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The two-loop massless off-shell QCD operator matrix elements to finite terms

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Abstract

We calculate the unpolarized and polarized two–loop massless off–shell operator matrix elements in QCD to $O(\varepsilon)$ in the dimensional parameter in an automated way. Here we use the method of arbitrary high Mellin moments and difference ring theory, based on integration-by-parts relations. This method also constitutes one way to compute the QCD anomalous dimensions. The presented higher order contributions to these operator matrix elements occur as building blocks in the corresponding higher order calculations up to four–loop order. All contributing quantities can be expressed in terms of harmonic sums in Mellin–N space or by harmonic polylogarithms in z–space. We also perform comparisons to the literature.

1. Introduction

The unpolarized and polarized anomalous dimensions of the local twist–2 operators in Quantum Chromodynamics (QCD) play a fundamental role in the description of the scaling violations of the deep–inelastic structure functions. Their measurement provides one of the safest ways to measure the strong coupling constant $\alpha_s(M_Z^2) = 4\pi a_s$ [1]. While at first order, there is a wide

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variety of possibilities to calculate the anomalous dimensions, see e.g. [2], at higher orders only a few efficient methods are known. These are based either on the calculation of massless off–shell operator matrix elements (OMEs) [3–20], the calculation of the forward Compton amplitude of a space–like virtual gauge boson on a massless on–shell parton [21–25] and on massive on–shell OMEs [26–41]. All these methods have advantages and disadvantages and form complimentary ways to compute the anomalous dimensions.

To perform phenomenological analyses of the deep–inelastic world data also the process dependent massless and massive Wilson coefficients have to be calculated [21,26,33,39,41–70]. This is also necessary to account for the specific scaling violations implied by massive quarks, such as charm and bottom.

In this paper we present the results for the unpolarized and polarized massless off-shell OMEs up to two–loop order and to corrections of $O(\varepsilon)$ (at $O(a_s)$ of $O(\varepsilon^2)$), with the dimensional parameter $\varepsilon = D - 4$, including non–gauge invariant contributions. A part of these expansion coefficients of the various OMEs contribute to the physical OMEs up to four–loop order. Others emerge due to the kinematic breaking of gauge invariance both in the unpolarized and polarized off–shell case. One characteristics is the emergence of additional OMEs related to the breaking of the equation of motion (eom) and of new non–gauge invariant OMEs, also with new unphysical anomalous dimensions. These quantities play a role in the calculation of the unpolarized anomalous dimensions due to mixing. We extend earlier work of Refs. [19,20] and perform the calculation in an automated way, applying methods which have been developed by us solving a series of massive three–loop problems and other applications during the last decade.

The theoretical basis for these calculations has been laid out in a series of papers describing the situation holding in the off–shell case, breaking gauge invariance, cf. [4,15,19,20,71–78]. Compared to the method of the forward Compton amplitude, the method of massless off–shell OMEs needs no precautions as reference to Higgs and gravitational subsidiary fields. The use of massive on–shell OMEs, as also the method of the forward Compton amplitude do not encounter gauge invariance problems, on the other hand. However, the massive OMEs allow to derive only the contributions $\propto T_F$ of the anomalous dimensions, except of going to one order higher in the coupling constant. A new challenge in the present approach is to master the breaking of gauge invariance to the respective perturbative order. We present the results in Mellin–N space, because the expressions are more compact than in momentum fraction z space. We also perform a detailed comparison to the literature [15,19,20,79] and correct results given there, including Feynman rules, and also all non–gauge invariant terms. A by–product of the present calculation is the calculation of the (by now well–known) unpolarized physical anomalous dimensions to two–loop order. We present all contributing expansion coefficients up to $O(a_s^2)$ to the depth being needed in future four–loop calculations, extending the level previously attempted in Refs. [19,20].

The paper is organized as follows. In Section 2 we give a brief outline of the formalism, including the renormalization, and the main steps of the calculation of the different off–shell OMEs. We then turn to the calculation of the OMEs of the so-called alien operators in Section 3, which also lead to additional anomalous dimensions. These operator matrix elements contribute via mixing to the renormalization of the unpolarized singlet operators, which we discuss in Section 4. In Sections 5–7 we present the expansion coefficients of the unpolarized and the polarized standard OMEs and those for transversity for non–negative powers in the dimensional parameter ε. In Section 8 we compare to and correct partial results given previously in Refs. [19,20]. Section 9 contains the conclusions. In Appendix A we provide Feynman rules for the alien operators, if not previously being presented in Refs. [38,80,81], and list the contributing polynomials appearing in the standard OMEs in Appendix B. Our results both in Mellin–N space and z–space, are given in ancillary files in computer–readable form. In z–space we use the decomposition given in [82], Eqs. (45,46).

2. The formalism

The massless off–shell OMEs are defined by

$$\hat{A}_{ij}^{l} = \langle j(p)|O_i^l|j(p)\rangle, \qquad \text{with} \quad i, j = q, g, \tag{1}$$

where l further labels the type of the OME and p denotes the off–shell momentum with $p^2 < 0$. The twist–2 local physical operators[1] are given by

$$O^{\mathsf{NS}}_{q,r;\mu_1\ldots\mu_N} = i^{N-1}\mathbf{S}\left[\overline{\psi}\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}\frac{\lambda_r}{2}\psi\right] - \text{ trace terms}, \tag{2}$$

$$O^{\mathsf{S}}_{q;\mu_1\ldots\mu_N} = i^{N-1}\mathbf{S}\left[\overline{\psi}\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}\psi\right] - \text{ trace terms}, \tag{3}$$

$$O^{\mathsf{S}}_{g;\mu_1\ldots\mu_N} = 2i^{N-2}\mathbf{SSp}\left[F^a_{\mu_1\alpha}D_{\mu_2}\ldots D_{\mu_{N-1}}F^{\alpha,a}_{\mu_N}\right] - \text{ trace terms}, \tag{4}$$

in the unpolarized case. In the polarized case the operators are

$$O^{\mathsf{NS}}_{q,r;\mu_1\ldots\mu_N} = i^{N-1}\mathbf{S}\left[\overline{\psi}\gamma_5\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}\frac{\lambda_r}{2}\psi\right] - \text{ trace terms}, \tag{5}$$

$$O^{\mathsf{S}}_{q;\mu_1\ldots\mu_N} = i^{N-1}\mathbf{S}\left[\overline{\psi}\gamma_5\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}\psi\right] - \text{ trace terms}, \tag{6}$$

$$O^{\mathsf{S}}_{g;\mu_1\ldots\mu_N} = 2i^{N-2}\mathbf{SSp}\left[\frac{1}{2}\varepsilon_{\mu_1\alpha\beta\gamma}F^{\beta\gamma,a}D_{\mu_2}\ldots D_{\mu_{N-1}}F^{\alpha,a}_{\mu_N}\right] - \text{ trace terms}. \tag{7}$$

For transversity [83] the following local non–singlet operator contributes

$$O^{\mathsf{NS}}_{q,r;\mu\mu_1\ldots\mu_N} = i^{N-1}\mathbf{S}\left[\overline{\psi}\sigma_{\mu\mu_1}D_{\mu_2}\ldots D_{\mu_N}\frac{\lambda_r}{2}\psi\right] - \text{ trace terms}, \tag{8}$$

where $\sigma_{\mu\nu} = (i/2)[\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu]$. Here Δ denotes a light–like vector, $\Delta.\Delta = 0$. ψ denotes the quark field, λ_r, $r \in [1, N_F^2 - 1]$ the $SU(N_F)$ matrices, γ_μ a Dirac matrix, D_μ the covariant derivative and A_ν^a the gluon fields, with $F_{\mu\nu}^a$ the gluonic field strength tensor and $\varepsilon_{\alpha\beta\gamma\delta}$ the Levi–Civita pseudo–tensor. For further notations we refer to Ref. [82].

Furthermore, the relative normalization between the quark–singlet and gluon external states has to be fixed. We adopt the same convention as in the massive on–shell case [80] by demanding 4–momentum conservation

$$\int_0^1 dx x\,[\Sigma(x) + G(x)] = 1, \tag{9}$$

[1] The unphysical local operators are defined in Section 3 below.

where Σ is the quark singlet distribution

$$\Sigma(x) = \sum_{k=1}^{N_F} \left[q_k(x) + \bar{q}_k(x)\right], \tag{10}$$

with q the quark, $\bar{q}$ the antiquark distributions and G is the gluon distribution. In this way the operators are also fixed in the polarized case, replacing γ_μ by $\gamma_\mu\gamma_5$, etc., cf. [81].

We apply the following projectors to determine the different contributions to the OMEs. For external quark fields one uses in the unpolarized case

$$\hat{A}_{iq} = \left[\not{\Delta}\hat{A}_{iq}^{\rm phys} + \not{p}\frac{\Delta.p}{p^2}\hat{A}_{iq}^{\rm eom}\right](\Delta.p)^{N-1}, \quad i = q, g. \tag{11}$$

The quarkonic projections are obtained by

$$\hat{A}_{iq}^{\rm phys} = \frac{1}{4(\Delta.p)^N} \,{\rm tr}\left[\left(\not{p} - \frac{p^2}{\Delta.p}\not{\Delta}\right)\hat{A}_{iq}\right] \tag{12}$$

$$\hat{A}_{iq}^{\rm eom} = \frac{1}{(4\Delta.p)^N} \,{\rm tr}\left[\not{\Delta}\hat{A}_{iq}\right]. \tag{13}$$

For external gluons the decomposition is as follows [19]

$$\hat{A}_{ig,\mu\nu} = \hat{A}_{ig}^{\rm phys} T_{\mu\nu}^{(1)} + \hat{A}_{ig}^{\rm eom} T_{\mu\nu}^{(2)} + \hat{A}_{ig}^{\rm ngi} T_{\mu\nu}^{(3)}, \tag{14}$$

where

$$T_{\mu\nu}^{(1)} = \frac{1}{2}[1 + (-1)^N]\left[g_{\mu\nu} - \frac{p_\mu\Delta_\nu + \Delta_\mu p_\nu}{\Delta.p} + \frac{\Delta_\mu\Delta_\nu p^2}{(\Delta.p)^2}\right](\Delta.p)^N, \tag{15}$$

$$T_{\mu\nu}^{(2)} = \frac{1}{2}[1 + (-1)^N]\left[\frac{p_\mu p_\nu}{p^2} - \frac{p_\mu\Delta_\nu + \Delta_\mu p_\nu}{\Delta.p} + \frac{\Delta_\mu\Delta_\nu p^2}{(\Delta.p)^2}\right](\Delta.p)^N, \tag{16}$$

$$T_{\mu\nu}^{(3)} = \frac{1}{2}[1 + (-1)^N]\left[-\frac{p_\mu\Delta_\nu + \Delta_\mu p_\nu}{2\Delta.p} + \frac{\Delta_\mu\Delta_\nu p^2}{(\Delta.p)^2}\right](\Delta.p)^N. \tag{17}$$

Later one more tensor structure is needed

$$T_{\mu\nu}^{(4)} = \frac{1}{2}[1 + (-1)^N]\left[\frac{p_\mu\Delta_\nu + \Delta_\mu p_\nu}{2\Delta.p}\right](\Delta.p)^N, \tag{18}$$

with [19]

$$\begin{array}{rclcrcl} p^\mu T_{\mu\nu}^{(i)} &=& 0, & (i = 1, 2), & p^\mu T_{\mu\nu}^{(i)} &\neq& 0, \quad (i = 3, 4) \\ p^\mu p^\nu T_{\mu\nu}^{(i)} &=& 0, & (i = 1, 2, 3), & p^\mu p^\nu T_{\mu\nu}^{(4)} &\neq& 0. \end{array} \tag{19}$$

This tensor decomposition implies the choice of a physical gauge for the external gluon lines, with a propagator

$$D^{\mu\nu}(k^2) = i\frac{d^{\mu\nu}(k)}{k^2 + i0}, \quad d_{\mu\nu}(k) = -g^{\mu\nu} - n^2\frac{k^\mu k^\nu}{(k.n)^2} + \frac{n^\mu k^\nu + n^\nu k^\mu}{k.n}, \tag{20}$$

and $n^2 \le 0$.

The resulting Ward identities are

$$p^\mu \hat{A}_{qg,\mu\nu} = \frac{1}{2}[1+(-1)^N]\left[-p_\nu + \frac{\Delta_\nu p^2}{\Delta.p}\right](\Delta.p)^N \hat{A}_{qg}^{\rm ngi}, \tag{21}$$

$$p^\mu p^\nu \hat{A}_{qg,\mu\nu} = 0. \tag{22}$$

The gluonic projections are given by

$$\hat{A}_{ig}^{\rm phys} = \frac{1}{D-2}\left[g_{\mu\nu} + \frac{p^2}{(\Delta.p)^2}\Delta_\mu\Delta_\nu - \frac{p_\mu\Delta_\nu + p_\nu\Delta_\mu}{\Delta.p}\right]\hat{A}_{ig}^{\mu\nu}, \tag{23}$$

$$\hat{A}_{ig}^{\rm eom} = \frac{p^2}{(\Delta.p)^2}\Delta_\mu\Delta_\nu\hat{A}_{ig}^{\mu\nu}, \tag{24}$$

$$\hat{A}_{ig}^{\rm ngi} = \left[\frac{p_\mu p_\nu}{4p^2} - \frac{p_\mu\Delta_\nu + p_\nu\Delta_\mu}{2\Delta.p}\right]\hat{A}_{ig}^{\mu\nu}, \tag{25}$$

$$\hat{A}_{ig}^{\rm wi} = \frac{p_\mu p_\nu}{4p^2}\hat{A}_{ig}^{\mu\nu}. \tag{26}$$

In the polarized case the OMEs $\Delta\hat{A}_{iq}^{1}$, $i = q, g$ are given by

$$\Delta\hat{A}_{iq} = \left[\gamma_5 \not\Delta \Delta\hat{A}_{iq}^{\rm phys} + \gamma_5 \not p \frac{\Delta.p}{p^2}\Delta\hat{A}_{iq}^{\rm eom}\right](\Delta.p)^{N-1}, \tag{27}$$

and the OMEs with external gluon lines read

$$\Delta\hat{A}_{ig,\mu\nu} = \varepsilon_{\mu\nu\alpha\beta}\frac{\Delta^\alpha p^\beta}{\Delta.p}\Delta\hat{A}_{ig}^{\rm phys}, \quad i = q, g. \tag{28}$$

To treat γ_5 in $D = 4+\varepsilon$ dimensions we use the Larin scheme [20,86] and perform the replacement

$$\not p \gamma_5 = \frac{i}{6}\epsilon_{\mu\nu\rho\sigma} p^\mu \gamma^\nu \gamma^\rho \gamma^\sigma, \tag{29}$$

while contracting the occurring ϵ–tensors in D dimensions using

$$\epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\lambda\tau\gamma} = -{\rm Det}\left[g_\omega^\beta\right], \qquad \beta = \alpha, \lambda, \tau, \gamma; \quad \omega = \mu, \nu, \rho, \sigma\ . \tag{30}$$

In this scheme the projections onto the different terms are given by

$$\begin{aligned}\Delta\hat{A}_{iq}^{\rm phys} = &-\frac{1}{4(D-2)(D-3)}\epsilon_{\mu\nu\rho\sigma}p^\rho\Delta^\sigma {\rm tr}\left[\not p \gamma^\mu\gamma^\nu\Delta\hat{A}_{iq}\right](\Delta.p)^{-N-1} \\ &-\frac{p^2}{4(D-2)(D-3)}\epsilon_{\mu\nu\rho\sigma}p^\rho\Delta^\sigma {\rm tr}\left[\not\Delta \gamma^\mu\gamma^\nu\Delta\hat{A}_{iq}\right](\Delta.p)^{-N-2},\end{aligned} \tag{31}$$

$$\Delta\hat{A}_{iq}^{\rm eom} = \frac{p^2}{4(D-2)(D-3)}\epsilon_{\mu\nu\rho\sigma}p^\rho\Delta^\sigma {\rm tr}\left[\not\Delta \gamma^\mu\gamma^\nu\Delta\hat{A}_{iq}\right](\Delta.p)^{-N-2}, \tag{32}$$

$$\Delta\hat{A}_{ig}^{\rm phys} = \frac{1}{(D-2)(D-3)}\epsilon_{\mu\nu\rho\sigma}\Delta^\rho p^\sigma(\Delta.p)^{-N-1}\Delta\hat{A}_{ig}^{\mu\nu}. \tag{33}$$

There is no mixing between the physical and the ngi and the alien operators, because no symmetric rank two tensor contributes.

For transversity the tensor decomposition for the OME reads

$$\hat{A}_{qq,\mu}^{\rm NS,tr} = \Delta^\rho \sigma_{\mu\rho} \tilde{A}_{qq}^{\rm NS,tr} + c_1 \Delta_\mu + c_2 p_\mu + c_3 \gamma_\mu \not{p} + c_4 \Delta_\mu \not{\Delta} \not{p} + c_5 p_\mu \not{\Delta} \not{p}. \tag{34}$$

The physical OME can be extracted using the following projector

$$\tilde{A}_{qq}^{\rm NS,tr} = \frac{i}{4(D-2)} {\rm tr}\left[\left(-p^\mu \not{\Delta} \not{p} + \Delta.p \gamma^\mu \not{p} + i p^2 \Delta_\rho \sigma^{\mu\rho}\right) \hat{A}_{qq,\mu}^{\rm NS,tr}\right] (\Delta.p)^{-N-2}. \tag{35}$$

In the following we will insert the wave function renormalization [84] and perform a partial renormalization of the coupling constant [85] and the gauge parameter [84], cf. [82] for details. We will then denote the partly renormalized OMEs by $(\Delta)\tilde{A}_{ij}$.

The following representations hold for $\tilde{A}_{ij}$ and $(\Delta)\tilde{A}_{ij}$. The different partly renormalized OMEs are given by

$$\tilde{A}_{ij}^{\rm phys} = \delta_{ij} + \sum_{k=1}^{\infty} S_\varepsilon^k \left(\frac{-p^2}{\mu^2}\right)^{k\varepsilon/2} a_s^k \tilde{A}_{ij}^{\rm phys,k} \tag{36}$$

$$\tilde{A}_{ij}^{\rm eom} = \sum_{k=1}^{\infty} S_\varepsilon^k \left(\frac{-p^2}{\mu^2}\right)^{k\varepsilon/2} a_s^k \tilde{A}_{ij}^{\rm eom,k} \tag{37}$$

$$\tilde{A}_{ij}^{\rm ngi} = \sum_{k=1}^{\infty} S_\varepsilon^k \left(\frac{-p^2}{\mu^2}\right)^{k\varepsilon/2} a_s^k \tilde{A}_{ij}^{\rm ngi,k}, \tag{38}$$

with μ the factorization scale and the spherical factor S_ε is given by

$$S_\varepsilon = \exp\left[\frac{\varepsilon}{2}(\gamma_E - \ln(4\pi))\right], \tag{39}$$

where γ_E is the Euler-Mascheroni number. Analogous relations hold in the polarized case and the OMEs, cf. [82,83], and those of transversity, which are flavor non–singlet quantities. The structure of the partial amplitudes up to two–loop order is

$$\tilde{A}_{ij}^{\rm phys,1} = \frac{1}{\varepsilon}\gamma_{ij}^{(0)} + a_{ij}^{(1,0)} + \varepsilon a_{ij}^{(1,1)} + \varepsilon^2 a_{ij}^{(1,2)} + O(\varepsilon^3), \quad ij = qq \text{ NS}, qg, gq, gg, \tag{40}$$

$$\tilde{A}_{ij}^{\rm eom,1} = b_{ij}^{(1,0)} b + \varepsilon b_{ij}^{(1,1)} + \varepsilon^2 b_{ij}^{(1,2)} + O(\varepsilon^3) \quad ij = qq \text{ NS}, qg, gq, gg, \tag{41}$$

$$\tilde{A}_{ij}^{\rm ngi,1} = \frac{1}{\varepsilon}\gamma_{gA}^{(0)} + c_{gA}^{(1,0)} + \varepsilon c_{gA}^{(1,1)} + \varepsilon^2 c_{gA}^{(1,2)} + O(\varepsilon^3), \quad ij = gg, \tag{42}$$

$$\tilde{A}_{ij}^{\rm phys,2} = \frac{1}{\varepsilon^2} a_{ij}^{(2,-2)} + \frac{1}{\varepsilon} a_{ij}^{(2,-1)} + a_{ij}^{(2,0)} + \varepsilon a_{ij}^{(2,1)} + O(\varepsilon^2), \\ ij = qq \text{ NS}, qq \text{ PS}, qg, gq, gg, \tag{43}$$

$$\tilde{A}_{ij}^{\rm eom,2} = \frac{1}{\varepsilon} b_{ij}^{(2,-1)} + b_{ij}^{(2,0)} + \varepsilon b_{ij}^{(2,1)} + O(\varepsilon^2), \quad ij = qq \text{ NS}, qq \text{ PS}, qg, gq, gg, \tag{44}$$

$$\tilde{A}_{ij}^{\rm ngi,2} = \frac{1}{\varepsilon^2} c_{ij}^{(2,-2)} + \frac{1}{\varepsilon} c_{ij}^{(2,-1)} + c_{ij}^{(2,0)} + \varepsilon c_{ij}^{(2,1)} + O(\varepsilon^2), \quad ij = qg, gg. \tag{45}$$

Here the contributions $e^{(k,-1(-2))}$ with $e = a, b, c$ contain the LO and NLO anomalous dimensions, cf. [19,20,81,82]. Structures which lead to new anomalous dimensions are dealt with in Section 3. In the polarized case there are no ngi contributions, but the eom terms for $ij = qq$ NS, PS and gq to two–loop order. All results in the polarized case are presented in the Larin scheme [20,86].

There are also eom contributions in the transversity cases, which we will not deal with in the present paper, since the tensor composition in this case is even richer, cf. [83], but the anomalous dimensions come only from the physical part, cf. [82]. Note that our definitions differ in part from those in [19,20]. The corresponding mapping is obtained by performing the renormalization, which is carried out to 2–loop order in the present paper. The anomalous dimensions obey the following expansion in the strong coupling constant

$$\gamma_{ij}^{\rm a} = \sum_{k=1}^{\infty} a_s^k \gamma_{ij}^{(k-1),\rm a}. \tag{46}$$

Let us now turn to technical aspects of the calculation. The Feynman diagrams for the different operator insertions are generated by `QGRAF` [80,87]. The spinor and Lorentz–algebra is performed using `FORM` [88]. The different operator insertions are resummed using generating functions [89] either for even or odd integer moments, cf. [82], implied by the respective crossing relations [90,91]. In this way the local operators reappear in terms of propagators and one may derive the integration–by–parts relations [92] for the corresponding quantities, for which we use the package `Crusher` [93]. For part of the calculation we performed the reduction using `Litered` [94] and solved the differential equations by using the method of Refs. [95,96]. The general solution followed the route described in Refs. [81,82]. The relations between master integrals obtained by the reduction using `Crusher` allow to generated a sufficiently large number of Mellin moments by using the method of arbitrary high moments [97] for the different color and zeta factors, which formed the basis for the method of guessing [98,99], implemented in `Sage` [100,101], to determine the corresponding recurrences. Those were solved by applying difference-ring theory [102–111] as implemented in the package `Sigma` [112,113]. The generated recurrences in the present case factorize all at first order, unlike the case in a series of massive higher order calculations. First one obtains solutions in terms of also cyclotomic harmonic sums [114] because of separating even and odd moments. All the results can be written in terms of harmonic sums [115,116]

$$S_{b,\vec{a}}(N) = \sum_{k=1}^{N} \frac{(\mathrm{sign}(b))^k}{k^{|b|}} S_{\vec{a}}(k), \quad S_{\emptyset} = 1, \quad a, b_i \in \mathbb{Z}\backslash\{0\}, \tag{47}$$

for which one maps first

$$S_{b,\vec{a}}(2N+1) = \frac{1}{2N+1} S_{\vec{a}}(2N+1) + S_{b,\vec{a}}(2N) \tag{48}$$

recursively and then applies the `HarmonicSums` [114–122] command `Synchronize` and finally the command `TransformToBasis[`*expr*`, Online → True]` is applied. All results are now obtained in terms of harmonic sums of argument N only.

In the following section we turn now to the calculation of the matrix elements of the so–called alien operators, which contribute via mixing to the (ultraviolet) operator renormalization in the unpolarized case and discuss in a subsequent section the mixing relations to two–loop order in explicit form.

3. The OMEs of the alien operators

These operator matrix elements contribute in the unpolarized case and have been discussed in detail in Refs. [15,19,79]. The OMEs containing gluonic operators mix with these OMEs. The operators are given by

$$O_A^{\mu_1,\ldots,\mu_N} = i^{N-2}\mathbf{SSp}\Bigg[F_\alpha^{a,\mu_1} D_\alpha \partial^{\mu_2}\ldots\partial^{\mu_{N-1}} A_a^{\mu_N}$$
$$+igf^{abc} F_\alpha^{a,\mu_1} \sum_{i=2}^{N-1} \kappa_i \partial^\alpha \left\{ \left(\partial^{\mu_2}\ldots\partial^{\mu_{i-2}} A_b^{\mu_{i-1}}\right)\left(\partial^{\mu_2}\ldots\partial^{\mu_{N-1}} A_c^{\mu_N}\right)\right\}$$
$$+O(g^3)\Bigg] - \text{ trace terms },\tag{49}$$

$$O_\omega^{\mu_1,\ldots,\mu_N} = i^{N-2}\mathbf{SSp}\Bigg[\xi^a \partial^{\mu_1}\ldots\partial^{\mu_{N-1}}\overline{\omega}^a$$
$$-igf^{abc}\xi^a \sum_{i=2}^{N-1} \eta_i \partial^{\mu_1} \left\{ \left(\partial^{\mu_2}\ldots\partial^{\mu_{i-2}}\overline{\omega}_b\right)\left(\partial^{\mu_2}\ldots\partial^{\mu_{N-1}} A_c^{\mu_N}\right)\right\}$$
$$+O(g^3)\Bigg] - \text{ trace terms },\tag{50}$$

$$O_B^{\mu_1,\ldots,\mu_N} = i^{N-1}\mathbf{S}\left[g\overline{\psi}_k \gamma_{\mu_1} (T_a)^{kl} A^{a,\mu_2}\partial^{\mu_3}\ldots\partial_{\mu_N}\psi_l + O(g^3)\right] - \text{ trace terms},\tag{51}$$

with

$$\kappa_i = \frac{(-1)^i}{8} + \frac{3}{8}\left[\frac{(n-2)!}{(i-1)!(n-i-1)!} - \frac{(n-2)!}{i!(N-i-2)!}\right],\tag{52}$$
$$\eta_i = \frac{(-1)^i}{4} + \frac{1}{4}\left[3\frac{(n-2)!}{(i-1)!(n-i-1)!} + \frac{(n-2)!}{i!(N-i-2)!}\right],\tag{53}$$

and ξ, $\overline{\omega}$ denote the ghost and antighost respectively. The Feynman rules for these operator insertions are summarized in Appendix A.

To one loop order the OMEs are given by, after performing partial renormalization,

$$\tilde{A}_{\rm Aq}^{\rm phys} = a_s S_\varepsilon \left(\frac{-p^2}{\mu^2}\right)^{\varepsilon/2}\left[\frac{1}{\varepsilon}\gamma_{\rm Aq}^{(0)} + a_{\rm Aq}^{(1,0)} + \varepsilon a_{\rm Aq}^{(1,1)} + \varepsilon^2 a_{\rm Aq}^{(1,2)}\right],\tag{54}$$

$$\tilde{A}_{\rm Aq}^{\rm eom} = a_s S_\varepsilon \left(\frac{-p^2}{\mu^2}\right)^{\varepsilon/2}\left[b_{\rm Aq}^{(1,0)} + \varepsilon b_{\rm Aq}^{(1,1)} + \varepsilon^2 b_{\rm Aq}^{(1,2)}\right],\tag{55}$$

$$\tilde{A}_{\rm Bq}^{\rm phys} = a_s S_\varepsilon \left(\frac{-p^2}{\mu^2}\right)^{\varepsilon/2}\left[-\frac{1}{\varepsilon}\gamma_{\rm Aq}^{(0)} + a_{\rm Bq}^{(1,0)} + \varepsilon a_{\rm Bq}^{(1,1)} + \varepsilon^2 a_{\rm Bq}^{(1,2)}\right],\tag{56}$$

$$\tilde{A}_{\rm Bq}^{\rm eom} = a_s S_\varepsilon \left(\frac{-p^2}{\mu^2}\right)^{\varepsilon/2}\left[b_{\rm Bq}^{(1,0)} + \varepsilon b_{\rm Bq}^{(1,1)} + \varepsilon^2 b_{\rm Bq}^{(1,2)}\right],\tag{57}$$

$$\tilde{A}_{\rm Ag}^{\rm phys} = a_s S_\varepsilon \left(\frac{-p^2}{\mu^2}\right)^{\varepsilon/2}\left[\frac{1}{\varepsilon}\gamma_{\rm Ag}^{(0)} + a_{\rm Ag}^{(1,0)} + \varepsilon a_{\rm Ag}^{(1,1)} + \varepsilon^2 a_{\rm Ag}^{(1,2)}\right],\tag{58}$$

$$\tilde{A}_{\rm Ag}^{\rm eom} = a_s S_\varepsilon \left(\frac{-p^2}{\mu^2}\right)^{\varepsilon/2}\left[b_{\rm Ag}^{(1,0)} + \varepsilon b_{\rm Ag}^{(1,1)} + \varepsilon^2 b_{\rm Ag}^{(1,2)}\right],\tag{59}$$

$$\tilde{A}_{\rm Ag}^{\rm ngi} = 1 + a_s S_\varepsilon \left(\frac{-p^2}{\mu^2}\right)^{\varepsilon/2} \left[\frac{1}{\varepsilon}\gamma_{\rm AA}^{(0)} c_{\rm Ag}^{(1,0)} + \varepsilon c_{\rm Ag}^{(1,1)} + \varepsilon^2 c_{\rm Ag}^{(1,2)}\right], \tag{60}$$

$$\tilde{A}_{\rm Ag}^{\rm wi} = a_s S_\varepsilon \left(\frac{-p^2}{\mu^2}\right)^{\varepsilon/2} \left[d_{\rm Ag}^{(1,0)} + \varepsilon d_{\rm Ag}^{(1,1)} + \varepsilon^2 d_{\rm Ag}^{(1,2)}\right], \tag{61}$$

$$\tilde{A}_{\omega\rm g}^{\rm phys} = a_s S_\varepsilon \left(\frac{-p^2}{\mu^2}\right)^{\varepsilon/2} \left[-\frac{1}{\varepsilon}\gamma_{\rm Ag}^{(0)} + a_{\omega\rm g}^{(1,0)} + \varepsilon a_{\omega\rm g}^{(1,1)} + \varepsilon^2 a_{\omega\rm g}^{(1,2)}\right], \tag{62}$$

$$\tilde{A}_{\omega\rm g}^{\rm eom} = a_s S_\varepsilon \left(\frac{-p^2}{\mu^2}\right)^{\varepsilon/2} \left[+b_{\omega\rm g}^{(1,0)} + \varepsilon b_{\omega\rm g}^{(1,1)} + \varepsilon^2 b_{\omega\rm g}^{(1,2)}\right], \tag{63}$$

$$\tilde{A}_{\omega\rm g}^{\rm ngi} = a_s S_\varepsilon \left(\frac{-p^2}{\mu^2}\right)^{\varepsilon/2} \left[\frac{1}{\varepsilon}\gamma_{\omega\rm A}^{(0)} + c_{\omega\rm A}^{(1,0)} + \varepsilon c_{\omega\rm A}^{(1,1)} + \varepsilon^2 c_{\omega\rm A}^{(1,2)}\right], \tag{64}$$

$$\tilde{A}_{\omega\rm g}^{\rm wi} = a_s S_\varepsilon \left(\frac{-p^2}{\mu^2}\right)^{\varepsilon/2} \left[d_{\rm Ag}^{(1,0)} + \varepsilon d_{\rm Ag}^{(1,1)} + \varepsilon^2 d_{\rm Ag}^{(1,2)}\right], \tag{65}$$

$$\tilde{A}_{\rm qg}^{\rm ngi} = a_s^2 S_\varepsilon^2 \left(\frac{-p^2}{\mu^2}\right)^{\varepsilon} \left[\frac{1}{\varepsilon^2}\gamma_{\rm qg}^{(0)}\gamma_{\rm gA}^{(0)} + \frac{1}{\varepsilon}\gamma_{\rm qg}^{(0)} a_{\rm gA}^{(1,0)} + c_{qg}^{(2,0)} + \varepsilon c_{qg}^{(2,1)}\right], \tag{66}$$

$$\tilde{A}_{\rm gg}^{\rm ngi} = a_s S_\varepsilon \left(\frac{-p^2}{\mu^2}\right)^{\varepsilon/2} \left[\frac{1}{\varepsilon}\gamma_{\rm gA}^{(0)} + c_{gg}^{(1,0)} + \varepsilon c_{gg}^{(1,1)} + \varepsilon^2 c_{gg}^{(1,2)}\right]. \tag{67}$$

Here we listed also those contributions to the OMEs given in Section 2 for which new anomalous dimensions contribute to two–loop order. The latter read

$$\gamma_{Aq}^{(0)} = -\frac{8C_F}{(N-1)N}, \tag{68}$$

$$\gamma_{AA}^{(0)} = -C_A\left[\frac{16 + 46N + N^2 - 12N^3 - 3N^4}{2(N-1)N(1+N)(2+N)} + 2\xi + 6S_1\right], \tag{69}$$

$$\gamma_{Ag}^{(0)} = -\frac{2C_A}{(1+N)(2+N)}, \tag{70}$$

$$\gamma_{\omega A}^{(0)} = -\frac{C_A(N-1)(4+N)(6+N)}{6N(1+N)(2+N)}, \tag{71}$$

$$\gamma_{gA}^{(0)} = \frac{4C_A}{(N-1)N}. \tag{72}$$

The higher order expansion terms are given by

$$a_{Ag}^{(1,0)} = C_A\Biggl(\xi^2\frac{1}{2N} + \xi\frac{(1-2N)}{(N-1)N} - \frac{Q_9}{(N-1)N(1+N)^2(2+N)^2} + \frac{S_1}{(1+N)(2+N)}\Biggr), \tag{73}$$

$$a_{Ag}^{(1,1)} = C_A\Biggl(-\xi^2\left[\frac{N+1}{4N^2} + \frac{S_1}{4N}\right] + \xi\left[\frac{1-N+N^3}{2(N-1)^2N^2} + \frac{(-1+2N)S_1}{2(N-1)N}\right]$$

$$
\begin{aligned}
&+\frac{S_1 Q_9}{2(N-1)N(1+N)^2(2+N)^2}-\frac{Q_{17}}{(N-1)^2N^2(1+N)^3(2+N)^3}\\
&-\frac{S_1^2}{4(1+N)(2+N)}\\
&-\frac{3S_2}{4(1+N)(2+N)}+\frac{\zeta_2}{4(1+N)(2+N)}\Biggr),
\end{aligned} \tag{74}
$$

$$
\begin{aligned}
a_{Ag}^{(1,2)} = C_A\Biggl(&\xi^2\Biggl[\frac{1+N}{8N^3}+\frac{(1+N)S_1}{8N^2}+\frac{S_1^2}{16N}+\frac{3S_2}{16N}-\frac{\zeta_2}{16N}\Biggr]\\
&+\xi\Biggl[\frac{1-2N+N^2-N^3}{4(N-1)^3N^3}+\frac{\big(-1+N-N^3\big)S_1}{4(N-1)^2N^2}+\frac{(1-2N)S_1^2}{8(N-1)N}\\
&-\frac{3(-1+2N)S_2}{8(N-1)N}+\frac{(-1+2N)\zeta_2}{8(N-1)N}\Biggr]\\
&-\frac{S_1^2Q_9}{8(N-1)N(1+N)^2(2+N)^2}-\frac{3S_2Q_9}{8(N-1)N(1+N)^2(2+N)^2}\\
&+\frac{Q_{20}}{2(N-1)^3N^3(1+N)^4(2+N)^4}+\Biggl(\frac{Q_{17}}{2(N-1)^2N^2(1+N)^3(2+N)^3}\\
&+\frac{3S_2}{8(1+N)(2+N)}\Biggr)S_1+\frac{S_1^3}{24(1+N)(2+N)}+\frac{7S_3}{12(1+N)(2+N)}\\
&+\Biggl(\frac{Q_9}{8(N-1)N(1+N)^2(2+N)^2}-\frac{S_1}{8(1+N)(2+N)}\Biggr)\zeta_2\\
&-\frac{7\zeta_3}{12(1+N)(2+N)}\Biggr),
\end{aligned} \tag{75}
$$

$$
b_{Ag}^{(1,0)} = C_A\Biggl(\xi\Biggl[\frac{-1+9N-6N^2}{2(N-1)N}+\frac{3}{2}S_1\Biggr]-\frac{2\big(4+7N+2N^2\big)}{N(1+N)(2+N)}+\frac{(-2+N)\xi^2}{4N}\Biggr), \tag{76}
$$

$$
\begin{aligned}
b_{Ag}^{(1,1)} = C_A\Biggl(&\xi^2\Biggl[\frac{-2+2N+6N^2-5N^3}{8(N-1)N^2}+\frac{(1+N)S_1}{4N}\Biggr]+\xi\Biggl[\frac{Q_7}{4(N-1)^2N^2}\\
&+\frac{(1-3N)S_1}{4(N-1)N}-\frac{3}{8}S_1^2-\frac{9}{8}S_2\Biggr]-\frac{Q_3}{N^2(1+N)^2(2+N)^2}\\
&+\frac{\big(4+7N+2N^2\big)S_1}{N(1+N)(2+N)}\Biggr),
\end{aligned} \tag{77}
$$

$$
\begin{aligned}
b_{Ag}^{(1,2)} = C_A\Biggl(&\xi^2\Biggl[\frac{Q_{10}}{16(N-1)^2N^3}+\frac{\big(2-2N-N^3\big)S_1}{16(N-1)N^2}+\frac{(-1-N)S_1^2}{16N}-\frac{3(1+N)S_2}{16N}\\
&+\frac{(2-N)\zeta_2}{32N}\Biggr]+\xi\Biggl[\frac{Q_{11}}{8(N-1)^3N^3}+\Biggl(\frac{1+8N-13N^2+6N^3}{8(N-1)^2N^2}+\frac{9}{16}S_2\Biggr)S_1
\end{aligned}
$$

$$
+\frac{(-1+3N)S_1^2}{16(N-1)N}+\frac{1}{16}S_1^3+\frac{3(-1+3N)S_2}{16(N-1)N}+\frac{7}{8}S_3
+\left(\frac{1-9N+6N^2}{16(N-1)N}-\frac{3}{16}S_1\right)\zeta_2\Bigg]
+\frac{S_1 Q_3}{2N^2(1+N)^2(2+N)^2}+\frac{Q_{13}}{N^3(1+N)^3(2+N)^3}+\frac{\big(-4-7N-2N^2\big)S_1^2}{4N(1+N)(2+N)}
-\frac{3\big(4+7N+2N^2\big)S_2}{4N(1+N)(2+N)}+\frac{\big(4+7N+2N^2\big)\zeta_2}{4N(1+N)(2+N)}\Bigg), \tag{78}
$$

$$
c_{Ag}^{(1,0)}=C_A\Bigg(\frac{\xi^2}{2}+\xi\left[\frac{-2+2N-N^2}{2(N-1)N}-S_1\right]+\frac{Q_{16}}{12(N-1)^2N^2(1+N)^2(2+N)^2}
+\frac{\big(8+20N-N^2-3N^3\big)S_1}{2(N-1)N(1+N)(2+N)}+\frac{3}{2}S_1^2+\frac{9}{2}S_2\Bigg), \tag{79}
$$

$$
c_{Ag}^{(1,1)}=C_A\Bigg(-\frac{\xi^2}{2}+\xi\Bigg[\frac{-2+8N-7N^2+2N^3}{4(N-1)^2N^2}+\frac{\big(2-2N+N^2\big)S_1}{4(N-1)N}
+\frac{1}{4}S_1^2+\frac{3}{4}S_2+\frac{1}{4}\zeta_2\Bigg]
+\frac{Q_{21}}{72(N-1)^3N^3(1+N)^3(2+N)^3}
+\left(\frac{Q_{15}}{8(N-1)^2N^2(1+N)^2(2+N)^2}-\frac{9}{4}S_2\right)S_1
+\frac{\big(-8-20N+N^2+3N^3\big)S_1^2}{8(N-1)N(1+N)(2+N)}-\frac{1}{4}S_1^3
+\frac{3\big(-8-20N+N^2+3N^3\big)S_2}{8(N-1)N(1+N)(2+N)}-\frac{7}{2}S_3
+\left(\frac{Q_1}{16(N-1)N(1+N)(2+N)}+\frac{3}{4}S_1\right)\zeta_2\Bigg), \tag{80}
$$

$$
c_{Ag}^{(1,2)}=C_A\Bigg(\xi^2\left[\frac{1}{2}-\frac{\zeta_2}{16}\right]+\xi\Bigg[\frac{Q_2}{8(N-1)^3N^3}
+\left(\frac{2-8N+7N^2-2N^3}{8(N-1)^2N^2}-\frac{3}{8}S_2\right)S_1-\frac{7}{12}S_3
+\frac{\big(-2+2N-N^2\big)S_1^2}{16(N-1)N}-\frac{3\big(2-2N+N^2\big)S_2}{16(N-1)N}
+\left(\frac{2-2N+N^2}{16(N-1)N}+\frac{S_1}{8}\right)\zeta_2-\frac{1}{24}S_1^3
$$

$$
\begin{aligned}
&-\frac{7}{12}\zeta_3\Biggr] - \frac{3S_2Q_{15}}{32(N-1)^2N^2(1+N)^2(2+N)^2}\\
&+\frac{Q_{23}}{432(N-1)^4N^4(1+N)^4(2+N)^4}\\
&+\Biggl(\frac{Q_{19}}{16(N-1)^3N^3(1+N)^3(2+N)^3}\\
&-\frac{3\bigl(-8-20N+N^2+3N^3\bigr)S_2}{16(N-1)N(1+N)(2+N)}+\frac{7}{4}S_3\Biggr)S_1\\
&+\Biggl(-\frac{Q_{15}}{32(N-1)^2N^2(1+N)^2(2+N)^2}+\frac{9}{16}S_2\Biggr)S_1^2\\
&+\frac{\bigl(8+20N-N^2-3N^3\bigr)S_1^3}{48(N-1)N(1+N)(2+N)}\\
&+\frac{1}{32}S_1^4+\frac{27}{32}S_2^2-\frac{7\bigl(-8-20N+N^2+3N^3\bigr)S_3}{24(N-1)N(1+N)(2+N)}+\frac{45}{16}S_4\\
&+\Biggl(-\frac{Q_{16}}{96(N-1)^2N^2(1+N)^2(2+N)^2}+\frac{\bigl(-8-20N+N^2+3N^3\bigr)S_1}{16(N-1)N(1+N)(2+N)}\\
&-\frac{3}{16}S_1^2-\frac{9}{16}S_2\Biggr)\zeta_2+\Biggl(-\frac{7Q_1}{48(N-1)N(1+N)(2+N)}-\frac{7}{4}S_1\Biggr)\zeta_3\Biggr),
\end{aligned} \tag{81}
$$

$$
d_{Ag}^{(1,0)} = -\frac{C_A}{4N}, \tag{82}
$$

$$
d_{Ag}^{(1,1)} = C_A\Biggl(\frac{1}{8N^2}+\frac{S_1}{8N}\Biggr), \tag{83}
$$

$$
d_{Ag}^{(1,2)} = C_A\Biggl(-\frac{1}{16N^3}-\frac{S_1}{16N^2}-\frac{S_1^2}{32N}-\frac{3S_2}{32N}+\frac{\zeta_2}{32N}\Biggr), \tag{84}
$$

$$
a_{Aq}^{(1,0)} = C_F\Biggl(-\frac{2(-2+N)(-1+3N)}{(N-1)^2N^2}+\frac{(-4+N)\xi}{2(N-1)N}+\frac{4S_1}{(N-1)N}\Biggr), \tag{85}
$$

$$
\begin{aligned}
a_{Aq}^{(1,1)} = C_F\Biggl(&\xi(N-4)\Biggl[\frac{\bigl(1-3N+N^2\bigr)}{4(N-1)^2N^2}-\frac{S_1}{4(N-1)N}\Biggr]\\
&-\frac{(N-2)\bigl(-1+4N-6N^2+N^3\bigr)}{(N-1)^3N^3}\\
&+\frac{(N-2)(-1+3N)S_1}{(N-1)^2N^2}-\frac{S_1^2}{(N-1)N}-\frac{3S_2}{(N-1)N}+\frac{\zeta_2}{(N-1)N}\Biggr),
\end{aligned} \tag{86}
$$

$$
a_{Aq}^{(1,2)} = C_F\Biggl(\xi(N-4)\Biggl[-\frac{\bigl(-1+4N-6N^2+2N^3\bigr)}{8(N-1)^3N^3}
$$

$$
-\frac{\big(1-3N+N^2\big)S_1}{8(N-1)^2N^2}+\frac{S_1^2}{16(N-1)N}
+\frac{3S_2}{16(N-1)N}-\frac{\zeta_2}{16(N-1)N}\Bigg]+\frac{(N-2)Q_4}{2(N-1)^4N^4}+\Bigg(\frac{3S_2}{2(N-1)N}
+\frac{(N-2)\big(-1+4N-6N^2+N^3\big)}{2(N-1)^3N^3}\Bigg)S_1-\frac{(N-2)(-1+3N)S_1^2}{4(N-1)^2N^2}
+\frac{S_1^3}{6(N-1)N}-\frac{3(N-2)(-1+3N)S_2}{4(N-1)^2N^2}
+\frac{7S_3}{3(N-1)N}+\Bigg(\frac{(N-2)(-1+3N)}{4(N-1)^2N^2}-\frac{S_1}{2(N-1)N}\Bigg)
\times\zeta_2-\frac{7\zeta_3}{3(N-1)N}\Bigg), \tag{87}
$$

$$
b_{Aq}^{(1,0)}=-\xi\frac{C_F}{N}, \tag{88}
$$

$$
b_{Aq}^{(1,1)}=\xi C_F\Bigg(\frac{1-N}{2N^2}+\frac{S_1}{2N}\Bigg), \tag{89}
$$

$$
b_{Aq}^{(1,2)}=\xi C_F\Bigg(\frac{N-1}{4N^3}+\frac{(N-1)S_1}{4N^2}-\frac{S_1^2}{8N}-\frac{3S_2}{8N}+\frac{\zeta_2}{8N}\Bigg), \tag{90}
$$

$$
a_{Bq}^{(1,0)}=C_F\Bigg(\xi\frac{2}{(N-1)N}+\frac{4\big(1-3N+N^2\big)}{(N-1)^2N^2}-\frac{4S_1}{(N-1)N}\Bigg), \tag{91}
$$

$$
a_{Bq}^{(1,1)}=C_F\Bigg(\xi\Bigg[\frac{1-3N+N^2}{(N-1)^2N^2}-\frac{S_1}{(N-1)N}\Bigg]+\frac{2\big(1-4N+6N^2-2N^3\big)}{(N-1)^3N^3}
-\frac{2\big(1-3N+N^2\big)S_1}{(N-1)^2N^2}+\frac{S_1^2}{(N-1)N}+\frac{3S_2}{(N-1)N}-\frac{\zeta_2}{(N-1)N}\Bigg), \tag{92}
$$

$$
a_{Bq}^{(1,2)}=C_F\Bigg(\xi\Bigg[\frac{1-4N+6N^2-2N^3}{2(N-1)^3N^3}+\frac{\big(-1+3N-N^2\big)S_1}{2(N-1)^2N^2}
+\frac{S_1^2}{4(N-1)N}+\frac{3S_2}{4(N-1)N}
-\frac{\zeta_2}{4(N-1)N}\Bigg]+\frac{Q_5}{(N-1)^4N^4}+\Bigg(\frac{-1+4N-6N^2+2N^3}{(N-1)^3N^3}-\frac{3S_2}{2(N-1)N}\Bigg)S_1
+\frac{\big(1-3N+N^2\big)S_1^2}{2(N-1)^2N^2}-\frac{S_1^3}{6(N-1)N}+\frac{3\big(1-3N+N^2\big)S_2}{2(N-1)^2N^2}-\frac{7S_3}{3(N-1)N}
+\Bigg(\frac{-1+3N-N^2}{2(N-1)^2N^2}+\frac{S_1}{2(N-1)N}\Bigg)\zeta_2+\frac{7\zeta_3}{3(N-1)N}\Bigg), \tag{93}
$$

$$a_{\omega g}^{(1,0)} = C_A\left(\frac{-4-N+N^2}{(1+N)^2(2+N)^2} - \frac{S_1}{(1+N)(2+N)}\right), \tag{94}$$

$$a_{\omega g}^{(1,1)} = C_A\left(\frac{8+5N-3N^2-2N^3}{(1+N)^3(2+N)^3} + \frac{\left(4+N-N^2\right)S_1}{2(1+N)^2(2+N)^2} + \frac{S_1^2}{4(1+N)(2+N)} + \frac{3S_2}{4(1+N)(2+N)} - \frac{\zeta_2}{4(1+N)(2+N)}\right), \tag{95}$$

$$a_{\omega g}^{(1,2)} = C_A\left(\frac{Q_6}{(1+N)^4(2+N)^4} + \left(\frac{-8-5N+3N^2+2N^3}{2(1+N)^3(2+N)^3} - \frac{3S_2}{8(1+N)(2+N)}\right)S_1 + \frac{\left(-4-N+N^2\right)S_1^2}{8(1+N)^2(2+N)^2} - \frac{S_1^3}{24(1+N)(2+N)} + \frac{3\left(-4-N+N^2\right)S_2}{8(1+N)^2(2+N)^2} - \frac{7S_3}{12(1+N)(2+N)} + \left(\frac{4+N-N^2}{8(1+N)^2(2+N)^2} + \frac{S_1}{8(1+N)(2+N)}\right)\zeta_2 + \frac{7\zeta_3}{12(1+N)(2+N)}\right), \tag{96}$$

$$b_{\omega g}^{(1,0)} = \frac{2C_A}{(1+N)(2+N)}, \tag{97}$$

$$b_{\omega g}^{(1,1)} = C_A\left(\frac{-4-N+N^2}{(1+N)^2(2+N)^2} - \frac{S_1}{(1+N)(2+N)}\right), \tag{98}$$

$$b_{\omega g}^{(1,2)} = C_A\left(\frac{8+5N-3N^2-2N^3}{(1+N)^3(2+N)^3} + \frac{\left(4+N-N^2\right)S_1}{2(1+N)^2(2+N)^2} + \frac{S_1^2}{4(1+N)(2+N)} + \frac{3S_2}{4(1+N)(2+N)} - \frac{\zeta_2}{4(1+N)(2+N)}\right), \tag{99}$$

$$c_{\omega g}^{(1,0)} = C_A\left(-\frac{Q_{12}}{36N^2(1+N)^2(2+N)^2} + \frac{\left(-4+2N+N^2\right)S_1}{2N(1+N)(2+N)}\right), \tag{100}$$

$$c_{\omega g}^{(1,1)} = C_A\left(-\frac{S_1 Q_8}{8N^2(1+N)^2(2+N)^2} + \frac{Q_{18}}{216N^3(1+N)^3(2+N)^3} + \frac{\left(4-2N-N^2\right)S_1^2}{8N(1+N)(2+N)} - \frac{3\left(-4+2N+N^2\right)S_2}{8N(1+N)(2+N)} + \frac{(N-1)(4+N)(6+N)\zeta_2}{48N(1+N)(2+N)}\right), \tag{101}$$

$$c_{\omega g}^{(1,2)} = C_A\left(\frac{S_1^2 Q_8}{32N^2(1+N)^2(2+N)^2} + \frac{3S_2 Q_8}{32N^2(1+N)^2(2+N)^2} + \frac{Q_{22}}{1296N^4(1+N)^4(2+N)^4}\right.$$

$$+\Bigg(\frac{Q_{14}}{16N^3(1+N)^3(2+N)^3}+\frac{3\big(-4+2N+N^2\big)S_2}{16N(1+N)(2+N)}\Bigg)S_1$$
$$+\frac{\big(-4+2N+N^2\big)S_1^3}{48N(1+N)(2+N)}$$
$$+\frac{7\big(-4+2N+N^2\big)S_3}{24N(1+N)(2+N)}+\Bigg(\frac{Q_{12}}{288N^2(1+N)^2(2+N)^2}$$
$$+\frac{\big(4-2N-N^2\big)S_1}{16N(1+N)(2+N)}\Bigg)\zeta_2$$
$$-\frac{7(N-1)(4+N)(6+N)\zeta_3}{144N(1+N)(2+N)}\Bigg), \tag{102}$$

$$d_{\omega g}^{(1,0)}=\frac{C_A}{4N}, \tag{103}$$

$$d_{\omega g}^{(1,1)}=C_A\big(-\frac{1}{8N^2}-\frac{S_1}{8N}\big), \tag{104}$$

$$d_{\omega g}^{(1,2)}=C_A\Bigg(\frac{1}{16N^3}+\frac{S_1}{16N^2}+\frac{S_1^2}{32N}+\frac{3S_2}{32N}-\frac{\zeta_2}{32N}\Bigg), \tag{105}$$

with the polynomials

$$Q_1=-3N^4-12N^3+N^2+46N+16, \tag{106}$$
$$Q_2=-3N^4+13N^3-19N^2+10N-2, \tag{107}$$
$$Q_3=-N^4-13N^3-30N^2-24N-8, \tag{108}$$
$$Q_4=2N^4-10N^3+10N^2-5N+1, \tag{109}$$
$$Q_5=3N^4-10N^3+10N^2-5N+1, \tag{110}$$
$$Q_6=3N^4+10N^3+4N^2-17N-16, \tag{111}$$
$$Q_7=12N^4-30N^3+25N^2-8N-1, \tag{112}$$
$$Q_8=-3N^5-14N^4-11N^3-24N^2-60N-16, \tag{113}$$
$$Q_9=-2N^5-13N^4-40N^3-53N^2-28N-8, \tag{114}$$
$$Q_{10}=12N^5-26N^4+17N^3-6N^2+4N-2, \tag{115}$$
$$Q_{11}=-24N^6+72N^5-64N^4-3N^3+29N^2-7N-1, \tag{116}$$
$$Q_{12}=-8N^6-21N^5+22N^4+3N^3+184N^2+540N+144, \tag{117}$$
$$Q_{13}=N^6-6N^5-38N^4-71N^3-66N^2-36N-8, \tag{118}$$
$$Q_{14}=-3N^7-23N^6-15N^5+7N^4-194N^3-372N^2-168N-32, \tag{119}$$
$$Q_{15}=-3N^7+12N^6+94N^5+16N^4-231N^3-164N^2-44N+32, \tag{120}$$
$$Q_{16}=-16N^8-55N^7-68N^6-154N^5+64N^4+629N^3$$
$$+428N^2+132N-96, \tag{121}$$
$$Q_{17}=-N^8-7N^7+3N^6+69N^5+148N^4+154N^3+78N^2-4N-8, \tag{122}$$

$$Q_{18} = -52N^9 - 468N^8 - 1635N^7 - 2655N^6 - 3027N^5 - 2061N^4 + 4822N^3 \\ +10044N^2 + 4536N + 864, \quad (123)$$

$$Q_{19} = 11N^{10} + 34N^9 - 157N^8 - 422N^7 + 69N^6 + 798N^5 + 1025N^4 + 670N^3 \\ -308N^2 - 56N + 64, \quad (124)$$

$$Q_{20} = -3N^{11} - 33N^{10} - 85N^9 - 7N^8 + 363N^7 + 897N^6 \\ +1085N^5 + 527N^4 - 96N^3 \\ -88N^2 + 16N + 16, \quad (125)$$

$$Q_{21} = 92N^{12} + 552N^{11} + 729N^{10} - 1226N^9 - 1623N^8 + 3246N^7 + 2599N^6 \\ -5526N^5 - 10329N^4 - 6766N^3 + 2772N^2 + 504N - 576, \quad (126)$$

$$Q_{22} = 320N^{12} + 3840N^{11} + 19840N^{10} + 57357N^9 + 99480N^8 + 114390N^7 \\ +103660N^6 + 40845N^5 - 103420N^4 - 176904N^3 - 103680N^2 - 34992N \\ -5184, \quad (127)$$

$$Q_{23} = -544N^{16} - 4352N^{15} - 10880N^{14} + 513N^{13} + 43712N^{12} + 41684N^{11} \\ -67480N^{10} - 119386N^9 - 6592N^8 + 133644N^7 + 221992N^6 + 105361N^5 \\ -75664N^4 - 1512N^3 + 21600N^2 + 1296N - 3456. \quad (128)$$

The eom part of A_{Bq} vanishes. In the above expressions the QCD color factor are $C_F = (N_c^2 - 1)/(2N_c)$, $C_A = N_c$, $T_F = 1/2$ for $SU(N_c)$ and $N_c = 3$ for QCD; (For interpretation of the colors, the reader is referred to the web version of this article.) N_F denotes the number of massless quark flavors. $\zeta_k, \;\; k \in \mathbb{N}, \;\; k \geq 2$ are the values of Riemann's ζ function at integer arguments.

4. The operator mixing in the unpolarized singlet case

In the calculation of the off–shell OMEs in the unpolarized singlet case mixing between the physical and alien operators occurs, which we discuss in the following. In Mellin N–space the Z–factor for the (ultraviolet) renormalization of the local operators up to $O(a_s^2)$ reads

$$Z_{ij}^S = \delta_{ij} + a_s S_\varepsilon \frac{\gamma_{ij}^{(0)}}{\varepsilon} + a_s^2 S_\varepsilon^2 \left[\frac{1}{\varepsilon^2} \left(\frac{1}{2} \gamma_{il}^{(0)} \gamma_{lj}^{(0)} + \beta_0 \gamma_{ij}^{(0)} \right) + \frac{1}{2\varepsilon} \gamma_{ij}^{(1)} \right] + O(a_s^3) \, . \quad (129)$$

The renormalized physical OMEs are obtained by[2]

$$A_{ij}^{\rm phys} = (Z_{ik}^S)^{-1} \tilde{\tilde{A}}_{kj}, \quad (130)$$

with[3]

[2] Note that we focus on the physical projection of the OMEs in this section, since the QCD anomalous dimensions are extracted from them. Similar relations can also be derived for the other projections.

[3] The occurrence of the factor of $-1/2$ in (134) is due to a convention used in Ref. [19], which is, however, not correctly implemented in Ref. [19].

$$\tilde{\tilde{A}}_{qq}^{\rm PS} = \tilde{A}_{qq}^{\rm PS,phys}, \tag{131}$$

$$\tilde{\tilde{A}}_{qg} = \tilde{A}_{qg}^{\rm phys}, \tag{132}$$

$$\tilde{\tilde{A}}_{gq} = \tilde{A}_{gq}^{\rm phys} + \eta(\tilde{A}_{Aq} + \tilde{A}_{Bq}), \tag{133}$$

$$\tilde{\tilde{A}}_{gg} = \tilde{A}_{gg}^{\rm phys} - \frac{\eta}{2}(\tilde{A}_{Ag} + \tilde{A}_{\omega g}), \tag{134}$$

and

$$\eta = -a_s S_\varepsilon \frac{\gamma_{gA}^{(0)}}{\varepsilon} + O(a_s^2). \tag{135}$$

The individual contributions $\tilde{\tilde{A}}_{ij}$ are given by

$$\tilde{\tilde{A}}_{qq}^{\rm PS} = a_s^2 S_\varepsilon^2 \Biggl[\frac{1}{2\varepsilon^2} \gamma_{gq}^{(0)} \gamma_{qg}^{(0)} + \frac{1}{\varepsilon} \Biggl(\gamma_{qg}^{(0)} a_{gq}^{(1,0)} + \frac{\gamma_{qq}^{\rm PS,(1)}}{2} \Biggr) + a_{qq}^{\rm PS,(2,0)} + \varepsilon\, a_{qq}^{\rm PS,(2,1)} \Biggr] + O(a_s^3)\,, \tag{136}$$

$$\tilde{\tilde{A}}_{qg} = a_s S_\varepsilon \Biggl[\frac{\gamma_{qg}^{(0)}}{\varepsilon} + a_{qg}^{(1,0)} + \varepsilon\, a_{qg}^{(1,1)} + \varepsilon^2\, a_{qg}^{(1,2)} \Biggr] + a_s^2 S_\varepsilon^2 \Biggl[\frac{1}{2\varepsilon^2} \gamma_{qg}^{(0)} \Biggl(\gamma_{gg}^{(0)} + \gamma_{qq}^{(0)} + 2\beta_0 \Biggr) + \frac{1}{\varepsilon} \Biggl(\gamma_{qg}^{(0)} a_{gg}^{(1,0)} + \gamma_{qq}^{(0)} a_{qg}^{(1,0)} + \frac{\gamma_{qg}^{(1)}}{2} \Biggr) + a_{qg}^{(2,0)} + \varepsilon\, a_{qg}^{(2,1)} \Biggr] + O(a_s^3)\,, \tag{137}$$

$$\tilde{\tilde{A}}_{gq} = a_s S_\varepsilon \Biggl[\frac{\gamma_{gq}^{(0)}}{\varepsilon} + a_{gq}^{(1,0)} + \varepsilon\, a_{gq}^{(1,1)} + \varepsilon^2\, a_{gq}^{(1,2)} \Biggr] + a_s^2 S_\varepsilon^2 \Biggl[\frac{1}{2\varepsilon^2} \gamma_{gq}^{(0)} \Biggl(\gamma_{gg}^{(0)} + \gamma_{qq}^{(0)} + 2\beta_0 \Biggr) + \frac{1}{\varepsilon} \Biggl(\gamma_{gg}^{(0)} a_{gq}^{(1,0)} + \gamma_{gq}^{(0)} a_{qq}^{(1,0)} + \frac{\gamma_{gq}^{(1)}}{2} \Biggr) + \frac{\delta_{gq}^{(-1)}}{\varepsilon} + a_{gq}^{(2,0)} + \delta_{gq}^{(0)} + \varepsilon\, a_{gq}^{(2,1)} + \varepsilon \delta_{gq}^{(1)} \Biggr] + O(a_s^3)\,, \tag{138}$$

$$\tilde{\tilde{A}}_{gg} = 1 + a_s S_\varepsilon \Biggl[\frac{\gamma_{gg}^{(0)}}{\varepsilon} + a_{gg}^{(1,0)} + \varepsilon\, a_{gg}^{(1,1)} + \varepsilon^2\, a_{gg}^{(1,2)} \Biggr] + a_s^2 S_\varepsilon^2 \Biggl[\frac{1}{2\varepsilon^2} \Biggl(\gamma_{gg}^{(0)} \left(\gamma_{gg}^{(0)} + 2\beta_0 \right) + \gamma_{gq}^{(0)} \gamma_{qg}^{(0)} \Biggr) + \frac{1}{\varepsilon} \Biggl(\gamma_{gg}^{(0)} a_{gg}^{(0)} + \gamma_{gq}^{(0)} a_{qg}^{(1,0)} + \frac{\gamma_{gg}^{(1)}}{2} \Biggr) + \frac{\delta_{gg}^{(-1)}}{\varepsilon} + a_{gg}^{(2,0)} + \delta_{gg}^{(0)} + \varepsilon\, a_{gg}^{(2,1)} + \varepsilon \delta_{gg}^{(1)} \Biggr] + O(a_s^3)\,. \tag{139}$$

In the flavor non–singlet cases, including transversity [82], and in the polarized singlet case [81] no mixing with the alien operators occurs. To two–loop order, these terms contribute in the renormalization of $\tilde{A}_{gq}$ and $\tilde{A}_{gg}$, since one has to consider the combinations (133), (134) in this case. Because of this the additional contributions

$$\delta_{gq}^{(i-1)} = -\gamma_{gA}^{(0)} \left(a_{Aq}^{(1,i)} + a_{Bq}^{(1,i)} \right), \quad i \geq 0\,, \tag{140}$$

$$\delta_{gg}^{(i-1)} = \frac{\gamma_{gA}^{(0)}}{2} \left(a_{Ag}^{(1,i)} + a_{\omega g}^{(1,i)} \right), \quad i \geq 0 \tag{141}$$

are present. The anomalous dimensions are obtained from the pole terms of (136)–(139).

5. The unpolarized OMEs

We now turn to the calculation of contributions to the unpolarized off–shell OMEs for non–negative order in the dimensional parameter ε. They are gauge dependent in general and are given by[4]

$$a_{qq}^{(1,0),\mathrm{NS}} = C_F\left[-\frac{P_{39}}{N^2(1+N)^2} + \frac{2S_1}{N(1+N)} - 2S_1^2 - 6S_2 + \xi\left(-\frac{1}{N} + S_1\right)\right], \tag{142}$$

$$\begin{aligned} a_{qq}^{(1,1),\mathrm{NS}} = C_F\Bigg[&\xi\left(\frac{1-N+2N^2}{2N^2} + \frac{(1-N)S_1}{2N} - \frac{1}{4}S_1^2 - \frac{3}{4}S_2\right) + \frac{P_{299}}{N^3(1+N)^3} \\ &+\left(-\frac{(N-1)\big(-1-N+N^2\big)}{N^2(1+N)^2} + 3S_2\right)S_1 - \frac{S_1^2}{2N(1+N)} \\ &+\frac{1}{3}S_1^3 - \frac{3S_2}{2N(1+N)} \\ &+\frac{14}{3}S_3 + \left(\frac{2+3N+3N^2}{4N(1+N)} - S_1\right)\zeta_2\Bigg], \end{aligned} \tag{143}$$

$$\begin{aligned} a_{qq}^{(1,2),\mathrm{NS}} = C_F\Bigg[&\xi\left(\frac{-1+N-4N^3}{4N^3} + \left(\frac{N-1}{4N^2} + \frac{3S_2}{8}\right)S_1 + \frac{(N-1)S_1^2}{8N} + \frac{1}{24}S_1^3 \right. \\ &\left.+\frac{3(N-1)S_2}{8N} + \frac{7}{12}S_3 + \left(\frac{1}{8N} - \frac{S_1}{8}\right)\zeta_2\right) + \frac{P_{469}}{2N^4(1+N)^4} \\ &+\left(-\frac{(N-1)\big(1+2N-2N^3+N^4\big)}{2N^3(1+N)^3} + \frac{3S_2}{4N(1+N)} - \frac{7}{3}S_3\right)S_1 \\ &-\left(\frac{(N-1)\big(1+N-N^2\big)}{4N^2(1+N)^2} + \frac{3}{4}S_2\right)S_1^2 + \frac{S_1^3}{12N(1+N)} - \frac{1}{24}S_1^4 \\ &+\frac{3(N-1)\big(-1-N+N^2\big)S_2}{4N^2(1+N)^2} - \frac{9}{8}S_2^2 + \frac{7S_3}{6N(1+N)} \\ &-\frac{15}{4}S_4 + \left(\frac{P_{39}}{8N^2(1+N)^2} - \frac{S_1}{4N(1+N)} + \frac{1}{4}S_1^2 + \frac{3}{4}S_2\right)\zeta_2 \\ &+\left(\frac{7}{3}S_1 - \frac{7\big(2+3N+3N^2\big)}{12N(1+N)}\right)\zeta_3\Bigg], \end{aligned} \tag{144}$$

$$a_{qq}^{(2,0),\mathrm{NS}} = \xi^2\Bigg[C_A C_F\left(\frac{1+2N}{4N} + \frac{(2-3N)S_1}{4N} - \frac{1}{4}S_1^2 - \frac{1}{4}S_2\right)$$

[4] The polynomials are listed in Appendix B.

$$
\begin{aligned}
&+C_F^2\left(\frac{1-N}{N}-\frac{S_1}{N}+\frac{1}{2}S_1^2+\frac{1}{2}S_2\right)\Bigg]\\
&+\xi\Bigg[C_F^2\Bigg(\frac{P_{186}}{N^3(1+N)^2}+\left(\frac{P_{132}}{N^2(1+N)^2}-16S_2\right)S_1-4S_1^3\\
&+\frac{\big(18+7N-5N^2\big)S_1^2}{2N(1+N)}+\frac{\big(22+25N+13N^2\big)S_2}{2N(1+N)}\Bigg)\\
&+C_AC_F\Bigg(\frac{-2-3N-7N^2}{2N^2}\\
&+\left(\frac{-2+11N+12N^2}{2N(2+N)}-\frac{(N-1)NS_2}{(1+N)(2+N)}\right)S_1+\frac{5}{4}S_1^2+\frac{(2+13N)S_2}{4(2+N)}\\
&-\frac{(N-1)NS_3}{(1+N)(2+N)}+\frac{2(N-1)NS_{2,1}}{(1+N)(2+N)}-\frac{6(N-1)N}{(1+N)(2+N)}\zeta_3\Bigg)\Bigg]\\
&+C_F\Bigg(N_FT_F\Bigg(\left(\left(\frac{4\big(-21+29N+73N^2+41N^3\big)}{27N(1+N)^2}+\frac{8}{3}S_2\right)S_1\right.\\
&+\frac{P_{286}}{54N^3(1+N)^3}\\
&+\frac{2\big(-6+17N+17N^2\big)S_1^2}{9N(1+N)}+\frac{2\big(-6+37N+37N^2\big)S_2}{9N(1+N)}+\frac{8}{9}S_1^3-\frac{8}{9}S_3\Bigg)\\
&+C_A\Bigg(\frac{S_2P_{166}}{18N^2(1+N)^2(2+N)}+\frac{P_{376}}{216N^3(1+N)^3}\\
&+\Bigg(\frac{P_{379}}{27N^3(1+N)^3(2+N)}\\
&-\frac{2\big(22+33N+14N^2\big)S_2}{3(1+N)(2+N)}-6S_3-4S_{2,1}\Bigg)S_1\\
&+\left(-\frac{11\big(-6+23N+23N^2\big)}{18N(1+N)}+S_2\right)S_1^2\\
&+\frac{2\big(72+58N+33N^2+2N^3\big)S_3}{9N(1+N)(2+N)}-\frac{22}{9}S_1^3\\
&-6S_2^2-29S_4+\left(-\frac{4(-3+N)}{N(1+N)^2}+\frac{16S_1}{N(1+N)}-8S_1^2-16S_2\right)S_{-2}-4S_{-2}^2\\
&+\left(\frac{20}{N(1+N)}-32S_1\right)S_{-3}-28S_{-4}\\
&+\frac{4N^2S_{2,1}}{(1+N)(2+N)}+6S_{3,1}-\frac{8S_{-2,1}}{N(1+N)}
\end{aligned}
$$

$$
\begin{aligned}
&+8S_{-2,2}+24S_{-3,1}+6S_{2,1,1}+16S_{-2,1,1}\\
&+\left(-\frac{12\left(2-N+N^2\right)}{N(2+N)}+12S_1\right)\zeta_3\Bigg)\Bigg)\\
&+C_F^2\Bigg[\frac{S_2P_{173}}{N^2(1+N)^2(2+N)}+\frac{P_{486}}{8N^4(1+N)^4}\\
&+\left(-\frac{2P_{414}}{N^3(1+N)^3(2+N)}-\frac{2(26+11N)\left(2+N+N^2\right)S_2}{N(1+N)(2+N)}+\frac{256}{3}S_3\right)S_1\\
&+\left(\frac{P_{29}}{N^2(1+N)^2}+48S_2\right)S_1^2\\
&-\frac{2\left(14+3N+3N^2\right)S_1^3}{3N(1+N)}+\frac{14}{3}S_1^4+30S_2^2\\
&-\frac{2\left(164+232N+237N^2+75N^3\right)S_3}{3N(1+N)(2+N)}\\
&+48S_4+\left(\frac{8(-3+N)}{N(1+N)^2}-\frac{32S_1}{N(1+N)}+16S_1^2+32S_2\right)S_{-2}+8S_{-2}^2\\
&+\left(-\frac{40}{N(1+N)}+64S_1\right)S_{-3}+56S_{-4}\\
&-\frac{4(-2+N)\left(2+N+N^2\right)S_{2,1}}{N(1+N)(2+N)}+\frac{16S_{-2,1}}{N(1+N)}\\
&-16S_{-2,2}-48S_{-3,1}-32S_{-2,1,1}\\
&+\left(-\frac{\left(2+3N+3N^2\right)^2}{2N^2(1+N)^2}+\frac{4\left(2+3N+3N^2\right)S_1}{N(1+N)}-8S_1^2\right)\zeta_2\\
&-\frac{48N}{(1+N)(2+N)}\zeta_3\Bigg],
\end{aligned}
\tag{145}
$$

$$
\begin{aligned}
a_{qq}^{(2,1),\mathrm{NS}}=\xi^2\Bigg[&C_F^2\Bigg(\frac{-1-N+3N^2}{2N^2}+\left(\frac{1-2N+3N^2}{2N^2}-\frac{5S_2}{4}\right)S_1\\
&+\frac{(3-2N)S_1^2}{4N}-\frac{1}{4}S_1^3\\
&+\frac{(5-2N)S_2}{4N}-S_3\Bigg)+C_AC_F\Bigg(\frac{-1-2N-13N^2}{8N^2}\\
&+\left(\frac{-2+5N^2}{8N^2}+\frac{5S_2}{8}\right)S_1\\
&+\frac{(-3+4N)S_1^2}{8N}+\frac{1}{8}S_1^3+\frac{(-5+7N)S_2}{8N}+\frac{1}{2}S_3+\left(\frac{1}{8N}-\frac{S_1}{8}\right)\zeta_2\Bigg)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\xi\Biggl[C_A C_F\Biggl(\frac{6+3N+10N^2+42N^3}{4N^3}+\frac{S_2 P_{17}}{8N^2(2+N)^2}\\
&+\frac{S_{2,1}P_{47}}{N(1+N)^2(2+N)^2}\\
&+\frac{S_3 P_{183}}{2N(1+N)^2(2+N)^2}+\Biggl(\frac{S_2 P_{176}}{8N(1+N)^2(2+N)^2}+\frac{P_{180}}{4N^3(2+N)^2}\\
&+\frac{3(N-1)N S_3}{(1+N)(2+N)}-\frac{2(N-1)N S_{2,1}}{(1+N)(2+N)}\Biggr)S_1+\Biggl(\frac{6-23N-31N^2}{8N(2+N)}\\
&+\frac{3(N-1)N S_2}{4(1+N)(2+N)}\Biggr)S_1^2-\frac{5}{8}S_1^3+\frac{\bigl(-1-N-N^2\bigr)S_2^2}{(1+N)(2+N)}\\
&+\frac{\bigl(-4-13N+5N^2\bigr)S_4}{4(1+N)(2+N)}\\
&+\frac{\bigl(2+7N-3N^2\bigr)S_{3,1}}{(1+N)(2+N)}+\frac{(N-1)N S_{2,1,1}}{(1+N)(2+N)}+\Biggl(-\frac{5}{8N}+\frac{5S_1}{8}\Biggr)\zeta_2\\
&+\frac{9(N-1)N\zeta_2^2}{5(1+N)(2+N)}+\Biggl(\frac{3P_{227}}{2N(1+N)^2(2+N)^2}+\frac{6\bigl(1+N+N^2\bigr)S_1}{(1+N)(2+N)}\Biggr)\zeta_3\Biggr)\\
&+C_F^2\Biggl(\frac{S_2 P_{13}}{4N^2(1+N)^2}+\frac{P_{410}}{2N^4(1+N)^3}+\Biggl(\frac{P_{292}}{2N^3(1+N)^3}\\
&+\frac{\bigl(-70-25N+19N^2\bigr)S_2}{4N(1+N)}+\frac{52}{3}S_3\Biggr)S_1\\
&+\Biggl(\frac{P_{23}}{4N^2(1+N)^2}+12S_2\Biggr)S_1^2+\frac{7\bigl(-6-N+3N^2\bigr)S_1^3}{12N(1+N)}+\frac{7}{6}S_1^4\\
&+\frac{15}{2}S_2^2+\frac{\bigl(-66-73N-45N^2\bigr)S_3}{6N(1+N)}\\
&+\Biggl(\frac{-2-3N-3N^2}{2N^2(1+N)}+\frac{\bigl(6+7N+3N^2\bigr)S_1}{2N(1+N)}-2S_1^2\Biggr)\zeta_2\\
&+\Biggl(\frac{3\bigl(2+3N+3N^2\bigr)}{N(1+N)}-12S_1\Biggr)\zeta_3\Biggr)\Biggr]\\
&+C_F\Biggl[C_A\Biggl(-\frac{2S_{2,1}P_{325}}{N^2(1+N)^2(2+N)^2}+\frac{S_3 P_{371}}{27N^2(1+N)^2(2+N)^2}\\
&+\frac{P_{432}}{2592N^4(1+N)^4}+\Biggl(\frac{S_2 P_{373}}{36N^2(1+N)^2(2+N)^2}\\
&+\frac{P_{550}}{162N^4(1+N)^4(2+N)^2}+5S_2^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2\big(-18+29N+57N^2+28N^3\big)S_3}{3N(1+N)(2+N)}\\
&+\frac{49}{2}S_4+\frac{4(-2+3N)S_{2,1}}{N(2+N)}-2S_{2,1,1}\Bigg)S_1\\
&+\Bigg(\frac{P_{428}}{108N^3(1+N)^3(2+N)}+S_3\\
&+\frac{\big(12+104N+147N^2+58N^3\big)S_2}{6N(1+N)(2+N)}+2S_{2,1}\Bigg)S_1^2\\
&+\Bigg(\frac{-198+491N+491N^2}{108N(1+N)}-\frac{1}{2}S_2\Bigg)S_1^3+\frac{11}{12}S_1^4\\
&+\Bigg(\frac{P_{488}}{108N^3(1+N)^3(2+N)^2}\\
&+18S_3+4S_{2,1}-8S_{-2,1}\Bigg)S_2+\frac{\big(-96+106N+231N^2+65N^3\big)S_2^2}{12N(1+N)(2+N)}\\
&+\frac{\big(-324-64N+147N^2+70N^3\big)S_4}{6N(1+N)(2+N)}+62S_5+\Bigg(-\frac{2P_{83}}{N^3(1+N)^3}\\
&+\Bigg(\frac{8\big(-1-3N+N^2\big)}{N^2(1+N)^2}+16S_2\Bigg)S_1-\frac{8S_1^2}{N(1+N)}+\frac{8}{3}S_1^3\\
&-\frac{16S_2}{N(1+N)}+\frac{64}{3}S_3+8S_{2,1}\Bigg)S_{-2}\\
&+\Bigg(-\frac{4}{N(1+N)}+4S_1\Bigg)S_{-2}^2+\Bigg(\frac{2\big(-2-15N+5N^2\big)}{N^2(1+N)^2}\\
&-\frac{32S_1}{N(1+N)}+16S_1^2+20S_2+16S_{-2}\Bigg)S_{-3}\\
&+\Bigg(-\frac{42}{N(1+N)}+56S_1\Bigg)S_{-4}+54S_{-5}\\
&+16S_{2,3}+20S_{2,-3}-\frac{2\big(-2-7N-9N^2+N^3\big)S_{3,1}}{N(1+N)(2+N)}-13S_{4,1}\\
&-\frac{4\big(-2-3N+N^2\big)S_{-2,1}}{N^2(1+N)^2}+\frac{4S_{-2,2}}{N(1+N)}\\
&-12S_{-2,3}+\frac{12S_{-3,1}}{N(1+N)}-28S_{-4,1}\\
&-\frac{2\big(-6+3N+9N^2+2N^3\big)S_{2,1,1}}{N(1+N)(2+N)}-8S_{2,1,-2}-18S_{2,2,1}
\end{aligned}
$$

$$
\begin{aligned}
&-6S_{3,1,1}+\frac{8S_{-2,1,1}}{N(1+N)}\\
&-8S_{-2,2,1}-24S_{-3,1,1}-3S_{2,1,1,1}-16S_{-2,1,1,1}\\
&+\Bigg(\frac{310+793N+1128N^2+513N^3}{72N(1+N)^2}\\
&-\frac{11\big(-6+23N+23N^2\big)S_1}{36N(1+N)}-\frac{11}{6}S_1^2-\frac{11}{6}S_2-2S_3\\
&+\left(\frac{2}{N(1+N)}-4S_1\right)S_{-2}\\
&-2S_{-3}+4S_{-2,1}\Bigg)\zeta_2+\left(\frac{18\big(2-N+N^2\big)}{5N(2+N)}-\frac{18}{5}S_1\right)\zeta_2^2\\
&+\Bigg(\frac{P_{265}}{2N^2(1+N)(2+N)^2}\\
&-\frac{2\big(22+33N+8N^2\big)S_1}{(1+N)(2+N)}-3S_1^2+3S_2+12S_{-2}\Bigg)\zeta_3\Bigg)\\
&+N_FT_F\Bigg(\frac{S_2P_6}{27N^2(1+N)^2}+\frac{P_{492}}{648N^4(1+N)^4}\\
&+\left(-\frac{2P_{278}}{81N^2(1+N)^3}-\frac{5\big(-6+13N+13N^2\big)S_2}{9N(1+N)}-\frac{16}{3}S_3\right)\\
&\times S_1+\left(\frac{P_{14}}{27N^2(1+N)^2}-\frac{10S_2}{3}\right)S_1^2+\frac{\big(18-31N-31N^2\big)S_1^3}{27N(1+N)}\\
&-\frac{1}{3}S_1^4-\frac{7}{3}S_2^2-\frac{8\big(-9+28N+28N^2\big)S_3}{27N(1+N)}-\frac{10}{3}S_4\\
&+\Bigg(\frac{-14-53N-96N^2-45N^3}{18N(1+N)^2}\\
&+\frac{\big(-6+17N+17N^2\big)S_1}{9N(1+N)}+\frac{2}{3}S_1^2+\frac{2}{3}S_2\Bigg)\zeta_2\\
&+\left(-\frac{2\big(2+3N+3N^2\big)}{N(1+N)}+8S_1\right)\times\zeta_3\Bigg)\Bigg]+{C_F}^2\Bigg(\frac{4S_{2,1}P_{210}}{N(1+N)^2(2+N)^2}\\
&+\frac{S_3P_{364}}{3N^2(1+N)^2(2+N)^2}+\frac{P_{520}}{32N^5(1+N)^5}\\
&+\Bigg(\frac{S_2P_{352}}{2N^2(1+N)^2(2+N)^2}+\frac{P_{541}}{N^4(1+N)^4(2+N)^2}-47S_2^2\\
&+\frac{2\big(76+60N+45N^2+11N^3\big)S_3}{N(1+N)(2+N)}-128S_4-\frac{16NS_{2,1}}{(1+N)(2+N)}\Bigg)S_1
\end{aligned}
$$

$$
\begin{aligned}
&+\Biggl(\frac{P_{412}}{2N^3(1+N)^3(2+N)}+\frac{\bigl(116+80N+45N^2+11N^3\bigr)S_2}{2N(1+N)(2+N)}-60S_3\Biggr)S_1^2\\
&+\Biggl(\frac{P_{123}}{6N^2(1+N)^2}-\frac{56S_2}{3}\Biggr)S_1^3+\frac{\bigl(10+N+N^2\bigr)S_1^4}{4N(1+N)}-S_1^5\\
&+\Biggl(-\frac{232}{3}S_3+16S_{-2,1}+\frac{P_{480}}{2N^3(1+N)^3(2+N)^2}\Biggr)S_2\\
&+\frac{\bigl(388+416N+361N^2+111N^3\bigr)S_4}{2N(1+N)(2+N)}\\
&+\frac{\bigl(204+196N+125N^2+47N^3\bigr)S_2^2}{4N(1+N)(2+N)}-108S_5\\
&+\Biggl(\frac{4P_{83}}{N^3(1+N)^3}+\Biggl(-\frac{16\bigl(-1-3N+N^2\bigr)}{N^2(1+N)^2}-32S_2\Biggr)S_1+\frac{16S_1^2}{N(1+N)}\\
&-\frac{16}{3}S_1^3+\frac{32S_2}{N(1+N)}-\frac{128}{3}S_3-16S_{2,1}\Biggr)S_{-2}\\
&+\Biggl(\frac{8}{N(1+N)}-8S_1\Biggr)S_{-2}^2+\Biggl(-\frac{4\bigl(-2-15N+5N^2\bigr)}{N^2(1+N)^2}\\
&+\frac{64S_1}{N(1+N)}-32S_1^2-40S_2-32S_{-2}\Biggr)S_{-3}\\
&+\Biggl(\frac{84}{N(1+N)}-112S_1\Biggr)S_{-4}-108S_{-5}\\
&-40S_{2,-3}+\frac{2\bigl(-12-7N^2+3N^3\bigr)S_{3,1}}{N(1+N)(2+N)}\\
&+\frac{8\bigl(-2-3N+N^2\bigr)S_{-2,1}}{N^2(1+N)^2}-\frac{8S_{-2,2}}{N(1+N)}\\
&+24S_{-2,3}-\frac{24S_{-3,1}}{N(1+N)}+56S_{-4,1}+\frac{4\bigl(-4+4N+N^2\bigr)S_{2,1,1}}{N(2+N)}+16S_{2,1,-2}\\
&-\frac{16S_{-2,1,1}}{N(1+N)}+16S_{-2,2,1}+48S_{-3,1,1}+32S_{-2,1,1,1}\\
&+\Biggl(\frac{P_{360}}{8N^3(1+N)^3}+\Biggl(\frac{P_{30}}{N^2(1+N)^2}+16S_2\Biggr)S_1\\
&-\frac{3\bigl(2+N+N^2\bigr)S_1^2}{N(1+N)}+4S_1^3-\frac{4\bigl(2+3N+3N^2\bigr)S_2}{N(1+N)}+4S_3\\
&+\Biggl(-\frac{4}{N(1+N)}+8S_1\Biggr)S_{-2}+4S_{-3}-8S_{-2,1}\Biggr)\zeta_2+\frac{72N\zeta_2^2}{5(1+N)(2+N)}
\end{aligned}
$$

$$+\Biggl(\frac{P_{294}}{3N^2(1+N)^2(2+N)^2}-\frac{8\bigl(14+37N+36N^2+15N^3\bigr)S_1}{3N(1+N)(2+N)} +\frac{92}{3}S_1^2-12S_2-24S_{-2}\Biggr)\zeta_3\Biggr), \tag{146}$$

$$b_{qq}^{(1,0),\mathrm{NS}}=C_F\Biggl[-\frac{4}{1+N}+\xi\frac{2}{N}\Biggr], \tag{147}$$

$$b_{qq}^{(1,1),\mathrm{NS}}=C_F\Biggl[\xi\Biggl(\frac{N-1}{N^2}-\frac{S_1}{N}\Biggr)+\Biggl(-\frac{2(N-1)}{(1+N)^2}+\frac{2S_1}{1+N}\Biggr)\Biggr], \tag{148}$$

$$b_{qq}^{(1,2),\mathrm{NS}}=C_F\Biggl[\xi\Biggl(\frac{1-N}{2N^3}+\frac{(1-N)S_1}{2N^2}+\frac{S_1^2}{4N}+\frac{3S_2}{4N}-\frac{\zeta_2}{4N}\Biggr) +\frac{2(N-1)}{(1+N)^3}+\frac{(N-1)S_1}{(1+N)^2} -\frac{S_1^2}{2(1+N)}-\frac{3S_2}{2(1+N)}+\frac{\zeta_2}{2(1+N)}\Biggr], \tag{149}$$

$$b_{qq}^{(2,0),\mathrm{NS}}=C_F\Biggl[N_FT_F\Biggl(\frac{16\bigl(-1+6N+4N^2\bigr)}{9N(1+N)^2}+\frac{8(N-1)S_1}{3N(1+N)}\Biggr) +C_A\Biggl(\Biggl(\frac{8S_2}{(1+N)(2+N)} -\frac{2\bigl(-12+8N+65N^2+23N^3\bigr)}{3N^2(1+N)(2+N)}\Biggr)S_1-\frac{4\bigl(-8+102N+77N^2\bigr)}{9N(1+N)^2} +\frac{4(-2+N)S_2}{N(2+N)}+\frac{8S_3}{(1+N)(2+N)} -\frac{16S_{2,1}}{(1+N)(2+N)}+\frac{48\zeta_3}{(1+N)(2+N)}\Biggr)\Biggr]$$

$$+C_F^2\Biggl(\frac{8P_{72}}{N^2(1+N)^3}+\Biggl(-\frac{4P_{114}}{N^2(1+N)^2(2+N)}-\frac{16S_2}{(1+N)(2+N)}\Biggr)S_1 +\frac{24S_1^2}{1+N}+\frac{8\bigl(2+7N+2N^2\bigr)S_2}{N(1+N)(2+N)} -\frac{16S_3}{(1+N)(2+N)}+\frac{32S_{2,1}}{(1+N)(2+N)} -\frac{96}{(1+N)(2+N)}\zeta_3\Biggr)+\xi\Biggl[C_F^2\Biggl(\frac{2(3+N)\bigl(2+N+N^2+4N^3\bigr)}{N^3(1+N)^2} +\frac{2\bigl(-4+3N+9N^2\bigr)S_1}{N^2(1+N)}-\frac{12S_1^2}{N}-\frac{12S_2}{N}\Biggr)$$

$$
\begin{aligned}
&+C_A C_F\Biggl(\frac{2+3N}{N^2}+\Biggl(\frac{2+7N}{N(2+N)}-\frac{2(N-1)S_2}{(1+N)(2+N)}\Biggr)S_1 \\
&+\frac{6S_2}{2+N}-\frac{2(N-1)S_3}{(1+N)(2+N)}+\frac{4(N-1)S_{2,1}}{(1+N)(2+N)}-\frac{12(N-1)}{(1+N)(2+N)}\zeta_3\Biggr)\Biggr] \\
&+\xi^2\Biggl[C_A C_F\Biggl(-\frac{1}{2N}-\frac{S_1}{N}\Biggr)+{C_F}^2\Biggl(-\frac{2}{N}+\frac{2S_1}{N}\Biggr)\Biggr],
\end{aligned}
\tag{150}
$$

$$
\begin{aligned}
b_{qq}^{(2,1),\mathrm{NS}}={}&{C_F}^2\Biggl[-\frac{8S_3P_{71}}{3N(1+N)^2(2+N)^2}+\frac{S_2P_{266}}{N^2(1+N)^2(2+N)^2}+\frac{2P_{321}}{N^3(1+N)^4} \\
&+\Biggl(-\frac{4S_2P_{113}}{N(1+N)^2(2+N)^2}-\frac{4P_{403}}{N^3(1+N)^3(2+N)^2}+\frac{48S_3}{(1+N)(2+N)} \\
&-\frac{32S_{2,1}}{(1+N)(2+N)}\Biggr)S_1+\Biggl(\frac{P_{137}}{N^2(1+N)^2(2+N)}+\frac{12S_2}{(1+N)(2+N)}\Biggr)S_1^2 \\
&-\frac{28S_1^3}{3(1+N)}-\frac{8S_2^2}{(1+N)(2+N)}-\frac{16\bigl(-4+2N+5N^2+N^3\bigr)S_{2,1}}{N(1+N)^2(2+N)^2} \\
&+\frac{28S_4}{(1+N)(2+N)}-\frac{64S_{3,1}}{(1+N)(2+N)}+\frac{16S_{2,1,1}}{(1+N)(2+N)}+\Biggl(\frac{8S_1}{1+N} \\
&-\frac{2\bigl(2+3N+3N^2\bigr)}{N(1+N)^2}\Biggr)\zeta_2+\frac{144\zeta_2^2}{5(1+N)(2+N)}+\Biggl(\frac{48S_1}{(1+N)(2+N)} \\
&-\frac{24\bigl(4-12N-13N^2+N^4\bigr)}{N(1+N)^2(2+N)^2}\Biggr)\zeta_3\Biggr]+C_F\Biggl[N_F T_F\Biggl(-\frac{8P_{95}}{27N^2(1+N)^3} \\
&-\frac{4\bigl(-3-7N+33N^2+13N^3\bigr)S_1}{9N^2(1+N)^2}-\frac{2(N-1)S_1^2}{N(1+N)}-\frac{10(N-1)S_2}{3N(1+N)} \\
&+\frac{2(N-1)\zeta_2}{3N(1+N)}\Biggr)+C_A\Biggl(\frac{2P_{155}}{27N^2(1+N)^3}+\frac{S_2P_{262}}{6N^2(1+N)^2(2+N)^2} \\
&+\Biggl(-\frac{2S_2P_{65}}{N(1+N)^2(2+N)^2}+\frac{P_{422}}{9N^3(1+N)^3(2+N)^2} \\
&-\frac{24S_3}{(1+N)(2+N)}+\frac{16S_{2,1}}{(1+N)(2+N)}\Biggr)S_1 \\
&+\Biggl(\frac{-12+8N+65N^2+23N^3}{2N^2(1+N)(2+N)}-\frac{6S_2}{(1+N)(2+N)}\Biggr)S_1^2 \\
&+\frac{4S_2^2}{(1+N)(2+N)}-\frac{4(3+2N)\bigl(-4-2N+N^2+N^3\bigr)S_3}{N(1+N)^2(2+N)^2} \\
&-\frac{14S_4}{(1+N)(2+N)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{8\big(-4+2N+5N^2+N^3\big)S_{2,1}}{N(1+N)^2(2+N)^2} \\
&+\frac{32S_{3,1}}{(1+N)(2+N)}-\frac{8S_{2,1,1}}{(1+N)(2+N)} \\
&+\frac{(5-17N)\zeta_2}{6N(1+N)}-\frac{72\zeta_2^2}{5(1+N)(2+N)}+\Bigg(-\frac{12P_{70}}{N(1+N)^2(2+N)^2} \\
&-\frac{24S_1}{(1+N)(2+N)}\Bigg)\zeta_3\Bigg)\Bigg]+\xi\Bigg[{C_F}^2\Bigg(\frac{P_{297}}{N^4(1+N)^3}+\Bigg(\frac{22S_2}{N} \\
&+\frac{(N-1)\big(-4-15N-11N^2+2N^3\big)}{N^3(1+N)^2}\Bigg)S_1 \\
&+\frac{\big(16-3N-21N^2\big)S_1^2}{2N^2(1+N)}+\frac{14S_1^3}{3N} \\
&+\frac{\big(8-13N-27N^2\big)S_2}{2N^2(1+N)}+\frac{28S_3}{3N}+\Bigg(\frac{2+3N+3N^2}{N^2(1+N)}-\frac{4S_1}{N}\Bigg)\zeta_2\Bigg) \\
&+C_AC_F\Bigg(\frac{-6-3N-10N^2}{2N^3}+\frac{4S_{2,1}P_{66}}{N(1+N)^2(2+N)^2} \\
&-\frac{2S_3P_{103}}{N(1+N)^2(2+N)^2} \\
&+\Bigg(\frac{S_2P_{40}}{N(1+N)^2(2+N)^2}+\frac{P_{59}}{2N^3(2+N)^2}+\frac{6(N-1)S_3}{(1+N)(2+N)} \\
&-\frac{4(N-1)S_{2,1}}{(1+N)(2+N)}\Bigg) \\
&\times S_1+\Bigg(-\frac{3(2+7N)}{4N(2+N)}+\frac{3(N-1)S_2}{2(1+N)(2+N)}\Bigg)S_1^2+\frac{(1-N)S_2^2}{(1+N)(2+N)} \\
&+\frac{\big(-48-4N-84N^2-35N^3\big)S_2}{4N^2(2+N)^2} \\
&+\frac{7(N-1)S_4}{2(1+N)(2+N)}-\frac{8(N-1)S_{3,1}}{(1+N)(2+N)} \\
&+\frac{2(N-1)S_{2,1,1}}{(1+N)(2+N)}+\frac{5\zeta_2}{4N}+\frac{18(N-1)\zeta_2^2}{5(1+N)(2+N)}+\Bigg(\frac{12P_{67}}{N(1+N)^2(2+N)^2} \\
&+\frac{6(N-1)S_1}{(1+N)(2+N)}\Bigg)\zeta_3\Bigg)\Bigg] \\
&+\xi^2\Bigg[{C_F}^2\Bigg(\frac{1+N}{N^2}+\frac{(-1+2N)S_1}{N^2}-\frac{3S_1^2}{2N}-\frac{5S_2}{2N}\Bigg)
\end{aligned}
$$

$$+C_A C_F\left(\frac{1+2N}{4N^2}+\frac{S_1}{2N^2}+\frac{3S_1^2}{4N}+\frac{5S_2}{4N}-\frac{\zeta_2}{4N}\right)\Bigg], \tag{151}$$

$$a_{qq}^{(2,0),\rm PS} = C_F N_F T_F\Bigg[-\frac{8S_1 P_{456}}{(N-1)^2N^3(1+N)^3(2+N)^2} +\frac{4P_{574}}{(N-1)^3N^4(1+N)^4(2+N)^3} +\frac{8\left(2+N+N^2\right)^2 S_1^2}{(N-1)N^2(1+N)^2(2+N)}+\frac{16\left(2+N+N^2\right)^2 S_2}{(N-1)N^2(1+N)^2(2+N)} -\frac{4\left(2+N+N^2\right)^2\zeta_2}{(N-1)N^2(1+N)^2(2+N)}\Bigg], \tag{152}$$

$$a_{qq}^{(2,1),\rm PS} = C_F N_F T_F\Bigg[\frac{4S_1^2 P_{456}}{(N-1)^2N^3(1+N)^3(2+N)^2} +\frac{8S_2 P_{456}}{(N-1)^2N^3(1+N)^3(2+N)^2} +\frac{2P_{606}}{(N-1)^4N^5(1+N)^5(2+N)^4}+\Bigg(-\frac{4P_{574}}{(N-1)^3N^4(1+N)^4(2+N)^3} -\frac{16\left(2+N+N^2\right)^2 S_2}{(N-1)N^2(1+N)^2(2+N)}\Bigg)S_1-\frac{8\left(2+N+N^2\right)^2 S_1^3}{3(N-1)N^2(1+N)^2(2+N)} -\frac{52\left(2+N+N^2\right)^2 S_3}{3(N-1)N^2(1+N)^2(2+N)}+\Bigg(-\frac{2P_{456}}{(N-1)^2N^3(1+N)^3(2+N)^2} +\frac{4\left(2+N+N^2\right)^2 S_1}{(N-1)N^2(1+N)^2(2+N)}\Bigg)\zeta_2+\frac{64\left(2+N+N^2\right)^2\zeta_3}{3(N-1)N^2(1+N)^2(2+N)}\Bigg], \tag{153}$$

$$b_{qq}^{(2,0),\rm PS} = C_F N_F T_F\Bigg[\frac{16P_{226}}{N^3(1+N)^3(2+N)^2}-\frac{32\left(2+N+N^2\right)S_1}{N^2(1+N)^2(2+N)}\Bigg], \tag{154}$$

$$b_{qq}^{(2,1),\rm PS} = C_F N_F T_F\Bigg[-\frac{16S_1 P_{226}}{N^3(1+N)^3(2+N)^2}+\frac{4P_{461}}{N^4(1+N)^4(2+N)^3} +\frac{16\left(2+N+N^2\right)S_1^2}{N^2(1+N)^2(2+N)}+\frac{32\left(2+N+N^2\right)S_2}{N^2(1+N)^2(2+N)} -\frac{8\left(2+N+N^2\right)\zeta_2}{N^2(1+N)^2(2+N)}\Bigg], \tag{155}$$

$$a_{qg}^{(1,0)} = N_F T_F\Bigg[\frac{4P_{194}}{N^2(1+N)^2(2+N)^2}+\frac{4\left(2+N+N^2\right)S_1}{N(1+N)(2+N)}\Bigg], \tag{156}$$

$$a_{qg}^{(1,1)} = N_F T_F\Bigg[-\frac{2S_1 P_{194}}{N^2(1+N)^2(2+N)^2}-\frac{2P_{391}}{N^3(1+N)^3(2+N)^3}$$

$$
-\frac{\big(2+N+N^2\big)S_1^2}{N(1+N)(2+N)}
-\frac{3\big(2+N+N^2\big)S_2}{N(1+N)(2+N)}+\frac{\big(2+N+N^2\big)\zeta_2}{N(1+N)(2+N)}\Biggr], \tag{157}
$$

$$
\begin{aligned}
a_{qg}^{(1,2)} = {} & N_F T_F\Biggl[\frac{S_1^2 P_{194}}{2N^2(1+N)^2(2+N)^2}+\frac{3S_2 P_{194}}{2N^2(1+N)^2(2+N)^2} \\
& +\frac{P_{502}}{N^4(1+N)^4(2+N)^4} \\
& +\Biggl(\frac{P_{391}}{N^3(1+N)^3(2+N)^3}+\frac{3\big(2+N+N^2\big)S_2}{2N(1+N)(2+N)}\Biggr)S_1 \\
& +\frac{\big(2+N+N^2\big)S_1^3}{6N(1+N)(2+N)}+\frac{7\big(2+N+N^2\big)S_3}{3N(1+N)(2+N)} \\
& +\Biggl(-\frac{P_{194}}{2N^2(1+N)^2(2+N)^2}+\frac{\big(-2-N-N^2\big)S_1}{2N(1+N)(2+N)}\Biggr)\zeta_2 \\
& -\frac{7\big(2+N+N^2\big)\zeta_3}{3N(1+N)(2+N)}\Biggr],
\end{aligned} \tag{158}
$$

$$
\begin{aligned}
a_{qg}^{(2,0)} = {} & C_A N_F T_F \xi^2\Biggl[\frac{P_{193}}{N^2(1+N)^2(2+N)^2}-\frac{\big(2+N+N^2\big)S_1}{N(1+N)(2+N)}\Biggr] \\
& +C_A N_F T_F \xi\Biggl[\frac{4S_1 P_{207}}{N^2(1+N)^2(2+N)^2}-\frac{4P_{317}}{N^3(1+N)^2(2+N)^2} \\
& +\frac{6\big(2+N+N^2\big)S_1^2}{N(1+N)(2+N)}+\frac{10\big(2+N+N^2\big)S_2}{N(1+N)(2+N)}\Biggr] \\
& +{N_F}^2{T_F}^2\Biggl[-\frac{16P_{256}}{27N^2(1+N)^2(2+N)^2}-\frac{80\big(2+N+N^2\big)S_1}{9N(1+N)(2+N)} \\
& +\frac{8\big(2+N+N^2\big)\zeta_2}{3N(1+N)(2+N)}\Biggr] \\
& +C_F N_F T_F\Biggl[\frac{4S_2 P_{250}}{N^2(1+N)^2(2+N)^2}+\frac{4S_1^2 P_{252}}{N^2(1+N)^2(2+N)^2} \\
& +\frac{2P_{536}}{N^4(1+N)^4(2+N)^3} \\
& +\Biggl(-\frac{4P_{462}}{N^3(1+N)^3(2+N)^3}-\frac{28\big(2+N+N^2\big)S_2}{N(1+N)(2+N)}\Biggr)S_1 \\
& -\frac{28\big(2+N+N^2\big)S_1^3}{3N(1+N)(2+N)}+\frac{64\big(2+N+N^2\big)S_3}{3N(1+N)(2+N)}-\frac{16\big(2+N+N^2\big)S_{2,1}}{N(1+N)(2+N)}
\end{aligned}
$$

$$
\begin{aligned}
&+\Biggl(-\frac{2\bigl(2+N+N^2\bigr)\bigl(2+3N+3N^2\bigr)}{N^2(1+N)^2(2+N)}+\frac{8\bigl(2+N+N^2\bigr)S_1}{N(1+N)(2+N)}\Biggr)\zeta_2\\
&-\frac{48\bigl(2+N+N^2\bigr)\zeta_3}{N(1+N)(2+N)}\Biggr]\\
&+C_A N_F T_F\Biggl[\frac{8S_1^2P_{107}}{(N-1)N^2(1+N)^2(2+N)^2}\\
&+\frac{8S_2P_{398}}{(N-1)N^2(1+N)^2(2+N)^2(3+N)}\\
&+\frac{4P_{604}}{27(N-1)^3N^4(1+N)^4(2+N)^4(3+N)}\\
&+\Biggl(-\frac{4\bigl(114+87N+70N^2+17N^3\bigr)S_2}{N(1+N)(2+N)(3+N)}\\
&+\frac{4P_{564}}{9(N-1)^2N^3(1+N)^3(2+N)^3(3+N)}\Biggr)S_1-\frac{28\bigl(2+N+N^2\bigr)S_1^3}{3N(1+N)(2+N)}\\
&-\frac{8\bigl(258+203N+163N^2+40N^3\bigr)S_3}{3N(1+N)(2+N)(3+N)}\\
&+\Biggl(-\frac{16P_{76}}{N(1+N)^2(2+N)^2(3+N)}-\frac{32\bigl(2+N+N^2\bigr)S_1}{N(1+N)(2+N)}\Biggr)S_{-2}\\
&-\frac{40\bigl(2+N+N^2\bigr)S_{-3}}{N(1+N)(2+N)}+\frac{16\bigl(12+6N+5N^2+N^3\bigr)S_{2,1}}{N(1+N)(2+N)(3+N)}\\
&+\frac{16\bigl(2+N+N^2\bigr)S_{-2,1}}{N(1+N)(2+N)}+\Biggl(-\frac{2\bigl(2+N+N^2\bigr)P_{115}}{3(N-1)N^2(1+N)^2(2+N)^2}\\
&+\frac{8\bigl(2+N+N^2\bigr)S_1}{N(1+N)(2+N)}\Biggr)\times\zeta_2+\frac{24\bigl(6+7N+3N^2\bigr)\zeta_3}{N(2+N)(3+N)}\Biggr],
\end{aligned}
\tag{159}
$$

$$
\begin{aligned}
a_{qg}^{(2,1)}=\;&C_A N_F T_F\xi^2\Biggl[-\frac{S_1P_{193}}{2N^2(1+N)^2(2+N)^2}\\
&+\frac{P_{451}}{2N^3(1+N)^3(2+N)^3}+\frac{\bigl(2+N+N^2\bigr)S_1^2}{4N(1+N)(2+N)}\\
&+\frac{3\bigl(2+N+N^2\bigr)S_2}{4N(1+N)(2+N)}+\frac{\bigl(-2-N-N^2\bigr)\zeta_2}{2N(1+N)(2+N)}\Biggr]\\
&+C_A N_F T_F\xi\Biggl[\frac{S_2P_{190}}{N^2(1+N)^2(2+N)^2}\\
&+\frac{S_1^2P_{195}}{N^2(1+N)^2(2+N)^2}+\frac{P_{499}}{N^4(1+N)^3(2+N)^3}
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg(\frac{2P_{458}}{N^3(1+N)^3(2+N)^3}-\frac{13\big(2+N+N^2\big)S_2}{N(1+N)(2+N)}\Bigg)S_1 \\
&-\frac{7\big(2+N+N^2\big)S_1^3}{3N(1+N)(2+N)}-\frac{38\big(2+N+N^2\big)S_3}{3N(1+N)(2+N)} \\
&+\Bigg(\frac{2(N-1)\big(2+N+N^2\big)}{N^2(1+N)(2+N)}+\frac{2\big(2+N+N^2\big)S_1}{N(1+N)(2+N)}\Bigg)\zeta_2 \\
&+\frac{12\big(2+N+N^2\big)\zeta_3}{N(1+N)(2+N)}\Bigg] \\
&+N_F{}^2T_F{}^2\Bigg[\frac{8S_1P_{256}}{27N^2(1+N)^2(2+N)^2} \\
&+\frac{8P_{477}}{81N^3(1+N)^3(2+N)^3}+\frac{20\big(2+N+N^2\big)S_1^2}{9N(1+N)(2+N)} \\
&+\frac{20\big(2+N+N^2\big)S_2}{3N(1+N)(2+N)} \\
&+\Bigg(-\frac{4(3+N)P_{100}}{9N^2(1+N)^2(2+N)^2}-\frac{4\big(2+N+N^2\big)S_1}{3N(1+N)(2+N)}\Bigg)\zeta_2 \\
&-\frac{56\big(2+N+N^2\big)\zeta_3}{9N(1+N)(2+N)}\Bigg]+C_FN_FT_F\Bigg(-\frac{16S_{2,1}P_{208}}{N^2(1+N)^2(2+N)^2(3+N)} \\
&-\frac{2P_{259}}{3N^2(1+N)^2(2+N)^2}S_1^3-\frac{2P_{344}}{3N^2(1+N)^2(2+N)^2(3+N)}S_3 \\
&+\frac{P_{597}}{N^5(1+N)^5(2+N)^4(3+N)}+\frac{P_{509}}{N^3(1+N)^3(2+N)^3(3+N)}S_2 \\
&+\Bigg(-\frac{2S_2P_{347}}{N^2(1+N)^2(2+N)^2(3+N)}-\frac{2P_{575}}{N^4(1+N)^4(2+N)^4(3+N)} \\
&+\frac{4\big(2+N+N^2\big)S_3}{N(1+N)(2+N)}+\frac{16\big(2+N+N^2\big)S_{2,1}}{N(1+N)(2+N)}\Bigg)S_1+\Bigg(\frac{P_{473}}{N^3(1+N)^3(2+N)^3} \\
&+\frac{19\big(2+N+N^2\big)S_2}{N(1+N)(2+N)}\Bigg)S_1^2+\frac{5\big(2+N+N^2\big)S_1^4}{2N(1+N)(2+N)}+\frac{15\big(2+N+N^2\big)S_2^2}{2N(1+N)(2+N)} \\
&-\frac{57\big(2+N+N^2\big)S_4}{N(1+N)(2+N)}+\frac{48\big(2+N+N^2\big)S_{3,1}}{N(1+N)(2+N)}-\frac{8\big(2+N+N^2\big)S_{2,1,1}}{N(1+N)(2+N)} \\
&+\Bigg(\frac{2P_{245}}{N^2(1+N)^2(2+N)^2}S_1+\frac{P_{394}}{N^3(1+N)^3(2+N)^2}-\frac{6\big(2+N+N^2\big)S_1^2}{N(1+N)(2+N)} \\
&-\frac{2\big(2+N+N^2\big)S_2}{N(1+N)(2+N)}\Bigg)\zeta_2+\frac{72\big(2+N+N^2\big)\zeta_2^2}{5N(1+N)(2+N)}
\end{aligned}
$$

$$+\Biggl(-\frac{16P_{339}}{3N^2(1+N)^2(2+N)^2(3+N)}$$
$$+\frac{16\bigl(2+N+N^2\bigr)S_1}{3N(1+N)(2+N)}\Biggr)\zeta_3\Biggr)$$
$$+C_A N_F T_F\Biggl(-\frac{4S_1^3 P_{131}}{3(N-1)N^2(1+N)^2(2+N)^2}$$
$$+\frac{8P_{324}S_{2,1}}{N^2(1+N)^2(2+N)^2(3+N)^2}$$
$$-\frac{4S_3 P_{466}}{3(N-1)N^2(1+N)^2(2+N)^2(3+N)^2}$$
$$+\frac{P_{571}S_2}{3(N-1)^2N^3(1+N)^3(2+N)^3(3+N)^2}$$
$$-\frac{2P_{619}}{81(N-1)^4N^5(1+N)^5(2+N)^5(3+N)^2}$$
$$-\Biggl(\frac{4P_{449}S_2}{(N-1)N^2(1+N)^2(2+N)^2(3+N)^2}$$
$$-\frac{8\bigl(102+73N+59N^2+14N^3\bigr)S_3}{N(1+N)(2+N)(3+N)}$$
$$+\frac{2P_{607}}{27(N-1)^3N^4(1+N)^4(2+N)^4(3+N)^2}$$
$$+\frac{8\bigl(18+7N+6N^2+N^3\bigr)S_{2,1}}{N(1+N)(2+N)(3+N)}\Biggr)S_1$$
$$+\Biggl(\frac{P_{555}}{9(N-1)^2N^3(1+N)^3(2+N)^3(3+N)}$$
$$+\frac{\bigl(246+181N+146N^2+35N^3\bigr)S_2}{N(1+N)(2+N)(3+N)}\Biggr)$$
$$\times S_1^2+\frac{5\bigl(2+N+N^2\bigr)S_1^4}{2N(1+N)(2+N)}+\frac{\bigl(426+387N+308N^2+79N^3\bigr)S_2^2}{2N(1+N)(2+N)(3+N)}$$
$$+\frac{\bigl(1182+929N+746N^2+183N^3\bigr)S_4}{N(1+N)(2+N)(3+N)}$$
$$+\Biggl(\frac{32S_1P_{206}}{N^2(1+N)^2(2+N)^2(3+N)}$$
$$+\frac{8P_{503}}{N^3(1+N)^3(2+N)^3(3+N)^2}+\frac{16\bigl(2+N+N^2\bigr)S_1^2}{N(1+N)(2+N)}$$
$$+\frac{32\bigl(2+N+N^2\bigr)S_2}{N(1+N)(2+N)}\Biggr)S_{-2}+\frac{8\bigl(2+N+N^2\bigr)S_{-2}^2}{N(1+N)(2+N)}$$

$$+\Biggl(\frac{8P_{249}}{N^2(1+N)^2(2+N)^2(3+N)}+\frac{64\bigl(2+N+N^2\bigr)S_1}{N(1+N)(2+N)}\Biggr)$$
$$\times S_{-3}+\frac{84\bigl(2+N+N^2\bigr)S_{-4}}{N(1+N)(2+N)}-\frac{4\bigl(102+53N+44N^2+9N^3\bigr)S_{3,1}}{N(1+N)(2+N)(3+N)}$$
$$+\frac{16\bigl(-4-12N-5N^2+N^3\bigr)S_{-2,1}}{N^2(1+N)^2(2+N)^2}-\frac{8\bigl(2+N+N^2\bigr)S_{-2,2}}{N(1+N)(2+N)}$$
$$-\frac{24\bigl(2+N+N^2\bigr)S_{-3,1}}{N(1+N)(2+N)}-\frac{4(N-1)\bigl(6+3N+N^2\bigr)S_{2,1,1}}{N(1+N)(2+N)(3+N)}$$
$$-\frac{16\bigl(2+N+N^2\bigr)}{N(1+N)(2+N)}\times S_{-2,1,1}+\Biggl(\frac{P_{336}S_1}{3(N-1)N^2(1+N)^2(2+N)^2}$$
$$+\frac{P_{542}}{9(N-1)^2N^3(1+N)^3(2+N)^3}$$
$$-\frac{6\bigl(2+N+N^2\bigr)S_1^2}{N(1+N)(2+N)}-\frac{14\bigl(2+N+N^2\bigr)S_2}{N(1+N)(2+N)}$$
$$-\frac{4\bigl(2+N+N^2\bigr)S_{-2}}{N(1+N)(2+N)}\Biggr)\zeta_2$$
$$-\frac{36\bigl(6+7N+3N^2\bigr)\zeta_2^2}{5N(2+N)(3+N)}+\Biggl(\frac{2P_{485}}{9(N-1)N^2(1+N)^2(2+N)^2(3+N)^2}$$
$$-\frac{8\bigl(96+116N+91N^2+25N^3\bigr)S_1}{3N(1+N)(2+N)(3+N)}\Biggr)\zeta_3\Biggr), \tag{160}$$

$$b_{qg}^{(1,0)}=\frac{16N_FT_F}{(1+N)(2+N)}, \tag{161}$$

$$b_{qg}^{(1,1)}=N_FT_F\Biggl[\frac{8\bigl(-4-N+N^2\bigr)}{(1+N)^2(2+N)^2}-\frac{8S_1}{(1+N)(2+N)}\Biggr], \tag{162}$$

$$b_{qg}^{(1,2)}=N_FT_F\Biggl[-\frac{8\bigl(-8-5N+3N^2+2N^3\bigr)}{(1+N)^3(2+N)^3}-\frac{4\bigl(-4-N+N^2\bigr)S_1}{(1+N)^2(2+N)^2}$$
$$+\frac{2S_1^2}{(1+N)(2+N)}$$
$$+\frac{6S_2}{(1+N)(2+N)}-\frac{2\zeta_2}{(1+N)(2+N)}\Biggr], \tag{163}$$

$$b_{qg}^{(2,0)}=-\frac{4C_AN_FT_F\xi^2}{(1+N)(2+N)}+C_AN_FT_F\xi\Biggl[\frac{2P_{253}}{(N-1)^2N(1+N)^2(2+N)^2}$$
$$+\frac{8\bigl(-2-3N+N^2\bigr)S_1}{(N-1)N(1+N)(2+N)}\Biggr]-\frac{320{N_F}^2{T_F}^2}{9(1+N)(2+N)}$$
$$+C_FN_FT_F\Biggl(-\frac{16P_{223}}{N^2(1+N)^3(2+N)^2}+\frac{32\bigl(4-4N+11N^2+9N^3\bigr)S_1}{N(1+N)^2(2+N)^2}$$

$$
-\frac{96S_1^2}{(1+N)(2+N)}-\frac{32S_2}{(1+N)(2+N)}\Biggr)+C_AN_FT_F
$$
$$
\times\Biggl[\frac{8P_{481}}{9N^3(1+N)^3(2+N)^3(3+N)}+\Biggl(-\frac{16(-2+N)P_{68}}{N^2(1+N)^2(2+N)^2(3+N)}
$$
$$
-\frac{16(N-1)S_2}{(1+N)(2+N)(3+N)}\Biggr)S_1-\frac{16(9+N)S_2}{(1+N)(2+N)(3+N)}
$$
$$
+\frac{16(7+N)S_3}{(1+N)(2+N)(3+N)}
$$
$$
+\Biggl(-\frac{32\bigl(6+7N+6N^2+N^3\bigr)}{N(1+N)^2(2+N)(3+N)}+\frac{64S_1}{(1+N)(2+N)}\Biggr)
$$
$$
\times S_{-2}+\frac{32S_{-3}}{(1+N)(2+N)}+\frac{32(N-1)S_{2,1}}{(1+N)(2+N)(3+N)}-\frac{64S_{-2,1}}{(1+N)(2+N)}
$$
$$
-\frac{192\zeta_3}{(2+N)(3+N)}\Biggr], \tag{164}
$$

$$
b_{qg}^{(2,1)}=C_AN_FT_F\xi^2\Biggl[\frac{2\bigl(8+7N+N^2\bigr)}{(1+N)^2(2+N)^2}+\frac{2S_1}{(1+N)(2+N)}\Biggr]
$$
$$
+C_AN_FT_F\xi\Biggl[-\frac{2S_1P_{251}}{(N-1)^2N(1+N)^2(2+N)^2}
$$
$$
+\frac{P_{539}}{2(N-1)^3N^3(1+N)^3(2+N)^3}
$$
$$
-\frac{8\bigl(-1-2N+N^2\bigr)S_1^2}{(N-1)N(1+N)(2+N)}-\frac{4\bigl(-4-7N+3N^2\bigr)S_2}{(N-1)N(1+N)(2+N)}
$$
$$
-\frac{2\bigl(2+N+N^2\bigr)\zeta_2}{(N-1)N(1+N)(2+N)}\Biggr]+{N_F}^2{T_F}^2\Biggl(\frac{32\bigl(116+99N+13N^2\bigr)}{27(1+N)^2(2+N)^2}
$$
$$
+\frac{160S_1}{9(1+N)(2+N)}-\frac{16\zeta_2}{3(1+N)(2+N)}\Biggr)+C_FN_FT_F
$$
$$
\times\Biggl(-\frac{8P_{505}}{N^3(1+N)^4(2+N)^3(3+N)}-\frac{16S_2P_{130}}{N(1+N)^2(2+N)^2(3+N)}
$$
$$
+\Biggl(\frac{16P_{404}}{N^2(1+N)^3(2+N)^3(3+N)}+\frac{16(17+11N)S_2}{(1+N)(2+N)(3+N)}\Biggr)S_1
$$
$$
-\frac{32\bigl(2-5N+6N^2+6N^3\bigr)S_1^2}{N(1+N)^2(2+N)^2}+\frac{112S_1^3}{3(1+N)(2+N)}
$$
$$
-\frac{64(15+N)S_3}{3(1+N)(2+N)(3+N)}
$$
$$
-\frac{64(-5+N)S_{2,1}}{(1+N)(2+N)(3+N)}+\Bigl(\frac{8\bigl(2+3N+3N^2\bigr)}{N(1+N)^2(2+N)}-\frac{32S_1}{(1+N)(2+N)}\Bigr)\zeta_2
$$

$$
+\frac{192(1+3N)\zeta_3}{(1+N)(2+N)(3+N)}\Biggr)+C_A N_F T_F\Biggl(\frac{4\zeta_2 P_{122}}{3N^2(1+N)^2(2+N)^2}
+\frac{8S_3 P_{214}}{N(1+N)^2(2+N)^2(3+N)^2}+\frac{4S_2 P_{327}}{N^2(1+N)^2(2+N)^2(3+N)^2}
-\frac{2P_{582}}{27N^4(1+N)^4(2+N)^4(3+N)^2}+\Biggl(-\frac{8P_{511}}{9N^3(1+N)^3(2+N)^3(3+N)^2}
-\frac{16\bigl(-64-47N-3N^2+2N^3\bigr)S_2}{(1+N)(2+N)^2(3+N)^2}+\frac{32(-3+N)S_3}{(1+N)(2+N)(3+N)}
-\frac{64S_{2,1}}{(2+N)(3+N)}\Biggr)S_1+\Biggl(\frac{4P_{235}}{N^2(1+N)^2(2+N)^2(3+N)}
+\frac{4(3+5N)S_2}{(1+N)(2+N)(3+N)}\Biggr)S_1^2-\frac{8(5+3N)S_2^2}{(1+N)(2+N)(3+N)}
-\frac{4(85+19N)S_4}{(1+N)(2+N)(3+N)}+\Biggl(\frac{32\bigl(12+2N+N^2+4N^3+N^4\bigr)S_1}{N(1+N)^2(2+N)^2(3+N)}
+\frac{32P_{314}}{N^2(1+N)^3(2+N)^2(3+N)^2}-\frac{32S_1^2}{(1+N)(2+N)}
-\frac{64S_2}{(1+N)(2+N)}\Biggr)S_{-2}-\frac{16S_{-2}^2}{(1+N)(2+N)}
+\Biggl(\frac{32P_{78}}{N(1+N)^2(2+N)^2(3+N)}-\frac{128S_1}{(1+N)(2+N)}\Biggr)S_{-3}
-\frac{112S_{-4}}{(1+N)(2+N)}+\frac{16\bigl(-74-43N+8N^2+5N^3\bigr)S_{2,1}}{(1+N)(2+N)^2(3+N)^2}
+\frac{96S_{-3,1}}{(1+N)(2+N)}
-\frac{16(-1+5N)S_{3,1}}{(1+N)(2+N)(3+N)}+\frac{64(3+2N)S_{-2,1}}{(1+N)^2(2+N)^2}+\frac{32S_{-2,2}}{(1+N)(2+N)}
+\frac{64S_{2,1,1}}{(1+N)(3+N)}+\frac{64S_{-2,1,1}}{(1+N)(2+N)}+\frac{288\zeta_2^2}{5(2+N)(3+N)}
+\Biggl(-\frac{48P_{102}}{N(1+N)(2+N)^2(3+N)^2}+\frac{48(N-1)S_1}{(1+N)(2+N)(3+N)}\Biggr)\zeta_3\Biggr), \qquad (165)
$$

$$
c_{qg}^{(2,0)}=C_A N_F T_F\Biggl[\frac{8S_1 P_{323}}{(N-1)^2N^3(1+N)^2(2+N)^2}
-\frac{4P_{531}}{(N-1)^3N^4(1+N)^3(2+N)^3}
-\frac{8\bigl(2+N+N^2\bigr)S_1^2}{(N-1)N^2(1+N)(2+N)}-\frac{16\bigl(2+N+N^2\bigr)S_2}{(N-1)N^2(1+N)(2+N)}
$$

$$+\frac{4\big(2+N+N^2\big)\zeta_2}{(N-1)N^2(1+N)(2+N)}\Bigg], \tag{166}$$

$$\begin{aligned}
c_{qg}^{(2,1)} = {} & C_A N_F T_F \Bigg[-\frac{4S_1^2 P_{323}}{(N-1)^2N^3(1+N)^2(2+N)^2} \\
& -\frac{8S_2 P_{323}}{(N-1)^2N^3(1+N)^2(2+N)^2} \\
& +\frac{P_{598}}{(N-1)^4N^5(1+N)^4(2+N)^4}+\Bigg(\frac{4P_{531}}{(N-1)^3N^4(1+N)^3(2+N)^3} \\
& +\frac{16\big(2+N+N^2\big)S_2}{(N-1)N^2(1+N)(2+N)}\Bigg)S_1+\frac{8\big(2+N+N^2\big)S_1^3}{3(N-1)N^2(1+N)(2+N)} \\
& +\frac{52\big(2+N+N^2\big)S_3}{3(N-1)N^2(1+N)(2+N)}+\Bigg(\frac{2P_{323}}{(N-1)^2N^3(1+N)^2(2+N)^2} \\
& -\frac{4\big(2+N+N^2\big)S_1}{(N-1)N^2(1+N)(2+N)}\Bigg)\zeta_2-\frac{64\big(2+N+N^2\big)\zeta_3}{3(N-1)N^2(1+N)(2+N)}\Bigg],
\end{aligned} \tag{167}$$

$$a_{gq}^{(1,0)} = C_F\Bigg[\frac{2P_{192}}{(N-1)^2N^2(1+N)^2}+\frac{2\big(2+N+N^2\big)S_1}{(N-1)N(1+N)}\Bigg], \tag{168}$$

$$\begin{aligned}
a_{gq}^{(1,1)} = {} & C_F\Bigg[-\frac{S_1P_{192}}{(N-1)^2N^2(1+N)^2}+\frac{P_{447}}{(N-1)^3N^3(1+N)^3}-\frac{\big(2+N+N^2\big)S_1^2}{2(N-1)N(1+N)} \\
& -\frac{3\big(2+N+N^2\big)S_2}{2(N-1)N(1+N)}+\frac{\big(2+N+N^2\big)\zeta_2}{2(N-1)N(1+N)}\Bigg],
\end{aligned} \tag{169}$$

$$\begin{aligned}
a_{gq}^{(1,2)} = {} & C_F\Bigg[\frac{S_1^2P_{192}}{4(N-1)^2N^2(1+N)^2}+\frac{3S_2P_{192}}{4(N-1)^2N^2(1+N)^2} \\
& +\frac{P_{529}}{2(N-1)^4N^4(1+N)^4} \\
& +\Bigg(-\frac{P_{447}}{2(N-1)^3N^3(1+N)^3}+\frac{3\big(2+N+N^2\big)S_2}{4(N-1)N(1+N)}\Bigg)S_1 \\
& +\frac{\big(2+N+N^2\big)S_1^3}{12(N-1)N(1+N)}+\frac{7\big(2+N+N^2\big)S_3}{6(N-1)N(1+N)} \\
& +\Bigg(-\frac{P_{192}}{4(N-1)^2N^2(1+N)^2}-\frac{\big(2+N+N^2\big)S_1}{4(N-1)N(1+N)}\Bigg)\zeta_2 \\
& -\frac{7\big(2+N+N^2\big)\zeta_3}{6(N-1)N(1+N)}\Bigg],
\end{aligned} \tag{170}$$

$$a_{gq}^{(2,0)} = \xi\Bigg[{C_F}^2\Bigg(-\frac{2S_1P_{200}}{(N-1)^2N^2(1+N)^2}$$

$$
-\frac{2P_{320}}{(N-1)^2N^3(1+N)^2}+\frac{3\big(2+N+N^2\big)S_1^2}{(N-1)N(1+N)}
$$
$$
+\frac{5\big(2+N+N^2\big)S_2}{(N-1)N(1+N)}\Bigg)+C_AC_F\Bigg(-\frac{P_{50}}{2(-2+N)(N-1)^3N^2}-\frac{2S_1}{(N-1)^2N}
$$
$$
+\frac{2(-3+N)S_{-2}}{(-2+N)(N-1)(1+N)}\Bigg)\Bigg]+C_F\Bigg[N_FT_F\Bigg(-\frac{8S_1P_{220}}{9(N-1)^2N^2(1+N)^2}
$$
$$
-\frac{8\big(2+N+N^2\big)S_2}{3(N-1)N(1+N)}-\frac{8P_{464}}{27(N-1)^3N^3(1+N)^3}+\frac{4\big(2+N+N^2\big)\zeta_2}{3(N-1)N(1+N)}\Bigg)
$$
$$
+C_A\Bigg(\frac{4S_2P_{254}}{3(N-1)N^2(1+N)^2(2+N)}+\frac{2S_1^2P_{334}}{(N-1)^2N^2(1+N)^2(2+N)}
$$
$$
+\frac{P_{603}}{27(-2+N)(N-1)^4N^4(1+N)^4(2+N)^3}
$$
$$
+\Bigg(-\frac{2P_{540}}{9(N-1)^3N^3(1+N)^3(2+N)^2}
$$
$$
-\frac{2\big(12+28N+23N^2+9N^3\big)S_2}{(N-1)N(1+N)(2+N)}\Bigg)S_1-\frac{4\big(-32-8N-3N^2+N^3\big)S_3}{3(N-1)N(1+N)(2+N)}
$$
$$
-\frac{14\big(2+N+N^2\big)S_1^3}{3(N-1)N(1+N)}
$$
$$
+\Bigg(\frac{4P_{62}}{(-2+N)(N-1)N(1+N)^2}-\frac{16\big(2+N+N^2\big)S_1}{(N-1)N(1+N)}\Bigg)
$$
$$
\times S_{-2}-\frac{20\big(2+N+N^2\big)S_{-3}}{(N-1)N(1+N)}-\frac{16\big(6+2N+N^2\big)S_{2,1}}{(N-1)N(1+N)(2+N)}
$$
$$
+\frac{8\big(2+N+N^2\big)S_{-2,1}}{(N-1)N(1+N)}
$$
$$
+\Bigg(-\frac{\big(2+N+N^2\big)P_{115}}{3(N-1)^2N^2(1+N)^2(2+N)}+\frac{4\big(2+N+N^2\big)S_1}{(N-1)N(1+N)}\Bigg)\zeta_2
$$
$$
-\frac{12\big(12+8N+3N^2\big)\zeta_3}{N(1+N)(2+N)}\Bigg)\Bigg]+C_F{}^2\Bigg[\frac{2S_1^2P_{97}}{(N-1)N^2(1+N)^2}
$$
$$
+\frac{2S_2P_{108}}{(N-1)N^2(1+N)^2}+\frac{P_{507}}{(N-1)^2N^4(1+N)^4}
$$
$$
+\Bigg(\frac{4P_{335}}{(N-1)^2N^3(1+N)^3}-\frac{2\big(38+23N+15N^2\big)S_2}{(N-1)N(1+N)}\Bigg)S_1
$$
$$
-\frac{14\big(2+N+N^2\big)S_1^3}{3(N-1)N(1+N)}-\frac{4\big(74+43N+31N^2\big)S_3}{3(N-1)N(1+N)}+\frac{32S_{2,1}}{(N-1)N}
$$
$$
+\Bigg(-\frac{\big(2+N+N^2\big)\big(2+3N+3N^2\big)}{(N-1)N^2(1+N)^2}+\frac{4\big(2+N+N^2\big)S_1}{(N-1)N(1+N)}\Bigg)\zeta_2
$$

$$+\frac{48\zeta_3}{1+N}\Bigg], \tag{171}$$

$$a_{gq}^{(2,1)} = \xi\Bigg[C_A C_F\Bigg(\frac{S_1 P_{50}}{2(-2+N)(N-1)^3N^2} + \frac{P_{397}}{8(-2+N)^2(N-1)^4N^3}$$
$$+\frac{S_1^2+2S_2}{(N-1)^2N}$$
$$+\frac{(3-N)S_3}{(-2+N)(N-1)(1+N)} + \Bigg(\frac{P_{60}}{(-2+N)^2(N-1)N(1+N)^2}$$
$$-\frac{2(-3+N)S_1}{(-2+N)(N-1)(1+N)}\Bigg)S_{-2}$$
$$-\frac{4(-3+N)S_{-3}}{(-2+N)(N-1)(1+N)} - \frac{\zeta_2}{2(N-1)^2N}$$
$$-\frac{3(-3+N)\zeta_3}{(-2+N)(N-1)(1+N)}\Bigg) + {C_F}^2\Bigg(\frac{S_1^2 P_{224}}{2(N-1)^2N^2(1+N)^2}$$
$$+\frac{S_2 P_{238}}{2(N-1)^2N^2(1+N)^2} + \frac{P_{500}}{(N-1)^3N^4(1+N)^3} + \Bigg(\frac{P_{459}}{(N-1)^3N^3(1+N)^3}$$
$$-\frac{13(2+N+N^2)S_2}{2(N-1)N(1+N)}\Bigg)S_1 - \frac{7(2+N+N^2)S_1^3}{6(N-1)N(1+N)} - \frac{19(2+N+N^2)S_3}{3(N-1)N(1+N)}$$
$$+\Bigg(\frac{-2-N-N^2}{(N-1)N^2(1+N)} + \frac{(2+N+N^2)S_1}{(N-1)N(1+N)}\Bigg)\zeta_2 + \frac{6(2+N+N^2)\zeta_3}{(N-1)N(1+N)}\Bigg)\Bigg]$$
$$+C_F\Bigg(N_F T_F\Bigg(-\frac{2S_2P_{215}}{9(N-1)^2N^2(1+N)^2} + \frac{2S_1^2P_{220}}{9(N-1)^2N^2(1+N)^2}$$
$$-\frac{4P_{562}}{81(N-1)^4N^4(1+N)^4}$$
$$+\Bigg(\frac{4P_{464}}{27(N-1)^3N^3(1+N)^3} + \frac{4(2+N+N^2)S_2}{3(N-1)N(1+N)}\Bigg)S_1$$
$$+\frac{16(2+N+N^2)S_3}{3(N-1)N(1+N)} + \Bigg(\frac{2P_{216}}{9(N-1)^2N^2(1+N)^2} - \frac{2(2+N+N^2)S_1}{3(N-1)N(1+N)}\Bigg)\zeta_2$$
$$-\frac{100(2+N+N^2)\zeta_3}{9(N-1)N(1+N)}\Bigg) + C_A\Bigg(\frac{S_1^3P_{296}}{3(N-1)^2N^2(1+N)^2(2+N)}$$
$$+\frac{4S_{-2,1}P_{205}}{(N-1)^2N^2(1+N)^2} - \frac{2S_3P_{474}}{3(-2+N)(N-1)^2N^2(1+N)^2(2+N)^2}$$
$$-\frac{4S_{2,1}P_{332}}{(N-1)^2N^2(1+N)^2(2+N)^2} + \frac{S_2P_{543}}{18(N-1)^3N^3(1+N)^3(2+N)^2}$$
$$+\frac{P_{618}}{324(-2+N)^2(N-1)^5N^5(1+N)^5(2+N)^4}$$

$$
\begin{aligned}
&+\Biggl(-\frac{2S_2P_{419}}{3(N-1)^2N^2(1+N)^2(2+N)^2}\\
&+\frac{P_{602}}{27(-2+N)(N-1)^4N^4(1+N)^4(2+N)^3}\\
&+\frac{4\bigl(-16+8N+9N^2+5N^3\bigr)S_3}{(N-1)N(1+N)(2+N)}+\frac{4\bigl(28+12N+7N^2+N^3\bigr)S_{2,1}}{(N-1)N(1+N)(2+N)}\Biggr)S_1\\
&+\Biggl(\frac{P_{545}}{18(N-1)^3N^3(1+N)^3(2+N)^2}+\frac{\bigl(20+68N+57N^2+23N^3\bigr)S_2}{2(N-1)N(1+N)(2+N)}\Biggr)S_1^2\\
&+\frac{5\bigl(2+N+N^2\bigr)S_1^4}{4(N-1)N(1+N)}+\frac{\bigl(188+124N+85N^2+23N^3\bigr)S_2^2}{4(N-1)N(1+N)(2+N)}\\
&+\frac{\bigl(-124-12N+5N^2+11N^3\bigr)S_4}{2(N-1)N(1+N)(2+N)}+\Biggl(-\frac{8S_1P_{308}}{(-2+N)(N-1)^2N^2(1+N)^2}\\
&-\frac{4P_{450}}{(-2+N)^2(N-1)^2N^3(1+N)^3}+\frac{8\bigl(2+N+N^2\bigr)S_1^2}{(N-1)N(1+N)}\\
&+\frac{16\bigl(2+N+N^2\bigr)S_2}{(N-1)N(1+N)}\Biggr)S_{-2}\\
&+\frac{4\bigl(2+N+N^2\bigr)S_{-2}^2}{(N-1)N(1+N)}\\
&+\Biggl(-\frac{2P_{329}}{(-2+N)(N-1)^2N^2(1+N)^2}+\frac{32\bigl(2+N+N^2\bigr)S_1}{(N-1)N(1+N)}\Biggr)\\
&\times S_{-3}+\frac{42\bigl(2+N+N^2\bigr)S_{-4}}{(N-1)N(1+N)}+\frac{2\bigl(108+44N+25N^2+3N^3\bigr)S_{3,1}}{(N-1)N(1+N)(2+N)}\\
&-\frac{4\bigl(2+N+N^2\bigr)S_{-2,2}}{(N-1)N(1+N)}-\frac{12\bigl(2+N+N^2\bigr)S_{-3,1}}{(N-1)N(1+N)}\\
&-\frac{2\bigl(36+20N+13N^2+3N^3\bigr)S_{2,1,1}}{(N-1)N(1+N)(2+N)}-\frac{8\bigl(2+N+N^2\bigr)}{(N-1)N(1+N)}\\
&\times S_{-2,1,1}+\Biggl(\frac{S_1P_{350}}{6(N-1)^2N^2(1+N)^2(2+N)}\\
&+\frac{P_{537}}{18(N-1)^3N^3(1+N)^3(2+N)^2}\\
&-\frac{3\bigl(2+N+N^2\bigr)S_1^2}{(N-1)N(1+N)}+\frac{\bigl(-2-N-N^2\bigr)S_2}{(N-1)N(1+N)}-\frac{2\bigl(2+N+N^2\bigr)S_{-2}}{(N-1)N(1+N)}\Biggr)\zeta_2\\
&+\frac{18\bigl(12+8N+3N^2\bigr)\zeta_2^2}{5N(1+N)(2+N)}+\Biggl(\frac{P_{440}}{9(-2+N)(N-1)^2N^2(1+N)^2(2+N)^2}\\
&+\frac{4\bigl(-64+8N+15N^2+11N^3\bigr)S_1}{3(N-1)N(1+N)(2+N)}\Biggr)\zeta_3\Biggr)\Biggr)+C_F^2\Biggl(\frac{S_1^3P_{32}}{3(N-1)N^2(1+N)^2}
\end{aligned}
$$

$$+\frac{S_3 P_{170}}{3(N-1)N^2(1+N)^2(2+N)}+\frac{S_2 P_{465}}{2(N-1)^2N^3(1+N)^3(2+N)}$$
$$+\frac{P_{573}}{2(N-1)^3N^5(1+N)^5}+\Biggl(\frac{S_2 P_{179}}{(N-1)N^2(1+N)^2(2+N)}$$
$$-\frac{2P_{559}}{(N-1)^3N^4(1+N)^4(2+N)}+\frac{2\big(62+43N+19N^2\big)S_3}{(N-1)N(1+N)}-\frac{32S_{2,1}}{(N-1)N}\Biggr)S_1$$
$$+\Biggl(\frac{P_{400}}{2(N-1)^2N^3(1+N)^3}+\frac{\big(86+55N+31N^2\big)S_2}{2(N-1)N(1+N)}\Biggr)S_1^2$$
$$+\frac{5\big(2+N+N^2\big)S_1^4}{4(N-1)N(1+N)}$$
$$+\frac{\big(110+39N+71N^2\big)S_2^2}{4(N-1)N(1+N)}+\frac{\big(286+171N+115N^2\big)S_4}{2(N-1)N(1+N)}$$
$$+\frac{8\big(26+N+3N^2\big)S_{2,1}}{(N-1)N(1+N)(2+N)}-\frac{64S_{3,1}}{(N-1)N}+\frac{16S_{2,1,1}}{(N-1)N}$$
$$+\Biggl(\frac{S_1 P_{92}}{(N-1)N^2(1+N)^2}$$
$$+\frac{P_{319}}{2(N-1)N^3(1+N)^3}-\frac{3\big(2+N+N^2\big)S_1^2}{(N-1)N(1+N)}-\frac{7\big(2+N+N^2\big)S_2}{(N-1)N(1+N)}\Biggr)\zeta_2$$
$$+\Biggl(\frac{2P_{263}}{3(N-1)N^2(1+N)^2(2+N)}-\frac{8\big(16-N+17N^2\big)S_1}{3(N-1)N(1+N)}\Biggr)\zeta_3$$
$$-\frac{72\zeta_2^2}{5(1+N)}\Biggr), \tag{172}$$

$$b_{gq}^{(1,0)}=-\frac{4C_F}{N(1+N)}, \tag{173}$$

$$b_{gq}^{(1,1)}=C_F\Biggl[-\frac{2(N-1)(1+2N)}{N^2(1+N)^2}+\frac{2S_1}{N(1+N)}\Biggr], \tag{174}$$

$$b_{gq}^{(1,2)}=C_F\Biggl[-\frac{(N-1)\big(-1-3N-3N^2+N^3\big)}{N^3(1+N)^3}+\frac{(N-1)(1+2N)S_1}{N^2(1+N)^2}$$
$$-\frac{S_1^2}{2N(1+N)}-\frac{3S_2}{2N(1+N)}+\frac{\zeta_2}{2N(1+N)}\Biggr], \tag{175}$$

$$b_{gq}^{(2,0)}=\xi\Biggl[{C_F}^2\Biggl(-\frac{4P_{221}}{(N-1)^2N^3(1+N)^2}+\frac{4\big(4+3N+N^2\big)S_1}{(N-1)N^2(1+N)}\Biggr)$$
$$+C_AC_F\Biggl(\frac{P_{49}}{(-2+N)(N-1)^2N^3}+\frac{4S_1}{(N-1)N^2}$$

$$
\begin{aligned}
&-\frac{4(N-1)S_{-2}}{(-2+N)N(1+N)}\Biggr)\Biggr]\\
&+C_F\Biggl[N_FT_F\Biggl(-\frac{16\big(-9-19N+2N^2\big)}{9N^2(1+N)^2}+\frac{16S_1}{3N(1+N)}\Biggr)\\
&+C_A\Biggl(-\frac{2P_{475}}{9(-2+N)(N-1)N^3(1+N)^3(2+N)^2}+\Biggl(-\frac{4(-2+N)S_2}{N(1+N)(2+N)}\\
&-\frac{4P_{268}}{3(N-1)N^3(1+N)^2(2+N)}\Biggr)S_1+\frac{24S_1^2}{N(1+N)}\\
&+\frac{4\big(-2+5N+5N^2\big)S_2}{N^2(1+N)(2+N)}\\
&-\frac{4(-2+N)S_3}{N(1+N)(2+N)}+\frac{8(-3+N)S_{-2}}{(-2+N)N(1+N)}+\frac{8(-2+N)S_{2,1}}{N(1+N)(2+N)}\\
&-\frac{24(-2+N)\zeta_3}{N(1+N)(2+N)}\Biggr)\Biggr]+C_F^2\Biggl(\frac{4P_{243}}{(N-1)^2N^2(1+N)^3}\\
&-\frac{4\big(-4+9N+3N^2\big)S_1}{(N-1)N^2(1+N)}+\frac{16S_2}{N(1+N)}\Biggr),
\end{aligned}
\tag{176}
$$

$$
\begin{aligned}
b_{gq}^{(2,1)}=\xi\Biggl[&C_F^2\Biggl(\frac{2S_1P_{240}}{(N-1)^2N^3(1+N)^2}-\frac{2P_{460}}{(N-1)^3N^4(1+N)^3}\\
&-\frac{\big(8+7N+N^2\big)S_1^2}{(N-1)N^2(1+N)}\\
&+\frac{\big(-16-13N-3N^2\big)S_2}{(N-1)N^2(1+N)}+\frac{2\big(2+N+N^2\big)\zeta_2}{(N-1)N^2(1+N)}\Biggr)\\
&+C_AC_F\Biggl(-\frac{S_1P_{49}}{(-2+N)(N-1)^2N^3}+\frac{P_{392}}{4(-2+N)^2(N-1)^3N^4}\\
&-\frac{2S_1^2}{(N-1)N^2}-\frac{4S_2}{(N-1)N^2}\\
&+\frac{2(N-1)S_3}{(-2+N)N(1+N)}+\Biggl(-\frac{2P_{204}}{(-2+N)^2(N-1)N^2(1+N)^2}\\
&+\frac{4(N-1)S_1}{(-2+N)N(1+N)}\Biggr)S_{-2}+\frac{8(N-1)S_{-3}}{(-2+N)N(1+N)}+\frac{\zeta_2}{(N-1)N^2}\\
&+\frac{6(N-1)\zeta_3}{(-2+N)N(1+N)}\Biggr)\Biggr]+C_F\Bigl(N_FT_F\Biggl(-\frac{8P_{139}}{27N^3(1+N)^3}\\
&+\frac{8\big(-9-19N+2N^2\big)S_1}{9N^2(1+N)^2}-\frac{4S_1^2}{3N(1+N)}-\frac{28S_2}{3N(1+N)}+\frac{4\zeta_2}{N(1+N)}\Biggr)
\end{aligned}
$$

$$
\begin{aligned}
&+C_A\Biggl(-\frac{4S_3P_{258}}{3(-2+N)N^2(1+N)^2(2+N)^2}+\frac{S_2P_{365}}{3(N-1)N^3(1+N)^2(2+N)^2}\\
&+\frac{P_{580}}{54(-2+N)^2(N-1)N^4(1+N)^4(2+N)^3}+\Biggl(-\frac{2S_2P_{138}}{N^2(1+N)^2(2+N)^2}\\
&-\frac{2P_{513}}{9(-2+N)(N-1)N^4(1+N)^3(2+N)^2}+\frac{12(-2+N)S_3}{N(1+N)(2+N)}\\
&-\frac{8(N-2)S_{2,1}}{N(1+N)(2+N)}\Biggr)S_1\\
&+\Biggl(\frac{P_{275}}{3(N-1)N^3(1+N)^2(2+N)}+\frac{3(N-2)S_2}{N(1+N)(2+N)}\Biggr)S_1^2\\
&-\frac{28S_1^3}{3N(1+N)}-\frac{2(-2+N)S_2^2}{N(1+N)(2+N)}+\frac{7(-2+N)S_4}{N(1+N)(2+N)}\\
&+\Biggl(\frac{4(-3+N)P_{58}}{(-2+N)^2(N-1)N^2(1+N)^2}-\frac{8(-3+N)S_1}{(-2+N)N(1+N)}\Biggr)S_{-2}\\
&-\frac{16(-3+N)S_{-3}}{(-2+N)N(1+N)}+\frac{8\bigl(-14-23N-3N^2+3N^3\bigr)S_{2,1}}{N(1+N)^2(2+N)^2}\\
&-\frac{16(-2+N)S_{3,1}}{N(1+N)(2+N)}\\
&+\frac{4(-2+N)S_{2,1,1}}{N(1+N)(2+N)}+\Biggl(\frac{P_{34}}{(N-1)N^2(1+N)^2(2+N)}+\frac{8S_1}{N(1+N)}\Biggr)\zeta_2\\
&+\frac{36(-2+N)\zeta_2^2}{5N(1+N)(2+N)}+\Biggl(-\frac{24P_{239}}{(-2+N)N^2(1+N)^2(2+N)^2}\\
&+\frac{12(-2+N)S_1}{N(1+N)(2+N)}\Biggr)\zeta_3\Biggr)\Biggr)+C_F^2\Biggl(\frac{S_2P_{36}}{(N-1)N^2(1+N)^2(2+N)}\\
&+\frac{2P_{472}}{(N-1)^3N^3(1+N)^4}\\
&+\Biggl(-\frac{2P_{341}}{(N-1)^2N^3(1+N)^2(2+N)}-\frac{48S_2}{N(1+N)(2+N)}\Biggr)\\
&\times S_1+\frac{\bigl(-12+7N+32N^2+5N^3\bigr)S_1^2}{(N-1)N^2(1+N)^2}-\frac{24(4+N)S_3}{N(1+N)(2+N)}\\
&-\frac{16(-4+N)S_{2,1}}{N(1+N)(2+N)}\\
&-\frac{4\bigl(2+N+N^2\bigr)\zeta_2}{(N-1)N(1+N)^2}+\frac{96(N-1)\zeta_3}{N(1+N)(2+N)}\Biggr),
\end{aligned}
\tag{177}
$$

$$
a_{gg}^{(1,0)}=-\frac{C_A\xi^2}{4}+C_A\xi\Biggl(\frac{-1+N}{N}+S_1\Biggr)+C_A\Biggl(-\frac{P_{442}}{9(N-1)^2N^2(1+N)^2(2+N)^2}
$$

$$+\frac{8(1+N+N^2)S_1}{(N-1)N(1+N)(2+N)} - 2S_1^2 - 6S_2\Bigg) - \frac{20N_FT_F}{9}, \tag{178}$$

$$a_{gg}^{(1,1)} = \frac{C_A\xi^2}{4} + C_A\xi\Bigg(\frac{1+N}{2N^2} + \frac{(1-N)S_1}{2N} - \frac{1}{4}S_1^2 - \frac{3}{4}S_2\Bigg) + C_A\Bigg(\frac{P_{570}}{27(N-1)^3N^3(1+N)^3(2+N)^3} + \Bigg(\frac{P_{395}}{(N-1)^2N^2(1+N)^2(2+N)^2} + 3S_2\Bigg) \times S_1 - \frac{2(1+N+N^2)S_1^2}{(N-1)N(1+N)(2+N)} + \frac{1}{3}S_1^3 - \frac{6(1+N+N^2)S_2}{(N-1)N(1+N)(2+N)} + \frac{14}{3}S_3 + \Bigg(\frac{P_{115}}{12(N-1)N(1+N)(2+N)} - S_1\Bigg)\zeta_2\Bigg) + N_FT_F\Bigg(\frac{56}{27} - \frac{\zeta_2}{3}\Bigg), \tag{179}$$

$$a_{gg}^{(1,2)} = C_A\xi^2\Bigg[-\frac{1}{4} + \frac{\zeta_2}{32}\Bigg] + C_A\xi\Bigg[\frac{-N-1}{4N^3} + \Bigg(\frac{-1-N}{4N^2} + \frac{3S_2}{8}\Bigg)S_1 + \frac{(N-1)S_1^2}{8N} + \frac{1}{24}S_1^3 + \frac{3(N-1)S_2}{8N} + \frac{7}{12}S_3 + \Bigg(\frac{1-N}{8N} - \frac{S_1}{8}\Bigg)\zeta_2\Bigg] + C_A\Bigg(-\frac{1}{24}S_1^4 + \frac{P_{608}}{162(N-1)^4N^4(1+N)^4(2+N)^4} - \frac{3S_2P_{395}}{4(N-1)^2N^2(1+N)^2(2+N)^2} + \Bigg(\frac{P_{530}}{2(N-1)^3N^3(1+N)^3(2+N)^3} + \frac{3(1+N+N^2)S_2}{(N-1)N(1+N)(2+N)} - \frac{7}{3}S_3\Bigg)S_1 + \Bigg(-\frac{P_{395}}{4(N-1)^2N^2(1+N)^2(2+N)^2} - \frac{3}{4}S_2\Bigg)S_1^2 + \frac{(1+N+N^2)S_1^3}{3(N-1)N(1+N)(2+N)} - \frac{9}{8}S_2^2 + \frac{14(1+N+N^2)S_3}{3(N-1)N(1+N)(2+N)} - \frac{15}{4}S_4 + \Bigg(\frac{P_{442}}{72(N-1)^2N^2(1+N)^2(2+N)^2} + \frac{(-1-N-N^2)S_1}{(N-1)N(1+N)(2+N)} + \frac{1}{4}S_1^2 + \frac{3}{4}S_2\Bigg)\zeta_2$$

$$+\Biggl(-\frac{7P_{115}}{36(N-1)N(1+N)(2+N)}$$
$$+\frac{7}{3}S_1\Biggr)\zeta_3\Biggr)+N_F T_F\Biggl(-\frac{164}{81}+\frac{5\zeta_2}{18}+\frac{7\zeta_3}{9}\Biggr), \tag{180}$$

$$a_{gg}^{(2,0)}=\frac{{C_A}^2\xi^4}{16}+{C_A}^2\xi^3\Biggl[\frac{4-3N}{16N}-\frac{S_1}{4}\Biggr]$$
$$+\xi^2\Biggl[\frac{2C_A N_F T_F}{3}+C_A^2\Biggl(\frac{S_1P_{105}}{4N^2(1+N)(2+N)}$$
$$+\frac{P_{556}}{12(-2+N)(N-1)^2N^3(1+N)^2(2+N)^2(3+N)}+\frac{3}{4}S_1^2+\frac{3}{2}S_2$$
$$+\frac{\bigl(4-N+N^2\bigr)S_{-2}}{4(-2+N)N(1+N)(3+N)}-\frac{1}{8}\zeta_3\Biggr)\Biggr]$$
$$+\xi\Biggl[C_A N_F T_F\Biggl(\frac{2\bigl(18+16N+7N^2\bigr)}{9N^2}$$
$$-\frac{4(-3+8N)S_1}{9N}-\frac{2}{3}S_1^2-\frac{14}{3}S_2\Biggr)+C_A^2\Biggl(\frac{S_3P_{48}}{2N(1+N)(2+N)(3+N)}$$
$$-\frac{S_{2,1}P_{48}}{N(1+N)(2+N)(3+N)}+\frac{S_2P_{359}}{12(N-1)N^2(1+N)(2+N)(3+N)}$$
$$+\frac{P_{577}}{18(-2+N)(N-1)^3N^3(1+N)^2(2+N)^2(3+N)}$$
$$-\frac{3\zeta_3P_{88}}{2N(1+N)(2+N)(3+N)}$$
$$+\Biggl(\frac{S_2P_{21}}{2N(1+N)(2+N)(3+N)}$$
$$+\frac{P_{547}}{18(N-1)^2N^3(1+N)^2(2+N)^2(3+N)}\Biggr)S_1$$
$$+\frac{\bigl(142+323N+2N^2-35N^3\bigr)S_1^2}{12(N-1)(1+N)(2+N)}-4S_1^3$$
$$+\frac{\bigl(-1-15N-N^2+N^3\bigr)S_{-2}}{(-2+N)(N-1)N(1+N)(3+N)}\Biggr)\Biggr]$$
$$+C_F N_F T_F\Biggl(-\frac{8S_1P_{401}}{(N-1)N^3(1+N)^3(2+N)^2}$$
$$+\frac{P_{572}}{3(N-1)N^4(1+N)^4(2+N)^3}$$
$$+\frac{8\bigl(2+N+N^2\bigr)^2S_1^2}{(N-1)N^2(1+N)^2(2+N)}+\frac{16\bigl(2+N+N^2\bigr)^2S_2}{(N-1)N^2(1+N)^2(2+N)}$$

$$+\frac{64S_{-2}}{(N-1)N(1+N)(2+N)}-\frac{4\big(2+N+N^2\big)^2\zeta_2}{(N-1)N^2(1+N)^2(2+N)}+16\zeta_3\Bigg)$$

$$+C_AN_FT_F\Bigg(\frac{2S_1^2P_{144}}{9(N-1)N(1+N)(2+N)}+\frac{2S_2P_{145}}{3(N-1)N(1+N)(2+N)}$$

$$-\frac{4P_{583}}{81(N-1)^3N^3(1+N)^3(2+N)^3}+\Bigg(\frac{4P_{476}}{9(N-1)^2N^2(1+N)^2(2+N)^2}$$

$$+\frac{8(3+2N)S_2}{2+N}\Bigg)S_1+\frac{16}{9}S_1^3+\frac{8(47+28N)S_3}{9(2+N)}-\frac{32S_{-2}}{(N-1)N(1+N)(2+N)}$$

$$-\frac{16(N-1)S_{2,1}}{3(2+N)}+\Bigg(\frac{4P_{115}}{9(N-1)N(1+N)(2+N)}-\frac{16}{3}S_1\Bigg)\zeta_2$$

$$-\frac{8(8+N)\zeta_3}{2+N}\Bigg)$$

$$+C_A^2\Bigg(-\frac{4S_1^3P_{116}}{9(N-1)N(1+N)(2+N)}+\frac{S_3P_{163}}{9(N-1)N(1+N)(2+N)(3+N)}$$

$$+\frac{2S_{2,1}P_{255}}{3(N-1)N(1+N)(2+N)(3+N)}$$

$$+\frac{S_2P_{433}}{6(N-1)N^2(1+N)^2(2+N)^2(3+N)}$$

$$+\frac{P_{616}}{162(-2+N)(N-1)^4N^4(1+N)^4(2+N)^4(3+N)}+\Bigg(\frac{238}{3}S_3-4S_{2,1}$$

$$+\frac{S_2P_{174}}{(N-1)N(1+N)(2+N)(3+N)}$$

$$+\frac{P_{585}}{9(N-1)^3N^3(1+N)^3(2+N)^3(3+N)}\Bigg)S_1$$

$$+\Bigg(\frac{P_{434}}{18(N-1)^2N^2(1+N)^2(2+N)^2}+49S_2\Bigg)S_1^2+\frac{14}{3}S_1^4+24S_2^2+19S_4$$

$$+\Bigg(\frac{2P_{408}}{(-2+N)(N-1)N^2(1+N)^2(2+N)^2(3+N)}$$

$$-\frac{8\big(10+9N+6N^2-N^3\big)S_1}{(N-1)N(1+N)(2+N)}$$

$$+8S_1^2+16S_2\Bigg)S_{-2}+4S_{-2}^2+\Bigg(\frac{4\big(-22-21N-18N^2+N^3\big)}{(N-1)N(1+N)(2+N)}+32S_1\Bigg)S_{-3}$$

$$+28S_{-4}+6S_{3,1}+\frac{8\big(6+5N+2N^2-N^3\big)S_{-2,1}}{(N-1)N(1+N)(2+N)}-8S_{-2,2}$$

$$
\begin{aligned}
&-24S_{-3,1}+6S_{2,1,1}-16S_{-2,1,1}+\Biggl(\frac{4S_1P_{115}}{3(N-1)N(1+N)(2+N)}\\
&-\frac{P_{115}^2}{18(N-1)^2N^2(1+N)^2(2+N)^2}-8S_1^2\Biggr)\zeta_2\\
&+\Biggl(\frac{-72-24N-83N^2-25N^3}{N(2+N)(3+N)}+12S_1\Biggr)\zeta_3\Biggr)\\
&+{N_F}^2{T_F}^2\Biggl(\frac{848}{81}-\frac{8\zeta_2}{9}\Biggr),
\end{aligned}
\tag{181}
$$

$$
\begin{aligned}
a_{gg}^{(2,1)}=&-\frac{1}{8}C_A^2\xi^4+C_A^2\xi^3\Biggl[\frac{-4-12N-N^2}{32N^2}+\frac{(-1+3N)S_1}{8N}\\
&+\frac{1}{16}S_1^2+\frac{3}{16}S_2+\frac{1}{16}\zeta_2\Biggr]\\
&+\xi^2\Biggl[C_A^2\Biggl(\frac{S_3P_{25}}{24(-2+N)N(1+N)(3+N)}+\frac{\zeta_3P_{46}}{24(-2+N)N(1+N)(3+N)}\\
&+\frac{S_1^2P_{187}}{8(N-1)N^2(1+N)(2+N)}+\frac{S_2P_{191}}{2(N-1)N^2(1+N)(2+N)}\\
&+\frac{P_{609}}{144(-2+N)^2(N-1)^3N^4(1+N)^3(2+N)^3(3+N)^2}\\
&+\Biggl(\frac{P_{557}}{8(-2+N)(N-1)^2N^3(1+N)^2(2+N)^2(3+N)}-\frac{5}{4}S_2\Biggr)S_1-\frac{5}{24}S_1^3\\
&+\Biggl(\frac{P_{202}}{4(-2+N)^2(N-1)N^2(1+N)^2(3+N)^2}\\
&+\frac{\bigl(-4+N-N^2\bigr)S_1}{4(-2+N)N(1+N)(3+N)}\Biggr)S_{-2}\\
&+\frac{\bigl(-4+N-N^2\bigr)S_{-3}}{2(-2+N)N(1+N)(3+N)}+\Biggl(\frac{P_{182}}{24(N-1)N^2(1+N)(2+N)}\\
&+\frac{3}{8}S_1\Biggr)\zeta_2+\frac{3}{80}\zeta_2^2\Biggr)+C_AN_FT_F\Biggl(-\frac{2}{3}+\frac{\zeta_2}{12}\Biggr)\Biggr]\\
&+\xi\Biggl[C_AN_FT_F\Biggl(\frac{-126-144N-50N^2-53N^3}{27N^3}\\
&+\Biggl(\frac{2\bigl(-27-24N+25N^2\bigr)}{27N^2}+\frac{7S_2}{3}\Biggr)S_1+\frac{(-3+8N)S_1^2}{9N}\\
&+\frac{1}{9}S_1^3+\frac{7(-3+8N)S_2}{9N}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{62}{9}S_3+\left(\frac{4-3N}{6N}-\frac{2S_1}{3}\right)\times\zeta_2-4\zeta_3\Bigg)\\
&+C_A^2\Bigg(\frac{S_{3,1}P_{43}}{2N(1+N)(2+N)(3+N)}+\frac{S_4P_{54}}{8N(1+N)(2+N)(3+N)}\\
&+\frac{9\zeta_2^2P_{88}}{20N(1+N)(2+N)(3+N)}+\frac{S_{2,1,1}P_{91}}{2N(1+N)(2+N)(3+N)}\\
&+\frac{S_2^2P_{135}}{4N(1+N)(2+N)(3+N)}+\frac{S_{2,1}P_{452}}{2(N-1)N(1+N)^2(2+N)^2(3+N)^2}\\
&+\frac{S_3P_{525}}{36(-2+N)(N-1)N^2(1+N)^2(2+N)^2(3+N)^2}\\
&+\frac{S_2P_{553}}{72(N-1)^2N^3(1+N)^2(2+N)^2(3+N)^2}\\
&+\frac{P_{613}}{108(-2+N)^2(N-1)^4N^4(1+N)^3(2+N)^3(3+N)^2}\\
&+\Bigg(\frac{S_{2,1}P_{44}}{N(1+N)(2+N)(3+N)}+\frac{S_3P_{153}}{6N(1+N)(2+N)(3+N)}\\
&+\frac{P_{605}}{108(-2+N)(N-1)^3N^4(1+N)^3(2+N)^3(3+N)^2}\\
&+\frac{S_2P_{508}}{24(N-1)N^2(1+N)^2(2+N)^2(3+N)^2}\Bigg)S_1\\
&+\Bigg(\frac{S_2P_{150}}{8N(1+N)(2+N)(3+N)}\\
&+\frac{P_{524}}{72(N-1)^2N^3(1+N)^2(2+N)^2(3+N)}\Bigg)S_1^2+\frac{7}{6}S_1^4\\
&-\frac{\left(370+773N-34N^2-101N^3\right)S_1^3}{72(N-1)(1+N)(2+N)}\\
&+\Bigg(\frac{P_{311}}{(-2+N)^2(N-1)N^2(1+N)^2(3+N)^2}\\
&+\frac{\left(1+15N+N^2-N^3\right)S_1}{(-2+N)(N-1)N(1+N)(3+N)}\Bigg)S_{-2}\\
&+\frac{2\left(1+15N+N^2-N^3\right)S_{-3}}{(N-2)(N-1)N(1+N)(3+N)}\\
&+\Bigg(\frac{P_{351}}{24(N-1)^2N^2(1+N)(2+N)}\\
&+\frac{\left(26+181N+70N^2+11N^3\right)S_1}{24(N-1)(1+N)(2+N)}-2S_1^2\Bigg)\zeta_2
\end{aligned}
$$

$$
\begin{aligned}
&+\Biggl(\frac{P_{534}}{(-2+N)(N-1)N^2(1+N)^2(2+N)^2(3+N)^2} \\
&-\frac{3S_1P_{90}}{2N(1+N)(2+N)(3+N)}\Biggr)\zeta_3\Biggr)\Biggr] \\
&+C_FN_FT_F\Biggl(-\frac{4S_3P_{120}}{3(N-1)N^2(1+N)^2(2+N)} \\
&-\frac{4\zeta_3P_{343}}{3(N-1)N^2(1+N)^2(2+N)}+\frac{4S_1^2P_{401}}{(N-1)N^3(1+N)^3(2+N)^2} \\
&+\frac{8S_2P_{401}}{(N-1)N^3(1+N)^3(2+N)^2} \\
&+\frac{P_{610}}{36(N-2)(N-1)N^5(1+N)^5(2+N)^4(3+N)} \\
&+\Biggl(-\frac{4P_{532}}{(N-1)N^4(1+N)^4(2+N)^3}-\frac{16\bigl(2+N+N^2\bigr)^2S_2}{(N-1)N^2(1+N)^2(2+N)}\Biggr)S_1 \\
&-\frac{8\bigl(2+N+N^2\bigr)^2S_1^3}{3(N-1)N^2(1+N)^2(2+N)} \\
&+\Biggl(\frac{8P_{399}}{(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)} \\
&-\frac{64S_1}{(N-1)N(1+N)(2+N)}\Biggr)S_{-2}-\frac{128S_{-3}}{(N-1)N(1+N)(2+N)} \\
&+\Biggl(\frac{P_{501}}{(N-1)N^3(1+N)^3(2+N)^2}+\frac{4\bigl(2+N+N^2\bigr)^2S_1}{(N-1)N^2(1+N)^2(2+N)}\Biggr)\zeta_2 \\
&-\frac{24}{5}\zeta_2^2\Biggr)+C_AN_FT_F\Biggl(\frac{S_1^3P_{26}}{9(N-1)N(1+N)(2+N)} \\
&+\frac{8S_{2,1}P_{330}}{9(N-1)N(1+N)(2+N)^2(3+N)} \\
&-\frac{4S_3P_{358}}{9(N-1)N(1+N)(2+N)^2(3+N)} \\
&+\frac{S_2P_{495}}{9(N-1)^2N^2(1+N)^2(2+N)^2(3+N)} \\
&+\frac{P_{615}}{54(-2+N)(N-1)^4N^4(1+N)^4(2+N)^4(3+N)} \\
&+\Biggl(\frac{S_2P_{291}}{3(N-1)N(1+N)(2+N)^2(3+N)} \\
&-\frac{2P_{589}}{27(N-1)^3N^3(1+N)^3(2+N)^3(3+N)}
\end{aligned}
$$

$$
-\frac{8(23+25N)S_3}{9(2+N)} - \frac{16S_{2,1}}{2+N}\Bigg)S_1 + \Bigg(\frac{P_{438}}{27(N-1)^2N^2(1+N)^2(2+N)^2}
$$
$$
-\frac{2(11+10N)S_2}{3(2+N)}\Bigg)S_1^2 - \frac{4}{9}S_1^4 - \frac{4(17+7N)S_2^2}{3(2+N)} - \frac{2(29+18N)S_4}{2+N}
$$
$$
+\Bigg(\frac{32S_1}{(N-1)N(1+N)(2+N)}
$$
$$
-\frac{4P_{399}}{(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}\Bigg)
$$
$$
\times S_{-2} + \frac{64S_{-3}}{(N-1)N(1+N)(2+N)} + \frac{8(-2+N)S_{3,1}}{2+N} + \frac{8(7+2N)S_{2,1,1}}{3(2+N)}
$$
$$
+\Bigg(\frac{S_1P_{128}}{3(N-1)N(1+N)(2+N)} + \frac{P_{439}}{9(N-1)^2N^2(1+N)^2(2+N)^2}
$$
$$
+2S_1^2 + 6S_2\Bigg)\zeta_2
$$
$$
+\frac{12(8+N)\zeta_2^2}{5(2+N)} + \Bigg(-\frac{2P_{367}}{27(N-1)N(1+N)(2+N)^2(3+N)}
$$
$$
+\frac{8(109+41N)S_1}{9(2+N)}\Bigg) \times \zeta_3\Bigg) + C_A^2\Bigg(\frac{S_1^4P_{117}}{9(N-1)N(1+N)(2+N)}
$$
$$
+\frac{S_{2,1,1}P_{178}}{3(N-1)N(1+N)(2+N)(3+N)}
$$
$$
+\frac{S_{3,1}P_{185}}{(N-1)N(1+N)(2+N)(3+N)}
$$
$$
+\frac{S_2^2P_{274}}{6(N-1)N(1+N)(2+N)(3+N)}
$$
$$
+\frac{S_4P_{279}}{4(N-1)N(1+N)(2+N)(3+N)}
$$
$$
+\frac{8S_{-2,1}P_{307}}{(N-1)^2N^2(1+N)^2(2+N)^2}
$$
$$
-\frac{2S_{2,1}P_{538}}{9(N-1)^2N^2(1+N)^2(2+N)^2(3+N)^2}
$$
$$
+\frac{S_3P_{566}}{9(-2+N)(N-1)^2N^2(1+N)^2(2+N)^2(3+N)^2}
$$
$$
+\frac{P_{620}}{648(-2+N)^2(N-1)^5N^5(1+N)^5(2+N)^5(3+N)^2}
$$
$$
+\Bigg(\frac{S_3P_{281}}{9(N-1)N(1+N)(2+N)(3+N)}
$$

$$
\begin{aligned}
&+\frac{S_2 P_{549}}{12(N-1)^2N^2(1+N)^2(2+N)^2(3+N)^2}\\
&+\frac{P_{617}}{27(-2+N)(N-1)^4N^4(1+N)^4(2+N)^4(3+N)^2}-42S_2^2-\frac{207}{2}S_4\\
&+\frac{2\big(-12+21N+7N^2\big)S_{2,1}}{N(2+N)(3+N)}-2S_{2,1,1}\Bigg)S_1\\
&+\Bigg(\frac{S_2 P_{276}}{12(N-1)N(1+N)(2+N)(3+N)}\\
&+\frac{P_{592}}{108(N-1)^3N^3(1+N)^3(2+N)^3(3+N)}-59S_3+2S_{2,1}\Bigg)S_1^2\\
&+\Bigg(\frac{P_{483}}{36(N-1)^2N^2(1+N)^2(2+N)^2}-\frac{115}{6}S_2\Bigg)S_1^3-S_1^5\\
&+\Bigg(\frac{P_{601}}{36(N-1)^3N^3(1+N)^3(2+N)^3(3+N)^2}\\
&-\frac{178}{3}S_3+4S_{2,1}+8S_{-2,1}\Bigg)S_2-46S_5\\
&+\Bigg(-\frac{2P_{576}}{(-2+N)^2(N-1)^2N^3(1+N)^3(2+N)^3(3+N)^2}\\
&+\Bigg(-\frac{2P_{471}}{(-2+N)(N-1)^2N^2(1+N)^2(2+N)^2(3+N)}-16S_2\Bigg)S_1\\
&-\frac{4\big(-10-9N-6N^2+N^3\big)S_1^2}{(N-1)N(1+N)(2+N)}-\frac{8}{3}S_1^3-\frac{8\big(-10-9N-6N^2+N^3\big)S_2}{(N-1)N(1+N)(2+N)}\\
&-\frac{64}{3}S_3-8S_{2,1}\Bigg)S_{-2}\\
&+\Bigg(-\frac{2\big(-10-9N-6N^2+N^3\big)}{(N-1)N(1+N)(2+N)}-4S_1\Bigg)S_{-2}^2\\
&+\Bigg(-\frac{4P_{467}}{(-2+N)(N-1)^2N^2(1+N)^2(2+N)^2(3+N)}\\
&-16S_1^2-20S_2-16S_{-2}\\
&-\frac{16\big(-10-9N-6N^2+N^3\big)S_1}{(N-1)N(1+N)(2+N)}\Bigg)S_{-3}\\
&-\Bigg(\frac{14\big(-14-13N-10N^2+N^3\big)}{(N-1)N(1+N)(2+N)}+56S_1\Bigg)S_{-4}\\
&-\frac{4\big(6+5N+2N^2-N^3\big)S_{-2,2}}{(N-1)N(1+N)(2+N)}-54S_{-5}
\end{aligned}
$$

$$+16S_{2,3} - 20S_{2,-3} - 13S_{4,1}$$
$$+12S_{-2,3} + \frac{12\big(-6-5N-2N^2+N^3\big)S_{-3,1}}{(N-1)N(1+N)(2+N)}$$
$$+28S_{-4,1} + 8S_{2,1,-2} - 18S_{2,2,1}$$
$$-6S_{3,1,1} + \frac{8\big(-6-5N-2N^2+N^3\big)S_{-2,1,1}}{(N-1)N(1+N)(2+N)}$$
$$+8S_{-2,2,1} + 24S_{-3,1,1} - 3S_{2,1,1,1}$$
$$+16S_{-2,1,1,1} + \Bigg(\frac{S_2 P_{22}}{2(N-1)N(1+N)(2+N)} + \frac{S_1^2 P_{33}}{2(N-1)N(1+N)(2+N)}$$
$$+\frac{P_{581}}{36(N-1)^3N^3(1+N)^3(2+N)^3}$$
$$+\Bigg(\frac{P_{437}}{12(N-1)^2N^2(1+N)^2(2+N)^2} + 16S_2\Bigg)$$
$$\times S_1 + 4S_1^3 + 2S_3$$
$$+\Bigg(-\frac{8\big(1+N+N^2\big)}{(N-1)N(1+N)(2+N)} + 4S_1\Bigg)S_{-2} + 2S_{-3} - 4S_{-2,1}\Bigg)$$
$$\times\zeta_2 + \Bigg(\frac{3\big(72+24N+83N^2+25N^3\big)}{10N(2+N)(3+N)} - \frac{18}{5}S_1\Bigg)\zeta_2^2$$
$$+\Bigg(\frac{P_{568}}{54(-2+N)(N-1)^2N^2(1+N)^2(2+N)^2(3+N)^2}$$
$$+\frac{S_1 P_{162}}{9(N-1)N(1+N)(2+N)(3+N)} + \frac{83}{3}S_1^2 - 9S_2 - 12S_{-2}\Bigg)\zeta_3\Bigg)$$
$$+N_F^2 T_F^2\Bigg(-\frac{1184}{81} + \frac{20\zeta_2}{9} + \frac{56\zeta_3}{27}\Bigg), \tag{182}$$

$$b_{gg}^{(1,0)} = \frac{2C_A\big(2-N+N^2\big)}{N(1+N)(2+N)} - \frac{C_A\xi}{N-1}, \tag{183}$$

$$b_{gg}^{(1,1)} = C_A\Bigg[\xi\Bigg(\frac{1+N-N^2}{2(N-1)^2N} + \frac{S_1}{2(N-1)}\Bigg)$$
$$-\frac{P_{196}}{N^2(1+N)^2(2+N)^2} - \frac{\big(2-N+N^2\big)S_1}{N(1+N)(2+N)}\Bigg], \tag{184}$$

$$b_{gg}^{(1,2)} = C_A\xi\Bigg[\frac{1-4N+2N^2}{4(N-1)^3N^2} + \frac{\big(-1-N+N^2\big)S_1}{4(N-1)^2N}$$
$$-\frac{S_1^2}{8(N-1)} - \frac{3S_2}{8(N-1)} + \frac{\zeta_2}{8(N-1)}\Bigg]$$

$$
\begin{aligned}
&+C_A\Biggl(\frac{S_1 P_{196}}{2N^2(1+N)^2(2+N)^2}+\frac{P_{388}}{2N^3(1+N)^3(2+N)^3}\\
&+\frac{\bigl(2-N+N^2\bigr)S_1^2}{4N(1+N)(2+N)}\\
&+\frac{3\bigl(2-N+N^2\bigr)S_2}{4N(1+N)(2+N)}-\frac{\bigl(2-N+N^2\bigr)\zeta_2}{4N(1+N)(2+N)}\Biggr),
\end{aligned}
\tag{185}
$$

$$
\begin{aligned}
b_{gg}^{(2,0)} = {} & \frac{{C_A}^2\xi^3}{4(N-1)}+{C_A}^2\xi^2\Biggl[\frac{P_{444}}{8(-2+N)(N-1)^2N^3(1+N)(2+N)(3+N)}\\
&+\frac{\bigl(-4-N-2N^2\bigr)S_1}{4(N-1)N^2}+\frac{\bigl(-2-3N+N^2\bigr)S_{-2}}{2(-2+N)N(1+N)(3+N)}\Biggr]\\
&+\xi\Biggl[C_A N_F T_F\Biggl(-\frac{4\bigl(-3-10N+4N^2\bigr)}{9(N-1)^2N}+\frac{4S_1}{3(N-1)}\Biggr)\\
&+C_A^2\Biggl(\frac{S_2 P_{126}}{4(N-1)N^2(2+N)(3+N)}\\
&+\frac{P_{561}}{36(-2+N)(N-1)^3N^3(1+N)^2(2+N)^2(3+N)}\\
&+\Biggl(\frac{P_{385}}{3(N-1)^2N^3(1+N)(2+N)(3+N)}\\
&+\frac{\bigl(-72-8N+3N^2+N^3\bigr)S_2}{2N(1+N)(2+N)(3+N)}\Biggr)S_1\\
&+\frac{3(3+8N)S_1^2}{4(N-1)N}+\frac{\bigl(-96-52N-21N^2-3N^3\bigr)S_3}{2N(1+N)(2+N)(3+N)}-\frac{2S_{-3}}{N}+\frac{4S_{-2,1}}{N}\\
&+\Biggl(\frac{P_{219}}{(-2+N)(N-1)N^2(1+N)(3+N)}-\frac{4S_1}{N}\Biggr)S_{-2}\\
&+\frac{\bigl(72+8N-3N^2-N^3\bigr)S_{2,1}}{N(1+N)(2+N)(3+N)}+\frac{3\bigl(-60+14N+15N^2+3N^3\bigr)\zeta_3}{N(1+N)(2+N)(3+N)}\Biggr)\Biggr]\\
&+C_F N_F T_F\Biggl(-\frac{32P_{234}}{(N-1)N^2(1+N)^3(2+N)^2}\\
&+\frac{64\bigl(2+N+N^2\bigr)S_1}{(N-1)N(1+N)^2(2+N)}\Biggr)\\
&+C_A N_F T_F\Biggl(-\frac{4P_{333}}{9(N-1)N^2(1+N)^2(2+N)^2}\\
&+\Biggl(-\frac{4P_{225}}{3(N-1)^2N^2(1+N)(2+N)}
\end{aligned}
$$

$$
+\frac{16S_2}{(1+N)(2+N)}\Bigg)S_1+\frac{8\big(6-13N+N^2\big)S_2}{3(N-1)N(2+N)}+\frac{16S_3}{(1+N)(2+N)}
-\frac{32S_{2,1}}{(1+N)(2+N)}+\frac{96\zeta_3}{(1+N)(2+N)}\Bigg)
+C_A^2\Bigg(\frac{P_{578}}{9(-2+N)(N-1)^2N^3(1+N)^3(2+N)^3(3+N)}
+\frac{S_2P_{175}}{3(N-1)N^2(1+N)(2+N)(3+N)}
+\Bigg(\frac{P_{512}}{3(N-1)^2N^3(1+N)^2(2+N)^2(3+N)}
+\frac{\big(18-22N-N^2+N^3\big)S_2}{N(1+N)(2+N)(3+N)}\Bigg)S_1+\frac{\big(42-26N+7N^2+5N^3\big)S_3}{N(1+N)(2+N)(3+N)}
-\frac{12\big(2-N+N^2\big)S_1^2}{N(1+N)(2+N)}+\Bigg(\frac{2P_{230}}{(-2+N)N^2(1+N)^2(2+N)(3+N)}
+\frac{8\big(2-N+N^2\big)S_1}{N(1+N)(2+N)}\Bigg)S_{-2}+\frac{4\big(2-N+N^2\big)S_{-3}}{N(1+N)(2+N)}
-\frac{2\big(18-22N-N^2+N^3\big)S_{2,1}}{N(1+N)(2+N)(3+N)}
-\frac{8\big(2-N+N^2\big)S_{-2,1}}{N(1+N)(2+N)}-\frac{6\big(-6+20N+5N^2+N^3\big)\zeta_3}{N(1+N)(2+N)(3+N)}\Bigg), \tag{186}
$$

$$
b_{gg}^{(2,1)}=C_A{}^2\xi^3\Bigg[\frac{-1+N-N^2}{8(N-1)^2N}-\frac{S_1}{8(N-1)}\Bigg]+C_A{}^2\xi^2\Bigg[\frac{\big(8+5N+6N^2\big)S_1^2}{16(N-1)N^2}
+\frac{S_1P_{453}}{8(-2+N)(N-1)^2N^3(1+N)(2+N)(3+N)}
+\frac{P_{596}}{32(-2+N)^2(N-1)^3N^4(1+N)^2(2+N)^2(3+N)^2}
+\frac{\big(16+7N+8N^2\big)S_2}{16(N-1)N^2}
+\frac{\big(2+3N-N^2\big)S_3}{4(-2+N)N(1+N)(3+N)}
+\Bigg(\frac{P_{302}}{2(-2+N)^2(N-1)N^2(1+N)^2(3+N)^2}
+\frac{\big(2+3N-N^2\big)S_1}{2(-2+N)N(1+N)(3+N)}\Bigg)S_{-2}
+\frac{\big(2+3N-N^2\big)S_{-3}}{(-2+N)N(1+N)(3+N)}+\frac{(2+N)\zeta_2}{8N^2}
$$

$$
\begin{aligned}
&-\frac{3\big(-2-3N+N^2\big)\zeta_3}{4(-2+N)N(1+N)(3+N)}\Bigg]\\
&+\xi\Bigg[C_A N_F T_F\Bigg(-\frac{2P_{127}}{27(N-1)^3N^2}\\
&+\frac{2\big(-3-10N+4N^2\big)S_1}{9(N-1)^2N}-\frac{S_1^2}{3(N-1)}-\frac{7S_2}{3(N-1)}+\frac{2\zeta_2}{3(N-1)}\Bigg)\\
&+C_A^2\Bigg(\frac{S_3P_{498}}{12(-2+N)(N-1)N^2(1+N)^2(2+N)^2(3+N)^2}\\
&+\frac{S_{2,1}P_{389}}{2(N-1)N(1+N)^2(2+N)^2(3+N)^2}\\
&+\frac{S_2P_{514}}{12(N-1)^2N^3(1+N)(2+N)^2(3+N)^2}\\
&+\frac{P_{611}}{432(-2+N)^2(N-1)^4N^4(1+N)^3(2+N)^3(3+N)^2}\\
&+\Bigg(\frac{P_{588}}{36(-2+N)(N-1)^3N^4(1+N)^2(2+N)^2(3+N)^2}\\
&+\frac{S_2P_{441}}{8(N-1)N^2(1+N)^2(2+N)^2(3+N)^2}+\frac{\big(228+46N+3N^2-N^3\big)S_3}{2N(1+N)(2+N)(3+N)}\\
&+\frac{\big(-60+14N+15N^2+3N^3\big)S_{2,1}}{N(1+N)(2+N)(3+N)}\Bigg)S_1\\
&+\Bigg(\frac{P_{421}}{12(N-1)^2N^3(1+N)(2+N)(3+N)}\\
&+\frac{\big(192-20N-33N^2-7N^3\big)S_2}{8N(1+N)(2+N)(3+N)}\Bigg)S_1^2\\
&+\frac{\big(-48+36N+27N^2+5N^3\big)S_2^2}{4N(1+N)(2+N)(3+N)}+\frac{\big(816+628N+291N^2+45N^3\big)S_4}{8N(1+N)(2+N)(3+N)}\\
&-\frac{7(3+8N)S_1^3}{24(N-1)N}+\Bigg(\frac{S_1P_{45}}{(-2+N)(N-1)N(1+N)(3+N)}\\
&+\frac{P_{454}}{2(-2+N)^2(N-1)N^3(1+N)^2(3+N)^2}+\frac{2S_1^2}{N}+\frac{4S_2}{N}\Bigg)S_{-2}+\frac{S_{-2}^2}{N}\\
&+\Bigg(\frac{P_{189}}{(-2+N)(N-1)N^2(1+N)(3+N)}+\frac{8S_1}{N}\Bigg)S_{-3}+\frac{7S_{-4}}{N}\\
&+\frac{\big(-138-5N+12N^2+3N^3\big)S_{3,1}}{N(1+N)(2+N)(3+N)}\\
&-\frac{2(-2+N)S_{-2,1}}{(N-1)N^2}-\frac{2S_{-2,2}}{N}-\frac{6S_{-3,1}}{N}
\end{aligned}
$$

$$
+\frac{\big(36-58N-39N^2-7N^3\big)S_{2,1,1}}{2N(1+N)(2+N)(3+N)}-\frac{4S_{-2,1,1}}{N}
+\Biggl(\frac{P_{171}}{24(N-1)^2N^2(1+N)(2+N)}
+\frac{(3+8N)S_1}{4(N-1)N}\Biggr)\zeta_2-\frac{9\big(-60+14N+15N^2+3N^3\big)\zeta_2^2}{10N(1+N)(2+N)(3+N)}
+\Biggl(-\frac{3P_{504}}{2(-2+N)(N-1)N^2(1+N)^2(2+N)^2(3+N)^2}
-\frac{3\big(-72-8N+3N^2+N^3\big)S_1}{2N(1+N)(2+N)(3+N)}\Biggr)\zeta_3\Biggr)\Biggr]
$$

$$
+{\color{blue}C_FN_FT_F}\Biggl(\frac{32S_1P_{234}}{(N-1)N^2(1+N)^3(2+N)^2}
-\frac{16P_{533}}{(-2+N)(N-1)N^3(1+N)^4(2+N)^3(3+N)}
-\frac{32\big(2+N+N^2\big)S_1^2}{(N-1)N(1+N)^2(2+N)}
-\frac{64\big(2+N+N^2\big)S_2}{(N-1)N(1+N)^2(2+N)}+\frac{256S_{-2}}{(-2+N)(N-1)(1+N)(2+N)(3+N)}
+\frac{16\big(2+N+N^2\big)\zeta_2}{(N-1)N(1+N)^2(2+N)}\Biggr)
$$

$$
+{\color{blue}C_AN_FT_F}\Biggl(\frac{32S_{2,1}P_{57}}{(N-1)N(1+N)^2(2+N)^2(3+N)}
-\frac{16S_3P_{305}}{3(N-1)N(1+N)^2(2+N)^2(3+N)}
+\frac{S_2P_{411}}{9(N-1)^2N^2(1+N)(2+N)^2(3+N)}
+\frac{2P_{579}}{27(-2+N)(N-1)^2N^3(1+N)^3(2+N)^3(3+N)}
+\Biggl(-\frac{4S_2P_{306}}{3(N-1)N(1+N)^2(2+N)^2(3+N)}
-\frac{48S_3}{(1+N)(2+N)}+\frac{32S_{2,1}}{(1+N)(2+N)}
+\frac{2P_{535}}{9(N-1)^3N^3(1+N)^2(2+N)^2(3+N)}\Biggr)S_1
+\Biggl(\frac{P_{242}}{3(N-1)^2N^2(1+N)(2+N)}
$$

$$
\begin{aligned}
&-\frac{12S_2}{(1+N)(2+N)}\Biggr)S_1^2-\frac{128S_{-2}}{(-2+N)(N-1)(1+N)(2+N)(3+N)}\\
&+\frac{8S_2^2}{(1+N)(2+N)}-\frac{28S_4}{(1+N)(2+N)}\\
&+\frac{64S_{3,1}}{(1+N)(2+N)}-\frac{16S_{2,1,1}}{(1+N)(2+N)}\\
&-\frac{(-2+N)\big(6-5N+5N^2\big)\zeta_2}{3(N-1)N(1+N)(2+N)}-\frac{144\zeta_2^2}{5(1+N)(2+N)}\\
&+\Biggl(\frac{8P_{304}}{(N-1)N(1+N)^2(2+N)^2(3+N)}-\frac{48S_1}{(1+N)(2+N)}\Biggr)\zeta_3\Biggr)\\
&+C_A^2\Biggl(-\frac{4S_{-2,1}P_{201}}{N^2(1+N)^2(2+N)^2}+\frac{S_{2,1}P_{393}}{(N-1)N(1+N)^2(2+N)^2(3+N)^2}\\
&+\frac{S_3P_{510}}{6(-2+N)(N-1)N^2(1+N)^2(2+N)^2(3+N)^2}\\
&+\frac{P_{612}}{108(-2+N)^2(N-1)^3N^4(1+N)^4(2+N)^4(3+N)^2}\\
&+\frac{S_2P_{523}}{36(N-1)^2N^3(1+N)^2(2+N)^2(3+N)^2}\\
&+\Biggl(\frac{P_{599}}{18(-2+N)(N-1)^3N^4(1+N)^2(2+N)^3(3+N)^2}\\
&+\frac{S_2P_{482}}{6(N-1)N^2(1+N)^2(2+N)^2(3+N)^2}+\frac{\big(-66+68N-N^2-5N^3\big)S_3}{N(1+N)(2+N)(3+N)}\\
&-\frac{2\big(-6+20N+5N^2+N^3\big)S_{2,1}}{N(1+N)(2+N)(3+N)}\Biggr)S_1\\
&+\Biggl(\frac{P_{494}}{12(N-1)^2N^3(1+N)^2(2+N)^2(3+N)}\\
&+\frac{\big(-30+62N+11N^2+N^3\big)S_2}{4N(1+N)(2+N)(3+N)}\Biggr)S_1^2-\frac{3\big(2+6N+3N^2+N^3\big)S_2^2}{2N(1+N)(2+N)(3+N)}\\
&+\frac{14\big(2-N+N^2\big)S_1^3}{3N(1+N)(2+N)}+\frac{\big(-438+206N-97N^2-59N^3\big)S_4}{4N(1+N)(2+N)(3+N)}\\
&+\Biggl(\frac{P_{560}}{(-2+N)^2(N-1)N^3(1+N)^3(2+N)^2(3+N)^2}\\
&+\frac{2S_1P_{315}}{(-2+N)N(1+N)^2(2+N)^2(3+N)}-\frac{4\big(2-N+N^2\big)S_1^2}{N(1+N)(2+N)}\\
&-\frac{8\big(2-N+N^2\big)S_2}{N(1+N)(2+N)}\Biggr)
\end{aligned}
$$

$$
\times S_{-2} - \frac{2\big(2-N+N^2\big)S_{-2}^2}{N(1+N)(2+N)} + \Bigg(\frac{2P_{396}}{(-2+N)N^2(1+N)^2(2+N)^2(3+N)}
- \frac{16\big(2-N+N^2\big)S_1}{N(1+N)(2+N)}\Bigg)S_{-3} - \frac{14\big(2-N+N^2\big)S_{-4}}{N(1+N)(2+N)}
+ \frac{2\big(30-43N-4N^2+N^3\big)}{N(1+N)(2+N)(3+N)}
\times S_{3,1} + \frac{4\big(2-N+N^2\big)S_{-2,2}}{N(1+N)(2+N)} + \frac{\big(18+16N+13N^2+5N^3\big)S_{2,1,1}}{N(1+N)(2+N)(3+N)}
+ \frac{12\big(2-N+N^2\big)S_{-3,1}}{N(1+N)(2+N)} + \frac{8\big(2-N+N^2\big)S_{-2,1,1}}{N(1+N)(2+N)}
+ \Bigg(-\frac{4\big(2-N+N^2\big)S_1}{N(1+N)(2+N)}
+ \frac{P_{356}}{12(N-1)N^2(1+N)^2(2+N)^2}\Bigg)\zeta_2 + \frac{9\big(-6+20N+5N^2+N^3\big)\zeta_2^2}{5N(1+N)(2+N)(3+N)}
+ \Bigg(\frac{P_{506}}{(-2+N)(N-1)N^2(1+N)^2(2+N)^2(3+N)^2}
- \frac{3\big(18-22N-N^2+N^3\big)S_1}{N(1+N)(2+N)(3+N)}\Bigg)\zeta_3\Bigg), \tag{187}
$$

$$
c_{gg}^{(1,0)} = C_A\left[\frac{2\big(1-4N+2N^2\big)}{(N-1)^2N^2} - \frac{2S_1}{(N-1)N}\right], \tag{188}
$$

$$
c_{gg}^{(1,1)} = C_A\Bigg[-\frac{P_{52}}{(N-1)^3N^3} + \frac{\big(-1+4N-2N^2\big)S_1}{(N-1)^2N^2} + \frac{S_1^2}{2(N-1)N} + \frac{3S_2}{2(N-1)N}
- \frac{\zeta_2}{2(N-1)N}\Bigg], \tag{189}
$$

$$
c_{gg}^{(1,2)} = C_A\Bigg(-\frac{\big(1-3N+N^2\big)\big(-1+3N-5N^2+2N^3\big)}{2(N-1)^4N^4} + \Bigg(\frac{P_{52}}{2(N-1)^3N^3}
- \frac{3S_2}{4(N-1)N}\Bigg)S_1 + \frac{\big(1-4N+2N^2\big)S_1^2}{4(N-1)^2N^2}
- \frac{S_1^3}{12(N-1)N} + \frac{3\big(1-4N+2N^2\big)S_2}{4(N-1)^2N^2}
- \frac{7S_3}{6(N-1)N} + \Bigg(\frac{-1+4N-2N^2}{4(N-1)^2N^2} + \frac{S_1}{4(N-1)N}\Bigg)\zeta_2 + \frac{7\zeta_3}{6(N-1)N}\Bigg), \tag{190}
$$

$$
c_{gg}^{(2,0)} = {C_A}^2\xi\Bigg[\frac{P_{402}}{4(-2+N)(N-1)^3N^3(1+N)^2}
$$

$$
\begin{aligned}
&+\Biggl(\frac{2-8N-21N^2+15N^4}{4(N-1)^2N^3(1+N)}+\frac{S_2}{2N(1+N)}\Biggr)\\
&\times S_1-\frac{15S_1^2}{8(N-1)N}+\frac{(4-31N)S_2}{8(N-1)N^2}\\
&+\frac{S_3}{2N(1+N)}+\frac{S_{-2}}{(-2+N)(N-1)N(1+N)}\\
&-\frac{S_{2,1}}{N(1+N)}+\frac{3\zeta_3}{N(1+N)}\Biggr]\\
&+C_AN_FT_F\Biggl(\frac{8P_{106}}{27(N-1)^3N^3}+\frac{8\bigl(-3+4N+2N^2\bigr)S_1}{9(N-1)^2N^2}\\
&+\frac{8S_2}{3(N-1)N}-\frac{4\zeta_2}{3(N-1)N}\Biggr)+C_A{}^2\Biggl(\frac{S_2P_4}{12(N-1)^2N^2(1+N)(2+N)}\\
&+\frac{S_1^2P_{11}}{4(N-1)^2N^2(1+N)(2+N)}\\
&+\frac{P_{587}}{54(-2+N)(N-1)^4N^4(1+N)^3(2+N)^3}\\
&+\Biggl(\frac{P_{478}}{9(N-1)^3N^3(1+N)^2(2+N)^2}\\
&+\frac{\bigl(226+399N+125N^2\bigr)S_2}{4(N-1)N(1+N)(2+N)}\Biggr)S_1\\
&+\frac{77S_1^3}{12(N-1)N}+\frac{\bigl(214+411N+125N^2\bigr)S_3}{6(N-1)N(1+N)(2+N)}\\
&+\Biggl(-\frac{2(-5+2N)}{(-2+N)(N-1)N}\\
&+\frac{8S_1}{(N-1)N}\Biggr)S_{-2}+\frac{10S_{-3}}{(N-1)N}-\frac{12S_{2,1}}{N(1+N)(2+N)}-\frac{4S_{-2,1}}{(N-1)N}\\
&+\Biggl(\frac{P_{143}}{6(N-1)^2N^2(1+N)(2+N)}-\frac{11S_1}{2(N-1)N}\Biggr)\zeta_2\\
&+\frac{36\zeta_3}{N(1+N)(2+N)}\Biggr),
\end{aligned}
\tag{191}
$$

$$
\begin{aligned}
c_{gg}^{(2,1)}=C_A{}^2\xi\Bigl(&\frac{S_2P_{20}}{16(N-1)^2N^3(1+N)}+\frac{S_3P_{154}}{24(-2+N)(N-1)N^2(1+N)^2}\\
&+\frac{P_{558}}{16(-2+N)^2(N-1)^4N^4(1+N)^3}+\Biggl(\frac{P_{446}}{4(-2+N)(N-1)^3N^4(1+N)^2}
\end{aligned}
$$

$$
+\frac{\big(-4+83N+146N^2+75N^3\big)S_2}{16(N-1)N^2(1+N)^2}-\frac{3S_3}{2N(1+N)}+\frac{S_{2,1}}{N(1+N)}\Bigg)S_1
$$

$$
+\Bigg(\frac{P_{24}}{16(N-1)^2N^3(1+N)}-\frac{3S_2}{8N(1+N)}\Bigg)S_1^2
$$

$$
+\frac{35S_1^3}{48(N-1)N}+\frac{S_2^2}{4N(1+N)}-\frac{7S_4}{8N(1+N)}
$$

$$
+\Bigg(\frac{-4-6N+N^2}{2(-2+N)^2N^2(1+N)^2}-\frac{S_1}{(-2+N)(N-1)N(1+N)}\Bigg)S_{-2}
$$

$$
-\frac{2S_{-3}}{(-2+N)(N-1)N(1+N)}+\frac{\big(-5-2N-N^2\big)S_{2,1}}{2(N-1)N(1+N)^2}+\frac{2S_{3,1}}{N(1+N)}
$$

$$
-\frac{S_{2,1,1}}{2N(1+N)}+\Bigg(\frac{-5-8N+3N^2+8N^3}{8(N-1)^2N^2(1+N)}-\frac{5S_1}{8(N-1)N}\Bigg)\zeta_2
$$

$$
-\frac{9\zeta_2^2}{10N(1+N)}
$$

$$
+\Bigg(-\frac{3P_{82}}{2(-2+N)(N-1)N^2(1+N)^2}-\frac{3S_1}{2N(1+N)}\Bigg)\zeta_3\Bigg)
$$

$$
+C_AN_FT_F\Bigg(\frac{4P_{340}}{81(N-1)^4N^4}+\Bigg(-\frac{4P_{106}}{27(N-1)^3N^3}-\frac{4S_2}{3(N-1)N}\Bigg)S_1
$$

$$
-\frac{2\big(-3+4N+2N^2\big)S_1^2}{9(N-1)^2N^2}+\frac{2\big(21-44N+2N^2\big)S_2}{9(N-1)^2N^2}-\frac{16S_3}{3(N-1)N}
$$

$$
+\Bigg(-\frac{2\big(9-20N+2N^2\big)}{9(N-1)^2N^2}+\frac{2S_1}{3(N-1)N}\Bigg)\zeta_2+\frac{100\zeta_3}{9(N-1)N}\Bigg)
$$

$$
+C_A^2\Bigg(\frac{S_1^3P_{156}}{24(N-1)^2N^2(1+N)(2+N)}+\frac{2S_{2,1}P_{309}}{(N-1)^2N^2(1+N)^2(2+N)^2}
$$

$$
+\frac{S_3P_{424}}{12(-2+N)(N-1)^2N^2(1+N)^2(2+N)^2}
$$

$$
+\frac{S_2P_{436}}{36(N-1)^3N^3(1+N)^2(2+N)^2}
$$

$$
+\frac{P_{614}}{648(-2+N)^2(N-1)^5N^5(1+N)^4(2+N)^4}
$$

$$
+\Bigg(\frac{S_2P_{375}}{24(N-1)^2N^2(1+N)^2(2+N)^2}
$$

$$
+\frac{P_{590}}{54(-2+N)(N-1)^4N^4(1+N)^3(2+N)^3}-\frac{3\big(70+165N+47N^2\big)S_3}{4(N-1)N(1+N)(2+N)}
$$

$$
-\frac{2\big(8-3N+N^2\big)S_{2,1}}{(N-1)N(1+N)(2+N)}\Bigg)S_1
$$

$$
\begin{aligned}
&+\Biggl(\frac{P_{435}}{36(N-1)^3N^3(1+N)^2(2+N)^2}-\frac{\bigl(514+951N+293N^2\bigr)S_2}{16(N-1)N(1+N)(2+N)}\Biggr)S_1^2\\
&-\frac{\bigl(1018+1287N+461N^2\bigr)S_2^2}{32(N-1)N(1+N)(2+N)}-\frac{55S_1^4}{32(N-1)N}\\
&+\frac{\bigl(-730-1515N-449N^2\bigr)S_4}{16(N-1)N(1+N)(2+N)}+\Biggl(\frac{P_{236}}{(-2+N)^2(N-1)^2N^3(1+N)}\\
&+\frac{2\bigl(-4+15N-11N^2+2N^3\bigr)S_1}{(-2+N)(N-1)^2N^2}-\frac{4S_1^2}{(N-1)N}-\frac{8S_2}{(N-1)N}\Biggr)S_{-2}\\
&-\frac{2S_{-2}^2}{(N-1)N}\\
&+\Biggl(\frac{2\bigl(-2+15N-16N^2+4N^3\bigr)}{(-2+N)(N-1)^2N^2}-\frac{16S_1}{(N-1)N}\Biggr)S_{-3}\\
&-\frac{3\bigl(10-5N+N^2\bigr)S_{3,1}}{(N-1)N(1+N)(2+N)}\\
&-\frac{21S_{-4}}{(N-1)N}+\frac{4(-1+2N)S_{-2,1}}{(N-1)^2N^2}+\frac{2S_{-2,2}}{(N-1)N}\\
&+\frac{3\bigl(4+N+N^2\bigr)S_{2,1,1}}{(N-1)N(1+N)(2+N)}\\
&+\frac{6S_{-3,1}}{(N-1)N}+\frac{4S_{-2,1,1}}{(N-1)N}+\Biggl(\frac{S_1P_9}{12(N-1)^2N^2(1+N)(2+N)}\\
&+\frac{P_{443}}{36(N-1)^3N^3(1+N)^2(2+N)^2}+\frac{33S_1^2}{8(N-1)N}\\
&+\frac{37S_2}{8(N-1)N}+\frac{S_{-2}}{(N-1)N}\Biggr)\zeta_2\\
&-\frac{54\zeta_2^2}{5N(1+N)(2+N)}+\Biggl(\frac{P_{382}}{9(-2+N)(N-1)^2N^2(1+N)^2(2+N)^2}\\
&+\frac{\bigl(176+129N+61N^2\bigr)S_1}{3(N-1)N(1+N)(2+N)}\Biggr)\zeta_3\Biggr).
\end{aligned}
\tag{192}
$$

6. The polarized OMEs

The expansion coefficients in the polarized flavor singlet case are calculated using the Larin scheme [20,86], as the complete calculation of the massless off–shell OMEs in the polarized case. By this one obtains the anomalous dimensions in the Larin scheme, which are finally transformed into the M scheme [20,25]. In the non–singlet case one could use a known Ward identity and obtain the anomalous dimension directly in the $\overline{\mathsf{MS}}$ scheme. However, the consistent renormalization of the singlet case requires to calculate the non-singlet also in the Larin scheme.

$$\Delta a_{qq}^{(1,0),\mathrm{NS}} = C_F \xi \left(-\frac{3}{N} + S_1\right) + C_F \left(-\frac{P_{38}}{N^2(1+N)^2} + \frac{2S_1}{N(1+N)} - 2S_1^2 - 6S_2\right), \tag{193}$$

$$\Delta a_{qq}^{(1,1),\mathrm{NS}} = C_F \xi \left(\frac{3+3N+2N^2}{2N^2} + \frac{(3-N)S_1}{2N} - \frac{1}{4}S_1^2 - \frac{3}{4}S_2\right) + C_F \Biggl(\frac{P_{298}}{N^3(1+N)^3} + \left(-\frac{(1+3N)\left(1+N+N^2\right)}{N^2(1+N)^2} + 3S_2\right)S_1 - \frac{S_1^2}{2N(1+N)} + \frac{1}{3}S_1^3 - \frac{3S_2}{2N(1+N)} + \frac{14}{3}S_3 + \left(\frac{2+3N+3N^2}{4N(1+N)} - S_1\right)\zeta_2\Biggr), \tag{194}$$

$$\Delta a_{qq}^{(1,2),\mathrm{NS}} = C_F \xi \Biggl[\frac{-3-3N-4N^3}{4N^3} + \left(-\frac{3(1+N)}{4N^2} + \frac{3S_2}{8}\right)S_1 + \frac{(-3+N)S_1^2}{8N} + \frac{1}{24}S_1^3 + \frac{3(N-3)S_2}{8N} + \frac{7}{12}S_3 + \left(\frac{3}{8N} - \frac{S_1}{8}\right)\zeta_2\Biggr] + C_F \Biggl(\frac{P_{470}}{2N^4(1+N)^4} + \left(\frac{P_{228}}{2N^3(1+N)^3} + \frac{3S_2}{4N(1+N)} - \frac{7}{3}S_3\right)S_1 + \left(\frac{(1+3N)\left(1+N+N^2\right)}{4N^2(1+N)^2} - \frac{3}{4}S_2\right)S_1^2 + \frac{S_1^3}{12N(1+N)} - \frac{1}{24}S_1^4 + \frac{3(1+3N)\left(1+N+N^2\right)S_2}{4N^2(1+N)^2} - \frac{9}{8}S_2^2 + \frac{7S_3}{6N(1+N)} - \frac{15}{4}S_4 + \left(\frac{P_{38}}{8N^2(1+N)^2} - \frac{S_1}{4N(1+N)} + \frac{1}{4}S_1^2 + \frac{3}{4}S_2\right)\zeta_2 + \left(-\frac{7\left(2+3N+3N^2\right)}{12N(1+N)} + \frac{7}{3}S_1\right)\zeta_3\Biggr), \tag{195}$$

$$\Delta a_{qq}^{(2,0),\mathrm{NS}} = \xi^2 \Biggl[C_A C_F \left(\frac{3+2N}{4N} - \frac{3(-2+N)S_1}{4N} - \frac{1}{4}S_1^2 - \frac{1}{4}S_2\right) + C_F^2 \left(\frac{3-N}{N} - \frac{3S_1}{N} + \frac{1}{2}S_1^2 + \frac{1}{2}S_2\right)\Biggr]$$

$$
\begin{aligned}
&+\xi\Biggl[C_F^2\Bigl(-\frac{(3+N)P_{101}}{N^3(1+N)^2}+\frac{\bigl(42+31N-5N^2\bigr)S_1^2}{2N(1+N)}\\
&+\Biggl(\frac{P_{133}}{N^2(1+N)^2}-16S_2\Biggr)S_1\\
&-4S_1^3+\frac{(2+N)(23+13N)S_2}{2N(1+N)}\Biggr)+C_AC_F\Biggl(\frac{-6-9N-7N^2}{2N^2}\\
&+\Biggl(\frac{3\bigl(-2-N+4N^2\bigr)}{2N(2+N)}-\frac{(-2+N)(N-1)S_2}{(1+N)(2+N)}\Biggr)S_1\\
&+\frac{5}{4}S_1^2+\frac{(-22+13N)S_2}{4(2+N)}\\
&-\frac{(-2+N)(N-1)S_3}{(1+N)(2+N)}+\frac{2(-2+N)(N-1)S_{2,1}}{(1+N)(2+N)}\\
&-\frac{6(-2+N)(N-1)\zeta_3}{(1+N)(2+N)}\Biggr)\Biggr]\\
&+C_F\Biggl(N_FT_F\Bigl(\frac{P_{285}}{54N^3(1+N)^3}\\
&+\Biggl(\frac{4\bigl(-3+29N+55N^2+41N^3\bigr)}{27N(1+N)^2}+\frac{8}{3}S_2\Biggr)S_1\\
&+\frac{2\bigl(-6+17N+17N^2\bigr)S_1^2}{9N(1+N)}+\frac{8}{9}S_1^3+\frac{2\bigl(-6+37N+37N^2\bigr)S_2}{9N(1+N)}-\frac{8}{9}S_3\Biggr)\\
&+C_A\Biggl(\frac{S_2P_{165}}{18N^2(1+N)^2(2+N)}+\frac{P_{551}}{216(N-1)N^4(1+N)^4(2+N)}\\
&+\Biggl(\frac{P_{380}}{27N^3(1+N)^3(2+N)}-\frac{2\bigl(34+33N+14N^2\bigr)S_2}{3(1+N)(2+N)}-6S_3-4S_{2,1}\Biggr)S_1\\
&+\Biggl(-\frac{11\bigl(-6+23N+23N^2\bigr)}{18N(1+N)}+S_2\Biggr)S_1^2\\
&+\frac{2\bigl(72+22N+33N^2+2N^3\bigr)S_3}{9N(1+N)(2+N)}\\
&-\frac{22}{9}S_1^3-6S_2^2-29S_4+\Biggl(\frac{4\bigl(-2-N-6N^2+N^3\bigr)}{(N-1)N(1+N)^2(2+N)}+\frac{16S_1}{N(1+N)}-8S_1^2\\
&-16S_2\Biggr)S_{-2}-4S_{-2}^2+\Biggl(\frac{20}{N(1+N)}-32S_1\Biggr)S_{-3}-28S_{-4}\\
&+\frac{4\bigl(4+N^2\bigr)S_{2,1}}{(1+N)(2+N)}
\end{aligned}
$$

$$
\begin{aligned}
&+6S_{3,1}-\frac{8S_{-2,1}}{N(1+N)}+8S_{-2,2}+24S_{-3,1}\\
&+\left(-\frac{12\big(2+5N+N^3\big)}{N(1+N)(2+N)}+12S_1\right)\zeta_3\\
&+6S_{2,1,1}+16S_{-2,1,1}\Bigg)\Bigg)+C_F^2\Bigg(\frac{P_{548}}{8(N-1)N^4(1+N)^4(2+N)}\\
&+\frac{S_2P_{172}}{N^2(1+N)^2(2+N)}+\Bigg(-\frac{2\big(52+40N+37N^2+11N^3\big)S_2}{N(1+N)(2+N)}+\frac{256}{3}S_3\\
&-\frac{2P_{415}}{N^3(1+N)^3(2+N)}\Bigg)S_1+\left(\frac{P_{27}}{N^2(1+N)^2}+48S_2\right)S_1^2\\
&-\frac{2\big(14+3N+3N^2\big)S_1^3}{3N(1+N)}\\
&+\frac{14}{3}S_1^4+30S_2^2-\frac{2\big(164+208N+237N^2+75N^3\big)S_3}{3N(1+N)(2+N)}+48S_4\\
&+\left(-\frac{8\big(-2-N-6N^2+N^3\big)}{(N-1)N(1+N)^2(2+N)}-\frac{32S_1}{N(1+N)}+16S_1^2+32S_2\right)S_{-2}\\
&+8S_{-2}^2\\
&+\left(-\frac{40}{N(1+N)}+64S_1\right)S_{-3}+56S_{-4}-\frac{4\big(-4+8N-N^2+N^3\big)S_{2,1}}{N(1+N)(2+N)}\\
&+\frac{16S_{-2,1}}{N(1+N)}-16S_{-2,2}-48S_{-3,1}-32S_{-2,1,1}+\Bigg(-\frac{\big(2+3N+3N^2\big)^2}{2N^2(1+N)^2}\\
&+\frac{4\big(2+3N+3N^2\big)S_1}{N(1+N)}-8S_1^2\Bigg)\zeta_2-\frac{48(-2+N)\zeta_3}{(1+N)(2+N)}\Bigg),
\end{aligned}
\tag{196}
$$

$$
\begin{aligned}
\Delta a_{qq}^{(2,1),\mathrm{NS}}=\xi^2\Bigg[&C_F^2\Bigg(\frac{3\big(-1-3N+N^2\big)}{2N^2}+\left(\frac{3\big(1+N^2\big)}{2N^2}-\frac{5S_2}{4}\right)S_1\\
&+\frac{(9-2N)S_1^2}{4N}-\frac{1}{4}S_1^3\\
&+\frac{(15-2N)S_2}{4N}-S_3\Bigg)+C_AC_F\Bigg(\frac{-3-12N-13N^2}{8N^2}\\
&+\left(\frac{-6-12N+5N^2}{8N^2}+\frac{5S_2}{8}\right)S_1+\frac{(-9+4N)S_1^2}{8N}\\
&+\frac{1}{8}S_1^3+\frac{(-15+7N)S_2}{8N}+\frac{1}{2}S_3+\left(\frac{3}{8N}-\frac{S_1}{8}\right)\zeta_2\Bigg)\Bigg]\\
&+\xi\Bigg[C_AC_F\Bigg(\frac{3\big(6+7N+16N^2+14N^3\big)}{4N^3}+\frac{S_2P_{18}}{8N^2(2+N)^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{S_3 P_{184}}{2N(1+N)^2(2+N)^2}\\
&+\Biggl(\frac{S_2 P_{177}}{8N(1+N)^2(2+N)^2}+\frac{P_{181}}{4N^3(2+N)^2}+\frac{3(-2+N)(N-1)S_3}{(1+N)(2+N)}\\
&-\frac{2(-2+N)(N-1)S_{2,1}}{(1+N)(2+N)}\Biggr)S_1\\
&+\Biggl(\frac{18+19N-31N^2}{8N(2+N)}+\frac{3(-2+N)(N-1)S_2}{4(1+N)(2+N)}\Biggr)S_1^2\\
&+\frac{\bigl(-2-N^2\bigr)S_2^2}{(1+N)(2+N)}+\frac{(-5+N)(-2+5N)S_4}{4(1+N)(2+N)}\\
&+\frac{\bigl(12-44N-51N^2-7N^3\bigr)S_{2,1}}{N(1+N)^2(2+N)^2}\\
&-\frac{3\bigl(2-5N+N^2\bigr)S_{3,1}}{(1+N)(2+N)}+\frac{(-2+N)(N-1)S_{2,1,1}}{(1+N)(2+N)}-\frac{5}{8}S_1^3\\
&+\Biggl(-\frac{15}{8N}+\frac{5S_1}{8}\Biggr)\zeta_2\\
&+\frac{9(-2+N)(N-1)\zeta_2^2}{5(1+N)(2+N)}+\Biggl(\frac{3P_{222}}{2N(1+N)^2(2+N)^2}\\
&+\frac{6\bigl(2+N^2\bigr)S_1}{(1+N)(2+N)}\Biggr)\zeta_3\Biggr)\\
&+C_F^2\Biggl(\frac{S_2 P_{12}}{4N^2(1+N)^2}+\frac{3P_{405}}{2N^4(1+N)^3}\\
&+\Biggl(\frac{\bigl(-158-113N+19N^2\bigr)S_2}{4N(1+N)}\\
&+\frac{P_{293}}{2N^3(1+N)^3}+\frac{52}{3}S_3\Biggr)S_1+\Biggl(-\frac{3P_{110}}{4N^2(1+N)^2}+12S_2\Biggr)S_1^2\\
&+\frac{7\bigl(-14-9N+3N^2\bigr)S_1^3}{12N(1+N)}+\frac{7}{6}S_1^4+\frac{15}{2}S_2^2+\frac{\bigl(-122-129N-45N^2\bigr)S_3}{6N(1+N)}\\
&+\Biggl(-\frac{3\bigl(2+3N+3N^2\bigr)}{2N^2(1+N)}+\frac{\bigl(14+15N+3N^2\bigr)S_1}{2N(1+N)}-2S_1^2\Biggr)\zeta_2\\
&+\Biggl(\frac{3\bigl(2+3N+3N^2\bigr)}{N(1+N)}-12S_1\Biggr)\zeta_3\Biggr)\Biggr]+C_F\Biggl(C_A\Biggl(-\frac{2S_{2,1}P_{326}}{N^2(1+N)^2(2+N)^2}\\
&+\frac{S_3 P_{425}}{27(N-1)N^2(1+N)^2(2+N)^2}+\frac{P_{593}}{2592(N-1)^2N^5(1+N)^5(2+N)^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\Biggl(\frac{S_2 P_{374}}{36N^2(1+N)^2(2+N)^2}+\frac{P_{569}}{162(N-1)N^4(1+N)^4(2+N)^2}+5S_2^2\\
&+\frac{2\bigl(-18+65N+57N^2+28N^3\bigr)S_3}{3N(1+N)(2+N)}+\frac{49}{2}S_4+\frac{4\bigl(-2-3N+3N^2\bigr)S_{2,1}}{N(1+N)(2+N)}\\
&-2S_{2,1,1}\Biggr)S_1+\Biggl(\frac{P_{427}}{108N^3(1+N)^3(2+N)}\\
&+\frac{\bigl(12+140N+147N^2+58N^3\bigr)S_2}{6N(1+N)(2+N)}\\
&+S_3+2S_{2,1}\Biggr)S_1^2+\Biggl(\frac{-198+491N+491N^2}{108N(1+N)}-\frac{1}{2}S_2\Biggr)S_1^3+\frac{11}{12}S_1^4\\
&+\Biggl(\frac{P_{489}}{108N^3(1+N)^3(2+N)^2}+18S_3+4S_{2,1}-8S_{-2,1}\Biggr)S_2\\
&+\frac{\bigl(-96+58N+231N^2+65N^3\bigr)S_2^2}{12N(1+N)(2+N)}\\
&+\frac{\bigl(-324+20N+147N^2+70N^3\bigr)S_4}{6N(1+N)(2+N)}\\
&+62S_5+\Biggl(-\frac{2P_{312}}{(N-1)^2N(1+N)^3(2+N)^2}\\
&+\Biggl(-\frac{8P_{73}}{(N-1)N^2(1+N)^2(2+N)}+16S_2\Biggr)S_1\\
&-\frac{8S_1^2}{N(1+N)}+\frac{8}{3}S_1^3-\frac{16S_2}{N(1+N)}+\frac{64}{3}S_3+8S_{2,1}\Biggr)S_{-2}\\
&+\Biggl(-\frac{4}{N(1+N)}+4S_1\Biggr)S_{-2}^2\\
&+\Biggl(-\frac{2P_{99}}{(N-1)N^2(1+N)^2(2+N)}-\frac{32S_1}{N(1+N)}\\
&+16S_1^2+20S_2+16S_{-2}\Biggr)S_{-3}+\Biggl(-\frac{42}{N(1+N)}+56S_1\Biggr)S_{-4}\\
&+54S_{-5}+16S_{2,3}+20S_{2,-3}-\frac{2\bigl(-2+9N-9N^2+N^3\bigr)S_{3,1}}{N(1+N)(2+N)}\\
&-13S_{4,1}+\frac{4(2+N)(1+3N)S_{-2,1}}{N^2(1+N)^2}\\
&+\frac{4S_{-2,2}}{N(1+N)}-12S_{-2,3}+\frac{12S_{-3,1}}{N(1+N)}-\frac{2\bigl(-6-N+9N^2+2N^3\bigr)S_{2,1,1}}{N(1+N)(2+N)}\\
&-28S_{-4,1}-8S_{2,1,-2}-18S_{2,2,1}-6S_{3,1,1}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{8S_{-2,1,1}}{N(1+N)}-8S_{-2,2,1}-24S_{-3,1,1}\\
&-3S_{2,1,1,1}-16S_{-2,1,1,1}+\Biggl(\frac{P_{369}}{72N^3(1+N)^3}-\frac{11\bigl(-6+23N+23N^2\bigr)S_1}{36N(1+N)}\\
&-\frac{11}{6}S_1^2-\frac{11}{6}S_2-2S_3\\
&+\Bigl(\frac{2}{N(1+N)}-4S_1\Bigr)S_{-2}-2S_{-3}+4S_{-2,1}\Biggr)\zeta_2\\
&+\Biggl(\frac{18\bigl(2+5N+N^3\bigr)}{5N(1+N)(2+N)}-\frac{18}{5}S_1\Biggr)\zeta_2^2+\Biggl(\frac{P_{418}}{2(N-1)N^2(1+N)^2(2+N)^2}\\
&-\frac{2\bigl(10+33N+8N^2\bigr)S_1}{(1+N)(2+N)}-3S_1^2+3S_2+12S_{-2}\Biggr)\zeta_3\Biggr)\\
&+N_FT_F\Biggl(\frac{S_2P_7}{27N^2(1+N)^2}+\frac{P_{493}}{648N^4(1+N)^4}\\
&+\Biggl(-\frac{2P_{277}}{81N^2(1+N)^3}-\frac{5\bigl(-6+13N+13N^2\bigr)S_2}{9N(1+N)}-\frac{16}{3}S_3\Biggr)\\
&\times S_1+\Biggl(\frac{P_{15}}{27N^2(1+N)^2}-\frac{10S_2}{3}\Biggr)S_1^2+\frac{\bigl(18-31N-31N^2\bigr)S_1^3}{27N(1+N)}-\frac{1}{3}S_1^4\\
&-\frac{7}{3}S_2^2-\frac{8\bigl(-9+28N+28N^2\bigr)S_3}{27N(1+N)}-\frac{10}{3}S_4\\
&+\Biggl(\frac{-2-53N-108N^2-45N^3}{18N(1+N)^2}\\
&+\frac{\bigl(-6+17N+17N^2\bigr)S_1}{9N(1+N)}+\frac{2}{3}S_1^2+\frac{2}{3}S_2\Biggr)\zeta_2\\
&+\Biggl(-\frac{2\bigl(2+3N+3N^2\bigr)}{N(1+N)}+8S_1\Biggr)\zeta_3\Biggr)\Biggr)\\
&+C_F^2\Biggl(\frac{4(3+N)S_{2,1}P_{64}}{N(1+N)^2(2+N)^2}+\frac{S_3P_{423}}{3(N-1)N^2(1+N)^2(2+N)^2}\\
&+\frac{P_{595}}{32(N-1)^2N^5(1+N)^5(2+N)^2}+\Biggl(\frac{P_{563}}{(N-1)N^4(1+N)^4(2+N)^2}\\
&+\frac{S_2P_{353}}{2N^2(1+N)^2(2+N)^2}-47S_2^2\\
&+\frac{2\bigl(76+36N+45N^2+11N^3\bigr)S_3}{N(1+N)(2+N)}-128S_4
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{16(-2+N)S_{2,1}}{(1+N)(2+N)}\Biggr)S_1 \\
&+\Biggl(\frac{\bigl(116+56N+45N^2+11N^3\bigr)S_2}{2N(1+N)(2+N)} \\
&+\frac{P_{413}}{2N^3(1+N)^3(2+N)}-60S_3\Biggr)S_1^2+\Biggl(\frac{P_{124}}{6N^2(1+N)^2}-\frac{56S_2}{3}\Biggr)S_1^3 \\
&+\frac{\bigl(10+N+N^2\bigr)S_1^4}{4N(1+N)}-S_1^5 \\
&+\Biggl(\frac{P_{479}}{2N^3(1+N)^3(2+N)^2}-\frac{232}{3}S_3+16S_{-2,1}\Biggr)S_2 \\
&+\frac{\bigl(204+228N+125N^2+47N^3\bigr)S_2^2}{4N(1+N)(2+N)} \\
&+\frac{\bigl(388+360N+361N^2+111N^3\bigr)S_4}{2N(1+N)(2+N)} \\
&-108S_5+\Biggl(\frac{4P_{312}}{(N-1)^2N(1+N)^3(2+N)^2} \\
&+\Biggl(\frac{16P_{73}}{(N-1)N^2(1+N)^2(2+N)} \\
&-32S_2\Biggr)S_1+\frac{16S_1^2}{N(1+N)}-\frac{16}{3}S_1^3+\frac{32S_2}{N(1+N)}-\frac{128}{3}S_3-16S_{2,1}\Biggr)S_{-2} \\
&+\Biggl(\frac{8}{N(1+N)}-8S_1\Biggr)S_{-2}^2+\Biggl(\frac{4P_{99}}{(N-1)N^2(1+N)^2(2+N)}+\frac{64S_1}{N(1+N)} \\
&-32S_1^2-40S_2-32S_{-2}\Biggr)S_{-3}+\Biggl(\frac{84}{N(1+N)}-112S_1\Biggr)S_{-4} \\
&-108S_{-5}-40S_{2,-3} \\
&+\frac{2\bigl(-12+32N-7N^2+3N^3\bigr)S_{3,1}}{N(1+N)(2+N)} \\
&-\frac{8(2+N)(1+3N)S_{-2,1}}{N^2(1+N)^2}-\frac{8S_{-2,2}}{N(1+N)} \\
&+24S_{-2,3}-\frac{24S_{-3,1}}{N(1+N)}+56S_{-4,1} \\
&+\frac{4\bigl(-4-4N+5N^2+N^3\bigr)S_{2,1,1}}{N(1+N)(2+N)}+16S_{2,1,-2} \\
&-\frac{16S_{-2,1,1}}{N(1+N)}+16S_{-2,2,1}+48S_{-3,1,1}+32S_{-2,1,1,1}
\end{aligned}
$$

$$
\begin{aligned}
&+\Biggl(\frac{P_{361}}{8N^3(1+N)^3}+\Biggl(\frac{P_{28}}{N^2(1+N)^2}+16S_2\Biggr)S_1\\
&-\frac{3\bigl(2+N+N^2\bigr)S_1^2}{N(1+N)}+4S_1^3-\frac{4\bigl(2+3N+3N^2\bigr)S_2}{N(1+N)}\\
&+4S_3+\Biggl(-\frac{4}{N(1+N)}+8S_1\Biggr)S_{-2}+4S_{-3}-8S_{-2,1}\Biggr)\zeta_2\\
&+\frac{72(-2+N)\zeta_2^2}{5(1+N)(2+N)}\\
&+\Biggl(\frac{P_{386}}{3(N-1)N^2(1+N)^2(2+N)^2}-\frac{8\bigl(14+55N+36N^2+15N^3\bigr)S_1}{3N(1+N)(2+N)}\\
&+\frac{92}{3}S_1^2-12S_2-24S_{-2}\Biggr)\zeta_3\Biggr),
\end{aligned} \tag{197}
$$

$$
\Delta b_{qq}^{(1,0),\mathrm{NS}}=\frac{2C_F\xi}{N}-\frac{4C_F}{1+N}, \tag{198}
$$

$$
\Delta b_{qq}^{(1,1),\mathrm{NS}}=-C_F\xi\Biggl(\frac{1+N}{N^2}+\frac{S_1}{N}\Biggr)+C_F\Biggl(\frac{2(3+N)}{(1+N)^2}+\frac{2S_1}{1+N}\Biggr), \tag{199}
$$

$$
\begin{aligned}
\Delta b_{qq}^{(1,2),\mathrm{NS}}=&\,C_F\xi\Biggl(\frac{1+N}{2N^3}+\frac{(1+N)S_1}{2N^2}+\frac{S_1^2}{4N}+\frac{3S_2}{4N}-\frac{\zeta_2}{4N}\Biggr)\\
&+C_F\Biggl(-\frac{2(3+N)}{(1+N)^3}-\frac{(3+N)S_1}{(1+N)^2}\\
&-\frac{S_1^2}{2(1+N)}-\frac{3S_2}{2(1+N)}+\frac{\zeta_2}{2(1+N)}\Biggr),
\end{aligned} \tag{200}
$$

$$
\begin{aligned}
\Delta b_{qq}^{(2,0),\mathrm{NS}}=&\,\xi^2\Biggl[C_AC_F\Biggl(-\frac{1}{2N}-\frac{S_1}{N}\Biggr)+C_F^2\Biggl(-\frac{2}{N}+\frac{2S_1}{N}\Biggr)\Biggr]\\
&+\xi\Biggl[C_F^2\Biggl(\frac{2P_{111}}{N^3(1+N)^2}+\frac{2\bigl(-4-5N+N^2\bigr)S_1}{N^2(1+N)}-\frac{12S_1^2}{N}-\frac{12S_2}{N}\Biggr)\\
&+C_AC_F\Biggl(\frac{2+3N}{N^2}+\Biggl(\frac{2+7N}{N(2+N)}-\frac{2(N-1)S_2}{(1+N)(2+N)}\Biggr)S_1\\
&+\frac{6S_2}{2+N}-\frac{2(N-1)S_3}{(1+N)(2+N)}+\frac{4(N-1)S_{2,1}}{(1+N)(2+N)}-\frac{12(N-1)\zeta_3}{(1+N)(2+N)}\Biggr)\Biggr]\\
&+C_F\Biggl(N_FT_F(\frac{16\bigl(-1+6N+4N^2\bigr)}{9N(1+N)^2}+\frac{8(N-1)S_1}{3N(1+N)}\Biggr)\\
&+C_A\Biggl(-\frac{4P_{271}}{9(N-1)N(1+N)^3(2+N)}
\end{aligned}
$$

$$
\begin{aligned}
&+\Biggl(-\frac{2\bigl(-12+8N+65N^2+23N^3\bigr)}{3N^2(1+N)(2+N)} \\
&+\frac{8S_2}{(1+N)(2+N)}\Biggr)S_1+\frac{4(-2+N)S_2}{N(2+N)}+\frac{32S_{-2}}{(N-1)(1+N)(2+N)} \\
&+\frac{8S_3}{(1+N)(2+N)}-\frac{16S_{2,1}}{(1+N)(2+N)}+\frac{48\zeta_3}{(1+N)(2+N)}\Biggr)\Biggr) \\
&+C_F^2\Biggl(-\frac{8P_{310}}{(N-1)N^2(1+N)^3(2+N)}+\Biggl(-\frac{4P_{81}}{N^2(1+N)^2(2+N)} \\
&-\frac{16S_2}{(1+N)(2+N)}\Biggr)S_1+\frac{24S_1^2}{1+N} \\
&+\frac{8\bigl(2+7N+2N^2\bigr)S_2}{N(1+N)(2+N)}-\frac{16S_3}{(1+N)(2+N)} \\
&-\frac{64S_{-2}}{(N-1)(1+N)(2+N)}+\frac{32S_{2,1}}{(1+N)(2+N)}-\frac{96\zeta_3}{(1+N)(2+N)}\Biggr),
\end{aligned}
\tag{201}
$$

$$
\begin{aligned}
\Delta b_{qq}^{(2,1),\mathrm{NS}} = \xi^2\Biggl[&C_F^2\Biggl(\frac{1+3N}{N^2}-\frac{S_1}{N^2}-\frac{3S_1^2}{2N}-\frac{5S_2}{2N}\Biggr) \\
&+C_AC_F\Biggl(\frac{1+4N}{4N^2}+\frac{(1+2N)S_1}{2N^2}+\frac{3S_1^2}{4N}+\frac{5S_2}{4N}-\frac{\zeta_2}{4N}\Biggr)\Biggr] \\
&+\xi\Biggl[C_F^2\Biggl(\frac{P_{295}}{N^4(1+N)^3}+\Biggl(\frac{P_{37}}{N^3(1+N)^2}+\frac{22S_2}{N}\Biggr)S_1 \\
&+\frac{\bigl(16+21N+3N^2\bigr)S_1^2}{2N^2(1+N)}+\frac{14S_1^3}{3N}+\frac{\bigl(8+11N-3N^2\bigr)S_2}{2N^2(1+N)}+\frac{28S_3}{3N} \\
&+\Biggl(\frac{2+3N+3N^2}{N^2(1+N)}-\frac{4S_1}{N}\Biggr)\zeta_2\Biggr) \\
&+C_AC_F\Biggl(\frac{-6-7N-16N^2}{2N^3}-\frac{4S_3P_{89}}{N(1+N)^2(2+N)^2} \\
&+\Biggl(\frac{P_{31}}{2N^3(2+N)^2}+\frac{S_2P_{42}}{N(1+N)^2(2+N)^2} \\
&+\frac{6(N-1)S_3}{(1+N)(2+N)}-\frac{4(N-1)S_{2,1}}{(1+N)(2+N)}\Biggr) \\
&\times S_1-\Biggl(\frac{3(2+7N)}{4N(2+N)}-\frac{3(N-1)S_2}{2(1+N)(2+N)}\Biggr)S_1^2
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\big(48+4N+132N^2+59N^3\big)S_2}{4N^2(2+N)^2} \\
&+\frac{(1-N)S_2^2}{(1+N)(2+N)}+\frac{7(N-1)S_4}{2(1+N)(2+N)}+\frac{8\big(-1+4N+5N^2+N^3\big)S_{2,1}}{N(1+N)^2(2+N)^2} \\
&-\frac{8(N-1)S_{3,1}}{(1+N)(2+N)}+\frac{2(N-1)S_{2,1,1}}{(1+N)(2+N)}+\frac{5\zeta_2}{4N}+\frac{18(N-1)\zeta_2^2}{5(1+N)(2+N)} \\
&+\left(\frac{12P_{77}}{N(1+N)^2(2+N)^2}+\frac{6(N-1)S_1}{(1+N)(2+N)}\right)\zeta_3\Bigg)\Bigg] \\
&+C_F\Bigg(N_FT_F\Bigg[-\frac{8P_{140}}{27N^2(1+N)^3}-\frac{4\big(-3-13N+33N^2+19N^3\big)S_1}{9N^2(1+N)^2} \\
&-\frac{2(N-1)S_1^2}{N(1+N)}-\frac{10(N-1)S_2}{3N(1+N)}+\frac{2(N-1)\zeta_2}{3N(1+N)}\Bigg] \\
&+C_A\Bigg[-\frac{4S_3P_{218}}{(N-1)N(1+N)^2(2+N)^2}+\frac{S_2P_{247}}{6N^2(1+N)^2(2+N)^2} \\
&+\frac{2P_{518}}{27(N-1)^2N^2(1+N)^4(2+N)^2}+\Bigg(\frac{P_{484}}{9(N-1)N^3(1+N)^3(2+N)^2} \\
&-\frac{2S_2P_{69}}{N(1+N)^2(2+N)^2}-\frac{24S_3}{(1+N)(2+N)}+\frac{16S_{2,1}}{(1+N)(2+N)}\Bigg)S_1 \\
&+\left(\frac{-12+8N+65N^2+23N^3}{2N^2(1+N)(2+N)}-\frac{6S_2}{(1+N)(2+N)}\right)S_1^2 \\
&+\frac{4S_2^2}{(1+N)(2+N)}-\frac{14S_4}{(1+N)(2+N)} \\
&+\left(-\frac{32\big(-3+5N+4N^2\big)}{(N-1)^2(1+N)^2(2+N)^2}-\frac{32S_1}{(N-1)(1+N)(2+N)}\right) \\
&\times S_{-2}-\frac{64S_{-3}}{(N-1)(1+N)(2+N)}+\frac{8\big(-4+6N+11N^2+3N^3\big)S_{2,1}}{N(1+N)^2(2+N)^2} \\
&+\frac{32S_{3,1}}{(1+N)(2+N)}-\frac{8S_{2,1,1}}{(1+N)(2+N)} \\
&+\frac{(5-17N)\zeta_2}{6N(1+N)}-\frac{72\zeta_2^2}{5(1+N)(2+N)} \\
&+\left(-\frac{12P_{213}}{(N-1)N(1+N)^2(2+N)^2}-\frac{24S_1}{(1+N)(2+N)}\right)\zeta_3\Bigg]\Bigg) \\
&+C_F^2\Bigg(-\frac{8S_3P_{212}}{3(N-1)N(1+N)^2(2+N)^2}+\frac{S_2P_{264}}{N^2(1+N)^2(2+N)^2} \\
&+\frac{2P_{528}}{(N-1)^2N^3(1+N)^4(2+N)^2}+\Big(-\frac{4S_2P_{112}}{N(1+N)^2(2+N)^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4P_{448}}{(N-1)N^3(1+N)^3(2+N)^2}\\
&+\frac{48S_3}{(1+N)(2+N)}-\frac{32S_{2,1}}{(1+N)(2+N)}\Biggr)S_1\\
&+\Biggl(\frac{P_{80}}{N^2(1+N)^2(2+N)}+\frac{12S_2}{(1+N)(2+N)}\Biggr)S_1^2\\
&-\frac{28S_1^3}{3(1+N)}-\frac{8S_2^2}{(1+N)(2+N)}\\
&+\frac{28S_4}{(1+N)(2+N)}\\
&+\Biggl(\frac{64\bigl(-3+5N+4N^2\bigr)}{(N-1)^2(1+N)^2(2+N)^2}+\frac{64S_1}{(N-1)(1+N)(2+N)}\Biggr)\\
&\times S_{-2}+\frac{128S_{-3}}{(N-1)(1+N)(2+N)}-\frac{16\bigl(-4+6N+11N^2+3N^3\bigr)S_{2,1}}{N(1+N)^2(2+N)^2}\\
&-\frac{64S_{3,1}}{(1+N)(2+N)}+\frac{16S_{2,1,1}}{(1+N)(2+N)}\\
&+\Biggl(-\frac{2\bigl(2+3N+3N^2\bigr)}{N(1+N)^2}+\frac{8S_1}{1+N}\Biggr)\zeta_2\\
&+\frac{144\zeta_2^2}{5(1+N)(2+N)}+\Biggl(-\frac{24P_{203}}{(N-1)N(1+N)^2(2+N)^2}\\
&+\frac{48S_1}{(1+N)(2+N)}\Biggr)\zeta_3\Biggr),
\end{aligned}
\tag{202}
$$

$$
\begin{aligned}
\Delta a_{qq}^{(2,0),\mathrm{PS}} = {}& C_F N_F T_F\Biggl(-\frac{8S_1P_{87}}{N^3(1+N)^3}+\frac{4P_{328}}{N^4(1+N)^4}\\
&+\frac{8(N-1)(2+N)S_1^2}{N^2(1+N)^2}+\frac{16(N-1)(2+N)S_2}{N^2(1+N)^2}-\frac{4(N-1)(2+N)\zeta_2}{N^2(1+N)^2}\Biggr),
\end{aligned}
\tag{203}
$$

$$
\begin{aligned}
\Delta a_{qq}^{(2,1),\mathrm{PS}} = {}& C_F N_F T_F\Biggl(\frac{4S_1^2P_{87}}{N^3(1+N)^3}+\frac{8S_2P_{87}}{N^3(1+N)^3}+\frac{2P_{457}}{N^5(1+N)^5}\\
&+\Biggl(-\frac{4P_{328}}{N^4(1+N)^4}-\frac{16(N-1)(2+N)S_2}{N^2(1+N)^2}\Biggr)S_1\\
&-\frac{8(N-1)(2+N)S_1^3}{3N^2(1+N)^2}-\frac{52(N-1)(2+N)S_3}{3N^2(1+N)^2}\\
&+\Biggl(-\frac{2P_{87}}{N^3(1+N)^3}+\frac{4(N-1)(2+N)S_1}{N^2(1+N)^2}\Biggr)\zeta_2
\end{aligned}
$$

$$+\frac{64(N-1)(2+N)\zeta_3}{3N^2(1+N)^2}\Bigg), \tag{204}$$

$$\Delta b_{qq}^{(2,0),\mathrm{PS}} = C_F N_F T_F \Bigg(-\frac{16\big(3+4N-6N^2+3N^3\big)}{N^3(1+N)^3} + \frac{32(N-1)S_1}{N^2(1+N)^2}\Bigg), \tag{205}$$

$$\Delta b_{qq}^{(2,1),\mathrm{PS}} = C_F N_F T_F \Bigg(-\frac{8P_{232}}{N^4(1+N)^4} + \frac{16\big(3+4N-6N^2+3N^3\big)S_1}{N^3(1+N)^3}$$
$$-\frac{16(N-1)S_1^2}{N^2(1+N)^2} - \frac{32(N-1)S_2}{N^2(1+N)^2} + \frac{8(N-1)\zeta_2}{N^2(1+N)^2}\Bigg), \tag{206}$$

$$\Delta a_{qg}^{(1,0)} = N_F T_F \Bigg(-\frac{4\big(1+3N-N^2+N^3\big)}{N^2(1+N)^2} + \frac{4(N-1)S_1}{N(1+N)}\Bigg), \tag{207}$$

$$\Delta a_{qg}^{(1,1)} = N_F T_F \Bigg(-\frac{2P_{51}}{N^3(1+N)^3} + \frac{2\big(1+3N-N^2+N^3\big)S_1}{N^2(1+N)^2}$$
$$+\frac{(1-N)S_1^2}{N(1+N)} - \frac{3(N-1)S_2}{N(1+N)}$$
$$+\frac{(N-1)\zeta_2}{N(1+N)}\Bigg), \tag{208}$$

$$\Delta a_{qg}^{(1,2)} = N_F T_F \Bigg(\frac{P_{198}}{N^4(1+N)^4} + \Bigg(\frac{P_{51}}{N^3(1+N)^3} + \frac{3(N-1)S_2}{2N(1+N)}\Bigg)S_1 + \frac{(N-1)S_1^3}{6N(1+N)}$$
$$+\frac{\big(-1-3N+N^2-N^3\big)S_1^2}{2N^2(1+N)^2} - \frac{3\big(1+3N-N^2+N^3\big)S_2}{2N^2(1+N)^2} + \frac{7(N-1)S_3}{3N(1+N)}$$
$$+\Bigg(\frac{1+3N-N^2+N^3}{2N^2(1+N)^2} + \frac{(1-N)S_1}{2N(1+N)}\Bigg)\zeta_2 - \frac{7(N-1)\zeta_3}{3N(1+N)}\Bigg), \tag{209}$$

$$\Delta a_{qg}^{(2,0)} = \xi^2 C_A N_F T_F \Bigg[\frac{1+5N-N^2-N^3}{N^2(1+N)^2} + \frac{(1-N)S_1}{N(1+N)}\Bigg]$$
$$+\xi C_A N_F T_F \Bigg[\frac{6(N-1)S_1^2}{N(1+N)} - \frac{4(N-1)(1+2N)\big(3+N^2\big)}{N^3(1+N)^2}$$
$$+\frac{4\big(1-5N-N^2+N^3\big)S_1}{N^2(1+N)^2} + \frac{10(N-1)S_2}{N(1+N)}\Bigg]$$
$$+N_F^2 T_F^2 \Bigg(-\frac{16\big(-15-73N+15N^2+13N^3\big)}{27N^2(1+N)^2}$$
$$-\frac{80(N-1)S_1}{9N(1+N)} + \frac{8(N-1)\zeta_2}{3N(1+N)}\Bigg)$$
$$+C_F N_F T_F \Bigg(\frac{2P_{407}}{N^4(1+N)^4} + \Bigg(-\frac{4P_{248}}{N^3(1+N)^3} - \frac{28(N-1)S_2}{N(1+N)}\Bigg)S_1$$

$$
\begin{aligned}
& -\frac{28(N-1)S_1^3}{3N(1+N)} \\
& +\frac{4(1+3N)\big(7-4N+3N^2\big)S_1^2}{N^2(1+N)^2}+\frac{4\big(1+8N-N^2+8N^3\big)S_2}{N^2(1+N)^2} \\
& +\frac{64(N-1)S_3}{3N(1+N)} \\
& -\frac{16(N-1)S_{2,1}}{N(1+N)}+\left(-\frac{2(N-1)\big(2+3N+3N^2\big)}{N^2(1+N)^2}+\frac{8(N-1)S_1}{N(1+N)}\right)\zeta_2 \\
& -\frac{48(N-1)\zeta_3}{N(1+N)}\Bigg)+C_A N_F T_F\Bigg(\frac{4P_{516}}{27(N-1)N^4(1+N)^4(2+N)} \\
& +\frac{8S_2P_{63}}{N^2(1+N)^2(2+N)} \\
& +\left(\frac{4P_{362}}{9N^3(1+N)^3(2+N)}-\frac{4\big(-38+19N+17N^2\big)S_2}{N(1+N)(2+N)}\right) \\
& \times S_1+\frac{8(-4+7N)S_1^2}{N^2(1+N)^2}-\frac{28(N-1)S_1^3}{3N(1+N)}-\frac{8\big(-86+43N+40N^2\big)S_3}{3N(1+N)(2+N)} \\
& +\left(-\frac{16\big(3-8N-N^2+2N^3\big)}{(N-1)N(1+N)^2(2+N)}-\frac{32(N-1)S_1}{N(1+N)}\right)S_{-2}-\frac{40(N-1)S_{-3}}{N(1+N)} \\
& +\frac{16\big(-4+2N+N^2\big)S_{2,1}}{N(1+N)(2+N)}+\frac{16(N-1)S_{-2,1}}{N(1+N)} \\
& -\Bigg(\frac{2(N-1)\big(24+11N+11N^2\big)}{3N^2(1+N)^2} \\
& -\frac{8(N-1)S_1}{N(1+N)}\Bigg)\zeta_2+\frac{24(-2+3N)\zeta_3}{N(2+N)}\Bigg), \qquad (210)
\end{aligned}
$$

$$
\begin{aligned}
\Delta a_{qg}^{(2,1)} = {} & \xi^2 C_A N_F T_F\Bigg[\frac{P_{217}}{2N^3(1+N)^3}-\frac{\big(1+5N-N^2-N^3\big)S_1}{2N^2(1+N)^2}+\frac{(N-1)S_1^2}{4N(1+N)} \\
& +\frac{3(N-1)S_2}{4N(1+N)}+\frac{(1-N)\zeta_2}{2N(1+N)}\Bigg] \\
& +\xi C_A N_F T_F\Bigg[-\frac{2P_{316}}{N^4(1+N)^3}+\Bigg(\frac{2P_{233}}{N^3(1+N)^3} \\
& -\frac{13(N-1)S_2}{N(1+N)}\Bigg)S_1+\frac{\big(-1+13N+N^2-N^3\big)S_1^2}{N^2(1+N)^2}-\frac{7(N-1)S_1^3}{3N(1+N)} \\
& +\frac{\big(-3+23N+3N^2-3N^3\big)S_2}{N^2(1+N)^2}-\frac{38(N-1)S_3}{3N(1+N)}+\Bigg(\frac{2(N-1)^2}{N^2(1+N)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2(N-1)S_1}{N(1+N)}\Big)\zeta_2+\frac{12(N-1)\zeta_3}{N(1+N)}\Big]\\
&+N_F^2T_F^2\Bigg(\frac{8\big(-15-73N+15N^2+13N^3\big)}{27N^2(1+N)^2}\\
&\times S_1+\frac{8P_{272}}{81N^3(1+N)^3}+\frac{20(N-1)S_1^2}{9N(1+N)}+\frac{20(N-1)S_2}{3N(1+N)}+\Bigg(-\frac{4(N-1)S_1}{3N(1+N)}\\
&-\frac{4\big(-3-19N+3N^2+7N^3\big)}{9N^2(1+N)^2}\Bigg)\zeta_2-\frac{56(N-1)\zeta_3}{9N(1+N)}\Bigg)\\
&+C_FN_FT_F\Bigg(-\frac{2S_3P_{129}}{3N^2(1+N)^2(2+N)}+\frac{S_2P_{348}}{N^3(1+N)^3(2+N)}\\
&+\frac{P_{497}}{N^5(1+N)^5}+\Bigg(-\frac{2S_2P_{141}}{N^2(1+N)^2(2+N)}\\
&-\frac{2P_{468}}{N^4(1+N)^4(2+N)}+\frac{4(N-1)S_3}{N(1+N)}\\
&+\frac{16(N-1)S_{2,1}}{N(1+N)}\Bigg)S_1+\Bigg(\frac{P_{260}}{N^3(1+N)^3}+\frac{19(N-1)S_2}{N(1+N)}\Bigg)S_1^2+\frac{5(N-1)S_1^4}{2N(1+N)}\\
&-\frac{2\big(17+40N-21N^2+20N^3\big)S_1^3}{3N^2(1+N)^2}-\frac{16\big(4+5N-4N^2+N^3\big)S_{2,1}}{N^2(1+N)^2(2+N)}\\
&+\frac{15(N-1)S_2^2}{2N(1+N)}-\frac{57(N-1)S_4}{N(1+N)}+\frac{48(N-1)S_{3,1}}{N(1+N)}-\frac{8(N-1)S_{2,1,1}}{N(1+N)}\\
&+\Bigg(\frac{P_{197}}{N^3(1+N)^3}+\frac{2\big(4+11N-6N^2+7N^3\big)S_1}{N^2(1+N)^2}\\
&-\frac{6(N-1)S_1^2}{N(1+N)}-\frac{2(N-1)S_2}{N(1+N)}\Bigg)\\
&\times\zeta_2+\frac{72(N-1)\zeta_2^2}{5N(1+N)}+\Bigg(-\frac{16P_{118}}{3N^2(1+N)^2(2+N)}+\frac{16(N-1)S_1}{3N(1+N)}\Bigg)\zeta_3\Bigg)\\
&+C_AN_FT_F\Bigg(\frac{8S_{2,1}P_{84}}{N^2(1+N)^2(2+N)^2}\\
&-\frac{4S_3P_{338}}{3(N-1)N^2(1+N)^2(2+N)^2}\\
&+\frac{S_2P_{383}}{3N^3(1+N)^3(2+N)^2}-\frac{2P_{591}}{81(N-1)^2N^5(1+N)^5(2+N)^2}\\
&+\Bigg(-\frac{4S_2P_{209}}{N^2(1+N)^2(2+N)^2}-\frac{2P_{544}}{27(N-1)N^4(1+N)^4(2+N)^2}
\end{aligned}
$$

$$
+\frac{8\big(-34+17N+14N^2\big)S_3}{N(1+N)(2+N)}-\frac{8\big(-6+3N+N^2\big)S_{2,1}}{N(1+N)(2+N)}\Bigg)S_1
$$
$$
+\Bigg(\frac{P_{290}}{9N^3(1+N)^3(2+N)}+\frac{\big(-82+41N+35N^2\big)S_2}{N(1+N)(2+N)}\Bigg)S_1^2
$$
$$
-\frac{4(-8+15N)S_1^3}{3N^2(1+N)^2}+\frac{5(N-1)S_1^4}{2N(1+N)}+\frac{\big(-142+71N+79N^2\big)S_2^2}{2N(1+N)(2+N)}
$$
$$
+\frac{\big(-394+197N+183N^2\big)S_4}{N(1+N)(2+N)}
$$
$$
+\Bigg(\frac{32S_1P_{61}}{(N-1)N^2(1+N)^2(2+N)}+\frac{8P_{313}}{(N-1)^2N(1+N)^3(2+N)^2}
$$
$$
+\frac{16(N-1)S_1^2}{N(1+N)}+\frac{32(N-1)S_2}{N(1+N)}\Bigg)S_{-2}
$$
$$
+\frac{8(N-1)S_{-2}^2}{N(1+N)}+\Bigg(\frac{8P_{104}}{(N-1)N^2(1+N)^2(2+N)}
$$
$$
+\frac{64(N-1)S_1}{N(1+N)}\Bigg)S_{-3}+\frac{84(N-1)S_{-4}}{N(1+N)}-\frac{4\big(-34+17N+9N^2\big)S_{3,1}}{N(1+N)(2+N)}
$$
$$
+\frac{16(1+3N)S_{-2,1}}{N^2(1+N)^2}-\frac{8(N-1)S_{-2,2}}{N(1+N)}-\frac{24(N-1)S_{-3,1}}{N(1+N)}
$$
$$
-\frac{4\big(2-N+N^2\big)S_{2,1,1}}{N(1+N)(2+N)}
$$
$$
-\frac{16(N-1)S_{-2,1,1}}{N(1+N)}+\Bigg(\frac{P_{273}}{9N^3(1+N)^3}-\frac{\big(48-61N-11N^3\big)S_1}{3N^2(1+N)^2}
$$
$$
-\frac{6(N-1)S_1^2}{N(1+N)}
$$
$$
-\frac{14(N-1)S_2}{N(1+N)}-\frac{4(N-1)S_{-2}}{N(1+N)}\Bigg)\zeta_2+\Bigg(\frac{2P_{366}}{9(N-1)N^2(1+N)^2(2+N)^2}
$$
$$
-\frac{8\big(-32+16N+25N^2\big)S_1}{3N(1+N)(2+N)}\Bigg)\zeta_3-\frac{36(-2+3N)\zeta_2^2}{5N(2+N)}\Bigg), \tag{211}
$$

$$
\Delta a_{gq}^{(1,0)}=C_F\Bigg(-\frac{2\big(-2-N+4N^2+N^3\big)}{N^2(1+N)^2}+\frac{2(2+N)S_1}{N(1+N)}\Bigg), \tag{212}
$$

$$
\Delta a_{gq}^{(1,1)}=C_F\Bigg(\frac{P_{53}}{N^3(1+N)^3}+\frac{\big(-2-N+4N^2+N^3\big)S_1}{N^2(1+N)^2}+\frac{(-2-N)S_1^2}{2N(1+N)}
$$
$$
-\frac{3(2+N)S_2}{2N(1+N)}+\frac{(2+N)\zeta_2}{2N(1+N)}\Bigg), \tag{213}
$$

$$\Delta a_{gq}^{(1,2)} = C_F\Biggl(\frac{P_{237}}{2N^4(1+N)^4} + \Biggl(-\frac{P_{53}}{2N^3(1+N)^3} + \frac{3(2+N)S_2}{4N(1+N)}\Biggr)S_1 + \frac{(2+N)S_1^3}{12N(1+N)} + \frac{\bigl(2+N-4N^2-N^3\bigr)S_1^2}{4N^2(1+N)^2} - \frac{3\bigl(-2-N+4N^2+N^3\bigr)S_2}{4N^2(1+N)^2} + \frac{7(2+N)S_3}{6N(1+N)} + \Biggl(\frac{-2-N+4N^2+N^3}{4N^2(1+N)^2} + \frac{(-2-N)S_1}{4N(1+N)}\Biggr)\zeta_2 - \frac{7(2+N)\zeta_3}{6N(1+N)}\Biggr), \tag{214}$$

$$\Delta a_{gq}^{(2,0)} = \xi\Biggl[C_F^2\Biggl(-\frac{6(2+N)\bigl(3+3N-N^2+N^3\bigr)}{N^3(1+N)^2} - \frac{2\bigl(10+15N+7N^2\bigr)S_1}{N^2(1+N)^2} + \frac{3(2+N)S_1^2}{N(1+N)} + \frac{5(2+N)S_2}{N(1+N)}\Biggr) + C_AC_F\Biggl(\frac{(2+3N)\bigl(2-N+N^2\bigr)}{(N-1)N^3(2+N)} + \frac{2(2+3N)S_{-2}}{(N-1)N(2+N)}\Biggr)\Biggr]$$
$$+C_F\Biggl(N_FT_F\Biggl(-\frac{8P_{257}}{27N^3(1+N)^3} - \frac{8\bigl(6+7N+2N^3\bigr)S_1}{9N^2(1+N)^2} - \frac{8(2+N)S_2}{3N(1+N)} + \frac{4(2+N)\zeta_2}{3N(1+N)}\Biggr) + C_A\Biggl(\frac{2S_2P_{134}}{3N^2(1+N)^2(2+N)} + \frac{2P_{515}}{27(N-1)N^4(1+N)^4(2+N)} + \Biggl(-\frac{2P_{337}}{9N^3(1+N)^3(2+N)} - \frac{2(2+3N)(10+3N)S_2}{N(1+N)(2+N)}\Biggr)S_1 - \frac{14(2+N)S_1^3}{3N(1+N)} + \frac{4\bigl(-1-2N+12N^2+3N^3\bigr)S_1^2}{N^2(1+N)^2} - \frac{4\bigl(-20+4N+N^2\bigr)S_3}{3N(1+N)(2+N)} + \Biggl(-\frac{8\bigl(5+3N+2N^2\bigr)}{(N-1)(1+N)^2(2+N)} - \frac{16(2+N)S_1}{N(1+N)}\Biggr)S_{-2} - \frac{20(2+N)S_{-3}}{N(1+N)} - \frac{64S_{2,1}}{N(1+N)(2+N)} + \frac{8(2+N)S_{-2,1}}{N(1+N)} + \Biggl(-\frac{(2+N)\bigl(24+11N+11N^2\bigr)}{3N^2(1+N)^2} + \frac{4(2+N)S_1}{N(1+N)}\Biggr)\zeta_2 - \frac{12\bigl(-4+12N+3N^2\bigr)\zeta_3}{N(1+N)(2+N)}\Biggr)\Biggr)$$

$$
\begin{aligned}
&+C_F^2\Biggl(\frac{P_{409}}{N^4(1+N)^4}+\Biggl(\frac{2P_{231}}{N^3(1+N)^3}-\frac{2(38+15N)S_2}{N(1+N)}\Biggr)S_1\\
&+\frac{2(2+N)\bigl(2+3N^2\bigr)S_1^2}{N^2(1+N)^2}-\frac{14(2+N)S_1^3}{3N(1+N)}\\
&+\frac{2\bigl(8+12N+22N^2+3N^3\bigr)S_2}{N^2(1+N)^2}-\frac{4(74+31N)S_3}{3N(1+N)}+\frac{32S_{2,1}}{N(1+N)}\\
&+\Biggl(-\frac{(2+N)\bigl(2+3N+3N^2\bigr)}{N^2(1+N)^2}+\frac{4(2+N)S_1}{N(1+N)}\Biggr)\zeta_2+\frac{48\zeta_3}{1+N}\Biggr),
\end{aligned}
\tag{215}
$$

$$
\begin{aligned}
\Delta a_{gq}^{(2,1)}=\xi\Biggl[&C_F^2\Biggl(\frac{3P_{199}}{N^4(1+N)^3}+\Biggl(\frac{P_{241}}{N^3(1+N)^3}-\frac{13(2+N)S_2}{2N(1+N)}\Biggr)S_1\\
&+\frac{3\bigl(6+9N+5N^2\bigr)S_1^2}{2N^2(1+N)^2}\\
&-\frac{7(2+N)S_1^3}{6N(1+N)}+\frac{\bigl(38+57N+29N^2\bigr)S_2}{2N^2(1+N)^2}-\frac{19(2+N)S_3}{3N(1+N)}\\
&+\Biggl(-\frac{3(2+N)}{N^2(1+N)}+\frac{(2+N)S_1}{N(1+N)}\Biggr)\zeta_2+\frac{6(2+N)\zeta_3}{N(1+N)}\Biggr)\\
&+C_AC_F\Biggl(\frac{P_{322}}{2(N-1)^2N^4(2+N)^2}-\frac{(2+3N)\bigl(2-N+N^2\bigr)S_1}{(N-1)N^3(2+N)}\\
&+\frac{(-2-3N)S_3}{(N-1)N(2+N)}+\Biggl(-\frac{2\bigl(10+3N+2N^2\bigr)}{(N-1)^2N(2+N)^2}\\
&-\frac{2(2+3N)S_1}{(N-1)N(2+N)}\Biggr)S_{-2}-\frac{4(2+3N)S_{-3}}{(N-1)N(2+N)}-\frac{3(2+3N)\zeta_3}{(N-1)N(2+N)}\Biggr)\Biggr]\\
&+C_F\Biggl(N_FT_F(\frac{4P_{416}}{81N^4(1+N)^4}+\Biggl(\frac{4P_{257}}{27N^3(1+N)^3}+\frac{4(2+N)S_2}{3N(1+N)}\Biggr)S_1\\
&+\frac{2\bigl(6+7N+2N^3\bigr)S_1^2}{9N^2(1+N)^2}+\frac{2\bigl(42+41N-24N^2+4N^3\bigr)S_2}{9N^2(1+N)^2}+\frac{16(2+N)S_3}{3N(1+N)}\\
&+\Biggl(-\frac{2\bigl(18+17N-12N^2+N^3\bigr)}{9N^2(1+N)^2}-\frac{2(2+N)S_1}{3N(1+N)}\Biggr)\zeta_2-\frac{100(2+N)\zeta_3}{9N(1+N)}\Biggr)\\
&+C_A\Biggl(\frac{4S_{2,1}P_{96}}{N^2(1+N)^2(2+N)^2}-\frac{2S_3P_{345}}{3(N-1)N^2(1+N)^2(2+N)^2}\\
&+\frac{S_2P_{390}}{18N^3(1+N)^3(2+N)^2}+\frac{P_{586}}{81(N-1)^2N^5(1+N)^5(2+N)^2}\\
&+\Biggl(\frac{S_2P_{169}}{3N^2(1+N)^2(2+N)^2}+\frac{P_{526}}{27(N-1)N^4(1+N)^4(2+N)^2}
\end{aligned}
$$

$$
+\frac{4\big(-4+20N+5N^2\big)S_3}{N(1+N)(2+N)}+\frac{4\big(20+4N+N^2\big)S_{2,1}}{N(1+N)(2+N)}\Bigg)S_1
+\Bigg(\frac{P_{363}}{18N^3(1+N)^3(2+N)}+\frac{\big(44+92N+23N^2\big)S_2}{2N(1+N)(2+N)}\Bigg)S_1^2+\frac{5(2+N)S_1^4}{4N(1+N)}
-\frac{2\big(-5-6N+28N^2+7N^3\big)S_1^3}{3N^2(1+N)^2}+\frac{\big(156+92N+23N^2\big)S_2^2}{4N(1+N)(2+N)}
+\frac{\big(-68+44N+11N^2\big)S_4}{2N(1+N)(2+N)}+\Bigg(\frac{16S_1P_{75}}{(N-1)N^2(1+N)^2(2+N)}
-\frac{4P_{303}}{(N-1)^2N(1+N)^3(2+N)^2}+\frac{8(2+N)S_1^2}{N(1+N)}+\frac{16(2+N)S_2}{N(1+N)}\Bigg)S_{-2}
+\frac{4(2+N)S_{-2}^2}{N(1+N)}+\Bigg(\frac{4P_{109}}{(N-1)N^2(1+N)^2(2+N)}+\frac{32(2+N)S_1}{N(1+N)}\Bigg)S_{-3}
+\frac{42(2+N)S_{-4}}{N(1+N)}+\frac{2\big(76+12N+3N^2\big)S_{3,1}}{N(1+N)(2+N)}-\frac{8(2+N)(1+2N)S_{-2,1}}{N^2(1+N)^2}
-\frac{4(2+N)S_{-2,2}}{N(1+N)}-\frac{12(2+N)S_{-3,1}}{N(1+N)}-\frac{2\big(28+12N+3N^2\big)S_{2,1,1}}{N(1+N)(2+N)}
-\frac{8(2+N)S_{-2,1,1}}{N(1+N)}
+\Bigg(\frac{P_{269}}{18N^3(1+N)^3}+\frac{\big(24+22N+129N^2+35N^3\big)S_1}{6N^2(1+N)^2}-\frac{3(2+N)S_1^2}{N(1+N)}
-\frac{(2+N)S_2}{N(1+N)}-\frac{2(2+N)S_{-2}}{N(1+N)}\Bigg)\zeta_2+\frac{18\big(-4+12N+3N^2\big)\zeta_2^2}{5N(1+N)(2+N)}
+\Bigg(\frac{P_{355}}{9(N-1)N^2(1+N)^2(2+N)^2}+\frac{4\big(-28+44N+11N^2\big)S_1}{3N(1+N)(2+N)}\Bigg)\zeta_3\Bigg)\Bigg)
+C_F^2\Bigg(\frac{S_3P_{41}}{3N^2(1+N)^2(2+N)}+\frac{S_2P_{331}}{2N^3(1+N)^3(2+N)}+\frac{P_{496}}{2N^5(1+N)^5}
+\Bigg(\frac{S_2P_{35}}{N^2(1+N)^2(2+N)}+\frac{P_{445}}{N^4(1+N)^4(2+N)}
+\frac{2(62+19N)S_3}{N(1+N)}-\frac{32S_{2,1}}{N(1+N)}\Bigg)
\times S_1+\Bigg(\frac{P_{188}}{2N^3(1+N)^3}+\frac{(86+31N)S_2}{2N(1+N)}\Bigg)S_1^2-\frac{(2+N)\big(4-N+6N^2\big)S_1^3}{3N^2(1+N)^2}
+\frac{5(2+N)S_1^4}{4N(1+N)}+\frac{(110+71N)S_2^2}{4N(1+N)}+\frac{(286+115N)S_4}{2N(1+N)}
$$

$$
\begin{aligned}
&-\frac{8(-3+N)(2+3N)S_{2,1}}{N(1+N)^2(2+N)} \\
&-\frac{64S_{3,1}}{N(1+N)}+\frac{16S_{2,1,1}}{N(1+N)}+\Biggl(\frac{(2+N)P_{94}}{2N^3(1+N)^3}+\frac{(2+N)\bigl(2+N+3N^2\bigr)S_1}{N^2(1+N)^2} \\
&-\frac{3(2+N)S_1^2}{N(1+N)}-\frac{7(2+N)S_2}{N(1+N)}\Biggr)\zeta_2-\frac{72\zeta_2^2}{5(1+N)} \\
&+\Biggl(-\frac{2P_{79}}{3N^2(1+N)^2(2+N)}-\frac{8(16+17N)S_1}{3N(1+N)}\Biggr)\zeta_3\Biggr), \qquad (216)
\end{aligned}
$$

$$
\Delta b_{gq}^{(1,0)}=\frac{4C_F}{N(1+N)}, \qquad (217)
$$

$$
\Delta b_{gq}^{(1,1)}=C_F\Biggl(\frac{2\bigl(-1-2N+N^2\bigr)}{N^2(1+N)^2}-\frac{2S_1}{N(1+N)}\Biggr), \qquad (218)
$$

$$
\begin{aligned}
\Delta b_{gq}^{(1,2)}=C_F\Biggl(&\frac{1+3N+3N^2-3N^3}{N^3(1+N)^3}+\frac{\bigl(1+2N-N^2\bigr)S_1}{N^2(1+N)^2}+\frac{S_1^2}{2N(1+N)} \\
&+\frac{3S_2}{2N(1+N)}-\frac{\zeta_2}{2N(1+N)}\Biggr), \qquad (219)
\end{aligned}
$$

$$
\begin{aligned}
\Delta b_{gq}^{(2,0)}=\xi\Biggl[&C_F^2\Biggl(-\frac{4(2+N)\bigl(-3-3N+2N^2\bigr)}{N^3(1+N)^2}+\frac{4(4+3N)S_1}{N^2(1+N)}\Biggr) \\
&+C_AC_F\Biggl(-\frac{2(1+N)\bigl(2-N+N^2\bigr)}{(N-1)N^3(2+N)}-\frac{4(1+N)S_{-2}}{(N-1)N(2+N)}\Biggr)\Biggr] \\
&+C_F\Biggl(N_FT_F\Bigl(-\frac{16\bigl(9+22N+N^2\bigr)}{9N^2(1+N)^2}-\frac{16S_1}{3N(1+N)}\Biggr) \\
&+C_A\Biggl(\frac{4P_{357}}{9(N-1)N^3(1+N)^3(2+N)}+\Biggl(\frac{4P_{146}}{3N^3(1+N)^2(2+N)} \\
&+\frac{4(-2+N)S_2}{N(1+N)(2+N)}\Biggr)S_1-\frac{24S_1^2}{N(1+N)}-\frac{4\bigl(-2+5N+5N^2\bigr)S_2}{N^2(1+N)(2+N)} \\
&+\frac{4(-2+N)S_3}{N(1+N)(2+N)}+\frac{8\bigl(3+N^2\bigr)S_{-2}}{(N-1)N(1+N)(2+N)}-\frac{8(-2+N)S_{2,1}}{N(1+N)(2+N)} \\
&+\frac{24(-2+N)\zeta_3}{N(1+N)(2+N)}\Biggr)\Biggr) \\
&+C_F^2\Biggl(\frac{4\bigl(-4-22N-5N^2+3N^3\bigr)}{N^2(1+N)^3}-\frac{4(-4+5N)S_1}{N^2(1+N)} \\
&-\frac{16S_2}{N(1+N)}\Biggr), \qquad (220)
\end{aligned}
$$

$$\Delta b_{gq}^{(2,1)} = \xi \Biggl[C_F^2 \Biggl(\frac{2P_{211}}{N^4(1+N)^3} - \frac{2(3+4N)\bigl(4+N-N^2\bigr)S_1}{N^3(1+N)^2} - \frac{(8+7N)S_1^2}{N^2(1+N)} - \frac{(16+13N)S_2}{N^2(1+N)} + \frac{2(2+N)\zeta_2}{N^2(1+N)} \Biggr)$$
$$+ C_A C_F \Biggl(\frac{P_{301}}{(N-1)^2 N^4 (2+N)^2} + \frac{2(1+N)\bigl(2-N+N^2\bigr)S_1}{(N-1)N^3(2+N)} + \frac{2(1+N)S_3}{(N-1)N(2+N)} + \Biggl(\frac{4\bigl(3+2N+N^2\bigr)}{(N-1)^2 N (2+N)^2} + \frac{4(1+N)S_1}{(N-1)N(2+N)} \Biggr) S_{-2} + \frac{8(1+N)S_{-3}}{(N-1)N(2+N)} + \frac{6(1+N)\zeta_3}{(N-1)N(2+N)} \Biggr) \Biggr]$$
$$+ C_F \Biggl(N_F T_F \Biggl(\frac{8P_{142}}{27N^3(1+N)^3} + \frac{8\bigl(9+22N+N^2\bigr)S_1}{9N^2(1+N)^2} + \frac{4S_1^2}{3N(1+N)} + \frac{28S_2}{3N(1+N)} - \frac{4\zeta_2}{N(1+N)} \Biggr)$$
$$+ C_A \Biggl(\frac{S_2 P_{167}}{3N^3(1+N)^2(2+N)^2} + \frac{2S_3 P_{261}}{3(N-1)N^2(1+N)^2(2+N)^2} - \frac{2P_{546}}{27(N-1)^2N^4(1+N)^4(2+N)^2} + \Biggl(-\frac{2P_{463}}{9(N-1)N^4(1+N)^3(2+N)^2} + \frac{2S_2 P_{136}}{N^2(1+N)^2(2+N)^2} - \frac{12(-2+N)S_3}{N(1+N)(2+N)} + \frac{8(-2+N)S_{2,1}}{N(1+N)(2+N)} \Biggr) S_1$$
$$+ \Biggl(\frac{P_{10}}{3N^3(1+N)^2(2+N)} - \frac{3(-2+N)S_2}{N(1+N)(2+N)} \Biggr) S_1^2 + \frac{2(-2+N)S_2^2}{N(1+N)(2+N)} + \frac{28S_1^3}{3N(1+N)} - \frac{7(-2+N)S_4}{N(1+N)(2+N)} + \Biggl(-\frac{8\bigl(3+N^2\bigr)\bigl(-3+5N+4N^2\bigr)}{(N-1)^2N(1+N)^2(2+N)^2} - \frac{8\bigl(3+N^2\bigr)S_1}{(N-1)N(1+N)(2+N)} \Biggr) S_{-2} - \frac{4\bigl(-24-42N-7N^2+5N^3\bigr)S_{2,1}}{N(1+N)^2(2+N)^2}$$
$$- \frac{16\bigl(3+N^2\bigr)S_{-3}}{(N-1)N(1+N)(2+N)} + \frac{16(-2+N)S_{3,1}}{N(1+N)(2+N)} - \frac{4(-2+N)S_{2,1,1}}{N(1+N)(2+N)} + \Biggl(\frac{16+11N+11N^2}{N^2(1+N)^2} - \frac{8S_1}{N(1+N)} \Biggr) \zeta_2 - \frac{36(-2+N)\zeta_2^2}{5N(1+N)(2+N)}$$
$$+ \Biggl(\frac{12P_{244}}{(N-1)N^2(1+N)^2(2+N)^2} - \frac{12(-2+N)S_1}{N(1+N)(2+N)} \Biggr) \zeta_3 \Biggr) \Biggr)$$

$$+C_F^2\Biggl(-\frac{2P_{246}}{N^3(1+N)^4}+\Biggl(-\frac{2P_{74}}{N^3(1+N)^2(2+N)}+\frac{48S_2}{N(1+N)(2+N)}\Biggr)S_1$$
$$+\frac{\bigl(-12-5N+11N^2\bigr)S_1^2}{N^2(1+N)^2}+\frac{\bigl(-40+30N+99N^2+49N^3\bigr)S_2}{N^2(1+N)^2(2+N)}$$
$$+\frac{24(4+N)S_3}{N(1+N)(2+N)}+\frac{16(-4+N)S_{2,1}}{N(1+N)(2+N)}$$
$$-\frac{4(2+N)\zeta_2}{N(1+N)^2}-\frac{96(N-1)\zeta_3}{N(1+N)(2+N)}\Biggr), \tag{221}$$

$$\Delta a_{gg}^{(1,0)}=-\frac{C_A\xi^2}{4}+\xi C_A\Biggl(\frac{-1+N}{N}+S_1\Biggr)-\frac{20T_FN_F}{9}$$
$$+C_A\Biggl(-\frac{P_{16}}{9N^2(1+N)^2}+\frac{8S_1}{N(1+N)}-2S_1^2-6S_2\Biggr), \tag{222}$$

$$\Delta a_{gg}^{(1,1)}=\frac{C_A\xi^2}{4}+C_A\xi\Biggl(\frac{1+N}{2N^2}+\frac{(1-N)S_1}{2N}-\frac{1}{4}S_1^2-\frac{3}{4}S_2\Biggr)$$
$$+C_A\Biggl(\frac{P_{288}}{27N^3(1+N)^3}+\Biggl(\frac{-4-9N+2N^2-N^3}{N^2(1+N)^2}+3S_2\Biggr)S_1$$
$$-\frac{2S_1^2}{N(1+N)}+\frac{1}{3}S_1^3-\frac{6S_2}{N(1+N)}+\frac{14}{3}S_3$$
$$+\Biggl(\frac{24+11N+11N^2}{12N(1+N)}-S_1\Biggr)\zeta_2\Biggr)+N_FT_F\Biggl(\frac{56}{27}-\frac{\zeta_2}{3}\Biggr), \tag{223}$$

$$\Delta a_{gg}^{(1,2)}=\xi^2C_A\Biggl[-\frac{1}{4}+\frac{\zeta_2}{32}\Biggr]$$
$$+\xi C_A\Biggl[\frac{-1-N}{4N^3}+\Biggl(\frac{-1-N}{4N^2}+\frac{3S_2}{8}\Biggr)S_1+\frac{(N-1)S_1^2}{8N}+\frac{1}{24}S_1^3$$
$$+\frac{3(N-1)S_2}{8N}+\frac{7}{12}S_3+\Biggl(\frac{1-N}{8N}-\frac{S_1}{8}\Biggr)\zeta_2\Biggr]+C_A\Biggl(\frac{P_{487}}{162N^4(1+N)^4}$$
$$+\Biggl(\frac{P_{56}}{2N^3(1+N)^3}+\frac{3S_2}{N(1+N)}-\frac{7}{3}S_3\Biggr)S_1$$
$$+\Biggl(\frac{4+9N-2N^2+N^3}{4N^2(1+N)^2}-\frac{3}{4}S_2\Biggr)S_1^2$$
$$+\frac{S_1^3}{3N(1+N)}-\frac{1}{24}S_1^4+\frac{3\bigl(4+9N-2N^2+N^3\bigr)S_2}{4N^2(1+N)^2}$$
$$-\frac{9}{8}S_2^2+\frac{14S_3}{3N(1+N)}$$

$$
-\frac{15}{4}S_4+\left(\frac{P_{16}}{72N^2(1+N)^2}-\frac{S_1}{N(1+N)}+\frac{1}{4}S_1^2+\frac{3}{4}S_2\right)\zeta_2
+\left(\frac{7}{3}S_1-\frac{7\big(24+11N+11N^2\big)}{36N(1+N)}\right)\zeta_3\Bigg)
+N_FT_F\left(-\frac{164}{81}+\frac{5\zeta_2}{18}+\frac{7\zeta_3}{9}\right), \qquad (224)
$$

$$
\begin{aligned}
\Delta a_{gg}^{(2,0)} = {}& \xi^4\frac{C_A^2}{16}+\xi^3C_A^2\left[\frac{4-3N}{16N}-\frac{S_1}{4}\right] \\
&+\xi^2\Bigg[\frac{2C_AN_FT_F}{3}+C_A^2\Bigg(\frac{\big(-10+7N+9N^2\big)S_1}{4N(1+N)} \\
&+\frac{P_{384}}{12(N-1)N^3(1+N)^2(2+N)}+\frac{3}{4}S_1^2 \\
&+\frac{3}{2}S_2+\frac{(2-N)S_{-2}}{4(N-1)N(2+N)}-\frac{1}{8}\zeta_3\Bigg)\Bigg] \\
&+\xi\Bigg[C_AN_FT_F\left(\frac{2\big(18+16N+7N^2\big)}{9N^2}-\frac{4(-3+8N)S_1}{9N}-\frac{2}{3}S_1^2-\frac{14}{3}S_2\right) \\
&+C_A^2\Bigg(\frac{S_2P_{147}}{12N^2(1+N)(2+N)}+\frac{P_{417}}{18(N-1)N^3(1+N)^2(2+N)} \\
&+\left(\frac{P_{368}}{18N^3(1+N)^2(2+N)}+\frac{\big(-2-60N-96N^2-33N^3\big)S_2}{2N(1+N)(2+N)}\right)S_1 \\
&+\frac{\big(216+37N-35N^2\big)S_1^2}{12N(1+N)}-4S_1^3+\frac{\big(-2+4N-N^3\big)S_3}{2N(1+N)(2+N)} \\
&+\frac{(-7+3N)S_{-2}}{(N-1)N(2+N)} \\
&+\frac{\big(2-4N+N^3\big)S_{2,1}}{N(1+N)(2+N)}-\frac{3\big(4-6N+3N^2+3N^3\big)\zeta_3}{2N(1+N)(2+N)}\Bigg)\Bigg] \\
&+C_FN_FT_F\Bigg(\frac{P_{527}}{3(N-1)N^4(1+N)^4(2+N)} \\
&-\frac{8(2+N)\big(3+6N-4N^2+3N^3\big)}{N^3(1+N)^3} \\
&\times S_1+\frac{8(N-1)(2+N)S_1^2}{N^2(1+N)^2}+\frac{16(N-1)(2+N)S_2}{N^2(1+N)^2} \\
&-\frac{64S_{-2}}{(N-1)N(1+N)(2+N)}-\frac{4(N-1)(2+N)\zeta_2}{N^2(1+N)^2}+16\zeta_3\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+C_A N_F T_F\Biggl(-\frac{2P_{491}}{81(N-1)N^3(1+N)^3(2+N)}\\
&+\Biggl(\frac{4P_{267}}{9N^2(1+N)^2(2+N)}+\frac{8(3+2N)S_2}{2+N}\Biggr)S_1\\
&+\frac{2\bigl(-48+37N+37N^2\bigr)S_1^2}{9N(1+N)}\\
&+\frac{16}{9}S_1^3+\frac{2\bigl(-68+52N+127N^2+39N^3\bigr)S_2}{3N(1+N)(2+N)}+\frac{8(47+28N)S_3}{9(2+N)}\\
&+\frac{32S_{-2}}{(N-1)N(1+N)(2+N)}-\frac{16(N-1)S_{2,1}}{3(2+N)}+\Biggl(\frac{4\bigl(24+11N+11N^2\bigr)}{9N(1+N)}\\
&-\frac{16}{3}S_1\Biggr)\zeta_2-\frac{8(8+N)\zeta_3}{2+N}\Biggr)+C_A^2\Biggl(\frac{P_{552}}{162(N-1)N^4(1+N)^4(2+N)}\\
&+\frac{S_2P_{164}}{6N^2(1+N)^2(2+N)}+\Biggl(\frac{\bigl(-400-269N-114N^2-41N^3\bigr)S_2}{N(1+N)(2+N)}\\
&+\frac{P_{378}}{9N^3(1+N)^3(2+N)}+\frac{238}{3}S_3-4S_{2,1}\Biggr)S_1\\
&+\Biggl(\frac{P_3}{18N^2(1+N)^2}+49S_2\Biggr)S_1^2\\
&-\frac{4\bigl(84+11N+11N^2\bigr)S_1^3}{9N(1+N)}+\frac{\bigl(-3144-2561N-1650N^2-589N^3\bigr)S_3}{9N(1+N)(2+N)}\\
&+\frac{14}{3}S_1^4+24S_2^2+19S_4\\
&+\Biggl(\frac{8\bigl(8-13N-3N^2+2N^3\bigr)}{(N-1)N(1+N)^2(2+N)}+\frac{8(-7+N)S_1}{N(1+N)}+8S_1^2+16S_2\Biggr)S_{-2}\\
&+\Biggl(\frac{4(-19+N)}{N(1+N)}+32S_1\Biggr)S_{-3}\\
&+\frac{2\bigl(48+11N+12N^2+13N^3\bigr)S_{2,1}}{3N(1+N)(2+N)}\\
&+4S_{-2}^2+28S_{-4}+6S_{3,1}-\frac{8(-3+N)S_{-2,1}}{N(1+N)}-8S_{-2,2}-24S_{-3,1}+6S_{2,1,1}\\
&-16S_{-2,1,1}\\
&+\Biggl(-\frac{\bigl(24+11N+11N^2\bigr)^2}{18N^2(1+N)^2}+\frac{4\bigl(24+11N+11N^2\bigr)S_1}{3N(1+N)}-8S_1^2\Biggr)\zeta_2\\
&+\Biggl(\frac{-24-8N-33N^2-25N^3}{N(1+N)(2+N)}+12S_1\Biggr)\zeta_3\Biggr)
\end{aligned}
$$

$$+N_F^2 T_F^2\left(\frac{848}{81}-\frac{8\zeta_2}{9}\right), \tag{225}$$

$$\Delta a_{gg}^{(2,1)} = -\xi^4\frac{1}{8}C_A^2$$
$$+\xi^3 C_A^2\left[\frac{-4-12N-N^2}{32N^2}+\frac{(-1+3N)S_1}{8N}+\frac{1}{16}S_1^2+\frac{3}{16}S_2+\frac{1}{16}\zeta_2\right]$$
$$+\xi^2\Bigg[C_A^2\Big(\frac{P_{565}}{72(N-1)^2N^4(1+N)^3(2+N)^2}$$
$$+\left(\frac{P_{387}}{8(N-1)N^3(1+N)^2(2+N)}-\frac{5}{4}S_2\right)S_1$$
$$+\frac{\left(7-3N-6N^2\right)S_1^2}{8N(1+N)}-\frac{5}{24}S_1^3$$
$$+\frac{\left(-6+59N-28N^2-28N^3\right)S_3}{24(N-1)N(2+N)}+\frac{\left(4-2N-3N^2\right)S_2}{2N(1+N)}$$
$$+\left(\frac{2-5N}{4(N-1)^2N(2+N)^2}+\frac{(-2+N)S_1}{4(N-1)N(2+N)}\right)S_{-2}$$
$$+\frac{(-2+N)S_{-3}}{2(N-1)N(2+N)}+\left(\frac{-21-13N-16N^2}{24N(1+N)}+\frac{3}{8}S_1\right)\zeta_2+\frac{3}{80}\zeta_2^2$$
$$+\frac{\left(-18+13N-2N^2-2N^3\right)\zeta_3}{24(N-1)N(2+N)}\Big)+C_AN_FT_F\left(-\frac{2}{3}+\frac{\zeta_2}{12}\right)\Bigg]$$
$$+\xi\Bigg[C_AN_FT_F\Bigg(\frac{-126-144N-50N^2-53N^3}{27N^3}$$
$$+\left(\frac{2\left(-27-24N+25N^2\right)}{27N^2}+\frac{7S_2}{3}\right)S_1$$
$$+\frac{(-3+8N)S_1^2}{9N}+\frac{1}{9}S_1^3+\frac{7(-3+8N)S_2}{9N}+\frac{62}{9}S_3$$
$$+\left(\frac{4-3N}{6N}-\frac{2S_1}{3}\right)\zeta_2-4\zeta_3\Bigg)$$
$$+C_A^2\Bigg(\frac{S_{2,1}P_{229}}{2N(1+N)^2(2+N)^2}+\frac{S_2P_{377}}{72N^3(1+N)^2(2+N)^2}$$
$$+\frac{S_3P_{381}}{36(N-1)N^2(1+N)^2(2+N)^2}+\frac{P_{554}}{108(N-1)^2N^4(1+N)^3(2+N)^2}$$
$$+\Bigg(\frac{S_2P_{346}}{24N^2(1+N)^2(2+N)^2}+\frac{P_{522}}{108(N-1)N^4(1+N)^3(2+N)^2}$$

$$
+\frac{\big(18+178N+321N^2+116N^3\big)S_3}{6N(1+N)(2+N)}
$$
$$
+\frac{\big(-2+2N-3N^2-2N^3\big)S_{2,1}}{N(1+N)(2+N)}\Bigg)S_1
$$
$$
+\Bigg(\frac{P_{287}}{72N^3(1+N)^2(2+N)}+\frac{\big(6+184N+294N^2+101N^3\big)S_2}{8N(1+N)(2+N)}\Bigg)S_1^2
$$
$$
+\frac{\big(-504-67N+101N^2\big)S_1^3}{72N(1+N)}+\frac{7}{6}S_1^4
$$
$$
+\frac{\big(-2+56N+78N^2+25N^3\big)S_2^2}{4N(1+N)(2+N)}
$$
$$
+\frac{\big(14-40N-18N^2+N^3\big)S_4}{8N(1+N)(2+N)}
$$
$$
+\Bigg(\frac{-5+17N}{(N-1)^2N(2+N)^2}+\frac{(7-3N)S_1}{(N-1)N(2+N)}\Bigg)
$$
$$
\times S_{-2}-\frac{2(-7+3N)S_{-3}}{(N-1)N(2+N)}+\frac{\big(-8+18N+3N^2-3N^3\big)S_{3,1}}{2N(1+N)(2+N)}
$$
$$
+\frac{\big(2+2N+9N^2+4N^3\big)S_{2,1,1}}{2N(1+N)(2+N)}+\Bigg(\frac{-96+37N-5N^2+54N^3}{24N^2(1+N)}
$$
$$
+\frac{\big(144+59N+11N^2\big)S_1}{24N(1+N)}-2S_1^2\Bigg)\zeta_2+\frac{9\big(4-6N+3N^2+3N^3\big)\zeta_2^2}{20N(1+N)(2+N)}
$$
$$
+\Bigg(\frac{P_{406}}{(N-1)N^2(1+N)^2(2+N)^2}
$$
$$
-\frac{3\big(-2+12N+12N^2+3N^3\big)S_1}{2N(1+N)(2+N)}\Bigg)\zeta_3\Bigg)\Bigg]
$$
$$
+C_FN_FT_F\Bigg(-\frac{4S_3P_{119}}{3(N-1)N^2(1+N)^2(2+N)}
$$
$$
-\frac{4\zeta_3P_{342}}{3(N-1)N^2(1+N)^2(2+N)}
$$
$$
+\frac{P_{600}}{36(N-1)^2N^5(1+N)^5(2+N)^2}+\Bigg(-\frac{4P_{455}}{(N-1)N^4(1+N)^4(2+N)}
$$
$$
-\frac{16(N-1)(2+N)S_2}{N^2(1+N)^2}\Bigg)S_1+\frac{4(2+N)\big(3+6N-4N^2+3N^3\big)S_1^2}{N^3(1+N)^3}
$$
$$
-\frac{8(N-1)(2+N)S_1^3}{3N^2(1+N)^2}+\frac{8(2+N)\big(3+6N-4N^2+3N^3\big)S_2}{N^3(1+N)^3}
$$

$$
\begin{aligned}
&+\Biggl(-\frac{32\bigl(4-11N-6N^2+N^3\bigr)}{(N-1)^2N(1+N)^2(2+N)^2}\\
&+\frac{64S_1}{(N-1)N(1+N)(2+N)}\Biggr)S_{-2}\\
&+\frac{128S_{-3}}{(N-1)N(1+N)(2+N)}\\
&+\Biggl(\frac{P_{300}}{N^3(1+N)^3}+\frac{4(N-1)(2+N)S_1}{N^2(1+N)^2}\Biggr)\zeta_2-\frac{24}{5}\zeta_2^2\Biggr)\\
&+C_AN_FT_F\Biggl(\frac{8S_{2,1}P_{93}}{9N(1+N)(2+N)^2}-\frac{4S_3P_{270}}{9(N-1)N(1+N)(2+N)^2}\\
&+\frac{S_2P_{289}}{9N^2(1+N)^2(2+N)^2}+\frac{P_{584}}{54(N-1)^2N^4(1+N)^4(2+N)^2}\\
&+\Biggl(-\frac{16S_{2,1}}{2+N}+\frac{S_2P_{19}}{3N(1+N)(2+N)^2}\\
&-\frac{2P_{517}}{27(N-1)N^3(1+N)^3(2+N)^2}-\frac{8(23+25N)S_3}{9(2+N)}\Biggr)S_1\\
&+\Biggl(\frac{P_{168}}{27N^2(1+N)^2(2+N)}-\frac{2(11+10N)S_2}{3(2+N)}\Biggr)S_1^2\\
&+\frac{\bigl(32-17N-17N^2\bigr)S_1^3}{9N(1+N)}\\
&-\frac{4}{9}S_1^4-\frac{4(17+7N)S_2^2}{3(2+N)}-\frac{2(29+18N)S_4}{2+N}\\
&+\Biggl(\frac{16\bigl(4-11N-6N^2+N^3\bigr)}{(N-1)^2N(1+N)^2(2+N)^2}\\
&-\frac{32S_1}{(N-1)N(1+N)(2+N)}\Biggr)S_{-2}-\frac{64S_{-3}}{(N-1)N(1+N)(2+N)}\\
&+\frac{8(N-2)S_{3,1}}{2+N}+\frac{8(7+2N)S_{2,1,1}}{3(2+N)}\\
&+\Biggl(\frac{P_8}{9N^2(1+N)^2}+\frac{\bigl(-24+19N+19N^2\bigr)S_1}{3N(1+N)}+2S_1^2+6S_2\Biggr)\\
&\times\zeta_2+\frac{12(8+N)\zeta_2^2}{5(2+N)}\\
&+\Biggl(-\frac{2P_{280}}{27(N-1)N(1+N)(2+N)^2}+\frac{8(109+41N)S_1}{9(2+N)}\Biggr)\zeta_3\Biggr)
\end{aligned}
$$

$$
\begin{aligned}
&+C_A^2\Biggl(\frac{S_3 P_{426}}{9(N-1)N^2(1+N)^2(2+N)^2}\\
&+\frac{P_{594}}{648(N-1)^2N^5(1+N)^5(2+N)^2}-\frac{2S_{2,1}P_{349}}{9N^2(1+N)^2(2+N)^2}\\
&+\Biggl(\frac{S_2 P_{370}}{12N^2(1+N)^2(2+N)^2}-42S_2^2-\frac{207}{2}S_4-2S_{2,1,1}\\
&+\frac{P_{567}}{27(N-1)N^4(1+N)^4(2+N)^2}\\
&+\frac{\big(4572+2891N+1227N^2+496N^3\big)S_3}{9N(1+N)(2+N)}\\
&+\frac{2\big(-4+7N+7N^2\big)S_{2,1}}{N(1+N)(2+N)}\Biggr)S_1\\
&+\Biggl(\frac{\big(2760+1709N+564N^2+211N^3\big)S_2}{12N(1+N)(2+N)}\\
&+\frac{P_{429}}{108N^3(1+N)^3(2+N)}-59S_3+2S_{2,1}\Biggr)S_1^2\\
&+\Biggl(\frac{P_{157}}{36N^2(1+N)^2}-\frac{115S_2}{6}\Biggr)S_1^3\\
&+\frac{\big(2024+1539N+968N^2+361N^3\big)S_4}{4N(1+N)(2+N)}\\
&+\frac{\big(1056+809N+414N^2+133N^3\big)S_2^2}{6N(1+N)(2+N)}\\
&+\frac{\big(90+11N+11N^2\big)S_1^4}{9N(1+N)}-S_1^5+\Biggl(\frac{P_{490}}{36N^3(1+N)^3(2+N)^2}\\
&-\frac{178}{3}S_3+4S_{2,1}+8S_{-2,1}\Biggr)S_2-46S_5\\
&+\Biggl(-\frac{4P_{318}}{(N-1)^2N(1+N)^3(2+N)^2}\\
&+\Biggl(-16S_2-\frac{8P_{55}}{(N-1)N^2(1+N)^2(2+N)}\Biggr)S_1\\
&-\frac{4(-7+N)S_1^2}{N(1+N)}-\frac{8}{3}S_1^3-\frac{8(-7+N)S_2}{N(1+N)}-\frac{64}{3}S_3-8S_{2,1}\Biggr)S_{-2}\\
&+\Biggl(-\frac{2(-7+N)}{N(1+N)}-4S_1\Biggr)S_{-2}^2
\end{aligned}
$$

$$
\begin{aligned}
&+\Biggl(-\frac{16(-7+N)S_1}{N(1+N)}-16S_1^2-20S_2-16S_{-2}\\
&-\frac{4P_{98}}{(N-1)N^2(1+N)^2(2+N)}\Biggr)S_{-3}\\
&+\Biggl(-\frac{14(-11+N)}{N(1+N)}-56S_1\Biggr)S_{-4}-54S_{-5}+16S_{2,3}-20S_{2,-3}\\
&+\frac{\big(-20+20N+7N^2-9N^3\big)S_{3,1}}{N(1+N)(2+N)}\\
&-13S_{4,1}-\frac{8\big(3+8N+N^2\big)S_{-2,1}}{N^2(1+N)^2}+\frac{4(-3+N)S_{-2,2}}{N(1+N)}\\
&+12S_{-2,3}+28S_{-4,1}\\
&+\frac{12(-3+N)S_{-3,1}}{N(1+N)}+\frac{\big(-60-85N-87N^2-26N^3\big)S_{2,1,1}}{3N(1+N)(2+N)}\\
&+8S_{2,1,-2}-18S_{2,2,1}\\
&-6S_{3,1,1}+\frac{8(-3+N)S_{-2,1,1}}{N(1+N)}+8S_{-2,2,1}+24S_{-3,1,1}\\
&-3S_{2,1,1,1}+16S_{-2,1,1,1}\\
&+\Biggl(\frac{P_{372}}{36N^3(1+N)^3}+\Biggl(\frac{P_5}{12N^2(1+N)^2}+16S_2\Biggr)S_1\\
&+\frac{\big(-48-11N-11N^2\big)S_1^2}{2N(1+N)}\\
&+4S_1^3+\frac{\big(-64-33N-33N^2\big)S_2}{2N(1+N)}+2S_3\\
&+\Biggl(-\frac{8}{N(1+N)}+4S_1\Biggr)S_{-2}+2S_{-3}-4S_{-2,1}\Biggr)\zeta_2\\
&+\Biggl(\frac{3\big(24+8N+33N^2+25N^3\big)}{10N(1+N)(2+N)}-\frac{18}{5}S_1\Biggr)\zeta_2^2\\
&+\Biggl(\frac{83}{3}S_1^2-9S_2+\frac{P_{430}}{54(N-1)N^2(1+N)^2(2+N)^2}\\
&-\frac{\big(1776+2665N+2382N^2+713N^3\big)S_1}{9N(1+N)(2+N)}-12S_{-2}\Biggr)\zeta_3\Biggr)\\
&+N_F^2T_F^2\Biggl(-\frac{1184}{81}+\frac{20\zeta_2}{9}+\frac{56\zeta_3}{27}\Biggr). \qquad (226)
\end{aligned}
$$

7. The transversity OMEs

In the case of transversity various OMEs contribute [82,83]. Here we only consider the expansion coefficients contributing to the physical projection $A_{qq}^{\mathrm{tr},\pm}$. The coefficients corresponding to the non–negative powers in ε are given by

$$a_{qq}^{(1,0),\mathrm{tr},+} = C_F \xi S_1 + C_F\big(7 - 2S_1^2 - 6S_2\big), \tag{227}$$

$$\begin{aligned} a_{qq}^{(1,1),\mathrm{tr},+} = {} & C_F \xi \left[\frac{-1+2N}{2N} - \frac{1}{2} S_1 - \frac{1}{4} S_1^2 - \frac{3}{4} S_2 \right] \\ & + C_F \left(\frac{-1-6N-7N^2}{N(1+N)} + \frac{1}{3} S_1^3 + 3 S_1 S_2 \right. \\ & \left. + \frac{14}{3} S_3 + \left(\frac{3}{4} - S_1 \right) \zeta_2 \right), \end{aligned} \tag{228}$$

$$\begin{aligned} a_{qq}^{(1,2),\mathrm{tr}+} = {} & C_F \xi \left[- \frac{(-1+2N)(1+2N)}{4N^2} + \left(\frac{1}{4N} + \frac{3S_2}{8} \right) S_1 \right. \\ & \left. + \frac{1}{8} S_1^2 + \frac{1}{24} S_1^3 + \frac{3}{8} S_2 + \frac{7}{12} S_3 - \frac{1}{8} S_1 \zeta_2 \right] \\ & + C_F \left(\frac{P_{121}}{2N^2(1+N)^2} + \left(\frac{1-N}{2N(1+N)} - \frac{7S_3}{3} \right) S_1 - \frac{1}{24} S_1^4 - \frac{3}{4} S_1^2 S_2 \right. \\ & \left. - \frac{9}{8} S_2^2 - \frac{15}{4} S_4 + \left(-\frac{7}{8} + \frac{S_1^2}{4} + \frac{3S_2}{4} \right) \zeta_2 + \left(-\frac{7}{4} + \frac{7S_1}{3} \right) \zeta_3 \right), \end{aligned} \tag{229}$$

$$\begin{aligned} a_{qq}^{(2,0),\mathrm{tr}+} = {} & \xi^2 \left[C_A C_F \left(\frac{1}{2} - \frac{3}{4} S_1 - \frac{1}{4} S_1^2 - \frac{1}{4} S_2 \right) + C_F^2 \left(-1 + \frac{S_1^2}{2} + \frac{S_2}{2} \right) \right] \\ & + \xi \left[C_F^2 \left(\frac{3-7N}{N} + \left(\frac{2(-2+11N)}{N} - 16 S_2 \right) S_1 - \frac{5}{2} S_1^2 - 4 S_1^3 + \frac{13}{2} S_2 \right) \right. \\ & + C_A C_F \left(-\frac{7}{2} + \left(\frac{3(3+2N)}{2+N} + \frac{(1-N)S_2}{2+N} \right) S_1 \right. \\ & \left. \left. + \frac{5}{4} S_1^2 + \frac{(14+13N)S_2}{4(2+N)} + \frac{(1-N)S_3}{2+N} + \frac{2(N-1)S_{2,1}}{2+N} - \frac{6(N-1)\zeta_3}{2+N} \right) \right] \\ & + C_F \left(N_F T_F \left(\frac{24-281N-281N^2}{18N(1+N)} + \left(\frac{164}{27} + \frac{8S_2}{3} \right) S_1 \right. \right. \\ & \left. + \frac{34}{9} S_1^2 + \frac{8}{9} S_1^3 + \frac{74}{9} S_2 - \frac{8}{9} S_3 \right) \\ & + C_A \left(\frac{-264+3625N+3625N^2}{72N(1+N)} \right. \end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{108-1538N-715N^2}{27N(2+N)}-\frac{2(19+14N)S_2}{3(2+N)}-6S_3-4S_{2,1}\right)S_1-32S_{-3}S_1\\
&+\left(-\frac{253}{18}+S_2\right)S_1^2-\frac{22}{9}S_1^3+\frac{(-1078-485N)S_2}{18(2+N)}-6S_2^2+\frac{2(31+2N)S_3}{9(2+N)}\\
&-29S_4+\left(\frac{12}{N(1+N)}-8S_1^2-16S_2\right)S_{-2}-4S_{-2}^2-28S_{-4}+\frac{4(N-1)S_{2,1}}{2+N}\\
&+6S_{3,1}+8S_{-2,2}+24S_{-3,1}+6S_{2,1,1}+16S_{-2,1,1}\\
&+\left(-\frac{12(N-1)}{2+N}+12S_1\right)\zeta_3\Bigg)\Bigg)\\
&+C_F^2\Bigg(\frac{48+493N+541N^2}{8N(1+N)}+\Bigg(-\frac{2\big(10+51N+77N^2+28N^3\big)}{N(1+N)(2+N)}+\frac{256}{3}S_3\\
&-\frac{2(26+11N)S_2}{2+N}\Bigg)S_1+64S_{-3}S_1+\Big(-14+48S_2\Big)S_1^2-2S_1^3+\frac{14}{3}S_1^4\\
&-\frac{2(54+29N)S_2}{2+N}+30S_2^2-\frac{2(54+25N)S_3}{2+N}+48S_4\\
&+\left(-\frac{24}{N(1+N)}+16S_1^2+32S_2\right)S_{-2}\\
&+8S_{-2}^2+56S_{-4}-\frac{4(-2+N)S_{2,1}}{2+N}-16S_{-2,2}-48S_{-3,1}-32S_{-2,1,1}\\
&+\left(-\frac{9}{2}+12S_1-8S_1^2\right)\zeta_2-\frac{48\zeta_3}{2+N}\Bigg), \qquad (230)
\end{aligned}
$$

$$
\begin{aligned}
a_{qq}^{(2,1),\mathrm{tr}+}=\xi^2\Bigg[&{C_F}^2\left(\frac{1+3N}{2N}+\left(\frac{-1+3N}{2N}-\frac{5S_2}{4}\right)S_1-\frac{1}{2}S_1^2-\frac{1}{4}S_1^3-\frac{1}{2}S_2-S_3\right)\\
&+C_AC_F\Bigg(\frac{1-13N}{8N}+\left(\frac{2+5N}{8N}+\frac{5S_2}{8}\right)S_1\\
&+\frac{1}{2}S_1^2+\frac{1}{8}S_1^3+\frac{7}{8}S_2+\frac{1}{2}S_3\\
&-\frac{1}{8}S_1\zeta_2\Bigg)\Bigg]+\xi\Bigg(C_AC_F\Bigg(\frac{-2-3N+42N^2}{4N^2}+\Bigg(\frac{-4-72N-83N^2-21N^3}{4N(2+N)^2}\\
&+\frac{\big(-132-244N-149N^2-33N^3\big)S_2}{8(1+N)(2+N)^2}+\frac{3(N-1)S_3}{2+N}-\frac{2(N-1)S_{2,1}}{2+N}\Bigg)S_1\\
&+\left(\frac{-44-31N}{8(2+N)}+\frac{3(N-1)S_2}{4(2+N)}\right)S_1^2-\frac{5}{8}S_1^3+\frac{\big(-160-218N-63N^2\big)S_2}{8(2+N)^2}\\
&+\frac{(-1-2N)S_2^2}{2(2+N)}+\frac{\big(-34-72N-52N^2-13N^3\big)S_3}{2(1+N)(2+N)^2}+\frac{(-11+5N)S_4}{4(2+N)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\left(6+4N-N^2\right)S_{2,1}}{(1+N)(2+N)^2}-\frac{3(-2+N)S_{3,1}}{2+N}+\frac{(N-1)S_{2,1,1}}{2+N}\\
&+\frac{5}{8}S_1\zeta_2+\frac{9(N-1)\zeta_2^2}{5(2+N)}\\
&+\left(\frac{3(N-1)\left(12+14N+3N^2\right)}{2(1+N)(2+N)^2}+\frac{3(1+2N)S_1}{2+N}\right)\zeta_3\Bigg)\\
&+C_F^2\Bigg(\frac{(1+3N)(-5+8N)}{2N^2}\\
&+\left(\frac{8-N-36N^2-31N^3}{2N^2(1+N)}+\frac{19}{4}S_2+\frac{52}{3}S_3\right)S_1\\
&+\left(-\frac{3(-2+5N)}{2N}+12S_2\right)S_1^2+\frac{7}{4}S_1^3+\frac{7}{6}S_1^4+\frac{(6-41N)S_2}{2N}\\
&+\frac{15}{2}S_2^2-\frac{15}{2}S_3\\
&+\left(\frac{3S_1}{2}-2S_1^2\right)\zeta_2+\left(9-12S_1\right)\zeta_3\Bigg)\Bigg)+C_F\Bigg(C_A\Bigg(\frac{S_3P_{159}}{27N(1+N)(2+N)^2}\\
&+\frac{P_1}{864N^2(1+N)^2}+\Bigg(\frac{\left(4564+8948N+5525N^2+1069N^3\right)S_2}{36(1+N)(2+N)^2}\\
&+\frac{P_{284}}{162N^2(1+N)(2+N)^2}+5S_2^2+\frac{2(29+28N)S_3}{3(2+N)}\\
&+\frac{49}{2}S_4+\frac{12S_{2,1}}{2+N}-2S_{2,1,1}\Bigg)S_1\\
&+4S_{-2}^2S_1+56S_{-4}S_1+\Bigg(\frac{-324+3094N+1385N^2}{108N(2+N)}+S_3+\frac{(89+58N)S_2}{6(2+N)}\\
&+2S_{2,1}\Bigg)S_1^2+\left(\frac{491}{108}-\frac{S_2}{2}\right)S_1^3+\frac{11}{12}S_1^4+\Bigg(18S_3+4S_{2,1}-8S_{-2,1}\\
&+\frac{-1080+12232N+11260N^2+2599N^3}{108N(2+N)^2}\Bigg)S_2+\frac{(166+65N)S_2^2}{12(2+N)}\\
&+\frac{7(11+10N)S_4}{6(2+N)}+62S_5+\Bigg(-\frac{12(1+2N)}{N^2(1+N)^2}+\left(-\frac{12}{N(1+N)}+16S_2\right)S_1\\
&+\frac{8}{3}S_1^3+\frac{64}{3}S_3+8S_{2,1}\Bigg)S_{-2}\\
&+\left(-\frac{24}{N(1+N)}+16S_1^2+20S_2+16S_{-2}\right)S_{-3}
\end{aligned}
$$

$$
\begin{aligned}
&+54S_{-5}-\frac{2\big(14+26N+17N^2+3N^3\big)S_{2,1}}{(1+N)(2+N)^2}\\
&+16S_{2,3}+20S_{2,-3}-\frac{2(-10+N)S_{3,1}}{2+N}\\
&-13S_{4,1}-12S_{-2,3}-28S_{-4,1}-\frac{2(7+2N)S_{2,1,1}}{2+N}-8S_{2,1,-2}\\
&-18S_{2,2,1}-6S_{3,1,1}-8S_{-2,2,1}-24S_{-3,1,1}-3S_{2,1,1,1}-16S_{-2,1,1,1}\\
&+\left(\frac{57}{8}-\frac{253}{36}S_1-4S_{-2}S_1-\frac{11}{6}S_1^2-\frac{11}{6}S_2-2S_3-2S_{-3}+4S_{-2,1}\right)\zeta_2\\
&+\left(\frac{18(N-1)}{5(2+N)}-\frac{18S_1}{5}\right)\zeta_2^2\\
&+\left(\frac{3P_{125}}{2N(1+N)(2+N)^2}-\frac{2(25+8N)S_1}{2+N}-3S_1^2+3S_2+12S_{-2}\right)\zeta_3\Bigg)\\
&+N_FT_F\left(\frac{P_{161}}{216N^2(1+N)^2}+\left(-\frac{2\big(-27+253N+226N^2\big)}{81N(1+N)}\right.\right.\\
&\left.-\frac{65}{9}S_2-\frac{16}{3}S_3\right)S_1\\
&+\left(-\frac{73}{27}-\frac{10S_2}{3}\right)S_1^2-\frac{31}{27}S_1^3-\frac{1}{3}S_1^4-\frac{155}{27}S_2\\
&-\frac{7}{3}S_2^2-\frac{224}{27}S_3-\frac{10}{3}S_4\\
&\left.+\left(-\frac{5}{2}+\frac{17}{9}S_1+\frac{2}{3}S_1^2+\frac{2}{3}S_2\right)\zeta_2+\Big(-6+8S_1\Big)\zeta_3\right)\Bigg)\\
&+{C_F}^2\left(\frac{P_2}{32N^2(1+N)^2}+\frac{2S_3P_{149}}{3N(1+N)(2+N)^2}\right.\\
&+\left(\frac{\big(112+214N+131N^2+27N^3\big)S_2}{(1+N)(2+N)^2}\right.\\
&\left.+\frac{P_{354}}{N^2(1+N)^2(2+N)^2}-47S_2^2+\frac{2(34+11N)S_3}{2+N}-128S_4-\frac{16S_{2,1}}{2+N}\right)S_1\\
&-8S_{-2}^2S_1-112S_{-4}S_1\\
&+\left(\frac{30+41N+63N^2+28N^3}{2N(1+N)(2+N)}+\frac{(34+11N)S_2}{2(2+N)}-60S_3\right)\\
&\times S_1^2+\left(\frac{7}{3}-\frac{56S_2}{3}\right)S_1^3+\frac{1}{4}S_1^4-S_1^5
\end{aligned}
$$

$$
\begin{aligned}
&+\Biggl(\frac{P_{152}}{2N(1+N)(2+N)^2}-\frac{232}{3}S_3+16S_{-2,1}\Biggr)S_2\\
&+\frac{(78+47N)S_2^2}{4(2+N)}+\frac{(250+111N)S_4}{2(2+N)}-108S_5\\
&+\Biggl(\frac{24(1+2N)}{N^2(1+N)^2}+\Biggl(\frac{24}{N(1+N)}-32S_2\Biggr)S_1-\frac{16}{3}S_1^3-\frac{128}{3}S_3-16S_{2,1}\Biggr)S_{-2}\\
&+\Biggl(\frac{48}{N(1+N)}-32S_1^2-40S_2-32S_{-2}\Biggr)S_{-3}-108S_{-5}\\
&+\frac{4\bigl(-2+3N+5N^2+N^3\bigr)S_{2,1}}{(1+N)(2+N)^2}\\
&-40S_{2,-3}+\frac{2(-10+3N)S_{3,1}}{2+N}+24S_{-2,3}\\
&+56S_{-4,1}+\frac{4(4+N)S_{2,1,1}}{2+N}+16S_{2,1,-2}\\
&+16S_{-2,2,1}+48S_{-3,1,1}+32S_{-2,1,1,1}\\
&+\Biggl(\frac{87}{8}+\bigl(-14+16S_2\bigr)S_1+8S_{-2}S_1-3S_1^2\\
&+4S_1^3-12S_2+4S_3+4S_{-3}-8S_{-2,1}\Biggr)\zeta_2+\frac{72\zeta_2^2}{5(2+N)}\\
&+\Biggl(-\frac{3P_{86}}{N(1+N)(2+N)^2}-\frac{8(7+5N)S_1}{2+N}+\frac{92}{3}S_1^2-12S_2-24S_{-2}\Biggr)\zeta_3\Biggr),
\end{aligned}
\tag{231}
$$

$$
a_{qq}^{(1,0),\mathrm{tr},-}=C_F\xi S_1+C_F\Biggl[7-2S_1^2-6S_2\Biggr],
\tag{232}
$$

$$
\begin{aligned}
a_{qq}^{(1,1),\mathrm{tr},-}=&C_F\xi\Biggl[\frac{-1+2N}{2N}-\frac{1}{2}S_1-\frac{1}{4}S_1^2-\frac{3}{4}S_2\Biggr]\\
&+C_F\Biggl(\frac{-1-6N-7N^2}{N(1+N)}+\frac{1}{3}S_1^3+3S_1S_2\\
&+\frac{14}{3}S_3+\Biggl(\frac{3}{4}-S_1\Biggr)\zeta_2\Biggr),
\end{aligned}
\tag{233}
$$

$$
\begin{aligned}
a_{qq}^{(1,2),\mathrm{tr},-}=&C_F\xi\Biggl[-\frac{(-1+2N)(1+2N)}{4N^2}+\Biggl(\frac{1}{4N}+\frac{3S_2}{8}\Biggr)S_1\\
&+\frac{1}{8}S_1^2+\frac{1}{24}S_1^3+\frac{3}{8}S_2+\frac{7}{12}S_3\\
&-\frac{1}{8}S_1\zeta_2\Biggr]+C_F\Biggl(\frac{P_{121}}{2N^2(1+N)^2}+\Biggl(\frac{1-N}{2N(1+N)}-\frac{7S_3}{3}\Biggr)S_1
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{24}S_1^4-\frac{3}{4}S_1^2S_2 \\
&-\frac{9}{8}S_2^2-\frac{15}{4}S_4+\left(-\frac{7}{8}+\frac{S_1^2}{4}+\frac{3S_2}{4}\right)\zeta_2+\left(-\frac{7}{4}+\frac{7S_1}{3}\right)\zeta_3\Bigg), \qquad (234)
\end{aligned}
$$

$$
\begin{aligned}
a_{qq}^{(2,0),\mathrm{tr},-}=\;&\xi^2\Bigg[C_AC_F\left(\frac{1}{2}-\frac{3}{4}S_1-\frac{1}{4}S_1^2-\frac{1}{4}S_2\right)+C_F{}^2\left(-1+\frac{S_1^2}{2}+\frac{S_2}{2}\right)\Bigg] \\
&+\xi\Bigg[C_F{}^2\left(\frac{3-7N}{N}+\left(\frac{2(-2+11N)}{N}-16S_2\right)S_1-\frac{5}{2}S_1^2-4S_1^3+\frac{13}{2}S_2\right) \\
&+C_AC_F\Bigg(-\frac{7}{2}+\left(\frac{3(3+2N)}{2+N}+\frac{(1-N)S_2}{2+N}\right)S_1+\frac{5}{4}S_1^2 \\
&+\frac{(14+13N)S_2}{4(2+N)}+\frac{(1-N)S_3}{2+N} \\
&+\frac{2(N-1)S_{2,1}}{2+N}-\frac{6(N-1)\zeta_3}{2+N}\Bigg)\Bigg] \\
&+C_F\Bigg(N_FT_F\Bigg(\frac{24-281N-281N^2}{18N(1+N)}+\left(\frac{164}{27}+\frac{8S_2}{3}\right)S_1 \\
&+\frac{34}{9}S_1^2+\frac{8}{9}S_1^3+\frac{74}{9}S_2-\frac{8}{9}S_3\Bigg) \\
&+C_A\Bigg(\frac{P_{283}}{72(N-1)N(1+N)^2(2+N)} \\
&+\Bigg(\frac{-108-1538N-2253N^2-715N^3}{27N(1+N)(2+N)} \\
&-\frac{2(19+14N)S_2}{3(2+N)}-6S_3-4S_{2,1}\Bigg)S_1 \\
&-32S_{-3}S_1+\left(-\frac{253}{18}+S_2\right)S_1^2-\frac{22}{9}S_1^3+\frac{(-1078-485N)S_2}{18(2+N)}-6S_2^2 \\
&+\frac{2(31+2N)S_3}{9(2+N)}-29S_4+\left(-\frac{4}{(N-1)(2+N)}-8S_1^2-16S_2\right)S_{-2}-4S_{-2}^2 \\
&-28S_{-4}+\frac{4(N-1)S_{2,1}}{2+N}+6S_{3,1}+8S_{-2,2}+24S_{-3,1}+6S_{2,1,1}+16S_{-2,1,1} \\
&+\left(-\frac{12(N-1)}{2+N}+12S_1\right)\zeta_3\Bigg)\Bigg)+C_F{}^2\Bigg(\frac{P_{282}}{8(N-1)N(1+N)^2(2+N)} \\
&+\left(-\frac{2\big(2+47N+77N^2+28N^3\big)}{N(1+N)(2+N)}-\frac{2(26+11N)S_2}{2+N}+\frac{256}{3}S_3\right)S_1
\end{aligned}
$$

$$
\begin{aligned}
&+64S_{-3}S_1+\Bigl(-14+48S_2\Bigr)S_1^2-2S_1^3+\frac{14}{3}S_1^4\\
&-\frac{2(54+29N)S_2}{2+N}+30S_2^2-\frac{2(54+25N)S_3}{2+N}\\
&+48S_4+\Bigl(\frac{8}{(N-1)(2+N)}+16S_1^2+32S_2\Bigr)S_{-2}\\
&+8S_{-2}^2+56S_{-4}-\frac{4(N-2)S_{2,1}}{2+N}\\
&-16S_{-2,2}-48S_{-3,1}-32S_{-2,1,1}\\
&+\Bigl(-\frac{9}{2}+12S_1-8S_1^2\Bigr)\zeta_2-\frac{48\zeta_3}{2+N}\Bigr), \qquad (235)
\end{aligned}
$$

$$
\begin{aligned}
a_{qq}^{(2,1),\mathrm{tr},-}=\xi^2\Biggl[&C_F{}^2\Bigl(\frac{1+3N}{2N}+\Bigl(\frac{-1+3N}{2N}-\frac{5S_2}{4}\Bigr)S_1-\frac{1}{2}S_1^2-\frac{1}{4}S_1^3-\frac{1}{2}S_2-S_3\Bigr)\\
&+C_AC_F\Bigl(\frac{1-13N}{8N}+\Bigl(\frac{2+5N}{8N}+\frac{5S_2}{8}\Bigr)S_1\\
&+\frac{1}{2}S_1^2+\frac{1}{8}S_1^3+\frac{7}{8}S_2+\frac{1}{2}S_3-\frac{1}{8}S_1\zeta_2\Bigr)\Biggr]\\
&+\xi\Biggl[C_AC_F\Bigl(\frac{-2-3N+42N^2}{4N^2}+\Bigl(\frac{-4-72N-83N^2-21N^3}{4N(2+N)^2}\\
&+\frac{\bigl(-132-244N-149N^2-33N^3\bigr)S_2}{8(1+N)(2+N)^2}+\frac{3(N-1)S_3}{2+N}-\frac{2(N-1)S_{2,1}}{2+N}\Bigr)S_1\\
&+\Bigl(\frac{-44-31N}{8(2+N)}+\frac{3(N-1)S_2}{4(2+N)}\Bigr)S_1^2-\frac{5}{8}S_1^3+\frac{\bigl(-160-218N-63N^2\bigr)S_2}{8(2+N)^2}\\
&+\frac{(-1-2N)S_2^2}{2(2+N)}+\frac{\bigl(-34-72N-52N^2-13N^3\bigr)S_3}{2(1+N)(2+N)^2}+\frac{(-11+5N)S_4}{4(2+N)}\\
&+\frac{\bigl(6+4N-N^2\bigr)S_{2,1}}{(1+N)(2+N)^2}-\frac{3(-2+N)S_{3,1}}{2+N}+\frac{(N-1)S_{2,1,1}}{2+N}\\
&+\frac{5}{8}S_1\zeta_2+\frac{9(N-1)\zeta_2^2}{5(2+N)}\\
&+\Bigl(\frac{3(N-1)\bigl(12+14N+3N^2\bigr)}{2(1+N)(2+N)^2}+\frac{3(1+2N)S_1}{2+N}\Bigr)\zeta_3\Bigr)\\
&+C_F^2\Bigl(\frac{(1+3N)(-5+8N)}{2N^2}\\
&+\Bigl(\frac{8-N-36N^2-31N^3}{2N^2(1+N)}+\frac{19}{4}S_2+\frac{52}{3}S_3\Bigr)S_1
\end{aligned}
$$

$$
\begin{aligned}
&+\left(-\frac{3(-2+5N)}{2N}+12S_2\right)S_1^2+\frac{7}{4}S_1^3+\frac{7}{6}S_1^4\\
&+\frac{(6-41N)S_2}{2N}+\frac{15}{2}S_2^2-\frac{15}{2}S_3\\
&+\left(\frac{3S_1}{2}-2S_1^2\right)\zeta_2+\left(9-12S_1\right)\zeta_3\Bigg)\Bigg]\\
&+C_F\Bigg(C_A\Bigg(\frac{S_3P_{158}}{27(N-1)(1+N)(2+N)^2}\\
&+\frac{P_{519}}{864(N-1)^2N^3(1+N)^3(2+N)^2}\\
&+\Bigg(\frac{P_{431}}{162(N-1)N^2(1+N)^2(2+N)^2}\\
&+\frac{\left(4564+8948N+5525N^2+1069N^3\right)S_2}{36(1+N)(2+N)^2}\\
&+5S_2^2+\frac{2(29+28N)S_3}{3(2+N)}+\frac{49}{2}S_4+\frac{12S_{2,1}}{2+N}-2S_{2,1,1}\Bigg)S_1+4S_{-2}^2S_1\\
&+56S_{-4}S_1+\Bigg(S_3+\frac{(89+58N)S_2}{6(2+N)}+2S_{2,1}\\
&+\frac{108+2986N+4479N^2+1385N^3}{108N(1+N)(2+N)}\Bigg)S_1^2+\left(\frac{491}{108}-\frac{S_2}{2}\right)S_1^3+\frac{11}{12}S_1^4\\
&+\left(\frac{P_{160}}{108N(1+N)(2+N)^2}+18S_3+4S_{2,1}-8S_{-2,1}\right)S_2+\frac{(166+65N)S_2^2}{12(2+N)}\\
&+\frac{7(11+10N)S_4}{6(2+N)}+62S_5+\Bigg(-\frac{4\left(-7-2N+2N^2+N^3\right)}{(N-1)^2(1+N)(2+N)^2}\\
&+\left(\frac{4}{(N-1)(2+N)}+16S_2\right)S_1+\frac{8}{3}S_1^3\\
&+\frac{64}{3}S_3+8S_{2,1}\Bigg)S_{-2}\\
&+\left(\frac{8}{(N-1)(2+N)}+16S_1^2+20S_2+16S_{-2}\right)S_{-3}+54S_{-5}\\
&-\frac{2\left(14+26N+17N^2+3N^3\right)S_{2,1}}{(1+N)(2+N)^2}+16S_{2,3}+20S_{2,-3}\\
&-\frac{2(-10+N)S_{3,1}}{2+N}-13S_{4,1}-12S_{-2,3}-28S_{-4,1}\\
&-\frac{2(7+2N)S_{2,1,1}}{2+N}-8S_{2,1,-2}
\end{aligned}
$$

$$
\begin{aligned}
&-18S_{2,2,1}-6S_{3,1,1}-8S_{-2,2,1}-24S_{-3,1,1}-3S_{2,1,1,1}-16S_{-2,1,1,1}\\
&+\Biggl(\frac{-8+57N+57N^2}{8N(1+N)}-\frac{253}{36}S_1-4S_{-2}S_1-\frac{11}{6}S_1^2-\frac{11}{6}S_2-2S_3-2S_{-3}\\
&+4S_{-2,1}\Biggr)\zeta_2+\Biggl(\frac{18(N-1)}{5(2+N)}-\frac{18S_1}{5}\Biggr)\zeta_2^2+\Biggl(-3S_1^2+3S_2+12S_{-2}\\
&+\frac{3(3+N)\bigl(-12-8N+11N^2+15N^3\bigr)}{2(N-1)(1+N)(2+N)^2}-\frac{2(25+8N)S_1}{2+N}\Biggr)\zeta_3\Biggr)\\
&+N_FT_F\Biggl(\frac{P_{161}}{216N^2(1+N)^2}\\
&+\Biggl(-\frac{2\bigl(-27+253N+226N^2\bigr)}{81N(1+N)}-\frac{65}{9}S_2-\frac{16}{3}S_3\Biggr)S_1\\
&+\Biggl(-\frac{73}{27}-\frac{10S_2}{3}\Biggr)S_1^2-\frac{31}{27}S_1^3-\frac{1}{3}S_1^4-\frac{155}{27}S_2-\frac{7}{3}S_2^2\\
&-\frac{224}{27}S_3-\frac{10}{3}S_4+\Biggl(-\frac{5}{2}\\
&+\frac{17}{9}S_1+\frac{2}{3}S_1^2+\frac{2}{3}S_2\Biggr)\zeta_2+\Biggl(-6+8S_1\Biggr)\zeta_3\Biggr)\Biggr)\\
&+C_F^2\Biggl(\frac{2S_3P_{148}}{3(N-1)(1+N)(2+N)^2}\\
&+\frac{P_{521}}{32(N-1)^2N^3(1+N)^3(2+N)^2}+\Biggl(\frac{\bigl(112+214N+131N^2+27N^3\bigr)S_2}{(1+N)(2+N)^2}\\
&+\frac{P_{420}}{(N-1)N^2(1+N)^2(2+N)^2}-47S_2^2\\
&+\frac{2(34+11N)S_3}{2+N}-128S_4-\frac{16S_{2,1}}{2+N}\Biggr)S_1\\
&-8S_{-2}^2S_1-112S_{-4}S_1+\Biggl(\frac{14+33N+63N^2+28N^3}{2N(1+N)(2+N)}+\frac{(34+11N)S_2}{2(2+N)}\\
&-60S_3\Biggr)S_1^2+\Biggl(\frac{7}{3}-\frac{56S_2}{3}\Biggr)S_1^3+\frac{1}{4}S_1^4-S_1^5\\
&+\Biggl(\frac{P_{151}}{2N(1+N)(2+N)^2}-\frac{232}{3}S_3+16S_{-2,1}\Biggr)S_2\\
&+\frac{(78+47N)S_2^2}{4(2+N)}+\frac{(250+111N)S_4}{2(2+N)}-108S_5
\end{aligned}
$$

$$
\begin{aligned}
&+\Biggl(\frac{8\bigl(-7-2N+2N^2+N^3\bigr)}{(N-1)^2(1+N)(2+N)^2}+\Biggl(-\frac{8}{(N-1)(2+N)}-32S_2\Biggr)S_1-\frac{16}{3}S_1^3\\
&-\frac{128}{3}S_3-16S_{2,1}\Biggr)S_{-2}\\
&+\Biggl(-\frac{16}{(N-1)(2+N)}-32S_1^2-40S_2-32S_{-2}\Biggr)S_{-3}\\
&-108S_{-5}+\frac{4\bigl(-2+3N+5N^2+N^3\bigr)S_{2,1}}{(1+N)(2+N)^2}\\
&-40S_{2,-3}+\frac{2(-10+3N)S_{3,1}}{2+N}\\
&+24S_{-2,3}+56S_{-4,1}+\frac{4(4+N)S_{2,1,1}}{2+N}\\
&+16S_{2,1,-2}+16S_{-2,2,1}+48S_{-3,1,1}\\
&+32S_{-2,1,1,1}+\Biggl(\frac{16+87N+87N^2}{8N(1+N)}\\
&+\Bigl(-14+16S_2\Bigr)S_1+8S_{-2}S_1-3S_1^2+4S_1^3\\
&-12S_2+4S_3+4S_{-3}-8S_{-2,1}\Biggr)\zeta_2+\frac{72\zeta_2^2}{5(2+N)}\\
&+\Biggl(\frac{92}{3}S_1^2-12S_2-24S_{-2}-\frac{3P_{85}}{(N-1)(1+N)(2+N)^2}\\
&-\frac{8(7+5N)S_1}{2+N}\Biggr)\zeta_3\Biggr). \qquad (236)
\end{aligned}
$$

In summary, the expressions shown in Sections 3–6 depend on the following 28 harmonic sums after algebraic reduction [118],

$$
\begin{aligned}
&\bigl\{S_{-5}, S_{-4}, S_{-3}, S_{-2}, S_1, S_2, S_3, S_4, S_5, S_{-4,1}, S_{-3,1}, S_{-2,1}, S_{-2,2}, S_{-2,3}, S_{2,-3},\\
&S_{2,1}, S_{2,3}, S_{3,1}, S_{4,1}, S_{-3,1,1}, S_{-2,1,1}, S_{-2,2,1},\\
&S_{2,1,-2}, S_{2,1,1}, S_{2,2,1}, S_{3,1,1}, S_{-2,1,1,1}, S_{2,1,1,1}\bigr\}. \qquad (237)
\end{aligned}
$$

After applying also the structural relations [121] the following 13 sum contribute,

$$
\begin{aligned}
&\{S_1, S_{-2,1}, S_{2,1}, S_{-3,1}, S_{4,1}, S_{-4,1}, S_{-2,1,1},\\
&S_{2,1,1}, S_{-3,1,1}, S_{-2,2,1}, S_{2,2,1}, S_{-2,1,1,1}, S_{2,1,1,1}\}. \qquad (238)
\end{aligned}
$$

8. Comparison to the literature

In Refs. [19,20], extending earlier results in Refs. [15,79], unpolarized and polarized OMEs have been calculated to $O(a_s^2\varepsilon^0)$, i.e. one order in ε less than in the present calculation and the OMEs of transversity were not considered. The calculation has been carried out in z–space. This has been slightly before harmonic sums [115,116] and harmonic polylogarithms [122] became the standard entities to represent different calculation steps and the final results in single scale calculations. Therefore classical polylogarithms and Nielsen integrals [123–128], partly with involved argument, were used. In [19,20] the expansion coefficients of the completely unrenormalized OMEs are discussed, while we perform the renormalization of the strong coupling and the gauge parameter first. Furthermore, the gauge parameter there is $\tilde{\xi} = 1 - \xi$, compared to the present case. In the following we summarize a series of differences to the present results which we have observed, both due to typographical and combinatoric errors. Comparing to [19,20] one first observes that the concrete results for the non–singlet OMEs differ by a global minus sign. For all the other OMEs we find a relative minus sign at $O(\alpha_s)$, but agree at $O(\alpha_s^2)$. The formal representations in terms of anomalous dimensions, i.e. Eqs. (2.8, 2.16, 2.31, 2.34) of Ref. [19] and Eqs. (2.18, 2.24, 2.26, 2.29) of Ref. [20], are, however, correct. Furthermore, the expressions for $\Delta \hat{A}_{iq}^{\rm phys}$ there are obtained as the sum $\Delta \hat{A}_{iq}^{\rm phys} + \Delta \hat{A}_{iq}^{\rm eom}$ of the present results in the Larin scheme.

After adjusting the signs we find also some specific differences. Let us define

$$\delta A_{ij}^{(k)}(x) = A_{ij}^{(k),\text{this calc.}}(x) - A_{ij}^{(k),[19]}(x). \tag{239}$$

$$\delta \Delta A_{ij}^{(k)}(x) = \Delta A_{ij}^{(k),\text{this calc.}}(x) - \Delta A_{ij}^{(k),[20]}(x). \tag{240}$$

We will express the differences in Mellin N space, after adjusting the signs as mentioned above. In the unpolarized case we find the following difference in the non-singlet physical case

$$\delta A_{qq}^{\rm NS,phys}(N) = a_s^2 \frac{C_F^2}{\varepsilon}\left(10 S_1^2 + 10 S_2 - \frac{20 S_1}{N}\right), \tag{241}$$

however, the constant part of the OME agree. For the gluonic OME we find the difference

$$\delta A_{gg}^{\rm phys}(N) = a_s C_A \left[\frac{20}{N^2} + \frac{20 S_1}{N} - 10 S_1^2 - 30 S_2 + 20\zeta_2 - \varepsilon C_A \left(\frac{\xi \zeta_2}{4} + \frac{2\zeta_3}{3}\right)\right] + a_s^2 C_A \zeta_3 \left[\frac{20}{9} C_A - \frac{16}{9} N_F T_F\right]. \tag{242}$$

Since these differences at $O(a_s \varepsilon)$ and $O(a_s^2)$ are independent of N they are probably due to wrong reducible contributions.In the pure-singlet and qg case we find bigger differences of

$$\delta A_{qq}^{\rm PS,phys}(N) = a_s^2 C_F N_F T_F \Biggl(-\frac{8\bigl(32 - 8N + 80N^2 + 314N^3 + 311N^4 + 120N^5 + 15N^6\bigr)}{3(-1+N)N(1+N)^3(2+N)^3} - \frac{8\bigl(44 + 36N + 19N^2 + 30N^3 + 15N^4\bigr)S_1}{3(-1+N)N(1+N)^2(2+N)^2} + \frac{8\bigl(-2 + 5N + 5N^2\bigr)\bigl(S_2 + S_1^2\bigr)}{(-1+N)N(1+N)(2+N)}\Biggr) \tag{243}$$

$$\delta A_{qg}^{\rm phys}(N) = a_s^2 N_F^2 T_F^2 \Biggl(\frac{64\bigl(-8 - 5N + 3N^2 + 2N^3\bigr)}{3(1+N)^3(2+N)^3} + \frac{32\bigl(-4 - N + N^2\bigr)S_1}{3(1+N)^2(2+N)^2} - \frac{16 S_1^2}{3(1+N)(2+N)} - \frac{16 S_2}{(1+N)(2+N)} + \frac{32\zeta_2}{3(1+N)(2+N)}\Biggr) \tag{244}$$

Note that the $O(\varepsilon^0)$ contribution in (242) is not in agreement with the earlier calculation [15,79], where the corresponding expansion has been given to $O(\varepsilon^0)$.

In the polarized case we find large differences in the constant parts of the non-singlet OMEs, which we do not give explicitly here. Further smaller differences are found in the OME qg

$$\delta\Delta A_{qg}^{\rm phys}(N) = a_s^2 N_F^2 T_F^2 \left(\frac{64\left(2+4N-3N^2\right)}{9N^2(1+N)^2} + \frac{32(4-N)S_1}{9N(1+N)}\right) \tag{245}$$

and the gluonic OME

$$\delta\Delta A_{gg}^{\rm phys}(N) = -a_s\varepsilon C_A\left(\frac{\xi\zeta_2}{4}+\frac{2\zeta_3}{3}\right) + a_s^2 C_A\zeta_3\left[\frac{20}{9}C_A - \frac{16}{9}N_F T_F\right] \tag{246}$$

The differences in the constant parts of the OMEs do not affect the extraction of the NLO splitting functions, but they are needed for the correct renormalization of the OMEs at $O(a_s^3)$ and therefore for the extraction of the splitting functions at the next loop order. Because of these aspect it is therefore rather difficult to use these older results in current calculations.

9. Conclusions

We have calculated the unpolarized and polarized massless off–shell twist–2 operator matrix elements in QCD to two–loop orders in an automated way. These quantities are of interest because they allow the direct calculation of the QCD anomalous dimensions. Contrary to this, on shell massive operator matrix elements only allow to compute the anomalous dimensions of contributions $\propto T_F$ in the same order in perturbation theory and require the corrections of one more order in the coupling constant to obtain the complete anomalous dimensions. The off–shellness implies gauge variant expressions in general. In particular, the equation of motion is not valid anymore for these terms and non–gauge invariant contributions are present, which may mix with the gauge invariant contributions. Despite of these technical complications, the method allows for a direct calculations of the anomalous dimensions. The on–shell calculation of the forward Compton amplitude also allows to compute the anomalous dimensions contributing to the deep–inelastic structure functions. However, special arrangements are necessary for the gluonic contributions, which require auxiliary Higgs- and graviton fields, also complicating this method.

We compared our results to those in Refs. [19,20] for the contributions to $O(a_s^2\varepsilon^0)$ and calculated newly the contributions of $O(a_s\varepsilon^2)$ and $O(a_s^2\varepsilon)$, which contribute to the calculation of the four–loop anomalous dimensions. In the comparison we have found a series of errors in the previous calculations [19,20]. If results of these papers, beyond the anomalous dimensions have been used in the literature, these results have to be corrected. In particular, as a by–product of the present calculation all unpolarized and the polarized anomalous dimensions and those for transversity are correctly obtained to two–loop order [8–18,32,34–36,38,81,82]. Our calculation has been performed in an algorithmic manner using well established methods having been applied in various massless and massive three–loop calculations before, as e.g. [36,38,129]. The present formalism allows extensions to higher loop calculations. Here, however, also new structures are expected to emerge and the complexity of the calculation will naturally be larger. The quantities to be computed are expressible in terms of the usual harmonic sums [115,116] and no other function spaces as e.g. [114,119,120,130] are required to derive and express the results such as the different expansion coefficients of the different massless OMEs and the anomalous dimensions.

CRediT authorship contribution statement

The authors have equally contributed in different fields to the results presented in the present paper.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Feynman rules

The alien operators introduce new Feynman rules for operator insertions. They have been given in Refs. [15,19,79]. We give them here fore completeness and correct typographical errors; particularly all sums have to start with the lower index 0.

$$O^{ab}_{A,\mu\nu}(p,-p) = \delta^{ab}\frac{1+(-1)^N}{2}\Big[2\Delta_\mu\Delta_\nu p^2 - (p_\mu\Delta_\nu + p_\nu\Delta_\mu)\Delta.p\Big](\Delta.p)^{N-2}, \quad (247)$$

$$O^{abc}_{A,\mu\nu\lambda}(p,q,k) = -ig\frac{1+(-1)^N}{2} f^{abc} V^{(3)}_{\mu\nu\lambda}(p,q,k), \quad (248)$$

$$O^{ab}_{\omega}(p,-p) = -\delta^{ab}(\Delta.p)^N, \quad (249)$$

$$O^{abc}_{\omega,\mu}(p,q,k) = igf^{abc}\tilde{V}^{(3)}_\mu(p,q,k), \quad (250)$$

$$\begin{aligned}
&V^{(3)}_{\mu\nu\lambda}(p,q,k) \\
&= \big[\Delta_\mu\Delta_\nu(q_\lambda - p_\lambda) + \Delta_\mu\Delta_\lambda(p_\nu - k_\nu) + g_{\nu\lambda}\Delta_\mu(\Delta.k - \Delta.q)\big](\Delta.p)^{N-2} \\
&\quad -\frac{1}{4}\Delta_\nu\Delta_\lambda(\Delta_\mu p^2 - p_\mu\Delta.p)\sum_{i=0}^{N-3}\Big[(-\Delta.q)^i(\Delta.k)^{N-3-i} - 3(-\Delta.k)^i(\Delta.p)^{N-3-i} \\
&\quad -3(-\Delta.p)^i(\Delta.q)^{N-3-i}\Big] + \left\{\begin{matrix} p\to q\to k\to p \\ \mu\to\nu\to\lambda\to\mu\end{matrix}\right\} + \left\{\begin{matrix} p\to k\to q\to p \\ \mu\to\lambda\to\nu\to\mu\end{matrix}\right\} \quad (251)
\end{aligned}$$

$$\begin{aligned}
\tilde{V}^{(3)}_\mu(p,q,k) = \Delta_\mu\Bigg[&\frac{1}{2}(\Delta.k)^{N-1} + \frac{1}{4}(\Delta.q - 3\Delta.k)(\Delta.q)^{N-2} \\
&+\frac{1}{4}(\Delta.k - \Delta.q)(\Delta.p)^{N-2} - \frac{3}{4}(\Delta.k)^2\sum_{i=0}^{N-3}(\Delta.k)^i(-\Delta.q)^{N-3-i}\Bigg]. \quad (252)
\end{aligned}$$

Appendix B. The polynomials

In the following we list the polynomials occurring in Eqs. (142)–(192), (193)–(226) and (227)–(236).

$$P_1 = -102241N^4 - 197090N^3 - 96529N^2 - 1680N + 1584 \quad (253)$$
$$P_2 = -2895N^4 - 5822N^3 - 2863N^2 - 512N - 192 \quad (254)$$
$$P_3 = -521N^4 - 1258N^3 + 439N^2 - 2280N - 144 \quad (255)$$
$$P_4 = -367N^4 - 602N^3 + 895N^2 + 866N - 648 \quad (256)$$
$$P_5 = -263N^4 - 574N^3 + 97N^2 - 360N + 96 \quad (257)$$
$$P_6 = -155N^4 - 265N^3 - 161N^2 + 75N + 18 \quad (258)$$
$$P_7 = -155N^4 - 175N^3 - 89N^2 + 57N + 18 \quad (259)$$
$$P_8 = -137N^4 - 289N^3 - 227N^2 - 291N - 60 \quad (260)$$
$$P_9 = -127N^4 - 140N^3 + 331N^2 + 260N - 180 \quad (261)$$
$$P_{10} = -101N^4 - 96N^3 + 260N^2 + 33N + 18 \quad (262)$$
$$P_{11} = -97N^4 - 78N^3 + 333N^2 + 218N - 160 \quad (263)$$
$$P_{12} = -82N^4 - 155N^3 - 126N^2 - 107N - 34 \quad (264)$$
$$P_{13} = -82N^4 - 137N^3 - 22N^2 - 5N - 18 \quad (265)$$
$$P_{14} = -73N^4 - 119N^3 - 103N^2 + 33N + 18 \quad (266)$$
$$P_{15} = -73N^4 - 65N^3 - 31N^2 + 51N + 18 \quad (267)$$
$$P_{16} = -67N^4 - 152N^3 - 31N^2 - 162N - 72 \quad (268)$$
$$P_{17} = -63N^4 - 171N^3 - 52N^2 + 4N + 48 \quad (269)$$
$$P_{18} = -63N^4 - 29N^3 + 260N^2 + 12N + 144 \quad (270)$$
$$P_{19} = -55N^4 - 273N^3 - 366N^2 - 28N + 152 \quad (271)$$
$$P_{20} = -49N^4 - 6N^3 + 67N^2 + 38N - 2 \quad (272)$$
$$P_{21} = -33N^4 - 195N^3 - 348N^2 - 182N - 30 \quad (273)$$
$$P_{22} = -33N^4 - 66N^3 - 31N^2 + 2N - 64 \quad (274)$$
$$P_{23} = -30N^4 - 39N^3 + 6N^2 - 19N - 22 \quad (275)$$
$$P_{24} = -29N^4 + 6N^3 + 47N^2 + 14N - 6 \quad (276)$$
$$P_{25} = -28N^4 - 56N^3 + 137N^2 + 171N - 12 \quad (277)$$
$$P_{26} = -17N^4 - 34N^3 + 49N^2 + 66N + 32 \quad (278)$$
$$P_{27} = -14N^4 - 64N^3 - 59N^2 - 57N - 14 \quad (279)$$
$$P_{28} = -14N^4 - 40N^3 - 27N^2 - 17N - 4 \quad (280)$$
$$P_{29} = -14N^4 - 40N^3 + 13N^2 - 9N - 14 \quad (281)$$
$$P_{30} = -14N^4 - 32N^3 - 3N^2 - N - 4 \quad (282)$$
$$P_{31} = -13N^4 - 37N^3 - 18N^2 - 28N + 24 \quad (283)$$
$$P_{32} = -13N^4 - 19N^3 - 50N^2 - 30N - 8 \quad (284)$$

$$P_{33} = -11N^4 - 22N^3 - 37N^2 - 26N - 48 \tag{285}$$
$$P_{34} = -11N^4 - 22N^3 - 5N^2 + 6N - 16 \tag{286}$$
$$P_{35} = -9N^4 - 38N^3 - 104N^2 - 64N - 32 \tag{287}$$
$$P_{36} = -9N^4 + 62N^3 + 125N^2 + 54N - 40 \tag{288}$$
$$P_{37} = -8N^4 - 21N^3 - 10N^2 + 3N + 4 \tag{289}$$
$$P_{38} = -7N^4 - 20N^3 - 15N^2 - 8N - 2 \tag{290}$$
$$P_{39} = -7N^4 - 16N^3 - 3N^2 - 2 \tag{291}$$
$$P_{40} = -5N^4 - 20N^3 - 33N^2 - 18N + 4 \tag{292}$$
$$P_{41} = -3N^4 - 157N^3 - 558N^2 - 324N - 104 \tag{293}$$
$$P_{42} = -3N^4 - 16N^3 - 35N^2 - 22N + 4 \tag{294}$$
$$P_{43} = -3N^4 - 6N^3 + 27N^2 + 46N - 120 \tag{295}$$
$$P_{44} = -2N^4 - 9N^3 - 7N^2 + 4N - 30 \tag{296}$$
$$P_{45} = -2N^4 - 6N^3 + 3N^2 + 20N + 17 \tag{297}$$
$$P_{46} = -2N^4 - 4N^3 + N^2 + 21N - 36 \tag{298}$$
$$P_{47} = -2N^4 - 3N^3 - 9N^2 - 8N + 4 \tag{299}$$
$$P_{48} = -N^4 - 3N^3 + 4N^2 + 10N - 30 \tag{300}$$
$$P_{49} = -N^4 - 2N^3 + 15N^2 - 22N + 12 \tag{301}$$
$$P_{50} = -N^4 + 2N^3 + 3N^2 - 2N - 4 \tag{302}$$
$$P_{51} = -N^4 + 4N^3 - 6N^2 - 4N - 1 \tag{303}$$
$$P_{52} = -N^4 + 6N^3 - 10N^2 + 5N - 1 \tag{304}$$
$$P_{53} = -N^4 + 7N^3 + 3N^2 - 3N - 2 \tag{305}$$
$$P_{54} = N^4 - 15N^3 - 94N^2 - 106N + 210 \tag{306}$$
$$P_{55} = N^4 - 12N^3 - 22N^2 + 21N + 6 \tag{307}$$
$$P_{56} = N^4 - 9N^3 + 15N^2 + 13N + 4 \tag{308}$$
$$P_{57} = N^4 - 6N^3 - 29N^2 - 20N + 6 \tag{309}$$
$$P_{58} = N^4 - 6N^3 + 7N^2 - 2N - 4 \tag{310}$$
$$P_{59} = N^4 - 5N^3 - 10N^2 - 28N + 24 \tag{311}$$
$$P_{60} = N^4 - 4N^3 + 7N^2 - 4N + 8 \tag{312}$$
$$P_{61} = N^4 - 2N^3 - 6N^2 + 4N + 1 \tag{313}$$
$$P_{62} = N^4 - 2N^3 + 11N^2 - 24N - 14 \tag{314}$$
$$P_{63} = N^4 - N^3 + 8N^2 + 25N - 14 \tag{315}$$
$$P_{64} = N^4 + 3N^3 + 16N^2 + 4N - 16 \tag{316}$$
$$P_{65} = N^4 + 4N^3 + 7N^2 - 4N - 12 \tag{317}$$
$$P_{66} = N^4 + 4N^3 + 9N^2 + 6N - 2 \tag{318}$$
$$P_{67} = N^4 + 5N^3 + 5N^2 + 3N + 4 \tag{319}$$

$$P_{68} = N^4 + 8N^3 + 14N^2 + 5N + 6 \quad (320)$$
$$P_{69} = N^4 + 8N^3 + 19N^2 + 4N - 12 \quad (321)$$
$$P_{70} = N^4 + 10N^3 + 29N^2 + 20N - 4 \quad (322)$$
$$P_{71} = N^4 + 20N^3 + 59N^2 + 70N + 36 \quad (323)$$
$$P_{72} = 2N^4 + N^3 + 4N^2 - N - 1 \quad (324)$$
$$P_{73} = 2N^4 + 2N^3 + N^2 - 7N - 2 \quad (325)$$
$$P_{74} = 2N^4 + 5N^3 - 9N^2 + 30N + 8 \quad (326)$$
$$P_{75} = 2N^4 + 5N^3 + 4N^2 - 4N - 2 \quad (327)$$
$$P_{76} = 2N^4 + 7N^3 - 2N^2 - 9N + 6 \quad (328)$$
$$P_{77} = 2N^4 + 7N^3 + 4N^2 + N + 4 \quad (329)$$
$$P_{78} = 2N^4 + 14N^3 + 29N^2 + 31N + 24 \quad (330)$$
$$P_{79} = 3N^4 - 237N^3 - 100N^2 + 20N - 64 \quad (331)$$
$$P_{80} = 3N^4 - 45N^3 - 82N^2 + 22N + 12 \quad (332)$$
$$P_{81} = 3N^4 - 9N^3 - 22N^2 + 10N + 4 \quad (333)$$
$$P_{82} = 3N^4 - N^3 - 12N^2 + 6N + 2 \quad (334)$$
$$P_{83} = 3N^4 + 2N^3 + 11N^2 - 4N - 4 \quad (335)$$
$$P_{84} = 3N^4 + 3N^3 + 12N^2 + 32N + 8 \quad (336)$$
$$P_{85} = 3N^4 + 4N^3 - 23N^2 + 4N + 36 \quad (337)$$
$$P_{86} = 3N^4 + 7N^3 - 32N^2 - 76N - 48 \quad (338)$$
$$P_{87} = 3N^4 + 8N^3 - 8N^2 + 3N + 6 \quad (339)$$
$$P_{88} = 3N^4 + 12N^3 + 3N^2 - 14N + 60 \quad (340)$$
$$P_{89} = 3N^4 + 13N^3 + 20N^2 + 10N - 1 \quad (341)$$
$$P_{90} = 3N^4 + 21N^3 + 48N^2 + 34N - 30 \quad (342)$$
$$P_{91} = 4N^4 + 21N^3 + 29N^2 + 8N + 30 \quad (343)$$
$$P_{92} = 5N^4 + 8N^3 + 19N^2 + 12N + 4 \quad (344)$$
$$P_{93} = 5N^4 + 16N^3 + 13N^2 + 62N + 84 \quad (345)$$
$$P_{94} = 5N^4 + 28N^3 + 22N^2 + 15N + 6 \quad (346)$$
$$P_{95} = 5N^4 + 35N^3 + 61N^2 - 17N - 3 \quad (347)$$
$$P_{96} = 6N^4 - 19N^3 - 56N^2 + 4N + 16 \quad (348)$$
$$P_{97} = 6N^4 + 9N^3 + 23N^2 + 14N + 4 \quad (349)$$
$$P_{98} = 7N^4 - 21N^3 - 61N^2 + 45N + 6 \quad (350)$$
$$P_{99} = 7N^4 - 14N^3 - N^2 - 20N - 4 \quad (351)$$
$$P_{100} = 7N^4 + 10N^3 + 13N^2 + 20N + 4 \quad (352)$$
$$P_{101} = 7N^4 + 23N^3 + 19N^2 + 15N + 6 \quad (353)$$
$$P_{102} = 7N^4 + 24N^3 - 9N^2 - 74N - 36 \quad (354)$$

$$P_{103} = 7N^4 + 28N^3 + 39N^2 + 18N - 2 \tag{355}$$
$$P_{104} = 8N^4 - 7N^3 - 36N^2 + 17N + 2 \tag{356}$$
$$P_{105} = 9N^4 + 25N^3 + 4N^2 - 16N - 8 \tag{357}$$
$$P_{106} = 10N^4 - 32N^3 + 109N^2 - 87N + 27 \tag{358}$$
$$P_{107} = 10N^4 + 11N^3 + 7N^2 + 12N + 8 \tag{359}$$
$$P_{108} = 10N^4 + 17N^3 + 77N^2 + 48N + 8 \tag{360}$$
$$P_{109} = 10N^4 + 19N^3 + 23N^2 - 8N - 4 \tag{361}$$
$$P_{110} = 10N^4 + 23N^3 + 38N^2 + 47N + 18 \tag{362}$$
$$P_{111} = 10N^4 + 25N^3 + 22N^2 + 17N + 6 \tag{363}$$
$$P_{112} = 10N^4 + 47N^3 + 69N^2 + 40N + 12 \tag{364}$$
$$P_{113} = 10N^4 + 51N^3 + 81N^2 + 48N + 12 \tag{365}$$
$$P_{114} = 11N^4 + 15N^3 - 6N^2 + 10N + 4 \tag{366}$$
$$P_{115} = 11N^4 + 22N^3 + 13N^2 + 2N + 24 \tag{367}$$
$$P_{116} = 11N^4 + 22N^3 + 73N^2 + 62N + 84 \tag{368}$$
$$P_{117} = 11N^4 + 22N^3 + 79N^2 + 68N + 90 \tag{369}$$
$$P_{118} = 12N^4 - 21N^3 + 2N^2 + 53N + 8 \tag{370}$$
$$P_{119} = 13N^4 + 26N^3 - 63N^2 - 76N + 52 \tag{371}$$
$$P_{120} = 13N^4 + 26N^3 + 89N^2 + 76N + 52 \tag{372}$$
$$P_{121} = 14N^4 + 28N^3 + 11N^2 + 2N + 1 \tag{373}$$
$$P_{122} = 14N^4 + 33N^3 + 31N^2 + 12 \tag{374}$$
$$P_{123} = 14N^4 + 56N^3 - 45N^2 + 25N + 34 \tag{375}$$
$$P_{124} = 14N^4 + 112N^3 + 123N^2 + 137N + 34 \tag{376}$$
$$P_{125} = 15N^4 + 71N^3 + 80N^2 - 4N - 48 \tag{377}$$
$$P_{126} = 16N^4 + 101N^3 + 201N^2 + 366N - 144 \tag{378}$$
$$P_{127} = 19N^4 - 44N^3 + 22N^2 + 93N - 27 \tag{379}$$
$$P_{128} = 19N^4 + 38N^3 - 43N^2 - 62N - 24 \tag{380}$$
$$P_{129} = 19N^4 + 80N^3 + 131N^2 - 142N - 112 \tag{381}$$
$$P_{130} = 19N^4 + 82N^3 + 105N^2 + 50N + 24 \tag{382}$$
$$P_{131} = 22N^4 + 23N^3 + 11N^2 + 24N + 16 \tag{383}$$
$$P_{132} = 22N^4 + 35N^3 + 6N^2 + 3N + 6 \tag{384}$$
$$P_{133} = 22N^4 + 41N^3 + 38N^2 + 37N + 14 \tag{385}$$
$$P_{134} = 23N^4 + 133N^3 + 166N^2 + 74N + 84 \tag{386}$$
$$P_{135} = 25N^4 + 153N^3 + 290N^2 + 166N - 30 \tag{387}$$
$$P_{136} = 26N^4 + 93N^3 + 109N^2 + 44N - 4 \tag{388}$$
$$P_{137} = 27N^4 + 27N^3 - 34N^2 + 22N + 12 \tag{389}$$

$$P_{138} = 27N^4 + 94N^3 + 105N^2 + 40N - 4 \quad (390)$$
$$P_{139} = 29N^4 - 8N^3 + 212N^2 + 186N + 63 \quad (391)$$
$$P_{140} = 29N^4 + 95N^3 + 91N^2 - 23N - 3 \quad (392)$$
$$P_{141} = 30N^4 + 31N^3 + 20N^2 + 109N + 26 \quad (393)$$
$$P_{142} = 32N^4 + 61N^3 + 305N^2 + 213N + 63 \quad (394)$$
$$P_{143} = 34N^4 + 71N^3 + 14N^2 - 41N + 30 \quad (395)$$
$$P_{144} = 37N^4 + 74N^3 - 85N^2 - 122N - 48 \quad (396)$$
$$P_{145} = 39N^4 + 88N^3 - 75N^2 - 120N - 28 \quad (397)$$
$$P_{146} = 41N^4 + 54N^3 - 56N^2 + 21N - 6 \quad (398)$$
$$P_{147} = 85N^4 + 315N^3 + 584N^2 + 606N + 12 \quad (399)$$
$$P_{148} = 94N^4 + 352N^3 + 243N^2 - 367N - 358 \quad (400)$$
$$P_{149} = 94N^4 + 446N^3 + 713N^2 + 418N + 72 \quad (401)$$
$$P_{150} = 101N^4 + 597N^3 + 1066N^2 + 558N + 90 \quad (402)$$
$$P_{151} = 116N^4 + 547N^3 + 807N^2 + 376N + 12 \quad (403)$$
$$P_{152} = 116N^4 + 547N^3 + 823N^2 + 440N + 76 \quad (404)$$
$$P_{153} = 116N^4 + 669N^3 + 1141N^2 + 552N + 270 \quad (405)$$
$$P_{154} = 137N^4 - 24N^3 - 399N^2 - 262N + 48 \quad (406)$$
$$P_{155} = 211N^4 + 1018N^3 + 1208N^2 - 118N - 24 \quad (407)$$
$$P_{156} = 221N^4 + 170N^3 - 793N^2 - 510N + 360 \quad (408)$$
$$P_{157} = 253N^4 + 674N^3 - 435N^2 + 1832N + 240 \quad (409)$$
$$P_{158} = 799N^4 + 3574N^3 + 3099N^2 - 3358N - 3790 \quad (410)$$
$$P_{159} = 799N^4 + 4373N^3 + 7256N^2 + 3250N - 648 \quad (411)$$
$$P_{160} = 2599N^4 + 13859N^3 + 23924N^2 + 12880N + 648 \quad (412)$$
$$P_{161} = 7865N^4 + 15346N^3 + 7529N^2 + 48N - 144 \quad (413)$$
$$P_{162} = -713N^5 - 3808N^4 - 5290N^3 + 256N^2 + 1563N - 2664 \quad (414)$$
$$P_{163} = -589N^5 - 2828N^4 - 4094N^3 - 3388N^2 - 3357N - 5040 \quad (415)$$
$$P_{164} = -531N^5 - 2210N^4 - 2251N^3 - 508N^2 - 752N + 768 \quad (416)$$
$$P_{165} = -485N^5 - 2156N^4 - 2647N^3 - 556N^2 + 636N + 144 \quad (417)$$
$$P_{166} = -485N^5 - 2084N^4 - 2647N^3 - 772N^2 + 492N + 144 \quad (418)$$
$$P_{167} = -203N^5 - 706N^4 - 676N^3 - 293N^2 - 372N + 12 \quad (419)$$
$$P_{168} = -185N^5 - 803N^4 - 763N^3 + 1070N^2 + 2241N + 378 \quad (420)$$
$$P_{169} = -71N^5 - 599N^4 - 1230N^3 - 598N^2 + 128N - 72 \quad (421)$$
$$P_{170} = -65N^5 - 282N^4 - 995N^3 - 2198N^2 - 1252N - 104 \quad (422)$$
$$P_{171} = -59N^5 - 154N^4 - 91N^3 + 106N^2 - 6N - 12 \quad (423)$$
$$P_{172} = -58N^5 - 248N^4 - 389N^3 - 293N^2 - 160N - 20 \quad (424)$$

$$P_{173} = -58N^5 - 232N^4 - 253N^3 - 29N^2 - 16N - 20 \tag{425}$$
$$P_{174} = -41N^5 - 196N^4 - 374N^3 - 604N^2 - 585N - 600 \tag{426}$$
$$P_{175} = -37N^5 + 340N^3 - 246N^2 + 81N + 54 \tag{427}$$
$$P_{176} = -33N^5 - 166N^4 - 321N^3 - 240N^2 - 52N - 16 \tag{428}$$
$$P_{177} = -33N^5 - 150N^4 - 209N^3 + 48N^2 + 140N - 48 \tag{429}$$
$$P_{178} = -26N^5 - 139N^4 - 181N^3 + 55N^2 - 33N - 252 \tag{430}$$
$$P_{179} = -24N^5 - 97N^4 - 221N^3 - 450N^2 - 280N - 32 \tag{431}$$
$$P_{180} = -21N^5 - 77N^4 - 51N^3 + 10N^2 + 28N - 24 \tag{432}$$
$$P_{181} = -21N^5 - 37N^4 + 55N^3 + 54N^2 + 84N - 72 \tag{433}$$
$$P_{182} = -16N^5 - 29N^4 - 2N^3 + 11N^2 - 12N + 12 \tag{434}$$
$$P_{183} = -13N^5 - 52N^4 - 70N^3 - 27N^2 + 4N - 4 \tag{435}$$
$$P_{184} = -13N^5 - 30N^4 + 30N^3 + 135N^2 + 88N - 12 \tag{436}$$
$$P_{185} = -9N^5 - 11N^4 + 61N^3 - 33N^2 - 92N - 12 \tag{437}$$
$$P_{186} = -7N^5 - 18N^4 - 26N^3 - 10N^2 - 5N - 6 \tag{438}$$
$$P_{187} = -6N^5 - 9N^4 + 16N^3 + 9N^2 - 14N - 8 \tag{439}$$
$$P_{188} = -4N^5 - 82N^4 - 99N^3 - 25N^2 - 42N - 24 \tag{440}$$
$$P_{189} = -4N^5 - 15N^4 + 6N^3 + 67N^2 + 22N - 36 \tag{441}$$
$$P_{190} = -3N^5 - 9N^4 + 21N^3 + 27N^2 - 8N + 12 \tag{442}$$
$$P_{191} = -3N^5 - 5N^4 + 8N^3 + 6N^2 - 5N - 4 \tag{443}$$
$$P_{192} = -2N^5 + N^4 - 3N^3 + 9N^2 + 5N - 2 \tag{444}$$
$$P_{193} = -N^5 - 5N^4 - 5N^3 - 9N^2 - 16N - 4 \tag{445}$$
$$P_{194} = -N^5 - 3N^4 - 9N^3 - 7N^2 + 8N + 4 \tag{446}$$
$$P_{195} = -N^5 - 3N^4 + 15N^3 + 17N^2 - 8N + 4 \tag{447}$$
$$P_{196} = -N^5 + 3N^4 + N^3 - 7N^2 + 8N + 4 \tag{448}$$
$$P_{197} = -N^5 + 5N^4 - 18N^3 - 21N^2 - 23N - 6 \tag{449}$$
$$P_{198} = -N^5 + 11N^4 - 10N^3 - 10N^2 - 5N - 1 \tag{450}$$
$$P_{199} = N^5 - 4N^3 + 22N^2 + 35N + 14 \tag{451}$$
$$P_{200} = N^5 + 4N^3 - 6N^2 - 5N - 2 \tag{452}$$
$$P_{201} = N^5 + 5N^3 + 2N^2 - 20N - 8 \tag{453}$$
$$P_{202} = N^5 - 23N^4 + 19N^3 + 51N^2 - 88N - 24 \tag{454}$$
$$P_{203} = N^5 - 5N^4 - 25N^3 - 7N^2 + 16N - 4 \tag{455}$$
$$P_{204} = N^5 - 5N^4 + 5N^3 - 7N^2 + 10N + 4 \tag{456}$$
$$P_{205} = N^5 - 3N^4 - 5N^3 - 13N^2 + 4 \tag{457}$$
$$P_{206} = N^5 + 3N^4 + 9N^2 + 23N + 6 \tag{458}$$
$$P_{207} = N^5 + 3N^4 - 3N^3 - 5N^2 - 4 \tag{459}$$

$$P_{208} = N^5 + 3N^4 + 19N^3 + 15N^2 - 38N - 24 \quad (460)$$
$$P_{209} = N^5 + 4N^4 + 25N^3 + 109N^2 + 100N - 52 \quad (461)$$
$$P_{210} = N^5 + 6N^4 + 13N^3 + 16N^2 - 4N - 16 \quad (462)$$
$$P_{211} = N^5 + 8N^4 + 12N^3 - 20N^2 - 35N - 14 \quad (463)$$
$$P_{212} = N^5 + 13N^4 + 15N^3 - 19N^2 - 46N - 36 \quad (464)$$
$$P_{213} = N^5 + 13N^4 + 31N^3 - N^2 - 24N + 4 \quad (465)$$
$$P_{214} = N^5 + 61N^4 + 321N^3 + 547N^2 + 326N + 72 \quad (466)$$
$$P_{215} = 2N^5 - 46N^4 - 7N^3 - 134N^2 - 25N + 42 \quad (467)$$
$$P_{216} = 2N^5 - 19N^4 - N^3 - 59N^2 - 13N + 18 \quad (468)$$
$$P_{217} = 2N^5 + 3N^4 - 4N^3 - 18N^2 - 6N - 1 \quad (469)$$
$$P_{218} = 2N^5 + 5N^4 + 2N^3 - 3N^2 + 6N + 12 \quad (470)$$
$$P_{219} = 2N^5 + 8N^4 - 3N^3 - 38N^2 - 9N + 24 \quad (471)$$
$$P_{220} = 2N^5 + 8N^4 + 5N^3 + 16N^2 - N - 6 \quad (472)$$
$$P_{221} = 3N^5 + 4N^3 - 14N^2 - 7N + 6 \quad (473)$$
$$P_{222} = 3N^5 - 12N^4 - 79N^3 - 60N^2 - 20N - 48 \quad (474)$$
$$P_{223} = 3N^5 - 8N^4 - 22N^3 - 9N^2 - 12N - 4 \quad (475)$$
$$P_{224} = 3N^5 - 2N^4 + 10N^3 - 20N^2 - 13N - 2 \quad (476)$$
$$P_{225} = 3N^5 + 4N^4 - 13N^3 - 36N^2 + 30N - 12 \quad (477)$$
$$P_{226} = 3N^5 + 6N^4 + 8N^3 + 3N^2 - 24N - 12 \quad (478)$$
$$P_{227} = 3N^5 + 8N^4 - 15N^3 - 32N^2 - 20N - 16 \quad (479)$$
$$P_{228} = 3N^5 + 11N^4 + 10N^3 + 10N^2 + 5N + 1 \quad (480)$$
$$P_{229} = 3N^5 + 14N^4 + 23N^3 + 16N^2 - 10N - 24 \quad (481)$$
$$P_{230} = 3N^5 + 15N^4 - 17N^3 + 73N^2 + 54N + 48 \quad (482)$$
$$P_{231} = 3N^5 + 40N^4 + 59N^3 + 28N^2 + 26N + 12 \quad (483)$$
$$P_{232} = 4N^5 - 14N^4 + 24N^3 - 8N^2 - 19N - 7 \quad (484)$$
$$P_{233} = 4N^5 + 3N^4 + 4N^3 + 10N^2 - 8N - 5 \quad (485)$$
$$P_{234} = 4N^5 + 8N^4 + 11N^3 + 9N^2 - 12N - 4 \quad (486)$$
$$P_{235} = 4N^5 + 21N^4 - 3N^3 - 60N^2 - 8N - 24 \quad (487)$$
$$P_{236} = 5N^5 - 12N^4 - 19N^3 + 38N^2 + 12N - 16 \quad (488)$$
$$P_{237} = 5N^5 - 10N^4 - 10N^3 + 5N + 2 \quad (489)$$
$$P_{238} = 5N^5 - 2N^4 + 18N^3 - 32N^2 - 23N - 6 \quad (490)$$
$$P_{239} = 5N^5 - N^4 - 20N^3 - 5N^2 - 4N - 4 \quad (491)$$
$$P_{240} = 5N^5 + 4N^4 + 4N^3 - 28N^2 - 13N + 12 \quad (492)$$
$$P_{241} = 5N^5 + 8N^4 + 12N^3 + 56N^2 + 85N + 34 \quad (493)$$
$$P_{242} = 5N^5 + 24N^4 - 59N^3 - 88N^2 + 82N - 36 \quad (494)$$

$$P_{243} = 7N^5 + N^4 + 7N^3 - 33N^2 - 2N + 4 \tag{495}$$
$$P_{244} = 7N^5 + 2N^4 - 19N^3 - 4N^2 - 6N - 4 \tag{496}$$
$$P_{245} = 7N^5 + 22N^4 + 47N^3 + 32N^2 - 36N - 16 \tag{497}$$
$$P_{246} = 7N^5 + 31N^4 - 84N^2 - 26N - 4 \tag{498}$$
$$P_{247} = 7N^5 + 310N^4 + 1137N^3 + 1142N^2 + 140N - 120 \tag{499}$$
$$P_{248} = 8N^5 - 11N^4 + 49N^3 - 31N^2 - 25N - 10 \tag{500}$$
$$P_{249} = 8N^5 + 27N^4 - 6N^3 - 9N^2 + 64N + 12 \tag{501}$$
$$P_{250} = 8N^5 + 31N^4 + 58N^3 + 43N^2 - 12N - 4 \tag{502}$$
$$P_{251} = 9N^5 + 19N^4 - 25N^3 - 31N^2 - 24N - 44 \tag{503}$$
$$P_{252} = 9N^5 + 27N^4 + 59N^3 + 33N^2 - 64N - 28 \tag{504}$$
$$P_{253} = 13N^5 + 7N^4 - 29N^3 - 3N^2 - 40N - 44 \tag{505}$$
$$P_{254} = 13N^5 + 31N^4 + 37N^3 - 46N^2 - 173N - 114 \tag{506}$$
$$P_{255} = 13N^5 + 38N^4 - 4N^3 + 58N^2 + 111N + 72 \tag{507}$$
$$P_{256} = 13N^5 + 67N^4 + 61N^3 + 119N^2 + 232N + 60 \tag{508}$$
$$P_{257} = 16N^5 + 64N^4 - 49N^3 - N^2 + 105N + 54 \tag{509}$$
$$P_{258} = 20N^5 - 27N^4 - 130N^3 + 48N^2 + 152N + 48 \tag{510}$$
$$P_{259} = 20N^5 + 59N^4 + 130N^3 + 67N^2 - 156N - 68 \tag{511}$$
$$P_{260} = 23N^5 - 28N^4 + 129N^3 - 114N^2 - 98N - 36 \tag{512}$$
$$P_{261} = 25N^5 - 50N^4 - 231N^3 - 52N^2 + 116N + 48 \tag{513}$$
$$P_{262} = 31N^5 + 358N^4 + 1065N^3 + 950N^2 + 44N - 120 \tag{514}$$
$$P_{263} = 33N^5 + 24N^4 + 463N^3 + 276N^2 - 308N + 64 \tag{515}$$
$$P_{264} = 33N^5 + 59N^4 - 132N^3 - 210N^2 + 40N + 40 \tag{516}$$
$$P_{265} = 45N^5 + 225N^4 + 370N^3 + 244N^2 - 56N + 96 \tag{517}$$
$$P_{266} = 49N^5 + 163N^4 + 84N^3 - 50N^2 + 72N + 40 \tag{518}$$
$$P_{267} = 51N^5 + 207N^4 + 193N^3 - 210N^2 - 421N - 66 \tag{519}$$
$$P_{268} = 53N^5 + 37N^4 - 122N^3 + 53N^2 + 117N + 6 \tag{520}$$
$$P_{269} = 53N^5 + 47N^4 + 67N^3 - 197N^2 + 486N + 360 \tag{521}$$
$$P_{270} = 67N^5 + 289N^4 + 179N^3 - 409N^2 - 238N + 220 \tag{522}$$
$$P_{271} = 77N^5 + 256N^4 + 83N^3 - 380N^2 - 340N + 16 \tag{523}$$
$$P_{272} = 80N^5 + 119N^4 - 152N^3 - 770N^2 - 264N - 45 \tag{524}$$
$$P_{273} = 119N^5 + 188N^4 - 380N^3 + 184N^2 - 339N - 180 \tag{525}$$
$$P_{274} = 133N^5 + 680N^4 + 1238N^3 + 1408N^2 + 1329N + 1692 \tag{526}$$
$$P_{275} = 137N^5 + 67N^4 - 392N^3 + 155N^2 + 303N + 18 \tag{527}$$
$$P_{276} = 211N^5 + 986N^4 + 2204N^3 + 4558N^2 + 4425N + 4032 \tag{528}$$
$$P_{277} = 226N^5 + 165N^4 - 786N^3 - 632N^2 - 186N - 9 \tag{529}$$

$$P_{278} = 226N^5 + 561N^4 + 348N^3 + 178N^2 - 168N - 63 \quad (530)$$
$$P_{279} = 361N^5 + 1690N^4 + 2392N^3 + 2254N^2 + 2567N + 3504 \quad (531)$$
$$P_{280} = 379N^5 + 922N^4 + 609N^3 + 3110N^2 + 1076N - 4152 \quad (532)$$
$$P_{281} = 496N^5 + 2219N^4 + 3857N^3 + 6889N^2 + 7167N + 7020 \quad (533)$$
$$P_{282} = 541N^5 + 1575N^4 + 461N^3 - 1415N^2 - 746N - 160 \quad (534)$$
$$P_{283} = 3625N^5 + 10875N^4 + 3505N^3 - 11691N^2 - 8282N + 816 \quad (535)$$
$$P_{284} = 4148N^5 + 21847N^4 + 37963N^3 + 20318N^2 - 1188N - 648 \quad (536)$$
$$P_{285} = -843N^6 - 3105N^5 - 4121N^4 - 1915N^3 + 40N^2 + 240N + 72 \quad (537)$$
$$P_{286} = -843N^6 - 2721N^5 - 2681N^4 - 1051N^3 - 440N^2 - 48N + 72 \quad (538)$$
$$P_{287} = -695N^6 - 2849N^5 - 4354N^4 - 4804N^3 - 4125N^2 - 603N + 54 \quad (539)$$
$$P_{288} = -202N^6 - 606N^5 - 633N^4 + 41N^3 - 405N^2 - 351N - 108 \quad (540)$$
$$P_{289} = -195N^6 - 1277N^5 - 2741N^4 - 978N^3 + 3881N^2 + 4532N + 900 \quad (541)$$
$$P_{290} = -130N^6 - 381N^5 + 50N^4 - 411N^3 - 1054N^2 + 1854N + 828 \quad (542)$$
$$P_{291} = -55N^6 - 383N^5 - 747N^4 + 59N^3 + 1434N^2 + 1012N - 168 \quad (543)$$
$$P_{292} = -31N^6 - 95N^5 - 74N^4 + 14N^3 + 3N^2 - 19N - 6 \quad (544)$$
$$P_{293} = -31N^6 - 69N^5 + 2N^4 + 66N^3 - N^2 - 33N - 14 \quad (545)$$
$$P_{294} = -27N^6 - 54N^5 + 159N^4 + 462N^3 + 908N^2 + 584N - 160 \quad (546)$$
$$P_{295} = -21N^6 - 77N^5 - 116N^4 - 104N^3 - 93N^2 - 55N - 14 \quad (547)$$
$$P_{296} = -21N^6 - 21N^5 + 19N^4 + 115N^3 + 314N^2 + 22N - 116 \quad (548)$$
$$P_{297} = -7N^6 - 25N^5 - 68N^4 - 48N^3 - 11N^2 - 19N - 14 \quad (549)$$
$$P_{298} = -7N^6 - 24N^5 - 32N^4 - 17N^3 - 10N^2 - 5N - 1 \quad (550)$$
$$P_{299} = -7N^6 - 20N^5 - 24N^4 - 5N^3 + 2N^2 - N - 1 \quad (551)$$
$$P_{300} = -N^6 - 3N^5 - 9N^4 - 5N^3 + 4N^2 - 30N - 12 \quad (552)$$
$$P_{301} = -N^6 + 2N^5 + 4N^4 + 6N^3 + 13N^2 + 12N - 12 \quad (553)$$
$$P_{302} = -N^6 + 7N^5 - 5N^4 - 25N^3 + 38N^2 + 38N + 12 \quad (554)$$
$$P_{303} = N^6 - 7N^5 - 15N^4 - 29N^3 - 50N^2 - 8N - 12 \quad (555)$$
$$P_{304} = N^6 - 6N^5 - 80N^4 - 90N^3 + 235N^2 + 264N - 36 \quad (556)$$
$$P_{305} = N^6 - 6N^5 - 65N^4 - 180N^3 - 200N^2 - 36N + 54 \quad (557)$$
$$P_{306} = N^6 - 6N^5 - 56N^4 - 234N^3 - 461N^2 - 216N + 108 \quad (558)$$
$$P_{307} = N^6 - 4N^5 - 18N^4 - 40N^3 - 35N^2 + 12N + 12 \quad (559)$$
$$P_{308} = N^6 - 4N^5 + 7N^4 - 19N^3 + 18N^2 + 9N - 4 \quad (560)$$
$$P_{309} = N^6 + 3N^5 + 17N^4 - 17N^3 - 74N^2 - 10N + 8 \quad (561)$$
$$P_{310} = N^6 + 6N^5 + 12N^4 + 14N^3 + 14N^2 - 13N - 2 \quad (562)$$
$$P_{311} = N^6 + 7N^5 + 13N^4 - 7N^3 + 36N^2 + 116N - 6 \quad (563)$$
$$P_{312} = N^6 + 8N^5 - 58N^4 - 108N^3 + 21N^2 + 20N + 20 \quad (564)$$

$$P_{313} = N^6 + 9N^5 - 56N^3 - 21N^2 + 55N - 36 \quad (565)$$
$$P_{314} = N^6 + 21N^5 + 103N^4 + 221N^3 + 272N^2 + 186N + 36 \quad (566)$$
$$P_{315} = 2N^6 - N^5 - 23N^4 + N^3 - 135N^2 - 280N + 68 \quad (567)$$
$$P_{316} = 2N^6 + 2N^5 + 7N^4 - 12N^3 + 8N^2 + 18N + 7 \quad (568)$$
$$P_{317} = 2N^6 + 7N^5 + 20N^4 + 24N^3 + 13N^2 + 24N + 12 \quad (569)$$
$$P_{318} = 2N^6 + 16N^5 + 9N^4 - 87N^3 - 37N^2 + 99N - 74 \quad (570)$$
$$P_{319} = 2N^6 + 16N^5 + 23N^4 + 42N^3 + 17N^2 + 16N + 12 \quad (571)$$
$$P_{320} = 3N^6 + 3N^4 - 7N^3 + 8N^2 + 7N - 6 \quad (572)$$
$$P_{321} = 3N^6 - 7N^5 + 11N^4 - 17N^3 + 2N^2 + 4N + 2 \quad (573)$$
$$P_{322} = 3N^6 - 5N^5 - 7N^4 - 21N^3 - 26N^2 - 28N + 24 \quad (574)$$
$$P_{323} = 3N^6 + 3N^5 - 4N^4 - 17N^3 - 37N^2 - 8N + 12 \quad (575)$$
$$P_{324} = 3N^6 + 19N^5 + 75N^4 + 157N^3 + 26N^2 - 192N - 72 \quad (576)$$
$$P_{325} = 3N^6 + 20N^5 + 49N^4 + 52N^3 - 8N^2 - 48N - 16 \quad (577)$$
$$P_{326} = 3N^6 + 20N^5 + 61N^4 + 96N^3 + 16N^2 - 64N - 16 \quad (578)$$
$$P_{327} = 4N^6 + 11N^5 + 30N^4 + 105N^3 + 50N^2 - 96N - 144 \quad (579)$$
$$P_{328} = 4N^6 + 18N^5 - 28N^4 + 24N^3 + 21N^2 - 21N - 14 \quad (580)$$
$$P_{329} = 5N^6 - 17N^5 + 53N^4 - 143N^3 + 66N^2 + 60N - 8 \quad (581)$$
$$P_{330} = 5N^6 + 26N^5 + 30N^4 + 40N^3 + 133N^2 + 234N + 396 \quad (582)$$
$$P_{331} = 6N^6 - 204N^5 - 735N^4 - 827N^3 - 376N^2 - 236N - 96 \quad (583)$$
$$P_{332} = 6N^6 + 9N^5 + 54N^4 + 41N^3 - 174N^2 - 96N + 16 \quad (584)$$
$$P_{333} = 6N^6 + 41N^5 - 118N^4 - 171N^3 - 14N^2 - 212N - 72 \quad (585)$$
$$P_{334} = 9N^6 + 9N^5 - 7N^4 - 47N^3 - 130N^2 - 6N + 52 \quad (586)$$
$$P_{335} = 9N^6 + 10N^5 - 7N^4 - 29N^3 + 8N^2 + 7N - 6 \quad (587)$$
$$P_{336} = 11N^6 + 33N^5 + 129N^4 + 131N^3 + 76N^2 + 100N + 96 \quad (588)$$
$$P_{337} = 11N^6 + 276N^5 + 398N^4 + 45N^3 + 626N^2 - 240N - 288 \quad (589)$$
$$P_{338} = 12N^6 - 3N^5 - 6N^4 + 413N^3 + 228N^2 - 684N + 112 \quad (590)$$
$$P_{339} = 12N^6 + 39N^5 + 71N^4 + 45N^3 - 229N^2 - 298N - 48 \quad (591)$$
$$P_{340} = 16N^6 - 222N^5 + 753N^4 - 1660N^3 + 1581N^2 - 846N + 189 \quad (592)$$
$$P_{341} = 17N^6 + 6N^5 - 20N^4 - 36N^3 - N^2 - 6N - 8 \quad (593)$$
$$P_{342} = 19N^6 + 57N^5 + 3N^4 - 89N^3 - 62N^2 - 8N - 64 \quad (594)$$
$$P_{343} = 19N^6 + 57N^5 + 3N^4 - 89N^3 - 46N^2 + 8N - 64 \quad (595)$$
$$P_{344} = 19N^6 + 175N^5 + 533N^4 + 473N^3 + 564N^2 + 1340N + 672 \quad (596)$$
$$P_{345} = 24N^6 + 152N^5 + 59N^4 - 265N^3 + 70N^2 + 28N - 248 \quad (597)$$
$$P_{346} = 29N^6 - 54N^5 - 1657N^4 - 5916N^3 - 7786N^2 - 3408N - 24 \quad (598)$$
$$P_{347} = 30N^6 + 181N^5 + 473N^4 + 635N^3 + 77N^2 - 520N - 156 \quad (599)$$

$$P_{348} = 37N^6 + 28N^5 + 9N^4 + 130N^3 - 32N^2 - 32N - 32 \quad (600)$$
$$P_{349} = 40N^6 + 231N^5 + 484N^4 + 780N^3 + 691N^2 - 480N - 576 \quad (601)$$
$$P_{350} = 47N^6 + 69N^5 + 21N^4 - 145N^3 - 500N^2 - 20N + 240 \quad (602)$$
$$P_{351} = 54N^6 - 5N^5 - 125N^4 + 3N^3 + 107N^2 - 226N + 120 \quad (603)$$
$$P_{352} = 54N^6 + 356N^5 + 729N^4 + 375N^3 + 2N^2 + 404N + 200 \quad (604)$$
$$P_{353} = 54N^6 + 436N^5 + 1297N^4 + 1919N^3 + 1954N^2 + 1332N + 200 \quad (605)$$
$$P_{354} = 56N^6 + 339N^5 + 718N^4 + 621N^3 + 208N^2 + 76N + 32 \quad (606)$$
$$P_{355} = 59N^6 - 510N^5 + 2329N^4 + 5494N^3 - 1212N^2 - 1576N - 1344 \quad (607)$$
$$P_{356} = 61N^6 + 36N^5 - 59N^4 - 6N^3 - 140N^2 - 456N + 96 \quad (608)$$
$$P_{357} = 62N^6 + 399N^5 + 416N^4 + 6N^3 + 440N^2 - 657N - 378 \quad (609)$$
$$P_{358} = 67N^6 + 490N^5 + 1046N^4 + 128N^3 - 1729N^2 - 1382N - 60 \quad (610)$$
$$P_{359} = 85N^6 + 485N^5 + 959N^4 + 805N^3 + 480N^2 + 534N - 180 \quad (611)$$
$$P_{360} = 87N^6 + 285N^5 + 293N^4 + 191N^3 + 88N^2 + 56N + 24 \quad (612)$$
$$P_{361} = 87N^6 + 333N^5 + 485N^4 + 463N^3 + 216N^2 + 56N - 8 \quad (613)$$
$$P_{362} = 94N^6 + 291N^5 + 4N^4 + 33N^3 + 460N^2 - 882N - 396 \quad (614)$$
$$P_{363} = 119N^6 + 1212N^5 + 1784N^4 + 171N^3 + 1706N^2 + 480N - 144 \quad (615)$$
$$P_{364} = 188N^6 + 1084N^5 + 2143N^4 + 1539N^3 + 464N^2 + 572N + 160 \quad (616)$$
$$P_{365} = 233N^6 + 593N^5 - 12N^4 - 497N^3 + 607N^2 + 792N + 12 \quad (617)$$
$$P_{366} = 293N^6 + 231N^5 - 125N^4 + 661N^3 - 1152N^2 - 2308N + 1104 \quad (618)$$
$$P_{367} = 379N^6 + 2059N^5 + 3375N^4 + 4937N^3 + 9470N^2 + 9948N + 13032 \quad (619)$$
$$P_{368} = 380N^6 + 1499N^5 + 2113N^4 + 1870N^3 + 1191N^2 + 9N - 18 \quad (620)$$
$$P_{369} = 513N^6 + 1845N^5 + 2269N^4 + 1187N^3 + 538N^2 + 288N + 144 \quad (621)$$
$$P_{370} = 743N^6 + 4724N^5 + 9411N^4 + 7922N^3 + 9340N^2 + 10208N - 768 \quad (622)$$
$$P_{371} = 799N^6 + 5280N^5 + 12295N^4 + 10866N^3 - 980N^2 - 6408N - 1728 \quad (623)$$
$$P_{372} = 848N^6 + 2727N^5 + 4023N^4 + 5921N^3 + 3069N^2 + 3900N + 1440 \quad (624)$$
$$P_{373} = 1069N^6 + 6630N^5 + 14503N^4 + 12330N^3 + 556N^2 - 3912N - 864 \quad (625)$$
$$P_{374} = 1069N^6 + 6702N^5 + 15079N^4 + 13698N^3 + 844N^2 - 4776N - 864 \quad (626)$$
$$P_{375} = 1087N^6 + 4415N^5 + 1621N^4 - 10867N^3 - 11540N^2 + 548N + 3504 \quad (627)$$
$$P_{376} = 10875N^6 + 36321N^5 + 36361N^4 + 15419N^3 + 7336N^2 + 1248N - 792 \quad (628)$$
$$P_{377} = -1859N^7 - 10827N^6 - 22970N^5 - 23466N^4 - 15995N^3 - 8151N^2 - 144N + 36 \quad (629)$$
$$P_{378} = -777N^7 - 3738N^6 - 5972N^5 + 10N^4 + 5057N^3 - 2044N^2 + 2112N + 1008 \quad (630)$$
$$P_{379} = -715N^7 - 3476N^6 - 5448N^5 - 2380N^4 + 1129N^3 + 174N^2 - 756N - 216 \quad (631)$$

$$P_{380} = -715N^7 - 3062N^6 - 3450N^5 + 518N^4 + 2803N^3 + 1182N^2 + 108N + 216 \quad (632)$$
$$P_{381} = -559N^7 - 2855N^6 - 5215N^5 - 3779N^4 + 1904N^3 + 7012N^2 + 4212N + 144 \quad (633)$$
$$P_{382} = -128N^7 - 786N^6 - 1347N^5 + 5412N^4 + 5805N^3 - 7692N^2 - 1252N + 2256 \quad (634)$$
$$P_{383} = -100N^7 - 485N^6 - 286N^5 + 985N^4 - 88N^3 - 710N^2 + 2568N + 984 \quad (635)$$
$$P_{384} = -44N^7 - 138N^6 - 86N^5 + 12N^4 + 37N^3 + 174N^2 + 51N + 6 \quad (636)$$
$$P_{385} = -41N^7 - 226N^6 - 220N^5 + 478N^4 + 123N^3 - 24N^2 + 162N - 108 \quad (637)$$
$$P_{386} = -27N^7 + 45N^6 - 363N^5 - 2217N^4 - 274N^3 + 1980N^2 - 168N + 160 \quad (638)$$
$$P_{387} = -12N^7 - 31N^6 + 14N^5 + 86N^4 + 34N^3 - 71N^2 - 24N - 4 \quad (639)$$
$$P_{388} = -5N^7 + 20N^5 - 14N^4 - 35N^3 + 30N^2 + 28N + 8 \quad (640)$$
$$P_{389} = -5N^7 - 46N^6 - 122N^5 + 48N^4 + 483N^3 + 678N^2 + 1700N + 1872 \quad (641)$$
$$P_{390} = -5N^7 + 1382N^6 + 5380N^5 + 7451N^4 + 6436N^3 + 2228N^2 - 5952N - 3168 \quad (642)$$
$$P_{391} = -N^7 - 12N^5 - 50N^4 - 35N^3 + 30N^2 + 28N + 8 \quad (643)$$
$$P_{392} = -N^7 - 7N^6 + 87N^5 - 281N^4 + 482N^3 - 548N^2 + 384N - 112 \quad (644)$$
$$P_{393} = -N^7 - 2N^6 - 2N^5 + 104N^4 + 987N^3 + 2102N^2 + 640N - 756 \quad (645)$$
$$P_{394} = -N^7 + N^6 - 10N^5 + 15N^4 + 97N^3 + 102N^2 + 76N + 24 \quad (646)$$
$$P_{395} = -N^7 + 4N^6 + 10N^5 + 6N^4 - 25N^3 - 62N^2 - 12N + 8 \quad (647)$$
$$P_{396} = N^7 - 5N^6 - 43N^5 - 19N^4 - 126N^3 - 440N^2 - 200N - 144 \quad (648)$$
$$P_{397} = N^7 + 3N^6 - 55N^5 + 165N^4 - 114N^3 - 172N^2 + 256N - 80 \quad (649)$$
$$P_{398} = N^7 + 3N^6 + 22N^5 + 97N^4 + 75N^3 + 22N^2 + 104N + 60 \quad (650)$$
$$P_{399} = N^7 + 5N^6 + 7N^5 - 33N^4 - 168N^3 - 100N^2 + 176N + 96 \quad (651)$$
$$P_{400} = 2N^7 - 40N^6 - 53N^5 + 61N^4 + 149N^3 - 49N^2 - 46N + 24 \quad (652)$$
$$P_{401} = 4N^7 + 16N^6 + 37N^5 + 36N^4 + 9N^3 - 10N^2 - 52N - 24 \quad (653)$$
$$P_{402} = 5N^7 - 4N^6 - 3N^5 - 5N^4 - 65N^3 + 67N^2 + 55N - 30 \quad (654)$$
$$P_{403} = 6N^7 + 26N^6 + 60N^5 + 112N^4 + 91N^3 - 29N^2 - 24N - 4 \quad (655)$$
$$P_{404} = 7N^7 + 13N^6 - 80N^5 - 133N^4 + 163N^3 + 226N^2 - 92N - 24 \quad (656)$$
$$P_{405} = 8N^7 + 45N^6 + 105N^5 + 138N^4 + 116N^3 + 95N^2 + 55N + 14 \quad (657)$$
$$P_{406} = 11N^7 + 55N^6 + 83N^5 + 40N^4 + 20N^3 - 38N^2 - 93N - 6 \quad (658)$$
$$P_{407} = 12N^7 + 22N^6 + 6N^5 - 140N^4 - 167N^3 - 170N^2 - 77N - 14 \quad (659)$$
$$P_{408} = 13N^7 + 33N^6 - 43N^5 - 77N^4 - 362N^3 - 868N^2 - 424N - 96 \quad (660)$$
$$P_{409} = 14N^7 + 62N^6 + 192N^5 + 332N^4 + 253N^3 + 187N^2 + 112N + 28 \quad (661)$$
$$P_{410} = 24N^7 + 79N^6 + 85N^5 + 86N^4 + 60N^3 + 17N^2 + 19N + 14 \quad (662)$$

$$P_{411} = 25N^7 + 445N^6 + 2127N^5 - 355N^4 - 9232N^3 - 1938N^2 + 5688N - 1080 \quad (663)$$

$$P_{412} = 28N^7 + 104N^6 + 224N^5 + 211N^4 - 80N^3 + 13N^2 + 106N + 48 \quad (664)$$

$$P_{413} = 28N^7 + 122N^6 + 344N^5 + 537N^4 + 436N^3 + 425N^2 + 242N + 80 \quad (665)$$

$$P_{414} = 28N^7 + 126N^6 + 236N^5 + 187N^4 - 14N^3 - 15N^2 + 24N + 12 \quad (666)$$

$$P_{415} = 28N^7 + 128N^6 + 252N^5 + 241N^4 + 102N^3 + 101N^2 + 72N + 28 \quad (667)$$

$$P_{416} = 32N^7 + 16N^6 + 246N^5 - 1198N^4 - 1034N^3 + 624N^2 + 1089N + 378 \quad (668)$$

$$P_{417} = 43N^7 - 143N^6 - 1196N^5 - 1067N^4 + 676N^3 - 248N^2 + 909N + 738 \quad (669)$$

$$P_{418} = 45N^7 + 249N^6 + 709N^5 + 1099N^4 - 186N^3 - 1204N^2 - 328N - 96 \quad (670)$$

$$P_{419} = 49N^7 + 134N^6 + 51N^5 - 509N^4 - 1376N^3 - 825N^2 + 616N + 564 \quad (671)$$

$$P_{420} = 56N^7 + 283N^6 + 383N^5 - 97N^4 - 465N^3 - 196N^2 - 28N - 32 \quad (672)$$

$$P_{421} = 101N^7 + 550N^6 + 421N^5 - 1555N^4 - 258N^3 + 57N^2 - 504N + 324 \quad (673)$$

$$P_{422} = 131N^7 + 1216N^6 + 3842N^5 + 4834N^4 + 1883N^3 - 398N^2 - 348N - 72 \quad (674)$$

$$P_{423} = 188N^7 + 904N^6 + 1443N^5 + 636N^4 - 339N^3 - 1236N^2 - 1724N - 160 \quad (675)$$

$$P_{424} = 361N^7 + 1201N^6 - 1633N^5 - 6185N^4 - 40N^3 + 6376N^2 + 1024N - 2544 \quad (676)$$

$$P_{425} = 799N^7 + 4697N^6 + 7231N^5 - 2293N^4 - 12926N^3 - 4348N^2 + 6408N + 1728 \quad (677)$$

$$P_{426} = 896N^7 + 4597N^6 + 5607N^5 - 2578N^4 - 5167N^3 - 2691N^2 - 3712N + 3696 \quad (678)$$

$$P_{427} = 1385N^7 + 5386N^6 + 3624N^5 - 6814N^4 - 10313N^3 - 3408N^2 + 360N - 216 \quad (679)$$

$$P_{428} = 1385N^7 + 6628N^6 + 10410N^5 + 5048N^4 - 1763N^3 - 96N^2 + 1872N + 648 \quad (680)$$

$$P_{429} = 3001N^7 + 14078N^6 + 23382N^5 - 4526N^4 - 18883N^3 + 42660N^2 + 7128N - 432 \quad (681)$$

$$P_{430} = 3331N^7 + 16277N^6 + 25231N^5 + 25871N^4 + 21142N^3 - 20332N^2 - 35952N - 23904 \quad (682)$$

$$P_{431} = 4148N^7 + 21847N^6 + 33491N^5 - 1529N^4 - 34939N^3 - 15782N^2 - 108N + 648 \quad (683)$$

$$P_{432} = -306723N^8 - 1247148N^7 - 1970434N^6 - 1388892N^5 - 354099N^4 - 91504N^3 - 51504N^2 - 2736N + 4752 \quad (684)$$

$$P_{433} = -531N^8 - 4334N^7 - 11238N^6 - 5704N^5 + 18857N^4 + 31974N^3 + 14368N^2 - 5952N - 4608 \quad (685)$$

$$P_{434} = -521N^8 - 2300N^7 + 350N^6 + 7048N^5 + 4991N^4 - 9164N^3 - 16628N^2$$

$$-480N+3744 \quad (686)$$

$$P_{435}=-436N^8+260N^7+4375N^6+680N^5-14813N^4-11410N^3 +8516N^2+4512N-3024 \quad (687)$$

$$P_{436}=-392N^8+1192N^7+7397N^6+2398N^5-17569N^4-12656N^3+14578N^2 +7680N-4248 \quad (688)$$

$$P_{437}=-263N^8-1100N^7-70N^6+3184N^5+2561N^4-2516N^3-4340N^2 -144N+960 \quad (689)$$

$$P_{438}=-185N^8-803N^7-352N^6+2677N^5+2636N^4+970N^3 +1897N^2-684N-972 \quad (690)$$

$$P_{439}=-137N^8-563N^7-346N^6+814N^5+695N^4-935N^3 -1172N^2+156N+192 \quad (691)$$

$$P_{440}=-103N^8+663N^7-485N^6+1901N^5+460N^4-14164N^3+10688N^2 +7136N-8256 \quad (692)$$

$$P_{441}=-70N^8-913N^7-4936N^6-14466N^5-24422N^4-23673N^3-13852N^2 -4356N+864 \quad (693)$$

$$P_{442}=-67N^8-286N^7-62N^6+716N^5+577N^4-718N^3 -1384N^2-216N+144 \quad (694)$$

$$P_{443}=-49N^8-583N^7-1331N^6-904N^5+757N^4+875N^3 -2329N^2-1512N+540 \quad (695)$$

$$P_{444}=-9N^8+9N^7+81N^6-59N^5-140N^4-238N^3 +428N^2+120N-144 \quad (696)$$

$$P_{445}=-7N^8-67N^7-257N^6-497N^5-426N^4 -268N^3-334N^2-232N-56 \quad (697)$$

$$P_{446}=-N^8-7N^7+9N^6+22N^5+30N^4-76N^3-30N^2+27N-6 \quad (698)$$

$$P_{447}=-N^8+5N^7-9N^6+21N^5-15N^4-23N^3+3N^2+5N-2 \quad (699)$$

$$P_{448}=N^8-3N^7-33N^6-15N^5+157N^4+178N^3-61N^2-28N-4 \quad (700)$$

$$P_{449}=N^8+9N^7+71N^6+387N^5+816N^4+404N^3+184N^2+804N+396 \quad (701)$$

$$P_{450}=2N^8-13N^7+58N^6-126N^5+34N^4+75N^3+46N^2-28N-16 \quad (702)$$

$$P_{451}=2N^8+15N^7+44N^6+56N^5+56N^4+105N^3+118N^2+44N+8 \quad (703)$$

$$P_{452}=3N^8+29N^7+102N^6+154N^5+69N^4 -161N^3-110N^2+994N+1224 \quad (704)$$

$$P_{453}=4N^8-13N^7-46N^6+96N^5+158N^4+N^3-452N^2-84N+144 \quad (705)$$

$$P_{454}=4N^8-3N^7-96N^6+4N^5+454N^4-221N^3-738N^2+468N+576 \quad (706)$$

$$P_{455}=4N^8+4N^7-8N^6+8N^5-93N^4-198N^3+43N^2+84N+28 \quad (707)$$

$$P_{456}=4N^8+10N^7+11N^6-19N^5-65N^4-63N^3-90N^2-4N+24 \quad (708)$$

$$P_{457} = 4N^8 + 24N^7 - 57N^6 + 109N^5 - 32N^4 - 141N^3 - 8N^2 + 75N + 30 \quad (709)$$

$$P_{458} = 4N^8 + 27N^7 + 90N^6 + 208N^5 + 300N^4 + 257N^3 + 190N^2 + 140N + 40 \quad (710)$$

$$P_{459} = 5N^8 + 2N^7 - 8N^6 + 3N^5 - 7N^4 + 14N^3 - 24N^2 - 11N + 10 \quad (711)$$

$$P_{460} = 7N^8 - 8N^7 + 6N^6 - 45N^5 + 27N^4 + 68N^3 - 30N^2 - 23N + 14 \quad (712)$$

$$P_{461} = 7N^8 + 13N^7 + 19N^6 + 27N^5 - 136N^4 - 118N^3 + 388N^2 + 392N + 112 \quad (713)$$

$$P_{462} = 8N^8 + 37N^7 + 127N^6 + 127N^5 - 175N^4 -104N^3 + 408N^2 + 304N + 80 \quad (714)$$

$$P_{463} = 8N^8 + 604N^7 + 1583N^6 - 656N^5 - 1330N^4 +2716N^3 - 837N^2 - 468N + 108 \quad (715)$$

$$P_{464} = 10N^8 + 4N^7 + 90N^6 - 116N^5 + 387N^4 + 187N^3 - 145N^2 - 39N + 54 \quad (716)$$

$$P_{465} = 12N^8 - 34N^7 - 287N^6 - 235N^5 + 347N^4 +653N^3 - 180N^2 - 132N + 96 \quad (717)$$

$$P_{466} = 12N^8 + 69N^7 + 272N^6 + 1450N^5 + 3076N^4 + 297N^3 - 784N^2 +3792N + 1800 \quad (718)$$

$$P_{467} = 14N^8 + 17N^7 - 104N^6 - 68N^5 - 252N^4 -289N^3 + 678N^2 + 268N + 24 \quad (719)$$

$$P_{468} = 14N^8 + 48N^7 + 83N^6 - 174N^5 - 518N^4 +146N^3 + 241N^2 + 176N + 44 \quad (720)$$

$$P_{469} = 14N^8 + 56N^7 + 81N^6 + 62N^5 + 13N^4 - 4N^3 - N^2 + 2N + 1 \quad (721)$$

$$P_{470} = 14N^8 + 56N^7 + 89N^6 + 74N^5 + 29N^4 + 20N^3 + 15N^2 + 6N + 1 \quad (722)$$

$$P_{471} = 17N^8 + 8N^7 - 188N^6 - 170N^5 - 153N^4 +362N^3 + 1380N^2 + 88N - 192 \quad (723)$$

$$P_{472} = 20N^8 - 23N^7 - 92N^5 + 148N^4 - 19N^3 - 4N^2 - 2N + 4 \quad (724)$$

$$P_{473} = 23N^8 + 110N^7 + 373N^6 + 376N^5 - 398N^4 - 16N^3 + 1504N^2 +1088N + 288 \quad (725)$$

$$P_{474} = 25N^8 + 21N^7 - 62N^6 - 495N^5 - 209N^4 + 1768N^3 + 464N^2 -1360N - 944 \quad (726)$$

$$P_{475} = 31N^8 + 528N^7 + 318N^6 - 2460N^5 - 3105N^4 + 3012N^3 + 6860N^2 -864N - 1728 \quad (727)$$

$$P_{476} = 51N^8 + 207N^7 + 64N^6 - 657N^5 - 598N^4 -58N^3 - 177N^2 + 148N + 156 \quad (728)$$

$$P_{477} = 80N^8 + 599N^7 + 1764N^6 + 2168N^5 + 2034N^4 + 4193N^3 +5014N^2 + 1932N + 360 \quad (729)$$

$$P_{478} = 85N^8 - 413N^7 - 1792N^6 + 22N^5 + 5300N^4 + 3589N^3 - 3791N^2 - 1920N +1188 \quad (730)$$

$$P_{479} = 116N^8 + 688N^7 + 1772N^6 + 2705N^5 + 2454N^4 + 1347N^3 + 784N^2 + 396N + 176 \quad (731)$$

$$P_{480} = 116N^8 + 738N^7 + 1936N^6 + 2615N^5 + 1502N^4 - 137N^3 - 192N^2 + 60N + 48 \quad (732)$$

$$P_{481} = 125N^8 + 1314N^7 + 5027N^6 + 9183N^5 + 9122N^4 + 4371N^3 + 846N^2 + 1620N + 648 \quad (733)$$

$$P_{482} = 136N^8 + 900N^7 + 1417N^6 - 1308N^5 - 4442N^4 - 3972N^3 - 3963N^2 - 2268N - 324 \quad (734)$$

$$P_{483} = 253N^8 + 1180N^7 - 518N^6 - 4088N^5 - 2571N^4 + 6716N^3 + 12532N^2 + 512N - 2784 \quad (735)$$

$$P_{484} = 269N^8 + 1889N^7 + 3838N^6 + 320N^5 - 6005N^4 - 4429N^3 + 170N^2 + 420N + 72 \quad (736)$$

$$P_{485} = 293N^8 + 1989N^7 + 6150N^6 + 11150N^5 + 8593N^4 + 1561N^3 + 12564N^2 + 20628N + 10800 \quad (737)$$

$$P_{486} = 541N^8 + 2100N^7 + 3278N^6 + 2212N^5 + 1357N^4 + 880N^3 + 528N^2 + 320N + 112 \quad (738)$$

$$P_{487} = 1214N^8 + 4856N^7 + 7284N^6 + 4937N^5 - 730N^4 + 1782N^3 + 2268N^2 + 1377N + 324 \quad (739)$$

$$P_{488} = 2599N^8 + 18886N^7 + 49546N^6 + 53608N^5 + 15163N^4 - 9566N^3 - 2256N^2 + 3528N + 432 \quad (740)$$

$$P_{489} = 2599N^8 + 18976N^7 + 44560N^6 + 25186N^5 - 41987N^4 - 65474N^3 - 33432N^2 - 8136N - 3024 \quad (741)$$

$$P_{490} = 3081N^8 + 21046N^7 + 53564N^6 + 46888N^5 - 26329N^4 - 55730N^3 - 29056N^2 - 41472N - 12384 \quad (742)$$

$$P_{491} = 3412N^8 + 13756N^7 + 15943N^6 + 1681N^5 - 3785N^4 - 5141N^3 - 19998N^2 - 8676N - 2376 \quad (743)$$

$$P_{492} = 23595N^8 + 94860N^7 + 146258N^6 + 100860N^5 + 22875N^4 + 4400N^3 + 2928N^2 - 144N - 432 \quad (744)$$

$$P_{493} = 23595N^8 + 102732N^7 + 177746N^6 + 152124N^5 + 54363N^4 + 5360N^3 - 1680N^2 - 1872N - 432 \quad (745)$$

$$P_{494} = -235N^9 - 324N^8 + 503N^7 - 1189N^6 + 4599N^5 + 6718N^4 - 8011N^3 - 1233N^2 + 2952N - 324 \quad (746)$$

$$P_{495} = -195N^9 - 1472N^8 - 3139N^7 + 1553N^6 + 11481N^5 + 9208N^4 + 4501N^3 + 5003N^2 - 2856N - 3348 \quad (747)$$

$$P_{496} = -28N^9 - 154N^8 - 353N^7 - 719N^6 - 1138N^5 - 865N^4 - 720N^3 - 611N^2$$

$$-300N - 60 \quad (748)$$

$$P_{497} = -16N^9 - 72N^8 - 29N^7 + 206N^6 + 835N^5 + 959N^4 + 985N^3 + 595N^2 + 203N + 30 \quad (749)$$

$$P_{498} = -5N^9 - 197N^8 - 1658N^7 - 6602N^6 - 9017N^5 + 15379N^4 + 57408N^3 + 56292N^2 + 12240N - 10368 \quad (750)$$

$$P_{499} = -4N^9 - 28N^8 - 111N^7 - 194N^6 - 117N^5 + 44N^4 + 290N^3 + 580N^2 + 424N + 112 \quad (751)$$

$$P_{500} = -3N^9 + 10N^8 - 8N^7 + 30N^6 - 36N^5 - 6N^4 + 62N^3 - 24N^2 - 23N + 14 \quad (752)$$

$$P_{501} = -N^9 - 6N^8 - 20N^7 - 38N^6 - 65N^5 - 60N^4 - 14N^3 + 20N^2 + 104N + 48 \quad (753)$$

$$P_{502} = -N^9 + 13N^8 + 30N^7 - 86N^6 - 277N^5 - 167N^4 + 112N^3 + 152N^2 + 80N + 16 \quad (754)$$

$$P_{503} = N^9 + 19N^8 + 108N^7 + 188N^6 - 213N^5 - 1111N^4 - 1512N^3 - 1192N^2 - 672N - 144 \quad (755)$$

$$P_{504} = 3N^9 + 44N^8 + 237N^7 + 342N^6 - 1093N^5 - 2454N^4 + 1513N^3 + 1948N^2 - 2556N - 288 \quad (756)$$

$$P_{505} = 4N^9 + 28N^8 + 176N^7 + 756N^6 + 1463N^5 + 1172N^4 + 403N^3 + 258N^2 + 116N + 24 \quad (757)$$

$$P_{506} = 8N^9 + 65N^8 + 302N^7 + 176N^6 - 3100N^5 - 4735N^4 + 4950N^3 + 6510N^2 - 648N + 1080 \quad (758)$$

$$P_{507} = 9N^9 + 17N^8 + 35N^7 - 42N^6 + 108N^5 - 22N^4 - 20N^3 + 31N^2 - 24N - 28 \quad (759)$$

$$P_{508} = 29N^9 + 91N^8 - 1840N^7 - 14648N^6 - 44203N^5 - 64085N^4 - 46458N^3 - 21846N^2 - 8568N + 1080 \quad (760)$$

$$P_{509} = 37N^9 + 287N^8 + 1021N^7 + 1977N^6 + 1334N^5 - 1200N^4 - 992N^3 + 1616N^2 + 1376N + 384 \quad (761)$$

$$P_{510} = 75N^9 + 170N^8 - 1700N^7 - 5464N^6 + 739N^5 + 15122N^4 + 16366N^3 + 4500 N^2 - 4752N + 2592 \quad (762)$$

$$P_{511} = 76N^9 + 1002N^8 + 5101N^7 + 14103N^6 + 25360N^5 + 29769N^4 + 17235N^3 + 4158N^2 + 5508N + 1944 \quad (763)$$

$$P_{512} = 93N^9 + 152N^8 - 173N^7 + 367N^6 - 1609N^5 - 2430N^4 + 2577N^3 + 843N^2 - 1080N + 108 \quad (764)$$

$$P_{513} = 142N^9 - 354N^8 - 1086N^7 + 3057N^6 + 3105N^5 - 8769N^4 - 6553N^3 + 4014N^2 + 3636N + 216 \quad (765)$$

$$P_{514} = 158N^9 + 1721N^8 + 6837N^7 + 10006N^6 - 3098N^5 - 14157N^4 + 4347N^3 + 3798N^2 - 5076N + 648 \tag{766}$$

$$P_{515} = 173N^9 + 741N^8 + 423N^7 + 303N^6 - 1644N^5 - 6816N^4 + 5098N^3 + 3450N^2 - 3672N - 2376 \tag{767}$$

$$P_{516} = 203N^9 + 756N^8 + 15N^7 - 5034N^6 - 1836N^5 + 8223N^4 - 5618N^3 + 807N^2 + 3024N + 1188 \tag{768}$$

$$P_{517} = 294N^9 + 1836N^8 + 3347N^7 - 890N^6 - 9772N^5 - 10517N^4 + 2975N^3 + 12595N^2 + 4344N + 972 \tag{769}$$

$$P_{518} = 673N^9 + 4111N^8 + 7483N^7 - 1707N^6 - 18990N^5 - 16638N^4 + 3722N^3 + 10442N^2 - 16N + 552 \tag{770}$$

$$P_{519} = -102241N^{10} - 503813N^9 - 586422N^8 + 611142N^7 + 1447647N^6 + 282675N^5 - 722888N^4 - 354868N^3 + 29568N^2 - 10944N - 6912 \tag{771}$$

$$P_{520} = -2895N^{10} - 14571N^9 - 28822N^8 - 30262N^7 - 15467N^6 - 8303N^5 - 7456N^4 - 5600N^3 - 3296N^2 - 1760N - 480 \tag{772}$$

$$P_{521} = -2895N^{10} - 14507N^9 - 17370N^8 + 17754N^7 + 43745N^6 + 7693N^5 - 23608N^4 - 13244N^3 - 4736N^2 + 512N + 512 \tag{773}$$

$$P_{522} = -1300N^{10} - 6471N^9 - 5898N^8 + 15033N^7 + 23049N^6 - 5703N^5 - 8642N^4 + 7941N^3 - 3429N^2 - 3888N - 324 \tag{774}$$

$$P_{523} = -1037N^{10} - 5395N^9 - 9095N^8 - 14470N^7 + 5890N^6 + 124193N^5 + 92793N^4 - 103428N^3 - 55323N^2 + 35964N - 972 \tag{775}$$

$$P_{524} = -695N^{10} - 4934N^9 - 10384N^8 - 1718N^7 + 15950N^6 + 17014N^5 + 5092N^4 - 10254N^3 - 16335N^2 - 8316N + 1620 \tag{776}$$

$$P_{525} = -559N^{10} - 5091N^9 - 14922N^8 - 5760N^7 + 49539N^6 + 92367N^5 + 63754N^4 + 51540N^3 + 84276N^2 + 43416N - 12960 \tag{777}$$

$$P_{526} = -335N^{10} - 2167N^9 - 2175N^8 + 2226N^7 - 4038N^6 + 8349N^5 + 35642N^4 - 8030N^3 - 12408N^2 + 4536N + 4320 \tag{778}$$

$$P_{527} = -55N^{10} - 275N^9 - 392N^8 - 62N^7 + 289N^6 + 481N^5 - 1006N^4 - 2376N^3 + 516N^2 + 1008N + 336 \tag{779}$$

$$P_{528} = N^{10} + N^9 - 8N^8 + 30N^7 + 241N^6 + 585N^5 + 232N^4$$

$$-460N^3+130N^2+8N+8 \quad (780)$$

$$P_{529}=3N^{10}-15N^9+38N^8-68N^7+28N^6+50N^5+8N^4-20N^3+5N^2 +5N-2 \quad (781)$$

$$P_{530}=5N^{10}+2N^9-17N^8-66N^7-107N^6+172N^5+431N^4+92N^3 -80N^2-16N+16 \quad (782)$$

$$P_{531}=5N^{10}+6N^9-20N^8-44N^7-10N^6+205N^5+447N^4+161N^3-194N^2 -36N+56 \quad (783)$$

$$P_{532}=6N^{10}+32N^9+101N^8+175N^7+236N^6+379N^5+327N^4 +208N^3+488N^2+400N+112 \quad (784)$$

$$P_{533}=7N^{10}+23N^9-13N^8-105N^7-315N^6-592N^5-193N^4 +100N^3-448N^2-208N-48 \quad (785)$$

$$P_{534}=11N^{10}+99N^9+273N^8+6N^7-1002N^6-723N^5+145N^4-3090N^3 -2307N^2+2592N+540 \quad (786)$$

$$P_{535}=11N^{10}+191N^9+323N^8-1574N^7-1837N^6 +2695N^5+1533N^4-2392N^3 -2262N^2+1368N+216 \quad (787)$$

$$P_{536}=12N^{10}+94N^9+350N^8+676N^7+1073N^6 +1944N^5+2467N^4+1840N^3 +1128N^2+512N+112 \quad (788)$$

$$P_{537}=20N^{10}+331N^9+775N^8+664N^7+950N^6+2083N^5+1587N^4+738N^3 +3436N^2+648N-864 \quad (789)$$

$$P_{538}=40N^{10}+391N^9+1292N^8+1360N^7-775N^6-1709N^5 -1826N^4-12102N^3-22635N^2-8100N+2592 \quad (790)$$

$$P_{539}=41N^{10}+37N^9-76N^8+74N^7-243N^6-427N^5+1358N^4 +1596N^3+200N^2-192N-64 \quad (791)$$

$$P_{540}=47N^{10}-155N^9-479N^8-290N^7 -1249N^6+2227N^5+9897N^4+2226N^3 -6200N^2-1128N+1584 \quad (792)$$

$$P_{541}=56N^{10}+456N^9+1470N^8+2491N^7 +2381N^6+890N^5-529N^4-329N^3 +162N^2+180N+40 \quad (793)$$

$$P_{542}=119N^{10}+664N^9+829N^8-182N^7-1183N^6-512N^5+1647N^4+4518N^3 +5356N^2-24N-864 \quad (794)$$

$$P_{543}=133N^{10}-601N^9-949N^8-970N^7-7451N^6+2705N^5 +28755N^4+8634N^3-24232N^2-5448N+5904 \quad (795)$$

$$P_{544} = 149N^{10} + 838N^9 + 1068N^8 - 5247N^7 - 8718N^6 + 12867N^5 + 4483N^4 -28114N^3 + 10794N^2 + 16848N + 5400 \quad (796)$$

$$P_{545} = 227N^{10} + 97N^9 - 497N^8 - 470N^7 - 3985N^6 + 7015N^5 + 31839N^4 +9462N^3 - 14768N^2 - 3000N + 3888 \quad (797)$$

$$P_{546} = 265N^{10} + 2057N^9 + 8292N^8 + 10767N^7 - 7962N^6 - 6501N^5 + 9680N^4 -23027N^3 - 15N^2 + 11736N + 5076 \quad (798)$$

$$P_{547} = 380N^{10} + 2639N^9 + 5326N^8 + 116N^7 - 9551N^6 - 8041N^5 - 484N^4 +3666N^3 + 5589N^2 + 3492N - 540 \quad (799)$$

$$P_{548} = 541N^{10} + 2801N^9 + 5288N^8 + 4554N^7 + 2293N^6 - 11N^5 - 5338N^4 -5232N^3 - 2800N^2 - 208N + 160 \quad (800)$$

$$P_{549} = 743N^{10} + 7696N^9 + 25621N^8 + 16138N^7 - 72099N^6 - 123940N^5 +17403N^4 + 207482N^3 + 156396N^2 - 31536N - 38016 \quad (801)$$

$$P_{550} = 4148N^{10} + 32491N^9 + 99529N^8 + 149344N^7 + 106516N^6 + 8455N^5 -38095N^4 - 6924N^3 + 20016N^2 + 15552N + 3888 \quad (802)$$

$$P_{551} = 10875N^{10} + 65463N^9 + 139552N^8 + 92374N^7 - 87581N^6 - 165293N^5 -63158N^4 - 1040N^3 - 552N^2 - 13104N - 5184 \quad (803)$$

$$P_{552} = 24535N^{10} + 124997N^9 + 228788N^8 + 184556N^7 + 118649N^6 - 32641N^5 -263792N^4 + 13608N^3 - 188532N^2 - 184248N - 57024 \quad (804)$$

$$P_{553} = -1859N^{11} - 18263N^{10} - 61876N^9 - 64472N^8 + 73286N^7 + 177838N^6 +43720N^5 - 72000N^4 - 46503N^3 - 55251N^2 - 48816N + 1620 \quad (805)$$

$$P_{554} = -377N^{11} - 207N^{10} + 7199N^9 + 20907N^8 + 12912N^7 - 22608N^6 -18769N^5 + 5463N^4 - 23573N^3 - 1287N^2 + 20880N + 9828 \quad (806)$$

$$P_{555} = -130N^{11} - 1031N^{10} - 2729N^9 - 2426N^8 + 2666N^7 + 7057N^6 - 8397N^5 -29316N^4 - 30218N^3 - 23388N^2 + 1080N + 3888 \quad (807)$$

$$P_{556} = -44N^{11} - 226N^{10} - 21N^9 + 1281N^8 + 1182N^7 - 1707N^6 - 2068N^5 +1228N^4 + 1635N^3 - 648N^2 + 252N + 432 \quad (808)$$

$$P_{557} = -12N^{11} - 55N^{10} + 22N^9 + 377N^8 + 238N^7 - 779N^6 - 624N^5 + 429N^4 +472N^3 + 268N^2 - 336N - 288 \quad (809)$$

$$P_{558} = -N^{11} - 84N^{10} + 367N^9 - 180N^8 - 1321N^7 + 2558N^6 - 1007N^5 - 2224N^4 +2306N^3 + 402N^2 - 888N + 280 \quad (810)$$

$$P_{559} = 2N^{11} + 5N^{10} + 31N^9 + 22N^8 - 39N^7 - 70N^6 - 65N^5 + 120N^4 + 15N^3 - 97N^2 + 28 \quad (811)$$

$$P_{560} = 3N^{11} + 30N^{10} - 117N^9 - 326N^8 + 145N^7 - 1822N^6 - 1399N^5 + 3918N^4 +3624N^3 + 3192N^2 - 4176N - 2304 \quad (812)$$

$$P_{561} = 11N^{11} - 470N^{10} - 393N^9 + 2532N^8 - 5715N^7 - 1926N^6 + 29881N^5$$

$$-2980N^4-9096N^3+18828N^2+6912N-1296 \quad (813)$$

$$P_{562}=16N^{11}-188N^{10}+263N^9-1632N^8+3240N^7-5097N^6-3418N^5 +740N^4+1348N^3-1329N^2-369N+378 \quad (814)$$

$$P_{563}=56N^{11}+382N^{10}+982N^9+1045N^8 -326N^7-2177N^6-1575N^5+832N^4 +743N^3-150N^2-444N-136 \quad (815)$$

$$P_{564}=94N^{11}+761N^{10}+2081N^9+1940N^8-1586N^7-5059N^6+1989N^5 +14322N^4+17366N^3+12660N^2-936N-2160 \quad (816)$$

$$P_{565}=473N^{11}+2347N^{10}+2874N^9-2235N^8-5646N^7-1068N^6+634N^5 -835N^4+1593N^3+891N^2+432N+108 \quad (817)$$

$$P_{566}=896N^{11}+7285N^{10}+12349N^9-37301N^8 -120535N^7-3191N^6+258447N^5 +197745N^4-122645N^3-171450N^2+19008N+35424 \quad (818)$$

$$P_{567}=2310N^{11}+16134N^{10}+38381N^9+22356N^8-67974N^7-128568N^6 +6525N^5+103266N^4-22874N^3+17052N^2+21600N+7344 \quad (819)$$

$$P_{568}=3331N^{11}+26270N^{10}+51231N^9-57642N^8-238659N^7-113214N^6 +13301N^5-284390N^4-137220N^3+669168N^2+259632N-399168 \quad (820)$$

$$P_{569}=4148N^{11}+22259N^{10}+16296N^9-96255N^8-182904N^7-14577N^6 +182032N^5+126841N^4+6672N^3-13968N^2+7776N+3888 \quad (821)$$

$$P_{570}=-202N^{12}-1212N^{11}-1953N^{10}+1966N^9+7125N^8+2994N^7-4181N^6 -8280N^5-9213N^4-868N^3+2160N^2+432N-432 \quad (822)$$

$$P_{571}=-100N^{12}-1085N^{11}-4256N^{10}-7025N^9-4126N^8+1735N^7+7416N^6 -11223N^5-71498N^4-100698N^3-49692N^2+9432N+9936 \quad (823)$$

$$P_{572}=-55N^{12}-495N^{11}-1688N^{10}-2586N^9-603N^8+3585N^7+6022N^6 +6528N^5+4364N^4+2496N^3+5856N^2+4800N+1344 \quad (824)$$

$$P_{573}=-14N^{12}-100N^{11}-19N^{10}+26N^9+541N^8-556N^7-62N^6+14N^5 -93N^4+72N^3+163N^2-40N-60 \quad (825)$$

$$P_{574}=7N^{12}+20N^{11}+8N^{10}-109N^9-186N^8+234N^7+496N^6+687N^5 +1355N^4+184N^3-488N^2-16N+112 \quad (826)$$

$$P_{575}=14N^{12}+174N^{11}+1019N^{10}+3349N^9+6946N^8+12660N^7+23003N^6 +26429N^5+6974N^4-14616N^3-14496N^2-6256N-1056 \quad (827)$$

$$P_{576}=25N^{12}+117N^{11}-32N^{10}-172N^9-23N^8-7795N^7-15834N^6+9242N^5 +38328N^4+23664N^3-2304N^2-12960N-4608 \quad (828)$$

$$P_{577}=43N^{12}-100N^{11}-1924N^{10}-1271N^9+11862N^8+9003N^7-21196N^6 -19013N^5+14107N^4+24017N^3-14988N^2-4428N+6480 \quad (829)$$

$$P_{578} = 75N^{12} + 652N^{11} - 443N^{10} - 5777N^9 + 5586N^8 + 22584N^7 - 11631N^6 \\ -45601N^5 + 18305N^4 + 47798N^3 - 42132N^2 - 10152N + 5184 \quad (830)$$

$$P_{579} = 96N^{12} + 409N^{11} - 233N^{10} - 5951N^9 - 10686N^8 + 21813N^7 + 42633N^6 \\ -33655N^5 - 60658N^4 - 6808N^3 - 20832N^2 - 15984N - 6048 \quad (831)$$

$$P_{580} = 559N^{12} + 1825N^{11} + 14894N^{10} + 7082N^9 - 131685N^8 - 209955N^7 \\ +210920N^6 + 820088N^5 + 405808N^4 - 635312N^3 - 336768N^2 + 129024N \\ +96768 \quad (832)$$

$$P_{581} = 848N^{12} + 5271N^{11} + 9132N^{10} - 2123N^9 - 19026N^8 - 3075N^7 + 30280N^6 \\ +23259N^5 - 4602N^4 + 2684N^3 + 20808N^2 + 2208N - 3456 \quad (833)$$

$$P_{582} = 1136N^{12} + 18501N^{11} + 135921N^{10} + 571962N^9 + 1481748N^8 + 2454501N^7 \\ +2689333N^6 + 1935348N^5 + 846054N^4 + 386856N^3 \\ +437616N^2 + 246240N + 54432 \quad (834)$$

$$P_{583} = 1706N^{12} + 10290N^{11} + 16191N^{10} - 13469N^9 - 48873N^8 - 7212N^7 \\ +63553N^6 + 59895N^5 + 5655N^4 - 20920N^3 - 4320N^2 + 4464N + 3024 \quad (835)$$

$$P_{584} = 7159N^{12} + 43302N^{11} + 82301N^{10} + 16020N^9 - 116257N^8 - 98698N^7 \\ +17295N^6 - 25720N^5 - 78706N^4 + 38456N^3 \\ +35936N^2 + 29664N + 7776 \quad (836)$$

$$P_{585} = -777N^{13} - 6846N^{12} - 19922N^{11} - 7090N^{10} + 67924N^9 + 103426N^8 \\ -5038N^7 - 136214N^6 - 191123N^5 - 110260N^4 + 78808N^3 + 98520N^2 \\ -5328N - 21600 \quad (837)$$

$$P_{586} = -535N^{13} - 3614N^{12} - 6137N^{11} + 22547N^{10} + 57627N^9 - 16644N^8 \\ +26215N^7 + 114761N^6 - 162448N^5 - 7034N^4 + 162498N^3 + 57492N^2 \\ -59400N - 29808 \quad (838)$$

$$P_{587} = -392N^{13} - 2048N^{12} - 7248N^{11} - 1420N^{10} + 36983N^9 + 54705N^8 \\ +5876N^7 - 82924N^6 - 90399N^5 + 138127N^4 + 192772N^3 - 36672N^2 \\ -47520N + 15120 \quad (839)$$

$$P_{588} = -172N^{13} - 1091N^{12} - 489N^{11} + 7770N^{10} + 7740N^9 - 11943N^8 \\ -24125N^7 - 56548N^6 + 64494N^5 + 168732N^4 - 31032N^3 - 97416N^2 \\ +2592N + 23328 \quad (840)$$

$$P_{589} = 294N^{13} + 2718N^{12} + 8114N^{11} + 3145N^{10} - 27706N^9 \\ -43252N^8 + 4735N^7 + 21551N^6 - 46438N^5 - 49016N^4 + 7181N^3 \\ +1998N^2 - 5796N - 1944 \quad (841)$$

$$P_{590} = 709N^{13} + 943N^{12} + 411N^{11} + 6029N^{10} - 5923N^9 \\ -80745N^8 - 93874N^7 + 173546N^6 + 312405N^5 - 104825N^4 - 238028N^3 \\ +75360N^2 + 59616N - 29808 \quad (842)$$

$$P_{591} = 1216N^{13} + 7013N^{12} + 7697N^{11} - 23330N^{10} - 88314N^9 -43341N^8 + 234716N^7 + 100126N^6 - 272138N^5 + 145814N^4 + 75675N^3 -20394N^2 - 47628N - 14904 \quad (843)$$

$$P_{592} = 3001N^{13} + 26082N^{12} + 77868N^{11} + 34682N^{10} - 258930N^9 -419730N^8 - 53068N^7 + 658734N^6 +1592601N^5 + 1335568N^4 - 374352N^3 -687528N^2 + 47952N + 163296 \quad (844)$$

$$P_{593} = -306723N^{14} - 2340885N^{13} - 6607171N^{12} - 6904449N^{11} +3987055N^{10} + 16593333N^9 + 13467975N^8 - 1608819N^7 - 8946032N^6 -4282236N^5 - 815200N^4 + 587280N^3 -209664N^2 - 478656N - 145152 \quad (845)$$

$$P_{594} = -171287N^{14} - 1211897N^{13} - 3068659N^{12} - 2667841N^{11} +1694631N^{10} + 4206249N^9 + 2089679N^8 + 2659013N^7 + 972772N^6 -3123356N^5 + 3261648N^4 + 415176N^3 - 2010528N^2 -1822176N - 476928 \quad (846)$$

$$P_{595} = -2895N^{14} - 20361N^{13} - 51007N^{12} - 42853N^{11} + 26203N^{10} +52297N^9 - 11709N^8 + 769N^7 +82896N^6 + 25092N^5 - 26432N^4 - 57760N^3 -25056N^2 + 1664 \quad (847)$$

$$P_{596} = -17N^{14} - 79N^{13} - 61N^{12} + 615N^{11} + 1747N^{10} - 2965N^9 -4979N^8 + 18209N^7 - 11626N^6 - 43220N^5 + 40632N^4 +29552N^3 - 18784N^2 - 5568N + 8064 \quad (848)$$

$$P_{597} = -16N^{14} - 248N^{13} - 1605N^{12} - 5865N^{11} - 13383N^{10} -21272N^9 - 34866N^8 - 72922N^7 - 124252N^6 - 137177N^5 - 102690N^4 -60360N^3 - 28240N^2 - 9168N - 1440 \quad (849)$$

$$P_{598} = -15N^{14} - 34N^{13} + 63N^{12} + 162N^{11} - 281N^{10} - 912N^9 + 187N^8 + 4578N^7 +8700N^6 + 3998N^5 - 3598N^4 - 864N^3 + 2160N^2 + 160N -480 \quad (850)$$

$$P_{599} = 65N^{14} - 1504N^{13} - 3135N^{12} + 19307N^{11} + 3465N^{10} - 98136N^9 +109420N^8 + 97027N^7 - 330420N^6 + 135406N^5 + 262941N^4 -271764N^3 - 39312N^2 + 86832N - 11664 \quad (851)$$

$$P_{600} = 1711N^{14} + 11977N^{13} + 29375N^{12} + 19397N^{11} - 36003N^{10} - 66801N^9 -29659N^8 + 56591N^7 + 168496N^6 + 73084N^5 - 124560N^4 - 50472N^3 +18576N^2 + 30240N + 8640 \quad (852)$$

$$P_{601} = 3081N^{14} + 36451N^{13} + 160822N^{12}$$

$$+265142N^{11}-191472N^{10}-1244948N^9$$
$$-1268974N^8+458058N^7+1945999N^6+1996529N^5+616232N^4$$
$$-1244976N^3-1177272N^2+47088N+220320 \quad (853)$$
$$P_{602}=-368N^{15}-496N^{14}-2334N^{13}-3335N^{12}+18115N^{11}+7032N^{10}-6186N^9$$
$$+190861N^8+72423N^7-543466N^6-338338N^5+327996N^4+185840N^3$$
$$-168672N^2-35424N+39744 \quad (854)$$
$$P_{603}=88N^{15}+857N^{14}+6099N^{13}+5941N^{12}-21569N^{11}-38661N^{10}-8499N^9$$
$$+7159N^8-60363N^7-9808N^6+57620N^5-280368N^4-138400N^3$$
$$+103200N^2+27648N-24192 \quad (855)$$
$$P_{604}=203N^{15}+2177N^{14}+8464N^{13}+11458N^{12}$$
$$-4055N^{11}-17152N^{10}-19631N^9$$
$$-30020N^8+33188N^7+227867N^6+421391N^5+272806N^4-95256N^3$$
$$-95616N^2+12528N+18144 \quad (856)$$
$$P_{605}=-1300N^{16}-11671N^{15}-23280N^{14}+72661N^{13}+294991N^{12}+6582N^{11}$$
$$-966464N^{10}-557054N^9+1291890N^8$$
$$+672533N^7-1214272N^6-94815N^5$$
$$+1410291N^4+287604N^3-798336N^2-241056N+58320 \quad (857)$$
$$P_{606}=8N^{16}+29N^{15}+26N^{14}-162N^{13}-168N^{12}+1130N^{11}+695N^{10}-2837N^9$$
$$-2364N^8-8153N^7-16755N^6-2799N^5+5774N^4-72N^3$$
$$-2560N^2+80N+480 \quad (858)$$
$$P_{607}=149N^{16}+2030N^{15}+10972N^{14}+27481N^{13}+26215N^{12}-21433N^{11}$$
$$-116177N^{10}-148583N^9+245582N^8+934607N^7+1604033N^6$$
$$+2493562N^5+1732746N^4-451152N^3$$
$$-502848N^2+49248N+85536 \quad (859)$$
$$P_{608}=1214N^{16}+9712N^{15}+24280N^{14}+1053N^{13}-88702N^{12}-97444N^{11}$$
$$+87242N^{10}+150014N^9-43498N^8-31956N^7+127858N^6+63877N^5$$
$$-7306N^4-1944N^3+16848N^2+1296N-2592 \quad (860)$$
$$P_{609}=946N^{17}+7532N^{16}+9325N^{15}-65608N^{14}-173636N^{13}+141572N^{12}$$
$$+759206N^{11}+131590N^{10}-1359784N^9-740252N^8+1131425N^7$$
$$+947158N^6-420762N^5-502920N^4+270144N^3+205344N^2-82080N$$
$$-72576 \quad (861)$$
$$P_{610}=1711N^{17}+22243N^{16}+115213N^{15}+273811N^{14}+126835N^{13}-922871N^{12}$$
$$-2451337N^{11}-2618887N^{10}-953086N^9$$
$$+204800N^8+50168N^7+280136N^6$$
$$+502080N^5+685152N^4+1756800N^3+1913472N^2$$

$$
\begin{aligned}
&+1009152N+207360 &(862)\\
P_{611} =\ &3331N^{17}+21215N^{16}+31660N^{15}-96196N^{14}-487514N^{13}-298606N^{12}\\
&+1898000N^{11}+1099708N^{10}-4381585N^{9}+2587711N^{8}+7099436N^{7}\\
&-10245920N^{6}-3591360N^{5}+14626872N^{4}+6996672N^{3}-2811456N^{2}\\
&-1472256N+217728 &(863)\\
P_{612} =\ &-1536N^{18}-11971N^{17}-25135N^{16}+108682N^{15}+571220N^{14}\\
&-186284N^{13}-3832060N^{12}-1817070N^{11}+10172986N^{10}+8764643N^{9}\\
&-14073029N^{8}-15329356N^{7}+17717986N^{6}+8414044N^{5}-22518432N^{4}\\
&+2633760N^{3}+7011360N^{2}+1031616N-870912 &(864)\\
P_{613} =\ &-377N^{18}-961N^{17}+11118N^{16}+40455N^{15}-69189N^{14}-398487N^{13}\\
&+70186N^{12}+1667114N^{11}+481053N^{10}-3152601N^{9}-1445880N^{8}\\
&+2408547N^{7}+2408989N^{6}-195499N^{5}-3562764N^{4}+127800N^{3}\\
&+2246832N^{2}+94608N-544320 &(865)\\
P_{614} =\ &2801N^{18}+24554N^{17}-2860N^{16}-142290N^{15}+30162N^{14}+286452N^{13}\\
&-1009366N^{12}-2154796N^{11}+2538563N^{10}+10301106N^{9}+1319154N^{8}\\
&-20831058N^{7}-3061678N^{6}+29370560N^{5}+4864952N^{4}-12022176N^{3}\\
&+1577952N^{2}+2702592N-777600 &(866)\\
P_{615} =\ &7159N^{18}+64779N^{17}+161014N^{16}\\
&-186916N^{15}-1372538N^{14}-1131058N^{13}\\
&+3030520N^{12}+4986920N^{11}-1784857N^{10}-6856273N^{9}-1718422N^{8}\\
&+2267620N^{7}-108620N^{6}-1548464N^{5}\\
&-833712N^{4}-281920N^{3}-190272N^{2}\\
&+165888N+103680 &(867)\\
P_{616} =\ &24535N^{18}+223137N^{17}+571468N^{16}\\
&-535672N^{15}-4455392N^{14}-3953404N^{13}\\
&+8875142N^{12}+15762002N^{11}-5430715N^{10}-26302661N^{9}-10457122N^{8}\\
&+15101398N^{7}+12223476N^{6}-11961968N^{5}-20488992N^{4}-1064736N^{3}\\
&+5389632N^{2}+476928N-870912 &(868)\\
P_{617} =\ &2310N^{19}+27684N^{18}+110213N^{17}+62581N^{16}-730907N^{15}-1801977N^{14}\\
&+261074N^{13}+5643644N^{12}+4379456N^{11}-4397272N^{10}-2762167N^{9}\\
&+2836323N^{8}-8997563N^{7}-14308283N^{6}+2977232N^{5}+8931012N^{4}\\
&-1797408N^{3}-4069872N^{2}+461376N+855360 &(869)\\
P_{618} =\ &-64N^{20}-5741N^{19}-9955N^{18}-64422N^{17}-2708N^{16}+645006N^{15}\\
&+483098N^{14}-1197548N^{13}-908800N^{12}+4112583N^{11}+2097697N^{10}\\
&-20613750N^{9}-4094356N^{8}+29920768N^{7}-21624688N^{6}-27765600N^{5}
\end{aligned}
$$

$$
\begin{aligned}
&+9469696N^4+9366528N^3-6156288N^2-1769472N+1244160 \quad (870)\\
P_{619} &= 1216N^{20}+19173N^{19}+121764N^{18}+379926N^{17}+472571N^{16}-429454N^{15}\\
&-1823753N^{14}-1180304N^{13}+576898N^{12}+244797N^{11}+2069530N^{10}\\
&+13099954N^9+35093959N^8+47476772N^7+20154875N^6-11470512N^5\\
&-5237892N^4+5866560N^3+3692304N^2-933120N-699840 \quad (871)\\
P_{620} &= -171287N^{24}-2068332N^{23}-7720102N^{22}+2768090N^{21}+87092589N^{20}\\
&+154633846N^{19}-244069214N^{18}-978466140N^{17}-208517821N^{16}\\
&+2389374224N^{15}+2200366326N^{14}-2505722606N^{13}-4151353097N^{12}\\
&+478650438N^{11}+2967438446N^{10}+669674480N^9-687107760N^8\\
&-1428144224N^7-1830398016N^6+16556928N^5+761041152N^4\\
&+29251584N^3-318380544N^2-19408896N+44789760. \quad (872)
\end{aligned}
$$

Appendix C. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.nuclphysb.2022.115794.

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