Same-hemisphere three-gluon-emission contribution to the zero-jettiness soft function at N3LO QCD

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We complete the calculation of the three-gluon-emission contribution to the same-hemisphere part of the zero-jettiness soft function at next-to-next-to-next-to-leading order in perturbative QCD.

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I. INTRODUCTION

The goal of this paper is to present the result for the same-hemisphere three-gluon-emission contribution to the zero-jettiness soft function at next-to-next-to-next-to-leading order (N3LO) in perturbative QCD. Soft functions are required ingredients for a fully-differential perturbative description of collider processes in the context of so-called slicing schemes [1,2]. The zero-jettiness soft function is defined using the jettiness variable [3,4],

$$\tau = \sum_{j=1}^{m} \min_{q\in\{n,\bar{n}\}} \left[\frac{2qk_j}{n\bar{n}}\right], \tag{1}$$

where k_j are four-momenta of final-state partons, and n and $\bar{n}$ are two lightlike four-vectors pointing in the direction of incoming partons. Currently, the zero-jettiness soft function is known up to next-to-next-to-leading order (NNLO) in QCD [5,6], and its extension to one higher order is a nontrivial problem.

Indeed, computation of the zero-jettiness soft function needs to overcome several technical challenges that were discussed in Ref. [7]. These challenges stem from the fact that the observable that defines the soft function—the so-called jettiness—involves Heaviside functions. These Heaviside functions are needed to distinguish between emissions of soft gluons into two hemispheres, defined relative to directions of incoming hard radiators.

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The presence of Heaviside functions complicates the application of generalized unitarity [8] and integration-by-parts (IBP) identities [9] to phase-space integrals. We have discussed in Ref. [7] how to overcome this problem and explained how to derive useful integration-by-parts relations for integrals with Heaviside functions. To show the efficacy of this method, we employed an eikonal function derived in Ref. [10], which describes emissions of three soft gluons and integrated it over soft-gluon phase space subject to zero-jettiness constraints. We restricted ourselves to contributions where all gluons are emitted into the same hemisphere.

Unfortunately, in Ref. [7], we have not completed the computation of this "same-hemisphere" contribution. Indeed, the representation of the eikonal function derived in Ref. [10] involves four terms, and in Ref. [7] we have fully integrated three of them. The fourth contribution can be written in the following way:

$$S_d = \int \mathrm{d}\Phi^{nnn}_{\theta\theta\theta} \omega^{(3),d}_{n\bar{n}}(k_1, k_2, k_3), \tag{2}$$

where $k_{1,2,3}$ are four-momenta of final-state gluons, $\Phi^{nnn}_{\theta\theta\theta}$ is the phase space subject to zero-jettiness conditions, cf. Eq. (4), and $\omega^{(3),d}_{n\bar{n}}(k_1, k_2, k_3)$ is the eikonal function defined in Eq. (5.49) in Ref. [7]. We note that the integral in Eq. (2) is the most complicated one, as it contains a propagator that depends on the relative orientation of four-momenta of all three soft gluons.

Although we described a possible way to calculate this contribution in Ref. [7], we did not complete its computation there. The goal of this paper is to compute the missing piece and to present the result for the same-hemisphere three-gluon-emission contribution to the zero-jettiness soft function.

The rest of the paper is organized as follows. In Sec. II we explain the general strategy that we used to integrate the function $\omega^{3,(d)}_{n\bar{n}}$. In Sec. III we describe the computation of the boundary condition for one of the master integrals that provides a contribution that has unusual sensitivity to an analytic regulator. In Sec. IV we discuss checks that we performed to ensure the correctness of the computation. In Sec. V we present the result for the same-hemisphere three-gluon-emission contribution to the zero-jettiness soft function and conclude in Sec. VI. Abelian contributions to our result are derived in the Appendix.

II. INTEGRATING $\omega_{n\bar{n}}^{3,(d)}$

The missing part of the same-hemisphere contribution to the zero-jettiness soft function, displayed in Eq. (2), requires integration of the function $\omega_{n\bar{n}}^{3,(d)}$, which contains terms with a propagator $1/k_{123}^2$, where $k_{123} = k_1 + k_2 + k_3$.

To integrate the function $\omega_{n\bar{n}}^{3,(d)}$, we consider a class of integrals,

$$I_{\theta\theta\theta} = \int \mathrm{d}\Phi_{\theta\theta\theta}^{nnn} \frac{(k_1 n)^\nu (k_2 n)^\nu (k_3 n)^\nu}{k_{123}^2 (k_1 k_2)(k_1 k_3)\cdots}, \tag{3}$$

where n and $\bar{n}$ are two lightlike four-vectors pointing in the direction of incoming partons, ellipses stand for eikonal propagators,[1] and $\mathrm{d}\Phi_{\theta\theta\theta}^{nnn}$ is the normalized phase-space measure. It is defined as follows [7]:

$$\mathrm{d}\Phi_{f_1 f_2 f_3}^{nnn} = N_\epsilon^{-3} \left(\prod_{i=1}^{3} [\mathrm{d}k_i] f_i(k_i) \right) \delta\left(1 - \sum_{i=1}^{3} k_i n\right), \tag{4}$$

where the normalization factor is

$$N_\epsilon = \frac{\Omega^{(d-2)}}{4(2\pi)^{d-1}} = \frac{(4\pi)^\epsilon}{16\pi^2 \Gamma(1-\epsilon)}, \tag{5}$$

the function f_i is either $\theta(k_i\bar{n} - k_i n)$ or $\delta(k_i\bar{n} - k_i n)$, and

$$[\mathrm{d}k_i] = \frac{\mathrm{d}^d k}{(2\pi)^d} 2\pi \delta(k_i^2) \theta(k_i^0). \tag{6}$$

We note that we set the jettiness variable τ to one since the final result is a uniform function of τ. We also note that in the spirit of integration by parts, all propagators that appear in the integrand in Eq. (3) can be raised to arbitrary integer powers.

Finally, as can be seen from Eq. (3), we introduced scalar products of the gluon four-momenta $k_{1,2,3}$ with the light-cone vector n raised to power ν into the integrand of $I_{\theta\theta\theta}$. As we explained in Ref. [7], they are required because some integrals, that appear in the course of the IBP reduction, contain divergences that are not regulated dimensionally; the analytic regulator ν is introduced to regulate them. To simplify the notation, we define the regulated phase-space measure to be

$$\mathrm{d}\Phi_{f_1 f_2 f_3}^{\nu} = \mathrm{d}\Phi_{f_1 f_2 f_3}^{nnn} (k_1 n)^\nu (k_2 n)^\nu (k_3 n)^\nu. \tag{7}$$

Unfortunately, even after integrals defined in Eq. (3) are reduced to master integrals, the master integrals with $1/k_{123}^2$ propagators appear to be too complicated for direct analytic integration. For this reason, as explained in Ref. [7], we derive differential equations satisfied by these integrals and solve them numerically. To do that, we introduce a mass parameter into the propagator that contains the momenta of all three gluons [11],[2]

$$\frac{1}{k_{123}^2} \to \frac{1}{k_{123}^2 + m^2}. \tag{8}$$

The appearance of the mass parameter m allows us to differentiate with respect to it and use integration by parts to derive differential equations for relevant integrals. We then fix boundary conditions at $m \to \infty$, solve differential equations *numerically* and determine relevant integrals at $m = 0$. This can be done by matching the numerical solution with the formal solution at $m = 0$ and then taking the $m \to 0$ limit in an appropriate manner. Since, as we explained in Ref. [7], the differential equations can be solved to an arbitrary precision as a matter of principle, and to more than 2000 digits in practice, we have used the high-precision numerical results for master integrals to find the analytic form of the solution by fitting them to a basis of transcendental constants and rational numbers.

To reiterate, once it is understood how to use generalized unitarity to write down integration-by-parts identities for integrals with θ functions, it becomes a fairly standard problem to derive differential equations for relevant integrals. However, great care is needed when choosing the basis of master integrals because of the analytic regulator ν; in essence, we need to find a basis that admits a simple $\nu \to 0$ limit. Ideally, this should happen for master integrals that appear in the soft function S_d, as well as in m-dependent differential equations that we need to solve.

To find a suitable basis, we use the following consideration. In spite of the fact that the reduction to master integrals performed *after* setting $\nu = 0$ in Eq. (3) is incorrect, it gives us a good idea about integrals that are independent of each other in the $\nu \to 0$ limit. Hence, to find a suitable basis for master integrals, we start by performing a reduction of integrals shown in Eq. (3) at $\nu = 0$. We then insist that the master integrals found in the course of such a reduction should be also chosen as master integrals for the reduction of integrals with $\nu \neq 0$, to the extent possible.

[1] These are all possible scalar products $q \cdot v$, where q is a linear combination of four-vectors of soft gluons, and v is one of the two light-cone vectors n or $\bar{n}$.

[2] Similar ideas that use auxiliary mass scales have been presented, for example, in Refs. [12–14].

We do this for integrals with and without auxiliary mass parameter m. With this choice of master integrals, we find that the integral of the soft function $\omega_{n\bar{n}}^{3,(d)}$ written in terms of master integrals, as well as the differential equations that these integrals obey, admit a simple $\nu \to 0$ limit.

For example, the result of the reduction of S_d to master integrals can be written in the following way:

$$S_d = \sum_\alpha c_\alpha(\nu) I_\alpha^\nu + \nu \sum_\alpha \tilde{c}_\alpha(\nu) \bar{I}_\alpha^\nu, \tag{9}$$

where the coefficients c_α and $\tilde{c}_\alpha$ are regular in the $\nu \to 0$ limit. The list of integrals $\{I_\alpha^\nu\}$ coincides with the list of master integrals for S_d that one obtains performing a reduction at $\nu = 0$. New integrals that appear in the reduction, which we denote as $\bar{I}_\alpha^\nu$ in the above equation, are multiplied with the parameter ν and, therefore, disappear if the naive $\nu \to 0$ limit is taken.

However, the naive $\nu \to 0$ limit in Eq. (9) cannot be taken because some of $\bar{I}_\alpha^\nu$ integrals are $1/\nu$ divergent and, therefore, need to be retained. Examples of such integrals without the $1/k_{123}^2$ propagator can be found in Ref. [7].

Unfortunately, many integrals among I_α^ν and $\bar{I}_\alpha^\nu$ contain the $1/k_{123}^2$ propagator; to study these integrals, we use Eq. (8) and turn them into integrals with the mass parameter m. We construct a system of differential equations with respect to the mass parameter m by including every integral with the $1/k_{123}^2$ propagator that appears in Eq. (9) and other integrals that are needed to close it. Such a system takes the following form:

$$\frac{\partial}{\partial m^2} \boldsymbol{J}^\nu = \boldsymbol{M}_1(\nu) \boldsymbol{J}^\nu + \nu \boldsymbol{N}_1(\nu) \bar{\boldsymbol{J}}^\nu,$$
$$\frac{\partial}{\partial m^2} \bar{\boldsymbol{J}}^\nu = \boldsymbol{M}_2(\nu) \bar{\boldsymbol{J}}^\nu + \boldsymbol{N}_2(\nu) \boldsymbol{J}^\nu. \tag{10}$$

In Eq. (11) matrices $\boldsymbol{M}_{1,2}$ and $\boldsymbol{N}_{1,2}$ are regular in the $\nu \to 0$ limit, $\boldsymbol{J}^\nu$ integrals are the master integrals that need to be considered for computing $\{I_\alpha^\nu\}$ integrals, and $\bar{\boldsymbol{J}}^\nu$ are integrals that are needed for computing $\{\bar{I}_\alpha{}^\nu\}$ integrals.

From the structure of differential equations in Eq. (11), it is clear that taking the limit $\nu \to 0$ at the early stages of the computation is beneficial since this makes the system of differential equations significantly simpler. However, this is only possible if we know which master integrals are singular in the $\nu \to 0$ limit and which master integrals are not. Unfortunately, it is not trivial to answer this question, and we use several approaches to clarify it.

First, from the computation of integrals of the eikonal functions $\omega^{a,b,c}$ reported in Ref. [7], we know $1/\nu$-divergent integrals that appear in cases when the propagator $1/k_{123}^2$ is absent. Upon inspection, we find that the only m-independent $1/\nu$-divergent integral that appears in the amplitude S_d and in the differential equations reads,

$$\bar{I}_1^\nu = \bar{J}_1^\nu = \int \frac{\mathrm{d}\Phi_{\theta\delta\theta}^\nu}{(k_1\cdot k_3)(k_1\cdot n)(k_{12}\cdot \bar{n})(k_3\cdot\bar{n})}. \tag{11}$$

This integral was computed in Ref. [7], and for this reason we do not discuss it here.

Second, to determine which of the integrals with the propagator $1/k_{123}^2$ are singular in the $\nu \to 0$ limit, we can study these integrals at *finite* values of m and then employ differential equations to determine the $1/\nu$ behavior of the corresponding $\bar{I}_\alpha^\nu$ integrals. To this end, we employed Mellin-Barnes representation of the relevant integrals and used public programs MB [15] and MBresolve [16] to numerically compute all m-dependent integrals that appear in the differential equations at finite values of m. We also used the program pySecDec [17,18] for cross-checks of the numerical computation. Upon doing that, we discovered yet another integral that is singular in the $\nu \to 0$ limit. It reads,[3]

$$\bar{J}_2^\nu = \int \frac{\mathrm{d}\Phi_{\theta\delta\theta}^\nu}{(k_{123}^2+m^2)(k_1\cdot k_3)(k_1\cdot n)(k_{12}\cdot\bar{n})}. \tag{12}$$

This integral is quite peculiar, and we explain in the next section why this is the case and how to compute it.

To proceed further, we rescale $\bar{\boldsymbol{J}}^\nu$ integrals that appear in Eq. (11) by a factor ν. Since, as we just explained, only two integrals $\bar{J}_1^\nu$ and $\bar{J}_2^\nu$ diverge in the $\nu \to 0$ limit, we need to keep them in $\mathcal{O}(\nu^0)$ part of the differential equations. Therefore, we combine $\boldsymbol{J}^\nu$ integrals together with $\nu\bar{J}_1^\nu$ and $\nu\bar{J}_2^\nu$ integrals into a vector $\boldsymbol{\mathcal{J}}^\nu$ and use remaining $\bar{J}^\nu$ integrals, rescaled by a parameter ν, to define a new vector $\bar{\boldsymbol{\mathcal{J}}}^\nu$,

$$\boldsymbol{\mathcal{J}}^\nu = (\boldsymbol{J}^\nu, \nu\bar{J}_1^\nu, \nu\bar{J}_2^\nu),$$
$$\bar{\boldsymbol{\mathcal{J}}}^\nu = (\nu\bar{J}_3^\nu, \cdots). \tag{13}$$

We then obtain a new system of differential equations,

$$\frac{\partial}{\partial m^2}\boldsymbol{\mathcal{J}}^\nu = \boldsymbol{\mathcal{M}}_1(\nu)\boldsymbol{\mathcal{J}}^\nu + \boldsymbol{\mathcal{N}}_1(\nu)\bar{\boldsymbol{\mathcal{J}}}^\nu,$$
$$\frac{\partial}{\partial m^2}\bar{\boldsymbol{\mathcal{J}}}^\nu = \boldsymbol{\mathcal{M}}_2(\nu)\bar{\boldsymbol{\mathcal{J}}}^\nu + \nu\boldsymbol{\mathcal{N}}_2(\nu)\boldsymbol{\mathcal{J}}^\nu. \tag{14}$$

It is straightforward to solve the above equation expanding in $\nu \to 0$ because all the matrices that appear there have smooth $\nu \to 0$ limits and because the integral vectors satisfy $\boldsymbol{\mathcal{J}}^\nu \sim \mathcal{O}(\nu^0)$, $\bar{\boldsymbol{\mathcal{J}}}^\nu \sim \mathcal{O}(\nu)$.

Working to order $\mathcal{O}(\nu^0)$, we can drop $\bar{\boldsymbol{\mathcal{J}}}^\nu$ integrals and set ν to zero in Eq. (14). This leads to a significant reduction in the number of integrals that appear in Eq. (14) and allows

[3]To obtain the integral representation for the integral $\bar{I}_2^\nu$, we only need to replace the propagator $1/(k_{123}^2+m^2)$ with $1/k_{123}^2$ in Eq. (12).

for solving the system of differential equations in a more efficient way.

As follows from the differential equation satisfied by $\nu\bar{J}_1^\nu$ and $\nu\bar{J}_2^\nu$, these integrals are independent of m through $\mathcal{O}(\nu^0)$. In addition, they also have to obey the following relation:

$$\lim_{\nu\to 0}\nu\bar{J}_1^\nu = \frac{1+6\epsilon}{1+4\epsilon}\lim_{\nu\to 0}\nu\bar{J}_2^\nu, \tag{15}$$

to cancel the $1/\nu$ singularity which is naively present in the rescaled differential equation. We will show in the following section that this condition is indeed satisfied.

We note that the above discussion applies to differential equations at finite values of m whereas, eventually, we are interested in the solutions at $m=0$. This limit is, potentially, nontrivial. Indeed, to find required values of integrals at $m=0$, we need to compute master integrals using the following sequence of limits: $m\to 0$, $\nu\to 0$, $\epsilon\to 0$.

However, as explained above, we would like to simplify differential equations by taking the $\nu\to 0$ limit *first*, and there are two problems that may arise if the order of limits is changed. First, since the mass parameter m can *also* serve as a regulator of collinear and soft singularities, $m\to 0$ and $\nu\to 0$ limits should not necessarily commute. Second, additional contributions can mix into the Taylor m^0 branch of the solution that we require, if the $\nu\to 0$ limit is taken before the $m\to 0$ limit. We will discuss these two problems now.

Suppose we take the $\nu\to 0$ limit for finite m integrals, but the resulting integrals are still not regulated dimensionally at $m=0$. This feature can be detected in the following way. The dependence of the integral on m and ϵ at small m has the following form:

$$\mathcal{J}\sim\sum_{n_1,n_2,n_3} c_{n_1n_2n_3} m^{n_1+n_2\epsilon}\ln^{n_3}(m). \tag{16}$$

We are interested in taking the $m\to 0$ limit of this solution at fixed ϵ. However, this is only possible if the coefficients $c_{n_10n_3}$ with $n_1<0$ and $n_1=0$, $n_3>0$ vanish so that there are no $1/m$ and $\log m$ terms that are not multiplied by additional powers of m^ϵ or sufficiently high powers of m. We have checked that this condition is satisfied for all $\nu=0$ integrals that we considered; this implies that the $m\to 0$ limit does not lead to divergencies in integrals that are not regulated dimensionally. It follows that indeed $\bar{I}_{1,2}^\nu$ are the only integrals in $\bar{I}_\alpha^\nu$ that contribute to the amplitude, all other integrals can be safely discarded.

The second problem concerns the possible mixing between different branches of integrals if the $\nu\to 0$ limit is taken too early. To see how this comes about, consider a general solution in the limit $m\to 0$, $\nu\to 0$, $\epsilon\to 0$,

$$\mathcal{J}\sim\sum_{n_1,n_2,n_3,n_4} c_{n_1n_2n_3n_4} m^{n_1+n_2\epsilon+n_4\nu}\ln^{n_3}(m). \tag{17}$$

If there are terms that correspond to $n_1=0$, $n_2=0$, $n_3=0$, $n_4\neq 0$, they will mix with the contribution $n_1=0$, $n_2=0$, $n_3=0$, $n_4=0$, i.e., the m^0 branch that we are interested in.

Hence, we need to understand if such contributions exist and, if they are there, then how to isolate and remove them. This can be done by studying the exact differential equation, Eq. (17), at small values of m but with full ν and ϵ dependence, and checking if $m^{n_4\nu}$ solutions without additional dependencies of exponents on ϵ and additional powers of m are possible. We find that this does not happen at $m=0$ and, therefore, the Taylor branch does not receive any unwanted contributions.

In summary, we can solve the differential equations, Eq. (17), as an expansion in ν. As discussed in Ref. [7], we can compute the boundary conditions at $m=\infty$ and then find values of integrals $\mathcal{J}$ at $m=0$ by discarding all terms that have nonanalytic dependencies on m at $m=0$. We present the results of such a computation in Sec. V; in the next section we describe computation of a peculiar boundary condition which illustrates that our worry about potential mixing of a Taylor and $m^{n_4\nu}$ branches is not unfounded.

III. INTEGRAL $\bar{J}_2^\nu$ WITH $1/\nu$ DIVERGENCE

We can illustrate some points discussed in the previous section by considering the integral $\bar{J}_2^\nu$ and its contribution to differential equations. As mentioned there, this integral is singular in the limit $\nu\to 0$. Multiplied by a factor ν, it appears in the differential equations for two sets of ν-regular integrals. We will consider one of them for the sake of example. The integrals in this sector read,

$$J_{a_1}^\nu = \int\frac{\mathrm{d}\Phi_{\theta\delta\theta}^\nu}{(k_{123}^2+m^2)(k_1\cdot k_3)(k_{23}\cdot\bar{n})},$$
$$J_{a_2}^\nu = \int\frac{\mathrm{d}\Phi_{\theta\delta\theta}^\nu}{(k_{123}^2+m^2)(k_1\cdot k_3)(k_{23}\cdot\bar{n})^2}. \tag{18}$$

The differential equations take the following form to order $\mathcal{O}(\nu^0)$:

$$\frac{\partial}{\partial m^2}\begin{pmatrix}J_{a_1}^\nu\\ J_{a_2}^\nu\end{pmatrix} = \begin{pmatrix}-\frac{2\epsilon}{m^2}-\frac{2(1+2\epsilon)}{4m^2+1} & \frac{1}{4m^2+1}\\ \frac{2\epsilon(1+2\epsilon)}{m^2}-\frac{8\epsilon(1+2\epsilon)}{4m^2+1} & \frac{1+4\epsilon}{m^2}-\frac{28\epsilon}{4m^2+1}\end{pmatrix}\begin{pmatrix}J_{a_1}^\nu\\ J_{a_2}^\nu\end{pmatrix}$$
$$-\begin{pmatrix}\frac{(1+4\epsilon)}{(1+6\epsilon)}\left(\frac{1}{m^2}-\frac{4}{4m^2+1}\right)\\ \frac{(1+4\epsilon)}{(1+6\epsilon)}\left(\frac{4\epsilon}{m^2}-\frac{16\epsilon}{4m^2+1}\right)\end{pmatrix}\nu\bar{J}_2^\nu$$
$$+\text{contributions from other } J^\nu \text{ integrals}. \tag{19}$$

It follows from the above equation that the integral $\nu\bar{J}_2^\nu$ plays a role of an inhomogeneous contribution to the differential equation that $J_{a_1}^\nu$ and $J_{a_2}^\nu$ satisfy. In fact, analyzing the homogeneous terms of the above equation in the $m\to\infty$ limit, we find that we do not need to compute the boundary conditions for these integrals and that the solution of the differential equation in this limit is obtained by integrating the inhomogeneous part. We remind the reader that $\nu\bar{J}_2^\nu$ is independent of m to order $\mathcal{O}(\nu^0)$, which allows us to write the result of the integration directly,

$$J_{a_1}^\nu = m^{-2}\left(\frac{4\epsilon+1}{4(6\epsilon+1)}\nu\bar{J}_2^\nu\right)+\dots, \tag{20}$$

$$J_{a_2}^\nu = m^{-2}\left(\frac{\epsilon(4\epsilon+1)}{6\epsilon+1}\nu\bar{J}_2^\nu\right)+\dots. \tag{21}$$

In Eqs. (20) and (21) dots stand for other contributions, including homogeneous and inhomogeneous ones.

To calculate $\lim_{\nu\to 0}\nu\bar{J}_2^\nu$ at $m=\infty$, we inspect the various contributions to the asymptotic behavior of the integral $\bar{J}_2^\nu$, discussed in Ref. [7], and find that they do not produce terms that are $1/\nu$ divergent. It turns out that the integral $\bar{J}_2^\nu$ provides an example of a situation where the analysis of different regions that contribute to $m\to\infty$ asymptotic behavior of integrals, performed in Ref. [7], is *incomplete* and that there is another region that needs to be considered. In fact, we have found that the following scaling of integration variables,

$$\begin{aligned} k_3\cdot\bar{n} &= \alpha_3 \sim m^2 \gg 1,\\ k_1\cdot\bar{n} &= \alpha_1 \sim 1,\\ k_1\cdot n &= \beta_1 \sim m^{-2} \ll 1, \end{aligned} \tag{22}$$

leads to a $1/\nu$-divergent $\mathcal{O}(1/m^0)$ contribution to $\bar{J}_2^\nu$.

To show this, we write an approximation to the integrand of $\bar{J}_2^\nu$ integral in the region defined by Eq. (22),

$$\bar{J}_2^\nu \sim \widetilde{J}_2^\nu = \int \frac{\mathrm{d}\Phi_{\theta\delta\theta}^\nu}{((k_3\cdot\bar{n})(k_2\cdot n)+m^2)(k_1\cdot k_3)(k_1\cdot n)(k_{12}\cdot\bar{n})}. \tag{23}$$

We use the Sudakov variables α_i and β_i (see Ref. [7] for details) and take into account the condition $\beta_1 \ll 1$ to remove β_1 from the "jettiness" delta-function $\delta(1-\beta_1-\beta_2-\beta_3)\to\delta(1-\beta_2-\beta_3)$. We then extend the integration over β_1 to infinity. We find

$$\begin{aligned} \widetilde{J}_2^\nu = {}& 2\int_0^\infty \mathrm{d}\beta_1 \int \mathrm{d}\beta_2\mathrm{d}\beta_3\mathrm{d}\alpha_1\mathrm{d}\alpha_3 \beta_1^{-\epsilon+\nu}\beta_2^{-2\epsilon+\nu}\beta_3^{-\epsilon+\nu}\alpha_1^{-\epsilon}\alpha_3^{-\epsilon}\frac{\delta(1-\beta_{23})\theta(\alpha_1-\beta_1)\theta(\alpha_3-\beta_3)}{(\alpha_3\beta_2+m^2)\beta_1(\alpha_1+\beta_2)}\\ &\times\left[\frac{\theta(\beta_1/\alpha_1-\beta_3/\alpha_3)}{\beta_1\alpha_3}{}_2F_1\left(1,1+\epsilon;1-\epsilon;\frac{\alpha_1\beta_3}{\alpha_3\beta_1}\right)+\frac{\theta(\beta_3/\alpha_3-\beta_1/\alpha_1)}{\beta_3\alpha_1}{}_2F_1\left(1,1+\epsilon;1-\epsilon;\frac{\alpha_3\beta_1}{\alpha_1\beta_3}\right)\right]. \end{aligned} \tag{24}$$

We change integration variables $\alpha_1=\beta_1/\xi_1$ and $\alpha_3=\beta_3/\xi_3$ and obtain

$$\begin{aligned} \widetilde{J}_2^\nu = {}& 2\int \mathrm{d}\beta_2\mathrm{d}\beta_3\mathrm{d}\xi_1\mathrm{d}\xi_3\beta_2^{-2\epsilon+\nu}\beta_3^{-2\epsilon+\nu}\xi_1^{\epsilon-1}\xi_3^{\epsilon-1}\frac{\delta(1-\beta_{23})\theta(1-\xi_1)\theta(1-\xi_3)}{\beta_3\beta_2+m^2\xi_3}\\ &\times\left(\xi_3\theta(\xi_1-\xi_3){}_2F_1\left[\{1,1+\epsilon\},\{1-\epsilon\};\frac{\xi_3}{\xi_1}\right]+\xi_1\theta(\xi_3-\xi_1){}_2F_1\left[\{1,1+\epsilon\},\{1-\epsilon\};\frac{\xi_1}{\xi_3}\right]\right)\\ &\times\int_0^\infty \mathrm{d}\beta_1\frac{\beta_1^{-2\epsilon+\nu-1}}{\beta_1+\beta_2\xi_1}. \end{aligned} \tag{25}$$

Integrating over β_1, we find

$$\int_0^\infty \mathrm{d}\beta_1\frac{\beta_1^{-2\epsilon+\nu-1}}{\beta_1+\beta_2\xi_1} = (\beta_2\xi_1)^{-2\epsilon+\nu-1}\Gamma(-2\epsilon+\nu)\Gamma(2\epsilon-\nu+1). \tag{26}$$

We use this result in Eq. (25), change variables $\xi_1=r\xi_3$, and arrive at

$$\widetilde{J}_2^\nu = 2\Gamma(-2\epsilon+\nu)\Gamma(2\epsilon-\nu+1)\int \mathrm{d}\beta_2\mathrm{d}\beta_3\beta_2^{-4\epsilon+2\nu-1}\beta_3^{-2\epsilon+\nu}\delta(1-\beta_{23})\int_0^1\frac{\mathrm{d}\xi_3\xi_3^{\nu-1}J(\nu,\xi_3)}{\beta_3\beta_2+m^2\xi_3}, \tag{27}$$

where

$$J(\nu,\xi_3) = \int_0^1 \mathrm{d}r[\theta(1-\xi_3/r)r^{\epsilon-\nu} + r^{-\epsilon+\nu-1}]_2F_1[\{1,1+\epsilon\},\{1-\epsilon\};r]. \tag{28}$$

It follows from the above expression that the ν pole originates from a singularity at $\xi_3 = 0$. It is straightforward to compute it since $J(\nu,\xi_3)$ is regular at $\xi_3 = 0$.[4] We find

$$\int_0^1 \mathrm{d}\xi_3 \frac{\xi_3^{\nu-1} J(\nu,\xi_3)}{\beta_3\beta_2 + m^2\xi_3} = \frac{J(0,0)}{\nu}(\beta_3\beta_2)^{-1} + \mathcal{O}(\nu^0). \tag{29}$$

Upon further integration, we obtain a $1/\nu$-divergent contribution to $\tilde{J}_2^\nu$. It reads,

$$\tilde{J}_2^\nu = \frac{C_2}{\nu} + \mathcal{O}(\nu^0), \tag{30}$$

where

$$C_2 = \frac{2\Gamma^2(-2\epsilon)\Gamma(-4\epsilon-1)\Gamma(2\epsilon+1)}{\Gamma(-6\epsilon-1)}\left(\frac{{}_3F_2[\{1,1+\epsilon,1+\epsilon\},\{1-\epsilon,2+\epsilon\};1]}{1+\epsilon} - \frac{{}_3F_2[\{1,-\epsilon,1+\epsilon\},\{1-\epsilon,1-\epsilon\};1]}{\epsilon}\right). \tag{31}$$

Using the result of the explicit computation of the $1/\nu$ pole of $\bar{J}_2^\nu$ and the result for $\bar{J}_1^\nu$ reported in Ref. [7], we find that they satisfy the relation shown in Eq. (15). As we pointed out earlier, this relation is needed to ensure the smooth $\nu \to 0$ limit of the differential equations.

The above result provides the required boundary condition at $m = \infty$ and allows us to start solving differential equations numerically. However, it is interesting to point out that, from the perspective of the differential equations, integral $\bar{J}_2^\nu$ provides an example of a contribution proportional to $m^{-2\nu}$ which, therefore, can mix with the Taylor branch of the required integrals if the $\nu \to 0$ limit is taken first.

Indeed, if we first apply the scaling defined in Eq. (22) to the integration variables in Eq. (24),

$$\alpha_3 = m^2\alpha_3', \qquad \beta_1 = m^{-2}\beta_1', \tag{32}$$

and perform the integrations over α_3' and β_1' in the same way as before, then we find that this integration region leads to an overall factor $(1/m)^{2\nu}$,

$$\nu\bar{J}_2^\nu \sim (1/m)^{2\nu}[C_2 + \mathcal{O}(\nu^1)]. \tag{33}$$

This result is identical to Eq. (30) through the leading order in ν. We also note that since in the limit $\nu \to 0$, $(1/m)^{2\nu}$ becomes 1; this region looks like a "normal" Taylor-expansion region,

$$\nu\bar{J}_2^\nu \sim C_2 + \mathcal{O}(\nu^1), \tag{34}$$

and, once the $\nu \to 0$ limit is taken, the two regions cannot be distinguished. This is an illustration of the problem of mixing between different m branches that we discussed in the previous section; however, in contrast to the discussion there, our example in this section refers to $m \to \infty$ limit.

We emphasize one more time that from the point of view of the differential equation, the contribution shown in Eqs. (20) and (21) comes from the $(1/m)^{2\nu}$ branch, but once the $\nu \to 0$ limit is taken it becomes indistinguishable from a regular Taylor part of the integral. If a similar situation had occurred at $m = 0$, then we should have identified and removed all the $m^{n_4\nu}$ regions in all integrals since the correct sequence of limits that is needed is $m \to 0, \nu \to 0, \epsilon \to 0$. However, as we mentioned earlier, an analysis of the differential equation at $m \to 0$ leads to the conclusion that there are no eigenvalues that vanish, if the $\nu \to 0$ limit is taken, so that the problem described above does not occur.

IV. CHECKS

Given the highly unusual nature of the integrals that need to be computed to obtain the zero-jettiness soft function and the complex interplay of the various infrared regulators, it is important to perform as many checks as possible to ensure correctness of the result.

The most comprehensive check that can be performed is the numerical computation of all m-dependent integrals, which appear in the differential equations, as well as their derivatives. We constructed Mellin-Barnes representation

[4]As with any analytic regulator, the $\nu \to 0$ limit should be computed keeping ϵ fixed.

of the relevant integrals using public programs MB [15] and MBresolve [16] for this purpose. We also used the program pySecDec [17,18] as an alternative for the numerical computation. Using these programs, we have calculated all integrals that appeared in the differential equations at *finite* values of m and checked them against numerical solutions of the differential equations. We found good agreement between these two results up to the default precision of the numerical programs that, in practice, can vary between 3 and 10 digits.

Next, we compared the solutions of the differential equations at $m = 0$ and $\nu = 0$ with the results of the direct numerical computation. Unfortunately, although this can be done for some integrals that contribute to S_d, there are many integrals for which the numerical integration becomes next to impossible. To enlarge the set of integrals at $m = 0$ that can be checked, we have derived linear relations between various integrals at $m = 0$, using the integration-by-parts identities, and checked that integrals obtained from m-dependent differential equations and extrapolated to $m = 0$ satisfy them. We found that all the master integrals which appear in S_d fulfill the $m = 0$ IBP relations to the full precision.

V. RESULTS

Solving differential equations and separating the Taylor branch at $m = 0$, we obtain the numerical result for the integral of the function $\omega_{n\bar{n}}^{3,(d)}$. Since we can determine the solution of the differential equation to, essentially, arbitrary precision, we can try to obtain the analytic result for S_d by fitting the numerical results to a linear combination of various transcendental and rational numbers. By making use of the PSLQ [19] and LLL [20] algorithms, and choosing the appropriate basis of transcendental numbers [21], we find the following result:

$$
\begin{aligned}
S_d = \int \mathrm{d}\Phi_{\theta\theta\theta}^{nnn}\omega_{n\bar{n}}^{(3),d}(k_1,k_2,k_3) &= \frac{12}{\epsilon^5} + \frac{142}{3\epsilon^4} + \frac{1}{\epsilon^3}\left(\frac{46\pi^2}{3} + \frac{628}{3}\right) + \frac{1}{\epsilon^2}\left(196\zeta_3 + \frac{650\pi^2}{9} + \frac{18161}{27}\right) \\
&+ \frac{1}{\epsilon}\left(\frac{397\pi^4}{45} + 1380\zeta_3 + \frac{6808\pi^2}{27} + \frac{165323}{81}\right) + \left(8982\zeta_5 - \frac{2146\zeta_3\pi^2}{3} + \frac{191\pi^4}{9} + 4224\mathrm{Li}_4\left(\frac{1}{2}\right)\right. \\
&\left. + 3696\zeta_3\ln(2) - 176\pi^2\ln^2(2) + 176\ln^4(2) + \frac{46184\zeta_3}{9} + \frac{66614\pi^2}{81} + 96\ln(2) + \frac{413971}{81}\right) \\
&+ \epsilon\left(2304\zeta_{-5,-1} - 4464\zeta_5\ln(2) - 8380\zeta_3^2 + \frac{46934\pi^6}{2835} - 6336G_R(0,0,r_2,1,-1)\right. \\
&- 6336G_R(0,0,1,r_2,-1) - 3168G_R(0,0,1,r_2,r_4) - 6336G_R(0,0,r_2,-1)\ln(2) + \frac{324215\zeta_5}{3} \\
&- 45056\mathrm{Li}_5\left(\frac{1}{2}\right) - 45056\mathrm{Li}_4\left(\frac{1}{2}\right)\ln(2) + 176\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\pi - 1056\zeta_3\mathrm{Li}_2\left(\frac{1}{4}\right) - \frac{9634\zeta_3\pi^2}{3} \\
&- 21824\zeta_3\ln^2(2) + 2112\zeta_3\ln(2)\ln(3) - 1584\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\ln(3) - \frac{4400\mathrm{Cl}_2(\frac{\pi}{3})\pi^3}{27} + \frac{88\pi^4\ln(2)}{45} \\
&- \frac{616\pi^4\ln(3)}{27} + \frac{11264\pi^2\ln^3(2)}{9} - \frac{22528\ln^5(2)}{15} + 8576\mathrm{Li}_4\left(\frac{1}{2}\right) + 7504\zeta_3\ln(2) + \frac{4646\pi^4}{27} \\
&- \frac{1072\pi^2\ln^2(2)}{3} + \frac{1072\ln^4(2)}{3} + \frac{496592\zeta_3}{27} - 32\pi^2\ln(2) + \frac{587380\pi^2}{243} - 384\ln^2(2) + 832\ln(2) \\
&\left. + \frac{7857076}{729} + \sqrt{3}\left(192\Im\left\{\mathrm{Li}_3\left(\frac{\exp(\mathrm{i}\pi/3)}{2}\right)\right\} + 160\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln(2) - 16\pi\ln^2(2) - \frac{560\pi^3}{81}\right)\right) + \mathcal{O}(\epsilon^2), \quad (35)
\end{aligned}
$$

where $\zeta_{-5,-1} \approx -0.029902$ is a multiple zeta value, and $\mathrm{Cl}_n(x)$ are Clausen functions. Note that $G_R(a_1, \ldots, a_w)$ is the real part of the multiple polylogarithm $G(a_1, \ldots, a_w; z)$ evaluated at $z = 1$ [21],

$$
G_R(a_1, \ldots, a_w) = \Re\{G(a_1, \ldots, a_w; 1)\}. \quad (36)
$$

Finally, $r_2 = \exp(-\mathrm{i}\pi/3)$ and $r_4 = \exp(-\mathrm{i}2\pi/3)$. We note that we have computed the master integrals to more than two thousand digits to check the validity of the analytic result.

Having obtained the result for the integral of the function $\omega_{n\bar{n}}^{3,(d)}$, we are now in position to present the complete result for the same-hemisphere *three-gluon*-emission contribution to the N3LO soft function. To this end, we write

$$S^{nnn} = \int \mathrm{d}\Phi^{nnn}_{\theta\theta\theta} |\boldsymbol{J}(k_1,k_2,k_3)|^2 = \tau^{-1-6\epsilon} \frac{N_\epsilon^3}{3!} [C_a^3 S^{nnn}_{1+1+1} + C_a^2 C_A S^{nnn}_{1+2} + C_a C_A^2 S^{nnn}_3], \tag{37}$$

where we reintroduced the dependence on τ, recovered the symmetry factor $1/3!$ and the normalization factor N_ϵ^3 [cf. Eq. (4)], and split the integral into three color factors following Eq. (7.10) in Ref. [10]. We also note that $C_a = C_{F,A}$ for the quark (gluon) soft function, respectively.

The computation of the Abelian contributions S^{nnn}_{1+1+1} and S^{nnn}_{1+2} is described in the Appendix. We obtain the maximally non-Abelian contribution S^{nnn}_3 by adding results obtained in Ref. [7], and the result of this paper is given in Eq. (35). We find

$$S^{nnn}_{1+1+1} = \frac{48\Gamma^3(1-2\epsilon)}{\epsilon^5 \Gamma(1-6\epsilon)}, \tag{38}$$

$$\begin{aligned} S^{nnn}_{1+2} = &-\frac{9\Gamma(1-4\epsilon)\Gamma(1-2\epsilon)}{\epsilon^2\Gamma(1-6\epsilon)} \times \left[\frac{8}{\epsilon^3} + \frac{44}{3\epsilon^2} + \frac{1}{\epsilon}\left(\frac{268}{9} - 8\zeta_2\right) + \left(\frac{1544}{27} + \frac{88}{3}\zeta_2 - 72\zeta_3\right)\right. \\ &+ \epsilon\left(\frac{9568}{81} + \frac{536\zeta_2}{9} + \frac{352}{3}\zeta_3 - 300\zeta_4\right) + \epsilon^2\left(\frac{55424}{243} + \frac{3520\zeta_2}{27} + \frac{2144\zeta_3}{9} + 352\zeta_4 + 96\zeta_2\zeta_3 - 1208\zeta_5\right) \\ &\left.+ \epsilon^3\left(\frac{297472}{729} + \frac{22592\zeta_2}{81} + \frac{14080\zeta_3}{27} + \frac{2144}{3}\zeta_4 - \frac{4576}{3}\zeta_2\zeta_3 + 3696\zeta_5 + 424\zeta_3^2 - 3596\zeta_6\right) + \mathcal{O}(\epsilon^4)\right], \end{aligned} \tag{39}$$

$$\begin{aligned} S^{nnn}_3 = &\frac{24}{\epsilon^5} + \frac{308}{3\epsilon^4} + \frac{1}{\epsilon^3}\left(-12\pi^2 + \frac{3380}{9}\right) + \frac{1}{\epsilon^2}\left(-1000\zeta_3 + \frac{440\pi^2}{9} + \frac{10048}{9}\right) \\ &+ \frac{1}{\epsilon}\left(-\frac{2377\pi^4}{45} + \frac{440\zeta_3}{3} + \frac{7192\pi^2}{27} + \frac{253252}{81}\right) \\ &+ \left(-28064\zeta_5 + \frac{1972\zeta_3\pi^2}{3} - \frac{638\pi^4}{15} + 4224\mathrm{Li}_4\left(\frac{1}{2}\right) + 3696\zeta_3\ln(2) - 176\pi^2\ln^2(2) + 176\ln^4(2)\right. \\ &\left.+ \frac{13208\zeta_3}{3} + \frac{78848\pi^2}{81} + 96\ln(2) + \frac{1925074}{243}\right) \\ &+ \epsilon\left(2304\zeta_{-5,-1} - 4464\zeta_5\ln(2) + 25784\zeta_3^2 - \frac{67351\pi^6}{567} - 6336G_R(0,0,r_2,1,-1)\right. \\ &- 6336G_R(0,0,1,r_2,-1) - 3168G_R(0,0,1,r_2,r_4) - 6336G_R(0,0,r_2,-1)\ln(2) + \frac{268895\zeta_5}{3} \\ &- 45056\mathrm{Li}_5\left(\frac{1}{2}\right) - 45056\mathrm{Li}_4\left(\frac{1}{2}\right)\ln(2) + 176\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\pi - 1056\zeta_3\mathrm{Li}_2\left(\frac{1}{4}\right) - 3982\zeta_3\pi^2 \\ &- 21824\zeta_3\ln^2(2) + 2112\zeta_3\ln(2)\ln(3) - 1584\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\ln(3) - \frac{4400\mathrm{Cl}_2(\frac{\pi}{3})\pi^3}{27} + \frac{88\pi^4\ln(2)}{45} \\ &- \frac{616\pi^4\ln(3)}{27} + \frac{11264\pi^2\ln^3(2)}{9} - \frac{22528\ln^5(2)}{15} + 8576\mathrm{Li}_4\left(\frac{1}{2}\right) + 7504\zeta_3\ln(2) + \frac{4174\pi^4}{27} \\ &- \frac{1072\pi^2\ln^2(2)}{3} + \frac{1072\ln^4(2)}{3} + \frac{554032\zeta_3}{27} - 32\pi^2\ln(2) + \frac{730378\pi^2}{243} - 384\ln^2(2) + 832\ln(2) \\ &\left.+ \frac{1408681}{81} + \sqrt{3}\left(192\Im\left\{\mathrm{Li}_3\left(\frac{\exp(i\pi/3)}{2}\right)\right\} + 160\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln(2) - 16\pi\ln^2(2) - \frac{560\pi^3}{81}\right)\right) + \mathcal{O}(\epsilon^2). \end{aligned} \tag{40}$$

VI. CONCLUSIONS

In this paper, we have discussed the computation of the same-hemisphere three-gluon-emission contribution to the zero-jettiness soft function at N3LO in perturbative QCD. We have used the approach of Ref. [7], which allows us to apply integration-by-parts technology and the method of differential equations to phase-space integrals that contain Heaviside functions. While the appearance of integrals that are not regulated dimensionally requires an analytic regulator, and thus complicates the use of differential equations, we have described a way to bypass this problem in an efficient way.

Finally, we note that the missing kinematic configuration, in which one of the three gluons is emitted into the opposite hemisphere can be computed in a similar fashion. Once a complete result for the three-gluon-emission contribution is known, the contribution that arises from the emission of a soft $q\bar{q}$ pair and a soft gluon can be computed in a straightforward way. Similarly, we expect that virtual corrections to the double real emissions can be dealt with using the same method. We leave both problems to future investigations.

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APPENDIX: COMPUTATION OF ABELIAN CONTRIBUTIONS

In this Appendix, we describe how to compute Abelian contributions to the zero-jettiness soft function, i.e., the first two terms in Eq. (37). The first, so-called fully Abelian contribution S^{nnn}_{1+1+1} reads,

$$S^{nnn}_{1+1+1} = \int \mathrm{d}\Phi^{nnn}_{\theta\theta\theta}\,\omega^{(1)}_{n\bar{n}}(k_1)\,\omega^{(1)}_{n\bar{n}}(k_2)\,\omega^{(1)}_{n\bar{n}}(k_3), \tag{A1}$$

where [10]

$$\omega^{(1)}_{n\bar{n}}(q) = \frac{4}{(nq)(\bar{n}q)}. \tag{A2}$$

Thanks to its fully factorized structure, the integral in Eq. (A1) is straightforward to compute. Using Sudakov variables α_i and β_i, we find

$$S^{nnn}_{1+1+1} = 64\int_0^\infty \left(\prod_{i=1}^{3} \mathrm{d}\alpha_i \mathrm{d}\beta_i (\alpha_i\beta_i)^{-1-\epsilon}\theta(\alpha_i-\beta_i)\right) \times \delta(1-\beta_{123}) = \frac{64}{\epsilon^3}\frac{\Gamma^3(-2\epsilon)}{\Gamma(-6\epsilon)}. \tag{A3}$$

Making the $1/\epsilon$ poles in the above equation explicit, yields Eq. (38).

The second Abelian contribution reads,

$$\begin{aligned} S^{nnn}_{1+2} &= \int \mathrm{d}\Phi^{nnn}_{\theta\theta\theta}[\omega^{(1)}_{n\bar{n}}(k_1)\omega^{(2)}_{n\bar{n}}(k_2,k_3) \\ &\quad + (k_1 \leftrightarrow k_2) + (k_1 \leftrightarrow k_3)] \\ &= 3\int \mathrm{d}\Phi^{nnn}_{\theta\theta\theta}[\omega^{(1)}_{n\bar{n}}(k_1)\omega^{(2)}_{n\bar{n}}(k_2,k_3)]. \end{aligned} \tag{A4}$$

We write it as

$$\begin{aligned} S^{nnn}_{1+2} &= 3N_\epsilon^{-1}\int [\mathrm{d}k_1]\theta(k_1\bar{n}-k_1 n)\omega^{(1)}_{n\bar{n}}(k_1) \\ &\quad\times N_\epsilon^{-2}\int\left(\prod_{i=2}^{3}[\mathrm{d}k_i]\theta(k_i\bar{n}-k_i n)\right) \\ &\quad\times \delta(1-k_{123}n)\omega^{(2)}_{n\bar{n}}(k_2,k_3). \end{aligned} \tag{A5}$$

The inner integral in Eq. (A5) over $[\mathrm{d}k_2][\mathrm{d}k_3]$ can be obtained from the same-hemisphere double-real gluon emission contribution to the NNLO soft function. We find

$$\begin{aligned} S^{nnn}_{1+2} &= 3\int_0^\infty \mathrm{d}\alpha_1 \mathrm{d}\beta_1 \theta(\alpha_1-\beta_1)\frac{4(\alpha_1\beta_1)^{-\epsilon}}{\alpha_1\beta_1} \\ &\quad\times C_2^{nn}(1-\beta_1)^{-1-4\epsilon} = \frac{12}{\epsilon}\frac{\Gamma(-4\epsilon)\Gamma(-2\epsilon)}{\Gamma(-6\epsilon)}C_2^{nn}, \end{aligned} \tag{A6}$$

where the factor C_2^{nn} can be extracted from Refs. [7,22]. Upon doing so, we obtain the result displayed in Eq. (39).

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