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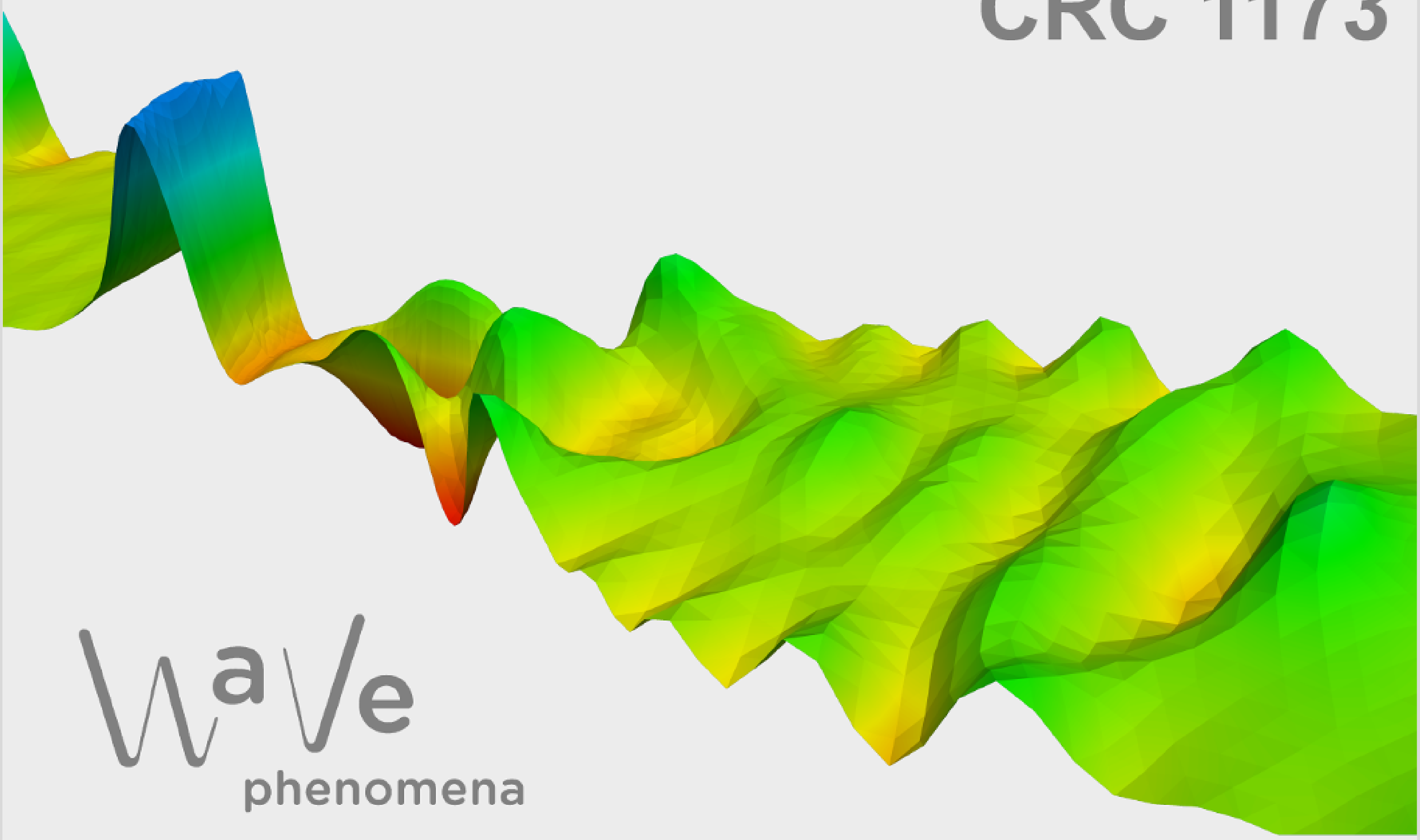

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NUMERICAL ANALYSIS FOR ELECTROMAGNETIC SCATTERING FROM NONLINEAR BOUNDARY CONDITIONS

JÖRG NICK

ABSTRACT. This work studies time-dependent electromagnetic scattering from obstacles whose interaction with the wave is fully determined by a nonlinear boundary condition. In particular, the boundary condition studied in this work enforces a power law type relation between the electric and magnetic field along the boundary. Based on time-dependent jump conditions of classical boundary operators, we derive a nonlinear system of time-dependent boundary integral equations that determines the tangential traces of the scattered electric and magnetic fields. These fields can subsequently be computed at arbitrary points in the exterior domain by evaluating a time-dependent representation formula.

Fully discrete schemes are obtained by discretising the nonlinear system of boundary integral equations with Runge–Kutta based convolution quadrature in time and Raviart–Thomas boundary elements in space. Error bounds with explicitly stated convergence rates are proven, under the assumption of sufficient regularity of the exact solution. The error analysis is conducted through novel techniques based on time-discrete transmission problems and the use of a new discrete partial integration inequality. Numerical experiments illustrate the use of the proposed method and provide empirical convergence rates.

1. INTRODUCTION

This work proposes and studies numerical schemes, which discretize Maxwell's equations in the context of wave scattering, where the interaction of the wave with the obstacle is governed by a nonlinear boundary condition.

Asymptotic analysis of small scale effects on the boundary of obstacles, typically arising from thin coatings around the scatterer, yield a large variety of boundary conditions of practical interest. Starting from [18], such asymptotic models have been studied extensively in the time-harmonic setting, for example in [21], [22] or [16].

When the material properties of the coating inhibit nonlinear phenomena, the derived boundary conditions may be nonlinear as well, as demonstrated in [19] and [20], which derive asymptotic models for thin ferromagnetic coatings. The presence of nonlinear phenomena naturally prohibits the use of time-harmonic techniques, which significantly complicates both the analysis and the numerical treatment of such problems. Consequently, the existing literature on nonlinear scattering problems is scarce and the numerical treatment of nonlinear scattering is rarely considered. The acoustic wave equation with nonlinear boundary conditions in the context of scattering has been analyzed in [11] and [6].

1.1. **Problem setting.** Let Ω denote an exterior Lipschitz domain, which is assumed to be the complement of one or several bounded domains. The total electric field $\boldsymbol{E}^{\text{tot}}(x,t)$ and the total magnetic field $\boldsymbol{H}^{\text{tot}}(x,t)$ are said to be solutions of

Maxwell's equations if

$$\begin{aligned} \varepsilon\, \partial_t \boldsymbol{E}^{\text{tot}} - \textbf{curl}\, \boldsymbol{H}^{\text{tot}} &= 0 \\ \mu\, \partial_t \boldsymbol{H}^{\text{tot}} + \textbf{curl}\, \boldsymbol{E}^{\text{tot}} &= 0 \end{aligned} \qquad \text{in the exterior domain } \Omega. \tag{1}$$

The permittivity ε and the permeability μ in Ω are known positive constants, which describe the material properties of the free space around the scatterer.

The total fields are initially, at time $t = 0$, assumed to have support away from the boundary Γ. The initial values are further assumed to be evaluations of incident fields $(\boldsymbol{E}^{\text{inc}}, \boldsymbol{H}^{\text{inc}})$, solutions to Maxwell's equations on the full space $\mathbb{R}^3$. This setting then allows for a formal decomposition of the total fields into unknown scattered fields, which initially vanish, and the known incidental fields.

Throughout the paper, the wave speed c is assumed to be set to one, which is always achieved by rescaling the time variable $t \to ct$. Consequently, the product of the permittivity ε and permeability μ is assumed to be normalized, since

$$\varepsilon\mu = c^{-2} = 1.$$

To completely avoid the occurence of the physical constants in the analysis, we further employ the rescaling $\mu\boldsymbol{H} \to \boldsymbol{H}$. This rescaled field $\mu\boldsymbol{H}$ is, in the context of physics literature, also referred to as the magnetic field $\boldsymbol{B}$. Applying this assumption and rescaling yields time-dependent Maxwell's equations without the physical constants ε and μ, which read

$$\begin{aligned} \partial_t \boldsymbol{E}^{\text{tot}} - \textbf{curl}\, \boldsymbol{H}^{\text{tot}} &= 0 \\ \partial_t \boldsymbol{H}^{\text{tot}} + \textbf{curl}\, \boldsymbol{E}^{\text{tot}} &= 0 \end{aligned} \qquad \text{in the exterior domain } \Omega. \tag{2}$$

The *nonlinear boundary condition* studied here enforces a nonlinear relation between the traces of the electromagnetic fields and reads

$$\boldsymbol{E}^{\text{tot}} \times \boldsymbol{\nu} + \boldsymbol{a}(\boldsymbol{H}^{\text{tot}} \times \boldsymbol{\nu}) \times \boldsymbol{\nu} = 0 \quad \text{on } \Gamma = \partial\Omega, \tag{3}$$

where $\boldsymbol{\nu}$ denotes the outer unit normal vector. Note that the rescaling of $\boldsymbol{H}$ with regards to the physical constant μ is, in this formulation of the boundary condition, assumed to be incorporated into the nonlinearity $\boldsymbol{a}$.

Despite the brevity of this formulation, serious challenges arise both in the numerical treatment and analysis of the described problem, due to the nonlinearity of the boundary condition. Throughout the paper, the nonlinearity is restricted to be a power law of the following type

$$\boldsymbol{a}(\boldsymbol{x}) = |\boldsymbol{x}|^{\alpha-1}\, \boldsymbol{x} \quad \text{for all } \boldsymbol{x} \in \mathbb{R}^3, \tag{4}$$

for some fixed $\alpha \in (0, 1]$. The limitation on this type of boundary condition has been motivated by [32] and [33], which present numerical analysis for this class of boundary conditions on bounded domains in the context of various electromagnetic phenomena. In altered form, this class of nonlinearities further appears in the evolution boundary condition studied in [36]. Well-posedness results and analysis for Maxwell's equations with such boundary conditions can be found in [17]. As $\boldsymbol{\nu}$ denotes the unit outward surface normal and the region of interest Ω is the exterior domain, the signs of $\boldsymbol{\nu}$ appearing in the boundary condition (3) differs in comparison to analysis conducted in the inner domain, as demonstrated for example by [36]. The boundary condition here differs from the mentioned literature by the positive factor μ, as a consequence of the rescaling of $\boldsymbol{H}$.

1.2. **Contributions of this paper.** The present paper gives, to the best of the author's knowledge, the first numerical analysis of time-dependent electromagnetic scattering with a nonlinear boundary condition. The derivation of the nonlinear time-dependent boundary integral equation results from the combination of techniques presented to treat the acoustic case in [11],[6] and [8], with the electromagnetic Calderón operator proposed in [24]. As such, the present boundary integral equations can be understood as a generalization of the case of linear boundary conditions for electromagnetic scattering, which was presented in [30].

While the formulation of the boundary integral equation is the consequence of earlier techniques applied to the present problem, the stability and error analysis conducted in this paper builds on novel techniques and yields new results. A particular challenge is the power-law form of the nonlinear boundary condition, which does not fulfill a monotonicity condition as strong as assumed to conduct numerical analysis in the acoustic case [6]. The present error analysis includes the following new ideas.

- Energy techniques based on time-discrete transmission problems, where errors and defects are rewritten through Green's formula in terms of discrete fields away from the boundary, were derived and utilized.
- The stability analysis was conducted in the presence of a weak monotonicity condition fulfilled by the nonlinearity (as provided by Lemma 1). This difficulty is circumvented by an a priori estimate on the numerical solution and a series of Hölder inequalities.
- A new discrete partial integration inequality for Runge-Kutta convolution quadrature discretizations based on Radau IIA multistage methods is shown and utilized.
- Time-harmonic bounds with superior dependence on the Laplace parameter s of the potential operators for the time-harmonic Maxwell's equations, in the context of the functional analytic setting of the nonlinearity, are shown and employed to obtain pointwise error bounds of the numerical solution away from the boundary.

Combining these techniques yields error bounds with explicit convergence rates under regularity assumptions on the exact solution. Finally, the present paper demonstrates, to the best of the author's knowledge, the first numerical experiments for the present problem.

1.3. **Overview of the paper.** The mathematical content of this paper starts from the next section, which provides the functional analytic framework surrounding Maxwell's equations and the nonlinear boundary condition. Time-harmonic bounds for the potential operators in terms of norms associated to the functional analytic setting of the nonlinearity (namely $\boldsymbol{L}^p$-spaces) are shown.

Time-dependent nonlinear boundary integral equations are derived in Section 3 and a continuous stability result is formulated and proven.

Section 4 introduces the Runge-Kutta convolution quadrature and gives a time-discrete scheme. Further introduced are time-discrete transmission problems, the foundation of the subsequent error analysis. A novel bound, reminiscent of partial integration, for difference formulas of convolution quadrature based on Runge-Kutta multistage methods is shown. Time-discrete jump conditions of the time-discrete Calderón operator are shown, preparing the main result.

The error analysis is then conducted in Section 5, which starts with the introduction and application of boundary elements. An intermediate result gives bounds for the fully discrete numerical solution, without assumptions on the exact solution, in terms of the incidental wave. This result, combined with the preparation in the previous section, enables the error analysis, under assumptions on the exact solution.

Finally, Section 6 presents numerical experiments. Convergence plots give empirical error rates and visualize an example simulation of a scattered wave for a given scatterer.

2. Framework and analytical background

We start with a formulation of the electromagnetic nonlinear scattering problem in terms of the *scattered fields* $\boldsymbol{E}$ and $\boldsymbol{H}$, effectively reformulating the problem of interest (1) – (3) with vanishing initial conditions. Initial values of the total fields are derived from given incident electric and magnetic fields $(\boldsymbol{E}^{\mathrm{inc}}, \boldsymbol{H}^{\mathrm{inc}})$, which solve Maxwell's equations in $\mathbb{R}^3$ and have initial support away from the boundary $\Gamma = \partial\Omega$. Our objective is the construction of scattered fields $(\boldsymbol{E}^{\mathrm{scat}}, \boldsymbol{H}^{\mathrm{scat}})$, solutions of Maxwell's equations with vanishing initial conditions, such that the total fields $(\boldsymbol{E}^{\mathrm{tot}}, \boldsymbol{H}^{\mathrm{tot}})$ satisfy a specified nonlinear boundary condition. These total fields should be constructed in such a way, that the evaluation of the fields at arbitrary points $\boldsymbol{x} \in \Omega$ and times $0 \le t \le T$ is computationally viable, which is practically achieved through a representation formula.

As the scattered fields are the unknowns to be computed, they are denoted without an index, by writing $(\boldsymbol{E}, \boldsymbol{H}) = (\boldsymbol{E}^{\mathrm{scat}}, \boldsymbol{H}^{\mathrm{scat}})$.

Let $\left(\boldsymbol{E}^{\mathrm{inc}}, \boldsymbol{H}^{\mathrm{inc}}\right)$ denote incidental waves, solutions to the time-dependent Maxwell's equations on the complete space $\mathbb{R}^3$, with initial support in the exterior domain Ω away from the boundary Γ. The quantities of interest are the scattered fields $\boldsymbol{E} = \boldsymbol{E}^{\mathrm{tot}} - \boldsymbol{E}^{\mathrm{inc}}$ and $\boldsymbol{H} = \boldsymbol{H}^{\mathrm{tot}} - \boldsymbol{H}^{\mathrm{inc}}$, which solve the following initial–boundary value problem of Maxwell's equations:

$$\partial_t \boldsymbol{E} - \mathbf{curl}\, \boldsymbol{H} = 0 \qquad \text{in} \quad \Omega, \tag{5}$$

$$\partial_t \boldsymbol{H} + \mathbf{curl}\, \boldsymbol{E} = 0 \qquad \text{in} \quad \Omega, \tag{6}$$

$$\boldsymbol{E} \times \boldsymbol{\nu} + \boldsymbol{a}(\boldsymbol{H} \times \boldsymbol{\nu} + \boldsymbol{H}^{\mathrm{inc}} \times \boldsymbol{\nu}) \times \boldsymbol{\nu} = -\boldsymbol{E}^{\mathrm{inc}} \times \boldsymbol{\nu} \qquad \text{on } \Gamma. \tag{7}$$

As the initial support of the incidental waves is away from the boundary, the initial values in Ω for both $\boldsymbol{E}$ and $\boldsymbol{H}$ vanish.

Asymptotic conditions for $|\boldsymbol{x}| \to \infty$ are not necessary, as the finite wave speed $c = 1$ implies that the fields $(\boldsymbol{E}, \boldsymbol{H})$ have bounded support at any time t.

For the derivation of a weak formulation and subsequently boundary integral equations it is crucial to give a functional analytic framework that is appropriate for the nonlinear scattering problem. The functional analytic setting of the nonlinear boundary condition (7) needs to reconcile the properties of the tangential trace γ_T and the nonlinear operator associated to the composition with the nonlinearity $\boldsymbol{a}$. The following section starts with the description of appropriate spaces for γ_T.

2.1. Tangential trace and trace space $\mathbf{X}_\Gamma$.

For a continuous vector field defined on the closure of the exterior domain, $\boldsymbol{v} : \overline{\Omega} \to \mathbb{C}^3$, we define the *tangential trace*

$$\gamma_T \boldsymbol{v} = \boldsymbol{v}|_\Gamma \times \boldsymbol{\nu} \qquad \text{on } \Gamma,$$

where $\boldsymbol{\nu}$ denotes the outer unit surface normal.

Green's formula for the **curl** operator yields, for sufficiently regular vector fields $\boldsymbol{u}, \boldsymbol{v} : \overline{\Omega} \to \mathbb{C}^3$, the identity

$$\int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \boldsymbol{v} - \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = \int_\Gamma (\gamma_T \boldsymbol{u} \times \boldsymbol{\nu}) \cdot \gamma_T \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}, \tag{8}$$

where the Euclidean inner product on $\mathbb{C}^3$ is denoted by the dot $\cdot$, defined by $\boldsymbol{a} \cdot \boldsymbol{b} = \overline{\boldsymbol{a}}^\top \boldsymbol{b}$ for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^3$. The skew-hermitian sesquilinear form on the right-hand side, also referred to as the *anti-symmetric pairing*, is denoted for continuous tangential vector fields on the boundary $\boldsymbol{\phi}, \boldsymbol{\psi} : \Gamma \to \mathbb{C}^3$ by

$$[\boldsymbol{\phi}, \boldsymbol{\psi}]_\Gamma = \int_\Gamma (\boldsymbol{\phi} \times \boldsymbol{\nu}) \cdot \boldsymbol{\psi} \, \mathrm{d}\sigma. \tag{9}$$

Plugging solutions of Maxwell's equations into the Green's formula, i.e. setting $\boldsymbol{u} = \boldsymbol{E}$ and $\boldsymbol{v} = \boldsymbol{H}$ yields, for the exterior domain Ω, the identity

$$\begin{aligned} [\gamma_T \boldsymbol{H}, \gamma_T \boldsymbol{E}]_\Gamma &= \int_\Omega \mathbf{curl}\, \boldsymbol{H} \cdot \boldsymbol{E} - \boldsymbol{H} \cdot \mathbf{curl}\, \boldsymbol{E} \, \mathrm{d}\boldsymbol{x} \\ &= \frac{1}{2} \partial_t \int_\Omega |\boldsymbol{E}|^2 + |\boldsymbol{H}|^2 \, \mathrm{d}\boldsymbol{x}. \end{aligned} \tag{10}$$

In the following, we describe the functional analytic setting of the tangential trace γ_T as it has been derived in [1] for smooth domains and in [14] for Lipschitz domains. A natural setting for the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$ is provided by

$$\boldsymbol{H}(\mathbf{curl}, \Omega) = \{ \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \, : \, \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \}.$$

The tangential trace γ_T extends to a surjective bounded linear operator from this space into a trace space $\mathbf{X}_\Gamma$, namely

$$\gamma_T : \, \boldsymbol{H}(\mathbf{curl}, \Omega) \to \mathbf{X}_\Gamma.$$

The Hilbert space $\mathbf{X}_\Gamma$, completed with the appropriate norm $\| \cdot \|_{\mathbf{X}_\Gamma}$, is occasionally referred to as the *proper trace space*. More background on the functional analytical setting of γ_T is described in the surveys [15, Sect. 2.2] and [27, Sect. 5.4].

The proper trace space $\mathbf{X}_\Gamma$ consists of those functions in the tangential subspace of the Sobolev space $\boldsymbol{H}^{-1/2}(\Gamma)$ with surface divergence in $H^{-1/2}(\Gamma)$, i.e.

$$\mathbf{X}_\Gamma = \{ \boldsymbol{\phi} \in \boldsymbol{H}^{-1/2}(\Gamma) \, : \, \boldsymbol{\phi} \times \boldsymbol{\nu} = 0, \ \mathrm{div}_\Gamma \, \boldsymbol{\phi} \in H^{-1/2}(\Gamma) \}.$$

For a precise formulation of the Sobolev space $H^{-1/2}(\Gamma)$ (in particular for Lipschitz domains) we refer to the cited publications above.

The anti-symmetric pairing $[\cdot, \cdot]_\Gamma$ extends to a non-degenerate continuous sesquilinear form on $\mathbf{X}_\Gamma \times \mathbf{X}_\Gamma$. As a consequence, $\mathbf{X}_\Gamma$ becomes its own dual by installing $[\cdot, \cdot]_\Gamma$ as the anti-duality.

The treatment of the nonlinear generalized impedance boundary conditions requires the combination of the natural setting of Maxwell's equations described above with an appropriate setting of the nonlinearity $\boldsymbol{a}$. The subsequent section describes some basic properties of $\boldsymbol{a}$ and gives a fitting functional analytic framework of the associated nonlinear operator.

2.2. **Functional analytical setting of the nonlinear function $\boldsymbol{a}$.** Let $\alpha \in (0,1]$ be a given constant, then we repeat the power-law type nonlinearity from the boundary condition discussed in [33] and occuring in the nonlinear evolution equation described in [36] and write

$$\boldsymbol{a}(\boldsymbol{x}) = |\boldsymbol{x}|^{\alpha-1}\,\boldsymbol{x}, \quad \text{for } \boldsymbol{x} \in \mathbb{R}^3. \tag{11}$$

This nonlinearity is positive, in the sense that for any $\boldsymbol{x} \in \mathbb{R}^3$, we trivially obtain

$$\boldsymbol{x} \cdot \boldsymbol{a}(\boldsymbol{x}) = |\boldsymbol{x}|^{1+\alpha}. \tag{12}$$

This identity plays a crucial role in stability estimates regarding both the continuous problem (5)–(7) and the numerical scheme proposed in the subsequent sections. To derive error estimates, a stronger form of positivity is necessary and almost provided by an intermediate result from [33]: the nonlinearity $\boldsymbol{a}$ is *monotone*, namely for arbitrary $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$ it holds that

$$(\boldsymbol{u}-\boldsymbol{v}) \cdot (\boldsymbol{a}(\boldsymbol{u}) - \boldsymbol{a}(\boldsymbol{v})) \geq 0. \tag{13}$$

The following Lemma gives bounds which will in particular include a stronger form of the monotonicity (13), crucial in the error analysis in the following sections.

Lemma 1 (Pointwise bounds on $\boldsymbol{a}$)**.** *For $\alpha \in (0,1]$ the nonlinearity $\boldsymbol{a}(\boldsymbol{x}) = |\boldsymbol{x}|^{\alpha-1}\boldsymbol{x}$ fulfills a positivity condition stronger than the monotonicity, namely for arbitrary $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$ it holds that*

$$(\boldsymbol{u}-\boldsymbol{v}) \cdot (\boldsymbol{a}(\boldsymbol{u}) - \boldsymbol{a}(\boldsymbol{v})) \geq \alpha(|\boldsymbol{v}| + |\boldsymbol{u}|)^{\alpha-1}\,|\boldsymbol{u}-\boldsymbol{v}|^2. \tag{14}$$

Furthermore, $\boldsymbol{a}$ is Hölder continuous, as the bound

$$|\boldsymbol{a}(\boldsymbol{u}) - \boldsymbol{a}(\boldsymbol{v})| \leq 2\,|\boldsymbol{u}-\boldsymbol{v}|^{\alpha}, \tag{15}$$

holds for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$.

Proof. The coercivity result builds upon an argument of the proof from [36, Lemma 2.1], which is repeated here for the convenience of the reader. The Jacobian of the nonlinearity $\boldsymbol{a} : \mathbb{R}^3 \to \mathbb{R}^3$, in the following denoted by $\boldsymbol{D}\boldsymbol{a} : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$, is derived from standard differentiation rules and has the explicit form

$$\boldsymbol{D}\boldsymbol{a}(\boldsymbol{x}) = (\alpha-1)\,|\boldsymbol{x}|^{\alpha-3}\,\boldsymbol{x}\boldsymbol{x}^T + |\boldsymbol{x}|^{\alpha-1}\,\boldsymbol{I}_3 \quad \text{for } \boldsymbol{x} \in \mathbb{R}^3 \setminus \{0\}, \tag{16}$$

where $\boldsymbol{I}_3 \in \mathbb{R}^{3\times 3}$ denotes the identity matrix.

The Jacobian is positive definite for all $\boldsymbol{x} \in \mathbb{R}^3 \setminus \{0\}$, which is a consequence of the Cauchy-Schwartz inequality for any $\boldsymbol{y} \in \mathbb{R}^3$ we can apply the Cauchy-Schwartz inequality to obtain the estimate

$$\begin{aligned} \boldsymbol{y}^T\boldsymbol{D}\boldsymbol{a}(\boldsymbol{x})\boldsymbol{y} &= (\alpha-1)\,|\boldsymbol{x}|^{\alpha-3}\,(\boldsymbol{x}^T\boldsymbol{y})^2 + |\boldsymbol{x}|^{\alpha-1}\,|\boldsymbol{y}|^2 \\ &\geq \Big((\alpha-1)\,|\boldsymbol{x}|^{\alpha-3}\,|\boldsymbol{x}|^2 + |\boldsymbol{x}|^{\alpha-1}\Big)\,|\boldsymbol{y}|^2 = \alpha\,|\boldsymbol{x}|^{\alpha-1}\,|\boldsymbol{y}|^2. \end{aligned}$$

The stated strong monotonicity (14) of the nonlinearity $\boldsymbol{a}$ is now a consequence of the positive definiteness of the Jacobian in combination with the fundamental theorem of calculus, which yields

$$\begin{aligned}(\boldsymbol{u}-\boldsymbol{v})\cdot(\boldsymbol{a}(\boldsymbol{u})-\boldsymbol{a}(\boldsymbol{v})) &= (\boldsymbol{u}-\boldsymbol{v})\cdot\int_0^1 \boldsymbol{D}\boldsymbol{a}(\boldsymbol{v}+\theta(\boldsymbol{u}-\boldsymbol{v}))(\boldsymbol{u}-\boldsymbol{v})\,\mathrm{d}\theta \\ &\geq \alpha\int_0^1 |\boldsymbol{v}+\theta(\boldsymbol{u}-\boldsymbol{v})|^{\alpha-1}\,|\boldsymbol{u}-\boldsymbol{v}|^2\,\mathrm{d}\theta \\ &\geq \alpha(|\boldsymbol{v}|+|\boldsymbol{u}|)^{\alpha-1}\,|\boldsymbol{u}-\boldsymbol{v}|^2\,.\end{aligned}$$

For a proof of the Hölder continuity we refer the reader to [36, Lemma 6.4]. □

Pointwise bounds of the nonlinearity imply bounds on the nonlinear operator defined by the composition with $\boldsymbol{a}$, namely the operator $\boldsymbol{u}\mapsto\boldsymbol{a}\circ\boldsymbol{u}$ for arbitrary $\boldsymbol{u}:\Gamma\to\mathbb{R}^3$. A natural space for this nonlinear operator is given by the tangential $\boldsymbol{L}^p$ space for $p>1$ on the boundary Γ, which reads

$$\boldsymbol{L}_T^p(\Gamma) = \{\boldsymbol{u}\in\boldsymbol{L}^p(\Gamma)\mid \boldsymbol{u}\cdot\boldsymbol{\nu}=0\}\,,$$

complete with the associated norm $\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Gamma)}$ inherited from the full space $\boldsymbol{L}^p(\Gamma)$.

The following result clarifies the relation of the nonlinear operator $\boldsymbol{a}$ with the tangential space $\boldsymbol{L}_T^p(\Gamma)$.

Lemma 2. *(Setting of $\boldsymbol{a}$) The nonlinear operator defined by the composition with the nonlinearity $\boldsymbol{a}(\boldsymbol{x})=|\boldsymbol{x}|^{\alpha-1}\boldsymbol{x}$ for $\alpha\in(0,1]$ is a well-posed bijective operator*

$$\boldsymbol{a}:\boldsymbol{L}_T^{1+\alpha}(\Gamma)\to\boldsymbol{L}_T^{\frac{1+\alpha}{\alpha}}(\Gamma). \tag{17}$$

Proof. The well-posedness of $\boldsymbol{a}$ on the stated spaces follows by observing

$$\|\boldsymbol{a}(\boldsymbol{u})\|_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)} = \left\||\boldsymbol{u}|^{\alpha-1}\boldsymbol{u}\right\|_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)} = \left\||\boldsymbol{u}|^{\alpha}\right\|_{L^{\frac{1+\alpha}{\alpha}}(\Gamma)} = \|\boldsymbol{u}\|^{\alpha}_{\boldsymbol{L}^{1+\alpha}(\Gamma)}\,, \tag{18}$$

which in particular implies the left-hand side is bounded for any $\boldsymbol{u}\in\boldsymbol{L}_T^{1+\alpha}(\Gamma)$. The well-posedness of the operator defined through the composition with the inverse of $\boldsymbol{a}$, which has the closed form

$$\boldsymbol{a}^{-1}(\boldsymbol{x}) = |\boldsymbol{x}|^{\frac{1-\alpha}{\alpha}}\,\boldsymbol{x},$$

is readily apparent by the same argument, proving that $\boldsymbol{a}$ is a bijection. □

As the boundary $\Gamma=\partial\Omega$ of the scatterer is a bounded surface, we have the following chain of dense inclusions

$$\boldsymbol{L}_T^{1+\alpha}(\Gamma)\subset\boldsymbol{L}_T^2(\Gamma)\subset\boldsymbol{L}_T^{\frac{1+\alpha}{\alpha}}(\Gamma). \tag{19}$$

Choosing $\boldsymbol{L}_T^2$ as the pivot space, these spaces are dual to each other, since the reciprocal of their exponents add to one. More precisely, the $\boldsymbol{L}^2$-scalar product, denoted by $(\cdot,\cdot)_\Gamma$, is a continuous hermitian bilinear form on $\boldsymbol{L}_T^{1+\alpha}(\Gamma)\times\boldsymbol{L}_T^{\frac{1+\alpha}{\alpha}}(\Gamma)$ and makes these spaces their respective dual. The continuity is a consequence of the Hölder inequality, which guarantees for boundary functions $\boldsymbol{u}$ and $\boldsymbol{v}$ of appropriate regularity the bound

$$(\boldsymbol{u},\boldsymbol{v})_\Gamma\leq\|\boldsymbol{u}\|_{\boldsymbol{L}^{1+\alpha}(\Gamma)}\,\|\boldsymbol{v}\|_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)}\,. \tag{20}$$

Consequently, when understood in the setting of Lemma 2, the nonlinear operator $\boldsymbol{a}$ maps into the dual of its domain, where the anti-duality between both spaces is explicitly given by the extension of the $\boldsymbol{L}^2$- pairing.

Turning towards the composition of the nonlinearity $\boldsymbol{a}$ with traces of solutions to Maxwell's equations, we introduce the dense subspace

$$\mathbf{V}_\Gamma = \mathbf{X}_\Gamma \cap \boldsymbol{L}_T^{1+\alpha}(\Gamma) \subset \mathbf{X}_\Gamma,$$

equipped with the norm $\|\boldsymbol{\phi}\|_{\mathbf{V}_\Gamma} = \|\boldsymbol{\phi}\|_{\mathbf{X}_\Gamma} + \|\boldsymbol{\phi}\|_{\boldsymbol{L}^{1+\alpha}(\Gamma)}$.

2.3. **Temporal Sobolev spaces and convolutions.** Let V be an arbitrary Banach space and $L(s)\colon V \to V'$ be an analytic family of bounded linear operators for $\operatorname{Re} s > 0$. Assume further that L is *polynomially bounded* in the sense that there exists a real κ, and for every $\sigma > 0$ there exists a positive constant $M_\sigma < \infty$, such that

$$\|L(s)\|_{V' \leftarrow V} \le M_\sigma \frac{|s|^\kappa}{(\operatorname{Re} s)^\nu}, \qquad \operatorname{Re} s \ge \sigma > 0. \tag{21}$$

This polynomial bound (in terms of the parameter s) ensures that the inverse Laplace transform of $L(s)$ is a distribution of finite order of differentiation, vanishing on the negative real half-line $t < 0$. Throughout the paper, we use the operational calculus notation, which reads for any sufficiently regular function $g : [0, T] \to V$ with vanishing initial conditions

$$L(\partial_t) g = (\mathcal{L}^{-1} L) * g, \tag{22}$$

thus defining a shorthand for the temporal convolution of the inverse Laplace transform of L with g. For two families of operators $A(s)$ and $B(s)$, following the described constraints and mapping into compatible spaces for their composition to be well-posed, the associativity of convolution and the product rule of Laplace transforms yield the composition rule $B(\partial_t)A(\partial_t)g = (BA)(\partial_t)g$.

Let V denote a Hilbert space and further let $r \in \mathbb{R}$. We denote the Sobolev space of real order r and V-valued functions on $\mathbb{R}$ by $\boldsymbol{H}^r(\mathbb{R}, V)$. Furthermore, we expand this notation to finite intervals $(0, T)$ by writing

$$H_0^r(0, T; V) = \{g|_{(0,T)} \,:\, g \in H^r(\mathbb{R}, V) \ \text{ with } \ g = 0 \ \text{ on } \ (-\infty, 0)\}.$$

For integer order $r \ge 0$ the natural norm on $H_0^r(0, T; V)$ is equivalent to the norm $\|\partial_t^r g\|_{L^2(0,T;V)}$. Temporal convolutions as defined by the Heaviside notations now fulfill the following result [25, Lemma 2.1]: Let $L(s)$ be an analytic family of polynomially bounded operators in the half-plane $\operatorname{Re} s > 0$. Then, $L(\partial_t)$ extends by density to a bounded temporal linear operator

$$L(\partial_t) : H_0^{r+\kappa}(0, T; V) \to H_0^r(0, T; V') \tag{23}$$

for arbitrary real r. It should be noted that the inclusion $H_0^r(0, T; V') \subset C^{r-1}([0, T]; V')$ further implies pointwise bounds for sufficient order $r \ge 1$.

2.4. **Weak formulation of the nonlinear boundary condition.** Let $\boldsymbol{\phi}$ denote an arbitrary continuous tangential vector field on Γ. Taking the anti-symmetric product $[\cdot, \cdot]_\Gamma$ of the boundary condition (7) with $\boldsymbol{\phi}$ yields

$$[\boldsymbol{\phi}, \gamma_T \boldsymbol{E}]_\Gamma + [\boldsymbol{\phi}, \boldsymbol{a}(\gamma_T \boldsymbol{H} + \gamma_T \boldsymbol{H}^{\text{inc}}) \times \boldsymbol{\nu}]_\Gamma = -[\boldsymbol{\phi}, \gamma_T \boldsymbol{E}^{\text{inc}}]_\Gamma. \tag{24}$$

Noting that the cross product with the unit normal $\boldsymbol{\nu}$ in the nonlinear term simplifies to a $\boldsymbol{L}^2$- product, leaves the following *weak formulation of the boundary condition* (7): the tangential traces $\gamma_T\boldsymbol{E} \in \boldsymbol{L}^2(0,T;\mathbf{X}_\Gamma)$ and $\gamma_T\boldsymbol{H} \in \boldsymbol{L}^2(0,T;\mathbf{V}_\Gamma)$, the boundary data of solutions $\boldsymbol{E},\boldsymbol{H} \in \boldsymbol{L}^2(0,T;\boldsymbol{H}(\mathbf{curl},\Omega)) \cap \boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega)^3)$ to the Maxwell's equations in Ω with zero initial conditions fulfill the weak boundary condition

$$[\boldsymbol{\phi},\gamma_T\boldsymbol{E}]_\Gamma + (\boldsymbol{\phi},\boldsymbol{a}(\gamma_T\boldsymbol{H}+\gamma_T\boldsymbol{H}^{\mathrm{inc}}))_\Gamma = [\gamma_T\boldsymbol{E}^{\mathrm{inc}},\boldsymbol{\phi}]_\Gamma \qquad \text{for all } \boldsymbol{\phi} \in \mathbf{V}_\Gamma, \tag{25}$$

for almost every $t \in (0,T)$ All terms appearing in this formulation are well-defined under the stated regularity assumptions.

2.5. **Time-harmonic Maxwell's equations.** Our interest lies in time-dependent problems, the study of which requires time-dependent potential and boundary operators. This section gives a short introduction into time-harmonic operators and their properties in the Laplace domain, whose implications on the time domain will be discussed in subsequent sections.

The *time-harmonic Maxwell's equations* read, for $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$,

$$s\widehat{\boldsymbol{E}} - \mathbf{curl}\,\widehat{\boldsymbol{H}} = 0 \quad \text{in } \Omega, \tag{26}$$

$$s\widehat{\boldsymbol{H}} + \mathbf{curl}\,\widehat{\boldsymbol{E}} = 0 \quad \text{in } \Omega. \tag{27}$$

We recall basic notations associated with the boundary integral operators for the time-harmonic Maxwell's equations, following [15, 27]. The *fundamental solution* reads

$$G(s,\boldsymbol{x}) = \frac{e^{-s|\boldsymbol{x}|}}{4\pi\,|\boldsymbol{x}|}, \qquad \operatorname{Re} s > 0,\ \boldsymbol{x} \in \mathbb{R}^3 \setminus \{0\}.$$

Let $\boldsymbol{\varphi}$ denote a regular complex-valued function on the boundary Γ. The electromagnetic *single layer potential* operator $\mathcal{S}(s)$, applied to $\boldsymbol{\varphi}$ and evaluated at $\boldsymbol{x} \in \mathbb{R}^3 \setminus \Gamma$, is defined by

$$\boldsymbol{\mathcal{S}}(s)\boldsymbol{\varphi}(\boldsymbol{x}) = -s\int_\Gamma G(s,\boldsymbol{x}-\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + s^{-1}\nabla\int_\Gamma G(s,\boldsymbol{x}-\boldsymbol{y})\operatorname{div}_\Gamma\boldsymbol{\varphi}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y},$$

and the electromagnetic *double layer potential* operator $\boldsymbol{\mathcal{D}}(s)$ is defined by

$$\boldsymbol{\mathcal{D}}(s)\boldsymbol{\varphi}(\boldsymbol{x}) = \mathbf{curl}\int_\Gamma G(s,\boldsymbol{x}-\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}.$$

Any outgoing solution to the time-harmonic Maxwell's equations are recovered from their tangential traces by the representation formulas

$$\widehat{\boldsymbol{E}} = -\,\boldsymbol{\mathcal{S}}(s)\big(\gamma_T\widehat{\boldsymbol{H}}\big) + \boldsymbol{\mathcal{D}}(s)\big(-\gamma_T\widehat{\boldsymbol{E}}\big) \quad \text{in } \Omega, \tag{28}$$

$$\widehat{\boldsymbol{H}} = -\boldsymbol{\mathcal{D}}(s)\big(\gamma_T\widehat{\boldsymbol{H}}\big) \;-\; \boldsymbol{\mathcal{S}}(s)\big(-\gamma_T\widehat{\boldsymbol{E}}\big) \quad \text{in } \Omega. \tag{29}$$

Despite our interest in the boundary value problem, which is purely formulated on the exterior domain Ω, it will prove to be useful to employ techniques from the theory of transmission problems. These formulations are posed on the full space $\mathbb{R}^3$, which is assumed to consist of the disjoint composition $\mathbb{R}^3 = \Omega^- \,\dot\cup\, \Gamma \,\dot\cup\, \Omega^+$. Whenever the framework of transmission problems is employed, Ω^+ denotes the exterior domain of interest (elsewhere referred to as Ω), whereas Ω^- denotes the bounded interior domain.

Quantities defined purely on the exterior domain, such as the unkown scattered fields, are naturally extended by zero in the inside of the scatterer Ω^-.

We define *jumps* and *averages*, which are bounded operators from $\boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)$ into the trace space $\mathbf{X}_\Gamma$, by

$$[\gamma_T] = \gamma_T^+ - \gamma_T^-, \qquad \{\gamma_T\} = \tfrac{1}{2}\left(\gamma_T^+ + \gamma_T^-\right).$$

Using the average trace operator we define the electromagnetic *single and double layer boundary operators* as the composition with the potential operators

$$\boldsymbol{V}(s) = \{\gamma_T\} \circ \boldsymbol{\mathcal{S}}(s), \qquad \boldsymbol{K}(s) = \{\gamma_T\} \circ \boldsymbol{\mathcal{D}}(s).$$

Building on the boundary operators, we define the *Calderón operator* as introduced in [24], with a sign corrected in [29]:

$$\boldsymbol{C}(s) = \begin{pmatrix} -\boldsymbol{V}(s) & \boldsymbol{K}(s) \\ -\boldsymbol{K}(s) & -\boldsymbol{V}(s) \end{pmatrix} = \{\gamma_T\} \circ \begin{pmatrix} -\boldsymbol{\mathcal{S}}(s) & \boldsymbol{\mathcal{D}}(s) \\ -\boldsymbol{\mathcal{D}}(s) & -\boldsymbol{\mathcal{S}}(s) \end{pmatrix}.$$

The jump relations of the boundary integral operators now imply the following central identity.

Any solution to the time-harmonic Maxwell's equations $\widehat{\boldsymbol{E}}, \widehat{\boldsymbol{H}} \in \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)$ with an asymptotic condition on the exterior domain Ω^+ fulfill

$$\boldsymbol{C}(s) \begin{pmatrix} [\gamma_T]\widehat{\boldsymbol{H}} \\ -[\gamma_T]\widehat{\boldsymbol{E}} \end{pmatrix} = \begin{pmatrix} \{\gamma_T\}\widehat{\boldsymbol{E}} \\ \{\gamma_T\}\widehat{\boldsymbol{H}} \end{pmatrix}. \tag{30}$$

The notation associated to the skew-hermitian pairing $[\cdot,\cdot]_\Gamma$ is extended from $\mathbf{X}_\Gamma \times \mathbf{X}_\Gamma$ to ${\mathbf{X}_\Gamma}^2 \times {\mathbf{X}_\Gamma}^2$ componentwise:

$$\left[\begin{pmatrix} \boldsymbol{\varphi} \\ \boldsymbol{\psi} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{pmatrix} \right]_\Gamma = [\boldsymbol{\varphi}, \boldsymbol{\xi}]_\Gamma + [\boldsymbol{\psi}, \boldsymbol{\eta}]_\Gamma.$$

The electromagnetic potential and boundary operators extend to bounded operators on their respective spaces, with the trace space $\mathbf{X}_\Gamma$ installed on the boundary and $\boldsymbol{H}(\mathbf{curl}, \Omega)$ installed on the domain Ω. Crucial for an analysis of the time-dependent analogues of these operators are bounds where the $s-$dependency of the constants is explicitly known, effectively demanding polynomial bounds of the type (21). Estimates explicit in s have been derived in [2, Theorem 4.4] and were sharpened in [30, Lemma 3.4].

Lemma 3. [30, Lemma 3.2 and Lemma 3.8] *The electromagnetic single and double-layer potential operators $\boldsymbol{\mathcal{S}}(s)$ and $\boldsymbol{\mathcal{D}}(s)$ extend to bounded linear operators from $\mathbf{X}_\Gamma$ to $\boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)$, which are bounded for $\operatorname{Re} s > 0$ by*

$$\|\boldsymbol{\mathcal{S}}(s)\|_{\boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3\setminus\Gamma) \leftarrow \mathbf{X}_\Gamma} \le C_\Gamma \frac{|s|^2 + 1}{\operatorname{Re} s}, \qquad \|\boldsymbol{\mathcal{D}}(s)\|_{\boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3\setminus\Gamma) \leftarrow \mathbf{X}_\Gamma} \le C_\Gamma \frac{|s|^2 + 1}{\operatorname{Re} s}.$$

Point evaluations away from the boundary fulfill time-harmonic bounds of the same structure with an additional factor, which exponentially decays with respect to the real part of s. Bounds of this type are shown in [2] for smooth domains and in [30] for Lipschitz domains. In the context of this work, slightly different norms are of interest, namely operator norms which derive from the $\boldsymbol{L}^p(\Gamma)$ space. The following Lemma gives such bounds.

Lemma 4. *Let $\boldsymbol{x} \in \Omega$ be a point away from the boundary, assumed for this statement to be smooth, with distance $d = \operatorname{dist}(\boldsymbol{x}, \Gamma) > 0$. There exists a positive constant C independent of s and $\boldsymbol{\varphi}$, such that the pointwise bounds*

$$|(\boldsymbol{\mathcal{S}}(s)\boldsymbol{\varphi})\,(\boldsymbol{x})| \leq C\,|s|\,e^{-d\operatorname{Re}s}\,\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^p(\Gamma)},$$
$$|(\boldsymbol{\mathcal{D}}(s)\boldsymbol{\varphi})\,(\boldsymbol{x})| \leq C\,|s|\,e^{-d\operatorname{Re}s}\,\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^p(\Gamma)},$$

holds for all $1 \leq p < \infty$. By densitiy, the potential operators extend to linear bounded operators of the type $\boldsymbol{\mathcal{S}}_{\boldsymbol{x}}(s) : \boldsymbol{L}^p(\Gamma) \to \mathbb{C}^3$, fulfilling the bound above.

Proof. The bounds are a direct consequence of Hölders inequality. To show the bound for the second integral of the single-layer potential operator, we additionally use partial integration on the surface to rewrite

$$\int_\Gamma G(s, \boldsymbol{x}-\boldsymbol{y})\operatorname{div}_\Gamma \boldsymbol{\varphi}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = -\int_\Gamma \left(\nabla_\Gamma G(s,\boldsymbol{x}-\boldsymbol{y})\right)\boldsymbol{\varphi}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}.$$

□

3. Maxwell's equations with nonlinear boundary conditions

This section combines the framework of temporal Sobolev spaces and convolutions with the electromagnetic time-harmonic operators, to enable a treatment of the time-dependent nonlinear scattering problem. A fundamental building block of our analysis is the time-dependent representation formula, which reads:

Let $\boldsymbol{E}, \boldsymbol{H} \in \boldsymbol{L}^2(0,T;\boldsymbol{H}(\mathbf{curl},\Omega))$ be time-dependent Maxwell solutions with vanishing initial conditions associated with their boundary densities $\boldsymbol{\varphi}, \boldsymbol{\psi} \in \boldsymbol{L}^2(0,T;\mathbf{X}_\Gamma)$ defined via

$$\boldsymbol{\varphi} = \gamma_T \boldsymbol{H}, \qquad \boldsymbol{\psi} = -\gamma_T \boldsymbol{E}. \tag{31}$$

The electromagnetic fields $(\boldsymbol{E}, \boldsymbol{H})$ and their respective boundary data $(\boldsymbol{\varphi}, \boldsymbol{\psi})$ then fulfill the time-dependent representation formulas

$$\boldsymbol{E} = -\boldsymbol{\mathcal{S}}(\partial_t)\boldsymbol{\varphi} + \boldsymbol{\mathcal{D}}(\partial_t)\boldsymbol{\psi}, \tag{32}$$
$$\boldsymbol{H} = -\boldsymbol{\mathcal{S}}(\partial_t)\boldsymbol{\varphi} - \boldsymbol{\mathcal{D}}(\partial_t)\boldsymbol{\psi}. \tag{33}$$

The time-dependent analogues to the boundary integral operators and consequently the Calderón operator are defined via the Heaviside notation of operational calculus, in the same way as the appearing time-dependent potential operators from (32)–(33). Applying the inverse Laplace transform and the convolution theorem to (30) yields the jump conditions of the time-dependent Calderón operator

$$\boldsymbol{C}(\partial_t)\begin{pmatrix}\gamma_T\boldsymbol{H}\\ -\gamma_T\boldsymbol{E}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\gamma_T\boldsymbol{E}\\ \gamma_T\boldsymbol{H}\end{pmatrix}. \tag{34}$$

Note that at this point we implicitly extended $\boldsymbol{E}$ and $\boldsymbol{H}$ by zero in the interior domain. These jump conditions have particularly been in the treatment of several linear and nonlinear boundary conditions in the context of hyperbolic problems [11, 6, 8, 30]. Following the combined ideas of these previous papers, we start by adding a symmetric block operator on both sides and arrive at

$$\boldsymbol{C}_{\mathrm{imp}}(\partial_t)\begin{pmatrix}\gamma_T\boldsymbol{H}\\ -\gamma_T\boldsymbol{E}\end{pmatrix} = \begin{pmatrix}\gamma_T\boldsymbol{E}\\ 0\end{pmatrix}, \qquad \boldsymbol{C}_{\mathrm{imp}}(\partial_t) = \boldsymbol{C}(\partial_t) + \begin{pmatrix}0 & -\frac{1}{2}\boldsymbol{I}\\ -\frac{1}{2}\boldsymbol{I} & 0\end{pmatrix}. \tag{35}$$

Testing both sides with $(\boldsymbol{\eta},\boldsymbol{\xi}) \in \mathbf{V}_\Gamma \times \mathbf{X}_\Gamma$ yields

$$\left[\begin{pmatrix}\boldsymbol{\eta}\\ \boldsymbol{\xi}\end{pmatrix}, \boldsymbol{C}_{\text{imp}}(\partial_t)\begin{pmatrix}\boldsymbol{\varphi}\\ \boldsymbol{\psi}\end{pmatrix}\right]_\Gamma = [\boldsymbol{\eta}, \gamma_T \boldsymbol{E}]_\Gamma\,.$$

Inserting the weak formulation of the nonlinear boundary condition (25) on the right-hand side and rearranging all unknown terms to the left-hand side yields the weak formulation of the boundary integral equation studied throughout the rest of this paper.

Boundary integral equation: *Find $(\boldsymbol{\varphi},\boldsymbol{\psi}) \in \boldsymbol{L}^2(0,T;\mathbf{V}_\Gamma \times \mathbf{X}_\Gamma)$ such that, for all $(\boldsymbol{\eta},\boldsymbol{\xi}) \in \mathbf{V}_\Gamma \times \mathbf{X}_\Gamma$ it holds that*

$$\left[\begin{pmatrix}\boldsymbol{\eta}\\ \boldsymbol{\xi}\end{pmatrix}, \boldsymbol{C}_{\text{imp}}(\partial_t)\begin{pmatrix}\boldsymbol{\varphi}\\ \boldsymbol{\psi}\end{pmatrix}\right]_\Gamma + (\boldsymbol{\eta}, \boldsymbol{a}(\boldsymbol{\varphi} + \gamma_T \boldsymbol{H}^{\text{inc}}))_\Gamma = [\gamma_T \boldsymbol{E}^{\text{inc}}, \boldsymbol{\eta}]_\Gamma. \tag{36}$$

Solutions of the time-dependent boundary integral equation coincide with Maxwell's solution fulfilling the weak form of the nonlinear boundary condition (25), which is proved in the next section.

Remark 3.1. *The boundary integral equation above can be shifted in the frequency domain in the following sense. Let $\sigma > 0$ be some constant and let $\widetilde{\boldsymbol{C}_{imp}}(s) = \boldsymbol{C}_{imp}(s+\sigma)$. Then,* (36) *is equivalent to the boundary integral equation*

$$\left[\begin{pmatrix}\boldsymbol{\eta}\\ \boldsymbol{\xi}\end{pmatrix}, \widetilde{\boldsymbol{C}_{imp}}(\partial_t)\begin{pmatrix}\widetilde{\boldsymbol{\varphi}}\\ \widetilde{\boldsymbol{\psi}}\end{pmatrix}\right]_\Gamma + e^{-\sigma t}(\boldsymbol{\eta}, \boldsymbol{a}(e^{\sigma t}\widetilde{\boldsymbol{\varphi}} + \gamma_T \boldsymbol{H}^{\text{inc}}))_\Gamma = e^{-\sigma t}[\gamma_T \boldsymbol{E}^{inc}, \boldsymbol{\eta}]_\Gamma, \tag{37}$$

where the boundary densities are shifted via $\widetilde{\boldsymbol{\varphi}} = e^{-\sigma t}\boldsymbol{\varphi}$ and $\widetilde{\boldsymbol{\psi}} = e^{-\sigma t}\boldsymbol{\psi}$. Although this boundary integral equation is equivalent to (36)*, their numerical discretization differ. In particular, parts of the subsequent error analysis only holds for the discretization of this shifted boundary integral equation, though numerical experiments indicate that the shift is not necessary for practical computations.*

3.1. **Well-posedness of the boundary integral equations.** To prepare our investigations into the stability of the time-dependent nonlinear boundary integral equations, we introduce the following time-dependent transmission problem. This result is the central property of the potential operators and rigorously associates a transmission problem to any pair of time-dependent densities in $\mathbf{X}_\Gamma$ with sufficient temporal regularity.

Let $(\boldsymbol{\varphi},\boldsymbol{\psi}) \in \boldsymbol{H}_0^k(0,T;\mathbf{X}_\Gamma \times \mathbf{X}_\Gamma)$ denote boundary densities, which are not necessary boundary data of solutions to the time-dependent Maxwell's equations. The representation formulas (32)–(33) define fields $\boldsymbol{E},\boldsymbol{H} \in \boldsymbol{H}_0^{k-2}(0,T;\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3 \setminus \Gamma))$, which are solutions to the time-dependent transmission problem

$$\partial_t \boldsymbol{E} - \mathbf{curl}\,\boldsymbol{H} = 0 \qquad \text{in} \quad \mathbb{R}^3 \setminus \Gamma, \tag{38}$$

$$\partial_t \boldsymbol{H} + \mathbf{curl}\,\boldsymbol{E} = 0 \qquad \text{in} \quad \mathbb{R}^3 \setminus \Gamma, \tag{39}$$

$$[\gamma_T]\boldsymbol{H} = \boldsymbol{\varphi}\,, \tag{40}$$

$$-[\gamma_T]\boldsymbol{E} = \boldsymbol{\psi}\,. \tag{41}$$

The time-dependent Maxwell's equations (38)-(39) hold by construction of the potential operators, whereas (40)–(41) are consequences of the jump conditions of the potential operators.

Applying this result to solutions of the boundary integral equation gives the following theorem, a stability result of the boundary integral equation, which bounds solutions $(\boldsymbol{\varphi}, \boldsymbol{\psi})$ and their respective fields $(\boldsymbol{E}, \boldsymbol{H})$ in terms of the incident fields. Solutions of the nonlinear scattering problem (5)–(7) and the boundary integral equation are shown to coincide. Consequently, we obtain a stability and uniqueness result for solutions of the nonlinear scattering problem, however, proving existence of solutions is beyond the scope of this paper.

Theorem 1. *(Well-posedness of the boundary integral equation)*

Let $(\boldsymbol{\varphi}, \boldsymbol{\psi}) \in \boldsymbol{L}^2_0(0, T; \mathbf{V}_\Gamma \times \mathbf{X}_\Gamma)$ *be a solution of the nonlinear boundary integral equation* (36)*. The solution pair* $(\boldsymbol{\varphi}, \boldsymbol{\psi})$ *is, in the sense of* (31)*, the boundary data of unique electromagnetic fields* $\boldsymbol{E}$ *and* $\boldsymbol{H}$*, which are strong solutions of Maxwell's equations and fulfill the weak formulation of the boundary condition* (25)*.*

Furthermore, there exists a constant $C > 0$ *such that the following bound on the boundary densities holds*

$$\int_0^T \|\boldsymbol{\varphi}\|^{1+\alpha}_{\boldsymbol{L}^{1+\alpha}(\Gamma)} + \|\boldsymbol{\psi}\|^{\frac{1+\alpha}{\alpha}}_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)} \,\mathrm{d}t \le C \int_0^T \left\|\gamma_T \boldsymbol{H}^{\mathrm{inc}}\right\|^{1+\alpha}_{\boldsymbol{L}^{1+\alpha}(\Gamma)} + \left\|\gamma_T \boldsymbol{E}^{\mathrm{inc}}\right\|^{\frac{1+\alpha}{\alpha}}_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)} \,\mathrm{d}t,$$

under the assumption that all terms on the right-hand side are finite. Additionally, the pointwise (in time) norms of the electromagnetic fields $\boldsymbol{E}, \boldsymbol{H} \in \boldsymbol{C}(0, T; \boldsymbol{L}^2(\Omega))$ *are bounded by the same estimate, namely for all* $t \in [0, T]$ *we have*

$$\|\boldsymbol{E}(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{H}(t)\|^2_{\boldsymbol{L}^2(\Omega)} \le C \int_0^t \left\|\gamma_T \boldsymbol{H}^{\mathrm{inc}}\right\|^{1+\alpha}_{\boldsymbol{L}^{1+\alpha}(\Gamma)} + \left\|\gamma_T \boldsymbol{E}^{\mathrm{inc}}\right\|^{\frac{1+\alpha}{\alpha}}_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)} \,\mathrm{d}t'.$$

The constant $C > 0$ *depends in both cases only on* α*.*

Proof. The uniqueness is a direct consequence of the monotonicity of $\boldsymbol{a}$ and the time-domain Calderón operator $\boldsymbol{C}(\partial_t)$, which is transported from the Laplace domain [24, Lemma 3.1] to the time domain via [9, Lemma 2.2].

We continue with the connection of that solution to the nonlinear scattering problem. Let $\boldsymbol{E}, \boldsymbol{H} \in \boldsymbol{H}^{-2}_0(0, T; \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma))$ be solutions to the associated transmission problem (38)–(41) defined by the time-dependent representation formulas.

The jump conditions of the temporal Calderón operator imply

$$\begin{aligned} \boldsymbol{C}_{\mathrm{imp}}(\partial_t) \begin{pmatrix} \boldsymbol{\varphi} \\ \boldsymbol{\psi} \end{pmatrix} &= \boldsymbol{C}(\partial_t) \begin{pmatrix} \boldsymbol{\varphi} \\ \boldsymbol{\psi} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \boldsymbol{\psi} \\ \boldsymbol{\varphi} \end{pmatrix} \\ &= \begin{pmatrix} \{\gamma_T \boldsymbol{E}\} \\ \{\gamma_T \boldsymbol{H}\} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -[\gamma_T \boldsymbol{E}] \\ [\gamma_T \boldsymbol{H}] \end{pmatrix} = \begin{pmatrix} \gamma_T^+ \boldsymbol{E} \\ \gamma_T^- \boldsymbol{H} \end{pmatrix}. \end{aligned} \tag{42}$$

In particular, both the trace theorem and the time-harmonic bounds on the Calderón operator in combination with [25, Lemma 2.1] show that $\gamma_T^+ \widehat{\boldsymbol{E}}$ and $\gamma_T^- \widehat{\boldsymbol{H}}$ are elements in $\boldsymbol{H}^{-2}_0(0, T; \mathbf{X}_\Gamma)$. Setting $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ pairwise to zero reduces the weak formulation (36) to

$$[\boldsymbol{\eta}, \gamma_T^+ \boldsymbol{E}]_\Gamma + (\boldsymbol{\eta}, \boldsymbol{a}(\boldsymbol{\varphi} + \gamma_T \boldsymbol{H}^{\mathrm{inc}}))_\Gamma = [\gamma_T \boldsymbol{E}^{\mathrm{inc}}, \boldsymbol{\eta}]_\Gamma, \quad \text{for all } \boldsymbol{\eta} \in \mathbf{V}_\Gamma, \tag{43}$$

$$[\boldsymbol{\xi}, \gamma_T^- \boldsymbol{H}]_\Gamma = 0, \quad \text{for all } \boldsymbol{\xi} \in \mathbf{X}_\Gamma. \tag{44}$$

The second equation implies $\gamma_T^- \boldsymbol{H} = 0$ for all $t \in [0,T]$, which plugged into the integrated Green's formula (10) yields

$$\frac{1}{2}\int_{\Omega^-} |\boldsymbol{E}|^2 + |\boldsymbol{H}|^2 \,\mathrm{d}x = -\partial_t^{-1}\left[\gamma_T^- \boldsymbol{H}, \gamma_T^- \boldsymbol{E}\right]_\Gamma = 0.$$

Consequently, the electromagnetic waves $\boldsymbol{E}$ and $\boldsymbol{H}$ vanish in the inner domain Ω^- and the boundary densities are given by the boundary data of the outer fields, namely (40)-(41) imply $\boldsymbol{\varphi} = \gamma_T^+ \boldsymbol{H}$ and $\boldsymbol{\psi} = -\gamma_T^+ \boldsymbol{E}$. Plugging these identites into (43) shows that $\boldsymbol{E}$ and $\boldsymbol{H}$, restricted to the outer domain Ω^+, are the desired fields which fulfill the nonlinear boundary condition (7).

We turn our attention to the stated bounds, starting with Green's formula (10) on the exterior domain Ω^+, which reads in terms of the boundary densities

$$\frac{1}{2}\left(\|\boldsymbol{E}\|^2_{\boldsymbol{L}^2(\Omega^+)} + \|\boldsymbol{H}\|^2_{\boldsymbol{L}^2(\Omega^+)}\right) = \partial_t^{-1}\left[\gamma_T^+ \boldsymbol{H}, \gamma_T^+ \boldsymbol{E}\right]_\Gamma = \partial_t^{-1}\left[\boldsymbol{\varphi}, \gamma_T^+ \boldsymbol{E}\right]_\Gamma.$$

By testing (43) with $\boldsymbol{\varphi}$ and integrating both sides, we arrive at the term on the right-hand side, in which we insert Green's formula as described above to arrive at

$$\frac{1}{2}\left(\|\boldsymbol{E}(t)\|^2_{\boldsymbol{L}^2(\Omega^+)} + \|\boldsymbol{H}(t)\|^2_{\boldsymbol{L}^2(\Omega^+)}\right) + \int_0^t (\boldsymbol{\varphi}, \boldsymbol{a}(\boldsymbol{\varphi} + \gamma_T \boldsymbol{H}^{\mathrm{inc}}))_\Gamma \,\mathrm{d}t' = \int_0^t [\gamma_T \boldsymbol{E}^{\mathrm{inc}}, \boldsymbol{\varphi}]_\Gamma \,\mathrm{d}t',$$

for all $t \in [0,T]$.

Continuing with the nonlinear term on the left-hand side, we introduce an intermediate term, which is subsequently estimated via the Hölder inequality and the bound (18) on $\boldsymbol{a}$, which yields

$$\begin{aligned}
(\boldsymbol{\varphi}, \boldsymbol{a}(\boldsymbol{\varphi} + \gamma_T \boldsymbol{H}^{\mathrm{inc}}))_\Gamma &= \left\|\boldsymbol{\varphi} + \gamma_T \boldsymbol{H}^{\mathrm{inc}}\right\|^{1+\alpha}_{\boldsymbol{L}^{1+\alpha}(\Gamma)} - (\gamma_T \boldsymbol{H}^{\mathrm{inc}}, \boldsymbol{a}(\boldsymbol{\varphi} + \gamma_T \boldsymbol{H}^{\mathrm{inc}}))_\Gamma \\
&\geq \left\|\boldsymbol{\varphi} + \gamma_T \boldsymbol{H}^{\mathrm{inc}}\right\|^{1+\alpha}_{\boldsymbol{L}^{1+\alpha}(\Gamma)} - \left(\left\|\gamma_T \boldsymbol{H}^{\mathrm{inc}}\right\|_{\boldsymbol{L}^{1+\alpha}(\Gamma)} \left\|\boldsymbol{\varphi} + \gamma_T \boldsymbol{H}^{\mathrm{inc}}\right\|^{\alpha}_{\boldsymbol{L}^{1+\alpha}(\Gamma)}\right) \\
&\geq \left\|\boldsymbol{\varphi} + \gamma_T \boldsymbol{H}^{\mathrm{inc}}\right\|^{1+\alpha}_{\boldsymbol{L}^{1+\alpha}(\Gamma)} - \left(C\left\|\gamma_T \boldsymbol{H}^{\mathrm{inc}}\right\|^{1+\alpha}_{\boldsymbol{L}^{1+\alpha}(\Gamma)} + \frac{1}{2}\left\|\boldsymbol{\varphi} + \gamma_T \boldsymbol{H}^{\mathrm{inc}}\right\|^{1+\alpha}_{\boldsymbol{L}^{1+\alpha}(\Gamma)}\right),
\end{aligned}$$

where the final estimate is obtained by the generalized Young's inequality. Applying the same dual estimate on the resulting right-hand side again and absorbing the term depending on $\boldsymbol{\varphi}$ leads to

$$\begin{aligned}
&\|\boldsymbol{E}\|^2_{\boldsymbol{L}^2(\Omega^+)} + \|\boldsymbol{H}\|^2_{\boldsymbol{L}^2(\Omega^+)} + \int_0^T \|\boldsymbol{\varphi}\|^{1+\alpha}_{\boldsymbol{L}^{1+\alpha}(\Gamma)} \,\mathrm{d}t \\
&\qquad \leq C\int_0^T \left\|\gamma_T \boldsymbol{E}^{\mathrm{inc}}\right\|^{\frac{1+\alpha}{\alpha}}_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)} + \left\|\gamma_T \boldsymbol{H}^{\mathrm{inc}}\right\|^{1+\alpha}_{\boldsymbol{L}^{1+\alpha}(\Gamma)} \,\mathrm{d}t.
\end{aligned}$$

It remains to show the stated bound on $\boldsymbol{\psi}$, which is obtained by inserting $\boldsymbol{\psi} = -\gamma_T^+ \boldsymbol{E}$ into (43), which yields for arbitrary $\boldsymbol{\eta} \in \mathbf{V}_\Gamma$ the estimate

$$\begin{aligned}
[\boldsymbol{\eta}, \boldsymbol{\psi}]_\Gamma &= (\boldsymbol{\eta}, \boldsymbol{a}(\boldsymbol{\varphi} + \gamma_T \boldsymbol{H}^{\mathrm{inc}}))_\Gamma - [\gamma_T \boldsymbol{E}^{\mathrm{inc}}, \boldsymbol{\eta}]_\Gamma \\
&\leq \|\boldsymbol{\eta}\|_{\boldsymbol{L}^{1+\alpha}(\Gamma)}\left(\left\|\boldsymbol{\varphi} + \gamma_T \boldsymbol{H}^{\mathrm{inc}}\right\|^{\alpha}_{\boldsymbol{L}^{1+\alpha}(\Gamma)} + \left\|\gamma_T \boldsymbol{E}^{\mathrm{inc}}\right\|_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)}\right),
\end{aligned}$$

where the Hölder inequality has been used to estimate both summands. The density of $\mathbf{V}_\Gamma \subset \boldsymbol{L}^{1+\alpha}(\Gamma)$ finally implies

$$\|\boldsymbol{\psi}\|_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)} = \sup_{\boldsymbol{\eta}\in\mathbf{V}_\Gamma} \frac{[\boldsymbol{\eta},\boldsymbol{\psi}]_\Gamma}{\|\boldsymbol{\eta}\|_{\boldsymbol{L}^{1+\alpha}(\Gamma)}} \leq \left\|\boldsymbol{\varphi} + \gamma_T \boldsymbol{H}^{\mathrm{inc}}\right\|^{\alpha}_{\boldsymbol{L}^{1+\alpha}(\Gamma)} + \left\|\gamma_T \boldsymbol{E}^{\mathrm{inc}}\right\|_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)}.$$

Taking both sides to the power of $\frac{1+\alpha}{\alpha}$ yields the stated result. □

4. Semi-discretization in time by Runge–Kutta convolution quadrature

4.1. Runge-Kutta convolution quadrature.

A Runge-Kutta method with m-stages is uniquely determined by its coefficients, which are collected in the *Butcher tableau*

$$\mathscr{A} = (a_{ij})_{i,j=1}^m, \quad \boldsymbol{b} = (b_1, \dots, b_m)^T, \quad \text{and} \quad \boldsymbol{c} = (c_1, \dots, c_m)^T.$$

The stability function of the Runge–Kutta method is given by $R(z) = 1 + zb^T(I - z\mathscr{A})^{-1}\mathbb{1}$, where $\mathbb{1} = (1, 1, \dots, 1)^T \in \mathbb{R}^m$. For more details about Runge-Kutta methods in general we refer to [23].

Runge–Kutta methods have been used extensively to construct convolution quadrature methods c.f. [26, 5, 7, 10, 4, 11, 6]. Crucial for the treatment of wave propagation and scattering problems is the A-stability of the underlying time scheme, which prohibits the use of multistep methods of order larger than 2.

A-stable Runge-Kutta methods of arbitrary order yield effective convolution quadrature schemes, which often outperform their counterparts based on multistep methods [3].

The *Runge–Kutta differentiation symbol* is defined by:

$$\Delta(\zeta) = \left(\mathscr{A} + \frac{\zeta}{1-\zeta}\mathbb{1}\boldsymbol{b}^T\right)^{-1} \in \mathbb{C}^{m\times m}, \qquad \zeta \in \mathbb{C} \text{ with } |\zeta| < 1. \tag{45}$$

This expression is well-defined for $|\zeta| < 1$ if $R(\infty) = 1 - \boldsymbol{b}^T\mathscr{A}^{-1}\mathbb{1}$ satisfies $|R(\infty)| \leq 1$. In fact, the Sherman–Morrison formula yields for RadauIIA methods (see, e.g., [23, Section IV.5])

$$\Delta(\zeta) = \mathscr{A}^{-1} - \frac{\zeta}{1 - R(\infty)\zeta}\mathscr{A}^{-1}\mathbb{1}\boldsymbol{b}^T\mathscr{A}^{-1} = \mathscr{A}^{-1}(\boldsymbol{I}_m - \zeta\mathbb{1}\boldsymbol{e}_m^T),$$

with $\boldsymbol{e}_m^T = (0, ..., 1) \in \mathbb{R}^m$ and $\boldsymbol{I}_m \in \mathbb{R}^{m\times m}$ denoting the identity matrix. The Runge–Kutta convolution quadrature weights are operators $\mathbf{W}_n(K) : V^m \to (V')^m$ defined by formally replacing the argument s in $L(s)$ by the Runge–Kutta differentiation symbol $\Delta(\zeta)/\tau$, and then expand the operator-valued matrix function to the power series

$$L\Big(\frac{\Delta(\zeta)}{\tau}\Big) = \sum_{n=0}^\infty \mathbf{W}_n(L)\zeta^n.$$

The convolution quadrature approximation of the temporal operator $L(\partial_t)$ is then given by the discrete convolution

$$\big(L(\partial_t^\tau)\boldsymbol{g}\big)^n = \sum_{j=0}^n \mathbf{W}_{n-j}(L)\boldsymbol{g}^j$$

for any sequence $\boldsymbol{g} = (\boldsymbol{g}^n)_{n\in\mathbb{N}} \in V^m$.

The sequences often arise from function values, for which we introduce the following notation. Let $g : [0, T] \to V$ be a time-dependent function, then we denote the vector of the evaluations at the stages by $\boldsymbol{g}_n = \big(g(t_n + c_i\tau)\big)_{i=1}^m$.

Generally, we will associate sequences with functions whenever notationally convenient, where sequences often are denoted by an additional superscript τ.

In particular, if $c_s = 1$, as is the case with Radau IIA methods [23, Section IV.5], then the continuous convolution at t_{n+1} is approximated by the last component of the discrete block convolution:

$$\big(L(\partial_t)g\big)(t_{n+1}) \approx \mathbf{e}_m^T \big(L(\partial_t^\tau)\boldsymbol{g}\big)^n,$$

where $\mathbf{e}_m = (0, \dots, 0, 1)^T \in \mathbb{R}^m$ is the m-th unit vector.

The following convolution quadrature approximation result from [7, Theorem 3], formulated for the stages of the Radau IIA method in [6, Theorem 4.2], yields efficient bounds for temporal defects originating from the employed time discretization.

Proposition 1. [7, Theorem 3] *Let L satisfy (21) and consider the Runge–Kutta convolution quadrature based on the Radau IIA method with $m \geq 2$ stages. Let $r > \max(m + 1 + \kappa, m + 1)$ and $g \in C^r([0, T], V)$ satisfy $g(0) = g'(0) = ... = g^{(r-1)}(0) = 0$. Then, there exists a $\tau_0 > 0$, such that for $0 < \tau \leq \tau_0$ and $t_n = n\tau \in [0, T]$ the following error bound holds:*

$$\begin{aligned}
&\big\| (L(\partial_t^\tau)\boldsymbol{g})^n - (L(\partial_t)g(t_n + c_i\tau))_{i=1}^m \big\| \\
&\qquad \leq C\tau^{\min(m+1, m+1-\kappa+\nu)} \left(\left\| g^{(r)}(0) \right\| + \int_0^t \left\| g^{(r+1)}(\lambda) \right\| \mathrm{d}\lambda \right).
\end{aligned}$$

This section transports results from the time continuous domain onto the time discrete regime. A useful tool for that purpose is the use of generating functions, which we introduce in the following. Let $\Phi_n \in V^m$ for all $n \in \mathbb{N}$ denote a sequence with finite support and m components in the Banach space V. Let furthermore $\sigma > 0$ be a real, constant value and $\rho = e^{-\sigma\tau}$ a weight which converges against one for $N \to \infty$. Operating on the complex contour $\mathbb{S}_\rho = \{z \in \mathbb{C} \mid |z| = \rho\}$, the generating function is denoted by an additional hat $\widehat{\Phi} : \mathbb{S}_\rho \to V^m$ and defined by the expression

$$\widehat{\Phi} : \; \zeta \mapsto \sum_{n=0}^{\infty} \Phi_n \, \zeta^n.$$

Bilinear forms are extended to $\boldsymbol{V}^m \times \boldsymbol{V}^m$ by weighting with the diagonal weight matrix $\mathscr{B} = \mathrm{diag}(\boldsymbol{b}_1, ..., \boldsymbol{b}_m)$, which yields for the scalar product $\cdot$ installed on $\boldsymbol{V}$ the extended definition

$$\boldsymbol{u} \cdot \boldsymbol{v} = (\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{b}} = \overline{\boldsymbol{u}}^T \mathscr{B} \boldsymbol{v} = \sum_{i=1}^m \boldsymbol{b}_i \boldsymbol{u}_i \cdot \boldsymbol{v}_i \qquad \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}^m,$$

where $\cdot$ on the right-hand side denotes the underlying dot product on $\boldsymbol{V}$ (which might be vector valued, for instance when $\boldsymbol{V} = \boldsymbol{L}^2(\Omega)$). In the same way, we extend the skew symmetric pairing (9). With regards to this positive bilinear form, the following result holds.

Lemma 5. *(Discrete partial integration) Let $(\boldsymbol{u}_n)_{i=1}^m$ and $(\boldsymbol{v}_n)_{i=1}^m$ be vector-valued sequences in $\mathbb{R}^m$ and consider the $m-$ stage RadauIIA Runge-Kutta method. For any $\epsilon > 0$, there exists a positive constant C independent of τ,$\boldsymbol{u}$ and $\boldsymbol{v}$ such that the following estimate holds*

$$\sum_{n=0}^{\infty} \rho^n \boldsymbol{u}_n \cdot \boldsymbol{v}_n \leq \sum_{n=0}^{\infty} \epsilon \rho^n \left| \left((\partial_t^\tau)^{-1} \boldsymbol{u} \right)_n \right|^2 + C\rho^n \left| (\partial_t^\tau \boldsymbol{v})_n \right|^2 .$$

Here, ∂_t^τ and $(\partial_t^\tau)^{-1}$ denote the convolution quadrature discretization of the temporal convolutions $L(\partial_t)$ corresponding to $L(s) = s$ and $L(s) = s^{-1}$, respectively.

Proof. We start by applying Parseval's theorem to the left-hand side of the stated bound, where we denote the integral contour containing all complex values with absolute value ρ by S_ρ, which yields for all positive $\widetilde{\epsilon}$

$$\begin{aligned}
\sum_{n=0}^{\infty} \rho^n \left(\boldsymbol{u}^n, \boldsymbol{v}^n \right)_{\boldsymbol{b}} &= \int_{S_\rho} \left(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{v}} \right)_{\boldsymbol{b}} \,\mathrm{d}\zeta \\
&= \int_{S_\rho} \left(\overline{\Delta(\zeta)^{-1} \widehat{\boldsymbol{u}}} \right)^T \left(\Delta(\overline{\zeta})^T \mathscr{B} \Delta(\zeta)^{-1} \Delta(\zeta) \widehat{\boldsymbol{v}} \right) \mathrm{d}\zeta \\
&\leq \int_{S_\rho} \left| \Delta(\overline{\zeta})^T \mathscr{B} \Delta(\zeta)^{-1} \right| \left(\frac{\widetilde{\epsilon}}{2} \left| \tau \Delta(\zeta)^{-1} \widehat{\boldsymbol{u}} \right|^2 + \frac{1}{2\widetilde{\epsilon}} \left| \frac{\Delta(\zeta)}{\tau} \widehat{\boldsymbol{v}} \right|^2 \right) \mathrm{d}\zeta,
\end{aligned}$$

where the last inequality holds for all $\widetilde{\epsilon} > 0$. The matrix appearing in the additional factor consisting of the matrix norm is bounded by applying the Sherman-Morrison formula and the triangle inequality, which yields

$$\begin{aligned}
\left| \Delta(\overline{\zeta})^T \mathscr{B} \Delta(\zeta)^{-1} \right| &= \left| \left(\boldsymbol{I}_m - \overline{\zeta}\, \boldsymbol{e}_m \, \mathbb{1}^T \right) \mathscr{A}^{-T} \mathscr{B} \left(\mathscr{A} + \frac{\zeta}{1-\zeta} \mathbb{1} \boldsymbol{b}^T \right) \right| \\
&\leq (1 + m^{1/2}) \left| \mathscr{A}^{-T} \mathscr{B} \mathscr{A} \right| + \frac{\left| \left(\boldsymbol{I}_m - \overline{\zeta}\, \boldsymbol{e}_m \, \mathbb{1}^T \right) \mathscr{A}^{-T} \boldsymbol{b} \boldsymbol{b}^T \right|}{|1-\zeta|}.
\end{aligned}$$

The first summand is independent of ζ and bounded. To estimate the second summand, we exploit $\boldsymbol{c}_m = 1$ and consequently $\mathscr{A}^{-T} \boldsymbol{b} = \boldsymbol{e}_m$, which implies

$$\frac{\left| \left(\boldsymbol{I}_m - \overline{\zeta}\, \boldsymbol{e}_m \, \mathbb{1}^T \right) \mathscr{A}^{-T} \boldsymbol{b} \boldsymbol{b}^T \right|}{|1-\zeta|} = \frac{\left| \left(\boldsymbol{e}_m - \overline{\zeta}\, \boldsymbol{e}_m \, \mathbb{1}^T \boldsymbol{e}_m \right) \boldsymbol{b}^T \right|}{|1-\zeta|} = \frac{\left| \left(\boldsymbol{e}_m - \overline{\zeta}\, \boldsymbol{e}_m \right) \boldsymbol{b}^T \right|}{|1-\zeta|} = \left| \boldsymbol{e}_m \boldsymbol{b}^T \right|.$$

The given statement is finally deduced by applying the estimate in the inequality above to obtain a constant C only depending on m and ϵ, such that

$$\begin{aligned}
\sum_{n=0}^{\infty} \rho^n \left(\boldsymbol{u}^n, \boldsymbol{v}^n \right)_{\boldsymbol{b}} &\leq \int_{S_\rho} \epsilon \left| \tau \Delta(\zeta)^{-1} \widehat{\boldsymbol{u}} \right|^2 + C \left| \frac{\Delta(\zeta)}{\tau} \widehat{\boldsymbol{v}} \right|^2 \mathrm{d}\theta \\
&= \sum_{n=0}^{\infty} \epsilon \rho^n \left| \left((\partial_t^\tau)^{-1} \boldsymbol{u} \right)_n \right|^2 + C\rho^n \left| (\partial_t^\tau \boldsymbol{v})_n \right|^2 .
\end{aligned}$$

□

The convolution quadrature scheme based on RaudauIIA- methods preserves central properties of the temporal operator $L(\partial_t)$, which is of particular importance to establish a discrete coercivity property, as described in the dedicated paper [6, Theorem 3.1]. The following Lemma restates this result for the standard discrete differential operator ∂_t^τ, which is sufficient for the subsequent error analysis.

Lemma 6. *The convolution quadrature discretization based on the two-stage Radau IIA method fulfills*

$$\tau \sum_{n=0}^{\infty} e^{-2n\tau/T} \langle f_n, (\partial_t^\tau f)_n \rangle \geq \frac{\tau}{2T} \sum_{n=0}^{\infty} e^{-2n\tau/T} \|f_n\|_V^2 ,$$

for every sequence $(f_n)_{n\geq 0}$ in V with finitely many non-zero entries. Furthermore, for $m > 2$ stages the left-hand side remains positive, namely

$$\sum_{n=0}^{\infty} e^{-2n\tau/T} \langle f_n, (\partial_t^\tau f)_n \rangle \geq 0.$$

The anti-duality $\langle \cdot, \cdot \rangle$ on V in both statements is extended to V^m with the weight matrix $\mathscr{B}$.

4.2. **Auxiliary result: Time-discrete transmission problem.** The following Lemma describes a discrete variant of the continuous transmission problem (38)–(39) and relates an arbitrary sequence in the trace space ${\mathbf{X}_\Gamma}^2$ with the solution to a corresponding transmission problem.

Lemma 7. *Let $\boldsymbol{\varphi}_n \in {\mathbf{X}_\Gamma}^m$ and $\boldsymbol{\psi}_n \in {\mathbf{X}_\Gamma}^m$ for $n \in \mathbb{N}$ denote sequences. We define fields $\boldsymbol{E}_n^\tau, \boldsymbol{H}_n^\tau \in \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)^m$ for $n \in \mathbb{N}$ by applying the discrete representation formulas to the sequences via*

$$\boldsymbol{E}^\tau = -\boldsymbol{\mathcal{S}}(\partial_t^\tau)\boldsymbol{\varphi} + \boldsymbol{\mathcal{D}}(\partial_t^\tau)\boldsymbol{\psi}, \tag{46}$$

$$\boldsymbol{H}^\tau = -\boldsymbol{\mathcal{D}}(\partial_t^\tau)\boldsymbol{\varphi} - \boldsymbol{\mathcal{S}}(\partial_t^\tau)\boldsymbol{\psi}. \tag{47}$$

These fields are exact solutions to the following discrete transmission problem:

$$\partial_t^\tau \boldsymbol{E}^\tau - \mathbf{curl}\, \boldsymbol{H}^\tau = 0 \quad in \quad \mathbb{R}^3 \setminus \Gamma, \tag{48}$$

$$\partial_t^\tau \boldsymbol{H}^\tau + \mathbf{curl}\, \boldsymbol{E}^\tau = 0 \quad in \quad \mathbb{R}^3 \setminus \Gamma, \tag{49}$$

$$[\gamma_T]\boldsymbol{H}^\tau = \boldsymbol{\varphi}, \tag{50}$$

$$-[\gamma_T]\boldsymbol{E}^\tau = \boldsymbol{\psi}. \tag{51}$$

Proof. The generating function of the representation formula yields formulas for the generating functions $\widehat{\boldsymbol{E}}^\tau(\zeta)$ and $\widehat{\boldsymbol{H}}^\tau(\zeta)$, which are of the form

$$\widehat{\boldsymbol{E}}^\tau(\zeta) = -\boldsymbol{\mathcal{S}}\left(\frac{\Delta(\zeta)}{\tau}\right)\widehat{\boldsymbol{\varphi}}(\zeta) + \boldsymbol{\mathcal{D}}\left(\frac{\Delta(\zeta)}{\tau}\right)\widehat{\boldsymbol{\psi}}(\zeta).$$

The construction and jump conditions of the time-harmonic potential operators $\boldsymbol{\mathcal{S}}(s)$ and $\boldsymbol{\mathcal{D}}(s)$ yield the following time-harmonic transmission problem for the generating functions

$$\begin{aligned}
\frac{\Delta(\zeta)}{\tau}\widehat{\boldsymbol{E}}^\tau(\zeta) - \mathbf{curl}\, \widehat{\boldsymbol{H}}^\tau(\zeta) &= 0 \qquad \text{in} \quad \mathbb{R}^3 \setminus \Gamma, \\
\frac{\Delta(\zeta)}{\tau}\widehat{\boldsymbol{H}}^\tau(\zeta) + \mathbf{curl}\, \widehat{\boldsymbol{E}}^\tau(\zeta) &= 0 \qquad \text{in} \quad \mathbb{R}^3 \setminus \Gamma, \\
[\gamma_T]\widehat{\boldsymbol{H}}^\tau(\zeta) &= \widehat{\boldsymbol{\varphi}}(\zeta) \quad \text{on} \quad \Gamma, \\
-[\gamma_T]\widehat{\boldsymbol{E}}^\tau(\zeta) &= \widehat{\boldsymbol{\psi}}(\zeta) \quad \text{on} \quad \Gamma.
\end{aligned}$$

Coefficient comparison now yields the result as stated. $\square$

The well posedness result of Proposition 1 was enabled through the jump conditions of $\boldsymbol{C}(\partial_t)$, which is a natural property to transport to the time-discrete Calderón operator $\boldsymbol{C}(\partial_t^\tau)$. The following Lemma provides such a statement and proves it, again by making use of generating functions and time-harmonic identities.

Lemma 8. *Let $\boldsymbol{C}(\partial_t^\tau)$ denote the convolution quadrature approximation of the time-dependent Calderón operator $\boldsymbol{C}(\partial_t)$, based on the Radau IIA method with m-stages. Let furthermore $\boldsymbol{\varphi}_n \in \mathbf{X}_\Gamma{}^m$ and $\boldsymbol{\psi}_n \in \mathbf{X}_\Gamma{}^m$ for $n \in \mathbb{N}$ denote arbitrary sequences, with associated discrete fields $\boldsymbol{E}^\tau$ and $\boldsymbol{H}^\tau$ with support on $\mathbb{R}^3 \setminus \Gamma$, defined via* (46)–(47)*. The Calderón operator then fulfills the jump conditions*

$$\boldsymbol{C}(\partial_t^\tau)\begin{pmatrix}\boldsymbol{\varphi}\\ \boldsymbol{\psi}\end{pmatrix} = \begin{pmatrix}\{\gamma_T \boldsymbol{E}^\tau\}\\ \{\gamma_T \boldsymbol{H}^\tau\}\end{pmatrix}.$$

Furthermore, for all $n \geq 0$ the weak form, derived by applying the tested anti-symmetric pairing to the Calderón operator, fulfills the identity

$$\left[\begin{pmatrix}\boldsymbol{\varphi}_n\\ \boldsymbol{\psi}_n\end{pmatrix}, \left[\boldsymbol{C}(\partial_t^\tau)\begin{pmatrix}\boldsymbol{\varphi}\\ \boldsymbol{\psi}\end{pmatrix}\right]^n\right]_\Gamma = \int_{\mathbb{R}^3\setminus\Gamma} \boldsymbol{E}_n^\tau \cdot (\partial_t^\tau \boldsymbol{E}^\tau)_n + (\partial_t^\tau \boldsymbol{H}^\tau)_n \cdot \boldsymbol{H}_n^\tau \,\mathrm{d}\boldsymbol{x}. \tag{52}$$

Proof. The generating function of the sequence on the left-hand side gives the stated jump conditions by employing the time-harmonic jump conditions of the Calderón operator, which gives

$$\left(\widehat{\boldsymbol{C}(\partial_t^\tau)\begin{pmatrix}\boldsymbol{\varphi}\\ \boldsymbol{\psi}\end{pmatrix}}\right) = \boldsymbol{C}\left(\frac{\Delta(\zeta)}{\tau}\right)\begin{pmatrix}\widehat{\boldsymbol{\varphi}}(\zeta)\\ \widehat{\boldsymbol{\psi}}(\zeta)\end{pmatrix} = \begin{pmatrix}\{\gamma_T \widehat{\boldsymbol{E}}^\tau\}\\ \{\gamma_T \widehat{\boldsymbol{H}}^\tau\}\end{pmatrix}.$$

Inserting the discrete jump conditions into the left-hand side of (52) and applying the relations of the boundary densities and their respective fields, as given by (50)–(51), yields

$$\begin{aligned}\left[\begin{pmatrix}\boldsymbol{\varphi}_n\\ \boldsymbol{\psi}_n\end{pmatrix}, \left[\boldsymbol{C}(\partial_t^\tau)\begin{pmatrix}\boldsymbol{\varphi}^\tau\\ \boldsymbol{\psi}^\tau\end{pmatrix}\right]^n\right]_\Gamma &= \left[\begin{pmatrix}[\gamma_T]\boldsymbol{H}_n^\tau\\ -[\gamma_T]\boldsymbol{E}_n^\tau\end{pmatrix}, \begin{pmatrix}\{\gamma_T\}\boldsymbol{E}_n^\tau\\ \{\gamma_T\}\boldsymbol{H}_n^\tau\end{pmatrix}\right]_\Gamma \\ &= [\gamma_T^+ \boldsymbol{H}_n^\tau, \gamma_T^+ \boldsymbol{E}_n^\tau]_\Gamma - [\gamma_T^- \boldsymbol{H}_n^\tau, \gamma_T^- \boldsymbol{E}_n^\tau]_\Gamma.\end{aligned}$$

Finally, applying Green's formula (8) and inserting the discretized Maxwell's equations (48)–(49) completes the proof by

$$\begin{aligned}[\gamma_T^+ \boldsymbol{H}_n^\tau, \gamma_T^+ \boldsymbol{E}_n^\tau]_\Gamma - [\gamma_T^- \boldsymbol{H}_n^\tau, \gamma_T^- \boldsymbol{E}_n^\tau]_\Gamma &= \int_{\mathbb{R}^3\setminus\Gamma} \boldsymbol{E}_n^\tau \cdot \mathbf{curl}\, \boldsymbol{H}_n^\tau - \mathbf{curl}\, \boldsymbol{E}_n^\tau \cdot \boldsymbol{H}_n^\tau \,\mathrm{d}\boldsymbol{x} \\ &= \int_{\mathbb{R}^3\setminus\Gamma} \boldsymbol{E}_n^\tau \cdot (\partial_t^\tau \boldsymbol{E}^\tau)_n + (\partial_t^\tau \boldsymbol{H}^\tau)_n \cdot \boldsymbol{H}_n^\tau \,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

□

Remark 4.1. *Of particular importance is the combination of this result with Lemma 6, which implies a coercivity for $m = 2$. For $m > 2$ no such result is known, but by introducing a positive shift σ and setting $\widetilde{\boldsymbol{C}}(s) = \boldsymbol{C}(s + \sigma)$, as it appears in the shifted boundary integral equation* (37)*, one obtains*

$$\begin{aligned}\left[\begin{pmatrix}\boldsymbol{\varphi}_n\\ \boldsymbol{\psi}_n\end{pmatrix}, \left[\widetilde{\boldsymbol{C}}(\partial_t^\tau)\begin{pmatrix}\boldsymbol{\varphi}\\ \boldsymbol{\psi}\end{pmatrix}\right]^n\right]_\Gamma = \int_{\mathbb{R}^3\setminus\Gamma} &\boldsymbol{E}_n^\tau \cdot (\partial_t^\tau \boldsymbol{E}^\tau)_n + \sigma\,|\boldsymbol{E}_n^\tau|^2 \\ &+ (\partial_t^\tau \boldsymbol{H}^\tau)_n \cdot \boldsymbol{H}_n^\tau + \sigma\,|\boldsymbol{H}_n^\tau|^2 \,\mathrm{d}\boldsymbol{x},\end{aligned}$$

thus recovering the crucial positivity of the discrete Calderón operator (in combination with the second part of Lemma 6). Numerical results indicate that such a shift is not necessary, but the subsequent error analysis depends on such a positivity result.

4.3. **Convolution quadrature for the nonlinear boundary integral equation.** Discretizing the temporal Calderón operator in the boundary integral equation (36) with Runge–Kutta based convolution quadrature yields the following semi-discrete scheme.

Time-discrete boundary integral equation: *Find* $(\boldsymbol{\varphi}^n, \boldsymbol{\psi}^n) = (\boldsymbol{\varphi}_i^n, \boldsymbol{\psi}_i^n)_{i=1}^m \in {\mathbf{V}_\Gamma}^m \times {\mathbf{X}_\Gamma}^m$, *such that for all* $(\boldsymbol{\eta}, \boldsymbol{\xi}) \in \mathbf{V}_\Gamma \times \mathbf{X}_\Gamma$ *and* $n \leq N$ *it holds*

$$\left[\begin{pmatrix}\boldsymbol{\eta}\\ \boldsymbol{\xi}\end{pmatrix}, \left[\boldsymbol{C}_{\mathrm{imp}}(\partial_t^\tau)\begin{pmatrix}\boldsymbol{\varphi}^\tau\\ \boldsymbol{\psi}^\tau\end{pmatrix}\right]^n\right]_\Gamma + \left(\boldsymbol{\eta}, \boldsymbol{a}\left(\boldsymbol{\varphi}_n^\tau + \gamma_T \boldsymbol{H}_n^{\mathrm{inc}}\right)\right)_\Gamma = [\gamma_T \boldsymbol{E}_n^{\mathrm{inc}}, \boldsymbol{\eta}]_\Gamma. \tag{53}$$

The numerical solution can then be evaluated by

$$\boldsymbol{E}^\tau = -\boldsymbol{\mathcal{S}}(\partial_t^\tau)\boldsymbol{\varphi}^\tau + \boldsymbol{\mathcal{D}}(\partial_t^\tau)\boldsymbol{\psi}^\tau, \tag{54}$$

$$\boldsymbol{H}^\tau = -\boldsymbol{\mathcal{D}}(\partial_t^\tau)\boldsymbol{\varphi}^\tau - \boldsymbol{\mathcal{S}}(\partial_t^\tau)\boldsymbol{\psi}^\tau. \tag{55}$$

To limit the extent of the present paper we abstain from giving a convergence analysis of the semi-discretization and directly move on to the space discretization. The proof of error bounds for the full discretization from Theorem 2 is readily reduced to the semi-discretization and predicts convergence rates of order m in the norms that are stated there.

5. Full discretization

We turn our attention to the development and analysis of fully discrete schemes. To achieve this, we start by an introduction of the Galerkin space discretization, suitable for the problem at hand.

5.1. **Boundary element method.** Restricting the time-discrete boundary integral equation (53) to finite dimensional subspaces $\boldsymbol{V}_h \subset \mathbf{V}_\Gamma$ and $\boldsymbol{X}_h \subset \mathbf{X}_\Gamma$, corresponding to piecewise polynomials defined on a family of triangulations with mesh width h, reveals the full discretization. As boundary element spaces, we employ Raviart–Thomas elements of order $k \geq 0$ (c.f. [31]) for the discretization of $\boldsymbol{V}_h$ and $\boldsymbol{X}_h$ respectively, which are defined on the unit triangle $\widehat{K}$ as reference element by

$$\mathrm{RT}_k(\widehat{K}) = \left\{\boldsymbol{x} \mapsto \boldsymbol{p}_1(\boldsymbol{x}) + p_2(\boldsymbol{x})\boldsymbol{x} \,:\, \boldsymbol{p}_1 \in P_k(\widehat{K})^2,\ p_2 \in P_k(\widehat{K})\right\},$$

where $P_k(\widehat{K})$ is the polynomial space of degree k on $\widehat{K}$. This definition naturally extends to arbitrary grids by piecewise pull-back to the reference element.

The following approximation result holds for Raviart-Thomas elements and are obtained in this form from the results collected in Lemma 14 and Theorem 15 of [15]; see also the original references [12, Section III.3.3] and [13]. Here, we use the same notation $\boldsymbol{H}_\times^p(\Gamma) = \gamma_T \boldsymbol{H}^{p+1/2}(\Omega)$ for boundary data of higher regularity, as in [15].

Lemma 9. *Let* $\boldsymbol{X}_h = \boldsymbol{V}_h$ *be the k-th order Raviart–Thomas boundary element space on* Γ. *For every* $\boldsymbol{\xi} \in \boldsymbol{H}_\times^{k+1}(\Gamma)$ *the best-approximation error is bounded by*

$$\inf_{\boldsymbol{\xi}_h \in \boldsymbol{X}_h} \|\boldsymbol{\xi}_h - \boldsymbol{\xi}\|_{\mathbf{X}_\Gamma} + h^{1/2}\|\boldsymbol{\xi}_h - \boldsymbol{\xi}\|_{\boldsymbol{L}^2(\Gamma)} \leq Ch^{k+3/2}\|\boldsymbol{\xi}\|_{\boldsymbol{H}_\times^{k+1}(\Gamma)}.$$

The $\boldsymbol{L}^{1+\alpha}(\Gamma)$ norm naturally arises in estimates derived from the boundary integral equation (e.g. in Proposition 2 and Theorem 1). On bounded domains, as is the case for Γ, this norm is estimated by the $\boldsymbol{L}^2(\Gamma)$ norm as a consequence of the Hölder inequality with $q = \frac{2}{1+\alpha}$ and $p = \frac{2}{1-\alpha}$, which implies

$$\|\boldsymbol{u}\|^{1+\alpha}_{\boldsymbol{L}^{1+\alpha}(\Gamma)} = \int_\Gamma |\boldsymbol{u}|^{1+\alpha}\,\mathrm{d}\boldsymbol{x} \le \|1\|_{L^{\frac{2}{1-\alpha}}(\Gamma)} \left\| |\boldsymbol{u}|^{1+\alpha} \right\|_{L^{\frac{2}{1+\alpha}}(\Gamma)} = C_{\alpha,\Gamma} \|\boldsymbol{u}\|^{1+\alpha}_{\boldsymbol{L}^2(\Gamma)}. \tag{56}$$

5.2. **Fully discretized boundary integral equation.** The fully discrete scheme of the boundary integral equation (36) is now given by combining the convolution quadrature method, used to discretize the temporal Calderón operator, with the boundary element method.

Full discretization of the boundary integral equation: *Find $\boldsymbol{\varphi}_h^\tau, \boldsymbol{\psi}_h^\tau \in \boldsymbol{H}_0^k(0,T;\boldsymbol{V}_h \times \boldsymbol{X}_h)$, such that for all $(\boldsymbol{\eta}_h, \boldsymbol{\xi}_h) \in \boldsymbol{V}_h \times \boldsymbol{X}_h$ and $n \le N$ the following scheme holds*

$$\left[\begin{pmatrix} \boldsymbol{\eta}_h \\ \boldsymbol{\xi}_h \end{pmatrix}, \left[\boldsymbol{C}_{\mathrm{imp}}(\partial_t^\tau) \begin{pmatrix} \boldsymbol{\varphi}_h^\tau \\ \boldsymbol{\psi}_h^\tau \end{pmatrix} \right]^n \right]_\Gamma + \left(\boldsymbol{\eta}_h, \boldsymbol{a}\left(\boldsymbol{\varphi}_h^n + \gamma_T \boldsymbol{H}_n^{\mathrm{inc}} \right) \right)_\Gamma = [\gamma_T \boldsymbol{E}_n^{\mathrm{inc}}, \boldsymbol{\eta}_h]_\Gamma. \tag{57}$$

The remaining part of this section is devoted to the derivation of error bounds of this scheme.

Our investigations into the errors of fully discrete solutions start with a stability result, which is desirable on its own but in particular takes a central role in the subsequent derivation of rate specific error bounds.

5.3. **Full discretization: Unconditional bounds on the numerical solution.** The following Proposition bounds the numerical solution in terms of the incidental waves, without making any assumptions on the regularity of the exact solution.

Proposition 2. *Consider $\boldsymbol{\varphi}_n^{\tau,h} \in \boldsymbol{V}_h^m$ for all $n \ge 0$, solution to the fully discrete scheme (57), where Radau IIA based Runge-Kutta convolution quadrature with m-stages in time and arbitrary boundary element spaces $\boldsymbol{V}_h$ and $\boldsymbol{X}_h$ have been employed. Then, the numerical solution is bounded by*

$$\sum_{n=0}^N \left\| \boldsymbol{\varphi}_n^{\tau,h} \right\|^{1+\alpha}_{\boldsymbol{L}^{1+\alpha}(\Gamma)} \le C \sum_{n=0}^N \left\| \gamma_T \boldsymbol{E}_n^{inc} \right\|^{\frac{1+\alpha}{\alpha}}_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)} + \left\| \gamma_T \boldsymbol{H}_n^{\mathrm{inc}} \right\|^{1+\alpha}_{\boldsymbol{L}^{1+\alpha}(\Gamma)},$$

where the constant C depends only on α.

Proof. Starting from weighted summation of the discretized scheme tested with the numerical solution $(\boldsymbol{\varphi}^{\tau,h}, \boldsymbol{\psi}^{\tau,h})$, completed for $n > N$ by enforcing the scheme with incidental waves $\boldsymbol{H}^{\mathrm{inc}}$ extended by 0, we obtain

$$\begin{aligned} &\sum_{n=0}^N e^{-\sigma n\tau} \left(\left[\begin{pmatrix} \boldsymbol{\varphi}_n^{\tau,h} \\ \boldsymbol{\psi}_n^{\tau,h} \end{pmatrix}, \left[\boldsymbol{C}_{\mathrm{imp}}(\partial_t^\tau) \begin{pmatrix} \boldsymbol{\varphi}_h^\tau \\ \boldsymbol{\psi}_h^\tau \end{pmatrix} \right]^n \right]_\Gamma + \left(\boldsymbol{\varphi}_n^{\tau,h}, \boldsymbol{a}\left(\boldsymbol{\varphi}_n^{\tau,h} + \gamma_T \boldsymbol{H}_n^{\mathrm{inc}} \right) \right)_\Gamma \right) \\ &\quad = \sum_{n=0}^N e^{-\sigma n\tau} [\gamma_T \boldsymbol{E}_n^{\mathrm{inc}}, \boldsymbol{\varphi}_n^\tau]_\Gamma. \end{aligned} \tag{58}$$

The positivity of the time-discrete operator $\boldsymbol{C}_{\mathrm{imp}}(\partial_t^\tau)$, seen as a direct consequence of Lemma 6 applied to the right-hand side of (52), implies

$$\sum_{n=0}^N e^{-\sigma n\tau} \left[\begin{pmatrix} \boldsymbol{\varphi}_n^{\tau,h} \\ \boldsymbol{\psi}_n^{\tau,h} \end{pmatrix}, \left[\boldsymbol{C}_{\mathrm{imp}}(\partial_t^\tau) \begin{pmatrix} \boldsymbol{\varphi}_n^{\tau,h} \\ \boldsymbol{\psi}_n^{\tau,h} \end{pmatrix} \right]^n \right]_\Gamma \ge 0.$$

The corresponding term in the expression (58) is therefore neglected, to bound the remaining term on the left-hand side by the right-hand side. Rewriting the summands of the second term yields

$$\left(\boldsymbol{\varphi}_n^{\tau,h}, \boldsymbol{a}\left(\boldsymbol{\varphi}_n^{\tau,h}+\gamma_T\boldsymbol{H}_n^{\mathrm{inc}}\right)\right)_\Gamma = \left\|\boldsymbol{\varphi}_n^{\tau,h}+\gamma_T\boldsymbol{H}_n^{\mathrm{inc}}\right\|_{\boldsymbol{L}^{1+\alpha}(\Gamma)}^{1+\alpha} - \left(\gamma_T\boldsymbol{H}_n^{\mathrm{inc}}, \boldsymbol{a}\left(\boldsymbol{\varphi}_n^{\tau,h}+\gamma_T\boldsymbol{H}_n^{\mathrm{inc}}\right)\right)_\Gamma .$$

Rearranging gives a constant $C>0$, such that the following intermediate inequality holds

$$\begin{aligned}\sum_{n=0}^{N} e^{-\sigma n\tau}\left\|\boldsymbol{\varphi}_n^{\tau,h}+\gamma_T\boldsymbol{H}_n^{\mathrm{inc}}\right\|_{L^{1+\alpha}(\Gamma)}^{1+\alpha} \leq & C\sum_{n=0}^{N} e^{-\sigma n\tau}[\gamma_T\boldsymbol{E}_n^{\mathrm{inc}},\boldsymbol{\varphi}_n^{\tau,h}]_\Gamma \\ & + C\sum_{n=0}^{N} e^{-\sigma n\tau}\left(\gamma_T\boldsymbol{H}_n^{\mathrm{inc}}, \boldsymbol{a}\left(\boldsymbol{\varphi}_n^{\tau,h}+\gamma_T\boldsymbol{H}_n^{\mathrm{inc}}\right)\right)_\Gamma .\end{aligned}$$

The stability bound is obtained by estimating the terms on the right-hand side by subsequently applying the Hölderlin inequality and Young's inequality. We start with the first term, which is estimated for all $n\in\mathbb{N}$ and $\epsilon>0$ by

$$[\gamma_T\boldsymbol{E}_n^{\mathrm{inc}},\boldsymbol{\varphi}_n^{\tau,h}]_\Gamma \leq \left\|\gamma_T\boldsymbol{E}_n^{\mathrm{inc}}\right\|_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)}\left\|\boldsymbol{\varphi}_n^{\tau,h}\right\|_{\boldsymbol{L}^{1+\alpha}(\Gamma)} \leq C\left\|\gamma_T\boldsymbol{E}_n^{\mathrm{inc}}\right\|_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)}^{\frac{1+\alpha}{\alpha}} + \epsilon\left\|\boldsymbol{\varphi}_n^{\tau,h}\right\|_{\boldsymbol{L}^{1+\alpha}(\Gamma)}^{1+\alpha} .$$

Choosing ϵ small enough enables the absorption of the term depending on the numerical solution $\boldsymbol{\varphi}_n^{\tau,h}$.

The last summand regarding $\boldsymbol{\varphi}_n^{\tau,h}$ is bounded by the same chain of inequalities via

$$\begin{aligned}\left(\gamma_T\boldsymbol{H}_n^{\mathrm{inc}}, \boldsymbol{a}\left(\boldsymbol{\varphi}_n^{\tau,h}+\gamma_T\boldsymbol{H}_n^{\mathrm{inc}}\right)\right)_\Gamma &\leq C\left\|\gamma_T\boldsymbol{H}_n^{\mathrm{inc}}\right\|_{\boldsymbol{L}^{1+\alpha}(\Gamma)}^{1+\alpha} + \epsilon\left\|\boldsymbol{a}\left(\boldsymbol{\varphi}_n^{\tau,h}+\gamma_T\boldsymbol{H}_n^{\mathrm{inc}}\right)\right\|_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)}^{\frac{1+\alpha}{\alpha}} \\ &= C\left\|\gamma_T\boldsymbol{H}_n^{\mathrm{inc}}\right\|_{\boldsymbol{L}^{1+\alpha}(\Gamma)}^{1+\alpha} + \epsilon\left\|\boldsymbol{\varphi}_n^{\tau,h}+\gamma_T\boldsymbol{H}_n^{\mathrm{inc}}\right\|_{\boldsymbol{L}^{1+\alpha}(\Gamma)}^{1+\alpha} .\end{aligned}$$

□

5.4. **Main result: Error bounds for the full discretization.**

Theorem 2. *Let $(\boldsymbol{\varphi},\boldsymbol{\psi})$ be the solution of the boundary integral equation* (36)*, assumed to be of regularity*

$$(\boldsymbol{\varphi},\boldsymbol{\psi}) \in \boldsymbol{H}_0^{m+5}\left(0,T;\mathbf{X}_\Gamma{}^2\right) \cap \boldsymbol{H}_0^3\left(0,T;\boldsymbol{H}_\times^{k+1}(\Gamma)^2\right).$$

Furthermore, let $\gamma_T\boldsymbol{H}(t)^{\mathrm{inc}} \in \boldsymbol{L}^{1+\alpha}(\Gamma)$ for all $t\in[0,T]$. Consider the fully discrete boundary densities $(\boldsymbol{\varphi}_n^{\tau,h},\boldsymbol{\psi}_n^{\tau,h}) \in \boldsymbol{V}_h^m \times \boldsymbol{X}_h^m$ for all $n\leq N$, solutions to the the fully-discrete boundary integral equations (57)*, discretized by*

- *Radau IIA based Runge-Kutta convolution quadrature with m-stages in time, and*
- *Raviart-Thomas boundary elements of order k in space.*

For $m>2$, we assume the scheme to be applied to the shifted boundary integral equation (37) *by some positive shift $\sigma>0$. The error of the m-stage Radau IIA semi-discretization, denoted by $\boldsymbol{e}_{\boldsymbol{\varphi}} = \boldsymbol{\varphi}_h^\tau - \Pi_h\boldsymbol{\varphi}$ and $\boldsymbol{e}_{\boldsymbol{\psi}} = \boldsymbol{\psi}_h^\tau - \Pi_h\boldsymbol{\psi}$, fulfill the bounds*

$$\left(\tau\sum_{n=0}^{N}\left\|\left(\left(\partial_t^\tau\right)^{-1}\boldsymbol{e}_{\boldsymbol{\varphi}}\right)_n\right\|_{\mathbf{X}_\Gamma}^2 + \left\|\left(\left(\partial_t^\tau\right)^{-1}\boldsymbol{e}_{\boldsymbol{\psi}}\right)_n\right\|_{\mathbf{X}_\Gamma}^2\right)^{1/2} \leq C\left(\tau^m + h^{\alpha(k+1)}\right).$$

Errors in the electromagnetic fields, defined through the discrete representation formulas (54)–(55)*, are bounded via*

$$\left(\tau \sum_{n=0}^{N} \left\| \boldsymbol{E}_n^{\tau,h} - \boldsymbol{E}(t_n) \right\|^2_{\boldsymbol{L}^2(\Omega)} + \left\| \boldsymbol{H}_n^{\tau,h} - \boldsymbol{H}(t_n) \right\|^2_{\boldsymbol{L}^2(\Omega)} \right)^{1/2} \leq C \left(\tau^m + h^{\alpha(k+1)}\right),$$

where the notation $\boldsymbol{E}(t_n) = (\boldsymbol{E}(t_n + c_i\tau))_{i=1}^m$ has been applied to the electromagnetic fields. The constants in the error bounds depend on higher Sobolev norms of the exact solution $(\boldsymbol{\varphi}, \boldsymbol{\psi})$, the shift σ for $m > 2$, the boundary Γ and polynomially on the final time T.

Proof. Throughout this proof, whenever an expression holds for arbitrary $n \in \mathbb{N}$ (or respectively at all time points t_n), we omit the index for notational convenience. For the sake of presentation, we further assume that $m = 2$, thus allowing for the first coercivity property of Lemma 6. The proof readily generalizes to $m > 2$, by means of Remark 3.1 and the resulting positivity as described in Remark 4.1.

We start by inserting a projection of the exact solution, which yields a sequence of defects $\mathbf{d} = (\mathbf{d}_n)_{n \geq 0}$, where each of the sequence elements fulfill $\mathbf{d}_n \in \boldsymbol{V}_h^m \times \boldsymbol{X}_h^m$ for all $n \geq 0$, such that the perturbed boundary integral equation

(59)
$$\left[\begin{pmatrix} \boldsymbol{\eta}_h \\ \boldsymbol{\xi}_h \end{pmatrix}, \boldsymbol{C}_{\text{imp}}(\partial_t^\tau) \begin{pmatrix} \Pi_h \boldsymbol{\varphi} \\ \Pi_h \boldsymbol{\psi} \end{pmatrix} \right]_\Gamma + \left(\boldsymbol{\eta}_h, \boldsymbol{a}\left(\Pi_h \boldsymbol{\varphi} + \gamma_T \boldsymbol{H}^{\text{inc}} \right) \right)_\Gamma = [\gamma_T \boldsymbol{E}^{\text{inc}}, \boldsymbol{\eta}_h]_\Gamma + \left[\begin{pmatrix} \boldsymbol{\eta}_h \\ \boldsymbol{\xi}_h \end{pmatrix}, \mathbf{d} \right]_\Gamma$$

holds. Furthermore, we define fields associated with the projected boundary densities through the discrete representation formulas via

$$\begin{pmatrix} \boldsymbol{E}_\Pi^{\tau,h} \\ \boldsymbol{H}_\Pi^{\tau,h} \end{pmatrix} = \begin{pmatrix} -\boldsymbol{\mathcal{S}}(\partial_t^\tau)\Pi_h \boldsymbol{\varphi} + \boldsymbol{\mathcal{D}}(\partial_t^\tau)\Pi_h \boldsymbol{\psi} \\ -\boldsymbol{\mathcal{D}}(\partial_t^\tau)\Pi_h \boldsymbol{\varphi} - \boldsymbol{\mathcal{S}}(\partial_t^\tau)\Pi_h \boldsymbol{\psi} \end{pmatrix} = \boldsymbol{\mathcal{W}}(\partial_t^\tau) \begin{pmatrix} \Pi_h \boldsymbol{\varphi} \\ \Pi_h \boldsymbol{\psi} \end{pmatrix}. \tag{60}$$

These intermediate fields approximate the exact fields $\boldsymbol{E}$ and $\boldsymbol{H}$ at least in the stated order due to

$$\begin{pmatrix} \boldsymbol{E}_\Pi^{\tau,h} - \boldsymbol{E} \\ \boldsymbol{H}_\Pi^{\tau,h} - \boldsymbol{H} \end{pmatrix} = \left(\boldsymbol{\mathcal{W}}(\partial_t^\tau) - \boldsymbol{\mathcal{W}}(\partial_t) \right) \begin{pmatrix} \Pi_h \boldsymbol{\varphi} \\ \Pi_h \boldsymbol{\psi} \end{pmatrix} + \boldsymbol{\mathcal{W}}(\partial_t) \begin{pmatrix} \Pi_h \boldsymbol{\varphi} - \boldsymbol{\varphi} \\ \Pi_h \boldsymbol{\psi} - \boldsymbol{\psi} \end{pmatrix},$$

which implies, due to the time-harmonic bounds of Lemma 3 and the general convolution quadrature approximation results of Proposition 1, the existence of a constant C depending only the surface Γ and polynomially on the final time T, such that for all $n \leq N$ we have the bound

$$\begin{aligned} &\left\| \boldsymbol{E}_\Pi^{n,h} - \boldsymbol{E}(t_n) \right\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} + \left\| \boldsymbol{H}_\Pi^{n,h} - \boldsymbol{H}(t_n) \right\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \\ &\quad \leq C \left(\tau^m \left\| \begin{pmatrix} \boldsymbol{\varphi} \\ \boldsymbol{\psi} \end{pmatrix} \right\|_{\boldsymbol{H}_0^{m+5}(0,T;\mathbf{X}_\Gamma{}^2)} + h^{k+3/2} \left\| \begin{pmatrix} \boldsymbol{\varphi} \\ \boldsymbol{\psi} \end{pmatrix} \right\|_{\boldsymbol{H}_0^3(0,T;\boldsymbol{H}_\times^{k+1}(\Gamma)^2)} \right). \end{aligned} \tag{61}$$

Subtracting the perturbed scheme from the full discretization yields, by testing with $\boldsymbol{e}_{\boldsymbol{\varphi}}^n = \boldsymbol{\varphi}_n^{\tau,h} - \Pi_h \boldsymbol{\varphi}(t_n)$ and $\boldsymbol{e}_{\boldsymbol{\psi}}^n = \boldsymbol{\psi}_n^{\tau,h} - \Pi_h \boldsymbol{\psi}(t_n)$, the following error equation

(62)
$$\left[\begin{pmatrix} \boldsymbol{e}_{\boldsymbol{\varphi}}^n \\ \boldsymbol{e}_{\boldsymbol{\psi}}^n \end{pmatrix}, \left[\boldsymbol{C}_{\mathrm{imp}}(\partial_t^\tau) \begin{pmatrix} \boldsymbol{e}_{\boldsymbol{\varphi}} \\ \boldsymbol{e}_{\boldsymbol{\psi}} \end{pmatrix} \right]^n \right]_\Gamma + \left(\boldsymbol{e}_{\boldsymbol{\varphi}}^n, \boldsymbol{a}\left(\boldsymbol{\varphi}_n^{\tau,h} + \gamma_T \boldsymbol{H}_n^{\mathrm{inc}} \right) - \boldsymbol{a}\left(\Pi_h \boldsymbol{\varphi}(t_n) + \gamma_T \boldsymbol{H}_n^{\mathrm{inc}} \right) \right)_\Gamma$$
$$= \left[\begin{pmatrix} \boldsymbol{e}_{\boldsymbol{\varphi}}^n \\ \boldsymbol{e}_{\boldsymbol{\psi}}^n \end{pmatrix}, \mathbf{d}^n \right]_\Gamma .$$

Note that the term with the nonlinearity $\boldsymbol{a}$ is readily estimated from below, by the pointwise monotonicity estimate of $\boldsymbol{a}$ from Lemma 1. The first summand, corresponding to the time-discrete Calderón operator, is estimated from below by applying the second identity of Lemma 8 to the term, which yields

$$\left[\begin{pmatrix} \boldsymbol{e}_{\boldsymbol{\varphi}}^n \\ \boldsymbol{e}_{\boldsymbol{\psi}}^n \end{pmatrix}, \left[\boldsymbol{C}_{\mathrm{imp}}(\partial_t^\tau) \begin{pmatrix} \boldsymbol{e}_{\boldsymbol{\varphi}} \\ \boldsymbol{e}_{\boldsymbol{\psi}} \end{pmatrix} \right]^n \right]_\Gamma = \int_{\mathbb{R}^3\setminus\Gamma} (\boldsymbol{E}^{\tau,h} - \boldsymbol{E}_\Pi^{\tau,h})_n \cdot \left(\partial_t^\tau \left(\boldsymbol{E}^{\tau,h} - \boldsymbol{E}_\Pi^{\tau,h} \right) \right)_n$$
$$+ \left(\boldsymbol{H}^{\tau,h} - \boldsymbol{H}_\Pi^{\tau,h} \right)_n \cdot \left(\partial_t^\tau \left(\boldsymbol{H}^{\tau,h} - \boldsymbol{H}_\Pi^{\tau,h} \right) \right)_n \,\mathrm{d}\boldsymbol{x}.$$

To employ the coercivity of the discrete operator ∂_t^τ, given by Lemma 6, a weighted summation on both sides is necessary.

Consequently, summation of the numerical scheme, weighted at the time point t_n with $\rho^n = e^{-2n\tau/T}$ and inserting this identity into the left-hand side yields a positive constant C, such that

$$\sum_{n=0}^N \rho^n \left(\left\| \boldsymbol{E}_n^{\tau,h} - \boldsymbol{E}_\Pi^{n,h} \right\|_{\boldsymbol{L}^2(\mathbb{R}^3\setminus\Gamma)}^2 + \left\| \boldsymbol{H}_n^{\tau,h} - \boldsymbol{H}_\Pi^{n,h} \right\|_{\boldsymbol{L}^2(\mathbb{R}^3\setminus\Gamma)}^2 \right)$$
$$+ \sum_{n=0}^N \rho^n \int_\Gamma \left(\left| \boldsymbol{\varphi}_n^{\tau,h} + \gamma_T \boldsymbol{H}^{\mathrm{inc}} \right| + \left| \Pi_h \boldsymbol{\varphi}(t_n) + \gamma_T \boldsymbol{H}^{\mathrm{inc}} \right| \right)^{\alpha-1} \left| \boldsymbol{e}_{\boldsymbol{\varphi}}^n \right|^2 \,\mathrm{d}\boldsymbol{x}$$
$$\leq C \sum_{n=0}^N \rho^n \left[\begin{pmatrix} \boldsymbol{e}_{\boldsymbol{\varphi}}^n \\ \boldsymbol{e}_{\boldsymbol{\psi}}^n \end{pmatrix}, \mathbf{d}^n \right]_\Gamma .$$

We turn towards the estimation of the defect, by subtracting the exact boundary integral equation (36) from the perturbed equation (59) to obtain

$$\left[\begin{pmatrix} \boldsymbol{e}_{\boldsymbol{\varphi}}^n \\ \boldsymbol{e}_{\boldsymbol{\psi}}^n \end{pmatrix}, \mathbf{d}^n \right]_\Gamma = \left[\begin{pmatrix} \boldsymbol{e}_{\boldsymbol{\varphi}}^n \\ \boldsymbol{e}_{\boldsymbol{\psi}}^n \end{pmatrix}, \left[\boldsymbol{C}_{\mathrm{imp}}(\partial_t^\tau) \begin{pmatrix} \Pi_h \boldsymbol{\varphi} \\ \Pi_h \boldsymbol{\psi} \end{pmatrix} - \boldsymbol{C}_{\mathrm{imp}}(\partial_t) \begin{pmatrix} \boldsymbol{\varphi} \\ \boldsymbol{\psi} \end{pmatrix} \right]^n \right]_\Gamma \qquad \text{(A)}$$
$$+ \left(\boldsymbol{e}_{\boldsymbol{\varphi}}^n, \boldsymbol{a}\left(\Pi_h \boldsymbol{\varphi}(t_n) + \gamma_T \boldsymbol{H}_n^{\mathrm{inc}} \right) - \boldsymbol{a}\left(\boldsymbol{\varphi}(t_n) + \gamma_T \boldsymbol{H}_n^{\mathrm{inc}} \right) \right)_\Gamma \quad \text{(B)}.$$

The defect has been split into two parts, first into a temporal defect where the approximation of the time-dependent Calderón operator enters and secondly a nonlinear defect depending on the nonlinearity $\boldsymbol{a}$. In the following, we estimate these terms successively.

(A) We start with the temporal defect, for which applying the jump conditions of both the discrete and the continuous time-dependent Calderón operator yields

$$\left[\begin{pmatrix} \boldsymbol{e}_{\boldsymbol{\varphi}} \\ \boldsymbol{e}_{\boldsymbol{\psi}} \end{pmatrix}, \boldsymbol{C}_{\mathrm{imp}}(\partial_t^\tau) \begin{pmatrix} \Pi_h \boldsymbol{\varphi} \\ \Pi_h \boldsymbol{\psi} \end{pmatrix} - \boldsymbol{C}_{\mathrm{imp}}(\partial_t) \begin{pmatrix} \boldsymbol{\varphi} \\ \boldsymbol{\psi} \end{pmatrix} \right]_\Gamma = \left[\begin{pmatrix} \boldsymbol{e}_{\boldsymbol{\varphi}} \\ \boldsymbol{e}_{\boldsymbol{\psi}} \end{pmatrix}, \begin{pmatrix} \gamma_T^+ \boldsymbol{E}_\Pi^{\tau,h} - \gamma_T^+ \boldsymbol{E} \\ \gamma_T^- \boldsymbol{H}_\Pi^{\tau,h} \end{pmatrix} \right]_\Gamma .$$

Writing the numerical solution and the projected exact solution in terms of the jumps of their respective fields and sorting for the inner and outer fields (in the left argument of the duality) yields

(63)

$$\left[\begin{pmatrix}\boldsymbol{\varphi}_h^\tau - \Pi_h\boldsymbol{\varphi} \\ \boldsymbol{\psi}_h^\tau - \Pi_h\boldsymbol{\psi}\end{pmatrix}, \begin{pmatrix}\gamma_T^+ \boldsymbol{E}_\Pi^{\tau,h} - \gamma_T^+\boldsymbol{E} \\ \gamma_T^- \boldsymbol{H}_\Pi^{\tau,h}\end{pmatrix}\right]_\Gamma = \left[\begin{pmatrix}\gamma_T^+(\boldsymbol{H}^{\tau,h} - \boldsymbol{H}_\Pi^{\tau,h}) \\ -\gamma_T^+(\boldsymbol{E}^{\tau,h} - \boldsymbol{E}_\Pi^{\tau,h})\end{pmatrix}, \begin{pmatrix}\gamma_T^+ \boldsymbol{E}_\Pi^{\tau,h} - \gamma_T^+\boldsymbol{E} \\ \gamma_T^- \boldsymbol{H}_\Pi^{\tau,h}\end{pmatrix}\right]_\Gamma$$
$$- \left[\begin{pmatrix}\gamma_T^-(\boldsymbol{H}^{\tau,h} - \boldsymbol{H}_\Pi^{n,h}) \\ -\gamma_T^-(\boldsymbol{E}^{\tau,h} - \boldsymbol{E}_\Pi^{\tau,h})\end{pmatrix}, \begin{pmatrix}\gamma_T^+ \boldsymbol{E}_\Pi^{\tau,h} - \gamma_T^+\boldsymbol{E} \\ \gamma_T^- \boldsymbol{H}_\Pi^{\tau,h}\end{pmatrix}\right]_\Gamma .$$

These terms, which correspond to errors in the inner and outer domains respectively, are bounded successively starting with the first summand. Rewriting the only inner trace appearing in the first summand via the discrete transmission problem yields

$$\gamma_T^- \boldsymbol{H}_\Pi^{\tau,h} = \gamma_T^+ \boldsymbol{H}_\Pi^{\tau,h} - \Pi_h\boldsymbol{\varphi} = \left(\gamma_T^+ \boldsymbol{H}_\Pi^{\tau,h} - \gamma_T^+\boldsymbol{H}\right) + (\boldsymbol{\varphi} - \Pi_h\boldsymbol{\varphi}) .$$

Inserting this identity into the first summand yields

$$\begin{aligned}
&\left[\begin{pmatrix}\gamma_T^+(\boldsymbol{H}^{\tau,h} - \boldsymbol{H}_\Pi^{\tau,h}) \\ -\gamma_T^+(\boldsymbol{E}^{\tau,h} - \boldsymbol{E}_\Pi^{\tau,h})\end{pmatrix}, \begin{pmatrix}\gamma_T^+ \boldsymbol{E}_\Pi^{\tau,h} - \gamma_T^+\boldsymbol{E} \\ \gamma_T^- \boldsymbol{H}_\Pi^{\tau,h}\end{pmatrix}\right]_\Gamma \\
&\quad = \left[\gamma_T^+(\boldsymbol{H}^{\tau,h} - \boldsymbol{H}_\Pi^{\tau,h}), \gamma_T^+(\boldsymbol{E}_\Pi^{\tau,h} - \boldsymbol{E})\right]_\Gamma && \text{(i)} \\
&\quad - \left[\gamma_T^+(\boldsymbol{E}^{\tau,h} - \boldsymbol{E}_\Pi^{\tau,h}), \gamma_T^+\left(\boldsymbol{H}_\Pi^{\tau,h} - \boldsymbol{H}\right)\right]_\Gamma && \text{(ii)} \\
&\quad - \left[\gamma_T^+(\boldsymbol{E}^{\tau,h} - \boldsymbol{E}_\Pi^{\tau,h}), \boldsymbol{\varphi} - \Pi_h\boldsymbol{\varphi}\right]_\Gamma . && \text{(iii)}
\end{aligned}$$

The next paragraphs are dedicated to the successive estimation of the terms (i)–(iii).

(i) Applying Green's formula to the first summand yields

$$\begin{aligned}
&\left[\gamma_T^+(\boldsymbol{H}^{\tau,h} - \boldsymbol{H}_\Pi^{\tau,h}),\ \gamma_T^+(\boldsymbol{E}_\Pi^{\tau,h} - \boldsymbol{E})\right]_\Gamma \\
&= \int_\Omega \mathbf{curl}(\boldsymbol{H}^{\tau,h} - \boldsymbol{H}_\Pi^{\tau,h}) \cdot (\boldsymbol{E}_\Pi^{\tau,h} - \boldsymbol{E}) - (\boldsymbol{H}^{\tau,h} - \boldsymbol{H}_\Pi^{\tau,h}) \cdot \mathbf{curl}(\boldsymbol{E}_\Pi^{\tau,h} - \boldsymbol{E})\,\mathrm{d}\boldsymbol{x} \\
&\leq \int_{\Omega^+} \left(\partial_t^\tau \boldsymbol{E}^{\tau,h} - \partial_t^\tau \boldsymbol{E}_\Pi^{\tau,h}\right) \cdot \left(\boldsymbol{E}_\Pi^{\tau,h} - \boldsymbol{E}\right)\,\mathrm{d}\boldsymbol{x} \\
&+ \left\|\boldsymbol{H}^{\tau,h} - \boldsymbol{H}_\Pi^{\tau,h}\right\|_{\boldsymbol{L}^2(\Omega+)} \left\|\mathbf{curl}\,\boldsymbol{E}_\Pi^{\tau,h} - \mathbf{curl}\,\boldsymbol{E}\right\|_{\boldsymbol{L}^2(\Omega+)} .
\end{aligned}$$

Summation over n, applying the discrete integration bound of Lemma 5 to the first summand and the Cauchy-Schwartz inequality to the second summand consequently implies, for arbitrary $\epsilon > 0$ the existence of a positive constant C, such that

$$\sum_{n=0}^{N}\rho^n \left[\gamma_T^+(\boldsymbol{H}_n^{\tau,h}-\boldsymbol{H}_\Pi^{n,h}),\, \gamma_T^+(\boldsymbol{E}_\Pi^{n,h}-\boldsymbol{E}(t_n))\right]_\Gamma$$
$$\leq \sum_{n=0}^{N}\rho^n \left(\epsilon \left\|\left(\boldsymbol{E}^{\tau,h}-\boldsymbol{E}_\Pi^{\tau,h}\right)_n\right\|^2_{\boldsymbol{L}^2(\Omega)} + C\left\|\left(\partial_t^\tau \boldsymbol{E}_\Pi^{\tau,h}-\partial_t^\tau \boldsymbol{E}\right)_n\right\|^2_{\boldsymbol{L}^2(\Omega)}\right)$$
$$+\sum_{n=0}^{N}\rho^n \left(\epsilon \left\|\left(\boldsymbol{H}^{\tau,h}-\boldsymbol{H}_\Pi^{\tau}\right)_n\right\|^2_{\boldsymbol{L}^2(\Omega)} + C\left\|\left(\boldsymbol{E}_\Pi^{\tau,h}-\boldsymbol{E}\right)_n\right\|^2_{\boldsymbol{H}(\mathbf{curl},\Omega)}\right).$$

Choosing $\epsilon > 0$ small enough enables the absorption of error terms depending on the numerical solution $\boldsymbol{E}_n^{\tau,h}$ and $\boldsymbol{H}_n^{\tau,h}$. The error of the intermediate field $\boldsymbol{E}_\Pi^{\tau,h}$ in the $\boldsymbol{H}(\mathbf{curl},\Omega)$ norm is bounded by (61) in the desired order. The remaining defect term, which is numerically differentiated, is rewritten by exploiting (48) and introducing an intermediate term, to obtain

$$\begin{aligned}\left\|\left(\partial_t^\tau \boldsymbol{E}_\Pi^{\tau,h}-\partial_t^\tau \boldsymbol{E}\right)_n\right\|_{\boldsymbol{L}^2(\Omega^+)} &\leq \left\|\left(\mathbf{curl}\,\boldsymbol{H}_\Pi^{\tau,h}-\mathbf{curl}\,\boldsymbol{H}\right)_n\right\|_{\boldsymbol{L}^2(\Omega^+)} + \|(\partial_t \boldsymbol{E}-\partial_t^\tau \boldsymbol{E})_n\|_{\boldsymbol{L}^2(\Omega^+)}\\ &\leq C(\tau^m+h^{k+3/2}),\end{aligned}$$

where the bound is the consequence of (61) and Proposition 1 respectively.

(ii) We repeat the argument structure and again apply Green's formula to obtain

$$\begin{aligned}&\left[\gamma_T^+\left(\boldsymbol{E}^{\tau,h}-\boldsymbol{E}_\Pi^{\tau,h}\right),\gamma_T^+\left(\boldsymbol{H}_\Pi^{\tau,h}-\boldsymbol{H}\right)\right]_\Gamma\\ &=\int_{\Omega^+}\mathbf{curl}\left(\boldsymbol{E}^{\tau,h}-\boldsymbol{E}_\Pi^{\tau,h}\right)\cdot\left(\boldsymbol{H}_\Pi^{\tau,h}-\boldsymbol{H}\right)-\left(\boldsymbol{E}^{\tau,h}-\boldsymbol{E}_\Pi^{\tau,h}\right)\cdot\mathbf{curl}\left(\boldsymbol{H}_\Pi^{\tau,h}-\boldsymbol{H}\right)\mathrm{d}\boldsymbol{x}\\ &\leq-\int_{\Omega^+}\left(\partial_t^\tau \boldsymbol{H}^{\tau,h}-\partial_t^\tau \boldsymbol{H}_\Pi^{\tau,h}\right)\cdot\left(\boldsymbol{H}_\Pi^{\tau,h}-\boldsymbol{H}\right)\mathrm{d}\boldsymbol{x}\\ &+\left\|\boldsymbol{E}^{\tau,h}-\boldsymbol{E}_\Pi^{\tau,h}\right\|_{\boldsymbol{L}^2(\Omega^+)}\left\|\mathbf{curl}\,\boldsymbol{H}_\Pi^{\tau,h}-\mathbf{curl}\,\boldsymbol{H}\right\|_{\boldsymbol{L}^2(\Omega^+)}.\end{aligned}$$

Applying the discrete integration bound of Lemma 5 to the first summand consequently leads to the estimate

$$\sum_{n=0}^{N}\rho^n\left[\gamma_T^+\left(\boldsymbol{E}_n^{\tau,h}-\boldsymbol{E}_\Pi^{n,h}\right),\gamma_T^+\left(\boldsymbol{H}_\Pi^{n,h}-\boldsymbol{H}(t_n)\right)\right]_\Gamma$$
$$\leq\sum_{n=0}^{N}\rho^n\left(\epsilon\left\|\boldsymbol{H}_n^{\tau,h}-\boldsymbol{H}_\Pi^{n,h}\right\|^2_{\boldsymbol{L}^2(\Omega^+)}+C\left\|\left(\partial_t^\tau \boldsymbol{H}_\Pi^{\tau,h}-\partial_t^\tau \boldsymbol{H}\right)_n\right\|^2_{\boldsymbol{L}^2(\Omega^+)}\right)$$
$$+\rho^n\left(\epsilon\left\|\boldsymbol{E}_n^{\tau,h}-\boldsymbol{E}_\Pi^{n,h}\right\|^2_{\boldsymbol{L}^2(\Omega^+)}+C\left\|\boldsymbol{H}_\Pi^{n,h}-\boldsymbol{H}(t_n)\right\|^2_{\boldsymbol{H}(\mathbf{curl},\Omega^+)}\right),$$

where $\epsilon > 0$ is chosen small enough to absorb the terms depending on the numerical solution. Applying the discrete identity (49) yields further

$$\begin{aligned}\left\|\left(\partial_t^\tau \boldsymbol{H}_\Pi^{\tau,h}-\partial_t^\tau \boldsymbol{H}\right)_n\right\|_{\boldsymbol{L}^2(\Omega^+)} &\leq\left\|\mathbf{curl}\,\boldsymbol{E}_\Pi^{n,h}-\mathbf{curl}\,\boldsymbol{E}(t_n)\right\|_{\boldsymbol{L}^2(\Omega^+)}+\|(\partial_t \boldsymbol{H}-\partial_t^\tau \boldsymbol{H})_n\|_{\boldsymbol{L}^2(\Omega^+)}\\ &\leq C(\tau^m+h^{k+3/2}).\end{aligned}$$

(iii) We use the discrete partial integration bound to obtain

$$\begin{aligned}\sum_{n=0}^{N} e^{-n\sigma\tau} &\left[\gamma_T^+(\boldsymbol{E}_n^{\tau,h} - \boldsymbol{E}_\Pi^{n,h}), \boldsymbol{\varphi}(t_n) - \Pi_h\boldsymbol{\varphi}(t_n)\right]_\Gamma \\ &\leq \sum_{n=0}^{N} e^{-n\sigma\tau}\left(\epsilon\left\|\left(\left(\partial_t^\tau\right)^{-1}\gamma_T^+(\boldsymbol{E}^{\tau,h} - \boldsymbol{E}_\Pi^{\tau,h})\right)_n\right\|_{\mathbf{X}_\Gamma}^2 + C\left\|\partial_t^\tau\left(\boldsymbol{\varphi} - \Pi_h\boldsymbol{\varphi}\right)(t_n)\right\|_{\mathbf{X}_\Gamma}^2\right).\end{aligned} \tag{64}$$

The second summand is of the required order, seen by splitting the discrete time derivative into $\partial_t^\tau = \partial_t - (\partial_t - \partial_t^\tau)$ and applying Lemma 1 to obtain

$$\left\|\left(\partial_t^\tau - \partial_t\right)\left(\boldsymbol{\varphi} - \Pi_h\boldsymbol{\varphi}\right)\right\|_{\mathbf{X}_\Gamma} \leq C\tau^m \left\|\boldsymbol{\varphi}\right\|_{\boldsymbol{H}_0^{m+3}(0,T;\mathbf{X}_\Gamma)}.$$

Applying the trace theorem to the first summand yields

$$\begin{aligned}\left\|\left(\left(\partial_t^\tau\right)^{-1}\gamma_T^+(\boldsymbol{E}^{\tau,h} - \boldsymbol{E}_\Pi^{\tau,h})\right)_n\right\|_{\mathbf{X}_\Gamma}^2 &\leq \left\|\left[\left(\partial_t^\tau\right)^{-1}\boldsymbol{E}^{\tau,h} - \left(\partial_t^\tau\right)^{-1}\boldsymbol{E}_\Pi^{\tau,h}\right]^n\right\|_{\boldsymbol{H}(\mathbf{curl},\Omega^+)}^2 \\ &= \int_{\Omega^+}\left|\left(\left(\partial_t^\tau\right)^{-1}\boldsymbol{E}_h^\tau - \left(\partial_t^\tau\right)^{-1}\boldsymbol{E}_\Pi^{\tau,h}\right)_n\right|^2 + \left|\boldsymbol{H}_h^n - \boldsymbol{H}_\Pi^{n,h}\right|^2 \mathrm{d}\boldsymbol{x}.\end{aligned}$$

Furthermore, by [6, Theorem 3.1] with $L(s) = 1$ and $R(s) = s^{-1}$, we estimate the discrete integral in the $\boldsymbol{L}^2$-norm with a constant C depending only on the final time T, such that

$$\sum_{n=0}^{\infty}\rho^n\int_{\Omega^+}\left|\left(\left(\partial_t^\tau\right)^{-1}\boldsymbol{E}_h^\tau - \left(\partial_t^\tau\right)^{-1}\boldsymbol{E}_\Pi^{\tau,h}\right)_n\right|^2 \mathrm{d}\boldsymbol{x} \leq C\sum_{n=0}^{\infty}\rho^n\left\|\boldsymbol{E}_n^{\tau,h} - \boldsymbol{E}_\Pi^{n,h}\right\|_{\boldsymbol{L}^2(\Omega^+)}^2.$$

Choosing ϵ small enough in (64) allows for the absorption of the remaining term to the left-hand side.

We turn our attention to the second summand of (63), consisting mostly of defects in term of the traces of the inner domain Ω^-. Structurally, the process of estimation is identical, starting from rewriting the only remaining term depending on outer traces by

$$\gamma_T^+\boldsymbol{E}_\Pi^{\tau,h} - \gamma_T^+\boldsymbol{E} = \gamma_T^-\boldsymbol{E}_\Pi^{\tau,h} - \Pi_h\boldsymbol{\psi} - \gamma_T^+\boldsymbol{E} = \left(\gamma_T^-\boldsymbol{E}_\Pi^{\tau,h} - \gamma_T^-\boldsymbol{E}\right) + \left(\boldsymbol{\psi} - \Pi_h\boldsymbol{\psi}\right),$$

where we exploited that the exact solution vanishes in the inner domain, i.e. $\gamma_T^-\boldsymbol{E} = 0$. Inserting this identity on the right argument of the second summand of (63) yields

$$\begin{aligned}&\left[\begin{pmatrix}\gamma_T^-(\boldsymbol{H}_n^{\tau,h} - \boldsymbol{H}_\Pi^{n,h}) \\ -\gamma_T^-(\boldsymbol{E}_n^{\tau,h} - \boldsymbol{E}_\Pi^{n,h})\end{pmatrix}, \begin{pmatrix}\gamma_T^+\boldsymbol{E}_\Pi^{n,h} - \gamma_T^+\boldsymbol{E} \\ \gamma_T^-\boldsymbol{H}_\Pi^{n,h}\end{pmatrix}\right]_\Gamma \\ &\quad= \left[\gamma_T^-(\boldsymbol{H}_n^{\tau,h} - \boldsymbol{H}_\Pi^{n,h}), \gamma_T^-\left(\boldsymbol{E}_\Pi^{n,h} - \boldsymbol{E}\right)\right]_\Gamma && \text{(iv)} \\ &\quad+ \left[\gamma_T^-(\boldsymbol{H}_n^{\tau,h} - \boldsymbol{H}_\Pi^{n,h}), \boldsymbol{\psi} - \Pi_h\boldsymbol{\psi}\right]_\Gamma && \text{(v)} \\ &\quad- \left[\gamma_T^-(\boldsymbol{E}_n^{\tau,h} - \boldsymbol{E}_\Pi^{n,h}), \gamma_T^-\left(\boldsymbol{H}_\Pi^{n,h} - \boldsymbol{H}(t_n)\right)\right]_\Gamma. && \text{(vi)}\end{aligned}$$

These terms depending on the inner traces are bounded precisely by the arguments presented to estimate (i)–(iii) respectively.

(B) We introduce the following notation for the nonlinear defect:

$$\mathbf{d}_{\boldsymbol{a}} = \boldsymbol{a}\left(\Pi_h \boldsymbol{\varphi} + \gamma_T \boldsymbol{H}^{\mathrm{inc}}\right) - \boldsymbol{a}\left(\boldsymbol{\varphi} + \gamma_T \boldsymbol{H}^{\mathrm{inc}}\right).$$

Rewriting the term of interest (B) by means of a multiplicative intermediate term in combination with Hölder's and Young's inequalities yields

$$\begin{aligned}
\left(\boldsymbol{\varphi}^{\tau,h} - \Pi_h \boldsymbol{\varphi}, \mathbf{d}_a\right)_\Gamma \leq C \int_\Gamma &\left|\boldsymbol{\varphi}^{\tau,h} - \Pi_h \boldsymbol{\varphi}\right| \left(\left|\boldsymbol{\varphi}^{\tau,h} + \gamma_T \boldsymbol{H}^{\mathrm{inc}}\right| + \left|\Pi_h \boldsymbol{\varphi} + \gamma_T \boldsymbol{H}^{\mathrm{inc}}\right|\right)^{\frac{\alpha-1}{2}} \\
&\left(\left|\boldsymbol{\varphi}^{\tau,h} + \gamma_T \boldsymbol{H}^{\mathrm{inc}}\right| + \left|\Pi_h \boldsymbol{\varphi} + \gamma_T \boldsymbol{H}^{\mathrm{inc}}\right|\right)^{\frac{1-\alpha}{2}} \mathbf{d}_{\boldsymbol{a}} \, \mathrm{d}\boldsymbol{x} \\
\leq \epsilon &\left\| \left(\left|\boldsymbol{\varphi}^{\tau,h} + \gamma_T \boldsymbol{H}^{\mathrm{inc}}\right| + \left|\Pi_h \boldsymbol{\varphi} + \gamma_T \boldsymbol{H}^{\mathrm{inc}}\right|\right)^{\frac{\alpha-1}{2}} \left(\boldsymbol{\varphi}^{\tau,h} - \Pi_h \boldsymbol{\varphi}\right) \right\|_{\boldsymbol{L}^2(\Gamma)}^2 \\
&+ C \int_\Gamma \left(\left|\boldsymbol{\varphi}^{\tau,h} + \gamma_T \boldsymbol{H}^{\mathrm{inc}}\right|^{1-\alpha} + \left|\Pi_h \boldsymbol{\varphi} + \gamma_T \boldsymbol{H}^{\mathrm{inc}}\right|^{1-\alpha} \right) |\mathbf{d}_{\boldsymbol{a}}|^2 \, \mathrm{d}\boldsymbol{x}.
\end{aligned}$$

The constant ϵ is chosen small enough for the first summand to be absorbed in the left-hand side.

All quantities in the factor multiplied with the defect in the integrand are bounded by the stated regularity assumptions on $\boldsymbol{\varphi}$ and $\gamma_T \boldsymbol{H}^{\mathrm{inc}}$, with the exception of $\boldsymbol{\varphi}^{\tau,h}$. Consequently, these understood terms are dropped and the rest of the proof focuses on this critical factor containing $\boldsymbol{\varphi}^{\tau,h}$. The key to estimate this term are the already established stability bounds of Proposition 2. Summation on both sides and applying the Hölder inequality, with the parameters $p = \frac{1+\alpha}{1-\alpha}$ and $q = \frac{1+\alpha}{2\alpha}$, repeatedly in space and time yields for the remaining terms

$$\begin{aligned}
\sum_{n=0}^{\infty} \rho^n \int_\Gamma \left|\boldsymbol{\varphi}_n^{\tau,h}\right|^{1-\alpha} |\mathbf{d}_{\boldsymbol{a}}^n|^2 \, \mathrm{d}\boldsymbol{x} &\leq \sum_{n=0}^{\infty} \rho^n \left\|\boldsymbol{\varphi}_n^{\tau,h}\right\|_{\boldsymbol{L}^{1+\alpha}(\Gamma)}^{1-\alpha} \|\mathbf{d}_{\boldsymbol{a}}^n\|_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)}^2 \\
&\leq \left(\sum_{n=0}^{\infty} \rho^n \left\|\boldsymbol{\varphi}_n^{\tau,h}\right\|_{\boldsymbol{L}^{1+\alpha}(\Gamma)}^{1+\alpha}\right)^{\frac{1-\alpha}{1+\alpha}} \left(\sum_{n=0}^{\infty} \rho^n \|\mathbf{d}_{\boldsymbol{a}}^n\|_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)}^{\frac{1+\alpha}{\alpha}}\right)^{\frac{2\alpha}{1+\alpha}}.
\end{aligned}$$

The final remaining factor depending on the numerical solution is bound by the already established bounds from Proposition 2.

An error rate in terms of the mesh width h is now readily obtained by applying the Hölder continuity of $\boldsymbol{a}$. For $\boldsymbol{\varphi}^{\tau,h}$ of the stated regularity we obtain

$$\begin{aligned}
\|\mathbf{d}_{\boldsymbol{a}}\|_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)}^{\frac{1+\alpha}{\alpha}} &= \left\|\boldsymbol{a}\left(\Pi_h \boldsymbol{\varphi} + \gamma_T \boldsymbol{H}^{\mathrm{inc}}\right) - \boldsymbol{a}\left(\boldsymbol{\varphi} + \gamma_T \boldsymbol{H}^{\mathrm{inc}}\right)\right\|_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)}^{\frac{1+\alpha}{\alpha}} \\
&\leq \|\Pi_h \boldsymbol{\varphi} - \boldsymbol{\varphi}\|_{\boldsymbol{L}^{1+\alpha}(\Gamma)}^{1+\alpha} \leq C \|\Pi_h \boldsymbol{\varphi} - \boldsymbol{\varphi}\|_{\boldsymbol{L}^2(\Gamma)}^{1+\alpha} \leq C h^{(1+\alpha)(k+1)}.
\end{aligned}$$

Inserting this estimate above yields the stated result. □

5.5. **Pointwise error bounds.** In the context of retarded boundary integral equations error estimates are often shown for points $\boldsymbol{x}$ away from the boundary. To derive such bounds with reasonable error rates, the following approach is taken. Firstly, alternate error bounds on the boundary densities are derived.

Those results bound the error of approximations of the densities $\boldsymbol{\varphi}^{\tau,h}$ and a slightly modified electric trace $\widetilde{\boldsymbol{\psi}}^{\tau,h}$ with regards of the norms $\boldsymbol{L}^{1+\alpha}(\Gamma)$ and $\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)$, which are the natural norms for the present setting. Employing the time-harmonic bounds of the potential operators described in Lemma 4 then yields pointwise error bounds away from the boundary.

Theorem 3. *Consider the setting of Theorem 2 under the assumptions stated therein and let Γ further be smooth. Furthermore, consider the alternative approximations $(\widetilde{\boldsymbol{\varphi}}^{\tau,h}, \widetilde{\boldsymbol{\psi}}^{\tau,h})$ of the boundary densities, deriven from the fully discrete solution $\boldsymbol{\varphi}^{\tau,h}$ and defined through*

$$\widetilde{\boldsymbol{\varphi}}^{\tau,h} = \boldsymbol{\varphi}^{\tau,h}, \qquad \widetilde{\boldsymbol{\psi}}^{\tau,h} = \boldsymbol{a}(\boldsymbol{\varphi}^{\tau,h} + \gamma_T \boldsymbol{H}^{\mathrm{inc}}) \times \boldsymbol{\nu} + \gamma_T \boldsymbol{E}^{\mathrm{inc}}.$$

Fully discrete electromagnetic fields $\boldsymbol{E}^{\tau,h}$ and $\boldsymbol{H}^{\tau,h}$ are then defined for these boundary densities through the discrete representation formulas. These numerical solutions then fulfill, for any $\boldsymbol{x} \in \Omega$ away from the boundary, the error bound

$$\left(\tau \sum_{n=0}^{N} \left| \boldsymbol{E}_n^{\tau,h}(\boldsymbol{x}) - \boldsymbol{E}(\boldsymbol{x}, t_n) \right|^2 + \left| \boldsymbol{H}_n^{\tau,h}(\boldsymbol{x}) - \boldsymbol{H}(\boldsymbol{x}, t_n) \right|^2 \right)^{1/2} \leq C\tau^{-1} \left(\tau^m + h^{\alpha(k+1)} \right)^{\frac{2\alpha}{1+\alpha}},$$

where the constant C depends on higher Sobolev norms of the exact solution, the boundary Γ, the point $\boldsymbol{x}$, on α and polynomially on the final time T. To give this simplified version of the error bound, the mild step size restriction $\tau \leq Ch^{4\alpha(k+1)}$ was assumed.

Proof. (i) *Properties of $\boldsymbol{a}^{-1}$*

The nonlinearity $\boldsymbol{a}^{-1}(\boldsymbol{x}) = |\boldsymbol{x}|^{\frac{1-\alpha}{\alpha}} \boldsymbol{x}$ fulfills, by [35, Lemma 2.3.16], the following positivity property

$$(\boldsymbol{a}^{-1}(\boldsymbol{x}) - \boldsymbol{a}^{-1}(\boldsymbol{y})) \cdot (\boldsymbol{x} - \boldsymbol{y}) \geq c\, |\boldsymbol{x} - \boldsymbol{y}|^{\frac{1+\alpha}{\alpha}}.$$

In particular, this identity implies a positivity condition of $\boldsymbol{a}$, which reads

$$\begin{aligned}(\boldsymbol{x} - \boldsymbol{y}) \cdot (\boldsymbol{a}(\boldsymbol{x}) - \boldsymbol{a}(\boldsymbol{y})) &= (\boldsymbol{a}^{-1}(\boldsymbol{a}(\boldsymbol{x})) - \boldsymbol{a}^{-1}(\boldsymbol{a}(\boldsymbol{y}))) \cdot (\boldsymbol{a}(\boldsymbol{x}) - \boldsymbol{a}(\boldsymbol{y})) \\ &\geq c\, |\boldsymbol{a}(\boldsymbol{x}) - \boldsymbol{a}(\boldsymbol{y})|^{\frac{1+\alpha}{\alpha}}.\end{aligned}$$

Furthermore, applying the fundamental theorem of calculus yields, by using the closed form (16) of the Jacobian of the nonlinearity $\boldsymbol{a}^{-1}$, directly the following bound from above

$$\left| \boldsymbol{a}^{-1}(\boldsymbol{x}) - \boldsymbol{a}^{-1}(\boldsymbol{y}) \right| \leq \int_0^1 \left| \boldsymbol{D}\boldsymbol{a}^{-1}_{\boldsymbol{x}+\theta(\boldsymbol{y}-\boldsymbol{x})} \right| |\boldsymbol{x} - \boldsymbol{y}| \, \mathrm{d}\theta \leq C\, |\boldsymbol{x} - \boldsymbol{y}| \left(|\boldsymbol{x}|^{\frac{1-\alpha}{\alpha}} + |\boldsymbol{x} - \boldsymbol{y}|^{\frac{1-\alpha}{\alpha}} \right).$$

(ii) *Convergence of densities in $\boldsymbol{L}^p$-spaces*

The identities for the inverse of the nonlinearity $\boldsymbol{a}$ imply convergence results for the boundary densities in their respective $\boldsymbol{L}^p$-setting, by modifying the proof of Theorem 2.

Applying the positivity to the nonlinear term on the left-hand side of the error equation (62) yields

$$\begin{aligned}&\left(\boldsymbol{\varphi}_n^{\tau,h} - \Pi_h \boldsymbol{\varphi}(t_n), \boldsymbol{a}\left(\boldsymbol{\varphi}_n^{\tau,h} + \gamma_T \boldsymbol{H}_n^{\mathrm{inc}} \right) - \boldsymbol{a}\left(\Pi_h \boldsymbol{\varphi}(t_n) + \gamma_T \boldsymbol{H}_n^{\mathrm{inc}} \right) \right)_\Gamma \\ &\quad \geq \left\| \boldsymbol{a}\left(\boldsymbol{\varphi}_n^{\tau,h} + \gamma_T \boldsymbol{H}_n^{\mathrm{inc}} \right) - \boldsymbol{a}\left(\Pi_h \boldsymbol{\varphi}(t_n) + \gamma_T \boldsymbol{H}_n^{\mathrm{inc}} \right) \right\|_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)}^{\frac{1+\alpha}{\alpha}}.\end{aligned}$$

The error analysis of Theorem 2 then yields, under the stated conditions there, a constant C independent of h and τ, such that

$$\tau \sum_{n=0}^{N} \left\| \boldsymbol{a}\left(\boldsymbol{\varphi}_n^{\tau,h} + \gamma_T \boldsymbol{H}_n^{\mathrm{inc}} \right) - \boldsymbol{a}\left(\Pi_h \boldsymbol{\varphi}(t_n) + \gamma_T \boldsymbol{H}_n^{\mathrm{inc}} \right) \right\|_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)}^{\frac{1+\alpha}{\alpha}} \leq C \left(\tau^m + h^{\alpha(k+1)} \right)^2.$$

Furthermore, the defect due to the projection Π_h is bounded due to the Hölder continuity of the nonlinearity $\boldsymbol{a}$ by

$$\left\|\boldsymbol{a}\left(\Pi_h\boldsymbol{\varphi}+\gamma_T\boldsymbol{H}^{\mathrm{inc}}\right)-\boldsymbol{a}\left(\boldsymbol{\varphi}+\gamma_T\boldsymbol{H}^{\mathrm{inc}}\right)\right\|_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)}\leq\|\Pi_h\boldsymbol{\varphi}-\boldsymbol{\varphi}\|^{\alpha}_{\boldsymbol{L}^{1+\alpha}(\Gamma)}\leq Ch^{\alpha(k+1)}.$$

Successively applying the Hölder inequality and inserting the boundary condition (7) into $\boldsymbol{\psi}$ yields finally

(65)
$$\begin{aligned}
&\left(\tau\sum_{n=0}^{N}\left\|\widetilde{\boldsymbol{\psi}}_n^{\tau,h}-\boldsymbol{\psi}(t_n)\right\|^2_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)}\right)^{\frac{1}{2}}\leq T^{\frac{1-\alpha}{1+\alpha}}\left(\tau\sum_{n=0}^{N}\left\|\widetilde{\boldsymbol{\psi}}_n^{\tau,h}-\boldsymbol{\psi}(t_n)\right\|^{\frac{1+\alpha}{\alpha}}_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)}\right)^{\frac{\alpha}{1+\alpha}}\\
&=T^{\frac{1-\alpha}{1+\alpha}}\left(\tau\sum_{n=0}^{N}\left\|\boldsymbol{a}(\boldsymbol{\varphi}_n^{\tau,h}+\gamma_T\boldsymbol{H}^{\mathrm{inc}})-\boldsymbol{a}(\boldsymbol{\varphi}(t_n)+\gamma_T\boldsymbol{H}^{\mathrm{inc}})\right\|^{\frac{1+\alpha}{\alpha}}_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)}\right)^{\frac{\alpha}{1+\alpha}}\\
&\leq C\left(\tau^m+h^{\alpha(k+1)}\right)^{\frac{2\alpha}{1+\alpha}}.
\end{aligned}$$

We turn towards the estimation of the error of $\boldsymbol{\varphi}^{\tau,h}$, in terms of the $\boldsymbol{L}^{1+\alpha}(\Gamma)$ norm, which is bounded from above via

$$\begin{aligned}
\left|\boldsymbol{\varphi}^{\tau,h}-\boldsymbol{\varphi}\right|&=\left|\boldsymbol{a}^{-1}(\boldsymbol{a}(\boldsymbol{\varphi}^{\tau,h}+\gamma_T\boldsymbol{H}^{\mathrm{inc}}))-\boldsymbol{a}^{-1}(\boldsymbol{a}(\boldsymbol{\varphi}+\gamma_T\boldsymbol{H}^{\mathrm{inc}}))\right|\\
&\leq C\left|\boldsymbol{a}(\boldsymbol{\varphi}^{\tau,h}+\gamma_T\boldsymbol{H}^{\mathrm{inc}})-\boldsymbol{a}(\boldsymbol{\varphi}+\gamma_T\boldsymbol{H}^{\mathrm{inc}})\right|\left|\boldsymbol{a}(\boldsymbol{\varphi}+\gamma_T\boldsymbol{H}^{\mathrm{inc}})\right|^{\frac{1-\alpha}{\alpha}}\\
&\quad+C\left|\boldsymbol{a}(\boldsymbol{\varphi}^{\tau,h}+\gamma_T\boldsymbol{H}^{\mathrm{inc}})-\boldsymbol{a}(\boldsymbol{\varphi}+\gamma_T\boldsymbol{H}^{\mathrm{inc}})\right|^{\frac{1}{\alpha}}.
\end{aligned}$$

Note that the nonlinear defect arising on the right-hand side is simply the pointwise error of $\widetilde{\boldsymbol{\psi}}^{\tau,h}$. Taking both sides to the power of $1+\alpha$ and integrating over the boundary Γ yields

$$\begin{aligned}
\left\|\boldsymbol{\varphi}^{\tau,h}-\boldsymbol{\varphi}\right\|^{1+\alpha}_{\boldsymbol{L}^{1+\alpha}(\Gamma)}\leq C&\left\|\left|\widetilde{\boldsymbol{\psi}}^{\tau,h}-\boldsymbol{\psi}\right|\left|\boldsymbol{\varphi}+\gamma_T\boldsymbol{H}^{\mathrm{inc}}\right|^{1-\alpha}\right\|^{1+\alpha}_{\boldsymbol{L}^{1+\alpha}(\Gamma)}\\
&+C\left\|\widetilde{\boldsymbol{\psi}}^{\tau,h}-\boldsymbol{\psi}\right\|^{\frac{1+\alpha}{\alpha}}_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)}.
\end{aligned}$$

The first summand is effectively rewritten by plugging the boundary condition into $\boldsymbol{\psi}$, which gives the estimate

$$\begin{aligned}
\left\|\left|\widetilde{\boldsymbol{\psi}}^{\tau,h}-\boldsymbol{\psi}\right|\left|\boldsymbol{\varphi}+\gamma_T\boldsymbol{H}^{\mathrm{inc}}\right|^{1-\alpha}\right\|^{1+\alpha}_{\boldsymbol{L}^{1+\alpha}(\Gamma)}&=\int_\Gamma\left|\widetilde{\boldsymbol{\psi}}^{\tau,h}-\boldsymbol{\psi}\right|^{1+\alpha}\left|\boldsymbol{\varphi}+\gamma_T\boldsymbol{H}^{\mathrm{inc}}\right|^{1-\alpha^2}\mathrm{d}\boldsymbol{x}\\
&\leq\left\|\left|\widetilde{\boldsymbol{\psi}}^{\tau,h}-\boldsymbol{\psi}\right|^{1+\alpha}\right\|_{\boldsymbol{L}^{\frac{1}{\alpha}}(\Gamma)}\left\|\left|\boldsymbol{\varphi}+\gamma_T\boldsymbol{H}^{\mathrm{inc}}\right|^{1-\alpha^2}\right\|_{\boldsymbol{L}^{\frac{1}{1-\alpha}}(\Gamma)}\\
&=\left\|\widetilde{\boldsymbol{\psi}}^{\tau,h}-\boldsymbol{\psi}\right\|^{1+\alpha}_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)}\left\|\boldsymbol{\varphi}+\gamma_T\boldsymbol{H}^{\mathrm{inc}}\right\|^{1-\alpha^2}_{\boldsymbol{L}^{1+\alpha}(\Gamma)}.
\end{aligned}$$

The factor depending only on the exact solution is independent of h and τ and bounded due to the regularity assumptions on $\boldsymbol{\varphi}$. These estimates imply bounds on the error of the numerical approximation of the boundary density $\boldsymbol{\varphi}$ via

$$\left(\tau\sum_{n=0}^{N}\left\|\boldsymbol{\varphi}_n^{\tau,h}-\boldsymbol{\varphi}(t_n)\right\|^2_{\boldsymbol{L}^{1+\alpha}(\Gamma)}\right)^{\frac{1}{2}} \leq C\left(\tau\sum_{n=0}^{N}\left\|\widetilde{\boldsymbol{\psi}}_n^{\tau,h}-\boldsymbol{\psi}(t_n)\right\|^2_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)}\right)^{\frac{1}{2}} + C\left(\tau\sum_{n=0}^{N}\left\|\widetilde{\boldsymbol{\psi}}_n^{\tau,h}-\boldsymbol{\psi}(t_n)\right\|^{\frac{2}{\alpha}}_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)}\right)^{\frac{1}{2}}.$$

The first summand on the right-hand side has already been bounded by (65). As a consequence of Minkowski's inequality for $p=\frac{2}{1+\alpha}$, the following estimate holds

$$\tau^{\frac{2}{1+\alpha}}\sum_{n=0}^{N}\left\|\widetilde{\boldsymbol{\psi}}_n^{\tau,h}-\boldsymbol{\psi}(t_n)\right\|^{\frac{2}{\alpha}}_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)} \leq \left(\tau\sum_{n=0}^{N}\left\|\widetilde{\boldsymbol{\psi}}_n^{\tau,h}-\boldsymbol{\psi}(t_n)\right\|^{\frac{1+\alpha}{\alpha}}_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)}\right)^{\frac{2}{1+\alpha}} \leq C\left(\tau^m+h^{\alpha(k+1)}\right)^{\frac{4}{\alpha+1}}.$$

Rearranging and taking the square root on both sides yields the estimate

$$\left(\tau\sum_{n=0}^{N}\left\|\widetilde{\boldsymbol{\psi}}_n^{\tau,h}-\boldsymbol{\psi}(t_n)\right\|^{\frac{2}{\alpha}}_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)}\right)^{\frac{1}{2}} \leq C\tau^{\frac{\alpha-1}{2(1+\alpha)}}\left(\tau^m+h^{\alpha(k+1)}\right)^{\frac{2}{1+\alpha}} \leq C\left(\tau^m+h^{\alpha(k+1)}\right)^{\frac{2\alpha}{1+\alpha}},$$

where the final estimate holds for the mild step size restriction $\tau \leq Ch^{4\alpha(k+1)}$, since then

$$\tau^{\frac{\alpha-1}{2(1+\alpha)}}\left(\tau^m+h^{\alpha(k+1)}\right)^{\frac{2}{1+\alpha}-\frac{2\alpha}{1+\alpha}} \leq 2\left(\tau^{-\frac{1}{2}}\left(\tau^{2m}+h^{2\alpha(k+1)}\right)\right)^{\frac{1-\alpha}{1+\alpha}} \leq C.$$

Overall, we obtain the complete error bound

$$\left(\tau\sum_{n=0}^{N}\left\|\boldsymbol{\varphi}_n^{\tau,h}-\boldsymbol{\varphi}(t_n)\right\|^2_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)}\right)^{\frac{1}{2}} + \left(\tau\sum_{n=0}^{N}\left\|\widetilde{\boldsymbol{\psi}}_n^{\tau,h}-\boldsymbol{\psi}(t_n)\right\|^2_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)}\right)^{\frac{1}{2}} \leq C\left(\tau^m+h^{\alpha(k+1)}\right)^{\frac{2\alpha}{1+\alpha}}. \tag{66}$$

(iii) *Pointwise error bound* Finally, the bound of Lemma 4, formulated for the combined block potential operator defined in (60) reads

$$\left\|\boldsymbol{\mathcal{W}}_{\boldsymbol{x}}(s)\right\|_{\mathbb{C}^3\times\mathbb{C}^3\leftarrow\boldsymbol{L}^{1+\alpha}(\Gamma)\times\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Gamma)} \leq C\left|s\right|.$$

The stated result is now given by [7, Lemma 5.2] in combination with the error bound (66). □

6. Implementation and numerical experiments

The proposed scheme (57) has been realized in Python by making use of the boundary element library Bempp [34] to discretize the boundary integral operators with Raviart-Thomas elements. All codes used to generate the figures of this section are distributed through [28].

Consider two unit cubes, shifted from the origin, such that they are separated by a gap of length $l = 0.5$. An electric planar wave illuminates the scatterers, and is determined by the closed form

$$\boldsymbol{E}^{\text{inc}}(t, x) = e^{-c(t-x_3-t_0)^2}\boldsymbol{e}_1, \tag{67}$$

with the orientation of $\boldsymbol{e_1} = (1, 0, 0)^T$, $t_0 = -2$ and $c = 100$. This incidental wave is scattered from both cubes, where the nonlinear boundary condition with $\alpha = 0.5$ is employed at the boundary. We observe the scattered wave then until the final time $T = 3$, which captures most of the interaction of the incident field with the cubes.

In this setting, we employ the full discretization of the boundary integral equation and evaluate the time-discrete representation formula to obtain approximations of the scattered wave away from the boundary.

In order to quantify the error of this approximation, the electric field has been computed at a single point away from the boundary (for our computations we used the origin $\boldsymbol{P} = (0, 0, 0)$). The error is then estimated by computing a reference solution, for which $N = 256$ time steps using the 3-stage Radau IIA based Runge–Kutta convolution quadrature method were used in combination with a 0-th order Raviart–Thomas boundary element discretization with 6228 degrees of freedom, which corresponds to the mesh width $h = 2^{-7/2}$. Mutually fixing the spatial resolution h, or respectively the time step size τ, then yields the convergence plots in Figure 1 and Figure 2.

In view of Theorem 2, we expect the error to decay with at least the order $\mathcal{O}(\tau^2 + h^{1/2})$, though the order reductions from the error bounds from Theorem 3 could reduce the convergence rates of the point evaluations. Furthermore, the assumed regularity of the solutions is unlikely to hold due to the low regularity of the scatterer.

In practice, the space convergence rate seems to be higher than the expected order $\mathcal{O}(h^{1/2})$ and more accurately described by $\mathcal{O}(h)$. The sharp increase in accuracy for the final data points might be explained by the comparable parameters used to compute the reference solution. The observed time convergence is more accurately captured by the theory, although the predicted order reductions for point evaluations (from Theorem 3) are not observed either. Asymptotically, the errors seem to approach the order $\mathcal{O}(\tau^2)$, though the empirical orders are below this expected order of convergence.

Overall, good convergence properties are observed despite the low regularity of the scatterer. Moreover, this convergence behavior is observed despite many underlying approximations during the implementation, such as the quadrature for the boundary integral operators, hierarchical matrix compression, iterative solution of the linear systems, Newton's method to solve the nonlinear system at each time step and the trapezoidal rule underlying the convolution quadrature method.

The paper concludes with Figure 3, a visualization of the wave, which shows the $z = 0.5$ plane at several time points.

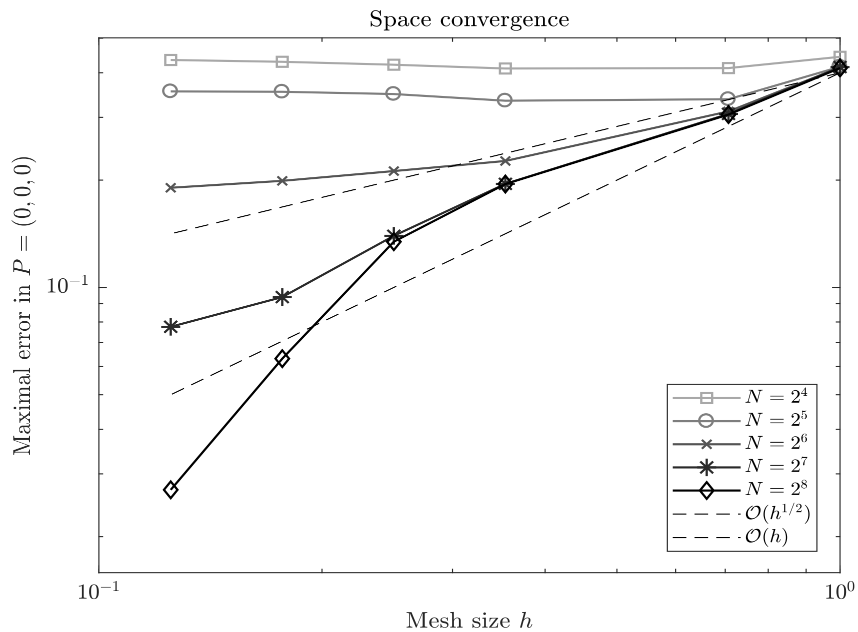

FIGURE 1. Space convergence plot of the fully discrete system, for 0th order Raviart–Thomas boundary elements and the 2-stage Radau IIA based Runge–Kutta convolution quadrature method. The grids generated by the mesh generator embedded in the boundary element library Bempp coincide for $h = 2^{-1/2}$ and $h = 2^{-1}$, which causes the larger gap between the 4-th and the 5-th data point.

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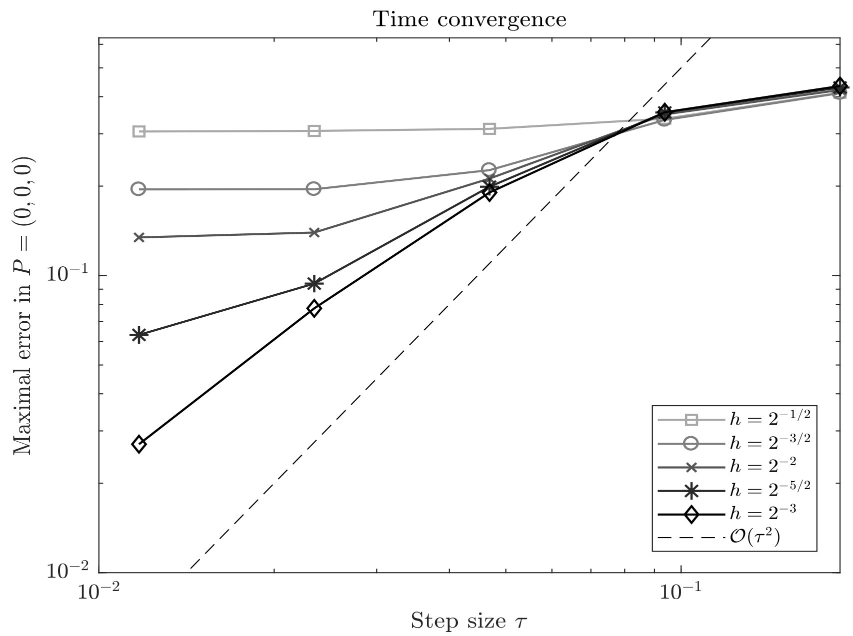

FIGURE 2. Time convergence plot of the fully discrete system, for 0th order Raviart–Thomas boundary elements and the 2-stage Radau IIA based Runge–Kutta convolution quadrature method.

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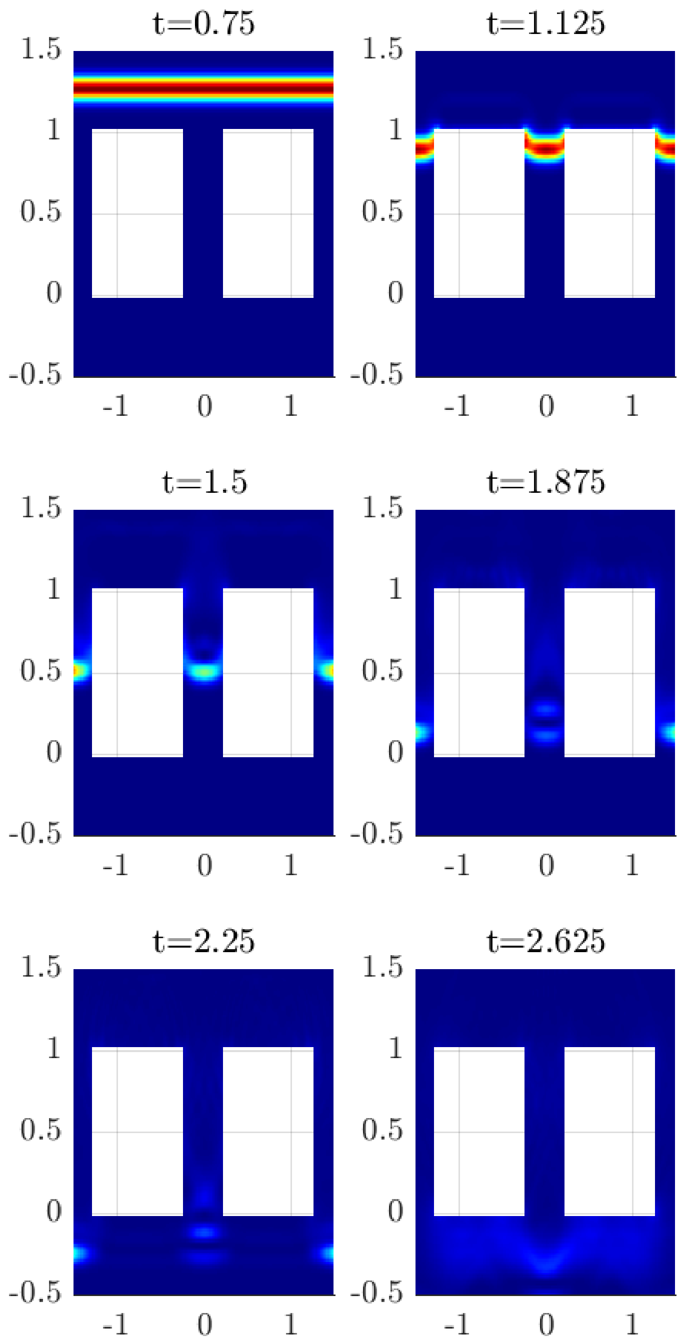

FIGURE 3. 3D-scattering arising from two unit cubes, with a gap between them at several time points. The approximation to the scattered wave was computed with $N = 256$ time steps until the final time $T = 3$ using the 3-stage Radau IIA method, in combination with 0th-order Raviart-Thomas boundary elements corresponding to the mesh width $h = 2^{-5/2}$, which results in a boundary element space with 1620 degrees of freedom.

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