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ORIGINAL PAPER

Entropies of non-positively curved metric spaces

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Abstract

We show the equivalences of several notions of entropy, like a version of the topological entropy of the geodesic flow and the Minkowski dimension of the boundary, in metric spaces with convex geodesic bicombings satisfying a uniform packing condition. Similar estimates will be given in case of closed subsets of the boundary of Gromov-hyperbolic metric spaces with convex geodesic bicombings.

Contents

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1 Introduction

This paper is devoted to the investigation of different asymptotic quantities associated to a metric space, some of them classical and widely studied. As we will see in a minute these invariants have different nature: dynamical, measure-theoretic and combinatorial. The purpose is two-fold:

- to show the relations between these invariants, especially to understand when they are equal;
- to develop interesting tools and techniques to simplify the computation of these invariants.

The second goal is extremely useful for applications: it is important to have flexible and easy to compute definitions in order to study these invariants in concrete cases. In the forthcoming paper [9] we will use this flexibility to extend Otal-Peigné's Theorem ([22]) to a large class of metric spaces. A simplified version of the techniques developed on this paper were used to show the continuity of the critical exponent of fundamental groups of a large class of compact metric spaces under Gromov-Hausdorff convergence, see [10].
We are mainly interested on metric spaces satisfying weak bounds on the curvature. As upper bound we consider a very weak convexity condition: the existence of a convex geodesic bicombing. A geodesic bicombing on a metric space X is a map $\sigma\colon X \times X \times [0, 1] \to X$ such that for all $x, y \in X$ the function $\sigma_{xy}(\cdot) = \sigma(x, y, \cdot)$ is a geodesic (parametrized proportionally to arc-length) from x to y. A bicombing σ is convex if for all $x, y, x', y' \in X$ the map $t \mapsto d(\sigma_{xy}(t), \sigma_{x'y'}(t))$ is convex. Among metric spaces admitting a convex bicombing there are CAT(0)-spaces, Busemann convex spaces and all normed vector spaces. The interest in this condition is given by its stability under limits ([13, 15]), while it is not the case for Busemann convex spaces. Given a bicombing σ, every curve σ_{xy} is called a σ-geodesic. The bicombing is geodesically complete if any σ-geodesic can be extended to a bigger σ-geodesic. This notion coincides with the usual geodesic completeness in case of Busemann convex metric spaces. A GCB-space is a couple (X, σ) where X is a complete metric space and σ is a geodesically complete, convex, geodesic bicombing on X.

As a lower bound on the curvature we take a uniform packing condition: a metric space X is said to be P_0-packed at scale r_0 if for all $x \in X$ the cardinality of a maximal $2r_0$-separated subset of $B(x, 3r_0)$ is at most P_0. This uniform packing condition interacts very well with the weak convexity property given by a geodesically complete, convex, geodesic bicombing, implying a uniform control of the packing condition *at every scale* (see Proposition 2.1 and [13]). Sometimes, especially in the second part of the paper where relative versions of the invariants will be studied, we will impose also a Gromov-hyperbolicity condition on our metric space.

1.1 Lipschitz-topological entropy of the geodesic flow

The topological entropy of the geodesic flow has been intensively studied in case of Riemannian manifolds, especially in the negatively curved setting. If such a manifold is denoted by $\bar{M} = M/\Gamma$, where M is its universal cover and Γ is its fundamental group, then the set of parametrized geodesic lines is identified with the unit tangent bundle $S\bar{M}$. Probably the most important invariant associated to the geodesic flow is its topological entropy, denoted $h_{\text{top}}(\bar{M})$. It equals the Hausdorff dimension of the limit set of Γ and the critical exponent of Γ (see [22, 27]). Moreover if $\bar{M}$ is compact then it coincides also with the volume entropy of M ([21]), while this is no more true in general, even when $\bar{M}$ has finite volume (cp. [16]): we will come back to these examples at the end of the introduction.

In case of GCB-metric spaces (X, σ) we restrict the attention to σ-geodesic lines. The topological entropy of the σ-geodesic flow is defined as the topological entropy (in the sense of Bowen, cp. [6, 19]) of the dynamical system $(\mathrm{Geod}_\sigma(X), \Phi_t)$, where $\mathrm{Geod}_\sigma(X)$ is the space of parametrized σ-geodesic lines, endowed with the topology of uniform convergence on compact subsets, and Φ_t is the reparametrization flow. It is:

$$h_{\mathrm{top}}(\mathrm{Geod}_\sigma(X)) = \inf_{\mathfrak{d}} \sup_{K \subseteq \mathrm{Geod}_\sigma(X)} \lim_{r\to 0} \lim_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}_{\mathfrak{d}^T}(K, r),$$

where the infimum is taken *among all metrics on* $\mathrm{Geod}_\sigma(X)$ inducing its topology, the supremum is taken among all compact subsets of $\mathrm{Geod}_\sigma(X)$, $\mathfrak{d}^T$ is the distance $\mathfrak{d}^T(\gamma, \gamma') = \max_{t\in[0,T]} \mathfrak{d}(\Phi_t(\gamma), \Phi_t(\gamma'))$ and $\mathrm{Cov}_{\mathfrak{d}^T}(K, r)$ is the minimal number of balls (with respect to the metric $\mathfrak{d}^T$) of radius r needed to cover K. We remark that in case of Busemann convex (or CAT(0)) metric spaces the space of σ-geodesic lines coincides with the set of geodesic lines. This flow has no recurrent geodesics, so applying the variational principle (cp. [19]) it is straightforward to conclude that its topological entropy is zero (Lemma 4.1). Looking carefully at the proof of the variational principle it turns out that the metrics on $\mathrm{Geod}_\sigma(X)$ almost realizing the infimum in the definition of the topological entropy are restriction to $\mathrm{Geod}_\sigma(X)$ of metrics on its one-point compactification. In particular they are no the natural ones to consider in this setting. That is why, in Sect. 4, we will restrict the attention to the class of *geometric metrics* $\mathfrak{d}$: those with the property that the evaluation map $E\colon (\mathrm{Geod}_\sigma(X), \mathfrak{d}) \to (X, d)$ defined as $E(\gamma) = \gamma(0)$ is Lipschitz. Notice that for a geometric metric two geodesic lines are not close if they are distant at time 0. Accordingly the *Lipschitz-topological entropy* of the geodesic flow is defined as

$$h_{\mathrm{Lip\text{-}top}}(\mathrm{Geod}_\sigma(X)) = \inf_{\mathfrak{d}} \sup_{K \subseteq \mathrm{Geod}_\sigma(X)} \lim_{r\to 0} \lim_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}_{\mathfrak{d}^T}(K, r),$$

where *the infimum is taken only among the geometric metrics of* $\mathrm{Geod}_\sigma(X)$. Although the definition of the Lipschitz-topological entropy is quite complicated, its computation can be remarkably simplified. Indeed one of the most used metric on $\mathrm{Geod}_\sigma(X)$ (see for instance [5]) is:

$$d_{\mathrm{Geod}}(\gamma, \gamma') = \int_{-\infty}^{+\infty} d(\gamma(s), \gamma'(s)) \frac{1}{2e^{|s|}} ds$$

that induces the topology of $\mathrm{Geod}_\sigma(X)$ and is geometric, and it turns out that it realizes the infimum in the definition of the Lipschitz-topological entropy.

Theorem A (Extract from Theorem 4.2 & Proposition 4.3). *Let* (X, σ) *be a* GCB*-space that is* P_0*-packed at scale* r_0*. Then*

$$h_{\mathrm{Lip\text{-}top}}(\mathrm{Geod}_\sigma(X)) = \lim_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}_{d^T_{\mathrm{Geod}}}(\mathrm{Geod}_\sigma(x), r_0),$$

where $\mathrm{Geod}_\sigma(x)$ *is the set of* σ*-geodesic lines passing through* x *at time* 0.

Therefore the infimum in the definition of the Lipschitz topological entropy is actually realized by the metric d_{Geod} and the supremum among the compact sets can be replaced by a fixed (relatively small) compact set. Moreover also the scale r can be fixed to be r_0 (or any other positive real number).
Actually the result of Theorem A is still valid for a whole family of metrics on $\mathrm{Geod}_\sigma(X)$: this flexibility will be one of the fundamental ingredient in the main result of [9].

1.2 Volume and covering entropy

The second definition of entropy we consider (see Sect. 3.2) is the volume entropy. If X is a metric space equipped with a measure μ it is classical to consider the exponential growth rate of the volume of balls, namely:

$$h_\mu(X) := \lim_{T\to+\infty} \frac{1}{T}\log \mu(\overline{B}(x,T)).$$

It is called the *volume entropy* of X with respect to the measure μ and it does not depend on the choice of the basepoint $x \in X$ by triangular inequality. This invariant has been studied intensively in case of complete Riemannian manifolds with non positive sectional curvature, where μ is the Riemannian volume on the universal cover. It is related to other interesting invariants as the simplicial volume of the manifold (see [18]), [8]), a macroscopical condition on the scalar curvature (cp. [24]) and the systole in case of compact, non-geometric 3-manifolds (cp. [12]). Moreover the infimum of the volume entropy among all the possible Riemannian metrics of volume 1 on a fixed closed manifold is a subtle homotopic invariant (see [1, 7] for general considerations and [2, 23] for the computation of the minimal volume entropy in case of, respectively, closed n-dimensional manifolds supporting a locally symmetric metric of negative curvature and 3-manifolds).
A measure μ is called H-homogeneous at scale r if

$$\frac{1}{H} \leq \mu(\overline{B}(x,r)) \leq H$$

for every $x \in X$. Among homogeneous measures there is a remarkable example: the volume measure μ_X of a complete, geodesically complete, CAT(0) metric space X that is P_0-packed at scale r_0 (see [14] and [20] for a description of the measure). If X is a Riemannian manifold of non-positive sectional curvature then μ_X coincides with the Riemannian volume, up to a universal multiplicative constant.
A more combinatorial and intrinsic version of the volume entropy of a generic metric space is the *covering entropy*, defined as:

$$h_{\mathrm{Cov}}(X) := \lim_{T\to+\infty} \frac{1}{T}\log \mathrm{Cov}(\overline{B}(x,T), r),$$

where x is a point of X and $\mathrm{Cov}(\overline{B}(x,T), r)$ is the minimal number of balls of radius r needed to cover $\overline{B}(x,T)$. It does not depend on x but it can depend on the choice of r. This is not the case when X is a GCB-space that is P_0-packed at scale r_0, as follows by Proposition 3.1. Moreover it is always finite (cp.Lemma 3.3).

1.3 Minkowski dimension of the boundary

The expression of the Lipschitz-topological entropy given by Theorem A suggests the possibility to relate that invariant to some property of the boundary at infinity of X. For simplicity we suppose X is also Gromov-hyperbolic, so that the boundary at infinity is metrizable. If we denote by $(\cdot,\cdot)_x$ the Gromov product based on x then the generalized visual ball of center $z \in \partial X$ and radius ρ is $B(z,\rho) = \{z' \in \partial X \text{ s.t. } (z,z')_x > \log\frac{1}{\rho}\}$. The *visual Minkowski dimension* of the Gromov boundary ∂X is:

$$\mathrm{MD}(\partial X) = \lim_{T\to+\infty} \frac{1}{T}\log \mathrm{Cov}(\partial X, e^{-T}),$$

where $\mathrm{Cov}(\partial X, e^{-T})$ is the minimal number of generalized visual balls of radius e^{-T} needed to cover ∂X. If the generalized visual balls are metric balls for some visual metric $D_{x,a}$ then we refind the usual definition of Minkowski dimension of the metric space $(\partial X, D_{x,a})$, once the obvious change of variable $\rho = e^{-T}$ is made. This invariant is presented in Sect. 5.1.

1.4 Equality of the entropies

One of our main results is:

Theorem B *Let (X, σ) be a* GCB*-space that is P_0-packed at scale r_0. Then*

$$h_{\text{Lip-top}}(\mathrm{Geod}_\sigma(X)) = h_\mu(X) = h_{\mathrm{Cov}}(X),$$

where μ is every homogeneous measure on X. Moreover if X is also δ-hyperbolic then the quantities above coincide also with $\mathrm{MD}(\partial X)$.

Actually something more is true but in order to state it we need to recall the notion of equivalent asymptotic behaviour of two functions introduced in [10]. Given $f, g\colon [0, +\infty) \to \mathbb{R}$ we say that *f and g have the same asymptotic behaviour*, and we write $f \asymp g$, if for all $\varepsilon > 0$ there exists $T_\varepsilon \geq 0$ such that if $T \geq T_\varepsilon$ then $|f(T) - g(T)| \leq \varepsilon$. The function T_ε is called the *threshold function*. Usually we will write $f \underset{P_0, r_0, \delta, \ldots}{\asymp} g$ meaning that the threshold function can be expressed only in terms of ε and $P_0, r_0, \delta, \ldots$

Theorem C *Let (X, σ) be a* GCB*-space that is P_0-packed at scale r_0. Then the functions defining the quantities of Theorem B have the same asymptotic behaviour and the threshold functions depend only on P_0, r_0, δ and the homogeneous constants of μ.*

Therefore not only all the introduced quantities define the same number, but all of them also have the same asymptotic behaviour. This means that if one can control the rate of convergence to the limit of one of these quantities then also the rate of convergence of all the other quantities is bounded. We remark that, differently from many of the papers in the literature, we do not require any group action on our metric spaces.
The control of the asymptotic behaviour of the function defining the Minkowski dimension is the main ingredient of the continuity theorem proved in [10]. The same ideas can be used to show similar continuity statements in more general settings, but we won't explore these applications here.

1.5 Entropies of the closed subsets of the boundary

In case X is δ-hyperbolic it is possible to define the versions of all the different notions of entropies relative to subsets of the boundary ∂X. For every subset $C \subseteq \partial X$ we denote by $\mathrm{Geod}_\sigma(C)$ the set of parametrized σ-geodesic lines with endpoints belonging to C and with $\text{QC-Hull}(C)$ the union of the points belonging to the geodesics joining any two points of C. Actually the hyperbolicity assumption (or at least a visibility assumption on ∂X) is necessary since otherwise the sets $\mathrm{Geod}_\sigma(C)$ and QC-Hull(C) could be empty. The numbers

$$h_{\mathrm{Cov}}(C) = \lim_{T\to+\infty} \frac{1}{T}\log \mathrm{Cov}(\overline{B}(x,T)\cap \text{QC-Hull}(C), r_0)$$

$$h_{\text{Lip-top}}(\mathrm{Geod}_\sigma(C)) = \inf_{\mathrm{d}} \sup_{K\subseteq \mathrm{Geod}_\sigma(C)} \lim_{r\to+\infty}\lim_{T\to+\infty}\frac{1}{T}\log \mathrm{Cov}_{\mathrm{d}^T}(K,r)$$

$$\mathrm{MD}(C) = \lim_{T\to+\infty}\frac{1}{T}\log \mathrm{Cov}(C, e^{-T})$$

are called, respectively, covering entropy of C, Lipschitz-topological entropy of $\mathrm{Geod}_\sigma(C)$ and visual Minkowski dimension of C. The volume entropy of C with respect to a measure μ is

$$h_\mu(C) = \sup_{\tau\geq 0}\lim_{T\to+\infty}\frac{1}{T}\log \mu(\overline{B}(x,T)\cap \overline{B}(\text{QC-Hull}(C),\tau)),$$

where $\overline{B}(Y,\tau)$ is the closed τ-neighbourhood of $Y\subseteq X$. If μ is H-homogeneous at scale r then the volume entropy can be computed taking $\tau = r$ in place of the supremum (Proposition 6.3). For instance when X is a Riemannian manifold with pinched negative curvature then the the Riemannian volume μ_X is $H(r)$-homogeneous at every scale $r > 0$, so the definition does not depend on τ at all. Most of the relations of Theorem B remain true for subsets of the boundary, but the asymptotic behaviour of the different functions involved in the definitions of the entropies depend also on the choice of the basepoint $x\in X$. The best possible choice, $x\in$ QC-Hull(C), allows us to give again uniform asymptotic estimates.

Theorem D *Let (X,σ) be a δ-hyperbolic* GCB*-space that is P_0-packed at scale r_0 and let $C\subseteq\partial X$. Then*

$$h_{\mathrm{Cov}}(C) = \mathrm{MD}(C) = h_\mu(C)$$

for every homogeneous measure μ on X. All the functions defining the quantities above have the same asymptotic behaviour and the threshold functions can be expressed only in terms of P_0, r_0, δ and the homogeneous constants of μ, if the basepoint x belongs to QC-Hull(C).

The proof of this result does not follow by the same arguments of Theorem B, indeed it will be based heavily on the Gromov-hyperbolicity of X. The relation between the Lipschitz-topological entropy of $\mathrm{Geod}_\sigma(C)$ and the other definitions of entropy is more complicated. We have

Theorem E *Let (X,σ) be a δ-hyperbolic* GCB*-space that is P_0-packed at scale r_0 and let $C\subseteq\partial X$. Then*

(i) if C is closed then $h_{\text{Lip-top}}(\mathrm{Geod}_\sigma(C)) = h_{\mathrm{Cov}}(C)$ and the functions defining these two quantities have the same asymptotic behaviour with thresholds function depending only on P_0, r_0, δ.
More precisely, if $x\in$ QC-Hull(C) *then*

$$h_{\text{Lip-top}}(\mathrm{Geod}_\sigma(C)) = \lim_{T\to+\infty}\frac{1}{T}\log \mathrm{Cov}_{d^T_{\mathrm{Geod}}}(\mathrm{Geod}_\sigma(\overline{B}(x,22\delta),C), r_0),$$

where $\mathrm{Geod}_\sigma(\overline{B}(x,22\delta),C)$ *is the set of geodesic lines with endpoints in C and passing through $\overline{B}(x,22\delta)$ at time* 0.

(ii) if C is not closed then

$$h_{\text{Lip-top}}(\text{Geod}_\sigma(C)) = \sup_{C' \subseteq C} h_{\text{Lip-top}}(\text{Geod}_\sigma(C')) \leq h_{\text{Cov}}(C),$$

where the supremum is taken among the closed subsets of C.

The inequality in (ii) can be strict, as shown in [9].

1.6 Differences of the invariants for geometrically finite groups

In this last part of the introduction we restrict the attention to the case of Riemannian manifolds $\bar{M} = M/\Gamma$ with pinched negative sectional curvature. If Γ is geometrically finite then the limit set of Γ is the union of the radial limit set $\Lambda_{\mathrm{r}}(\Gamma)$ and the bounded parabolic points. The latter is a countable set, therefore the Hausdorff dimension of the limit set coincides with the Hausdorff dimension of the radial limit set. So by Bishop-Jones' Theorem it holds (here HD denotes the Hausdorff dimension):

$$\text{HD}(\Lambda(\Gamma)) = \text{HD}(\Lambda_r(\Gamma)) = h_\Gamma. \tag{1.1}$$

We remark that this is not true if Γ is not geometrically finite, even when M is the hyperbolic space.

Example 1.1 In general it can happen $\text{HD}(\Lambda_{\mathrm{r}}(\Gamma))) < \text{HD}(\Lambda(\Gamma))$. Indeed let Γ be a cocompact group of $\mathbb{H}^2$ and let Γ' be a normal subgroup of Γ such that Γ/Γ' is not amenable. Let $F \subseteq \Lambda(\Gamma')$ be the subsets of points z that are fixed by some $g \in \Gamma'$. For every $z \in F$ and every $h \in \Gamma$ we have $hz = hgz = g'hz$ for some $g' \in \Gamma'$ since Γ' is normal. Then hz is fixed by g' and so it belongs to F, i.e. F is Γ-invariant. By minimality of $\Lambda(\Gamma)$ we get $\Lambda(\Gamma') = \Lambda(\Gamma)$, so $\text{HD}(\Lambda(\Gamma')) = \text{HD}(\Lambda(\Gamma))$. But by the growth tightness of Γ (cp. for instance [25]) we have

$$\text{HD}(\Lambda_{\mathrm{r}}(\Gamma')) = h_{\Gamma'} < h_\Gamma = \text{HD}(\Lambda(\Gamma)) = \text{HD}(\Lambda(\Gamma')).$$

However if M is the hyperbolic space and Γ is geometrically finite then even something more is true, indeed by [26]:

$$h_\Gamma = \text{HD}(\Lambda(\Gamma)) = \text{MD}(\Lambda(\Gamma)). \tag{1.2}$$

This equality fails to be true for geometrically finite (actually of finite covolume) groups of manifolds with pinched, but variable, negative curvature. Indeed we have:

Example 1.2 In [16] it is presented an example of a smooth Riemannian manifold M with pinched negative sectional curvature admitting a (non-uniform) lattice (i.e. a group of isometries Γ with $\text{Vol}(M/\Gamma) < +\infty$) such that $h_\Gamma < h_{\mu_M}(M)$ (recall that μ_M denotes the Riemannian volume of M). We observe that since Γ is a lattice then $\Lambda(\Gamma) = \partial M$, so $h_{\mu_M}(M) = \text{MD}(\Lambda(\Gamma))$ by Theorem B, while $h_\Gamma = \text{HD}(\Lambda_{\mathrm{r}}(\Gamma)) = \text{HD}(\Lambda(\Gamma))$ by (1.1).

The example above is due to a relevant variation of the curvature of M. Indeed in [17] is shown that for non-uniform lattices Γ of asymptotically $1/4$-pinched manifolds with negative curvature M it holds $h_{\mu_M}(M) = h_\Gamma$. The general situation in the geometrically finite case is:

$$\begin{aligned} &\text{HD}(\Lambda(\Gamma)) = h_\Gamma = h_{\text{top}}(M/\Gamma) \\ &h_{\text{Lip-top}}(\text{Geod}(\Lambda(\Gamma))) = h_{\text{Cov}}(\Lambda(\Gamma)) = h_{\mu_M}(\Lambda(\Gamma)) = \text{MD}(\Lambda(\Gamma)), \end{aligned} \tag{1.3}$$

where the equalities in the second line follow by Theorem D and Theorem E, while the equalities in the first line are consequences of (1.1) and Otal-Peigne's variational principle [22]. Moreover it is clear that the first line is always less than or equal to the second one, since the Hausdorff dimension is always smaller than or equal to the Minkowski dimension. The relations in (1.3) allow us to give new interpretations of the phenomena occurring in Example 1.2, i.e. the possible difference between the critical exponent of the group and the volume entropy of $\Lambda(\Gamma)$:

- *measure-theoretic interpretation:* it can be seen as the difference between the Hausdorff and the Minkowski dimension of the limit set $\Lambda(\Gamma)$, so it is related to the fractal structure of the limit set;
- *dynamical interpretation:* it can be seen as the difference between the topological entropy of the geodesic flow of the quotient and the Lipschitz-topological entropy of $\mathrm{Geod}(\Lambda(\Gamma))$.
- *combinatorial interpretation:* it can be seen as the difference between h_Γ and $h_{\mathrm{Cov}}(\Lambda(\Gamma))$, where the former counts the exponential growth rate of an orbit while the latter counts the exponential growth rate of the cardinality of r-nets, for some (any) $r > 0$. Here the difference arises in terms of sparsity of the orbit.

2 GCB-spaces and space of geodesics

Throughout the paper X will denote a metric space with metric d. The open (resp.closed) ball of radius r and center x is denoted by $B(x, r)$ (resp. $\overline{B}(x, r)$), while the metric sphere of center x and radius R is denoted by $S(x, R)$. We use the notation $A(x, r, r')$ to denote the closed annulus of center x and radii $0 < r < r'$, i.e. the set of points $y \in X$ such that $r \leq d(x, y) \leq r'$. A geodesic segment is an isometry $\gamma\colon I \to X$ where $I = [a, b]$ is a a bounded interval of $\mathbb{R}$. The points $\gamma(a), \gamma(b)$ are called the endpoints of γ. A metric space X is said geodesic if for all couple of points $x, y \in X$ there exists a geodesic segment whose endpoints are x and y. We will denote any geodesic segment between two points x and y, with an abuse of notation, as $[x, y]$. A geodesic ray is an isometry $\gamma\colon [0, +\infty) \to X$ while a geodesic line is an isometry $\gamma\colon \mathbb{R} \to X$.
Let Y be any subset of a metric space X:
– a subset S of Y is called r-*dense* if $\forall y \in Y\ \exists z \in S$ such that $d(y, z) \leq r$;
– a subset S of Y is called r-*separated* if $\forall y, z \in S$ it holds $d(y, z) > r$.
The packing number of Y at scale r is the maximal cardinality of a $2r$-separated subset of Y and it is denoted by $\mathrm{Pack}(Y, r)$. The covering number of Y is the minimal cardinality of a r-dense subset of Y and it is denoted by $\mathrm{Cov}(Y, r)$. The following inequalities are classical:

$$\mathrm{Pack}(Y, 2r) \leq \mathrm{Cov}(Y, 2r) \leq \mathrm{Pack}(Y, r). \tag{2.1}$$

The packing and the covering functions of X are respectively

$$\mathrm{Pack}(R, r) = \sup_{x \in X} \mathrm{Pack}(\overline{B}(x, R), r), \qquad \mathrm{Cov}(R, r) = \sup_{x \in X} \mathrm{Cov}(\overline{B}(x, R), r).$$

They take values on $[0, +\infty]$. By (2.1) it holds

$$\mathrm{Pack}(R, 2r) \leq \mathrm{Cov}(R, 2r) \leq \mathrm{Pack}(R, r). \tag{2.2}$$

Let $C_0, P_0, r_0 > 0$. We say that a metric space X is P_0-*packed at scale* r_0 if $\mathrm{Pack}(3r_0, r_0) \leq P_0$, that is every ball of radius $3r_0$ contains no more than P_0 points that are $2r_0$-separated. The space X is C_0-*covered at scale* r_0 if $\mathrm{Cov}(3r_0, r_0) \leq C_0$, that is every ball of radius $3r_0$ can be covered by at most C_0 balls of radius r_0.

A geodesic bicombing on a metric space X is a map $\sigma\colon X \times X \times [0,1] \to X$ with the property that for all $(x, y) \in X \times X$ the map $\sigma_{xy}\colon t \mapsto \sigma(x, y, t)$ is a geodesic from x to y parametrized proportionally to arc-length, i.e. $d(\sigma_{xy}(t), \sigma_{xy}(t')) = |t - t'| d(x, y)$ for all $t, t' \in [0, 1]$, $\sigma_{xy}(0) = x$, $\sigma_{xy}(1) = y$.
When X is equipped with a geodesic bicombing then for all $x, y \in X$ we will denote by $[x, y]$ the geodesic σ_{xy} parametrized by arc-length.
A geodesic bicombing is:

- *convex* if the map $t \mapsto d(\sigma_{xy}(t), \sigma_{x'y'}(t))$ is convex on $[0, 1]$ for all $x, y, x', y' \in X$;
- *consistent* if for all $x, y \in X$, for all $0 \leq s \leq t \leq 1$ and for all $\lambda \in [0, 1]$ it holds $\sigma_{pq}(\lambda) = \sigma_{xy}((1-\lambda)s + \lambda t)$, where $p := \sigma_{xy}(s)$ and $q := \sigma_{xy}(t)$;
- *reversible* if $\sigma_{xy}(t) = \sigma_{yx}(1-t)$ for all $t \in [0, 1]$.

For instance every convex metric space in the sense of Busemann (so also any CAT(0) metric space) admits a unique convex, consistent, reversible geodesic bicombing.
Given a geodesic bicombing σ we say that a geodesic (segment, ray, line) γ is a σ-geodesic (segment, ray, line) if for all $x, y \in \gamma$ we have that $[x, y]$ coincides with the subsegment of γ between x and y.
A geodesic bicombing is *geodesically complete* if every σ-geodesic segment is contained in a σ-geodesic line. A couple (X, σ) is said a GCB-space if σ is a convex, consistent, reversible, geodesically complete geodesic bicombing on the complete metric space X. The packing condition has a controlled behaviour in GCB-spaces.

Proposition 2.1 (Proposition 3.2 of [13]) *Let (X, σ) be a* GCB*-space that is P_0-packed at scale r_0. Then:*

(i) for all $r \leq r_0$, the space X is P_0-packed at scale r and is proper;
(ii) for every $0 < r \leq R$ and every $x \in X$ it holds:

$$\mathrm{Pack}(R, r) \leq P_0(1 + P_0)^{\frac{R}{r} - 1}, \text{ if } r \leq r_0;$$
$$\mathrm{Pack}(R, r) \leq P_0(1 + P_0)^{\frac{R}{r_0} - 1}, \text{ if } r > r_0;$$
$$\mathrm{Cov}(R, r) \leq P_0(1 + P_0)^{\frac{2R}{r} - 1}, \text{ if } r \leq 2r_0;$$
$$\mathrm{Cov}(R, r) \leq P_0(1 + P_0)^{\frac{R}{r_0} - 1}, \text{ if } r > 2r_0.$$

Basic examples of GCB-spaces that are P_0-packed at scale r_0 are:

i) complete and simply connected Riemannian manifolds with sectional curvature pinched between two nonpositive constants $\kappa' \leq \kappa < 0$;
ii) simply connected M^κ-complexes, with $\kappa \leq 0$, without free faces and *bounded geometry* (i.e., with valency at most V_0, size at most S_0 and positive injectivity radius);
iii) complete, geodesically complete, CAT(0) metric spaces X with dimension at most n and volume of balls of radius R_0 bounded above by V.

For further details on the second and the third class of examples we refer to [14].
When (X, σ) is a proper GCB-space we can consider the *space of parametrized geodesic lines* of X,

$$\mathrm{Geod}(X) = \{\gamma\colon \mathbb{R} \to X \text{ isometry}\},$$

endowed with the topology of uniform convergence on compact subsets of $\mathbb{R}$, and its subset $\mathrm{Geod}_\sigma(X)$ made of elements whose image is a σ-geodesic line. By the continuity of σ (due to its convexity, see [13, 15]) we have that $\mathrm{Geod}_\sigma(X)$ is closed in $\mathrm{Geod}(X)$. There is a natural action of $\mathbb{R}$ on $\mathrm{Geod}(X)$ defined by reparametrization:

$$\Phi_t\gamma(\cdot) = \gamma(\cdot + t)$$

for every $t \in \mathbb{R}$. It is easy to see it is a continuous action, i.e. $\Phi_t \circ \Phi_s = \Phi_{t+s}$ for all $t, s \in \mathbb{R}$ and for every $t \in \mathbb{R}$ the map Φ_t is a homeomorphism of $\mathrm{Geod}(X)$. Moreover the action restricts as an action on $\mathrm{Geod}_\sigma(X)$. This action on $\mathrm{Geod}_\sigma(X)$ is called the *σ-geodesic flow* on X. The *evaluation map* $E\colon \mathrm{Geod}(X) \to X$, which is defined as $E(\gamma) = \gamma(0)$, is continuous and proper ([5], Lemma 1.10), so its restriction to $\mathrm{Geod}_\sigma(X)$ has the same properties. Moreover this restriction is surjective since σ is assumed geodesically complete. The topology on $\mathrm{Geod}_\sigma(X)$ is metrizable. Indeed we can construct a family of metrics on $\mathrm{Geod}_\sigma(X)$ with the following method.
Let $\mathcal{F}$ be the class of continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

(a) $f(s) > 0$ for all $s \in \mathbb{R}$;
(b) $f(s) = f(-s)$ for all $s \in \mathbb{R}$;
(c) $\int_{-\infty}^{+\infty} f(s)ds = 1$;
(d) $\int_{-\infty}^{+\infty} 2|s|f(s)ds = C(f) < +\infty$.

For every $f \in \mathcal{F}$ we define the distance on $\mathrm{Geod}_\sigma(X)$:

$$f(\gamma, \gamma') = \int_{-\infty}^{+\infty} d(\gamma(s), \gamma'(s)) f(s) ds. \tag{2.3}$$

We remark that the choice of $f = \frac{1}{2e^{|s|}}$ gives exactly the distance d_{Geod}. We are motivated to study the whole class $\mathcal{F}$ because of the applications in further works as [9].

Lemma 2.2 *The expression defined in (2.3) satisfies these properties:*

(i) it is a well defined distance on $\mathrm{Geod}_\sigma(X)$*;*
(ii) for all $\gamma, \gamma' \in \mathrm{Geod}_\sigma(X)$ *it holds* $f(\gamma, \gamma') \leq d(\gamma(0), \gamma(0)) + C(f)$*;*
(iii) for all $\gamma, \gamma' \in \mathrm{Geod}_\sigma(X)$ *it holds* $d(\gamma(0), \gamma'(0)) \leq f(\gamma, \gamma')$*;*
(iv) it induces the topology of $\mathrm{Geod}_\sigma(X)$.

Proof For all $\gamma, \gamma' \in \mathrm{Geod}_\sigma(X)$ we have

$$\begin{aligned} d(\gamma(s), \gamma'(s)) &\leq d(\gamma(s), \gamma(0)) + d(\gamma(0), \gamma'(0)) + d(\gamma'(0), \gamma'(s)) \\ &\leq 2|s| + d(\gamma(0), \gamma'(0)), \end{aligned}$$

so

$$\int_{-\infty}^{+\infty} d(\gamma(s), \gamma'(s)) f(s) ds \leq d(\gamma(0), \gamma'(0)) + \int_{-\infty}^{+\infty} 2|s| f(s) dt < +\infty.$$

This shows (ii) and that the integral in (2.3) is finite. From the properties of the integral and the positivity of f it is easy to prove that (2.3) defines a distance. The proof of (iii) follows from the convexity of σ and the symmetry of f. Indeed for all $\gamma, \gamma' \in \mathrm{Geod}_\sigma(X)$ the function $g(s) = d(\gamma(s), \gamma'(s))$ is convex. This means that for all $S, S' \in \mathbb{R}$ and for all $\lambda \in [0, 1]$ it holds

$$g(\lambda S + (1-\lambda)S') \leq \lambda g(S) + (1-\lambda) g(S').$$

We take $s \geq 0$ and we use the inequality above with $S = s$, $S' = -s$ and $\lambda = \frac{1}{2}$, obtaining

$$d(\gamma(0), \gamma'(0)) = g(0) \leq \frac{1}{2} g(-s) + \frac{1}{2} g(s) = \frac{d(\gamma(s), \gamma'(s)) + d(\gamma(-s), \gamma'(-s))}{2}.$$

We can now estimate the distance between γ and γ' as

$$
\begin{aligned}
f(\gamma,\gamma') &= \int_{-\infty}^{0} d(\gamma(s),\gamma'(s))f(s)ds + \int_{0}^{+\infty} d(\gamma(s),\gamma'(s))f(s)ds \\
&= \int_{0}^{+\infty} \big(d(\gamma(-s),\gamma'(-s)) + d(\gamma(s),\gamma'(s))\big) f(s)ds \geq d(\gamma(0),\gamma'(0)),
\end{aligned}
$$

where we used the symmetry of f. This concludes the proof of (iii).
If a sequence γ_n converges to γ_∞ uniformly on compact subsets then it is clear that for every $T \geq 0$ it holds

$$\lim_{n\to+\infty} \int_{-T}^{+T} d(\gamma_n(s),\gamma_\infty(s))f(s)ds = 0.$$

For every $\varepsilon > 0$ we pick $T_\varepsilon \geq 0$ such that $\int_{T_\varepsilon}^{+\infty} 2|s|f(s) < \varepsilon$. Then it is easy to conclude, using the properties of f, that

$$\lim_{n\to+\infty} \int_{-\infty}^{+\infty} d(\gamma_n(s),\gamma_\infty(s))f(s)ds \leq 2\varepsilon.$$

By the arbitrariness of ε we conclude that the sequence γ_n converges to γ_∞ with respect to the metric f.
Now suppose the sequence γ_n converges to γ_∞ with respect to f and suppose it does not converge uniformly on compact subsets to γ_∞. Therefore there exists $T \geq 0$, $\varepsilon_0 > 0$ and a subsequence γ_{n_j} such that $d(\gamma_{n_j}(t_j),\gamma_\infty(t_j)) > 6\varepsilon_0$ for every j, where $t_j \in [-T,T]$. We can suppose $t_j \to t_\infty$ and so $d(\gamma_{n_j}(t_\infty),\gamma_\infty(t_\infty)) > 4\varepsilon_0$ for every j. For all $t \in [t_\infty - \varepsilon_0, t_\infty + \varepsilon_0]$ we get $d(\gamma_{n_j}(t),\gamma_\infty(t)) > 2\varepsilon_0$. Therefore, if we set $m = \min_{t\in[t_\infty-\varepsilon_0,t_\infty+\varepsilon_0]} f(s) > 0$, we obtain

$$\int_{-\infty}^{+\infty} d(\gamma_{n_j}(s),\gamma_\infty(s))f(s)ds > 4\varepsilon_0^2 m$$

for every j, which is a contradiction. □

A metric $\mathfrak{d}$ on $\mathrm{Geod}_\sigma(X)$ inducing the topology of uniform convergence on compact subsets is said to be *geometric* if the evaluation map E is Lipschitz with respect to this metric. Any metric induced by $f \in \mathcal{F}$ is geometric by Lemma 2.2.(iii).

3 Covering and volume entropy

In this section we will introduce the first two types of entropy: the covering entropy, defined in terms of the covering functions, and the volume entropy of a measure.

3.1 Covering entropy

Let (X,σ) be a GCB-space that is P_0-packed at scale r_0. It is natural to define the *upper covering entropy* of X as the number

$$\overline{h_{\mathrm{Cov}}}(X) = \limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}(\overline{B}(x,T), r_0),$$

where x is any point of X. The *lower covering entropy* is defined taking the limit inferior instead of the limit superior and it is denoted by $\underline{h_{\mathrm{Cov}}}(X)$.
By triangular inequality it is easy to show that the definitions of upper and lower covering entropy do not depend on the point $x \in X$. In the next proposition, which is essentially Proposition 3.2 of [10], we can see that they do not depend on r_0 too and moreover we can replace the covering function with the packing function.

Proposition 3.1 *Let (X, σ) be a* GCB*-space that is P_0-packed at scale r_0 and let $x \in X$. Then*

$$\frac{1}{T} \log \mathrm{Cov}(\overline{B}(x, T), r) \underset{P_0, r_0, r, r'}{\asymp} \frac{1}{T} \log \mathrm{Pack}(\overline{B}(x, T), r')$$

for all $r, r' > 0$. In particular any of these functions can be used in the definition of the upper and lower covering entropy.

Proof For all $0 < r \leq r'$ and $x \in X$ clearly $\mathrm{Cov}(\overline{B}(x, T), r) \geq \mathrm{Cov}(\overline{B}(x, T), r')$ and $\mathrm{Cov}(\overline{B}(x, T), r) \leq \mathrm{Cov}(\overline{B}(x, T), r') \cdot \sup_{y \in X} \mathrm{Cov}(\overline{B}(y, r'), r)$. By Proposition 2.1 we have $\sup_{y \in X} \mathrm{Cov}(\overline{B}(y, r'), r) = \mathrm{Cov}(r', r)$ which is a finite number depending only on P_0, r_0, r, r'. Therefore we obtain

$$\frac{1}{T} \log \mathrm{Cov}(\overline{B}(x, T), r) \underset{P_0, r_0, r, r'}{\asymp} \frac{1}{T} \log \mathrm{Cov}(\overline{B}(x, T), r').$$

Now the thesis follows from (2.1). □

The upper and lower covering entropies can also be computed using the covering function of the metric spheres.

Proposition 3.2 *Let (X, σ) be a* GCB*-space that is P_0-packed at scale r_0 and $x \in X$. Then for all $r > 0$*

$$\frac{1}{T} \log \mathrm{Cov}(\overline{B}(x, T), r) \underset{P_0, r_0, r}{\asymp} \frac{1}{T} \log \mathrm{Cov}(S(x, T), r)$$

Proof Clearly it holds that $\mathrm{Cov}(S(x, T), r) \leq \mathrm{Cov}(\overline{B}(x, T), r)$. The other estimate is more involved. We divide the ball $\overline{B}(x, T)$ into annuli $A(x, kr, (k+1)r)$ with $k = 0, \ldots, \frac{T}{r} - 1$. We easily obtain

$$\mathrm{Cov}(\overline{B}(x, T), 2r) \leq \sum_{k=0}^{\frac{T}{r}-1} \mathrm{Cov}(A(x, kr, (k+1)r), 2r).$$

Now we claim that for any k it holds that

$$\mathrm{Cov}(A(x, kr, (k+1)r), 2r) \leq \mathrm{Cov}(S(x, T), r).$$

Indeed let $\{y_1, \ldots, y_N\}$ be a set of points realizing $\mathrm{Cov}(S(x, T), r)$. For all $i = 1, \ldots, N$ we consider the σ-geodesic segment $\gamma_i = [x, y_i]$ and we call x_i the point along this geodesic segment at distance kr from x. Then $x_i \in A(x, kr, (k+1)r)$ for every $i = 1, \ldots, N$. We claim that $\{x_1, \ldots, x_N\}$ is a $2r$-dense subset of $A(x, kr, (k+1)r)$. We take any $y \in A(x, kr, (k+1)r)$ and we consider the σ-geodesic segment $[x, y]$. We extend this geodesic to a σ-geodesic segment $\gamma = [x, y']$, where y' is at distance T from x. Then there exists i such that $d(y', y_i) = d(\gamma(T), \gamma_i(T)) \leq r$. By convexity of σ we have $d(\gamma(t), \gamma_i(t)) \leq r$, where $t = d(x, y)$. Therefore we conclude that $d(y, x_i) \leq d(y, \gamma_i(t)) + d(\gamma_i(t), x_i) \leq 2r$. This ends the proof of the claim, so $\mathrm{Cov}(\overline{B}(x, T), 2r) \leq \frac{T}{r}\mathrm{Cov}(S(x, T), r)$. The thesis follows from these estimates and Proposition 3.1. □

Combining Proposition 2.1 and Proposition 3.1 we can find an uniform upper bound to the covering entropy, see also Proposition 3.2 of [10].

Lemma 3.3 *Let (X,σ) be a* GCB-*space that is P_0-packed at scale r_0. Then*

$$\overline{h_{\mathrm{Cov}}}(X) \leq \frac{\log(1+P_0)}{r_0}.$$

Proof For every $x \in X$ it holds $\mathrm{Pack}(\overline{B}(x,R), r_0) \leq P_0(1+P_0)^{\frac{R}{r_0}-1}$. The thesis follows immediately. □

3.2 Volume entropy of homogeneous measures

Let (X,σ) be a GCB-space that is P_0-packed at scale r_0. The *upper volume entropy* of a measure μ on X is defined as

$$\overline{h_\mu}(X) = \limsup_{T\to+\infty} \frac{1}{T}\log \mu(\overline{B}(x,T)),$$

while the *lower volume entropy* $\underline{h_\mu}(X)$ is defined taking the limit inferior. These definitions do not depend on the choice of the point $x \in X$.
A measure μ on X is called *H-homogeneous at scale $r > 0$* if

$$\frac{1}{H} \leq \mu(\overline{B}(x,r)) \leq H$$

for all $x \in X$. We remark that the condition must hold only at scale r.

Proposition 3.4 *Let (X,σ) be a* GCB-*space that is P_0-packed at scale r_0 and let μ be a measure on X which is H-homogeneous at scale r. Then*

$$\frac{1}{T}\log\mu(\overline{B}(x,T)) \underset{P_0,r_0,H,r}{\asymp} \frac{1}{T}\log\mathrm{Cov}(\overline{B}(x,T),r).$$

In particular the upper (resp. lower) volume entropy of μ coincides with the upper (resp. lower) covering entropy of X.

Proof For all $x \in X$ it holds $\mu(\overline{B}(x,T)) \leq H\cdot\mathrm{Cov}(\overline{B}(x,T),r)$ and $\mu(\overline{B}(x,T)) \geq \frac{1}{H}\cdot\mathrm{Pack}(\overline{B}(x,T-r),r)$.
By Proposition 3.1 and since $\frac{T-r}{T} \underset{r}{\asymp} 1$ we have the thesis. □

Remark 3.5 The proof of the proposition shows another fact: if a measure is H-homogeneous at scale r then it is $H(r')$-homogeneous at scale r' for all $r' \geq r$ and $H(r')$ depends only on H, P_0, r_0, r and r'.

We provide here an example of a homogeneous measure. If X is a complete, geodesically complete, CAT(0) metric space that is P_0-packed at scale r_0 then the natural measure on X satisfies

$$c \leq \mu_X(\overline{B}(x,r_0)) \leq C$$

for all $x \in X$, where c and C are constants depending only on P_0 and r_0 (Theorem 4.9 of [14]). The following result follows immediately.

Corollary 3.6 *Let X be a complete, geodesically complete,* CAT(0) *metric space. If it is P_0-packed at scale r_0 for some P_0 and r_0 then $\overline{h_{\mathrm{Cov}}}(X) = \overline{h_{\mu_X}}(X)$. The same holds for the lower entropies.*

4 Lipschitz-topological entropy

Let (X, σ) be a GCB-space that is P_0-packed at scale r_0. The space $\mathrm{Geod}_\sigma(X)$ is locally compact but not compact. The topological entropy of the geodesic flow can be defined (see [6, 19]) as

$$\overline{h_{\text{top}}}(\mathrm{Geod}_\sigma(X)) = \inf_{\mathrm{d}} \sup_K \lim_{r\to 0} \limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}_{\mathrm{d}^T}(K, r),$$

where the infimum is taken among all metrics d inducing the topology of $\mathrm{Geod}_\sigma(X)$, the supremum is taken among all compact subsets of $\mathrm{Geod}_\sigma(X)$ and $\mathrm{Cov}_{\mathrm{d}^T}(K, r)$ is the covering function of the compact subset K at scale r with respect to the metric d^T defined by

$$\mathrm{d}^T(\gamma, \gamma') = \max_{t\in[0,T]} \mathrm{d}(\Phi_t\gamma, \Phi_t\gamma').$$

By the variational principle this quantity equals the measure-theoretic entropy defined as the supremum of the entropies of the flow-invariant probability measures on $\mathrm{Geod}_\sigma(X)$ (cp. [19], Lemma 1.5). An easy computation shows that the topological entropy is always zero.

Lemma 4.1 *There are no flow-invariant probability measures on* $\mathrm{Geod}_\sigma(X)$. *In particular the topological entropy of the geodesic flow is* 0.

Proof Suppose there is a flow-invariant probability measure μ on $\mathrm{Geod}_\sigma(X)$. For $x \in X$ and $R \geq 0$ we define $A_R = \{\gamma \in \mathrm{Geod}_\sigma(X) \text{ s.t. } \gamma(0) \in \overline{B}(x, R)\}$. Clearly there exists $R \geq 0$ such that $\mu(A_R) > \frac{1}{2}$. By flow-invariance of μ we have that the set

$$\Phi_{2R+1}^{-1}(A_R) = \{\gamma \in \mathrm{Geod}_\sigma(X) \text{ s.t. } \gamma(2R+1) \in \overline{B}(x, R)\}$$

has measure $> \frac{1}{2}$. This implies that $\mu(A_R \cap \Phi_{2R+1}^{-1}(A_R)) > 0$, but this intersection is empty. □

Looking at the proof of the variational principle given in [19] we can observe that the sequence of metrics on $\mathrm{Geod}_\sigma(X)$ that approach the infimum in the definition of the topological entropy are the restriction to $\mathrm{Geod}_\sigma(X)$ of metrics defined on its one-point compactification. These metrics are not the natural ones on $\mathrm{Geod}_\sigma(X)$, since they are not geometric. We propose a more appropriate definition of topological entropy for proper GCB-spaces.

We define the *upper Lipschitz-topological entropy* of $\mathrm{Geod}_\sigma(X)$ as

$$\overline{h_{\text{Lip-top}}}(\mathrm{Geod}_\sigma(X)) = \inf_{\mathrm{d}} \sup_K \lim_{r\to 0} \limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}_{\mathrm{d}^T}(K, r),$$

where the infimum is now taken only among all geometric metrics on $\mathrm{Geod}_\sigma(X)$. The *lower Lipschitz-topological entropy* is defined by taking the limit inferior instead of the limit superior and it is denoted by $\underline{h_{\text{Lip-top}}}(\mathrm{Geod}_\sigma(X))$. The main result of this section is the following.

Theorem 4.2 *Let* (X, σ) *be a* GCB-*space that is* P_0-*packed at scale* r_0. *Then*

$$\overline{h_{\text{Lip-top}}}(\mathrm{Geod}_\sigma(X)) = \overline{h_{\text{Cov}}}(X).$$

The same holds for the lower entropies.

One of the two inequalities is easy. In order to prove the other one we will show that for the distances induced by the functions $f \in \mathcal{F}$ the definition of topological entropy can be heavily simplified.

4.1 Topological entropy for the distances induced by $f \in \mathcal{F}$

For a metric $f \in \mathcal{F}$ we denote by $\overline{h_f}$ the upper metric entropy of the σ-geodesic flow with respect to f, that is

$$\overline{h_f}(\mathrm{Geod}_\sigma(X)) = \sup_K \lim_{r\to 0} \limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}_{f^T}(K, r).$$

In the usual way it is defined the lower metric entropy with respect to f, $\underline{h_f}(\mathrm{Geod}(X))$. For a subset Y of X we denote by $\mathrm{Geod}_\sigma(Y)$ the set of σ-geodesic lines of X passing through Y at time 0.

Proposition 4.3 *Let (X, σ) be a* GCB*-space that is P_0-packed at scale r_0 and let $f \in \mathcal{F}$. Then*

(i) for all $x, y \in X$ it holds $\overline{h_f}(\mathrm{Geod}_\sigma(x)) = \overline{h_f}(\mathrm{Geod}_\sigma(y))$;
(ii) for all $x \in X$ and $R \geq 0$ it holds $\overline{h_f}(\mathrm{Geod}_\sigma(\overline{B}(x, R))) = \overline{h_f}(\mathrm{Geod}_\sigma(x))$;
(iii) for all $x \in X$ it holds $\overline{h_f}(\mathrm{Geod}_\sigma(X)) = \overline{h_f}(\mathrm{Geod}_\sigma(x)) \leq \overline{h_{\mathrm{Cov}}}(X)$;
(iv) for all $x \in X$ the function $r \mapsto \limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}_{f^T}(\mathrm{Geod}_\sigma(x), r)$ is constant.

The same conclusions hold for the lower Lipschitz-topological entropy.

The proposition is a consequence of the following fundamental key lemma.

Lemma 4.4 (Key Lemma) *Let $f \in \mathcal{F}$, $\gamma \in \mathrm{Geod}_\sigma(X)$ and $0 < r \leq r'$. Then*

$$\frac{1}{T} \log \mathrm{Cov}_{f^T}(\overline{B}_{f^T}(\gamma, r'), r) \underset{P_0, r_0, r, r', f}{\asymp} 0,$$

where $\overline{B}_{f^T}(\gamma, r')$ is the closed ball of center γ and radius r' with respect to the metric f^T. We remark that the convergence is uniform in γ.

Proof Let $P > 0$ depending only on f and r' such that

$$\int_{-\infty}^{-P} 2|u| f(u)du + \int_{P}^{+\infty} 2|u| f(u)du < \frac{r}{4}.$$

We fix $\varepsilon > 0$ and $T \geq \frac{P}{\varepsilon}$. Let $E_T = \{x_1, \ldots, x_N\}$ be a maximal $\frac{r}{16}$-separated subset of $B(\gamma(T), r' + \varepsilon T)$, so it is also $\frac{r}{16}$-dense, and $\{y_1, \ldots, y_M\}$ be a $\frac{r}{16}$-dense subset of $B(\gamma(-P), r' + 2P)$. For every $i = 1, \ldots, M$ and $j = 1, \ldots, N$ we take a σ-geodesic line γ_{ij} extending the σ-geodesic segment $[y_i, x_j]$. We parametrize γ_{ij} in such a way that $\gamma_{ij}(-P) = y_i$. The claim is that $\{\gamma_{ij}\}_{i,j}$ is a r-dense subset of $\overline{B}_{f^T}(\gamma, r')$ with respect to the metric f^T. We fix $\gamma' \in \overline{B}_{f^T}(\gamma, r')$. This means

$$\max_{t\in[0,T]} f^t(\gamma', \gamma) = \max_{t\in[0,T]} f(\Phi_t(\gamma'), \Phi_t(\gamma)) \leq r'.$$

In particular for all $t \in [0, T]$ we get $d(\gamma'(t), \gamma(t)) \leq r'$, since

$$d(\gamma'(t), \gamma(t)) = d(\Phi_t(\gamma'), \Phi_t(\gamma)) \leq f(\Phi_t(\gamma'), \Phi_t(\gamma)) \leq r'.$$

Therefore $d(\gamma'(-P), \gamma(-P)) \leq r' + 2P$. Moreover

$$d(\gamma'(T + \varepsilon T), \gamma(T)) \leq d(\gamma'(T + \varepsilon T), \gamma'(T)) + d(\gamma'(T), \gamma(T)) \leq \varepsilon T + r'.$$

Thus there exists x_j such that $d(x_j, \gamma'(T+\varepsilon T)) \leq \frac{r}{16}$ and y_i such that $d(y_i, \gamma'(-P)) \leq \frac{r}{16}$. We have $d(\gamma_{ij}(-P), \gamma'(-P)) \leq \frac{r}{16}$, so if we denote with t_j the time such that $\gamma_{ij}(t_j) = x_j$ it holds that $|t_j - (T+\varepsilon T)| \leq \frac{r}{8}$. Then

$$\begin{aligned} d(\gamma_{ij}(T+\varepsilon T), \gamma'(T+\varepsilon T)) &\leq d(\gamma_{ij}(T+\varepsilon T), \gamma_{ij}(t_j)) + d(\gamma_{ij}(t_j), \gamma'(T+\varepsilon T)) \\ &\leq \frac{r}{8} + \frac{r}{16} < \frac{r}{4}. \end{aligned}$$

From the convexity of σ we have $d(\gamma'(u), \gamma_{ij}(u)) < \frac{r}{4}$ for all $u \in [-P, (1+\varepsilon)T]$. For $t \in [0, T]$ we have

$$\begin{aligned} f^t(\gamma', \gamma_{ij}) &= \int_{-\infty}^{+\infty} d(\gamma'(u), \gamma_{ij}(u)) f(u-t)du \\ &\leq \int_{-\infty}^{-P} \left(\frac{r}{4} + 2|u+P|\right) f(u-t)du + \\ &\quad + \int_{-P}^{(1+\varepsilon)T} \frac{r}{4} f(u-t)du + \\ &\quad + \int_{(1+\varepsilon)T}^{+\infty} \left(\frac{r}{4} + 2|u-(1+\varepsilon)T|\right) f(u-t)du. \end{aligned}$$

The first term can be estimated as follows

$$\begin{aligned} \int_{-\infty}^{-P} \left(\frac{r}{4} + 2|u+P|\right) f(u-t)du &\leq \frac{r}{4} + \int_{-\infty}^{-P-t} 2|v+t+P| f(v)dv \\ &\leq \frac{r}{4} + \int_{-\infty}^{-P} 2|v| f(v)dv. \end{aligned}$$

The second term is less than or equal to $\frac{r}{4}$. The third term can be controlled in this way:

$$\begin{aligned} \int_{(1+\varepsilon)T}^{+\infty} \left(\frac{r}{4} + 2|u-(1+\varepsilon)T|\right) f(u-t)du &\leq \frac{r}{4} + \int_{(1+\varepsilon)T-t}^{+\infty} 2|v-(1+\varepsilon)T+t| f(v)dv \\ &\leq \frac{r}{4} + \int_{(1+\varepsilon)T-t}^{+\infty} 2|v| f(v)dv \\ &\leq \frac{r}{4} + \int_{P}^{+\infty} 2|v| f(v)dv. \end{aligned}$$

The last inequality follows from $T \geq \frac{P}{\varepsilon}$. Therefore

$$f^t(\gamma', \gamma_{ij}) \leq \frac{r}{4} + \frac{r}{4} + \frac{r}{4} + \int_{-\infty}^{-P} 2|v| f(v)dv + \int_{P}^{+\infty} 2|v| f(v)dv \leq r.$$

We conclude that

$$\mathrm{Cov}_{f^T}(\overline{B}_{f^T}(\gamma, r'), r) \leq \mathrm{Cov}\left(r' + 2P, \frac{r}{16}\right) \cdot \# E_T.$$

From Proposition 2.1, if $\rho = \min\left\{r_0, \frac{r}{16}\right\}$, we get $\# E_T \leq P_0(1+P_0)^{\frac{r'+\varepsilon T}{\rho} - 1}$. Thus

$$\begin{aligned} \frac{1}{T} \log \mathrm{Cov}_{f^T}(\overline{B}_{f^T}(\gamma, r'), r) &\leq \frac{1}{T} K(P_0, r_0, r, r', f) \cdot \frac{\varepsilon T}{\rho} \log(1+P_0) \\ &= \varepsilon \cdot K'(P_0, r_0, r, r', f). \end{aligned}$$

Here K, K' are constants depending only on P_0, r_0, r, r', f and not on ε or γ. So from the arbitrariness of ε we achieve the proof. □

The computation of $\overline{h_f}$ requires to consider the supremum among all compact subsets of $\text{Geod}_\sigma(X)$. We notice that given a compact subset $K \subseteq \text{Geod}_\sigma(X)$, the set $E(K)$ is compact since E is continuous. In particular it is bounded, hence contained in a ball $\overline{B}(x, R)$ centered at a reference point $x \in X$. We observe also that the set $\text{Geod}_\sigma(\overline{B}(x, R))$ is compact since the evaluation map E is proper. We conclude that any compact subset of $\text{Geod}_\sigma(X)$ is contained in a compact subset of the form $\text{Geod}_\sigma(\overline{B}(x, R))$ and therefore in order to compute $\overline{h_f}$ it is enough to take the supremum among these sets. The main consequence of Lemma 4.4 is the following result, which is the key ingredient in the proof of Proposition 4.3.

Corollary 4.5 *Let $f \in \mathcal{F}$, $x \in X$, $R \geq 0$ and $0 < r \leq r'$. Then*

$$\frac{1}{T}\log \text{Cov}_{f^T}(\text{Geod}_\sigma(\overline{B}(x, R)), r) \underset{P_0, r_0, r, r', f}{\asymp} \frac{1}{T}\log \text{Cov}_{f^T}(\text{Geod}_\sigma(\overline{B}(x, R)), r').$$

Proof The quantity $\frac{1}{T}\log \text{Cov}_{f^T}(\text{Geod}_\sigma(\overline{B}(x, R)), r)$ is

$$\begin{aligned}
&\leq \frac{1}{T}\log \text{Cov}_{f^T}(\text{Geod}_\sigma(\overline{B}(x, R)), r') \cdot \sup_{\gamma \in \text{Geod}_\sigma(X)} \text{Cov}_{f^T}(\overline{B}_{f^T}(\gamma, r'), r) \\
&= \frac{1}{T}\Big(\log \text{Cov}_{f^T}(\text{Geod}_\sigma(\overline{B}(x, R)), r') + \log \sup_{\gamma \in X} \text{Cov}_{f^T}(\overline{B}_{f^T}(\gamma, r'), r)\Big)
\end{aligned}$$

The conclusion follows by Lemma 4.4. □

Proof of Proposition 4.3.(ii) Let $\varepsilon > 0$ and $T > \frac{R}{\varepsilon}$. Let $\gamma_1, \ldots, \gamma_N$ be a r-dense subset of $\text{Geod}_\sigma(x)$ with respect to the metric $f^{(2+\varepsilon)T}$. The claim is that $\{\gamma_i\}$ is a K-dense subset of $\text{Geod}_\sigma(\overline{B}(x, R))$ with respect to f^T, where K depends only on r, R and f. We consider a σ-geodesic line $\gamma \in \text{Geod}_\sigma(\overline{B}(x, R))$. Then there exists a σ-geodesic line $\gamma' \in \text{Geod}_\sigma(x)$ extending the σ-geodesic segment $[x, \gamma((1+\varepsilon)T)]$. We call $t_{\gamma'}$ the time such that $\gamma'(t_{\gamma'}) = \gamma((1+\varepsilon)T)$. Then

$$\begin{aligned}
t_{\gamma'} = d(x, \gamma((1+\varepsilon)T)) &\leq d(x, \gamma(0)) + d(\gamma(0), \gamma((1+\varepsilon)T)) \\
&\leq R + (1+\varepsilon)T \leq (1+2\varepsilon)T
\end{aligned}$$

since $T \geq \frac{R}{\varepsilon}$. Moreover $|t_{\gamma'} - (1+\varepsilon)T| \leq R$. We know there exists γ_i such that $\max_{t\in[0,(1+2\varepsilon)T]} f(\Phi_t\gamma', \Phi_t\gamma_i) \leq r$. In particular $d(\gamma'(t_{\gamma'}), \gamma_i(t_{\gamma'})) \leq r$. Then $d(\gamma((1+\varepsilon)T), \gamma_i(t_{\gamma'})) \leq r$ and in conclusion

$$d(\gamma((1+\varepsilon)T), \gamma_i((1+\varepsilon)T)) \leq d(\gamma((1+\varepsilon)T), \gamma_i(t_{\gamma'})) + d(\gamma_i(t_{\gamma'}), \gamma_i((1+\varepsilon)T)) \leq r + R.$$

From the convexity of σ we have $d(\gamma(t), \gamma_i(t)) \leq R + r$ for all $t \in [0, (1+\varepsilon)T]$. We have to estimate $f^t(\gamma, \gamma_i) = \int_{-\infty}^{+\infty} d(\gamma(u), \gamma_i(u)) f(u-t)du$ for every $t \in [0, T]$. Since

$d(\gamma(0), \gamma_i(0)) \leq R$ and $d(\gamma((1+\varepsilon)T), \gamma_i((1+\varepsilon)T)) \leq r + R$ then

$$\begin{aligned}\int_{-\infty}^{+\infty} d(\gamma(u), \gamma_i(u)) f(u-t)du &\leq \int_{-\infty}^{0} (R + 2|u|) f(u-t)du + \\ &+ \int_{0}^{(1+\varepsilon)T} (R + r) f(u-t)du + \\ &+ \int_{(1+\varepsilon)T}^{+\infty} (R + r + 2|u - (1+\varepsilon)T|) f(u-t)du \\ &\leq R + \int_{-\infty}^{-t} 2|v+t| f(v)dv + (R + r) \\ &+ \int_{(1+\varepsilon)T-t}^{+\infty} (R + r + 2|v - (1+\varepsilon)T + t|) f(v)dv.\end{aligned}$$

We conclude that the above quantity is less than or equal to

$$3R + 2r + \int_{-\infty}^{0} 2|v| f(v)dv + \int_{0}^{+\infty} 2|v| f(v)dv \leq 3R + 2r + C(f) = K(R, r, f).$$

By the previous corollary $\overline{h_f}(\mathrm{Geod}_\sigma(\overline{B}(x, R)))$can be computed as

$$\limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}_{f^T}(\mathrm{Geod}_\sigma(\overline{B}(x, R)), K)$$

which is

$$\begin{aligned}&\leq \limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}_{f^{(1+2\varepsilon)T}}(\mathrm{Geod}_\sigma(x), r) \\ &= (1 + 2\varepsilon) \limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}_{f^T}(\mathrm{Geod}_\sigma(x), r).\end{aligned}$$

Since this is true for all $\varepsilon > 0$ then we obtain the thesis. □

Proof of Proposition 4.3.(i) We have $y \in \overline{B}(x, R)$, where $R = d(x, y)$, so $\mathrm{Geod}_\sigma(y) \subseteq \mathrm{Geod}_\sigma(\overline{B}(x, R))$. Therefore

$$\overline{h}_f(\mathrm{Geod}_\sigma(y)) \leq \overline{h}_f(\mathrm{Geod}_\sigma(\overline{B}(x, R))) = \overline{h}_f(\mathrm{Geod}_\sigma(x)).$$

The other inequality can be proved in the same way. □

Finally we achieve the proof of the remaining parts of Proposition 4.3.

Proof of Proposition 4.3.(iii) & (iv) The equality in (iii) follows directly from (ii), so

$$\overline{h_f}(\mathrm{Geod}_\sigma(X)) = \limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}_{f^T}(\mathrm{Geod}_\sigma(x), r_0),$$

where x is a point of X. We fix $T > 0$ and we consider a r_0-separated subset E_T of $S(x_0, T)$ of maximal cardinality, which is also r_0-dense. For all $y \in E_T$ we consider a σ-geodesic line γ_y extending the σ-geodesic segment $[x_0, y]$ such that $\gamma_y(0) = x_0$ and $\gamma_y(T) = y$. We claim that $\{\gamma_y\}_{y\in E_T}$ is a $(r_0 + C(f))$-dense subset of $\mathrm{Geod}_\sigma(x)$ with respect to f^T. We take a σ-geodesic line $\gamma \in \mathrm{Geod}_\sigma(x)$. Then there exists $y \in E_T$ such that $d(\gamma(T), y) = d(\gamma(T), \gamma_y(T)) \leq r_0$. From the convexity of σ it holds $d(\gamma(u), \gamma_y(u)) \leq r_0$

for all $u \in [0, T]$. Moreover $d(\gamma(u), \gamma_y(u)) \leq r_0 + 2|u - T|$ for all $u \in [T, +\infty)$ and $d(\gamma(u), \gamma_y(u)) \leq 2|u|$ for all $u \in (-\infty, 0]$. Then for all $t \in [0, T]$ we get

$$\begin{aligned} f^t(\gamma, \gamma_y) &= \int_{-\infty}^{+\infty} d(\gamma(u), \gamma_y(u)) f(u-t)du \\ &\leq \int_{-\infty}^{0} 2|u| f(u-t)du + \int_0^T r_0 f(u-t)du + \\ &+ \int_T^{+\infty} (r_0 + 2|u-T|) f(u-t)du \leq r_0 + C(f). \end{aligned}$$

The last inequality follows from similar estimates given in the proofs of Lemma 4.4. Therefore applying Corollary 4.5 we have

$$\limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}_{f^T}(\mathrm{Geod}_\sigma(x), r_0) \leq \limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}(S(x, T), r_0).$$

This, together with Proposition 3.2, proves (iii). We observe that (iv) is exactly Corollary 4.5 with $R = 0$. □

4.2 Proof of Theorem 4.2

We are ready to give the

Proof of Theorem 4.2 Proposition 4.3.(iii) shows that $\overline{h_{\text{Lip-top}}}(\mathrm{Geod}_\sigma(X))$ is less than or equal to $\overline{h_{\text{Cov}}}(X)$.
In order to prove the other inequality we fix a geometric metric $\mathfrak{d}$ on $\mathrm{Geod}_\sigma(X)$ and we denote by M the Lipschitz constant with respect to $\mathfrak{d}$ of the evaluation map E. Then we have

$$\sup_K \lim_{r\to 0} \limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}_{\mathfrak{d}^T}(K, r) \geq \limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}_{\mathfrak{d}^T}(\mathrm{Geod}_\sigma(x), r_0),$$

where $x \in X$. We fix $T \geq 0$ and we consider a set $\gamma_1, \ldots, \gamma_N$ realizing $<$ $\mathrm{Cov}_{\mathfrak{d}^T}(\mathrm{Geod}_\sigma(x), r_0) >$. The claim is that $\gamma_i(T)$ is a Mr_0-dense subset of $S(x, T)$. Indeed we take a point $y \in S(x, T)$ and we extend the σ-geodesic segment $[x, y]$ to a σ-geodesic line $\gamma \in \mathrm{Geod}_\sigma(x)$. Then there exists γ_i such that $\mathfrak{d}^T(\gamma, \gamma_i) \leq r_0$. Since the evaluation map is M-Lipschitz we have

$$d(y, \gamma_i(T)) = d(\gamma(T), \gamma_i(T)) = d(\Phi_T\gamma(0), \Phi_T\gamma_i(0)) \leq L\mathfrak{d}(\Phi_T\gamma, \Phi_T\gamma_i) \leq Mr_0.$$

Therefore

$$\limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}_{\mathfrak{d}^T}(\mathrm{Geod}_\sigma(x), r_0) \geq \limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}(S(x, T), Mr_0)$$

and the conclusion follows by Proposition 3.2. □

Remark 4.6 By Proposition 4.3 and Theorem 4.2 the upper Lipschitz-topological entropy of X can be computed as

$$\overline{h_{\text{Lip-top}}}(X) = \limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}_{f^T}(\mathrm{Geod}_\sigma(x), r)$$

independently of $f \in \mathcal{F}$, $x \in X$ and $r > 0$. Moreover

$$\frac{1}{T} \log \mathrm{Cov}_{f^T}(\mathrm{Geod}_\sigma(x), r_0) \underset{P_0, r_0, f}{\asymp} \frac{1}{T} \log \mathrm{Cov}(\overline{B}(x, T), r_0)$$

by the proofs of Theorem 4.2 and Proposition 4.3 and by Proposition 3.2.

5 Gromov-hyperbolic metric spaces

In the second part of the paper we will study the versions of the entropies introduced in the first part relative to subsets of the boundary at infinity. In order to have meaningful definitions we will consider Gromov-hyperbolic metric spaces.
Let X be a geodesic space. Given three points $x, y, z \in X$, the *Gromov product* of y and z with respect to x is defined as

$$(y, z)_x = \frac{1}{2}\big(d(x, y) + d(x, z) - d(y, z)\big).$$

The space X is said δ-*hyperbolic* if for every four points $x, y, z, w \in X$ the following *4-points condition* hold:

$$(x, z)_w \geq \min\{(x, y)_w, (y, z)_w\} - \delta. \tag{5.1}$$

The space X is *Gromov hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$.
Let X be a proper, δ-hyperbolic metric space x be a point of X.
The *Gromov boundary* of X is defined as the quotient

$$\partial X = \{(z_n)_{n\in\mathbb{N}} \subseteq X \mid \lim_{n,m\to+\infty} (z_n, z_m)_x = +\infty\} /_\sim,$$

where $(z_n)_{n\in\mathbb{N}}$ is a sequence of points in X and $\approx$ is the equivalence relation defined by $(z_n)_{n\in\mathbb{N}} \sim (z'_n)_{n\in\mathbb{N}}$ if and only if $\lim_{n,m\to+\infty}(z_n, z'_m)_x = +\infty$.
We will write $z = [(z_n)] \in \partial X$ for short, and we say that (z_n) *converges* to z. This definition does not depend on the basepoint x.
There is a natural topology on $X \cup \partial X$ that extends the metric topology of X. The Gromov product can be extended to points $z, z' \in \partial X$ by

$$(z, z')_x = \sup_{(z_n),(z'_n)} \liminf_{n,m\to+\infty} (z_n, z'_m)_x$$

where the supremum is taken among all sequences such that $(z_n) \approx z$ and $(z'_n) \approx z'$. For every $z, z', z'' \in \partial X$ it continues to hold

$$(z, z')_x \geq \min\{(z, z'')_x, (z', z'')_x\} - \delta. \tag{5.2}$$

Moreover for all sequences $(z_n), (z'_n)$ converging to z, z' respectively it holds

$$(z, z')_x - \delta \leq \liminf_{n,m\to+\infty} (z_n, z'_m)_x \leq (z, z')_x. \tag{5.3}$$

The Gromov product between a point $y \in X$ and a point $z \in \partial X$ is defined in a similar way and it satisfies a condition analogue of (5.3).
Every geodesic ray ξ defines a point $\xi^+ = [(\xi(n))_{n\in\mathbb{N}}]$ of the Gromov boundary ∂X: we say that ξ *joins* $\xi(0) = y$ *to* $\xi^+ = z$, and we denote it by $[y, z]$. Moreover for every $z \in \partial X$ and every $x \in X$ it is possible to find a geodesic ray ξ such that $\xi(0) = x$ and $\xi^+ = z$. Indeed if (z_n) is a sequence of points converging to z then, by properness of X, the sequence of geodesics $[x, z_n]$ converges to a geodesic ray ξ which has the properties above (cp. Lemma III.3.13 of [4]). We denote any of these geodesic rays as $\xi_{xz} = [x, z]$ even if it is possibly not unique.

Given different points $z = [(z_n)], z' = [(z'_n)] \in \partial X$ there always exists a geodesic line γ joining z to z', i.e. such that $\gamma|_{[0,+\infty)}$ and $\gamma|_{(-\infty,0]}$ join $\gamma(0)$ to z, z' respectively. We call z and z' the *positive* and *negative endpoints* of γ, respectively, denoted γ^+ and γ^-.
Here we recall some basic properties of Gromov-hyperbolic metric spaces.

Lemma 5.1 (Projection Lemma, cp. Lemma 3.2.7 of [11]) *Let X be a δ-hyperbolic metric space and let $x, y, z \in X$. For every geodesic segment $[y, z]$ we have $(y, z)_x \geq d(x, [y, z]) - 4\delta$.*

We recall that a $(1, \nu)$-quasigeodesic is a curve $\alpha\colon I \to X$ such that

$$|t - t'| - \nu \leq d(\alpha(t), \alpha(t')) \leq |t - t'| + \nu$$

for all t, t' belonging to the interval I. As an immediate consequence of the previous lemma and Proposition 2.7 of [13] we get:

Lemma 5.2 *Let X be a δ-hyperbolic metric space, $x \in X$ and ξ be a geodesic ray such that $\xi(0)$ is a projection of x on ξ. Then*

(i) for all $T \geq 0$ any curve $\alpha = [x, \xi(0)] \cup \xi|_{[0,T]}$ is a $(1, 4\delta)$-quasigeodesic. Moreover, if γ is a geodesic segment $[x, \xi(T)]$, then $d(\alpha(t), \gamma(t)) \leq 72\delta$ for all possibles t;
(ii) any curve $\alpha = [x, \xi(0)] \cup \xi$ is a $(1, 4\delta)$-quasigeodesic. Moreover, if ξ' is a geodesic ray $[x, \xi^+]$, then $d(\alpha(t), \xi'(t)) \leq 72\delta$ for all $t \geq 0$.

Furthermore:

Lemma 5.3 *Let X be a proper, δ-hyperbolic metric space. Let γ be a geodesic line and $x \in X$ with $S := d(\gamma(0), x)$. Let x' be a projection of x on γ. Then*

(i) there exists an orientation of γ such that $[x, x'] \cup [x', \gamma^+]$ is a $(1, 4\delta)$-quasigeodesic, where the second segment is the subsegment of γ;
(ii) with respect to the orientation of (i) then every geodesic ray $\xi = [x, \gamma^+]$ satisfies $d(\xi(S + t), \gamma(t)) \leq 76\delta$ for all $t \geq 0$;
(iii) for all orientations of γ every geodesic ray $\xi = [x, \gamma^+]$ satisfies $d(\xi(S + t), \gamma(t)) \leq 2S + 76\delta$ for all $t \geq 0$.

By an orientation of a geodesic line γ we simply mean a unit speed parametrization.

Proof We can choose a parametrization of γ for which x' belongs to the negative ray $\gamma|_{(-\infty,0]}$. We take a geodesic ray $\xi = [x, \gamma^+]$. By Lemma 5.2 the path $\alpha = [x, x'] \cup [x', \gamma^+]$ is a $(1, 4\delta)$-quasigeodesic and moreover it satisfies $d(\xi(S + t), \alpha(S + t)) \leq 72\delta$ for every $t \geq 0$. Furthermore the time t_0 such that $\alpha(t_0) = \gamma(0)$ is between S and $S + 4\delta$ implying $d(\xi(S+t), \gamma(t)) \leq 76\delta$. For the second part of the proof we assume to be in the situation above and we consider a geodesic ray $\xi = [x, \gamma^-]$. By Lemma 5.2 the path $\alpha = [x, x'] \cup [x', \gamma^-]$, where the second segment is a subsegment of γ, satisfies $d(\xi(S+t), \alpha(S+t)) \leq 72\delta$ for every $t \geq 0$. Furthermore for every $t \geq 0$ the point $\alpha(S + t)$ belongs to γ and $d(\alpha(S + t), \gamma(0)) \leq d(\alpha(S + t), x) + d(x, \gamma(0)) \leq 2S + t + 4\delta$. So $d(\xi(S + t), \gamma(t)) \leq 76\delta + 2S$. □

We remark that if $\gamma(0)$ is a projection of x on γ then the first part of the lemma holds for both the positive and negative ray of γ.
The *quasiconvex hull* of a subset C of ∂X is the union of all the geodesic lines joining two points of C and it is denoted by QC-Hull(C). The following is essentially Lemma 2.5 of [10].

Lemma 5.4 *Let X be a proper, δ-hyperbolic metric space. Let $x \in X$ and $C \subseteq \partial X$ be a subset with at least two points. Then for every $z \in C$ it exists a geodesic line γ with endpoints in C such that $d(\xi_{xz}(t), \gamma(t)) \leq 22\delta + d(x, \text{QC-Hull}(C)) =: L$ for every $t \geq 0$.*

Proof Let $x' \in$ QC-Hull(C) realizing $d(x, \text{QC-Hull}(C))$. By Lemma 2.5 of [10] we know there exists γ as desired such that $d(\xi_{x'z}(t), \gamma(t)) \leq 14\delta$ for every $t \geq 0$. Now the thesis follows by the fact that two geodesic rays with same endpoint are 8δ-close, see for instance Proposition 8.10 of [3]. □

Remark 5.5 Let $z, z' \in C \subseteq \partial X$. It is clear that the conclusion of Lemma 5.4 is true with $L = 22\delta + d(x, [z, z'])$.

Lemma 5.6 *Let (X, σ) be a proper, δ-hyperbolic* GCB*-space. Let* QC-Hull$_\sigma(C)$ *be the union of all σ-geodesic lines joining two points of C. Then for every $x \in$* QC-Hull(C) *there exists $x' \in$* QC-Hull$_\sigma(C)$ *with $d(x, x') \leq 8\delta$.*

Proof For all $z, z' \in \partial X$ there exists a σ-geodesic line joining them, see [13]. Moreover two parallel geodesics are at most at distance 8δ ([3], Proposition 8.10), hence the conclusion. □

5.1 Minkowski dimension

When X is a proper, δ-hyperbolic metric space we define the *generalized visual ball* of center $z \in \partial X$ and radius $\rho \geq 0$ to be

$$B(z, \rho) = \left\{ z' \in \partial X \text{ s.t. } (z, z')_x > \log \frac{1}{\rho} \right\}.$$

It is comparable to the metric balls of the visual metrics on ∂X, see Lemma 2.6 of [10]. Generalized visual balls are related to shadows. Let $x \in X$ be a basepoint. The shadow of radius $r > 0$ casted by a point $y \in X$ with center x is the set:

$$\text{Shad}_x(y, r) = \{z \in \partial X \text{ s.t. } [x, z] \cap B(y, r) \neq \emptyset \text{ for all rays } [x, z]\}.$$

Lemma 5.7 (Lemma 2.7 of [10]) *Let X be a proper, δ-hyperbolic metric space. Let $z \in \partial X$, $x \in X$ and $T \geq 0$. Then*

(i) $B(z, e^{-T}) \subseteq \text{Shad}_x(\xi_{xz}(T), 7\delta)$*;*
(ii) $\text{Shad}_x(\xi_{xz}(T), r) \subseteq B(z, e^{-T+r})$ *for all $r > 0$.*

The *upper* and *lower visual Minkowski dimension* of a subset C of ∂X was defined in [10] as

$$\overline{\text{MD}}(C) = \limsup_{T \to +\infty} \frac{1}{T} \log \text{Cov}(C, e^{-T}), \qquad \underline{\text{MD}}(C) = \liminf_{T \to +\infty} \frac{1}{T} \log \text{Cov}(C, e^{-T})$$

respectively, where $\text{Cov}(C, \rho)$ denotes the minimal number of generalized visual balls of radius ρ needed to cover C. Taking $C = \partial X$ we get

Proposition 5.8 *Let (X, σ) be a δ-hyperbolic* GCB*-space that is P_0-packed at scale r_0 and let $x \in X$. Then*

$$\frac{1}{T} \log \text{Cov}(\partial X, e^{-T}) \underset{P_0, r_0, \delta}{\asymp} \frac{1}{T} \log \text{Cov}(S(x, T), r_0).$$

In particular the upper (resp. lower) visual Minkowski dimension of ∂X equals the upper (resp. lower) covering entropy of X.

We need:

Lemma 5.9 ([10], Lemma 2.2) *Let X be a proper, δ-hyperbolic metric space, $z, z' \in \partial X$ and $x \in X$. Then*

(i) if $(z, z')_x \geq T$ then $d(\xi_{xz}(T-\delta), \xi_{xz'}(T-\delta)) \leq 4\delta$;
(ii) for all $b > 0$, if $d(\xi_{xz}(T), \xi_{xz'}(T)) < 2b$ then $(z, z')_x > T - b$.

Proof of Proposition 5.8 Let $z_1, \ldots, z_N$ be points of ∂X realizing $\mathrm{Cov}(\partial X, e^{-T})$, and let y_i be the point at distance T from x along one geodesic ray ξ_{xz_i}. We claim that $\{y_i\}$ covers $S(x, T)$ at scale 6δ. Indeed let $y \in S(x, T)$ and let $z \in \partial X$ be the point at infinity of a σ-geodesic ray ξ that extends the σ-geodesic $[x, y]$. We know there exists i such that $(z, z_i)_x > T$, then by Lemma 5.9 we get $d(y, y_i) \leq 6\delta$. This shows $\mathrm{Cov}(\partial X, e^{-T}) \geq \mathrm{Cov}(S(x, T), 6\delta)$.
Now let $\{y_i\}$ be points realizing $\mathrm{Cov}(S(x, T+\delta), \delta)$. For every i let $z_i \in \partial X$ be the point at infinity of a σ-geodesic ray ξ_i that extends the σ-geodesic $[x, y_i]$. For every $z \in \partial X$ we take a geodesic ray ξ_{xz}. We know it exists i such that $d(\xi_{xz}(T+\delta), y_i) \leq \delta < 2\delta$, therefore $(z, z_i)_x > T$ by Lemma 5.9. This shows $\mathrm{Cov}(\partial X, e^{-T}) \leq \mathrm{Cov}(S(x, T+\delta), \delta)$. The conclusion follows by Proposition 3.1. $\square$

Putting together Proposition 3.1, Proposition 3.4, Proposition 5.8, Theorem 4.2 and Proposition 4.3 we get the proof of Theorem B.

6 Entropies of subsets of the boundary

Let (X, σ) be a δ-hyperbolic GCB-space that is P_0-packed at scale r_0. In this section we will consider a subset C of ∂X and we define the relative version, with respect to C, of all the different definitions of entropies introduced in the previous sections. We observe that when $C = \partial X$ then we are in the case yet studied.

6.1 Covering and volume entropy

Let (X, σ) be a δ-hyperbolic GCB-space that is P_0-packed at scale r_0 and let C be a subset of ∂X. The *upper covering entropy* of C is defined as

$$\limsup_{T \to +\infty} \frac{1}{T} \log \mathrm{Cov}(\overline{B}(x, T) \cap \overline{B}(\text{QC-Hull}(C), \tau), r),$$

where $r > 0$, $\tau \geq 0$ and $x \in X$ and it is denoted by $\overline{h_{\mathrm{Cov}}}(C)$. The *lower covering entropy* of C, denoted by $\underline{h_{\mathrm{Cov}}}(C)$, is defined taking the limit inferior instead of the limit superior. These quantities do not depend on $x \in X$ as usual. The analogue of Proposition 3.1 holds.

Proposition 6.1 *Let (X, σ) be a δ-hyperbolic* GCB*-space that is P_0-packed at scale r_0, C be a subset of ∂X and $x \in X$. Then*

$$\frac{1}{T} \log \mathrm{Cov}(\overline{B}(x, T) \cap \overline{B}(\text{QC-Hull}(C), \tau), r) \underset{P_0, r_0, r, r', \tau, \tau'}{\asymp} \frac{1}{T} \log \mathrm{Pack}(\overline{B}(x, T) \cap \overline{B}(\text{QC-Hull}(C), \tau'), r')$$

for all $r, r' > 0$ and $\tau, \tau' \geq 0$. In particular any of these functions can be used in the definition of the upper and lower covering entropies of C.

Proof Once τ is fixed the asymptotic estimate can be proved exactly as in Proposition 3.1. Moreover for all $\tau \geq 0$ it is easy to prove that

$$\begin{aligned}&\mathrm{Cov}(\overline{B}(x,T)\cap\overline{B}(\text{QC-Hull}(C),\tau),r)\\&\quad\leq \mathrm{Cov}(\overline{B}(x,T)\cap\text{QC-Hull}(C),r_0)\cdot\mathrm{Cov}(r_0+\tau,r_0).\end{aligned}$$

and $\mathrm{Cov}(r_0+\tau, r_0)$ is uniformly bounded in terms of P_0, r_0 and τ by Proposition 2.1. This concludes the proof. □

Clearly when $C = \partial X$ we have $\overline{h_{\mathrm{Cov}}}(\partial X) = \overline{h_{\mathrm{Cov}}}(X)$. Moreover if C is a closed subset of ∂X then $\overline{h_{\mathrm{Cov}}}(C) \leq \overline{h_{\mathrm{Cov}}}(\partial X)$, so $\overline{h_{\mathrm{Cov}}}(C) \leq \frac{\log(1+P_0)}{r_0}$ by Lemma 3.3.
The analogue of Proposition 3.2 holds. We remark that in this case a dependence on δ appears.

Proposition 6.2 *Let (X,σ) be a δ-hyperbolic* GCB*-space that is P_0-packed at scale r_0, C be a subset of ∂X and $x \in X$. Then*

$$\frac{1}{T}\log\mathrm{Cov}(\overline{B}(x,T)\cap\text{QC-Hull}(C),r) \underset{P_0,r_0,r,\delta}{\asymp} \frac{1}{T}\log\mathrm{Cov}(S(x,T)\cap\text{QC-Hull}(C),r)$$

In particular any of these functions can be used in the definition of the upper and lower covering entropies of C.

Proof As in the proof of Proposition 3.2 one inequality is obvious, so we are going to prove the other. We divide the ball $\overline{B}(x,T)$ in the annulii $A(x,kr,(k+1)r)$ with $k = 0,\ldots,\frac{T}{r}-1$. Therefore we can estimate the quantity $\mathrm{Cov}(\overline{B}(x,T)\cap\text{QC-Hull}(C), 72\delta+2r)$ from above by

$$\sum_{k=0}^{\frac{T}{r}-1}\mathrm{Cov}(A(x,kr,(k+1)r)\cap\text{QC-Hull}(C),72\delta+2r).$$

We claim that every element of the sum is $\leq \mathrm{Cov}(S(x,T)\cap\text{QC-Hull}(C),r)$. Indeed let $y_1,\ldots,y_N$ be a set of points realizing $\mathrm{Cov}(S(x,T)\cap\text{QC-Hull}(C),r)$. For all $i=1,\ldots,N$ we consider the σ-geodesic segment $\gamma_i = [x,y_i]$ and we call x_i the point along this geodesic at distance kr from x. We want to show that $x_1,\ldots,x_N$ is a $(72\delta+2r)$-dense subset of $A(x,kr,(k+1)r)\cap\text{QC-Hull}(C)$. Given a point $y \in A(x,kr,(k+1)r)\cap\text{QC-Hull}(C)$ there exists a σ-geodesic line γ with endpoints in C containing y. We parametrize γ so that $\gamma(0)$ is a projection of x on γ and $y \in \gamma|_{[0,+\infty)}$. We take a point $y_T \in \gamma|_{[0,+\infty)}$ at distance T from x, so that $y_T \in S(x,T)\cap\text{QC-Hull}(C)$ and therefore there exists i such that $d(y_T,y_i) \leq r$. By Lemma 5.2 the path $\alpha = [x,\gamma(0)]\cup[\gamma(0),y_T]$, where the second geodesic is a subsegment of γ, is a $(1,4\delta)$-quasigeodesic and, if t_y denotes the real number such that $\alpha(t_y) = y$, it holds $t_y \in [kr,(k+1)r]$. By Lemma 5.2 we get $d(y,\gamma_i'(t_y)) \leq 72\delta$, where γ_i' is the σ-geodesic $[x,y_T]$. We conclude the proof of the claim since

$$d(y,x_i) \leq d(y,\gamma_i'(t_y)) + d(\gamma_i'(t_y),\gamma_i(t_y)) + d(\gamma_i(t_y),x_i) \leq 72\delta+2r,$$

from the convexity of σ. The thesis follows by Proposition 6.1. □

The *upper volume entropy* of C with respect to a measure μ is

$$\overline{h_\mu}(C) = \sup_{\tau\geq 0}\limsup_{T\to+\infty}\frac{1}{T}\log\mu(\overline{B}(x,T)\cap\overline{B}(\text{QC-Hull}(C),\tau)),$$

where $x \in X$. The *lower volume entropy* is defined by taking the limit inferior instead of the limit superior and it is denoted by $\underline{h_\mu}(C)$.

Proposition 6.3 *Let (X, σ) be a δ-hyperbolic* GCB*-space that is P_0-packed at scale r_0, let C be a subset of ∂X and let μ be a measure on X which is H-homogeneous at scale r. Then for all $\tau \geq r$ it holds*

$$\frac{1}{T}\log \mu(\overline{B}(x, T) \cap \overline{B}(\text{QC-Hull}(C), \tau)) \underset{H,P_0,r_0,r,\tau}{\asymp} \frac{1}{T}\log \text{Cov}(\overline{B}(x, T) \cap \text{QC-Hull}(C), r_0).$$

In particular the upper (resp. lower) volume entropy of C with respect to μ coincides with the upper (resp. lower) covering entropy of C and it can be computed using $\tau = r$ in place of the supremum.

Proof By Remark 3.5 we know that μ is $H(\tau)$-homogeneous at scale τ for all $\tau \geq r$, where $H(\tau)$ depends on P_0, r_0, τ, r, H. Therefore the proof of Proposition 3.4 works in this case. □

6.2 Lipschitz topological entropy

Let (X, σ) be a δ-hyperbolic GCB-space that is P_0-packed at scale r_0. For a subset C of ∂X and $Y \subseteq X$ we set

$$\text{Geod}_\sigma(Y, C) = \{\gamma \in \text{Geod}_\sigma(X) \text{ s.t. } \gamma^\pm \subseteq C \text{ and } \gamma(0) \in Y\}.$$

If $Y = X$ we simply write $\text{Geod}_\sigma(C)$. Clearly $\text{Geod}_\sigma(C)$ is a Φ-invariant subset of $\text{Geod}_\sigma(X)$, so the reparametrization flow is well defined on it. The *upper Lipschitz-topological entropy* of $\text{Geod}_\sigma(C)$ is defined as

$$\overline{h_{\text{Lip-top}}}(\text{Geod}_\sigma(C)) = \inf_{\mathrm{d}} \sup_K \lim_{r\to 0} \limsup_{T\to+\infty} \frac{1}{T}\log \text{Cov}_{\mathrm{d}^T}(K, r),$$

where the infimum is taken among all geometric metrics on $\text{Geod}_\sigma(C)$. The *lower Lipschitz-topological entropy* is defined taking the limit inferior instead of the limit superior and it is denoted by $\underline{h_{\text{Lip-top}}}(\text{Geod}_\sigma(C))$. In the following result we observe the difference between closed and non-closed subsets of ∂X.

Theorem 6.4 *Let (X, σ) be a δ-hyperbolic* GCB*-space that is P_0-packed at scale r_0 and C be a subset of ∂X. Then*

$$\overline{h_{\text{Lip-top}}}(\text{Geod}_\sigma(C)) = \sup_{C' \subseteq C} \overline{h_{\text{Cov}}}(C'),$$

where the supremum is among closed subsets C' of C. The same holds for the lower entropies.

We remark that the supremum of the covering entropies among the closed subsets of C can be strictly smaller than the covering entropy of C (see [9]), marking the distance between the equivalences of the different notions of entropies in case of non-closed subsets of the boundary. We start with an easy lemma.

Lemma 6.5 *Let (X, σ) and C be as in Theorem 6.4 and let $x \in X$. Then every compact subset of* $\text{Geod}_\sigma(C)$ *is contained in* $\text{Geod}_\sigma(\overline{B}(x, R), C')$ *for some $R \geq 0$ and some $C' \subseteq C$ closed. Moreover* $\text{Geod}_\sigma(\overline{B}(x_0, R), C')$ *is compact for all $R \geq 0$ and all closed $C' \subseteq C$.*

Proof We fix a compact subset K of $\text{Geod}_\sigma(C)$. The continuity of the evaluation map gives that $E(K)$ is contained in some ball $\overline{B}(x, R)$. Moreover the maps $+, - \colon \text{Geod}_\sigma(X) \to \partial X$, defined by $\gamma \mapsto \gamma^+, \gamma^-$ respectively, are continuous ([5], Lemma 1.6). This means that $C' = +(K) \cup -(K)$ is a closed subset of ∂X and clearly $K \subseteq \text{Geod}_\sigma(\overline{B}(x, R), C')$. By a similar argument, and since the evaluation map is proper, it follows that the set $\text{Geod}_\sigma(\overline{B}(x, R), C')$ is compact for all $R \geq 0$ and all $C' \subseteq C$ closed. □

For a metric $f \in \mathcal{F}$ and $C \subseteq \partial X$ we denote by $\overline{h_f}$ the upper metric entropy of $\text{Geod}_\sigma(C)$ with respect to f, that is

$$\overline{h_f}(\text{Geod}_\sigma(C)) = \sup_K \lim_{r\to 0} \limsup_{T\to+\infty} \frac{1}{T} \log \text{Cov}_{f^T}(K, r).$$

Taking the limit inferior instead of the limit superior we define the lower metric entropy of $\text{Geod}_\sigma(C)$ with respect to f, denoted by $\underline{h_f}(\text{Geod}_\sigma(C))$. The analogue of Proposition 4.3 is the following.

Proposition 6.6 *Let (X, σ) be as in Theorem 6.4, C' be a closed subset of ∂X, $f \in \mathcal{F}$, $x \in X$ and L be the constant given by Lemma 5.4. Then*

(i) $\overline{h_f}(\text{Geod}_\sigma(\overline{B}(x, R), C')) = \overline{h_f}(\text{Geod}_\sigma(\overline{B}(x, L), C'))$ *for all* $R \geq L$;
(ii) $\overline{h_f}(\text{Geod}_\sigma(C')) = \overline{h_f}(\text{Geod}_\sigma(\overline{B}(x, L), C')) \leq \overline{h_{\text{Cov}}}(C')$;
(iii) The function $r \mapsto \limsup_{T\to+\infty} \frac{1}{T} \log \text{Cov}_{f^T}(\text{Geod}_\sigma(\overline{B}(x, L), C'), r)$ *is constant.*

The same conclusions hold for the lower entropies.

We observe that applying the Key Lemma 4.4 we have directly the relative version of Corollary 4.5.

Corollary 6.7 *Let $f \in \mathcal{F}$, $x \in X$, $R \geq 0$ and $0 < r \leq r'$. Then*

$$\frac{1}{T} \log \text{Cov}_{f^T}(\text{Geod}_\sigma(\overline{B}(x, R), C'), r') \underset{P_0,r_0,r,r',f}{\asymp} \frac{1}{T} \log \text{Cov}_{f^T}(\text{Geod}_\sigma(\overline{B}(x, R), C'), r).$$

Proof of of Proposition 6.6 We fix $R \geq L$ and $T \geq 0$. We take a set $\gamma_1, \ldots, \gamma_N$ of σ-geodesic lines realizing $\text{Cov}_{f^T}(\text{Geod}_\sigma(\overline{B}(x, L), C'), r_0)$. Our aim is to show that $\gamma_1, \ldots, \gamma_N$ is a $(4R + 2L + C(f) + 76\delta + r_0)$-dense subset of $\text{Geod}_\sigma(\overline{B}(x, R), C')$. This, together with Corollary 6.7, will prove (i). We consider a σ-geodesic line $\gamma \in \text{Geod}_\sigma(\overline{B}(x, R), C')$, so $d(\gamma(0), x) =: S \leq R$. By Lemma 5.3 there exists a σ-geodesic ray ξ starting at x such that $d(\xi(S+t), \gamma(t)) \leq 2S + 76\delta$ for all $t \geq 0$ and in particular ξ^+ belongs to C. Now we apply Lemma 5.4 to find a σ-geodesic line $\gamma' \in \text{Geod}(C')$ such that $d(\xi(t), \gamma'(t)) \leq L$ for all $t \geq 0$. Clearly we have $\gamma' \in \text{Geod}_\sigma(\overline{B}(x, L), C')$ and $d(\gamma'(S+t), \gamma(t)) \leq 2S + L + 76\delta$ for all $t \geq 0$. Therefore $d(\gamma'(t), \gamma(t)) \leq 3S + L + 76\delta$ for all $t \geq 0$. This implies that for all $t \in [0, T]$ we have

$$f^t(\gamma, \gamma') \leq \int_{-\infty}^{-t} \big(d(\gamma(0), \gamma'(0)) + 2|s|\big) f(s)ds + \int_{-t}^{+\infty} \big(3S + L + 76\delta\big) f(s)ds.$$

Since $d(\gamma(0), \gamma'(0)) \leq L+S$ we get $f^t(\gamma, \gamma') \leq 4S+2L+C(f)+76\delta$ using the properties of f, and so $f^T(\gamma, \gamma') \leq 4R+2L+C(f)+76\delta$. Moreover, since $\gamma' \in \text{Geod}_\sigma(\overline{B}(x, L), C')$, there exists γ_i such that $f^T(\gamma', \gamma_i) \leq r_0$. This implies $f^T(\gamma, \gamma_i) \leq 4R + 2L + C(f) + 76\delta + r_0$.
We observe that (iii) follows directly from the previous corollary.

The first equality in (ii) follows by (i). In order to prove the inequality we fix $y_1, \ldots, y_N$ realizing $\mathrm{Cov}(S(x,T) \cap \text{QC-Hull}(C'), r_0)$. Up to change y_i with a point at distance at most 8δ from it we can suppose there are $\gamma_i \in \mathrm{Geod}_\sigma(C')$ such that $y_i \in \gamma_i$ and $y_1, \ldots, y_N$ is a $(8\delta + r_0)$-dense subset of $S(x,T) \cap \text{QC-Hull}(C')$, as follows by Lemma 5.6. By Lemma 5.3 there exists an orientation of γ_i such that, called $S_i = d(x, \gamma_i(0))$ and $T_i \geq 0$ such that $\gamma_i(T_i) = y_i$, we have $T \leq S_i + T_i \leq T + 4\delta$ and the σ-geodesic ray $\xi_i = [x, \gamma_i^+]$ satisfies $d(\xi_i(S_i + t), \gamma_i(t)) \leq 76\delta$ for all $t \geq 0$. By Lemma 5.4 there exists $\gamma_i' \in \mathrm{Geod}_\sigma(\overline{B}(x,L), C')$ such that $d(\gamma_i'(t), \xi_i(t)) \leq L$ for all $t \geq 0$. We claim that the set $\{\gamma_i'\}$ is $(6L + 176\delta + 2r_0 + 2C(f))$-dense in $\mathrm{Geod}_\sigma(\overline{B}(x,L), C')$. By (i) and (iii) this would imply the thesis. We fix $\gamma \in \mathrm{Geod}_\sigma(\overline{B}(x,L), C')$, so there exists $y \in S(x,T)$ and $T_y \in [T-L, T+L]$ such that $\gamma(T_y) = y$ and therefore $d(y, y_i) \leq 8\delta + r_0$ for some i. We observe that we have $d(\gamma_i'(S_i + T_i), y_i) \leq L + 76\delta$ and so $d(\gamma_i'(T), y_i) \leq L + 80\delta$. Moreover $d(\gamma(T), y_i) \leq L + 8\delta + r_0$ implying $d(\gamma(T), \gamma_i'(T)) \leq 2L + 88\delta + r_0$. Furthermore by definition $d(\gamma(0), \gamma_i'(0)) \leq 2L$, so by convexity of σ we get $d(\gamma(t), \gamma_i'(t)) \leq 2L + 88\delta + r_0$ for all $t \in [0, T]$. The thesis follows by the classical subdivision of the integral defining f into three parts, each estimated by the constants above. □

Proof of Theorem 6.4 We fix a geometric metric d on $\mathrm{Geod}_\sigma(C)$ and we denote by M the Lipschitz constant with respect to d of the evaluation map E. By Remark 5.5 the constant L given by Lemma 5.4 can be chosen independently of $C' \subseteq C$, once x is fixed. Clearly we have

$$\begin{aligned} &\sup_{R \geq 0, C' \subseteq C} \lim_{r \to 0} \limsup_{T \to +\infty} \frac{1}{T} \log \mathrm{Cov}_{\mathrm{d}^T}(\mathrm{Geod}_\sigma(\overline{B}(x,R), C'), r) \\ &\geq \sup_{C' \subseteq C} \limsup_{T \to +\infty} \frac{1}{T} \log \mathrm{Cov}_{\mathrm{d}^T}(\mathrm{Geod}_\sigma(\overline{B}(x,L), C'), r_0). \end{aligned}$$

We fix σ-geodesic lines $\gamma_1, \ldots, \gamma_N$ realizing $\mathrm{Cov}_{\mathrm{d}^T}(\mathrm{Geod}_\sigma(\overline{B}(x,L), C'), r_0)$. Since $d(\gamma_i(0), x) \leq L$ for all $i = 1, \ldots, N$ then there exists $t_i \in [T-L, T+L]$ such that $d(\gamma_i(t_i), x) = T$. We claim that the points $y_i = \gamma_i(t_i) \in S(x,T) \cap \text{QC-Hull}(C')$ are $(2L + 80\delta + Mr_0)$-dense. By Proposition 6.2 this would imply

$$\overline{h_{\text{Lip-top}}}(\mathrm{Geod}_\sigma(C)) \geq \sup_{C' \subseteq C} \overline{h_{\text{Lip-top}}}(\mathrm{Geod}_\sigma(C')) \geq \sup_{C' \subseteq C} \overline{h_{\mathrm{Cov}}}(C').$$

We fix $y \in S(x,T) \cap \text{QC-Hull}(C')$ and we select a geodesic line $\gamma \in \mathrm{Geod}(C')$ containing y. Up to replacing y with a point at distance at most 8δ we can suppose $\gamma \in \mathrm{Geod}_\sigma(C')$, as follows by Lemma 5.6. By Lemma 5.3, with an appropriate choice of the orientation of γ, the σ-geodesic ray $\xi = [x, \gamma^+]$ satisfies $d(\xi(S+t), \gamma(t)) \leq 76\delta$ for all $t \geq 0$, where $S = d(x, \gamma(0))$. By Lemma 5.4 there exists $\gamma' \in \mathrm{Geod}_\sigma(\overline{B}(x,L), C')$ such that $d(\xi(t), \gamma'(t)) \leq L$ for all $t \geq 0$, implying $d(\gamma'(S+t), \gamma(t)) \leq L + 76\delta$ for all $t \geq 0$. Denoting by T_y the real number such that $\gamma(T_y) = y$ we have by Lemma 5.3 that $T \leq S + T_y \leq T + 4\delta$. Therefore we apply the previous estimate with $t = T_y$ obtaining $d(\gamma'(T), y) \leq d(\gamma'(T), \gamma'(S + T_y)) + d(\gamma'(S+T_y), y) \leq L + 80\delta$. Moreover there exists $i \in \{1, \ldots, N\}$ such that $\mathrm{d}^T(\gamma', \gamma_i) \leq r_0$ and in particular $d(\gamma'(T), \gamma_i(T)) \leq Mr_0$. Therefore we get $d(y_i, y) \leq d(\gamma_i(t_i), \gamma_i(T)) + d(\gamma_i(T), y) \leq 2L + 80\delta + Mr_0$. Now, up to adding 8δ, we obtain the inequality. The other inequality follows by Proposition 6.6. Indeed we have

$$\overline{h_{\text{Lip-top}}}(\mathrm{Geod}_\sigma(C)) \leq \sup_{C' \subseteq C} \overline{h_f}(\mathrm{Geod}_\sigma(C')) \leq \sup_{C' \subseteq C} \overline{h_{\mathrm{Cov}}}(C').$$

□

Remark 6.8 Let (X, σ) be as in Theorem 6.4, $C \subseteq \partial X$ closed and $x \in \text{QC-Hull}(C)$. By the proof of Theorem 6.4, Lemma 5.4 and Remark 5.5 we obtain

$$\frac{1}{T}\log \mathrm{Cov}(S(x,T) \cap \text{QC-Hull}(C), r_0) \underset{P_0,r_0,\delta,f}{\asymp} \frac{1}{T}\log \mathrm{Cov}_{f^T}(\mathrm{Geod}_\sigma(\overline{B}(x,L),C), r_0)$$

for all $f \in \mathcal{F}$, where $L = 14\delta$.

6.3 Minkowski dimension

The relative version of Proposition 5.8 is:

Proposition 6.9 *Let (X, σ) be a δ-hyperbolic* GCB*-space that is P_0-packed at scale r_0, let C be a subset of ∂X, $x \in X$ and L be the constant given by Lemma 5.4. Then*

$$\frac{1}{T}\log \mathrm{Cov}(C, e^{-T}) \underset{P_0,r_0,\delta,L}{\asymp} \frac{1}{T}\log \mathrm{Cov}(S(x,T) \cap \text{QC-Hull}(C), r_0).$$

In particular the upper (resp. lower) Minkowski dimension of C equals the upper (resp. lower) covering entropy of C.

Proof We can suppose $T \geq L$. Let $z_1, \ldots, z_N$ be points realizing $\mathrm{Cov}(C, e^{-T})$. For every i we take a geodesic ray $[x, z_i]$. By Lemma 5.4 there exists a geodesic line γ_i with both endpoints in C such that $d(\gamma_i(t), \xi(t)) \leq L$ for every $t \geq 0$. We take a point $y_i = \gamma_i(t_i)$ with $t_i \geq 0$ such that $d(y_i, x) = T$. We know $|t_i - T| \leq 2L$. We claim the set $\{y_i\}$ covers $S(x,T) \cap \text{QC-Hull}(C)$ at scale $82\delta + 3L$. Indeed let $y \in S(x,T) \cap \text{QC-Hull}(C)$, i.e. there exists a geodesic line γ with both endpoints in C such that $y \in \gamma$. We parametrize γ so that $\gamma(0)$ is a projection of x on γ and $y \in \gamma|_{[0,+\infty)}$. We consider a geodesic ray $\xi = [x, \gamma^+]$. By Lemma 5.3 we know that $d(\xi(T), y) \leq 76\delta + 3L$. Moreover $\gamma^+ \in C$, so there is i such that $(z_i, \gamma^+) > T$. By Lemma 5.9 we get $d(y_i, \xi(T)) \leq 6\delta$, so $d(y, y_i) \leq 82\delta + 3L$. This shows

$$\mathrm{Cov}(S(x,T) \cap \text{QC-Hull}(C), 82\delta + 3L) \leq \mathrm{Cov}(C, e^{-T}).$$

Let $\{y_1, \ldots, y_N\}$ be points realizing $\mathrm{Cov}(S(x, T+2L+38\delta) \cap \text{QC-Hull}(C), \frac{L}{2})$. Therefore for every i there exists a geodesic line γ_i with both endpoints in C containing y_i. We parametrize each γ_i so that $\gamma_i(0)$ is a projection of x on γ_i and $y_i \in \gamma_i|_{[0,+\infty)}$. We claim that the set $\{\gamma_i^+\}$ covers C at scale e^{-T}. Indeed for every $z \in C$ we take a geodesic ray $\xi = [x, z]$ and we set $y = \xi(T + 2L + 38\delta)$. By Lemma 5.4 we know it exists a geodesic line γ with both endpoints in C such that $d(y, \gamma(T + 2L + 38\delta)) \leq L$. Moreover there is a point y' along γ which is at distance exactly $T + 2L + 38\delta$ from x and that satisfies $d(y, y') \leq 2L$. Now we know there exists i such that $d(y', y_i) \leq \frac{L}{2}$, moreover for a fixed geodesic ray $\xi_i = [x, z_i]$ it holds that $d(\xi_i(T + 2L + 38\delta), y_i) \leq L + 76\delta$ by Lemma 5.3. So in conclusion we get

$$d(\xi(T + 2L + 38\delta), \xi_i(T + 2L + 38\delta)) < 4L + 76\delta.$$

By Lemma 5.9 we conclude that $(z, z_i)_x > T$, i.e.

$$\mathrm{Cov}(C, e^{-T}) \leq \mathrm{Cov}\left(S(x, T + 2L + 38\delta) \cap \text{QC-Hull}(C), \frac{L}{2}\right).$$

Now the conclusion follows by Proposition 6.1. □

The proof of Theorems D and E follow by Proposition 6.1, Proposition 6.2, Proposition 6.3, Theorem 6.4, Proposition 6.6, Remark 6.8 and Proposition 6.9.

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