \ll_U z \ll_U y' \right\} \\ \subseteq\ & \left\{ (x', y') \,\middle|\, x' \ll_U y' \right\} \\ =\ & \ll_U\end{aligned}$$

demonstrate that $I_U^-(z) \times I_U^+(z)$ is an open neighborhood of (x, y) inside $\ll_U$ if ii) is true. As (x, y) was an arbitrary pair in $\ll_U$, this shows that $\ll_U$ is an open subset of $U \times U$. □

3.1.3 Timelike homotopies

In analogy to Definitions 2.4.2 and 2.4.5, we define:

Definition 3.1.7. Let X be a topological space with a system of relations $\ll_\bullet$. A *timelike homotopy (relative to the endpoints)* between paths c_0 and c_1 from x to y is a homotopy $(c_s)_{s\in[0,1]}$ with $c_s \in P^{\mathrm{t}}(X)(x, y)$ for all $s \in [0, 1]$.

Note that in a timelike homotopy all the paths c_s in between c_0 and c_1 have to be timelike too. Therefore, two paths might be homotopic to each other, but not timelike homotopic, see Example 3.5.4.

Definition 3.1.8. Let X be a topological space with a system of transitive relations $\ll_\bullet$ and let

$$\Pi^{\mathrm{t}}(X)(x, y) = P^{\mathrm{t}}(X)(x, y) \big/ \sim$$

be the set of timelike homotopy classes $[c]$ of paths from x to y, for any $x, y \in X$. The operation

$$\begin{aligned}\Pi^{\mathrm{t}}(X)(y, z) \times \Pi^{\mathrm{t}}(X)(x, y) &\to \Pi^{\mathrm{t}}(X)(x, z) \\ [d][c] &\mapsto [dc]\end{aligned}$$

is well-defined and turns $\Pi(X)$ into a semicategory, called the *fundamental semicategory of* $(X, \ll_\bullet)$.

For any subset $A \subseteq X$, called a *set of basepoints*, we define $\Pi^{\mathrm{t}}(X, A)$ to be the sub-semicategory of $\Pi^{\mathrm{t}}(X)$ with $\mathrm{Obj}\big(\Pi^{\mathrm{t}}(X, A)\big) = A$ and $\Pi^{\mathrm{t}}(X, A)(x, y) = \Pi^{\mathrm{t}}(X)(x, y)$ for all $x, y \in A$.

We can prove that $\Pi^{\mathrm{t}}(X)$ is a semicategory in much the same way as for the fundamental groupoid. The key observation is that the concatenation of timelike paths is timelike again. Note that $\Pi^{\mathrm{t}}(X)$ is in general not a category, as constant paths, which would represent identities, are in general not timelike.

The definition of $\Pi^{\mathrm{t}}(X, A)$ is only included for the sake of completeness. The reduction of the fundamental semicategory to a set of basepoints is not nearly as useful as for the fundamental groupoid. While the inclusion $\Pi(X, A) \to \Pi(X)$ has a left inverse and is an equivalence of categories if A has at least one point in every path-connected component of X (see [18, p. 231, p. 245]), the inclusion $\Pi^{\mathrm{t}}(X, A) \to \Pi^{\mathrm{t}}(X)$ has no left inverse except in some pathological cases. Furthermore, equivalence is not even defined for semicategories. For this reason, the semigroup $\Pi^{\mathrm{t}}(X, \{x\})$ may depend heavily on the choice of $x \in X$, while the fundamental group $\pi_1(X, x) = \Pi(X, \{x\})$ only depends on the path-connected component in which x lies.

The following definition is an analogue of Definition 2.4.6 and Lemma 2.4.7:

Definition 3.1.9. Let X be a topological space with a system of transitive relations.

An open subset $U \subseteq X$ is called *timelike simply connected* if $\Pi^{\mathrm{t}}(U)$ is simply connected, which means that for any $x, y \in U$ and any two paths $c_0, c_1 \in P^{\mathrm{t}}(U)(x, y)$ there is a timelike homotopy in U from c_0 to c_1.

X is called *locally timelike simply connected* if it has a topological basis of timelike simply connected open subsets.

The following lemma is an important observation on which all the main results in [8] rely on:

Lemma 3.1.10 ([8, Lemma 2.1])**.** *Every spacetime M is locally timelike simply connected.*

Proof sketch. There is a basis of so-called *convex normal neighborhoods U*, in which any two points $x, y \in U$ can be joined by a unique geodesic γ_{xy} in U. For fixed x, the geodesic γ_{xy} depends continuously on y and is timelike if $x \ll_U y$ holds, see [1, p. 54]. Hence, for any path $c \in P^{\mathrm{t}}(U)(x, y)$, the family $c_s := c|_{[s,1]}\, \gamma_{c(0)c(s)}$ is a timelike homotopy from $c = c_0$ to $\gamma_{xy} = c_1$. This shows that $\Pi^{\mathrm{t}}(U)(x, y)$ is either empty or equal to $\{\, [\gamma_{xy}] \,\}$. □

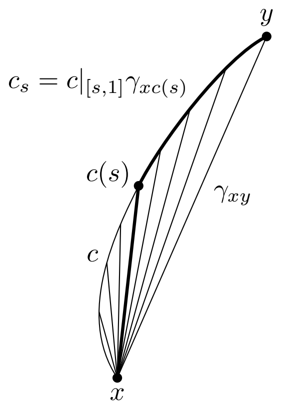


Figure 3.1: Illustration of the homotopy in the proof of Lemma 3.1.10 from the curve c to the geodesic (straight line) γ_{xy} from x to y. One of the intermediate paths c_s is drawn thicker.

3.1.4 Local strong causality

We have already seen in Lemma 2.2.10 that the manifold topology of a spacetime M can be reconstructed from the relation $\ll_M$ alone if M is strongly causal. In a general spacetime, the same is only possible locally. This fact is a "folk theorem" in Lorentzian geometry and is hard to attribute to a single author.

Lemma 3.1.11. *Any smooth spacetime is covered by open subsets that are strongly causal submanifolds.*

Proof. Let $x \in M$ be any point and $\phi\colon \widetilde{U} \to \mathbb{R}^n$ be a chart on a neighborhood $\widetilde{U}$ of x. Fix a constant Lorentzian metric h and a time orientation on $\mathbb{R}^n$ such that the cone of future-directed causal vectors of $h_{\phi(x)}$ in $T_{\phi(x)}M$ contains the one of $\phi_* g_x$. We denote this property by $h_{\phi(x)} > \phi_* g_x$. The set $U := \left\{ y \in \widetilde{U} \,\middle|\, h_{\phi(y)} > \phi_* g_y \right\}$ is then another open neighborhood of x.

If c is a future directed causal curve in (U, g), then $\phi \circ c$ is a future directed causal curve in $(\phi(U), \phi_* g)$ and therefore also in $(\mathbb{R}^n, h)$. This implies that any subset $V \subseteq U$ is causally convex in (U, g) if $\phi(V)$ is causally convex in $(\mathbb{R}^n, h)$. As the spacetime $(\mathbb{R}^n, h)$ is time-oriented isometric to Minkowski space and therefore strongly causal, this implies that (U, g) is also strongly causal. □

We can make a similar definition if we just have chronological relations:

Definition 3.1.12. Let X be a set with a transitive relation $\ll_X$. We call a subset $V \subseteq X$ *chronologically convex* in X if $I_X(x, y) \subseteq V$ holds for all $x, y \in V$.

We say that $(X, \ll_X)$ is *strongly chronological* if for any neighborhood $U \subseteq X$ of any $x \in X$ there exists a sub-neighborhood $V \subseteq U$ of x that is chronologically convex in X.

Lemma 3.1.13. *A smooth spacetime M is strongly chronological if and only if it is strongly causal.*

Corollary 3.1.14. *A spacetime M is covered by open subsets that are strongly chronological submanifolds.*

Proof of Lemma 3.1.13. If we revisit the proof of Lemma 2.2.10, we started by noting that every causally convex subset V is also chronologically convex because $\ll_X$ is a subrelation of $\leq_X$. In the rest of the proof, we actually showed the following implications:

$$\begin{aligned} & M \text{ is strongly causal.} \\ \Longrightarrow\ & M \text{ is strongly chronological.} \\ \Longrightarrow\ & \text{The Alexandrov topology on } M \text{ coincides with the manifold topology.} \\ \Longrightarrow\ & M \text{ is strongly causal.} \end{aligned}$$

□

Conceptually, the above proof works for any topological space X with two systems $\ll_\bullet$ and $\leq_\bullet$ of transitive relations that satisfy the following properties: There is a timelike path through any point, the relation $\ll_X$ is a subrelation of $\leq_X$, and $\ll_X$ and $\leq_X$ satisfy the push-up property as in Lemma 2.2.6.

3.1.5 Definition of a locally chronological space

Let us repeat all the properties discussed so far and turn them into a system of axioms:

Definition 3.1.15. A *locally chronological space* or *space with a local chronological structure* is a topological space X with a system of transitive relations $\ll_\bullet$ that induces a space of timelike paths $P^{\mathrm{t}}(X)$ and satisfies the following axioms:

(A1) For all open $U \subseteq X$ the relation $\ll_U$ is induced by its timelike paths, which means

$$\forall x, y \in U : x \ll_U y \iff P^{\mathrm{t}}(U)(x, y) \neq \emptyset.$$

(A2) For all open $U \subseteq X$ and $x \in U$ the sets $I^+_U(x)$ and $I^-_U(x)$ are open.

(A3) X is covered by open subsets U such that $(U, \ll_U)$ is strongly chronological.

(A4) $(X, \ll_\bullet)$ is locally timelike simply connected. This means that X has a basis B of open sets such that for any $U \in \mathsf{B}$ and any two paths in $P^{\mathrm{t}}(U)(x, y)$ there is a timelike homotopy in U between these paths.

Lemma 3.1.16. *A a topological space X with a system of transitive relations $\ll_\bullet$ is a locally chronological space if and only if there is a basis B of the topology of X such that*

(A1') The system $\ll_\bullet$ is a cosheaf of transitive relations and

$$\forall U \in \mathsf{B},\, x, y \in U : x \ll_U y \iff P^{\mathrm{t}}(U)(x,y) \neq \emptyset.$$

(A2') For all $U \in \mathsf{B}$ and $x \in U$ the sets $I^+_U(x)$ and $I^-_U(x)$ are open.

(A3') $(U, \ll_U)$ is strongly chronological for all $U \in \mathsf{B}$.

(A4') All sets $U \in \mathsf{B}$ are timelike simply connected.

Proof. We have already proven the equivalence (A1) $\iff$ (A1') in Lemma 3.1.4 and Lemma 3.1.5 ii).

Obviously, (A2) implies (A2'). For the reverse implication, let $U \subseteq X$ be any open set. As B is a basis, there are sets $U_i \in \mathsf{B}$ with $U = \bigcup_{i \in I} U_i$, and the cosheaf-property implies that

$$I^+_U(x) = \bigcup_{\substack{i \in I, \\ y \in U_i \cap I^+_U(x)}} I^+_{U_i}(y)$$

is the union of open sets, hence open.

The properties (A3') and (A4') imply Axioms (A3) and (A4), respectively, as a basis is also a cover. For the reverse implication, note that any open subset of a strongly chronological set is also strongly chronological. Therefore, there is a basis B consisting of all open sets that are simultaneously timelike simply convex and strongly chronological. ☐

3.2 Timelike boundary

Remember that we used the fact that there is a timelike curve through every point of a smooth spacetime in the proof of Lemma 2.2.10 and 3.1.13. The same is not true in a general locally chronological space (or in a smooth spacetime with boundary).

Definition 3.2.1. Let X be a topological space X with a system of transitive relations $\ll_\bullet$. The *(future* or *past) timelike boundary* of x is defined as

$$\begin{aligned} \partial^+ X &:= \left\{ x \in X \,\middle|\, I^+_X(x) = \emptyset \right\}, \\ \partial^- X &:= \left\{ x \in X \,\middle|\, I^-_X(x) = \emptyset \right\}, \\ \partial^\pm X &:= \partial^+ X \cup \partial^- X. \end{aligned}$$

On a Lorentzian spacetime with boundary, the timelike boundary is a subset of the manifold boundary because of the following property: Any timelike path that ends in $\partial^+ X$ or starts in $\partial^- X$ can not be extended further into the future or past, respectively.

In a locally chronological space, the sets

$$X \setminus \partial^+ X = \bigcup_{x \in X} I_X^-(x), \qquad X \setminus \partial^- X = \bigcup_{x \in X} I_X^+(x), \qquad X \setminus \partial^\pm X = \bigcup_{x,y \in X} I_X(x,y),$$

are unions of open subsets of X and therefore open, which means that $\partial^+ X$ and $\partial^- X$ are closed. More importantly, the sets $I_X(x,y)$ only generate a topology on $X \setminus \partial^\pm X$, not on the whole space X if $\partial^\pm X$ is nonempty. This fact poses a problem if we want to derive topological properties from the chronological structure. One could completely avoid this issue by postulating $\partial^\pm X = \emptyset$ as an additional axiom, which is what we will practically do in Chapter 4. However, such an axiom could be deemed unnecessarily restrictive as Lorentzian manifolds with boundary may have a non-empty timelike boundary.

A less restrictive approach would be to postulate $\partial^+ X \cap \partial^- X = \emptyset$. If this property holds, which is the case for Lorentzian manifolds with boundary, any point in X is the start or the endpoint of a timelike path $c\colon [0,1] \to X$ with $c((0,1)) \subseteq X \setminus \partial^\pm X$, so $X \setminus \partial^\pm X$ is an open and dense subset of X. Nevertheless, there are different possibilities for fixing the topology on the boundary, each with their own advantages and disadvantages, see [19] for a review in a slightly different context.

3.3 Morphisms of locally chronological spaces

To compare different locally chronological spaces, we need to define structure preserving maps between them:

Definition 3.3.1. Let $(X, \ll_\bullet)$ and $(Y, \lessdot_\bullet)$ be locally chronological spaces.
A map $f\colon X \to Y$ is a *morphism of locally chronological spaces* if f is continuous and

$$\forall \text{open } U \subseteq Y,\ x, y \in f^{-1}(U) : x \ll_{f^{-1}(U)} y \implies f(x) \lessdot_U f(y). \tag{3.3.1}$$

If " $\implies$ " can be replaced with " $\iff$ " and f is a homeomorphism, we call f an *isomorphism of locally chronological spaces.*

This definition is in line with the general principle from Definition 2.3.2 that an isomorphism is a morphism that has an inverse morphism. It is not hard to see that the class of locally chronological spaces together with their morphisms forms a category.

Morphisms are much easier to describe in terms of timelike paths:

Lemma 3.3.2. *Let* $(X, \ll_\bullet)$ *and* $(Y, \prec\!\!\prec_\bullet)$ *be locally chronological spaces. A continuous map* $f\colon X \to Y$ *is a morphism if and only if*

$$\forall c \in P^{\mathrm{t}}(X) : f \circ c \in P^{\mathrm{t}}(Y).$$

Proof. Let f be a morphism and $c \in P^{\mathrm{t}}(X)$. We need to check that $f \circ c$ is timelike in Y, so let $U \subseteq Y$ be an open set and $a, b \in [0,1]$ with $a < b$. Because f is a morphism and c is timelike, it follows that

$$\begin{aligned} f\left(c\big([a,b]\big)\right) \subseteq U &\implies c\big([a,b]\big) \subseteq f^{-1}(U) \\ &\implies c(a) \ll_{f^{-1}(U)} c(b) \\ &\implies f\big(c(a)\big) \prec\!\!\prec_U f\big(c(b)\big), \end{aligned}$$

which shows that $f \circ c$ is timelike in Y.

On the other hand, let $f\colon X \to Y$ be a continuous map that sends timelike paths in X to such in Y. Because of Axiom (A1) in Definition 3.1.15, we then have

$$\begin{aligned} x \ll_{f^{-1}(U)} y &\iff \exists c \in P^{\mathrm{t}}\left(f^{-1}(U)\right)(x, y) \\ &\implies f \circ c \in P^{\mathrm{t}}(U)\big(f(x), f(y)\big) \\ &\implies f(x) \prec\!\!\prec_U f(y). \end{aligned}$$ □

Morphisms also give us an alternative description of timelike paths:

Lemma 3.3.3. *The interval* $[0,1]$ *together with the relations*

$$a \ll_U b :\iff 0 \le a < b \le 1 \text{ and } [a,b] \subseteq U. \qquad \text{for all open } U \subseteq X$$

is a locally chronological space. A map $c\colon [0,1] \to X$ *to some locally chronological space* $(X, \prec\!\!\prec_\bullet)$ *is a timelike path if and only if it is a morphism of locally chronological spaces.*

The proof is a simple matter of checking the definitions. Note that the chronological structure on the interval $[0,1]$ is inherited from one-dimensional Minkowski space. A path $c\colon [0,1] \to [0,1]$ is timelike if and only if it is strictly monotonically increasing.

3.4 Constructions of locally chronological spaces

In this section we will demonstrate several ways to construct locally chronological spaces. In Subsection 3.4.1 we show two methods to construct these spaces "from scratch". In Subsection 3.4.2 we show, among other things, that subsets, products, and preimages of locally chronological spaces under local homeomorphisms are themselves locally chronological spaces.

3.4.1 General constructions

In order to construct a locally chronological space, we can either start by specifying a set of timelike paths or a (partial) system of relations, adjust these in such a way that Axiom (A1) in Definition 3.1.15 is satisfied, and then check the other axioms. For example, in a spacetime we can start by defining the set $\widehat{P^{\mathrm{t}}(X)}$ of piecewise continuously differentiable future-directed timelike curves and construct the rest from it:

Lemma 3.4.1. *Let X be a topological space and fix a set $\widehat{P^{\mathrm{t}}(X)} \subseteq P(X)$. Define*

$$\widehat{P^{\mathrm{t}}(U)}(x,y) := \widehat{P^{\mathrm{t}}(X)} \cap P(U)(x,y),$$
$$x \ll_U y \;:\Longleftrightarrow\; \widehat{P^{\mathrm{t}}(U)}(x,y) \neq \emptyset,$$

for all open $U \subseteq X$ and let $P^{\mathrm{t}}(U)$ be the set of timelike paths in U w.r.t. $\ll_\bullet$ as in Definition 3.1.1.

If $\widehat{P^{\mathrm{t}}(X)}$ contains all subpaths[2] and concatenations of paths in $\widehat{P^{\mathrm{t}}(X)}$, then $\ll_\bullet$ is a system of transitive relations satisfying Axiom (A1) in Definition 3.1.15 and $\widehat{P^{\mathrm{t}}(U)} \subseteq P^{\mathrm{t}}(U)$.

Proof. Because $\widehat{P^{\mathrm{t}}(X)}$ and $P(U)$ are closed under taking concatenations, so is $\widehat{P^{\mathrm{t}}(U)}$, hence the relations $\ll_U$ are transitive.

For any $c \in \widehat{P^{\mathrm{t}}(X)}$, any open $U \subseteq X$, $a, b \in [0,1]$ with $a < b$ and $c\big([a,b]\big) \subseteq U$ we have $c(a) \ll_U c(b)$ because $c|_{[a,b]}$ is a subpath of c and therefore lies in $\widehat{P^{\mathrm{t}}(U)}\big(c(a), c(b)\big)$. This shows that c is indeed timelike, hence $\widehat{P^{\mathrm{t}}(X)} \subseteq P^{\mathrm{t}}(X)$.

On the other hand, $x \ll_U y$ implies $\widehat{P^{\mathrm{t}}(U)}(x,y) \neq \emptyset$ and therefore $P^{\mathrm{t}}(U)(x,y) \neq \emptyset$, so Axiom (A1) in Definition 3.1.15 is satisfied. ☐

Note that the above construction only guarantees the validity of Axiom (A1). In fact, all the other axioms have to be verified separately, as Example 3.5.2 will show.

In some situations, it is more practical to start by defining a family of relations instead of a set of paths:

Lemma 3.4.2. *Let X be a topological space. Fix an open cover C of X and a transitive relation $\widehat{\ll}_U$ on every $U \in \mathsf{C}$.*

Let $\widehat{P^{\mathrm{t}}(X)}$ be the set of all paths $c\colon [0,1] \to X$ that satisfy

$$\forall U \in \mathsf{C},\ a, b \in [0,1],\ a < b,\ c\big([a,b]\big) \subseteq U : c(a) \mathrel{\widehat{\ll}_U} c(b). \tag{3.4.1}$$

[2] To simplify notation, a subpath $c|_{[a,b]}$ of a path $c\colon [0,1] \to x$ is understood to be reparametrized linearly to the interval $[0,1]$.

As in Lemma 3.4.1, we define $x \ll_V y :\Longleftrightarrow \widehat{P^{\mathrm{t}}(V)}(x,y) \neq \emptyset$ for any open $V \subseteq X$.

Then, $\widehat{P^{\mathrm{t}}(X)} = P^{\mathrm{t}}(X)$ is also the set of timelike paths w.r.t. $\ll_\bullet$, Axiom (A1) in Definition 3.1.15 is satisfied, and $\ll_U$ is a subrelation of $\widehat{\ll}_U$ for any $U \in \mathsf{C}$.

If $(U, \widehat{\ll}_U)$ is strongly chronological for every $U \in \mathsf{C}$, then $\ll_\bullet$ satisfies Axiom (A3) in Definition 3.1.15.

Proof. We can directly apply Lemma 3.4.1 to $\widehat{P^{\mathrm{t}}(X)}$, which implies that the relations $\ll_V$ are indeed transitive, they satisfy Axiom (A1), and $\widehat{P^{\mathrm{t}}(U)} \subseteq P^{\mathrm{t}}(U)$ holds.

For all $U \in \mathsf{C}$ and $x, y \in U$ with $x \ll_U y$ we have $\widehat{P^{\mathrm{t}}(U)}(x,y) \neq \emptyset$ and therefore $x \,\widehat{\ll}_U\, y$. It follows that any curve that is timelike w.r.t. $\ll_\bullet$ also satisfies equation (3.4.1), hence $P^{\mathrm{t}}(X) \subseteq \widehat{P^{\mathrm{t}}(X)}$. Both inclusions together imply $P^{\mathrm{t}}(X) = \widehat{P^{\mathrm{t}}(X)}$.

The last sentence of the lemma follows from the fact that the chronological diamonds $I_U(x,y)$ of $\ll_U$ are contained in the ones of $\widehat{\ll}_U$. □

3.4.2 Constructions from other locally chronological spaces

The two most basic ways to get a new locally chronological space from an existing one are by reversing the time direction or by taking open subsets:

Lemma 3.4.3. *If $(X, \ll_\bullet)$ is a locally chronological space, then reversing all relations yields another locally chronological space $(X, \gg_\bullet)$.*

Proof sketch. For any open $U \subset X$ and $x \in X$, the timelike future $I^+_U(x)$ in $(X, \ll_\bullet)$ is the timelike past $I^-_U(x)$ in $(X, \gg_\bullet)$ and vice versa. Furthermore, a path is timelike in $(X, \gg_\bullet)$ if and only if its reversed path is timelike in $(X, \ll_\bullet)$. A timelike homotopy in $(X, \ll_\bullet)$ induces a timelike homotopy in $(X, \gg_\bullet)$ between the reversed paths. Using these observations, the rest of the proof is straightforward. □

Lemma 3.4.4. *Let $V \subseteq X$ be an open subset of a locally chronological space $(X, \ll_\bullet)$. The* restriction of $\ll_\bullet$ to V*, which consists of the relations $\ll'_U := \ll_U$ for all open $U \subseteq V$, turns V into a locally chronological space. We call $(V, \ll'_\bullet)$ a* locally chronological subspace *of $(X, \ll_\bullet)$.*

The inclusion map $V \to X$ is a morphism of chronological spaces.

Proof. Any path $c\colon [0,1] \to V$ that is timelike in $(X, \ll_\bullet)$ is also timelike in $(V, \ll'_\bullet)$ because the latter space has fewer relations.

On the other hand, for any open $U \subseteq X$, the relation $\ll_U$ contains the relation $\ll_{U\cap V} = \ll'_{U\cap V}$ because $\ll_\bullet$ is a precosheaf. Therefore, if c is timelike w.r.t. $\ll'_\bullet$, it is also timelike w.r.t. $\ll_\bullet$.

We have just shown that the timelike paths in $(V, \ll'_\bullet)$ are exactly the timelike paths in $(X, \ll_\bullet)$ whose image happens to be contained in V. Furthermore, $\ll'_\bullet$ inherits the cosheaf-property from $\ll_\bullet$. Therefore, if B is a basis of $(X, \ll_\bullet)$ as in Lemma 3.1.16, then $\mathsf{B}' := \{U \in \mathsf{B} \,|\, U \subseteq V\}$ is such a basis for $(V, \ll'_\bullet)$. □

Note that the above lemma only holds for open subsets and cannot be generalized to arbitrary subsets. For example, take the two-dimensional Minkowski-space as a locally chronological space $(\mathbb{R}^2, \ll_\bullet)$. We can endow $V := \mathbb{R}^2 \setminus \mathbb{Q}^2 \subseteq \mathbb{R}^2$ with the subspace topology and define a system of relations on V by $x \ll'_{U\cap V} y \iff x \ll_U y$ for all open $U \subseteq \mathbb{R}^2$. However, $(V, \ll'_\bullet)$ is not locally timelike simply connected, as the only timelike homotopies in this space are reparametrizations.

The fact that all axioms in Definition 3.1.15 are local properties allows us to build up bigger locally chronological spaces from smaller ones:

Lemma 3.4.5. *Let $X = \bigcup_{i\in I} U_i$ be an open cover of a topological space X and let $(U_i, \ll_{i,\bullet})$ be locally chronological spaces.*

If for all $i, j \in I$ the restrictions of $\ll_{i,\bullet}$ and $\ll_{j,\bullet}$ to $U_i \cap U_j$ agree with one another, then there is a local chronological structure $\ll_\bullet$ on X such that $(U_i, \ll_{i,\bullet})$ are locally chronological subspaces of $(X, \ll_\bullet)$.

Proof. For an arbitrary open subset $U \subseteq X$, we define $\ll_U$ as the smallest transitive relation that contains $\ll_{i,V}$ for all $i \in I$ and open $V \subseteq U \cap U_i$. With this definition, it is immediately clear that $\ll_\bullet$ is a cosheaf of transitive relations.

If U happens to be a subset of U_j, any open $V \subseteq U \cap U_i$ is a subset of $U_j \cap U_i$, hence $\ll_{i,V} = \ll_{j,V}$, by assumption. As $\ll_{j,U}$ is itself the smallest transitive relation that contains all such $\ll_{j,V}$, this implies $\ll_U = \ll_{j,U}$. Therefore $\ll_{j,\bullet}$ is indeed the restriction of $\ll_\bullet$ to U_j.

If we combine the bases of all the spaces $(U_i, \ll_{i,\bullet})$ from Lemma 3.1.16, we get a basis of X with the same properties, hence $(X, \ll_\bullet)$ is indeed a locally chronological space. □

Lemma 3.4.6. *If $f\colon \hat{X} \to X$ is a local homeomorphism from a topological space $\hat{X}$ to a locally chronological space $(X, \ll_\bullet)$, there is a local chronological structure on $\hat{X}$ such that f becomes a local isomorphism of locally chronological spaces.*

Proof. Remember that f is a local homeomorphism if and only if $\hat{X}$ has an open cover $\hat{X} = \bigcup_{i \in I} U_i$ such that $f(U_i)$ is open and $f|_{U_i} \colon U_i \to f(U_i)$ is a homeomorphism for each $i \in I$. We can turn $f|_{U_i}$ into an isomorphism of locally chronological spaces $(U_i, \ll_{i,\bullet}) \to (f(U_i), \ll_\bullet)$ by defining the relations

$$x \ll_{i,V} y :\iff f(x) \ll_{f(V)} f(y)$$

for all $i \in I$ and open $V \subseteq U_i$.

For all $V \subseteq U_i \cap U_j$, we have $x \ll_{i,V} y \iff f(x) \ll_{f(V)} f(y) \iff x \ll_{j,V} y$, so we can apply Lemma 3.4.5 to finish the proof. □

Lemma 3.4.7. *Let $(X_1, \ll_{1,\bullet}), \dots, (X_n, \ll_{n,\bullet})$ be finitely many locally chronological spaces. The product space $\prod_i X_i$ together with the product relations*

$$x \ll_{\prod_i U_i} y :\iff \forall i \in \{1, \dots, n\} : p_i(x) \ll_{i,U_i} p_i(y)$$

for any open $U_1 \subseteq X_1, \dots, U_n \subseteq X_n$ is itself a locally chronological space and the canonical projections $p_k \colon \prod_i X_i \to X_k$ are morphisms.

Proof. For each $i \in \{1, \dots, n\}$, let B_i be a basis of X_i as in Lemma 3.1.16. As we already have defined relations on the open sets of the product basis

$$\mathsf{B} := \left\{ \prod_i U_i \,\middle|\, U_1 \in \mathsf{B}_1, \dots, U_n \in \mathsf{B}_n \right\}.$$

We can use the first half of the proof of Lemma 3.3.2 to see that any path $c \colon [0,1] \to \prod_i X_i$ that respects these relations is mapped to timelike paths $p_i \circ c \in P^{\mathrm{t}}(X_i)$ for all $i \in \{1, \dots, n\}$. The reverse implication is also true. If we have paths $c_i \in P^{\mathrm{t}}(X_i)(x_i, y_i)$, then

$$(c_1, \dots, c_n) \in P^{\mathrm{t}}\Big(\prod_i X_i\Big)\big((x_1, \dots, x_n), (y_1, \dots, y_n)\big)$$

holds. From this, it follows that we can use Lemma 3.4.2 to extend this system of relations to every open subset of $\prod_i X_i$ and that Axiom (A1) in Definition 3.1.15 is satisfied.

It is evident that the projections $p_k \colon \prod_i X_i \to X_k$ satisfy equation (3.3.1), so they are morphisms if $(\prod_i X_i, \ll)$ is a locally chronological space.

To finish the proof, we will now show that the product basis B satisfies the properties in Lemma 3.1.16. For the rest of the proof, fix sets $U_1 \in \mathsf{B}_1, \dots, U_n \in \mathsf{B}_n$ and let $U := \prod_i U_i \in \mathsf{B}$.

(A1') We already proved that Axiom (A1) in Definition 3.1.15 is satisfied, which implies (A1').

(A2') For any $x \in U$, the sets $I^+_U(x) = \prod_i I^+_{U_i}(x_i)$ and $I^-_U(x) = \prod_i I^-_{U_i}(x_i)$ are finite products of open sets and therefore open.

(A3') The product of chronologically convex subsets $V_i \subseteq U_i$ is chronologically convex in U because for any $x, y \in \prod_i V_i$ we have

$$I_U(x,y) = \prod_i I_{U_i}(x_i, y_i) \subseteq \prod_i V_i.$$

From this, it follows that U is strongly chronological.

(A4') We have to show that any two paths $c_0, c_1 \in P^{\mathrm{t}}\big(U\big)(x,y)$ are timelike homotopic to one another. We already know that $p_i \circ c_0$, $p_i \circ c_1 \in P^{\mathrm{t}}(U_i)(x_i, y_i)$ are timelike for all $i \in \{1, \dots, n\}$, and because $U_i \in \mathsf{U}_i$ is timelike simply connected, there is a timelike homotopy

$$\begin{aligned} H_i \colon [0,1] \times [0,1] &\to X_i \\ (t,s) &\mapsto c_{i,s}(t) \end{aligned}$$

from $c_{i,0} \coloneqq p_i \circ c_0$ to $c_{i,1} \coloneqq p_i \circ c_1$. We can put these together to get a timelike homotopy

$$\begin{aligned} (H_1, \dots, H_n) \colon [0,1] \times [0,1] &\to \prod_i X_i \\ (s,t) &\mapsto (c_{1,s}, \dots, c_{n,s}) \end{aligned}$$

from c_0 to c_1. □

Readers familiar with category theory can check that the above construction is indeed a product in the category of locally chronological spaces. Note that the above proof fails if we take an infinite product, as $\prod V_i = \bigcup p_i^{-1}(V_i)$ is an infinite intersection of open sets and therefore in general not open. It is not clear to the author if infinite products of locally chronological spaces exist apart from some very special cases, for example if all but finitely many factors have trivial topology.

The following Lemma is the analogue of a Lorentzian product metric in our setting:

Lemma 3.4.8. *Let (X, d_X) be a length space that has a basis B such that any $U \in \mathsf{B}$ has the following property: For any two points $x, y \in U$, there is a unique shortest path $\gamma_{x,y} \in P(U)(x,y)$, and $\gamma_{x,y}$ depends continuously on x and y.*

Then, $\mathbb{R} \times X$ has a local chronological structure $\ll_\bullet$ with

$$\begin{pmatrix} t_1 \\ x_1 \end{pmatrix} \ll_{\mathbb{R}\times X} \begin{pmatrix} t_2 \\ x_2 \end{pmatrix} :\Longleftrightarrow\ t_2 - t_2 > d_X(x_1, x_2).$$

For all $\begin{pmatrix} t_1 \\ x_1 \end{pmatrix}, \begin{pmatrix} t_2 \\ x_2 \end{pmatrix} \in \mathbb{R} \times X$.

Proof. We will check the four axioms in Definition 3.1.15:

(A1) We apply Lemma 3.4.2 to the cover $\mathsf{C} = \{\mathbb{R} \times X\}$ and the relation $\ll_{\mathbb{R}\times X}$ given in the lemma. This immediately implies that Axiom (A1) is satisfied and that a path is timelike if and only if it respects the single relation $\ll_{\mathbb{R}\times X}$. Remember that the application of Lemma 3.4.2 might make it necessary to replace $\ll_{\mathbb{R}\times X}$ with a subrelation. We will now prove that this is not actually the case here, by showing that $\ll_{\mathbb{R}\times X}$ is induced by timelike paths:

Any timelike path in X has a strictly monotonically increasing time coordinate and therefore can be reparametrized to the form

$$\begin{aligned} \tilde{c}\colon [0,1] &\to \mathbb{R} \times X, \\ \tau &\mapsto \begin{pmatrix} t_1 + \tau\,(t_2 - t_1) \\ c(\tau) \end{pmatrix} \end{aligned}$$

where c is some path in X. As $d_X\big(c(a), c(b)\big) \leq (b-a)(t_2 - t_1)$ holds for all $a, b \in [0,1]$ with $a < b$, we can easily see[3] $L(c) < t_2 - t_1$.

On the other hand, for any pair of points with $\binom{t_1}{x_1} \ll_{\mathbb{R}\times X} \binom{t_2}{x_2}$ we can find a path c from x_1 to x_2 of length $L(c) < t_2 - t_1$, because $d_X(x_1, x_2) < t_2 - t_1$ is the infimum of the lengths of all such paths. By reparametrizing c proportionally to its arc length, we can ensure $L\left(c|_{[0,\tau]}\right) = \tau\, L(c)$ for all $\tau \in [0,1]$. As the length functional L is additive, see Remark 2.1.3, we get the inequalities

$$\begin{aligned} d_X\big(c(a), c(b)\big) &\leq L\left(c|_{[a,b]}\right) \\ &= L\left(c|_{[0,b]}\right) - L\left(c|_{[0,a]}\right) \\ &= (b-a)\, L(c) \\ &< (b-a)\,(t_2 - t_1) \\ &= \big(t_1 + b\,(t_2 - t_1)\big) - \big(t_1 + a\,(t_2 - t_1)\big), \end{aligned}$$

which shows that the induced path $\tilde{c}$ is indeed a timelike path from $\binom{t_1}{x_1}$ to $\binom{t_2}{x_2}$.

(A2) Now, let $U \subseteq \mathbb{R} \times X$ be an arbitrary open subset and assume that $\binom{t_1}{x_1} \ll_U \binom{t_2}{x_2}$ holds. Our goal is to prove the existence of an open neighborhood of $\binom{t_2}{x_2}$ in $I_U^+\binom{t_1}{x_1}$. As U is open, there is an $\varepsilon > 0$ with

$$(t_2 - \varepsilon, t_2 + \varepsilon) \times B_\varepsilon(x_2) \subseteq U,$$

where $B_\varepsilon(x_2) = \{y \in X \mid d_X(x_2, y) < \varepsilon\}$ is the metric ball around x_2 with radius ε.

[3] Remember that d_X is a genuine metric, not a Lorentzian distance function, and $L(c)$ is the length of c in the sense of Definition 2.1.2.

By Axiom (A1), there is a timelike path in U from $\binom{t_1}{x_1}$ to $\binom{t_2}{x_2}$, which has to go through some point $\binom{t_3}{x_3} \in (t_2 - \varepsilon, t_2 + \varepsilon) \times B_\varepsilon(x_2)$ with $\binom{t_3}{x_3} \ll_U \binom{t_2}{x_2}$. We can now choose an $\varepsilon' > 0$ with $t_2 - t_3 > d_X(x_3, x_2) + 2\varepsilon'$. By definition of a length metric, there is a path c in X from x_2 to x_3 of length $L(c) < t_2 - 2\varepsilon' - t_3 < \varepsilon$. By the triangle inequality, this path stays within $B_\varepsilon(x_2)$ and the induced timelike path $\tilde{c}$ from $\binom{t_3}{x_3}$ to $\binom{t_2 - 2\varepsilon'}{x_2}$ therefore stays within $(t_2 - \varepsilon, t_2 + \varepsilon) \times B_\varepsilon(x_2) \subseteq U$. By an analogous argument, we can construct a timelike path in U from $\binom{t_2 - 2\varepsilon'}{x_2}$ to any point in

$$(t_2 - \varepsilon', t_2 + \varepsilon') \times B_{\varepsilon'}(x_2).$$

The latter set is therefore an open subset of $I_U^+\binom{t_3}{x_3} \subseteq I_U^+\binom{t_1}{x_1}$, which finishes the proof.

(A3) The whole space $\mathbb{R} \times X$ is strongly chronological because for any neighborhood U of a point $\binom{t}{x} \in \mathbb{R} \times X$, there is an $\varepsilon > 0$ with

$$I_{\mathbb{R}\times X}\left(\begin{pmatrix} t-\varepsilon \\ x \end{pmatrix}, \begin{pmatrix} t+\varepsilon \\ x \end{pmatrix}\right) \subseteq (t-\varepsilon, t+\varepsilon) \times B_\varepsilon(x) \subseteq U$$

and the chronological diamond $I_{\mathbb{R}\times X}$ is open and chronologically convex. This implies Axiom (A3).

(A4) The lemma asserts that B is a basis of X, hence $\{(a,b) \times U \mid (a,b) \subseteq \mathbb{R}, U \in \mathsf{B}\}$ is a basis of $\mathbb{R} \times X$.

For any $\binom{t_1}{x_1}, \binom{t_2}{x_2} \in (a,b) \times U$ with $\binom{t_1}{x_1} \ll_{(a,b)\times U} \binom{t_2}{x_2}$, there is a unique shortest path γ_{x_1,x_2} in U from x_1 to x_2, from which we construct the timelike path

$$\tilde{\gamma}_{\binom{t_1}{x_1},\binom{t_2}{x_2}} : [0,1] \to (a,b) \times U,$$
$$\tau \mapsto \begin{pmatrix} t_1 + \tau\,(t_2 - t_1) \\ \gamma_{x_1,x_2}(\tau) \end{pmatrix}$$

Just like the paths γ_{x_1,x_2}, the paths $\tilde{\gamma}_{\binom{t_1}{x_1},\binom{t_2}{x_2}}$ depend continuously on their start- and endpoints. We can now show that $(a,b) \times U$ is timelike simply connected in the same way as we did in Lemma 3.1.10. □

3.5 Examples and non-examples

The following examples of spaces with transitive relations are chosen to illustrate the interplay between the axioms in Definition 3.1.15.

Example 3.5.1 (Basic examples)**.**

i) Let X be an arbitrary topological space, $\ll_U$ be the empty relation, and $P^{\mathrm{t}}(X) = \emptyset$. This system satisfies all of the axioms in Definition 3.1.15 and has $\partial^+ X = \partial^- X = X$.

ii) Let X be an arbitrary topological space and define $P^{\mathrm{t}}(X) = P(X)$. To satisfy Axiom (A1), $\ll_U$ must be the equivalence relation whose equivalence classes $I_U^+(x) = I_U^-(x)$ are the path-connected components of U. The timelike boundary is $\partial^+ X = \partial^- X = \emptyset$.

   Axiom (A2) is satisfied if and only if X is locally path-connected, that is, if X has a basis of path-connected open subsets.

   Axiom (A3) is satisfied if and only if every path-connected component of X consists only of topologically indistinguishable points.

   Proof. Assume that X is locally strongly chronological and there is a path in X between topologically distinguishable points. By a covering argument, we can find a subpath that lies entirely in some strongly chronological open subset U and still joins topologically distinguishable points $x, y \in U$. This means that there is an open subset $U' \subseteq X$ that contains x but not y. As U is strongly chronological, there is a chronologically convex neighborhood V of x in $U \cap U'$. However, V must contain $I_U(x, x)$, which contains y, in contradiction to $y \notin U'$.

   On the other hand, if every path-connected component of X consists only of topologically indistinguishable points, then every open subset of X is a union of path-connected components and therefore chronologically convex in X. This means that the whole space $(X, \ll_X)$ is strongly chronological. □

   Axiom (A4) is satisfied if and only if X is locally simply connected.

iii) As a special case of the example above, let X be an arbitrary set with the discrete topology and $x \ll_U y \iff x = y \in U$. This satisfies all the axioms and $P(X) = P^{\mathrm{t}}(X)$ is the set of constant paths.

   Note that this is an example of a space that is strongly chronological but not chronological.

Example 3.5.2 (The shrinking wedge of circles as a counterexample)**.**

Let X be the shrinking wedge of circles,

$$X = \bigcup_{n=1}^{\infty} S_{\frac{1}{n}} \begin{pmatrix} \frac{1}{n} \\ 0 \end{pmatrix}$$

where $S_r(p)$ denotes the circle with center p and radius r in $\mathbb{R}^2$. We define $\widehat{P^{\mathrm{t}}(X)}$ to be the set of all paths in X that travel strictly counterclockwise through arbitrary many of the circles and apply the construction from Lemma 3.4.1.

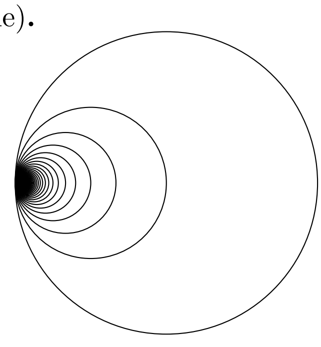


Figure 3.2

In the subspace topology of $\mathbb{R}^2$, any neighborhood U of 0 contains at least one full circle, hence $\widehat{P^{\mathrm{t}}(U)}(0,0)$ is nonempty and therefore $0 \ll_U 0$. It follows that the constant path in 0 is timelike w.r.t. $\ll_\bullet$, even though $\widehat{P^{\mathrm{t}}(X)}$ did not include constant paths.

By construction, this structure satisfies Axiom (A1) in Definition 3.1.15, but none of the other axioms are satisfied: For any neighborhood U of 0, the set $I_U(0,0) = I_U^+(0) \cap I_U^-(0)$ is the union of all circles that are completely contained in U. This is not an open subset of X unless $U = X$, so Axiom (A2) is violated. As $I_U(0,0)$ has to be contained in any chronologically convex neighborhood of 0, the set $(U, \ll_U)$ cannot be strongly chronological either, so X does not satisfy Axiom (A3). The constant path in 0 is not homotopic to any nonconstant timelike path from 0 to 0, which violates Axiom (A4).

Example 3.5.3. (The extended long ray as a counterexample) In this example, we will construct a space with a cosheaf of transitive relations that does not satisfy Axiom (A1) in Definition 3.1.15, but does satisfy the other three axioms.

Let X be a space with strict total order $<$ and the order topology, which is generated by the open intervals

$$(-\infty, b) \coloneqq \{ x \in X \mid x < b \}, \quad (a, \infty) \coloneqq \{ x \in X \mid x > a \}, \quad (a,b) \coloneqq (-\infty, b) \cap (a, \infty),$$

for all $a, b, \in X$. Note that $\pm\infty$ are only used to simplify notation; they are not actual elements of X. Suppose that X is connected and all closed intervals $[x,y] = (x,y) \cup \{x, y\}$ are compact. In analogy to Lemma 3.3.3, we define the relation

$$x \ll_U y :\Longleftrightarrow \; x < y \text{ and } [x,y] \subseteq U.$$

on any open set $U \subseteq X$. This relation is obviously transitive. If $U \subseteq V \subseteq X$ are open subsets, $x \ll_U y$ implies $x \ll_V y$. Hence, $\ll_\bullet$ is a precosheaf.

We will now show that $\ll_\bullet$ is a cosheaf: If we have an open cover $U = \bigcup_i U_i$ and $x \ll_U y$, we need to find a chain $x = x_0 \ll_{U_{i_1}} x_1 \ll_{U_{i_2}} \cdots \ll_{U_{i_k}} x_k = y$. As every open set is a union of open intervals, and $[x,y]$ is compact, we can choose finitely many intervals

$(a_1, b_1) \subseteq U_{i_1}, \dots, (a_k, b_k) \subseteq U_{i_k}$ that cover $[x, y]$. We can reorder the intervals such that $b_1 \leq \dots \leq b_k$ and we can also assume $a_1 < \dots < a_k$ without loss of generality. If the latter were not the case, there would be an $i \in \{1, \dots, k-1\}$ with $a_{i+1} \leq a_i < b_i \leq b_{i+1}$. In this case, we could remove the interval (a_i, b_i) and still have a cover of $[x, y]$.

For any $i \in \{1, \dots, k-1\}$, we can write X as a union of two open subsets:

$$\begin{aligned} X &= (-\infty, x) \cup [x, y] \cup (y, \infty) \\ &= \big((-\infty, x) \cup (a_1, b_1) \cup \dots \cup (a_i, b_i)\big) \cup \big((a_{i+1}, b_{i+1}) \cup \dots \cup (a_k, b_k) \cup (y, \infty)\big) \\ &= (-\infty, b_i) \cup (a_{i+1}, \infty). \end{aligned}$$

As X is connected, there is at least one point $x_i \in (-\infty, b_i) \cap (a_{i+1}, \infty) = (a_{i+1}, b_i)$ for all $i \in \{1, \dots k-1\}$. Because of $[x_{i-1}, x_i] \subseteq (a_i, b_i) \subseteq U_{i_i}$ we have found a chain

$$x = x_0 \ll_{U_{i_1}} x_1 \ll_{U_{i_2}} \cdots \ll_{U_{i_{k-1}}} x_{k-1} \ll_{U_{i_k}} = x_k = y,$$

which completes the proof that $\ll_\bullet$ is a cosheaf.

The relation $\ll_X$ is identical to the total order $<$ on X, so if X is not path-connected, there are points x, y with $x \ll_X y$ and $P^{\mathrm{t}}(X)(x, y) = \emptyset$, which implies that (A1) in Definition 3.1.15 is not satisfied. The extended long line (see [20, p. 71]) is an example of such a totally ordered, compact, connected, but not path-connected space.

It is not hard to check that Axioms (A2) and (A3) in Definition 3.1.15 are satisfied. A path $c\colon [0, 1] \to U$ from x to y in $U \subseteq X$ is timelike if and only if it is strictly monotonous with respect to $<$, which means it is a homeomorphism onto its image $[x, y]$. Any two timelike paths from x to y are therefore related by a strictly monotonous reparametrization and there is a timelike homotopy that interpolates between these two parametrizations. This implies that $(X, \ll_\bullet)$ also satisfies Axiom (A4).

The following example is meant to show some aspects of the fundamental semicategory that might be unintuitive at first sight.

Example 3.5.4.

This example is explained in greater detail in [8, Example 6.1]. We will only motivate the essential steps, as as the proofs involve the use of Morse theory, which is beyond the scope of this work.

Let $S = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + (z/2)^2 = 1 \}$ be an elongated rotational ellipsoid with the standard Riemannian metric g induced from $\mathbb{R}^3$.

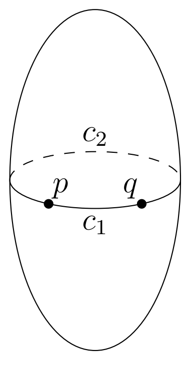


Figure 3.3

The points $p = (1, 0, 0), q = (0, 1, 0) \in S$ decompose the equator into two geodesics c_1 and c_2 of length $\ell_1 = \frac{1}{2}\pi$ and $\ell_2 = \frac{3}{2}\pi$, respectively, as shown in the figure to the right. The curve c_1 is the unique shortest geodesic from p to q, and c_2 is the unique second-shortest geodesic.

Now, we define the Lorentzian product space $X \coloneqq (\mathbb{R} \times S, -\mathrm{d}t^2 + g)$. As we have seen in the proof of Lemma 3.4.8, timelike paths $\tilde{c} \in P^{\mathrm{t}}(X)\big((t_1, p), (t_2, q)\big)$ correspond to paths $c \in P(S)(p, q)$ of length $L(c) < t_2 - t_1$. Therefore, the path c_1 induces a timelike path $\tilde{c}_1$ for $\ell_1 < t_2 - t_1$ and c_2 induces a second timelike path $\tilde{c}_2$ for $\ell_2 < t_2 - t_1$.

If we visualize paths as rubber strings with fixed ends on the surface S, we can deform the path c_1 into c_2 by pulling the string over one of the poles of the ellipsoid. This homotopy induces a timelike homotopy in X between $\tilde{c}_1$ and $\tilde{c}_2$ if the length of the rubber string stays below $t_2 - t_1$ during the deformation. However, we cannot deform c_1 to c_2 without stretching the string at least to some critical length $\ell_3 > \ell_2$ before letting it shorten again. The number of timelike homotopy classes is therefore

$$\left|\Pi^{\mathrm{t}}(X)\big((t_1, p), (t_2, q)\big)\right| = \begin{cases} 0 & \text{if } \quad\quad t_2 - t_1 \leq \ell_1 \\ 1 & \text{if } \ell_1 < t_2 - t_1 \leq \ell_2 \\ 2 & \text{if } \ell_2 < t_2 - t_1 \leq \ell_3 \\ 1 & \text{if } \ell_3 < t_2 - t_1 \end{cases}$$

Note that this means that X is not timelike simply connected despite being simply connected in the usual sense.

If we choose t_1, t_2 such that $\ell_2 < t_1 < \ell_3 < t_2$, there are morphisms

$$\begin{aligned} [a], [a'] &\in \Pi^{\mathrm{t}}(X)\big((0, p), (t_1, q)\big) \quad \text{with } [a] \neq [a'], \\ [b] &\in \Pi^{\mathrm{t}}(X)\big((t_1, q), (t_2, q)\big). \end{aligned}$$

As both $[b][a]$ and $[b][a']$ lie inside $\Pi^{\mathrm{t}}(X)\big((0, p), (t_2, q)\big)$, they are equal despite of $[a] \neq [a']$.

For readers familiar with category theory, this means that $[b]$ is not a monomorphism. Analogously, we can show that $[b]$ is not an epimorphism. This is of course in contrast to the fundamental groupoid, in which every morphism is an isomorphism.

4 Isomorphisms of locally chronological spaces and fundamental semicategories

The goal of this chapter is to show that the fundamental semicategory $\Pi^t(X)$ of a locally chronological space $(X, \ll_\bullet)$ contains enough information to reconstruct the topology of X and the local chronological structure $\ll_\bullet$ under the condition that the topology is regular and the timelike boundary $\partial^\pm X$ is empty. To be more precise, Theorem 4.3.2 will demonstrate that there is a one-to-one correspondence between isomorphisms of locally chronological spaces that satisfy these conditions and isomorphisms between their fundamental semicategories.

In this sense, the fundamental semicategory $\Pi^t(X)$ is a much more rigid algebraic invariant than the fundamental groupoid $\Pi(X)$. As the latter is invariant under homotopy eqivalences, it can only encode information on the homotopy type of X. In Section 4.1 we demonstrate this fact by means of a concrete example.

In Section 4.2, we will develop an algebraic description of the topology of a locally chronological space purely in terms of timelike homotopy classes of paths. Afterward, we will prove Theorem 4.3.2 in Section 4.3.

To further demonstrate the rigidity of the fundamental semicategories, we will look at the special case of smooth spacetimes in Section 4.4. In this case, we can reconstruct the Lorentzian metric up to a local scaling factor from its local chronological structure together with the differentiable structure of the manifold.

4.1 Comparison to the fundamental groupoid

Example 4.1.1. Let X and Y be path-connected and simply connected topological spaces, and let $f\colon X \to Y$ a bijective, not necessarily continuous map. For example, take any bijection between $X = \mathbb{R}^2$ and $Y = \mathbb{R}$.

As we have seen in Lemma 2.4.7, homotopy classes of paths in a simply connected space are uniquely determined by their endpoints. We denote

$$\Pi(X)(x, y) = \{ [c_{xy}] \} \quad \text{and} \quad \Pi(Y)(x', y') = \{ [d_{x'y'}] \}$$

for all $x, y \in X$ and $x', y' \in Y$.

The map $f\colon \operatorname{Obj}(\Pi(X)) \to \operatorname{Obj}(\Pi(Y))$ together with the morphism map

$$F\colon \operatorname{Mor}\big(\Pi(X)\big) \to \operatorname{Mor}\big(\Pi(Y)\big)$$
$$[c_{xy}] \mapsto \big[d_{f(x)f(y)}\big]$$

is then an invertible functor, hence an isomorphism of groupoids.

We have just seen that $\Pi(X)$ and $\Pi(Y)$ have isomorphic groupoid structures despite having very different topologies. In this sense, we cannot reconstruct the topology of X if we only know the groupoid structure of $\Pi(X)$—except in the case that X consists of only one point.

To encode more information about the topology, we could use higher groupoids, which are a generalization of higher homotopy groups. The idea is to introduce paths as morphisms, homotopies between paths as 2-morphisms, homotopies between homotopies as 3-morphisms and so on.

Analogously, we could define higher fundamental semicategories and they might be a useful tool. However, we will not develop these it in this work because the (1-)semicategory $\Pi^{\mathrm{t}}(X)$ already contains all information about the local chronological structure of X.

4.2 Refined Alexandrov topology

Most of the statements in this chapter are only true if the topology of our locally chronological space X is not “too wild”.

Definition 4.2.1. A topology on a space X is called *regular* if it satisfies one of the following equivalent conditions:

i) For any point $x \in X$ and any closed set F that does not contain x, there is an open neighborhood V of x and W of F such that $V \cap W = \emptyset$.

ii) A topology on a space X is regular if and only if every open neighborhood U of any point $x \in X$ contains an open neighborhood V of x with $\overline{V} \subseteq U$.

The equivalence of both statements is easily seen by defining $F \coloneqq X \setminus U$ or $U \coloneqq X \setminus F$, respectively, and using that complements of open subsets are closed and vice versa.

To reconstruct the topology of a locally chronological space X from $\Pi^{\mathrm{t}}(X)$, we need an algebraic description of open sets in X. A first observation is that we can retrieve the chronological diamonds

$$I_X(x,y) = \big\{\, z \in X \;\big|\; \exists [a] \in \Pi^{\mathrm{t}}(X)(x,z), [b] \in \Pi^{\mathrm{t}}(X)(z,y) \,\big\}\,.$$

We can refine these by restricting the homotopy classes:

Definition 4.2.2. Let $(X, \ll_\bullet)$ be a locally chronological space. For an open subset $U \subseteq X$ and a timelike homotopy class $[c] \in \Pi^{\mathrm{t}}(U)(x, y)$, let

$$I_U\big([c]\big) \coloneqq \big\{\, z \in U \,\big|\, \exists [a] \in \Pi^{\mathrm{t}}(U)(x, z), [b] \in \Pi^{\mathrm{t}}(U)(z, y) : [b][a] = [c] \,\big\}\,.$$

Theorem 4.2.3. *Let $(X, \ll_\bullet)$ be a locally chronological space with a regular topology. Then, any open subset of $X \setminus \partial^{\pm} X$ is a union of open sets $I_X\big([c]\big)$ for suitable $[c] \in \Pi^{\mathrm{t}}(X)$.*

Corollary 4.2.4. *If $(X, \ll_\bullet)$ is a locally chronological space with a regular topology and empty timelike boundary, then*

$$\mathsf{B} \coloneqq \big\{\, I_X\big([c]\big) \,\big|\, [c] \in \Pi^{\mathrm{t}}(X) \,\big\}$$

is a basis of the topology of X.

The proof for Theorem 4.2.3 is analogous to the one of [8, Theorem A], which is the corresponding statement in the setting of smooth spacetimes instead of locally chronological spaces. It is split up into two Lemmata, which also translate well to the setting of locally chronological spaces.

Note that $I_U\big([c]\big)$ is exactly the set of points z for which there is a timelike path in the homotopy class $[c]$ from x via z to y. This means that the image of all paths in the class $[c]$ is contained in $I_U\big([c]\big) \cup \{\, x, y \,\} \subseteq \overline{I_U\big([c]\big)}$.

Lemma 4.2.5. *Let $(X, \ll_\bullet)$ be a locally chronological space, $U \subseteq X$ a timelike simply connected open subset, $x, y \in U$, and $c \in P^{\mathrm{t}}(U)(x, y)$.*

Then we have $I_U(x, y) = I_U\big([c]\big) \subseteq I_X\big([c]\big)$. Moreover, the equality $I_U\big([c]\big) = I_X\big([c]\big)$ holds if $\overline{I_U(x, y)} \subseteq U$, where the overline denotes the closure in X.

Proof. Remember the equivalences

$$\Pi^{\mathrm{t}}(U)(x, y) \neq \emptyset \iff P^{\mathrm{t}}(U)(x, y) \neq \emptyset \iff x \ll_U y.$$

If U is timelike simply connected, the condition $[b][a] = [c]$ in Definition 4.2.2 is always fulfilled, as $[c]$ is the only element of $\Pi^{\mathrm{t}}(U)(x, y)$, so $I_U\big([c]\big)$ coincides with $I_U(x, y)$.

It is important to note that the equation $[b][a] = [c]$ in $\Pi^{\mathrm{t}}(U)(x, y)$ means that there is a timelike homotopy *in U* from ba to c. Any timelike homotopy in U is also a timelike homotopy in X, therefore $I_U\big([c]\big) \subseteq I_X\big([c]\big)$.

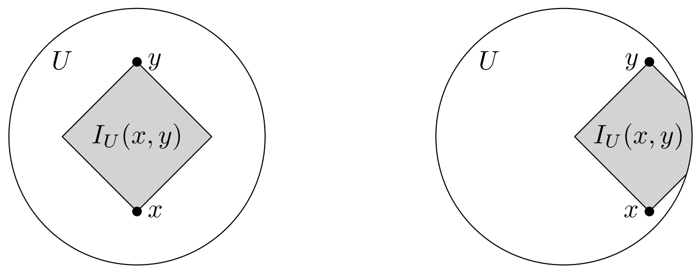


Figure 4.1: Illustration of chronological diamonds $I_U(x,y) \subseteq U$ in the proof of Lemma 4.2.5. In the left picture, $\overline{I_U(x,y)} \subseteq U$ holds and a timelike homotopy of paths from x to y cannot leave U without leaving $\overline{I_U(x,y)}$ first, which would lead to a contradiction.
The right picture shows a situation in which the diamond $I_U(x,y)$ touches the boundary of U, so $\overline{I_U(x,y)}$ is not contained in U.

If we now assume $I_U\big([c]\big) \neq I_X\big([c]\big)$ there must be a timelike homotopy $(c_s)_{s\in[0,1]}$ in X that starts in $c_0 = c$ and is not a timelike homotopy in U. Consider the sets

$$S_1 \coloneqq \Big\{ s \in [0,1] \,\Big|\, \forall t \in [0,1] : c_s(t) \in U \Big\},$$
$$S_2 \coloneqq \Big\{ s \in [0,1] \,\Big|\, \forall t \in [0,1] : c_s(t) \in \overline{I_U(x,y)} \Big\}.$$

By assumption, $0 \in S_1 \subsetneq [0,1]$. Since $[0,1]$ is compact, U is open and $\overline{I_U(x,y)}$ is closed, we see that S_1 is open in $[0,1]$ and S_2 is closed.

The images of all paths in $P^{\mathrm{t}}(U)(x,y)$ are contained in $\overline{I_U(x,y)}$, thus we have $S_1 \subseteq S_2$. The assumption $\overline{I_U(x,y)} \subseteq U$ implies $S_2 \subseteq S_1$, hence $S_2 = S_1 \neq [0,1]$ is a closed and open subset of $[0,1]$, which is the desired contradiction. □

Note that the assumption $\overline{I_U(x,y)} \subseteq U$ in the above lemma ensures that $I_U(x,y)$ does not touch the boundary of U, as illustrated in Figure 4.1.

Lemma 4.2.6. *Let $(X, \ll_\bullet)$ be a locally chronological space with a regular topology, and $c \in P^{\mathrm{t}}(X)$. For $t \in (0,1)$ and $0 < \varepsilon < \min(t, 1-t)$, let c_ε be the reparametrization of the path $c|_{[t-\varepsilon,t+\varepsilon]}$ to the unit interval.*
Then the sets $I_X\big([c_\varepsilon]\big)$ form an open neighborhood basis of $c(t)$.

Proof. Let U be a timelike simply connected and strongly chronological neighborhood of $c(t)$. This means there are arbitrarily a small neighborhoods V of $c(t)$ that are chronologically convex with respect to $\ll_U$. Since X has a regular topology, we can assume $\overline{V} \subseteq U$. For small enough ε the points $c(t \pm \varepsilon)$ lie in V, and therefore we have $I_U\big(c(t-\varepsilon), c(t+\varepsilon)\big) \subseteq V$. This implies $\overline{I_U\big(c(t-\varepsilon), c(t+\varepsilon)\big)} \subseteq \overline{V} \subseteq U$ and, by Lemma 4.2.5, $I_X\big([c_\varepsilon]\big) = I_U\big(c(t-\varepsilon), c(t+\varepsilon)\big)$. This shows that the sets $I_X\big([c_\varepsilon]\big)$ form a neighborhood basis of $c(t)$. □

Corollary 4.2.7. *For any $[c] \in \Pi^{\mathrm{t}}(X)$, the set $I_X\big([c]\big) \subseteq X$ is open.*

Proof. For any $[c] \in \Pi^{\mathrm{t}}(X)$ and $z \in I_X\big([c]\big)$, there is a path c in the class $[c]$ with $c(t) = z$ for some $t \in (0,1)$. For small enough $\varepsilon > 0$, the sets $I_X\big([c_\varepsilon]\big)$ are open due to Lemma 4.2.6 and we clearly have $I_X\big([c_\varepsilon]\big) \subseteq I_X\big([c]\big)$. We have just found an open neighborhood of any point $z \in I_X\big([c]\big)$ inside $I_X\big([c]\big)$, which shows that $I_X\big([c]\big)$ is open. □

Proof of Theorem 4.2.3. Every point $p \in X \setminus \partial^{\pm} X$ has a nonempty timelike future and past. By Axiom (A1) in Definition 3.1.15 this implies that there is a timelike path ending in p and another timelike path starting in p. We can concatenate these paths to get a timelike path c with $c\left(\frac{1}{2}\right) = p$ and apply Lemma 4.2.6 to finish the proof. □

4.3 Isomorphism rigidity of fundamental semicategories

In Lemma 2.4.8, we have seen that the assignment of fundamental groupoids to topological spaces is a functor **Top** → **Grpd**. The same statement, with the necessary changes, is true for fundamental semicategories:

Lemma 4.3.1. *Let X, Y be locally chronological spaces. Any morphism $f\colon X \to Y$ of locally chronological spaces induces a semifunctor with object map f and morphism map*

$$\begin{aligned} f_*\colon \Pi^{\mathrm{t}}(X) &\to \Pi^{\mathrm{t}}(Y) \\ [c] &\mapsto [f \circ c]. \end{aligned}$$

The object map $X \mapsto \Pi^{\mathrm{t}}(X)$ together with the morphism map $f \mapsto f_*$ forms a functor from the category of locally chronological spaces to the category of small semicategories. We call this functor Π^{t} the fundamental semicategory functor.

Proof. Lemma 3.3.2 implies that $f \circ c$ is a timelike path in Y if c is timelike in X. For the same reason, if $H\colon [0,1] \times [0,1] \to X$, $(t,s) \mapsto c_s(t)$ is a timelike homotopy in X, then the map $f \circ H\colon (t,s) \mapsto (f \circ c_s)(t)$ is a timelike homotopy from $f \circ c_0$ to $f \circ c_1$ in Y. Therefore, the assignment $[c] \mapsto [f \circ c]$ does not depend on the choice of representative c. It is the morphism map of a semifunctor $\Pi^{\mathrm{t}}(X) \to \Pi^{\mathrm{t}}(Y)$ because if $c \in P^{\mathrm{t}}(X)(x,y)$ and $d \in P^{\mathrm{t}}(X)(y,z)$, we have

$$f_*\big([d]\big) f_*\big([c]\big) = [f \circ d][f \circ c] = [(f \circ d)(f \circ c)] = [f \circ (dc)] = f_*\big([dc]\big).$$

If $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms of locally chronological spaces, then it is clear that $(g \circ f)_* = g_* \circ f_*$, which finishes the proof. □

The above lemma also implies that an isomorphism of locally chronological spaces induces an isomorphism of their fundamental semicategories. Surprisingly, the reverse is also true for a broad class of locally chronological spaces. This is the main result of this chapter:

Theorem 4.3.2. *Let $(X, \ll_\bullet)$ and $(Y, \lll_\bullet)$ be locally chronological spaces with a regular topology and empty timelike boundary.*

Any isomorphism $F\colon \Pi^{\mathrm{t}}(X) \to \Pi^{\mathrm{t}}(Y)$ of semicategories is induced by an isomorphism $f\colon X \to Y$ of locally chronological spaces.

Proof. We will first show that the object map $f\colon X = \mathrm{Obj}\big(\Pi^{\mathrm{t}}(X)\big) \to \mathrm{Obj}\big(\Pi^{\mathrm{t}}(Y)\big) = Y$ of an isomorphism F is a homeomorphism.

For any $[c] \in \Pi^{\mathrm{t}}(X)(x,y)$, we have

$$\begin{aligned} z \in I_X\big([c]\big) &\iff \exists [a] \in \Pi^{\mathrm{t}}(X)(x,z), [b] \in \Pi^{\mathrm{t}}(X)(z,y) : [b][a] = [c] \\ &\overset{(*)}{\iff} \exists [a'] \in \Pi^{\mathrm{t}}(Y)\big(f(x), f(z)\big), [b'] \in \Pi^{\mathrm{t}}(Y)\big(f(z), f(y)\big) : [b'][a'] = F\big([c]\big) \\ &\iff f(z) \in I_Y\big(F\big([c]\big)\big) \end{aligned}$$

hence $f\big(I_X\big([c]\big)\big) = I_Y\big(F\big([c]\big)\big)$. The starred equivalence is obtained by setting $[a'] \coloneqq F\big([a]\big)$ and $[b'] \coloneqq F\big([b]\big)$, or $[a] \coloneqq F^{-1}\big([a']\big)$ and $[b] \coloneqq F^{-1}\big([b']\big)$, and using that both F and F^{-1} are semifunctors. We have just shown that f and f^{-1} map the basis of X from Corollary 4.2.4 to a basis of Y and vice versa, hence f is a homeomorphism.

We will now show that f is a morphism of locally chronological spaces. Let $U \subseteq Y$ be open and $x, y \in f^{-1}(U)$ with $x \ll_{f^{-1}(U)} y$. Because of Lemma 4.2.6 and the regularity of the topology, we can subdivide any timelike path $c \in P^{\mathrm{t}}\big(f^{-1}(U)\big)(x,y)$ into finitely many subpaths $c_1, \ldots, c_n$ such that $\overline{I_X\big([c_i]\big)} \subseteq f^{-1}(U)$. To reach the desired conclusion $f(x) \lll_U f(y)$, we only need to show $f\big(c_i(0)\big) \lll_U f\big(c_i(1)\big)$ for any $i = 1, \ldots n$. This follows from the existence of the timelike homotopy class $F\big([c_i]\big) \in \Pi^{\mathrm{t}}(Y)\big(f(c_i(0)), f(c_i(1))\big)$, which consists of timelike paths in

$$\overline{I_Y\big(F\left([c_i]\right)\big)} = f\left(\overline{I_X\left([c_i]\right)}\right) \subseteq U.$$

Analogously, f^{-1} is also a morphism of locally chronological spaces.

By further refining our decomposition, we can ensure $\overline{I_X\big([c_i]\big)} \subseteq f^{-1}(U_i)$ for some timelike simply connected open subset $U_i \subseteq Y$. Therefore, there is a timelike homotopy from $f \circ c_i$ to any representative of $F\big([c_i]\big)$, which implies $f_*\big([c_i]\big) = [f \circ c_i] = F\big([c_i]\big)$. Taking everything together, we see

$$f_*\big([c]\big) = f_*\big([c_n]\big) \ldots f_*\big([c_1]\big) = F\big([c_n]\big) \ldots F\big([c_1]\big) = F\big([c]\big)$$

for any $[c] \in \Pi^{\mathrm{t}}(X)$.

□

Remark 4.3.3. While the object map of an invertible semifunctor is an isomorphism, the object map of a non-invertible semifunctor $F\colon \Pi^{\mathrm{t}}(X) \to \Pi^{\mathrm{t}}(Y)$ is not necessarily a morphism of locally chronological spaces.

For example, let $\mathbb{R}$ be the one-dimensional Minkowski space. A path in $\mathbb{R}$ is timelike if and only if it is continuous and strictly monotonously increasing and therefore $\mathbb{R}$ is timelike simply connected. For any $x, y \in X$ with $x < y$, we denote the only timelike homotopy class in $\Pi^{\mathrm{t}}(\mathbb{R})(x, y)$ by $[\gamma_{xy}]$. The object map

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \begin{cases} x & \text{if } x < 0 \\ x+1 & \text{if } x \geq 0 \end{cases} \end{aligned}$$

together with the morphism map

$$\begin{aligned} F\colon \operatorname{Mor}\big(\Pi^{\mathrm{t}}(\mathbb{R})\big) &\to \operatorname{Mor}\big(\Pi^{\mathrm{t}}(\mathbb{R})\big) \\ [\gamma_{xy}] &\mapsto \big[\gamma_{f(x),f(y)}\big] \end{aligned}$$

is a semifunctor, but f is not continuous.

4.4 Consequences for spacetime manifolds

We have just seen that a local chronological structure on topological space X (including its topology) can be reconstructed from the semicategory $\Pi^{\mathrm{t}}(X)$ under quite general conditions. In Chapter 3, we also discussed that smooth spacetimes are a special case of locally chronological spaces. Naturally, the question arises whether an isomorphism also preserves the differentiable structure or the geometry of spacetimes.

Definition 4.4.1. A *global conformal transformation* between two smooth spacetimes (X, g), (Y, h) is a time orientation preserving diffeomorphism $f\colon X \to Y$ such that f^*h is *conformal* to g. This means that there is a smooth function $\Omega\colon X \to (0, \infty)$ such that

$$h_p(f_*v, f_*w) = \Omega(p)\, g_p(v, w)$$

for all $p \in X$ and $v, w \in T_pM$.

Theorem 4.4.2. *A diffeomorphism $f\colon X \to Y$ between smooth spacetimes (X, g) and (Y, h) is a global conformal transformation if and only if it is an isomorphism of X and Y as locally chronological spaces.*

Proof. Any global conformal transformation f maps future-directed timelike vectors, and hence also curves, in X to such in Y. As f^{-1} is also a global conformal transformation, we have the equivalence $x \ll_U y \iff f(x) \ll_{f(U)} f(y)$. This implies that f is an isomorphism of locally chronological spaces.

On the other hand, if f maps future-directed timelike paths in X to such in Y, its differential also maps the future cone of g_p in T_pX to that of $h_{f(p)}$ in $T_{f(p)}Y$, for every $p \in X$. If this is the case, f is a conformal transformation, as described in [1, Section 2.3]. □

Corollary 4.4.3. *Let (X, g) and (Y, h) be two spacetimes and $F\colon \Pi^{\mathrm{t}}(X) \to \Pi^{\mathrm{t}}(Y)$ an isomorphism of semicategories. If the object map of F is a diffeomorphism, then it is a global conformal transformation.*

The lemma above means that if we are given a manifold X, a smooth atlas on X, and a local chronological structure that is induced by a time-oriented Lorentzian metric g, we can reconstruct g unambiguously up to a conformal factor.

Unfortunately, we cannot reconstruct the differentiable atlas from the local chronological structure alone because isomorphisms of locally chronological structures are not necessarily diffeomorphisms:

Example 4.4.4. Let $X = \mathbb{R}^2$. The Lorentzian metric $g = -(\mathrm{d}x_1\,\mathrm{d}x_2 + \mathrm{d}x_2\,\mathrm{d}x_1)$ on X is time-oriented by the constant vector field $V = (1, 1)$. This spacetime results from two-dimensional Minkowski space after a rotation of the coordinates by $\pi/4$.

It is not hard to check that the future cone of a point is its upper right quadrant,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \ll_X \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \iff y_1 > x_1 \text{ and } y_2 > x_2,$$

and a path $c\colon [0, 1] \to X, t \mapsto (c_1(t), c_2(t))$ is future-directed timelike if and only if both c_1 and c_2 are strictly monotonously increasing functions, as in Lemma 3.4.7. Therefore, both the continuous map $f\colon X \to X$, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \sqrt[3]{x_1} \\ \sqrt[3]{x_2} \end{pmatrix}$ and its inverse map $f^{-1}\colon \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} {x_1}^3 \\ {x_2}^3 \end{pmatrix}$ preserve timelike paths, hence f is an isomorphism of locally chronological spaces. However, f it is not differentiable, and therefore not a diffeomorphism.

5 Cosheaves and the theorem of Seifert-van Kampen

In algebraic topology, one of the most important and foundational theorems is the one of Seifert and van Kampen [21, 22]. It allows us to calculate the fundamental group $\pi_1(X)$ of a topological space $X = X_1 \cup X_2$ from the groups $\pi_1(X_1)$, $\pi_1(X_2)$, and $\pi_1(X_1 \cap X_2)$ under certain conditions. There is an even more powerful version for fundamental groupoids instead of groups. The goal of this chapter is to provide an analogous statement for fundamental semicategories of locally chronological spaces.

We will introduce some necessary constructions in Section 5.1 before reviewing the theorems for fundamental groups and groupoids in Section 5.2.

In Section 5.3, we will take a different point of view and regard Π^{t} and Π not only as functors from the category **Top** but restrict them to the category $\mathrm{Op}(X)$ of open subsets of a fixed topological space X. In this sense, the theorem of Seifert-van Kampen essentially states that the fundamental groupoid functor Π is a *cosheaf*, as it maps unions of open subsets to (generalized) pushouts. We will discuss the definition and some special cases of cosheaves, including the cosheaves of transitive relations from Definition 3.1.3.

Finally, in Section 5.4, we will prove a generalization of the theorem of Seifert-van Kampen for the fundamental semicategory functor of a locally chronological space.

5.1 Category theoretic constructions

In order to understand the theorem of Seifert-van Kampen, we first need to know the definition of a pushout. Coproducts and coequalizers will also be helpful along the way:

Definition 5.1.1. Let $\mathcal{C}$ be a category and I be some set.

i) Let $(X_i)_{i\in I}$ be a family of objects in $\mathcal{C}$.

   A *coproduct* of $(X_i)_{i\in I}$ is an object $\coprod_{i\in I} X_i$ with morphisms $\iota_k\colon X_k \to \coprod_{i\in I} X_i$ that satisfies the following universal property:

   If there is another object $K \in \mathrm{Obj}(\mathcal{C})$ with morphisms $f_k\colon X_k \to K$, there is exactly one morphism $\phi\colon \coprod_{i\in I} X_i \to K$ with $f_k = \phi \circ i_k$ for all $k \in I$.

The morphisms ι_k are called *canonical inclusions*. The universal property is often expressed by saying that the morphisms f_k "factor through $\coprod_{i\in I} X_i$".

ii) Let $X, Y \in \mathrm{Obj}(\mathcal{C})$ and let $f, g\colon X \to Y$ be two morphisms.

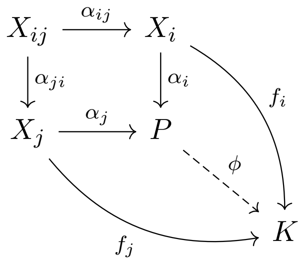
A *coequalizer* of f and g is an object $\mathrm{coeq}(f,g) \in \mathrm{Obj}(\mathcal{C})$ with a morphism $p\colon Y \to \mathrm{coeq}(f,g)$ such that $p \circ f = p \circ g$ that satisfies the following universal property:

If there is another object $K \in \mathrm{Obj}(\mathcal{C})$ with a morphism $q\colon Y \to K$ such that $q \circ f = q \circ g$, there is exactly one morphism $\phi\colon\ \mathrm{coeq}(f,g) \to K$ such that $q = \phi \circ p$.

iii) Let $(X_i)_{i\in I}$ and $(X_{ij})_{i,j\in I}$ be two families of objects in $\mathrm{Obj}(\mathcal{C})$ that satisfy $X_{ij} = X_{ji}$, together with morphisms $\alpha_{ij}\colon X_{ij} \to X_i$.

A *generalized pushout*[1] of the morphisms $\left(\alpha_{ij}\right)_{i,j\in I}$ is an object $P \in \mathrm{Obj}(\mathcal{C})$ with morphisms $\alpha_i\colon X_i \to P$ such that $\alpha_i \circ \alpha_{ij} = \alpha_j \circ \alpha_{ji}$ holds for every $i, j \in I$, and P satisfies the following universal property:

If there is another object K with morphisms $f_i\colon X_i \to K$ such that $f_i \circ \alpha_{ij} = f_j \circ \alpha_{ji}$, there is exactly one morphism $\phi\colon P \to K$ such that the diagram



commutes for every $i, j \in I$ with $i \neq j$.

For $I = \{1, 2\}$, the generalized pushout of α_{12} and α_{21} is just called the *pushout of* X_1 *and* X_2 *over* X_{12} and written as $P = X_1 \coprod_{X_{12}} X_2$, if the morphisms α_{12} and α_{21} are clear from the context.

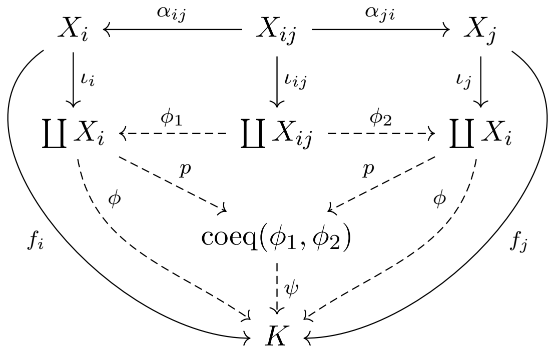
The three definitions above are instances of a more general universal construction called *colimit*. As every colimit, a coproduct, coequalizer, or pushout is unique up to an isomorphism in $\mathcal{C}$ if it exists. Therefore, it is justified to speak, for example, of "the" coequalizer of f and g.

For explicit constructions, it is often easier to express pushouts in terms of coproducts and coequalizers:

[1] The term "generalized pushout" is not commonly used in the literature. Pushouts are usually defined with only two objects.

Lemma 5.1.2. *In the situation of Definition 5.1.1 iii), the generalized pushout of* $\left(\alpha_{ij}\right)_{i,j\in I}$ *is the coequalizer of the two morphisms* $\coprod_{\substack{i,j\in I\\ i\neq j}} X_{ij} \to \coprod_{i\in I} X_i$*, induced by the maps* α_{ij} *or* α_{ji}*, respectively, if these coproducts and coequalizers exist.*

Proof. Let K be as in Definition 5.1.1 iii) and consider the commuting diagram

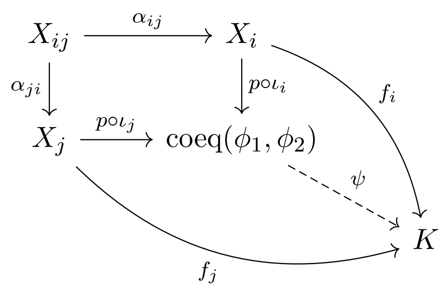


for all $i, j \in I$ with $i \neq j$, while ignoring all the dashed arrows at first.

By universality of the coproduct, the morphisms $\iota_i \circ \alpha_{ij}$ and $\iota_j \circ \alpha_{ji}$ both factor through $\coprod X_{ij}$ and give rise to the morphisms ϕ_1 and ϕ_2, respectively—these are the two induced morphisms that the lemma refers to. The morphisms f_i also factor through $\coprod X_i$ and give rise to a unique morphism ϕ with $\phi \circ \iota_i = f_i$ for every $i \in I$. Taking everything together, we see

$$\phi \circ \phi_1 \circ \iota_{ij} = \phi \circ \iota_i \circ \alpha_{ij} = f_i \circ \alpha_{ij} = f_j \circ \alpha_{ji} = \phi \circ \iota_j \circ \alpha_{ji} = \phi \circ \phi_2 \circ \iota_{ji},$$

but by universality of $\coprod X_{ij}$, there is exactly one morphism $\phi' \colon \coprod X_{ij} \to K$ such that $f_i \circ \alpha_{ij} = f_j \circ \alpha_{ji} = \phi' \circ \iota_{ij}$, which implies $\phi' = \phi \circ \phi_1 = \phi \circ \phi_2$. By definition of the coequalizer, there is exactly one morphism $\psi \colon \text{coeq}(\phi_1, \phi_2) \to K$ such that $\phi = \psi \circ p$. We have now proven that the above diagram, including all dashed arrows, commutes. Leaving out the coproducts yields the commuting diagram



for all $i, j \in I$ with $i \neq j$. Note that p does not depend on K or f_i.

On the other hand, if ψ is any morphism for that the second diagram commutes, we can define $\phi := \psi \circ p$ and get back the first commuting diagram. As ϕ is uniquely determined by the family (f_i) and ψ is uniquely determined by ϕ, this shows that ψ is unique. Therefore, $\operatorname{coeq}(\phi_1, \phi_2)$ is indeed the generalized pushout of $(\alpha_{ij})_{i,j\in I}$. □

As with all category theoretic constructions, it is instructive to look at examples from known categories.

Example 5.1.3 (Coproducts and pushouts in **Set** and **Top**).

i) The *coproduct* of a family $(X_i)_{i\in I}$ of sets (i.e. in the category **Set**) is their disjoint union, which can be formally defined as

$$\bigsqcup_{i\in I} X_i := \{\, (x,i) \mid i \in I, x \in X_i \,\}$$

with the canonical inclusions $\iota_k \colon X_k \to \bigsqcup_{i\in I} X_i$, $x \mapsto (x,k)$.

Proof. If K is any other set with maps $f_i \colon X_i \to K$, the map

$$\phi\colon \bigsqcup_{i\in I} X_i \to K$$
$$(x,i) \mapsto f_i(x)$$

is obviously the only one that satisfies $f_i = \phi \circ \iota_i$. □

ii) Let $(U_i)_{i\in I}$ be a family of subsets of a set X. The generalized pushout of the inclusion maps $\iota_{ij} \colon U_i \cap U_j \to U_i$ is the union $U := \bigcup_{i\in I} U_i \subseteq X$.

Proof. For brevity, we write $\iota_i \colon U_i \to U$ for the inclusion maps. If there is a set K with maps $f_i \colon U_i \to K$ that satisfy $f_i \circ \iota_{ij} = f_j \circ \iota_{ji}$, we have $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for all $i, j \in I$. Therefore, the map

$$\phi\colon \bigcup_{i\in I} U_i \to K$$
$$x \mapsto f_i(x) \quad \text{if } x \in U_i$$

is well-defined and the only map that satisfies $\phi \circ \iota_i = \phi|_{U_i} = f_i$ for all $i \in I$. □

iii) The same is true in the category **Top** if all the sets U_i are open because ϕ is continuous if all the maps $\phi|_{X_i} = f_i$ are continuous.

5.2 The theorem of Seifert-van Kampen for fundamental groups and groupoids

The Seifert-van Kampen theorem is often stated in its most simple form for the union of just two subsets, but there is also a variant for an arbitrary (not necessarily final) number of subsets:

Theorem 5.2.1 (Seifert-van Kampen, see [15, Theorem 1.20])**.**

Let $X = \bigcup_{i \in I} X_i$ be an open cover of a topological space X and fix a basepoint $x \in \bigcap_{i \in I} X_i$.

If X_i, $X_i \cap X_j$, and $X_i \cap X_j \cap X_k$ are path-connected for all $i, j, k \in I$, then $\pi_1(X)$ is the pushout of the group homomorphisms $\iota_{ij*} \colon \pi_1(X_i \cap X_j) \to \pi_1(X_i)$ induced by the inclusions $\iota_{ij} \colon X_i \cap X_j \to X_i$.

In the above theorem, the subsets need to be connected and contain the basepoint x because a fundamental group only depends on the path-connected component in which the basepoint lies. In other words, the fundamental groups contain no information about any other path-connected components, but this information is necessary to correctly reassemble $\pi_1(X, x)$. The usage of fundamental groupoids with multiple basepoints allows us to get rid off this restriction:

Theorem 5.2.2 (Seifert-van Kampen for unreduced groupoids, see [23])**.**

Let $X = \bigcup_{i \in I} X_i$ be an open cover of a topological space X.

The fundamental groupoid $\Pi(X)$ is the pushout of the functors $\iota_{ij*} \colon \Pi(X_i \cap X_j) \to \Pi(X_i)$ induced by the inclusions $\iota_{ij} \colon X_i \cap X_j \to X_i$.

For practical calculations, it is useful to restrict the number of objects in the groupoids by choosing a suitable set of basepoints:

Theorem 5.2.3 (Seifert-van Kampen for groupoids, see [23])**.**

Let $X = \bigcup_{i \in I} X_i$ be an open cover of a topological space X and fix a set $A \subseteq X$ that contains at least one point out of each path-connected component of X_i, $X_i \cap X_j$, and $X_i \cap X_j \cap X_k$ for all $i, j, k \in I$.

Then, $\Pi(X, A)$ is the pushout of the functors $\iota_{ij*} \colon \Pi(X_i \cap X_j, A \cap X_i \cap X_j) \to \Pi(X_i, A \cap X_i)$.

The proof for these two theorems, in the special case of only two subsets, is also nicely explained in [18].

5.3 Precosheaves and cosheaves

The attempt of generalizing the Seifert-van Kampen theorem naturally leads to the definition of a *cosheaf*:

Definition 5.3.1. Let $\mathcal{C}$ be a category and X a topological space. Let $\mathrm{Op}(X)$ be the category whose objects are the open subsets of X and whose morphisms are the inclusion maps $U \to V$ for any open $U \subseteq V \subseteq X$.

i) A *$\mathcal{C}$-valued precosheaf on X* is a functor $F\colon \mathrm{Op}(X) \to \mathcal{C}$.

ii) A *$\mathcal{C}$-valued cosheaf on X* is a functor $F\colon \mathrm{Op}(X) \to \mathcal{C}$ such that for any family $(U_i)_{i\in I}$ of open subsets of X, the object $F\big(\bigcup_{i\in I} U_i\big)$ is the generalized pushout of the morphisms $F(\iota_{ij})\colon F(U_i \cap U_j) \to F(U_i)$ induced by the inclusions $\iota_{ij}\colon U_i \cap U_j \to U_i$.

Note that $\mathrm{Op}(X)$ is a subcategory of **Top** because the subsets of X carry the subspace topology and the inclusion maps are continuous. If we restrict the functor $\Pi\colon \mathbf{Top} \to \mathbf{Grpd}$ from Lemma 2.4.8 to $\mathrm{Op}(X)$, Theorem 5.2.2 is equivalent to the statement "The functor $\Pi\colon \mathrm{Op}(X) \to \mathbf{Grpd}$ is a cosheaf".

Remark 5.3.2. For readers who are familiar with sheaves: If coproducts and coequalizers exist in the category $\mathcal{C}$, a functor $F\colon \mathrm{Op}(X) \to \mathcal{C}$ is a cosheaf if and only if for all covers $U = \bigcup_{i\in I} U_i$, the sequence

$$\coprod_{i,j\in I} F(U_i \cap U_j) \rightrightarrows \coprod_{i\in I} F(U_i) \to F(U)$$

is exact. This means that $F(U)$ is the coequalizer of the two morphisms induced by $F(\iota_{ij})$ and $F(\iota_{ji})$, respectively, as in Lemma 5.1.2. This property is dual to the definition of a sheaf.

The (pre-)cosheaves of transitive relations from Definition 3.1.3 are a special case of the general (pre-)cosheaves from Definition 5.3.1. To see this, we have to define the target category first, whose objects should be "sets with transitive relations". But instead of defining a new category, we can also identify these with small thin (or simply connected, see Definition 2.4.6) semicategories:

Remark 5.3.3. From now on, we regard any set with a transitive relation $(X, \ll)$ as a semicategory with object set X and total morphism set[2] $\ll$. A pair (x, y) is considered to be a morphism from x to y whenever $x \ll_X y$, and there is only one possible multiplication rule

[2]Remember that we adopted the definition that a relation $\ll$ is a subset of $X \times X$ and "$x \ll y$" is just an abbreviation for $(x, y) \in \ll$.

$(y, z)(x, y) := (x, z)$. The transitivity of $\ll$ ensures that the morphism (x, z) exists whenever the morphisms (x, y) and (y, z) exist.

If $(X, \ll_X)$ and $(Y, \ll_Y)$ are two sets with transitive relations, a map $f\colon X \to Y$ is the object map of a functor $(X, \ll_X) \to (Y, \ll_Y)$ if and only if

$$\forall x, y \in X : x \ll_X y \implies f(x) \ll_Y f(y).$$

If this is the case, the morphism map is $(x, y) \mapsto \big(f(x), f(y)\big)$, so any such functor is uniquely determined by its object map.

We can slightly generalize the last statement:

Lemma 5.3.4. *Let $\mathcal{C}, \mathcal{D}$ be semicategories, and let $\mathcal{D}$ be thin.*
A map $f\colon \operatorname{Obj}(\mathcal{C}) \to \operatorname{Obj}(\mathcal{D})$ is the object map of a semifunctor if and only if

$$\forall x, y \in \operatorname{Obj}(\mathcal{C}) : \mathcal{C}(x, y) \neq \emptyset \implies \mathcal{D}(f(x), f(y)) \neq \emptyset.$$

and in this case, the semifunctor is uniquely determined by its object map f.

Proof. For any $x, y \in \operatorname{Obj}(\mathcal{C})$, there is at most one map $\mathcal{C}(x, y) \to \mathcal{D}\big(f(x), f(y)\big)$ because $\mathcal{D}\big(f(x), f(y)\big)$ contains at most one element. Such a map exists if and only if the implication $\mathcal{C}(x, y) \neq \emptyset \implies \mathcal{D}\big(f(x), f(y)\big) \neq \emptyset$ holds.

All of these maps together form the morphism map of a semifunctor because for any morphisms $m_{xy} \in \mathcal{C}(x, y)$ and $m_{yz} \in \mathcal{C}(y, z)$, both $F(m_{yz})\, F(m_{xy})$ and $F(m_{yz}\, m_{xy})$ are in $\mathcal{D}(f(x), f(z))$, which contains at most one element, so they are equal. □

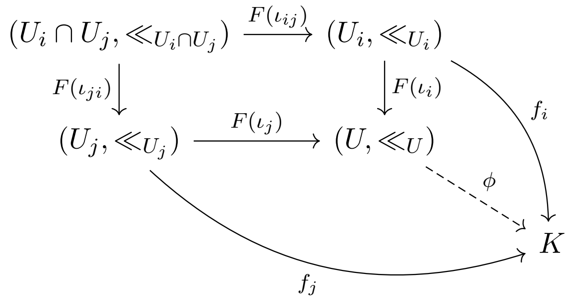
Lemma 5.3.5. *Let* **thinSemCat** *be the category of small thin semicategories (with semifunctors as morphisms).*

Any precosehaf $\ll_\bullet$ of transitive relations according to Definition 3.1.3 on a topological space X induces a **thinSemCat***-valued precosheaf according to Definition 5.3.1 with object map $U \mapsto (U, \ll_U)$ and the morphism map that is induced by the inclusion maps in $\operatorname{Op}(X)$.*

This precosheaf is a **thinSemCat***-valued cosheaf if and only if $\ll_\bullet$ is a cosheaf of transitive relations.*

Proof. Let $\ll_\bullet$ be a precosheaf of transitive relations and $U \subseteq V \subseteq X$ be open subsets. Remember that a precosheaf of transitive relations satisfies $x \ll_U y \implies x \ll_V y$ for all $x, y \in U$. By Remark 5.3.3, this implies that every inclusion map $\iota_{UV}\colon U \to V$ induces a semifunctor $F(\iota_{UV})\colon (U, \ll_U) \to (V, \ll_V)$ with object map ι_{UV}. If W is another open set with $V \subseteq W \subseteq X$, we have $F(\iota_{VW})F(\iota_{UV}) = F(\iota_{UW})$. Additionally, the identity $\iota_{UU} = \mathrm{id}_U$ gets mapped to $F(\mathrm{id}_U) = \mathrm{id}_{(U, \ll_U)}$, hence F is a functor.

Now let $\ll_\bullet$ be a cosheaf of transitive relations, $U = \bigcup_{i\in I} U_i$ be an open cover, K be another thin small semicategory with morphisms $f_i\colon (U_i, \ll_{U_i}) \to K$. We need to show that there is exactly one semifunctor ϕ such that the diagram



in **thinSemCat** commutes for all $i, j \in I$.

If we assume the existence of ϕ, its object map is the one with $\phi|_{U_i} = f_i$ for all $i \in I$, as in Example 5.1.3 ii). Because a semifunctor into a thin semicategory is uniquely determined by its object map, this implies that ϕ is unique if it exists.

For the existence proof of the semifunctor ϕ, we define the object map of ϕ as above by $\phi|_{U_i} = f_i$ for all $i \in I$. To apply Lemma 5.3.4, we only need to show that $K\big(\phi(x), \phi(y)\big) \neq \emptyset$ for any $x, y \in U$ with $x \ll_U y$. As $\ll_\bullet$ is a cosheaf of transitive relations, there is a chain of points $x_0, \dots, x_n \in U$ with

$$x = x_0 \ll_{U_{i_1}} x_1 \ll_{U_{i_2}} \cdots \ll_{U_{i_n}} x_n = y$$

for suitable $i_1, \dots, i_n \in I$. From this, it follows that there is a morphism

$$f_{i_n}(x_{n-1}, x_n) \dots f_{i_2}(x_1, x_2)\, f_{i_1}(x_0, x_1) \in K\big(f_{i_1}(x), f_{i_n}(y)\big) = K\big(\phi(x), \phi(y)\big),$$

so Lemma 5.3.4 implies that there is indeed a semifunctor ϕ with the required properties.

Now let F be a **thinSemCat**-valued cosheaf and let $\lessdot$ be a transitive relation that contains all relations $\ll_{U_i}$. If we set $K = (U, \lessdot)$ and let f_i be the semifunctor with object map ι_i, we can apply the argument from above again to see that the object map of ϕ is id_U. From this, it follows that $\ll_U$ is contained in $\lessdot$. Therefore, $\ll_U$ is the smallest relation that contains all relations $\ll_{U_i}$ and $\ll_\bullet$ is indeed a cosheaf of transitive relations. □

Note that in the proof above, we only showed that $(U, \ll_U)$ is a pushout in the category **thinSemCat**. In the category **SemCat**, the pushout of thin semicategories is in general not thin anymore.

In a similar fashion, we can see that the locally chronological space $(X, \ll_\bullet)$ that we constructed in Lemma 3.4.5 is actually the (generalized) pushout of $\big((U_i, \ll_{i,\bullet})\big)_{i\in I}$ over their pairwise intersections in the category of locally chronological spaces.

5.4 A generalization of Seifert-van Kampen for fundamental semicategories

We will now work toward the proof that the fundamental semicategory functor is a cosheaf of semicategories. This statement is analogous to the theorem of Seifert-van Kampen. For this purpose, the alternative definition of a path space that was discussed in Subsection 2.4.3 will be useful:

Definition 5.4.1. Let $(X, \ll_\bullet)$ be a locally chronological space and $P'(X)$ be the path space category as in Definition 2.4.10. A path $c \in P'(X)$ is called timelike if its reparametrization to the unit interval $\operatorname{rep}(c)$ is timelike.

We define $P'^{\mathrm{t}}(X)$ to be the sub-semicategory of $P'(X)$ with object set X and whose morphisms are the timelike paths.

Note that if $U \subseteq V \subseteq X$ are locally chronological subspaces of $(X, \ll_\bullet)$, the path space $P'^{\mathrm{t}}(U)$ is actually a sub-semicategory of $P'^{\mathrm{t}}(V)$ because of Lemma 3.4.4. The inclusion map $U \to V$ induces the inclusion semifunctor $P'^{\mathrm{t}}(U) \to P'^{\mathrm{t}}(V)$, which turns P'^{t} into a **semCat**-valued precosheaf on X. We will show that these path spaces actually form a cosheaf before proving that the fundamental semicategories form a cosheaf:

Lemma 5.4.2. *Let $X = \bigcup_{i \in I} X_i$ be an open cover of a locally chronological space $(X, \ll_\bullet)$.*

The semicategory $P'^{\mathrm{t}}(X)$ is the pushout in the category **semCat** *of the inclusion semifunctors $\iota_{ij} \colon P'^{\mathrm{t}}(X_i \cap X_j) \to P'^{\mathrm{t}}(X_i)$.*

Corollary 5.4.3. *For a locally chronological space X, the functor $P'^{\mathrm{t}} \colon \operatorname{Op}(X) \to$* **semCat** *is a cosheaf.*

Proof of Lemma 5.4.2. We are going to check the universal property of the pushout, so let K be a semicategory with semifunctors $f_i \colon X_i \to K$ such that $f_i \circ \iota_{ij} = f_j \circ \iota_{ji}$ for all $i, j \in I$. We need to show that there is a unique semifunctor $\phi \colon P'^{\mathrm{t}}(X) \to K$ that satisfies $f_i = \phi \circ \iota_i$ for all $i \in I$, where ι_i is the inclusion semifunctor $P'^{\mathrm{t}}(X_i) \to P'^{\mathrm{t}}(X)$.

By the same argument as in Example 5.1.3, there is only one possible object map for such a semifunctor:

$$\phi \colon \operatorname{Obj}\left(P'^{\mathrm{t}}(X)\right) = \bigcup_{i \in I} X_i \to \operatorname{Obj}(K)$$
$$x \mapsto f_i(x) \quad \text{if } x \in X_i.$$

Any timelike path $c \in P'^{\mathrm{t}}(X)$ has a compact trace in X, so by the Lebesgue covering lemma we can decompose its domain into finitely many intervals $[t_k, t_{k+1}]$ with $c\big([t_k, t_{k+1}]\big) \subseteq X_{i_k}$.

This leads to a decomposition $c = \iota_{i_n}(c_n) \cdots \iota_{i_1}(c_1)$ such that $c_k := c|_{[t_k, t_{k+1}]} \in P'^{\mathrm{t}}(X_{i_k})$. If a semifunctor ϕ with $f_i = \phi \circ \iota_i$ for all $i \in I$ exists, its morphism map has to satisfy

$$\phi(c) = \phi\big(\iota_{i_n}(c_n)\big) \cdots \phi\big(\iota_{i_1}(c_1)\big) = f_{i_n}(c_n) \cdots f_{i_1}(c_1),$$

which shows that it is uniquely determined.

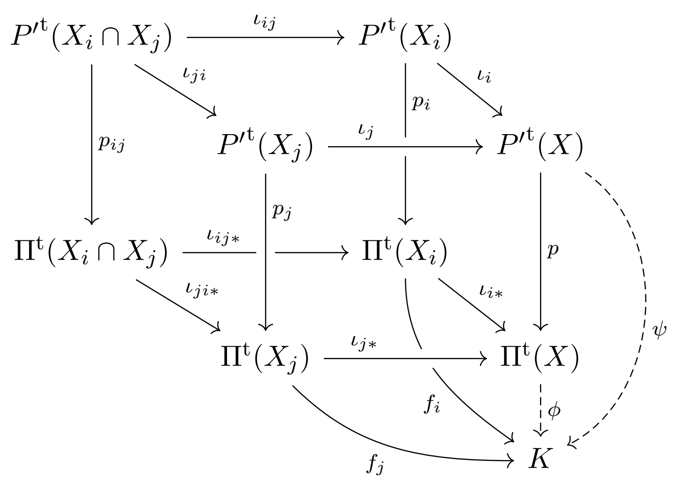
In order to prove the existence of ϕ, we take this equation as a definition for the morphism map of ϕ and show that $\phi(c)$ does not depend on which decomposition of c we choose:
If any path c_{i_k} lies in both $P'^{\mathrm{t}}(X_i)$ and $P'^{\mathrm{t}}(X_j)$, it also lies in $P'^{\mathrm{t}}(X_i \cap X_j)$, hence $f_i(c_{i_k}) = f_i\big(\iota_{ij}(c_{i_k})\big) = f_j\big(\iota_{ji}(c_{i_k})\big) = f_j(c_{i_k})$. For any two decompositions of c, we can find a common refinement of these two decompositions and see that these lead to the same image in $\mathcal{C}$, hence the morphism-map of ϕ is well defined. With these definitions, ϕ is indeed a functor because we can just concatenate decompositions of multiple paths in the defining equation. ☐

Theorem 5.4.4. *Let $X = \bigcup_{i \in I} X_i$ be an open cover of a locally chronological space $(X, \ll_\bullet)$.*

The fundamental semicategory $\Pi^{\mathrm{t}}(X)$ is the pushout of the semifunctors $\iota_{ij*} \colon \Pi^{\mathrm{t}}(X_i \cap X_j) \to \Pi^{\mathrm{t}}(X_i)$ induced by the inclusions $\iota_{ij} \colon X_i \cap X_j \to X_i$.

Corollary 5.4.5. *For a locally chronlogical space X, the fundamental semicategory functor $\Pi^{\mathrm{t}} \colon \mathrm{Op}(X) \to$ **semCat** is a cosheaf.*

Proof of Theorem 5.4.4. Let K be a semicategory with semifunctors $f_i \colon X_i \to K$ such that $f_i \circ \iota_{ij*} = f_j \circ \iota_{ji*}$ for all $i, j \in I$. Let us assume that there is a semifunctor $\phi \colon \Pi^{\mathrm{t}}(X) \to K$ with $f_i = \phi \circ \iota_{i*}$ for every $i \in I$. If we denote the quotient semifunctor by $p \colon P'^{\mathrm{t}}(X) \to \Pi^{\mathrm{t}}(X), c \mapsto [c]$ (and analogously p_i, p_{ij}) and define $\psi = \phi \circ p$, we get a commuting diagram



for any $i, j \in I$. As Lemma 5.4.2 states, $P'^{\mathrm{t}}(X)$ is the pushout of the inclusion semifunctors ι_{ij}, so there is exactly one semifunctor ψ with $f_i \circ p_i = \psi \circ \iota_i$ for all $i \in I$. The semifunctor ϕ is then determined by $\phi\big([c]\big) = \psi(c)$, so ϕ is unique if it exists.

For proving the existence of ϕ it is therefore sufficient to prove $\psi(c_0) = \psi(c_1)$ whenever $c_0, c_1 \in P'^{\mathrm{t}}(X)$ are equivalent. The functoriality of ϕ is then inherited from ψ. We can assume without loss of generality that c_0 and c_1 have the domain $[0,1]$ and there is a timelike homotopy

$$\begin{aligned} H\colon [0,1] \times [0,1] &\to X \\ (s,t) &\mapsto c_s(t) \end{aligned}$$

from c_0 to c_1.

We can use the Lebesgue covering lemma and Axiom (A4) in Definition 3.1.15 to find subdivisions

$$\begin{aligned} 0 = s_0 < \cdots < s_m = 1, \\ 0 = t_0 < \cdots < t_n \ = 1, \end{aligned}$$

such that for any values of $k \in \{0, \ldots, m-1\}$ and $\ell \in \{0, \ldots, n-2\}$ the rectangle $[s_k, s_{k+1}] \times [t_\ell, t_{\ell+2}]$ gets mapped by H into a timelike simply connected subset $U_{k,\ell} \subseteq X_{i_{k,\ell}}$, as indicated in the figure to the right.

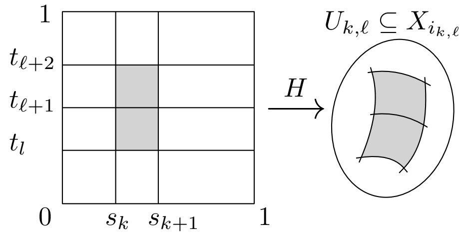


Figure 5.1

Note that in the bottom or top row, that is, for $\ell = 0$ or $\ell = n-1$, the rectangle $[s_k, s_{k+1}] \times [t_\ell, t_{\ell+1}]$ gets mapped into a single subset $U_{k,\ell}$ or $U_{k,\ell-1}$ while for every other ℓ it gets mapped into the intersection $U_{k,\ell-1} \cap U_{k,\ell}$. To increase readability, let us fix k and ℓ for the moment and abbreviate the target set or intersection to U.

By Lemma 3.1.6, the relation $\ll_U$ is an open subset of $U \times U$. Its preimage under the continuous map

$$\begin{aligned} f\colon [s_k, s_{k+1}] \times [s_k, s_{k+1}] &\to U \times U \\ (\sigma_1, \sigma_2) &\mapsto \big(c_{\sigma_1}(t_\ell), c_{\sigma_2}(t_{\ell+1})\big) \end{aligned}$$

contains the diagonal $\Delta \coloneqq \{\, (\sigma, \sigma) \mid \sigma \in [s_k, s_{k+1}] \,\}$ because $c_\sigma|_{[t_\ell, t_{\ell+1}]}$ is a timelike path in U. By compactness, there is an $\varepsilon > 0$ such that the ε-neighborhood of Δ is still contained in the open set $f^{-1}(\ll_U)$, which means that $c_{\sigma_1}(t_\ell) \ll_U c_{\sigma_2}(t_{\ell+1})$ whenever $|\sigma_2 - \sigma_1| < \varepsilon$. By refining our initial subdivision in such a way that $s_{k+1} - s_k < \varepsilon$ holds for all $k \in \{0, \ldots, m-1\}$, we can therefore ensure that there is a timelike *diagonal* path in U from $c_{s_k}(t_\ell)$ to $c_{s_{k+1}}(t_{\ell+1})$.

Putting all of the subpaths $c_{s_k}|_{[t_\ell,t_{\ell+1}]}$ and the diagonal paths together, we get a (not necessarily commuting) diagram of timelike paths

$$
\begin{array}{ccccccccc}
 & c_0(1) & = & c_{s_1}(1) & = & c_{s_2}(1) & = \dots = & c_1(1) & \\
 & \uparrow & \nearrow & \uparrow & \nearrow & \uparrow & & \uparrow & \\
 & c_0(t_{m-1}) & & c_{s_1}(t_{m-1}) & & c_{s_2}(t_{m-1}) & \cdots & c_1(t_{m-1}) & \\
 & \vdots & & \vdots & & \vdots & & \vdots & \\
c_0 & c_0(t_2) & & c_{s_1}(t_2) & & c_{s_2}(t_2) & \cdots & c_1(t_2) & c_1 \\
 & \uparrow & \nearrow & \uparrow & \nearrow & \uparrow & & \uparrow & \\
 & c_0(t_1) & & c_{s_1}(t_1) & & c_{s_2}(t_1) & \cdots & c_1(t_1) & \\
 & \uparrow & \nearrow & \uparrow & \nearrow & \uparrow & & \uparrow & \\
 & c_0(0) & = & c_{s_1}(0) & = & c_{s_2}(0) & = \dots = & c_1(0) &
\end{array}
$$

in $P'^{\mathrm{t}}(X)$. The equal signs do not denote identity morphisms, as these generally do not exist in $P'^{\mathrm{t}}(X)$. Instead, the top and bottom row each consists of only one object because $(c_s)_{s\in[0,1]}$ is a timelike homotopy relative to the endpoints.

For any $k \in \{0,\dots,m-1\}$ and $\ell \in \{0,\dots,n-2\}$ the arrows in the subdiagram

$$
\begin{array}{ccc}
c_{s_k}(t_{\ell+2}) & & c_{s_{k+1}}(t_{\ell+2}) \\
\uparrow & \nearrow & \uparrow \\
c_{s_k}(t_{\ell+1}) & & c_{s_{k+1}}(t_{\ell+1}) \\
\uparrow & \nearrow & \uparrow \\
c_{s_k}(t_\ell) & & c_{s_{k+1}}(t_\ell)
\end{array}
$$

are paths in the timelike simply connected subset $U_{k,\ell} \subseteq X_{i_{k,\ell}}$. As $p_{i_{k,\ell}}$ identifies paths that are timelike homotopic in $X_{i_{k,\ell}}$, the image of this subdiagram under $f_i \circ p_{i_{k,\ell}} = \psi \circ i_{k,\ell}$ commutes in K. Therefore, the image of the whole diagram under ψ also commutes and $\psi(c_0) = \psi(c_1)$ holds, which is what we wanted to prove. □

Up to the point where we subdivided the domain of the timelike homotopy H into rectangles, the above proof is analogous to the one of Theorem 5.2.2. The rest of the proof (see [18, pp. 243–245]) does not directly carry over to our setting because a timelike homotopy H only maps the vertical lines in the square $[0,1]\times[0,1]$ to timelike paths, while there is no restriction on the horizontal paths other than continuity.

To circumvent this problem, one could define *directed homotopies* in such a way that both horizontal and vertical lines of the square are mapped to timelike paths, as done in directed algebraic topology [5]. The drawback of this definition is that directed homotopies lead to a transitive but not reflexive relation.

6 Fundamental semicategories as topological spaces

In this final chapter, we will imbue the total morphism set $\mathrm{Mor}\big(\Pi^{\mathrm{t}}(X)\big)$ of the fundamental semicategory of a locally chronological space with a topology. The main result of this chapter is that the start and endpoint map $(s,e)\colon \mathrm{Mor}\big(\Pi^{\mathrm{t}}(X)\big) \to X \times X$ is a local homeomorphism if the space X has a regular topology and empty timelike boundary. By Lemma 3.4.6, there is also an induced local chronological structure on $\mathrm{Mor}\big(\Pi^{\mathrm{t}}(X)\big)$ itself.

As any covering map is also a local homeomorphism, this fact bears some resemblance to the fact that a universal covering of a topological space X can be constructed from the total morphism set of the fundamental group $\Pi(X)$, see [8, Lemma 1.2]. However, the start and endpoint map of the fundamental semicategory $\Pi^{\mathrm{t}}(X)$ is in general not a covering because the cardinality of $\Pi^{\mathrm{t}}(X)(x,y)$ depends on the position of $x, y \in X$.

In Section 6.1 we define a topology on $\mathrm{Mor}\big(\Pi^{\mathrm{t}}(X)\big)$ based on the compact-open topology on the path space and prove that the map (s,e) is open and continuous. Afterward, in Section 6.2 we derive an algebraic description of this topology and prove that (s,e) is a local homeomorphism.

6.1 Compact-open and quotient topologies

The *compact-open topology* is a commonly used topology on function spaces. We can use it for path spaces because a path is just a continuous function $[0,1] \to X$. For brevity, we will from now on use the same symbol $\Pi^{\mathrm{t}}(X)$ for the semicategory and the total morphism set of $\Pi^{\mathrm{t}}(X)$.

Definition 6.1.1. Let $(X, \ll_\bullet)$ be a locally chronological space.

For any open subset $U \subseteq X$ and compact subset $K \subseteq [0,1]$ we define

$$\Omega(K,U) \coloneqq \left\{ c \in P^t(X) \,\middle|\, c(K) \subseteq O \right\}.$$

The *compact-open topology* on $P^{\mathrm{t}}(X)$ is the topology generated by these sets. This means that the open sets in $P^{\mathrm{t}}(X)$ are unions of finite intersections of sets of the form $\Omega(K,U)$.

The quotient map $p\colon P^{\mathrm{t}}(X) \to \Pi^{\mathrm{t}}(X)$, $c \mapsto [c]$ induces a *quotient topology* on $\Pi^{\mathrm{t}}(X)$. This means that a subset $V \subseteq \Pi^{\mathrm{t}}(X)$ is open if its preimage $p^{-1}(V)$ is open in $P^{\mathrm{t}}(X)$ with respect to the compact-open topology.

The goal of this chapter is to prove that the start- and endpoint-map $(s,e)\colon \Pi^{\mathrm{t}}(X) \to X \times X$ is a local homeomorphism. A local homeomorphism is necessarily continuous and open, which means that preimages and images of open sets are open.

Lemma 6.1.2. *For any locally chronological space $(X, \ll_\bullet)$, the start- and endpoint-map $(s,e)\colon \Pi^{\mathrm{t}}(X) \to X \times X$ is open and continuous.*

Proof. To avoid confusion, we denote the start- and endpoint maps of $P^{\mathrm{t}}(X)$ and $\Pi^{\mathrm{t}}(X)$ by $(s,e)_{P^{\mathrm{t}}(X)}$ and $(s,e)_{\Pi^{\mathrm{t}}(X)}$, respectively. These two maps are related by

$$(s,e)_{P^{\mathrm{t}}(X)} = (s,e)_{\Pi^{\mathrm{t}}(X)} \circ p.$$

Any open set in $X \times X$ is a union of products $U_1 \times U_2$ of open subsets $U_1, U_2 \subseteq X$. The preimage of such a set is

$$p^{-1}\big((s,e)^{-1}_{\Pi^{\mathrm{t}}(X)}(U_1 \times U_2)\big) = (s,e)^{-1}_{P^{\mathrm{t}}(X)}(U_1 \times U_2) = \Omega\big(\{0\}, U_1\big) \cap \Omega\big(\{1\}, U_2\big)$$

and therefore open in $P^{\mathrm{t}}(X)$. This implies that $(s,e)^{-1}_{\Pi^{\mathrm{t}}(X)}(U_1 \times U_2)$ is open in $\Pi^{\mathrm{t}}(X)$ with respect to the quotient topology, hence $(s,e)_{\Pi^{\mathrm{t}}(X)}$ is continuous.

One can check that for any path $c \in P^{\mathrm{t}}(X)$ there is a neighborhood basis of c consisting of open sets $U := \bigcap_{i=1}^{k} \Omega\big([t_i, t_{i+1}], U_i\big)$ where $0 = t_1 < \cdots < t_k = 1$, and $U_i \subseteq X$ is an open neighborhood of $c\big([t_i, t_{i+1}]\big)$ for all $i \in \{1, \ldots, k-1\}$. By construction, any element of U is a concatenation of timelike paths in $U_1, \ldots U_{k-1}$. This means that a timelike path from x to y exists in U if and only if there are points $x_1, \ldots, x_k$ such that $x = x_1 \ll_{U_1} x_2 \ll_{U_2} \cdots \ll_{U_k} x_{k+1} = y$. As the chronological relations $\ll_{U_i}$ are open by Lemma 3.1.6, we see that $(s,e)_{P^{\mathrm{t}}(X)}(U)$ is open in $X \times X$. From this, it follows that the map $(s,e)_{P^{\mathrm{t}}(X)}$ is open.

If V is any open set in $\Pi^{\mathrm{t}}(X)$, its preimage $p^{-1}(V)$ is open in $P^{\mathrm{t}}(X)$ by definition of the quotient topology. Therefore, its image $(s,e)_{\Pi^{\mathrm{t}}(X)}(V) = (s,e)_{P^{\mathrm{t}}(X)}\big(p^{-1}(V)\big)$ is also open. □

Note that the above lemma only uses the Axioms (A1) and (A2) in Definition 3.1.15, but not Axioms (A3) and (A4). We also did not put any restrictions on the topology or timelike boundaries of X.

Corollary 6.1.3. *Let U be a timelike simply connected open subset of a locally chronological space. Then the map $(s,e)\colon \Pi^{\mathrm{t}}(U) \to U \times U$ is a homeomorphism onto its image $\ll_U \subseteq U \times U$.*

Proof. By definition of a timelike simply connected subset, the preimage of $(x, y) \in U \times U$ under (s, e) contains exactly one timelike homotopy class if $x \ll_U y$ holds and is empty otherwise. Therefore, (s, e) is an open and continuous bijection, hence a homeomorphism, onto its image $\ll_U$. ☐

The above Corollary implies that there is a continuous map

$$\{\,(x, y) \in U \times U \mid x \ll_U y\,\} \to \Pi^{\mathrm{t}}(U)$$

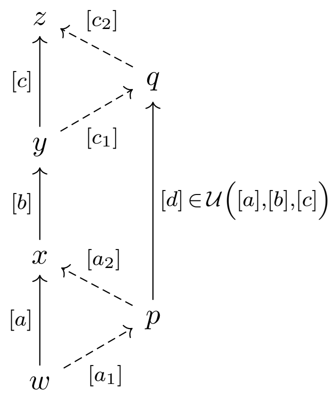
if U is a timelike simply connected set. We have already seen this kind of map in the proofs of Lemma 3.1.10 and Lemma 3.4.8 as $(x, y) \mapsto [\gamma_{xy}]$ or $(x, y) \mapsto [\tilde{\gamma}_{xy}]$, respectively.

6.2 A topological basis of the fundamental semicategory

The next step in proving that (s, e) is a local homeomorphism is to find a cover of $\Pi^{\mathrm{t}}(X)$ by open subsets on which the restriction of (s, e) is injective.

Definition 6.2.1. Let $(X, \ll_\bullet)$ be a locally chronological space, $w, x, y, z \in X$, $[a] \in \Pi^{\mathrm{t}}(X)(w, x)$, $[b] \in \Pi^{\mathrm{t}}(X)(x, y)$, and $[c] \in \Pi^{\mathrm{t}}(X)(y, z)$.

We define $\mathcal{U}\big([a], [b], [c]\big) \subseteq \Pi^{\mathrm{t}}(X)$ to be the set of all morphisms $[d]$ for which there are additional morphisms $[a_1], [a_2], [c_1], [c_2]$ (dashed arrows) that turn the following diagram into a commuting one:



The idea behind this definition is that the set $\mathcal{U}\big([a], [b], [c]\big)$ will be small if the sets $I_X([a])$ and $I_X([c])$ are small. To be more precise, we will see that the sets $\mathcal{U}\big([a], [b], [c]\big)$ form a basis of the topology of $\Pi^{\mathrm{t}}(X)$ if X has a regular topology and empty timelike boundary.

Lemma 6.2.2. *In the situation of Definition 6.2.1, the set* $\mathcal{U}\big([a], [b], [c]\big)$ *is open in* $\Pi^{\mathrm{t}}(X)$.

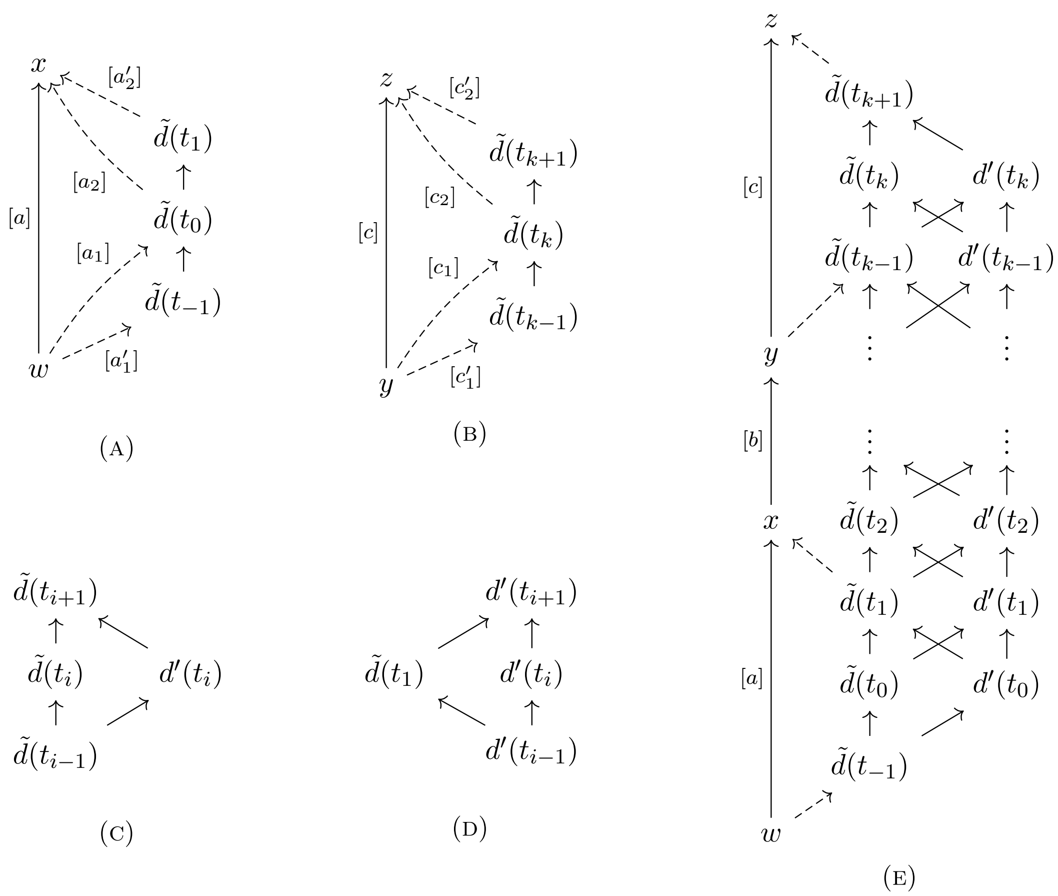


Figure 6.1: Commuting diagrams in $\Pi^{\mathrm{t}}(X)$ for the proof of Lemma 6.2.2. Unlabeled vertical arrows are timelike homotopy classes of paths $\tilde{d}|_{[t_i,t_{i+1}]}$ or $d'|_{[t_i,t_{i+1}]}$, respectively.

Proof. We will show that $p^{-1}\big(\mathcal{U}\big([a],[b],[c]\big)\big)$ is open in $P^{\mathrm{t}}(X)$ by constructing an open neighborhood $\mathcal{U}_d \subseteq p^{-1}\big(\mathcal{U}\big([a],[b],[c]\big)\big)$ of any path $d \in p^{-1}\big(\mathcal{U}\big([a],[b],[c]\big)\big)$.

As $[d] \in \mathcal{U}\big([a],[b],[c]\big)$, there are timelike paths $a_1, a_2, c_1, c_2 \in P^{\mathrm{t}}(X)$ such that the diagram in Definition 6.2.1 commutes. We define the path

$$\tilde{d}\colon [-1,2] \to X$$
$$t \mapsto \begin{cases} a_1(t+1) & \text{if } t \le 0 \\ d\,(t) & \text{if } 0 \le t \le 1 \\ c_2(t-1) & \text{if } 1 \le t \le 2. \end{cases}$$

Now we choose timelike simply connected open sets $U_0, \dots, U_{k-1} \subseteq X$ and

$$-1 < t_{-1} < t_0 = 0 < t_1 < \dots < t_k = 1 < t_{k+1} < 2$$

such that the following properties are satisfied:

i) For every $i \in \{1, \dots, k-1\}$ we have $c([t_{i-1}, t_{i+1}]) \subseteq U_i$.

ii) There are $[a_1'], [a_2'], [c_1'], [c_2'] \in \Pi^{\mathrm{t}}(X)$ such that the diagrams in Figures 6.1A and 6.1B commute.

We can achieve i) by using the Lebesgue covering lemma and ii) by choosing $t_{\pm 1}$ and $t_{k\pm 1}$ sufficiently close to 0 or 1, respectively. The set

$$\begin{aligned}\mathcal{U}_d \coloneqq\ & \Omega\Big(\{0\}, I^-_{U_1}\big(\tilde{d}(t_{-1})\big)\Big) \cap \Omega\Big(\{1\}, I^+_{U_k}\big(\tilde{d}(t_{k-1})\big)\Big) \\ & \cap \left(\bigcap_{i=2}^{k} \Omega\Big(\{t_i\}, I^+_{U_i \cap U_{i-1}}\big(\tilde{d}(t_{i-1})\big)\Big)\right) \\ & \cap \left(\bigcap_{i=1}^{k-2} \Omega\Big(\{t_i\}, I^-_{U_i \cap U_{i+1}}\big(\tilde{d}(t_{i+1})\big)\Big)\right) \\ & \cap \left(\bigcap_{i=1}^{k-1} \Omega\Big([t_{i-1}, t_{i+1}], U_i\Big)\right).\end{aligned}$$

is open in the compact-open topology and contains the path d. Now let $d' \in \mathcal{U}_d$. We need to show that $[d'] \in \mathcal{U}\big([a],[b],[c]\big)$:

The construction of $\mathcal{U}_d$ ensures that we can choose the diagonal arrows in Figure 6.1E in such a way that all arrows in the subdiagrams in Figures 6.1D and 6.1C are timelike homotopy classes of paths in U_i. As U_i is timelike simply connected, these subdiagrams commute in $\Pi^{\mathrm{t}}(X)$, which implies that the diagram in Figure 6.1E also commutes. From this, it follows that $[d']$ is an element of $\mathcal{U}\big([a],[b],[c]\big)$. □

Lemma 6.2.3. *Let $(X, \ll_\bullet)$ be a locally chronological space, $w, x, y, z \in X$, $[a] \in \Pi^{\mathrm{t}}(X)(w,x)$, $[b] \in \Pi^{\mathrm{t}}(X)(x,y)$, and $[c] \in \Pi^{\mathrm{t}}(X)(y,z)$.*

If there are timelike simply connected subsets $U_p, U_q \subseteq X$ such that $I_X([a]) \cup \{w, x\} \subseteq U_p$ and $I_X([c]) \cup \{y, z\} \subseteq U_q$, then the start and endpoint map (s,e) maps $\mathcal{U}\big([a],[b],[c]\big)$ homeomorphically onto $I_X([a]) \times I_X([c])$.

Proof. By Lemma 6.1.2, (s,e) is open and continuous and its restriction to the open set $\mathcal{U}([a],[b],[c])$ inherits these properties. We only need to show that (s,e) maps $\mathcal{U}([a],[b],[c])$ bijectively onto $I_X([a]) \times I_X([c])$.

By Definition 4.2.2, a point p lies in $I_X([a])$ if and only if there are timelike homotopy classes $[a_1] \in \Pi^{\mathrm{t}}(X)(w,p)$, $[a_2] \in \Pi^{\mathrm{t}}(X)(p,x)$ with $[a_2][a_1] = [a]$. Analogously, q lies in $I_X([c])$ if and only if there are $[c_1] \in \Pi^{\mathrm{t}}(X)(y,q)$ and $[c_2] \in \Pi^{\mathrm{t}}(X)(q,z)$ with $[c_2][c_1] = [c]$.
Therefore, any pair $(p,q) \in I_X([a]) \times I_X([c])$ has the preimage $[d] := [c_1][b][a_2]$ in $\mathcal{U}\big([a],[b],[c]\big)$. On the other hand, Definition 6.2.1 implies that for any $[d] \in \mathcal{U}\big([a],[b],[c]\big)$ that starts in p and ends in q, there are such classes $[a_1],[a_2],[c_1],[c_2]$, so we have

$$(s,e)\Big(\mathcal{U}\big([a],[b],[c]\big)\Big) = I_X([a]) \times I_X([c]).$$

Furthermore, a_1 and a_2 are subpaths of some path in the homotopy class $[a]$ and therefore lie completely in $I_X([a]) \cup \{w,x\} \subseteq U_p$. As U_p is timelike simply connected, this means that the homotopy class $[a_2]$ is uniquely determined by p and x. Analogously, the homotopy class $[c_1]$ is uniquely determined by y and q, hence the homotopy class $[d] = [c_1][b][a_2]$ is the unique preimage of (p,q) in $\mathcal{U}([a],[b],[c])$. This shows that $(s,e)|_{\mathcal{U}([a],[b],[c])}$ is injective. □

We are now in a position to prove the main result of this chapter:

Theorem 6.2.4. *Let $(X, \ll_\bullet)$ be a locally chronological space with a regular topology and empty future and past boundary. Then, the start- and endpoint map $(s,e)\colon \Pi^{\mathrm{t}}(X) \to X \times X$ is a local homeomorphism.*

Proof. For an arbitrary $d \in P^{\mathrm{t}}(X)$, let $U_p, U_q \subseteq X$ be timelike simply connected neighborhoods of $p := s(d)$ and $q := e(d)$, respectively.

As X has empty timelike boundary, the sets $I^-_X(p)$ and $I^+_X(q)$ are nonempty, so we can choose a timelike path a' that ends in p and a timelike path c' that starts in q. For any $\varepsilon \in \big(0,\frac{1}{2}\big) > 0$, we get a commuting diagram as in Definition 6.2.1 by defining $a_1 := a'|_{[1-\varepsilon,1]}$, $a_2 := d|_{[0,\varepsilon]}$, $b := d|_{[\varepsilon,1-\varepsilon]}$, $c_1 := d|_{[1-\varepsilon,1]}$, $c_2 := c'|_{[0,\varepsilon]}$, $a := a_2\,a_1$, and $c := c_2\,c_1$.

Because of Lemma 4.2.6, we can ensure

$$\begin{aligned} & I_X([a]) \cup \{w,x\} \subseteq \overline{I_X([a])} \subseteq U_p \\ \text{and} \qquad & I_X([c]) \cup \{y,z\} \subseteq \overline{I_X([c])} \subseteq U_q \end{aligned}$$

by choosing ε small enough. With these choices, Lemma 6.2.3 implies that $\mathcal{U}\big([a],[b],[c]\big)$ is an open neighborhood of $[d]$ that is mapped homeomorphically onto an open subset of $X \times X$. □

Corollary 6.2.5. *Let $(X, \ll_\bullet)$ be a locally chronological space with a regular topology and empty future and past boundary. The set*

$$\big\{\mathcal{U}\big([a],[b],[c]\big) \,\big|\, w,x,y,z \in X,\ [a] \in \Pi^{\mathrm{t}}(X)(w,x),\ [b] \in \Pi^{\mathrm{t}}(X)(x,y),\ [c] \in \Pi^{\mathrm{t}}(X)(y,z)\,\big\}$$

is a basis of the topology of $\Pi^{\mathrm{t}}(X)$.

Proof. The statement follows directly from Lemma 6.2.3 and the fact that

$$\{\, I_X([a]) \times I_X([c]) \,\big|\, [a], [b] \in \Pi^{\mathrm{t}}(X) \,\}$$

is a basis of the topology of $X \times X$ by Corollary 4.2.4. ☐

Corollary 6.2.6. *Let $(X, \ll_\bullet)$ be a locally chronological space with a regular topology and empty future and past boundary. There is a local chronological structure $⋘_\bullet$ on $\Pi^{\mathrm{t}}(X)$ such that the start- and endpoint map $(s, e)\colon \Pi^{\mathrm{t}}(X) \to X \times X$ is a local isomorphism of locally chronological spaces.*

Proof. By Lemmas 3.4.3 and 3.4.7, the product of the locally chronological spaces $(X, \gg_\bullet)$ and $(X, \ll_\bullet)$ yields a local chronological structure on $X \times X$. As (s, e) is a local homeomorphism, we can apply Lemma 3.4.6 to finish the proof. ☐

Note that we reversed the time-direction of one of the factors, which is not necessary for the proof. However, using the results of this chapter, it is possible to prove that the set $\mathcal{U}\big([a], [b], [c]\big)$ is the chronological diamond between $[b]$ and $[c][b][a]$ with respect to $⋘_\bullet$ in certain open subsets $U \subseteq \Pi^{\mathrm{t}}(X)$. It is not clear to the author if all chronological diamonds are of the form $\mathcal{U}\big([a], [b], [c]\big)$.

Remark 6.2.7. The topology of $\Pi^{\mathrm{t}}(X)$ is in general not regular, even if X has a regular topology.

In Example 3.5.4, there are two distinct timelike homotopy classes in $\Pi^{\mathrm{t}}(X)\big((0, p), (\ell_3, q)\big)$. Let U_1 be a neighborhood of the first one and U_2 be a neighborhood of the second one.

As X is a spacetime without boundary, Theorem 6.2.4 implies that $(s, e)(U_1)$ and $(s, e)(U_2)$ are neighborhoods of $\big((0, p), (\ell_3, q)\big) \in X \times X$. For a small enough $\varepsilon > 0$ the point $\big((0, p), (\ell_3 + \varepsilon, q)\big) \in X \times X$ is contained in both of them, but since the preimage of this point only contains one element, the neighborhoods U_1 and U_2 cannot be disjoint.

We have just proven that $\Pi^{\mathrm{t}}(X)$ is not a Hausdorff space, which implies that it is not regular.

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List of symbols

Chronological and causal structure

$\leq_\bullet$	causal relation	10
$\ll_\bullet$	chronological relation, or system of transitive relations	10, 27
$J_U^+(x), J_U^-(x)$	causal future and past	10
$J_U(x,y)$	causal diamond, $J_U(x,y) = J_U^+(x) \cap J_U^-(y)$	10
$I_U^+(x), I_U^-(x)$	chronological future and past	10, 30
$I_U(x,y)$	chronological diamond, $I_U(x,y) = I_U^+(x) \cap I_U^-(y)$	10, 30
$I_U([c])$	refined chronological diamond, for $[c] \in \Pi^{\mathrm{t}}(X)$	51
$\partial^+ X, \partial^- X$	future and past timelike boundary of a locally chronological space	35
$\mathcal{U}\big([a],[b],[c]\big)$	a special open set in $\Pi^{\mathrm{t}}(X)$; these sets form a topological basis.	73

Operators

$\bigsqcup_{i\in I} U_i$	disjoint union of sets or of topological spaces U_i	60
$X_1 \coprod_{X_0} X_2$	pushout of X_1 and X_2 over X_0	57

Path spaces and homotopy classes

$P(X)$	path space	20
$\pi_1(X)$	fundamental group	21
$\Pi(X)$	fundamental groupoid	22
$P^{\mathrm{t}}(X)$	space of timelike paths	27
$\Pi^{\mathrm{t}}(X)$	fundamental semicategory	31
$P'(X)$	alternative definition of a path space	24
$P'^{\mathrm{t}}(X)$	alternative definition of a space of timelike paths	65
$\Omega(K,U)$	generating set of the compact-open topology on $P^{\mathrm{t}}(U)$	71

Categories

$\mathrm{Obj}(\mathcal{C})$	class of objects of the semicategory $\mathcal{C}$	16
$\mathcal{C}(x, y)$	class of morphisms in the semicategory $\mathcal{C}$ from x to y	16
(s, e)	start- and endpoint map of a semicategory $\mathcal{C}$, defined by $(s, e)(m) = (x, y)$ if $m \in \mathcal{C}(x, y)$.	16
$\mathrm{Op}(X)$	open sets of a topological space X with inclusions	62
Set	sets with maps	16
Top	topological spaces with continuous maps	16
Grp	groups with group homomorphisms	16
Grpd	small groupoids with functors	18
Cat	small categories with functors	18
SemCat	small semicategories with semifunctors	18
thinSemCat	thin small semicategories with semifunctors	63

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