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Generalized Surgery on Riemannian Manifolds of Positive Ricci Curvature

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Philipp Reiser

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1. Referent:
Prof. Dr. Wilderich Tuschmann

2. Referent:
Prof. Dr. Enrico Leuzinger

3. Referent:
PD Dr. Fernando Galaz-García

Abstract

The surgery theorem of Wraith states that the existence of metrics of positive Ricci curvature is preserved under surgery if certain metric and dimensional conditions are satisfied. We generalize this theorem by relaxing the conditions on the dimensions involved and by generalizing the surgery construction itself. As applications we construct metrics of positive Ricci curvature on manifolds obtained by plumbing. Specifically, this construction provides an extension of a result of Burdick on the existence of metrics of positive Ricci curvature on connected sums of linear sphere bundles, and, moreover, it yields infinite families of new examples of manifolds with a metric of positive Ricci curvature in all dimensions divisible by 6.

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Contents

Introduction 1

Riemannian geometry studies connections between curvature and the shape of objects. In this context, the Ricci curvature appears naturally in many settings. For example, it is part of equations with far reaching geometric and topological consequences, like the Bochner formula for the Hodge Laplacian and the second variation formula of energy along hypersurfaces. Furthermore, the Ricci curvature is part of the Ricci flow equation, which, by the work of Hamilton and Perelman, has made substantial contributions to 3-dimensional geometry and topology. In physics, the Ricci curvature is of interest in general relativity as, for example, it is contained in the Einstein field equations.

Lower bounds on the Ricci curvature directly affect geometric quantities, like distance and volume, as well as topological quantities, like Betti numbers and the fundamental group. Therefore, the question, which manifolds admit a complete Riemannian metric with a given lower Ricci curvature bound, is of great interest. In this thesis, we are interested in the condition of positive Ricci curvature.

In dimension 2, where Ricci curvature coincides with the classical notion of Gaussian curvature, it follows from the Gauss-Bonnet formula that the only closed surfaces with positive Ricci curvature are the 2-dimensional sphere and the real projective plane. A similar result holds in dimension 3, where, as a consequence of the aforementioned Ricci flow techniques, a closed manifold with a Riemannian metric of positive Ricci curvature is finitely covered by the 3-dimensional sphere.

In higher dimensions, however, the situation is much less clear. Here the condition of positive Ricci curvature lies between the rather flexible and comparably well-understood condition of positive scalar curvature and the very restrictive condition of positive sectional curvature. By the theorem of Bonnet-Myers, a closed manifold that admits a metric of positive Ricci curvature has finite fundamental group. This result, together with obstructions to the existence of metrics of positive scalar curvature, is the only known topological obstruction for a manifold to admit a Riemannian metric of positive Ricci curvature. Thus, Riemannian metrics of positive Ricci curvature could potentially exist on a wide range of manifolds. However, when compared with the case of positive scalar curvature, examples of manifolds with a Riemannian metric of positive Ricci curvature are rare.

The majority of known methods to construct Riemannian metrics of positive Ricci curvature rely on the existence of certain additional structures, such as bundles structures, group actions, complex structures or Sasakian structures. An overview of these constructions is given in Section 3.1. While these methods produce many interesting examples of manifolds with a Riemannian metric of positive Ricci curvature, the necessity of the existence of these additional structures heavily restricts the range of applications.

Another approach to construct Riemannian metrics of positive Ricci curvature is surgery. Surgery, which will be introduced in Section 4.1, is a procedure to modify manifolds by cutting out and gluing in certain predefined parts. There exist different types of surgery, which are distinguished by a natural number, called the codimension of the surgery. Applying surgery to questions about curvature has been proven to be very successful in the case of positive scalar curvature, since it was shown independently by Gromov–Lawson [50] and Schoen–Yau [101], that the existence of Riemannian metrics of positive scalar curvature is preserved under any surgery of codimension at least 3. Since this result does not impose any assumptions on the Riemannian metric involved (except of having positive scalar curvature), it has far reaching consequences and establishes the existence of Riemannian metrics of positive scalar curvature on a wide range of manifolds, including all closed, simply-connected manifolds in dimensions 5, 6 and 7, see Section 3.4.

For positive Ricci curvature it is open whether a surgery theorem in the same generality holds. However, if one imposes additional assumptions on the Riemannian metrics involved, there exist surgery results for positive Ricci curvature, which can be divided into two categories: results on connected sums, which is surgery in codimension n (where n denotes the dimension of the manifold), by Perelman [90] and Burdick [21, 22, 23, 24], and results on higher surgeries by Sha–Yang [107] and Wraith [121, 122].

In the first case, to construct Riemannian metrics of positive Ricci curvature, Burdick introduced the notion of *core metrics*. These are Riemannian metrics of positive Ricci curvature, which, roughly speaking, contain a large round disc, see Section 5.2. Based on Perelman's work he then showed the following result.

Theorem 1.1 ([22, Theorem B], Proposition 5.2.6). *Let $M_1^n, \dots, M_k^n$ be manifolds that admit core metrics. If $n \geq 4$, then the connected sum $M_1 \# \dots \# M_k$ admits a Riemannian metric of positive Ricci curvature.*

Thus, the question for which manifolds their connected sum admits a Riemannian metric of positive Ricci curvature reduces to the question of which manifolds admit a core metric. Examples of manifolds with core metrics are the standard sphere, complex projective spaces and total spaces of certain linear sphere bundles, see Proposition 5.2.7.

For higher surgeries, Sha and Yang proved a surgery theorem for positive Ricci curvature, which was later extended by Wraith. For that, suppose we have the following:

$(S1)$ A Riemannian manifold (M^{p+q-1}, g_M) of positive Ricci curvature.

$(S2)$ An isometric embedding $\iota\colon S^{p-1}(\rho) \times D_R^q(N) \hookrightarrow M$, where $S^{p-1}(\rho)$ denotes the round $(p-1)$-sphere of radius $\rho > 0$ and $D_R^q(N)$ denotes the ball of radius $R > 0$ in $S^q(N)$.

$(S3)$ A smooth map $T\colon S^{p-1} \to SO(q)$, which induces a diffeomorphism $\tilde{T}\colon S^{p-1} \times S^{q-1} \to S^{p-1} \times S^{q-1}$ defined by $(x, y) \mapsto (x, T_x(y))$.

Theorem 1.2 ([122, Theorem 0.3], Theorem 5.5.2). *Under assumptions (S1)-(S3) let $p \geq q \geq 3$. Then there is a constant $\kappa = \kappa(p, q, R/N, T) > 0$, such that if $\frac{\rho}{N} < \kappa$, then the manifold*

$$\hat{M} = M \setminus \mathrm{im}(\iota)^\circ \cup_{\tilde{T}} (D^p \times S^{q-1})$$

admits a metric of positive Ricci curvature.

The condition $\frac{\rho}{N} < \kappa$ can be interpreted as requiring the disc D^q to be large compared to the sphere S^{p-1}. As a consequence, since it is not clear in a general setting whether for a given embedded sphere there exists a neighborhood of this form, one cannot apply this theorem without having some knowledge on the global structure of the manifold. The assumptions are for example satisfied for total spaces of linear sphere bundles, or, more generally, manifolds obtained by plumbing, which is a procedure that glues disc bundles to each other in a certain way, see Section 4.2.

Theorem 1.3 ([121, Theorems 2.2 and 2.3], Theorem 5.5.3). *Let W be the manifold obtained by plumbing linear n-disc bundles over n-spheres according to a simply-connected graph or by plumbing together two disc bundles over spheres (where fiber and base dimension may differ). If the fiber and base dimensions are at least 3, then ∂W admits a metric of positive Ricci curvature.*

The class of manifolds obtained as boundaries of plumbings as in this theorem contains many interesting examples, including all homotopy spheres that bound parallelizable manifolds and many highly-connected manifolds, see Theorem 3.1.12.

We generalize these theorems as follows:

Theorem A ([97, Theorem A]). *Under the assumptions (S1) and (S2) let B^p be a manifold with a core metric g_B, let $E \xrightarrow{\pi} B$ be a linear S^{q-1}-bundle, and let $r > 0$. If $p, q \geq 3$, then there is a constant $\kappa = \kappa(p, q, R/N, g_B, r) > 0$, such that if $\frac{\rho}{N} < \kappa$, then the manifold*

$$\hat{M} = M \setminus \operatorname{im}(\iota)^\circ \cup_\partial \pi^{-1}(B \setminus D^{p\circ})$$

admits a metric of positive Ricci curvature. This metric coincides outside a neighborhood of the gluing area with a submersion metric on E with totally geodesic and round fibers of radius r and with a scalar multiple of the metric g_M on M.

Note that under the assumptions $B = S^p$ and $p \geq q$ we precisely obtain Theorem 1.2 from Theorem A, with T being the clutching function for the bundle π. Thus, the generalization is two-fold: We relax the condition on the dimensions involved and we allow a wider range of manifolds to be glued in.

Similarly as in Theorem 1.3, we obtain the following consequence:

Theorem B ([97, Theorem B]). *Let W be the manifold obtained by plumbing according to a simply-connected graph with compact base manifolds. Suppose that the dimensions of base manifolds and fibers are all at least 3. Let B be one of the base manifolds and suppose that all other base manifolds admit core metrics. Then*

1. *If B admits a Riemannian metric of positive Ricci curvature, then ∂W admits a Riemannian metric of positive Ricci curvature.*
2. *If B admits a core metric and the fiber over B has dimension at least 4, then ∂W admits a core metric.*

Burdick [23] has shown that the total space of a linear S^p-bundle over a compact manifold B^q admits a core metric if $p, q \geq 3$ and B admits a core metric[1], see Proposition 5.2.7. As a consequence of Theorem B, we can extend this result.

Theorem C ([97, Theorem C]). *Let $E \to B^q$ be a linear S^p-bundle and suppose that*

- *$p = 2$ and $q \geq 4$, or*
- *$q = 2$ and $p \geq 4$.*

If B is closed and admits a core metric, then E admits a core metric.

Finally, we consider applications in dimension $6k$. As mentioned above, every closed, simply-connected 6-manifold admits a Riemannian metric of positive scalar curvature. For positive Ricci curvature, however, all known examples either have small Betti numbers, or have a simple cohomology ring structure, see Section 3.5. The only known examples of 6-manifolds with a core metric are S^6, $\mathbb{C}P^3$, $S^3 \times S^3$ and connected sums of copies of these manifolds.

[1] In [23] it is claimed that this result holds if $p \geq 3$ and $q \geq 2$. However, if $q = 2$ then, the proof given in [23] is not valid, see Remark 5.2.8.

We will apply Theorem B to extend the class of known examples. To identify the manifolds, we will use classification results that only hold in dimension 6. However, results, for which these classification results are not required, will also hold in any dimension $6k$.

To state our result, given a closed, simply-connected and oriented $6k$-dimensional manifold M with torsion-free homology, we have a trilinear form $\mu_M\colon H^{2k}(M) \times H^{2k}(M) \times H^{2k}(M) \to \mathbb{Z}$ defined by

$$\mu_M(x, y, z) = \langle x \smile y \smile z, [M] \rangle.$$

Further invariants we will consider are the k-th power of the second Stiefel-Whitney class

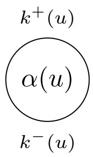
$$w_2(M)^k \in H^{2k}(M; \mathbb{Z}/2) \cong H^{2k}(M) \otimes \mathbb{Z}/2$$

and the k-th Pontryagin class

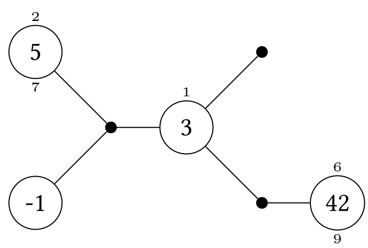
$$p_k(M) \in H^{4k}(M) \cong \mathrm{Hom}(H^{2k}(M), \mathbb{Z}).$$

In particular, these invariants are all defined on the cohomology group $H^{2k}(M)$. For $k = 1$, by the classification of Jupp [67], see also Theorem 6.1.1, these invariants already determine the diffeomorphism type of M up to connected sums with copies of $S^3 \times S^3$. Given a finitely generated free abelian group H, a symmetric trilinear form μ on H, an element $w \in H \otimes \mathbb{Z}/2$ and a linear form p on H, we call the system (H, μ, w, p) *admissible in dimension* $6k$, if it can be realized as the invariants of a closed, simply-connected $6k$-dimensional manifold with torsion-free homology.

In Section 6.2 we introduce the notion of *algebraic plumbing graphs*. These are bipartite graphs $G = (U, V, E, (\alpha, k^+, k^-))$, where U and V are the sets of vertices and $E \subseteq U \times V$ is the set of edges. Further, we have a labeling $(\alpha, k^+, k^-)\colon U \to \mathbb{Z} \times \mathbb{N}_0^2$ for vertices in U. We draw vertices $u \in U$ as follows:



If one of $k^+(u)$ and $k^-(u)$ vanishes, then we will omit it. Vertices in V will simply be drawn as dots. An example for such a graph is given as follows:

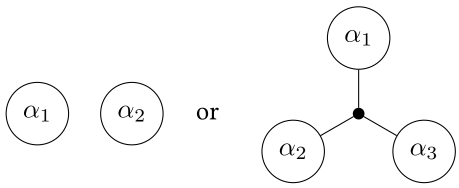


An algebraic plumbing graph G defines for every k a $6k$-dimensional manifold, denoted by $M_{\overline{G}^k}$. Important invariants, such as the cohomology group $H^{2k}(M_{\overline{G}^k})$, the trilinear form $\mu_{M_{\overline{G}^k}}$, and characteristic classes can be computed directly from the data provided by the algebraic plumbing graph if it is simply-connected. For example, if no vertex in v is a leaf, then $H^{2k}(M_{\overline{G}^k})$ has rank $|U| - |V|$ and $M_{\overline{G}^k}$ is spin if and only if $k^- = k^+ \equiv 0$. The fact, that the invariants can be obtained from the graph data if G is simply-connected, motivates defining invariants $(H_G, \mu_G^k, w_G, p_G^k)$ in a similar way for any algebraic plumbing graph G. We set $\mu_G = \mu_G^1$ and $p_G = p_G^1$.

Theorem D. *Let G be an algebraic plumbing graph.*

- *If $k = 1$, then the system of invariants (H_G, μ_G, w_G, p_G) is admissible in dimension 6.*
- *If every connected component of G is simply-connected, then the system of invariants $(H_G, \mu_G^k, w_G, p_G^k)$ is admissible in dimension $6k$ and realized by the manifold $M_{\overline{G}^k}$. Further, $M_{\overline{G}^k}$ admits a core metric.*
- *If $k = 1$ and every connected component of G is simply-connected, then any closed, simply-connected 6-manifold with torsion-free homology, whose invariants are equivalent to (H_G, μ_G, w_G, p_G), admits a core metric.*

Since different algebraic plumbing graphs can have equivalent systems of invariants, it is not clear a priori, how large the class of manifolds is that we obtain in this way. To analyze this further, we introduce a reduced form in Section 6.3 and conjecture that systems of invariants obtained from different reduced forms are indeed not equivalent, see Question 6.3.4. The difficulty lies in the problem that in general it is hard to determine whether two given trilinear forms are equivalent or not. In Section 6.4 we prove the conjecture for graphs G with $\mathrm{rank}(H_G) \leq 2$, except for the case where $\mathrm{rank}(H_G) = 2$ and $w_G = 0$. Here the reduced graphs are of the form



with $\alpha_i \neq 0$ in the second case. By using invariant theory of $\mathrm{SL}(2, \mathbb{C})$ we can show that for any graph of this form there exists at most one other graph of this form with an equivalent system of invariants, see Proposition 4.3.4. This fact, however, is sufficient to show that we obtain infinitely many diffeomorphism types in this way, so that infinitely many of these graphs define new examples of 6-manifolds with a Riemannian metric of positive Ricci curvature and of 6-manifolds with core metrics, see Remark 6.5.3. Further, by using the classification of Schmitt [100], we can analyze how large the class of 6-manifolds constructed in this way is within the class of all closed, simply-connected spin 6-manifolds M with torsion-free homology and $b_2(M) = 2$, see Proposition 6.5.4. An interesting subfamily of these graphs is given by certain graphs for which the corresponding 6-manifolds split as a connected sum, where one of the summands is a homotopy $\mathbb{C}P^3$, see Proposition 6.5.9.

For larger Betti numbers, using Theorem D, we have the following results.

Theorem E. *For every $k \in \mathbb{N}$ and for every odd $l \in \mathbb{N}$ sufficiently large there exists an infinite family M_j^{6k} of pairwise non-diffeomorphic closed $6k$-dimensional manifolds with torsion-free homology with the following properties:*

- *M_j is $(2k-1)$-connected with $b_{2k}(M_j) = l$,*
- *M_j does not split non-trivially as a connected sum,*
- *M_j is not diffeomorphic to the total space of a linear sphere bundle, a homogeneous space, a biquotient, a cohomogeneity one manifold or a Fano variety,*
- *M_j admits a core metric.*

Further, if $k = 1$ or k is even, then we can replace the conclusion that M_j is $(2k-1)$-connected by M_j being simply-connected and non-spin.

It follows that the manifolds M_j are new examples of manifolds with a metric of positive Ricci curvature.

We also consider limitations of this technique. In fact, the total space of any linear S^{2k}-bundle over $S^{2k} \times S^{2k}$, which is known to admit a core metric for $k \geq 2$ and a Riemannian metric of positive Ricci curvature for $k = 1$, cannot be constructed via an algebraic plumbing graph as in Theorem D, see Proposition 6.5.7.

This thesis is organized as follows: Chapter 2 summarizes the notation used. In Chapter 3 we give an overview over known constructions and obstructions for Riemannian metrics of positive Ricci curvature. We also briefly survey the situation for positive scalar and sectional curvature and explicitly consider the known examples of manifolds with a Riemannian metric of positive Ricci curvature in dimensions up to 6. In Chapter 4 we introduce surgery and the plumbing construction and analyze topological properties of manifolds obtained by plumbing. In Chapter 5 we consider surgery on manifolds with a Riemannian metric of positive Ricci curvature and prove Theorems A, B and C. Finally, we consider applications to $6k$-dimensional manifolds in Chapter 6 and give the proof of Theorems D and E. The appendices provide basic properties of Riemannian manifolds, fiber bundles, graph theory and invariant theory.

Notation and General Background 2

By $\mathbb{N}$ we denote the set of natural numbers and we use the convention that $0 \notin \mathbb{N}$. We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. As usual, $\mathbb{Z}$ denotes the ring of integers and $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the fields of rational, real and complex numbers, respectively.

Manifolds

For the definitions and basic properties of manifolds, vector bundles, Lie groups and Riemannian metrics we refer to [75], [76] and [94]. Basics of Riemannian geometry and of fiber bundles are introduced in Appendices A and B, respectively. Here, the term *manifold* denotes a smooth manifold, possibly with boundary. The boundary of a manifold M is denoted by ∂M. For $x \in M$ we denote by T_xM the tangents space at x and by TM the tangent bundle of M. By Int(M) we denote the interior of M. If $X \subseteq M$, we denote by X° the interior of X as a subspace of M. Note, that if X is a submanifold, these two notions of the interior do not necessarily coincide.

All maps between manifolds will be assumed to be smooth. For a manifold M we will write M^n to indicate that M has dimension n. By D^n we denote the closed n-dimensional disc and by $S^n = \partial D^{n+1}$ the n-dimensional sphere. Real, complex and quaternionic projective spaces are denoted by $\mathbb{R}P^n$, $\mathbb{C}P^n$ and $\mathbb{H}P^n$, respectively.

If M_1 and M_2 are two manifolds, such that there exists a diffeomorphism $\phi\colon \partial_c M_2 \to \partial_c M_1$ between boundary components $\partial_c M_i \subseteq \partial M_i$, we denote by

$$M_1 \cup_\phi M_2$$

the space obtained from $M_1 \sqcup M_2$ by gluing along ϕ. The smooth structures of M_1 and M_2 carry over to this space and turn it into a manifold in a natural way, see e.g. [58, Theorem 8.2.1]. If the identification ϕ is clear, then we also write

$$M_1 \cup_{\partial_c M_2} M_2,$$

and if $\partial_c M_i = \partial M_i$, we simply write

$$M_1 \cup_\partial M_2.$$

(Co-)Homology

If not stated otherwise, we use homology and cohomology with coefficients in $\mathbb{Z}$. For basic notions of (co-)homology we refer to [55]. If M is oriented, then $-M$ denotes M with the reversed orientation. If a compact manifold M^n is oriented with respect to a commutative ring R, we denote by

$[M, \partial M; R] \in H_n(M, \partial M; R)$ its fundamental class. We will leave out ∂M or R in this notation if $\partial M = \emptyset$ or $R = \mathbb{Z}$, respectively. Note that an orientation of M in the classical sense, which corresponds to an orientation with respect to $\mathbb{Z}$, induces an orientation with respect to any ring R, since in this case, we have

$$H_n(M, \partial M; R) \cong H_n(M, \partial M) \otimes R \cong R.$$

An important tool will be *Lefschetz duality*: If $A, B \subseteq \partial M$ are submanifolds such that $\partial A = \partial B = A \cap B$, then cap product with $[M, \partial M; R]$ defines an isomorphism

$$\cdot \cap [M, \partial M; R] \colon H^i(M, A; R) \to H_{n-i}(M, B; R),$$

see e.g. [55, Theorem 3.43].

The Lefschetz dual of $1 \in H_0(M; R) \cong R$ in $H^n(M, \partial M; R)$ is denoted by

$$[M, \partial M; R]^* \in H^n(M, \partial M; R).$$

In particular, we have $\langle [M, \partial M; R]^*, [M, \partial M; R] \rangle = 1 \in R$.

By

$$b_i(M) = \mathrm{rank}(H^i(M)) = \mathrm{rank}(H_i(M))$$

we denote the i-th *Betti number* of M. Finally,

$$\chi(M) = \sum_i (-1)^i b_i(M)$$

denotes the *Euler characteristic* of M.

Positive Ricci Curvature 3

In this chapter we give an introduction to positive Ricci curvature. We will first survey the known methods to construct metrics of positive Ricci curvature in Section 3.1 and topological obstructions to the existence of such metrics in Section 3.2. We then illustrate its connections to the related conditions of positive sectional and positive scalar curvature in Sections 3.3 and 3.4. Finally, in Section 3.5, we list all known examples of closed, simply-connected manifolds with a metric of positive Ricci curvature in dimensions up to 6.

3.1 Constructions of Metrics of Positive Ricci Curvature

Recall (cf. Appendix A), that for a Riemannian manifold (M^n, g) the Ricci curvature is a symmetric $(0, 2)$-tensor on M defined by

$$\mathrm{Ric}(u, v) = \mathrm{tr}_g(g(R(\cdot, u)v, \cdot))$$

for all $p \in M$, $u, v \in T_pM$, and (M, g) is said to have positive Ricci curvature, if $\mathrm{Ric}(v, v) > 0$ for all $v \in TM$.

In the following we will only consider results for compact manifolds, since this is our main focus.

The most basic example of a manifold with a metric of positive Ricci curvature is the sphere S^n. Indeed, the *round metric* of radius $r > 0$, denoted by $r^2 \cdot ds_n^2$, is the metric induced from the standard metric on $\mathbb{R}^{n+1}$ when we identify S^n with

$$\{v \in \mathbb{R}^{n+1} \mid \|v\| = r\}.$$

We will also write $S^n(r)$ for $(S^n, r^2 \cdot ds_n^2)$. If $n \geq 2$, then $r^2 ds_n^2$ has positive Ricci curvature. In fact, the Ricci tensor is given by

$$\mathrm{Ric} = \frac{n-1}{r^2} ds_n^2, \tag{3.1.1}$$

see e.g. [94, 4.2.1]. Further, for any finite group $\Gamma \subseteq \mathrm{O}(n)$ that acts freely on S^n, the quotient S^n/Γ with the induced metric of ds_n^2 is locally isometric to $S^n(1)$, hence it also has positive Ricci curvature. These spaces are called *spherical space forms*.

That the round metric has positive Ricci curvature is also a special case of the following result by Nash.

Theorem 3.1.1 ([86, Proposition 3.4]). *Let G be a compact Lie group and let $H \subseteq G$ be a closed subgroup. Then the homogeneous space G/H admits a metric of positive Ricci curvature if and only if $\pi_1(G/H)$ is finite. In fact, if $\pi_1(G/H)$ is finite, then every biinvariant metric on G induces a metric of positive Ricci curvature on G/H.*

The sphere S^n is the homogeneous space $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ (cf. Example B.1.3), hence Theorem 3.1.1 applies. Further applications are the projective spaces

$$\begin{aligned}
\mathbb{R}P^n &= \mathrm{O}(n+1)/(\mathrm{O}(n)\times\mathrm{O}(1)),\\
\mathbb{C}P^n &= \mathrm{U}(n+1)/(\mathrm{U}(n)\times\mathrm{U}(1)),\\
\mathbb{H}P^n &= \mathrm{Sp}(n+1)/(\mathrm{Sp}(n)\times\mathrm{Sp}(1)),\\
\mathbb{O}P^2 &= \mathrm{F}_4/\mathrm{Spin}(9).
\end{aligned}$$

A generalization of homogeneous spaces are *biquotients*. For a Lie group G and a closed subgroup $H \subseteq G\times G$ consider the action of H on G defined by

$$(h_1,h_2)\cdot g = h_1 g h_2^{-1}$$

for $g\in G$ and $(h_1,h_2)\in H$. If the action is free, then the quotient, denoted by $G/\!\!/H$, is a smooth manifold. A special case is where $H = H_1\times H_2$ for closed subgroups $H_1, H_2\subseteq G$. Then we also write $H_1\backslash G/H_2$. For H_1 the trivial group we then recover the definition of a homogeneous space. Theorem 3.1.1 was extended to biquotients by Schwachhöfer and Tuschmann.

Theorem 3.1.2 ([104, Theorem A]). *Let G be a compact Lie group and let $H \subseteq G\times G$ be a closed subgroup so that $G/\!\!/H$ is a biquotient. Then $G/\!\!/H$ admits a metric of positive Ricci curvature if and only if $\pi_1(G/\!\!/H)$ is finite. In fact, if $\pi_1(G/\!\!/H)$ is finite, then every biinvariant metric on G induces a metric of positive Ricci curvature on $G/\!\!/H$.*

If we decrease the degree of symmetry, we arrive at the notion of *cohomogeneity one manifolds*. Let G be a compact Lie group. Then a cohomogeneity one manifold for G is a manifold M with an action of G, so that the orbit space M/G is a 1-dimensional manifold (possibly with boundary). Grove and Ziller showed, that for cohomogeneity one manifolds a result similar to Theorem 3.1.1 holds.

Theorem 3.1.3 ([53, Theorem A]). *Let G be a compact Lie group and let M be a closed cohomogeneity one manifold for G. Then M admits a G-invariant metric of positive Ricci curvature if and only if $\pi_1(M)$ is finite.*

This theorem was extended by Schwachhöfer and Tuschmann to quotients by subgroups of G.

Theorem 3.1.4 ([104, Theorem B]). *Let G be a compact Lie group and let M be a closed cohomogeneity one manifold for G. Let $L\subseteq G$ be a subgroup that acts freely on M. Then M/L admits a $\mathrm{Norm}_G(L)$-invariant metric of positive Ricci curvature if and only if $\pi_1(M/L)$ is finite.*

Here $\mathrm{Norm}_G(L)=\{g\in G\mid gLg^{-1}=L\}$ is the normalizer of L.

Applications of Theorem 3.1.3 include *Brieskorn manifolds*. For $a_0,\dots,a_n\in\mathbb{N}$ the Brieskorn manifold $B(a_0,\dots,a_n)$ is the intersection of the zero set of the polynomial

$$z_0^{a_0}+\dots+z_n^{a_n}$$

in $\mathbb{C}^{n+1}$ with the unit sphere in $\mathbb{C}^{n+1}$. The space $B(a_0,\dots,a_n)$ has the structure of a smooth manifold of dimension $2n-1$, see [19]. The Brieskorn manifolds $B(d,2,\dots,2)$ are of cohomogeneity one for the group $\mathrm{O}(n)\times S^1$, hence Theorem 3.1.3 applies. This class contains the class of *Kervaire spheres*, which are certain odd-dimensional homotopy spheres that are exotic if they are of dimension $2n-1$ and $n+1$ is not a power of 2. By taking finite quotients of Kervaire spheres, one can construct manifolds that are homotopy equivalent, but not diffeomorphic to $\mathbb{R}P^{4k+1}$, which then also admit actions of cohomogeneity one and hence a metric of positive Ricci curvature, see [104, Section 7]. Note that this result has also been obtained in the context of Sasakian geometry, see Theorem 3.1.10 below and subsequent remarks.

It had already been shown previously by Cheeger [28, Example 4], by using the same description as a cohomogeneity one manifold, that all Kervaire spheres admit metrics of positive Ricci curvature. A further construction of metrics of positive Ricci curvature on Brieskorn manifolds was achieved by Hernandez [56], who considered the metric on $B(a_0, \dots, a_n)$ induced from the standard metric on $\mathbb{C}^{n+1}$. His main result is given as follows:

Theorem 3.1.5 ([56, Theorem III.4]). *Let $a_0, \dots, a_m \geq 2$. Then there exists an integer $N = N(a_0, \dots, a_m)$, such that, if we set $a_i = 2$ for $i > m$, the Brieskorn manifold $B(a_0, \dots, a_{m+p})$ admits a metric of positive Ricci curvature for all $p \geq N$.*

For example, in this way one obtains metrics of positive Ricci curvature on many exotic spheres that bound parallelizable manifolds, see [56, Theorem IV.1]

If the action is of higher cohomogeneity, i.e. the quotient M^{reg}/G, where M^{reg} denotes the union of the principal orbits, has dimension at least 2, there is no classification as in Theorems 3.1.1 and 3.1.3 known. The following result by Searle and Wilhelm gives a partial answer.

Theorem 3.1.6 ([105, Theorem A]). *Let G be a compact and connected Lie group and let (M, g) be a Riemannian manifold on which G acts isometrically and effectively. If the fundamental group of the principal orbits is finite and the induced metric on M^{reg}/G has Ricci curvature ≥ 1, then M admits a G-invariant metric of positive Ricci curvature.*

This theorem is motivated by a result of Lawson and Yau, see Theorem 3.4.14 below, which asserts that any closed manifold, on which a non-abelian Lie group acts effectively, admits a metric of positive scalar curvature.

We now consider metrics of positive Ricci curvature on fiber bundles. It is not hard to see that the product of two manifolds with metrics of positive Ricci curvature again has positive Ricci curvature, see Remark A.4. For fiber bundles in general there exist results going back to Poor [96] and Nash [86], which are consequences of the following more general result:

Theorem 3.1.7 ([47, Theorem 2.7.3], see also [27, Theorem 1.6]). *Let $\pi\colon E \to B$ be a fiber bundle with fiber F and structure group G. If F and B are compact and admit metrics of positive Ricci curvature, so that the action of G on F is isometric, then E admits a metric of positive Ricci curvature.*

Theorem 3.1.7 provides numerous ways to construct manifolds with metrics of positive Ricci curvature. Indeed, one can apply it to fiber bundles with fiber F and structure group G, where F is a homogeneous space or a cohomogeneity one manifold for G for which Theorem 3.1.1 or 3.1.3 applies. Of particular interest are linear sphere bundles, which are discussed in more detail in Section B.2.

In the special case of a principal bundle, all bundles admitting an invariant metric of positive Ricci curvature were determined by Gilkey, Park and Tuschmann [45], provided that the base manifold admits a metric of positive Ricci curvature.

Theorem 3.1.8 ([45, Theorem 0.1]). *Let G be a compact and connected Lie group and let $\pi\colon P \to B$ be a principal bundle, so that B admits a metric of positive Ricci curvature. Then P admits a metric of positive Ricci curvature if and only if $\pi_1(P)$ is finite.*

Note that the fiber G does not necessarily need to admit a metric of positive Ricci curvature, e.g. if G is a torus. By considering total spaces of principal torus bundles over simply-connected 4-manifolds, Corro and Galaz-García [31] constructed metrics of positive Ricci curvature on certain connected sums of sphere bundles.

Complex geometry can also be used to construct metrics of positive Ricci curvature. As a result of Yau's proof of the Calabi conjecture [125], we have the following theorem.

Theorem 3.1.9 ([125]). *Let M be a compact Kähler manifold. Then any closed, real $(1,1)$-form on M, whose cohomology class is the first Chern class of M, is the Ricci curvature of a Kähler metric on M. In particular, if the first Chern class can be represented by a closed, real $(1,1)$ form that is positive definite, then M admits a Kähler metric of positive Ricci curvature.*

Applications of Theorem 3.1.9 are given by so-called Fano varieties, which we will consider in Section 3.5.

Theorem 3.1.9 only applies to manifolds whose real dimension is even. In odd dimensions one can consider Sasakian geometry, which is a subfield of contact geometry, see e.g. [13]. For Sasakian manifolds, there exists a notion of positivity, and the analogue of Theorem 3.1.9, proven by Boyer, Galicki and Nakamaye [17], is given as follows.

Theorem 3.1.10 ([17]). *Let M be a compact manifold that admits a positive Sasakian structure. Then M admits a metric of positive Ricci curvature.*

Theorem 3.1.10 has turned out to provide many examples of manifolds with a metric of positive Ricci curvature, including rational homology spheres in dimension 5 [12, 14], all homotopy spheres that bound parallelizable manifolds, including all homotopy spheres in dimensions 7 and 11, and homotopy real projective spaces [18], and many odd-dimensional highly-connected manifolds which are homotopy equivalent to a connected sum of products of spheres [15]. Note that some of these results have independently been obtained via surgery techniques by Wraith [121] and Crowley–Wraith [33], see Theorem 3.1.12 below.

We also mention the techniques involving surgery, which are the main topic of this thesis. The process of surgery will be introduced in Section 4.1 and surgery in the context of positive Ricci curvature will be discussed in Chapter 5.

The surgery techniques for positive Ricci curvature were first introduced by Sha and Yang [107]. As a result of their surgery theorem they obtained the following manifolds with metrics of positive Ricci curvature.

Theorem 3.1.11 ([107, Theorem 1]). *Let $p, q \geq 2$. Then, for any $k \in \mathbb{N}$, the manifold $\#_k(S^p \times S^q)$ admits a metric of positive Ricci curvature.*

The family of manifolds in Theorem 3.1.11 were the first example of an infinite family of manifolds with a metric of positive Ricci curvature in a fixed dimension with arbitrarily large total Betti number. In particular, this shows that almost all of these manifolds cannot admit a metric of non-negative sectional curvature by Theorem 3.3.2 below. Theorem 3.1.11 got subsequently extended by Wraith [124] to connected sums of products of spheres, where the summands may differ from each other.

A modification of the surgery theorem of Sha and Yang is Theorem 1.2 and was developed by Wraith [122]. It led to the following applications.

Theorem 3.1.12 ([33],[121]). *The following manifolds admit metrics of positive Ricci curvature:*

1. *Every homotopy sphere that bounds a parallelizable manifold,*
2. *Up to connected sum with a homotopy sphere every highly connected manifold in dimension $4k-1$, $k \geq 2$, which is $(2k-1)$-parallelizable if $k \equiv 1 \mod 4$, including all 2-connected 7-manifolds,*
3. *Up to connected sum with a homotopy sphere every highly connected manifold in dimension $4k+1$, $k \geq 1$, that is $2k$-parallelizable and has torsion-free homology.*

For connected sums, which is a special case of surgery, Perelman [90] constructed a metric of positive Ricci curvature on any finite connected sum of copies of $\pm\mathbb{C}P^2$. For that he introduced a gluing technique for metrics of positive Ricci curvature.

Theorem 3.1.13 ([90]). *Let M_1^n, M_2^n be manifolds that admit metrics of positive Ricci curvature. Assume that there exists an isometry $\phi\colon \partial_c M_1 \to \partial_c M_2$ between compact boundary components $\partial_c M_1 \subseteq \partial M_1$ and $\partial_c M_2 \subseteq \partial M_2$. If the second fundamental forms $\mathrm{II}_{\partial_c M_i}$ satisfy*

$$\mathrm{II}_{\partial_c M_1} + \phi^* \mathrm{II}_{\partial_c M_2} \geq 0,$$

then $M_1 \cup_\phi M_2$ admits a metric of positive Ricci curvature that coincides with the original metrics on M_1 and M_2 outside an arbitrarily small neighborhood of the gluing area.

Perelman's ideas were later adapted by Burdick [21, 22, 23, 24], who introduced the notion of *core metrics* and showed that the connected sum of a finite number of manifolds with core metrics admits a metric with positive Ricci curvature, cf. Theorem 1.1. This approach, which we will discuss in detail in Section 5.2, combined with the surgery techniques, provides the most promising approach so far to construct metrics of positive Ricci curvature, since the manifolds constructed in this way do not need to have any additional structure, like admitting a certain group action, bundle structure or Kähler/Sasakian structure.

Finally, we mention two deformation results that allows to deform a metric of non-negative Ricci or sectional curvature to a metric of positive Ricci curvature. The first one, which is due to Ehrlich, uses so-called local convex deformations. For a given point these deformations decrease the Ricci curvatures at this point while increasing the Ricci curvatures on an annulus around this point. By a repeated application of this deformation, if the manifold has non-negative Ricci curvature and positive Ricci curvature at one point, one can spread the positivity of the Ricci curvatures to the whole manifold.

Theorem 3.1.14 ([41]). *Let M be a manifold that admits a complete metric of non-negative Ricci curvature. If there is a point in M at which all Ricci curvatures are positive, then M admits a complete metric of positive Ricci curvature.*

The second result, which is due to Böhm and Wilking, uses the Ricci flow. In fact, it shows that a metric of non-negative sectional curvature on a closed manifold evolves to a metric with positive Ricci curvature under the Ricci flow unless the original metric has a flat factor, which can only be the case if the manifold has an infinite fundamental group.

Theorem 3.1.15 ([9]). *Let M be a closed manifold that admits a metric of non-negative sectional curvature. If $\pi_1(M)$ is finite, then M admits a metric of positive Ricci curvature.*

3.2 Topological Obstructions to Positive Ricci Curvature

The main obstruction to the existence of metrics of positive Ricci curvature is the classical theorem of Bonnet–Myers.

Theorem 3.2.1 ([85], or [94, Theorem 6.3.3]). *A closed manifold that admits a metric of positive Ricci curvature has finite fundamental group.*

This shows that the existence of a metric of positive Ricci curvature has strong implications on the global structure of the manifold. For example, for any closed manifold M, the product $M \times S^1$ does not admit a metric of positive Ricci curvature.

Further, if M has finite, but non-trivial fundamental group, we have the following obstruction by Chen and Wu [29].

Theorem 3.2.2 ([29, Theorem A]). *There exists a constant $p(n) > 0$ such that for any closed manifold M^n that admits a metric of positive Ricci curvature, we have*

$$b_1(M, \mathbb{Z}/p) \leq n - 1$$

for all primes $p \geq p(n)$.

Any finitely presented group can be realized as the fundamental group of a closed 4-manifold, in fact one can always find such a manifold that admits a metric satisfying the weaker condition of positive scalar curvature, see e.g. [25, Corollary 2]. Thus, Theorem 3.2.2 provides numerous manifolds that cannot admit a metric of positive Ricci curvature, such as manifolds whose fundamental group is the product $(\mathbb{Z}/m\mathbb{Z})^m$ or the symmetric group S_m for m sufficiently large.

If M is simply-connected, then, besides the known obstructions for metrics with scalar curvature (see Section 3.4), there are no obstructions known. Hence, we can ask the following question.

Question 3.2.3. Let M be a closed, simply-connected manifold that admits a metric of positive scalar curvature. Does M admit a metric of positive Ricci curvature?

While the answer to Question 3.2.3 is affirmative in dimensions 2 and 3 (the only closed, simply-connected manifolds in these dimensions are spheres), it is expected to be negative in general. A counterexample would be given by a proof of the Stolz conjecture: Given a closed string manifold M^{4k}, Stolz [110] conjectured, that the so-called Witten genus of M vanishes. Provided the conjecture holds, Stolz showed that there are counterexamples to Question 3.2.3 in all dimensions $4k = 24$ or $4k \geq 32$.

For manifolds with boundary there exist topological obstructions if one additionally imposes conditions on the boundary. The first such result was proven by Lawson [71] and involves the mean curvature at the boundary.

Theorem 3.2.4 ([71, Theorem 1]). *Let M be a compact connected manifold with non-empty boundary that admits a metric of positive Ricci curvature. If the mean curvature at the boundary is positive, then ∂M is connected and the map $\pi_1(\partial M) \to \pi_1(M)$ induced by the inclusion is surjective.*

In particular, if a compact manifold M^n admits a metric of positive Ricci curvature and convex boundary, i.e. the second fundamental form on the boundary is positive definite, then there is only one boundary component. Further, Wang [118] showed, that the “degree of convexity ”of the boundary affects the topology of the interior. For that, let λ be the smallest eigenvalue of the second fundamental form of ∂M over all points in ∂M. Then define

$$\Lambda(M^n) = \lambda \left(\frac{\mathrm{vol}(\partial M)}{\omega_{n-1}} \right)^{\frac{1}{n-1}},$$

where ω_{n-1} is the volume of $S^{n-1}(1)$. The constant $\Lambda(M^n)$ can be interpreted as a measure of the convexity of the boundary, which is constructed so that it is invariant under scaling of the metric.

Theorem 3.2.5 ([118, Theorem 1′]). *Let (M^n, g), $n \geq 4$, be a compact, connected manifold. Then there exists a constant $\delta_n \in (0, 1)$, so that if*

1. $\mathrm{Ric}^g > 0$,
2. $\Lambda(M^n) > 1 - \delta_n$, *and*
3. $\left(\frac{\mathrm{Ric}^{\partial g}}{n-2} \right)^{\frac{1}{2}} \left(\frac{\mathrm{vol}(\partial M)}{\omega_{n-1}} \right)^{\frac{1}{n-1}} > 1 - \delta_n$,

then M is contractible.

For example, if ∂M is isometric to $S^{n-1}(r)$ for some $r > 0$, then the left-hand side of item (3) in Theorem 3.2.5 equals 1, hence the inequality holds. Then the theorem asserts, that if the boundary is sufficiently convex, then M must be contractible.

Finally, although these are not obstructions to the existence of metrics of positive Ricci curvature in general, we also discuss limitations of some of the techniques presented in Section 3.1. First we consider group actions.

Proposition 3.2.6. *There exists a constant $C(n)$ so that if M is a compact homogeneous space, biquotient or cohomogeneity one manifold of dimension n, then for any field $\mathbb{K}$, we have*

$$\sum_{i=0}^{n} b_i(M;\mathbb{K}) \leq C(n).$$

Proof. Since every homogeneous space, and more general, every biquotient admits a metric of non-negative sectional curvature (cf. [104, Section 2]), the claim follows directly from Theorem 3.3.2 below in this case.

A compact cohomogeneity one manifold M for a compact Lie group G can only be of one of the following forms, see e.g. [61]:

- $M/G \cong S^1$ and M has the structure of a fiber bundle over S^1 with fiber G/H for a closed subgroup $H \subseteq G$, or
- $M/G \cong [-1,1]$ and M is the union of tubular neighborhoods of the non-principal orbits $G/K_{\pm}$ over ± 1.

In the first case, M can alternatively be described as the mapping torus

$$T_f = [0,1] \times (G/H) \Big/ (0,x) \sim (1, f(x))$$

of a diffeomorphism $f\colon G/H \to G/H$. We consider $G/H \subseteq T_f$ via $G/H \cong \{0\} \times G/H \subseteq T_f$. Then we have for $i > 0$, where we consider cohomology with coefficients in $\mathbb{K}$,

$$H^i(T_f, G/H) = H^i(T_f/(G/H)) = H^i((G/H \times S^1)/(G/H)) = H^i(G/H \times S^1, G/H)$$

and

$$H^i(G/H \times S^1, G/H) \cong H^{i-1}(G/H)$$

by the long exact sequence of the pair $(G/H \times S^1, G/H)$. Hence, the long exact sequence of the pair $(T_f, G/H)$ is given by

$$\cdots \longrightarrow H^i(T_f, G/H) \longrightarrow H^i(T_f) \longrightarrow H^i(G/H) \longrightarrow \ldots,$$

so

$$\sum_{i=0}^{n} b_i(T_f) \leq \sum_{i=0}^{n} b_i(T_f, G/H) + b_i(G/H) = \sum_{i=0} b_{i-1}(G/H) + b_i(G/H) \leq 2C(n-1)$$

for the constant $C(n)$ in Theorem 3.3.2.

In the second case, there exists a long exact sequence

$$\ldots H^{i-l_- -1}(G/K_-) \longrightarrow H^i(M) \longrightarrow H^i(G/K_+) \longrightarrow H^{i-l_-}(G/K_-) \longrightarrow \ldots,$$

where $l_{\pm}$ denotes the dimension of $K_{\pm}/H$ and $H \subseteq G$ is the isotropy subgroup of the principal orbits, see e.g. [63]. Hence,

$$\sum_{i=0}^{n} b_i(M) \leq \sum_{i=0}^{n} b_{i-l_- -1}(G/K_-) + b_i(G/K_+) \leq 2C(n-1).$$

Alternatively, Schwachhöfer and Tuschmann showed that any closed cohomogeneity one manifold admits a metric of almost non-negative sectional curvature, see [103, Theorem A]. By Theorem 3.3.2 below, which also applies to metrics of almost non-negative sectional curvature, the claim follows. $\square$

For applications of Theorem 3.1.9 we have the following limitation.

Proposition 3.2.7 ([35, Theorem 2.1]). *In every dimension n there exist only finitely many diffeomorphism types of Fano varieties.*

For positive Sasakian structures an analogous result does not hold as we will see in Section 3.5. However, besides the fact that Sasakian structures only exist in odd dimensions, there exist additional obstructions:

Proposition 3.2.8. *Let M^{2n+1} be a Sasakian manifold. Then*

- *The top Stiefel-Whitney class $w_{2n+1}(M)$ vanishes, and*
- *If M is compact and simply-connected and the Sasakian structure is positive, then M is spin.*

Proof. By definition, M admits a unit length vector field, see e.g. [13, Definition-Theorem 6]. By [82, Proposition 4.4] this implies that $w_{2n+1}(M) = 0$, showing the first claim. For the second claim we refer to [17, Proposition 2.6]. □

3.3 Positive Sectional Curvature

In this section we briefly summarize the known results concerning the existence of metrics of positive sectional curvature on closed manifolds. We refer to [127] for a survey.

Recall (cf. Appendix A), that for a Riemannian manifold (M, g) and linearly independent vectors $v, w \in T_pM$, $p \in M$, the sectional curvature $\sec(u, v)$ is defined by

$$\sec(u, v) = \frac{g(R(u, v)v, u)}{g(u, u)g(v, v) - g(u, v)^2}$$

and that positivity of sec implies positivity of Ric.

There are strict topological obstructions to the existence of metrics of positive sectional curvature. First, the theorem of Bonnet-Myers (Theorem 3.2.1) can also be applied to metrics of positive sectional curvature, showing that closed manifolds with a metric of positive sectional curvature have finite fundamental group. Further obstructions are given as follows:

Theorem 3.3.1 (Synge [112],or [94, Theorem 6.3.6]). *Let M^n be a manifold that admits a metric of positive sectional curvature.*

1. *Suppose n is even. If M is orientable, then M is simply-connected. If M is not orientable, then $\pi_1(M) \cong \mathbb{Z}/2$.*
2. *Suppose n is odd. Then M is orientable.*

As a consequence, the product $\mathbb{R}P^n \times \mathbb{R}P^n$ cannot admit a metric of positive sectional curvature, which shows that the existence of metrics of positive sectional curvature is in general not preserved under Cartesian products. In the simply-connected case it is conjectured for the product $S^2 \times S^2$ that it does not admit a metric of positive sectional curvature, called the *Hopf conjecture*.

Theorem 3.3.2 (Gromov [48] and [49], or [94, Theorem 12.5.1]). *There is a constant $C(n)$ so that for any closed manifold M^n that admits a metric of non-negative sectional curvature, we have*

1. *$\pi_1(M)$ is generated by at most $C(n)$ elements, and*
2. *For any field $\mathbb{K}$, the inequality*
$$\sum_{i=0}^{n} b_i(M; \mathbb{K}) \leq C(n)$$
holds.

It is conjectured that $C(n)$ can be chosen to be 2^n. This value is attained for the torus T^n, which admits a flat metric, so in particular a metric of non-negative sectional curvature.

Examples of closed manifolds with a metric of positive sectional curvature are rare. In particular, all except two of the known examples are homogeneous spaces or biquotients, as we will describe now.

For a compact Lie group G and a closed subgroup $H \subseteq G$, a normal homogeneous metric on G/H, that is, a metric induced from a biinvariant metric on G, has non-negative sectional curvature (see e.g. [8, Proposition 7.87]). The simply-connected homogeneous manifolds with an invariant metric of positive sectional curvature have been classified by Berger [7], see also [119], Wallach [116] and Bérard-Bergery [6]. They include the rank-one symmetric spaces S^n, $\mathbb{C}P^n$, $\mathbb{H}P^n$ and $\mathbb{O}P^2$ as in the case of positive Ricci curvature. Further, all spherical space forms S^n/Γ have positive sectional curvature. In particular, if $\Gamma = \mathbb{Z}/2$ acts via the antipodal map, we obtain the real projective space $\mathbb{R}P^n$. In fact, in dimensions 2 and 3, spherical space forms are the only closed manifolds that admit a metric of positive sectional curvature. In dimension 2 this follows from the classification of surfaces together with Theorem 3.2.1. In dimension 3 it was shown by Hamilton [54]. In higher dimensions there exist additional examples. It is worth noting, however, that only in dimension 7 we obtain an infinite family, called the *Aloff-Wallach spaces* [1].

Further examples can be constructed via biquotients. As in the case for homogeneous spaces, a biinvariant metric on G induces a metric of non-negative sectional curvature on $G/\!\!/H$. By deforming this metric, Eschenburg [42, 43] obtained metrics of positive sectional curvature on a 6-dimensional biquotient, and on an infinite family of 7-dimensional biquotients, called the *Eschenburg spaces*. Finally, Bazaikin [5] constructed an infinite family of manifolds in dimension 13, called the *Bazaikin spaces* that admit a metric of positive sectional curvature.

The only further examples are due to Grove–Verdiani–Ziller [52] and Dearricott [36] given by a 7-dimensioanl orbifold fibration, and by Petersen–Wilhelm [95] (although not published yet) given by a 7-dimensional exotic sphere.

3.4 Positive Scalar Curvature

In this section we briefly summarize the known results concerning the existence of metrics of positive scalar curvature on closed manifolds. We refer to [98] for a survey.

Recall (cf. Appendix A), that for a Riemannian manifold (M^n, g) the scalar curvature scal is defined by

$$\mathrm{scal}_p = \sum_{i=1}^{n} \mathrm{Ric}(e_i, e_i)$$

for $p \in M$ and an orthonormal basis $(e_1, \dots, e_n)$ of T_pM, and that positivity of Ric implies positivity of scal.

The condition of positive scalar curvature is much weaker than the condition of positive sectional curvature, as can be seen both from the much weaker topological obstructions and the number of examples known as we will describe below.

There are several different obstructions to the existence of metrics of positive scalar curvature. We start with obstructions emerging from index theory of Dirac operators. One such obstruction is the alpha-invariant, which assigns to a spin manifold M^n an element $\alpha(M) \in \mathrm{KO}^{-n}(pt)$ in the real K-theory of a point. It only depends on the spin bordism class of M.

Theorem 3.4.1 (Hitchin [60]). *Let M be a closed spin manifold that admits a metric of positive scalar curvature. Then $\alpha(M) = 0$.*

If $n \equiv 0 \mod 4$, then $\mathrm{KO}^{-n}(pt) \cong \mathbb{Z}$, and the alpha-invariant $\alpha(M)$ coincides, up to a factor, with the $\hat{A}$-genus $\hat{A}(M)$, which can be computed from the Pontryagin classes of M. In this case, Theorem 3.4.1 had already been proven by Lichnerowicz [78].

Example 3.4.2. 1. The $K3$-surface is a closed, simply-connected spin 4-manifold with $\hat{A}(K3) = 2$. Hence, it does not admit a metric of positive scalar curvature.

2. Hitchin [60] (see also [73, Theorem II.8.13]) showed that in every dimension $n \geq 9$ with $n \equiv 1, 2 \mod 8$ there exist exotic spheres Σ^n with $\alpha(\Sigma) \neq 0$. Hence they do not admit a metric of positive scalar curvature.

For manifolds with non-trivial fundamental group, Gromov and Lawson [51] introduced the notion of *enlargeable* manifolds. For example, any closed manifold that admits a metric of non-positive sectional curvature, is enlargebale ([73, Theorem IV.5.4]). In particular, the torus T^n is enlargeable. Further, the connected sum of any manifold with an enlargeable manifold is enlargeable, and the product of two enlargeable manifolds is enlargeable ([73, Theorem IV.5.3]).

Theorem 3.4.3 ([51, Theorem A] or [73, Theorem IV.5.5]). *An enlargeable spin manifold admits no metric of positive scalar curvature.*

Note that there is a discrepancy in the definitions of enlargeable manifolds in [73] and [51] about whether the manifold, or a cover of it, is required to be spin. In any case, Theorem 3.4.3 only applies to manifolds that have a cover that is spin.

Theorems 3.4.1 and 3.4.3 require the manifolds to be spin. There exist many variations and generalizations of these result. However, in order to have some form of Dirac operator available, all of these results require the manifold, or a cover of it, to have some sort of generalized spin structure, such as spin^c, or pin structures. The next obstruction, which was developed by Schoen–Yau, is of different nature, and does not require the spin assumption.

Theorem 3.4.4 ([101]). *Let M^n be a closed oriented manifold that admits a metric of positive scalar curvature and let $H^{n-1} \subseteq M$ be an orientable stable minimal hypersurface. Then H also admits a metric of positive scalar curvature.*

Together with existence and non-existence results for hypersurfaces, one can apply Theorem 3.4.4 to show inductively that certain manifolds do not admit a metric of positive scalar curvature. The downside of this technique is that the existence of suitable smooth hypersurfaces in Theorem 3.4.4 is only guaranteed for $n \leq 7$. As an application one obtains that $M^n \# T^n$ for any closed manifold M^n with $n \leq 7$ does not admit a metric of positive scalar curvature, see [101, Corollary 2]. Note that a yet unpublished result [102] indicates, that this technique can be extended to all dimensions.

Finally, we note that there are obstructions exclusively for dimension 4, the *Seiberg-Witten invariants*. Every closed, oriented 4-manifold M admits a spin^c structure. If $b_2^+(M) \geq 2$, then for every spin^c structure ξ there exists an invariant $\mathrm{SW}(\xi) \in \mathbb{Z}$, which, roughly speaking, counts the number of solutions of certain non-linear equations involving the Dirac operator.

Theorem 3.4.5 ([120] or [87, Corollary 2.3.8]). *Let M be a closed, oriented 4-manifold with $b_2^+(M) \geq 2$. If M admits a metric of positive scalar curvautre, then $\mathrm{SW}(\xi) = 0$ for every spin^c structure ξ on M.*

For example, for any $m \geq 0$, the manifold $K3\#_m\overline{\mathbb{C}P}^2$ admits no metric of positive scalar curvature as it has a spin^c-structure with non-trivial Seiberg-Witten invariant, see e.g. [87, Corollaries 3.3.3 and 4.6.7].

On the positive side, a powerful tool to construct metrics of positive scalar curvature is surgery (for the definition of surgery see Section 4.1). It was developed independently by Schoen–Yau and Gromov–Lawson.

Theorem 3.4.6 ([101, Corollary 6], [50, Theorem A]). *Let M be a manifold that admits a complete metric of positive scalar curvature. Then any manifold obtained from M by surgery of codimension at least 3 also admits a complete metric of positive scalar curvature.*

We refer to Section 5.1 for a detailed discussion of Theorem 3.4.6.

A direct consequence of Theorem 3.4.6 with $q = n$ is the following (cf. Lemma 4.1.3):

Corollary 3.4.7. *Let M_1^n, M_2^n be manifolds of dimension at least 3 that admit a complete metric of positive scalar curvature. Then the connected sum $M_1 \# M_2$ admits a complete metric of positive scalar curvature.*

Theorem 3.4.6 has turned out to be very powerful to construct metrics of positive scalar curvature. For example, the following results are consequences of Theorem 3.4.6.

Theorem 3.4.8 ([50, Theorem B and subsequent comment]). *Every closed, simply-connected manifold in dimension 5, 6 and 7 admits a metric of positive scalar curvature.*

Theorem 3.4.9 ([50, Corollary C]). *Every, closed, simply-connected manifold of dimension at least 5, that is not spin, admits a metric of positive scalar curvature.*

Further, it was shown by Stolz [109], that in the simply-connected case the converse of Theorem 3.4.1 holds.

Theorem 3.4.10 ([109, Theorem A]). *A closed, simply-connected spin manifold M of dimension at least 5 admits a metric of positive scalar curvature if and only if its alpha-invariant $\alpha(M)$ vanishes.*

Together with Theorem 3.4.9, Theorem 3.4.10 provides a full classification of closed, simply-connected manifolds with a metric of positive scalar curvature in dimension at least 5. For the non-simply-connected case there is no such classification known. For conjectures in this direction we refer to [98, Section 1.3].

In view of the gluing techniques we will use for metrics of positive Ricci curvature, it is worth noting the following result by Gromov and Lawson.

Theorem 3.4.11 ([51, Theorem 5.7]). *Let M be a compact manifold that admits a metric with positive scalar curvature and positive mean curvature on the boundary. Then the double $M \cup_\partial (-M)$ admits a metric of positive scalar curvature.*

In fact, this result was generalized by Bär and Hanke (in a yet unpublished article). It can be seen as an analogue of Theorem 3.1.13 for positive scalar curvature.

Theorem 3.4.12 ([2, Theorem 42]). *Let M_1^n, M_2^n be manifolds that admit metrics of positive scalar curvature, so that there exists an isometry $\phi\colon \partial_c M_1 \to \partial_c M_2$ between compact boundary components $\partial_c M_1 \subseteq \partial M_1$ and $\partial_c M_2 \subseteq M_2$. If the mean curvatures $H_{\partial_c M_i}$ satisfy*

$$H_{\partial_c M_1} + H_{\partial_c M_2} \circ \phi \geq 0,$$

then $M_1 \cup_\phi M_2$ admits a metric of positive scalar curvature that coincides with the original metrics on M_1 and M_2 outside an arbitrarily small neighborhood of the gluing area.

A consequence of this gluing technique is the following result:

Corollary 3.4.13. *Let M^n, $n \geq 5$, be a closed, simply-connected manifold. Then $M \# (-M)$ admits a metric of positive scalar curvature.*

Proof. By [72, (1.1)], $M \setminus D^n$ admits a metric of positive sectional curvature and positive mean curvature on the boundary. Hence, by Theorem 3.4.11,

$$(M \setminus D^n) \cup_\partial (-M \setminus D^n) \cong M \# (-M)$$

admits a metric of positive scalar curvature.

Alternatively, if M is non-spin, then $M \# (-M)$ admits a metric of positive scalar curvature by Theorem 3.4.9. If M is spin, then, since α is a homomorphism on the spin bordism group, we have

$$\alpha(M \# (-M)) = \alpha(M) - \alpha(M) = 0,$$

so $M \# (-M)$ admits a metric of positive scalar curvature by Theorem 3.4.1. $\square$

Due to the great amount of flexibility of positive scalar curvature, there exist numerous other construction methods for such metrics. We mention one further tool, which appears in the context of group actions.

Theorem 3.4.14 ([74]). *Let M be a closed manifold that admits an effective action by compact, connected, non-abelian Lie group. Then M admits an invariant metric of positive scalar curvature.*

3.5 Positive Ricci Curvature in Low Dimensions

In this section we attempt to list all known examples of closed, simply-connected manifolds with a metric of positive Ricci curvature in dimensions up to 6.

In dimensions 2, since sec, Ric and scal are equal up to a constant factor, the classification of manifolds with a metric of positive sectional curvature carries over to the case of positive Ricci and scalar curvature, that is, the only manifolds admitting such a metric are S^2 and $\mathbb{R}P^2$. In dimension 3, Hamilton [54] showed that the only closed manifolds with a metric of positive Ricci curvature are spherical space forms, and hence these are also the only closed manifolds with a metric of positive sectional curvature in this dimension. For positive scalar curvature, it follows from Perelman's work on the Ricci flow with surgery [91, 92, 93], that a closed, orientable 3-manifold admits a metric of positive scalar curvature if and only if it is diffeomorphic to a connected sum of copies of $S^1 \times S^2$ and spherical space forms. In particular, the only closed, simply-connected manifolds in dimensions 2 and 3 that admit metrics of positive sectional, Ricci or scalar curvature, are the spheres S^2 and S^3 (these are also the only closed, simply-connected manifolds in these dimensions).

In dimension 4, we have the homogeneous spaces S^4, $S^2 \times S^2$ and $\mathbb{C}P^2$. By Theorem 3.1.7, the total space $S^2 \tilde{\times} S^2$ of the unique non-trivial linear S^2-bundle over S^2 also admits a metric of positive Ricci curvature. Further, we can construct metrics of positive Ricci curvature on any finite connected sum of copies of these manifolds.

Proposition 3.5.1. *Any finite connected sum of copies of $\pm\mathbb{C}P^2$, $S^2 \times S^2$ and $S^2 \tilde{\times} S^2$ admits a metric of positive Ricci curvature.*

Proof. First note that there exist diffeomorphisms

$$S^2 \tilde{\times} S^2 \cong \mathbb{C}P^2 \# (-\mathbb{C}P^2) \quad \text{and} \quad (S^2 \times S^2) \# (\pm\mathbb{C}P^2) \cong \mathbb{C}P^2 \# (-\mathbb{C}P^2) \# (\pm\mathbb{C}P^2),$$

see e.g. [69, Corollaries I.4.2 and I.4.3]. Thus, we only need to consider manifolds of the form

$$(S^2 \times S^2) \# \dots \# (S^2 \times S^2) \quad \text{and} \quad (\pm\mathbb{C}P^2) \# \dots \# (\pm\mathbb{C}P^2).$$

The first case is covered by Theorem 3.1.11, while the second case is covered by Perelman's gluing construction [90]. □

To the best of our knowledge, the manifolds in Proposition 3.5.1 are the only known examples in dimension 4. Indeed, any compact, simply-connected homogeneous space in dimension 4 is diffeomorphic to S^4, $\mathbb{C}P^2$ or $S^2 \times S^2$, see [46], while for biquotients the only additional diffeomorphism types appearing are $\mathbb{C}P^2 \# (-\mathbb{C}P^2)$ and $\mathbb{C}P^2 \# \mathbb{C}P^2$, see [37]. Further, every closed, simply-connected cohomogeneity one manifold in dimension 4 is diffeomorphic to S^4, $S^2 \times S^2$, $\mathbb{C}P^2$ or $\mathbb{C}P^2 \# (-\mathbb{C}P^2)$, see [61]. Finally, Fano varieties of (complex) dimension 2, also called *del Pezzo surfaces*, are either diffeomorphic to $S^2 \times S^2$ or to a finite connected sum of copies of $\pm\mathbb{C}P^2$, see e.g. [40, Corollary 8.1.17].

In dimension 5, closed, simply-connected manifolds were classified by Barden [3]. He defined closed, simply-connected 5-manifolds M_k for $k \in \{1, 2, \dots, \infty\}$ and X_j for $j \in \{-1, 0, \dots, \infty\}$ and showed that every closed, simply-connected 5-manifold M can uniquely be written as a connected sum

$$M \cong X_j \# M_{k_1} \# \dots \# M_{k_s}, \tag{3.5.1}$$

where $k_1 > 1$ and k_i divides k_{i+1} or $k_{i+1} = \infty$. The manifolds M_k are spin with $H_2(M_k) \cong (\mathbb{Z}/k)^2$ for $1 < k < \infty$, while the manifolds X_j are non-spin for $j \neq 0$ with $H_2(X_j) = (\mathbb{Z}/2^j)^2$ for $0 < j < \infty$. Further, we have $M_1 = X_0 = S^5$, $M_\infty = S^2 \times S^3$, $X_\infty = S^2 \tilde{\times} S^3$ and X_{-1} is the homogeneous space $\mathrm{SU}(3)/\mathrm{SO}(3)$, also called the *Wu manifold*, which satisfies $H_2(X_{-1}) \cong \mathbb{Z}/2$.

By Theorem 3.1.11, finite connected sums of copies of $M_\infty = S^2 \times S^3$ admit a metric of positive Ricci curvature. Further, Theorem 3.1.11 can be extended to connected sums of the form

$$(S^2 \tilde{\times} S^q)\#(S^2 \times S^q)\# \dots \#(S^2 \times S^q),$$

cf. [107, Theorem 5] or Section 5.5, showing that in dimension 5 we also obtain all manifolds of the form $X_\infty \# M_\infty \# \dots \# M_\infty$. In particular, all closed, simply-connected 5-manifolds with torsion-free homology admit a metric of positive Ricci curvature.

The only additional example we obtain via group actions is the Wu manifold X_{-1}, which is a homogeneous space. Indeed, every compact, simply-connected homogeneous space in dimension 5 is diffeomorphic to one of S^5, $S^2 \times S^3$ and $\mathrm{SU}(3)/\mathrm{SO}(3)$, see [46], and the only additional diffeomorphism type we obtain via biquotients is $S^2 \tilde{\times} S^3$, see [37]. Further, any closed, simply-connected 5-dimensional cohomogeneity one manifold is diffeomorphic to one of $S^2 \times S^3$, $S^2 \tilde{\times} S^3$ and $\mathrm{SU}(3)/\mathrm{SO}(3)$, see [61].

Concerning Sasakian geometry, rational homology spheres that admit a positive Sasakian structure have been classified by Kollár [70], see also [16, Corollary 10.2.20]. The classification asserts, that a closed, simply-connected 5-dimensional rational homology sphere admits a positive Sasakian structure if and only if M is one of

1. M_k, if k is not a multiple of 30,
2. nM_2 for all $n > 1$, or
3. $2M_3$, $3M_3$, $4M_3$, $2M_4$, $2M_5$.

Further, for manifolds with non-trivial second Betti number, there are positive Sasakian structures on the manifolds

4. $M_m\#(kM_\infty)$ for all $1 \leq k \leq 8$ and $m \geq 12$,
5. $M_m\#(2M_\infty)$ for all $m > 1$ not divisible by 3,
6. $M_m\#(3M_\infty)$ for all $m > 1$ not divisible by 2,
7. $M_m\#(4M_\infty)$ for all $m > 1$ not divisible by 2,
8. $(2M_2)\#(4M_\infty)$ and $(2M_2)\#(5M_\infty)$,
9. $M_m\#(kM_\infty)$ for all $m > 1$ and $6 \leq k \leq 8$,

see [16, Theorem 10.2.25 and Table B.4.2].

While Sasakian geometry produces many examples of 5-manifolds with a metric of positive Ricci curvature, there are limitations to this technique. In fact, besides a finite number of exceptional cases, the torsion subgroup of $H_2(M)$, where M is a closed, simply-connected 5-manifold with a positive Sasakian structure, is isomorphic to $(\mathbb{Z}/m)^2$ for some $m \in \mathbb{N}$, see [70, Theorem 1.4]. In particular, no manifold that contains X_{-1} as a summand in (3.5.1) can admit a positive Sasakian structure (which also follows from Proposition 3.2.8). Further, the manifolds $M_m\#(kM_\infty)$ cannot admit a positive Sasakian structure if $m \geq 12$ and $k \geq 9$.

Finally we consider the known examples in dimension 6. Here we follow [97, Section 5.1]. By Theorem 3.1.7, fiber bundles with homogeneous fibers admit a metric of positive Ricci curvature if both base and fiber admit a metric of positive Ricci curvature. We obtain the following list of manifolds that admit a metric of positive Ricci curvature:

1. Linear S^2-bundles over $B = \#_k(\pm\mathbb{C}P^2)$ (if $k = 0$ then $B = S^4$) or $B = \#_k(S^2 \times S^2)$ (the base B admits a metric of positive Ricci curvature by Proposition 3.5.1),
2. $S^3 \times S^3$,
3. $S^2 \tilde{\times} S^4$,
4. Projective bundles, i.e. $\mathbb{C}P^2$-bundles, over S^2.

Next, we list all closed simply-connected 6-dimensional homogeneous spaces, cohomogeneity one manifolds and biquotients that are not already contained in the list above, by using the classification results of Gorbatsevitch [46], Hoelscher [62], and DeVito [38], ordered by their second Betti number. These manifolds are:

5. S^6;
6. The oriented Grassmannian $\tilde{G}_2(\mathbb{R}^5) \cong \mathrm{SO}(5)/(\mathrm{SO}(3) \times \mathrm{SO}(2))$ (which is a homogeneous space);
7. The homogeneous space $\mathrm{SU}(3)/T^2$ and the biquotient $\mathrm{SU}(3)/\!\!/T^2$;
8. Biquotients of the form $(S^3)^3/\!\!/T^3$ that are diffeomorphic to a $(\mathbb{C}P^2\#\mathbb{C}P^2)$-bundle over S^2 or to one of 4 sporadic examples (see [38, Proposition 4.23], and note that the other families appearing in this proposition are already contained in the previous items).

Further, Fano 3-folds were classified by Iskovskih [65, 66] for $b_2 = 1$ and by Mori–Mukai [83, 84] for $b_2 \geq 2$. Their result can be summarized as follows (note that we omit the manifolds with $b_2 > 5$ as they are all diffeomorphic to a product of S^2 and a connected sum of copies of $\pm\mathbb{C}P^2$, i.e. they are contained in item 1):

9. 18 types of Fano 3-folds with $b_2 = 1$ and 83 types of Fano 3-folds with $2 \leq b_2 \leq 5$.

Finally, we have metrics of positive Ricci curvature on the following connected sums of sphere bundles:

10. $\#_k(S^2 \times S^4)\#_l(S^3 \times S^3)$, by a modification of Theorem 3.1.11, see [107, (5)] and Section 5.5,
11. $(S^2 \tilde{\times} S^4)\#_k(S^2 \times S^4)\#_{k+2}(S^3 \times S^3)$ by an application of Theorem 3.1.8, see [31].

Note that the manifolds in items 10 and 11, together with the linear S^2-bundles in item 1, are the only manifolds where the second Betti number is greater than 5.

In view of Question 3.2.3, dimensions 5 and 6 are of special interest, since there exist classifications in these dimensions (see Section 6.1 for the 6-dimensional classification) and further, since by Theorem 3.4.8 every closed, simply-connected manifold in these dimensions admits a metric of positive scalar curvature. Thus, potentially all closed, simply-connected 5- and 6-manifolds could admit a metric of positive Ricci curvature, and the classification results help to quantify any progress in this direction.

Surgery and Plumbing 4

This chapter provides an introduction to surgery and plumbing in Sections 4.1 and 4.2, respectively. Furthermore, in Section 4.3 we consider modifications of plumbing graphs that do not change the diffeomorphism type of the corresponding manifold and in Section 4.4 we calculate the fundamental group, characteristic classes and the cohomology ring of a manifold obtained by plumbing.

4.1 Surgery

The process of surgery relies on the fact, that the manifolds $S^{p-1} \times D^q$ and $D^p \times S^{q-1}$ have the same boundary $S^{p-1} \times S^{q-1}$.

Definition 4.1.1. Let $\varphi\colon S^{p-1} \times D^q \hookrightarrow \mathrm{Int}(M^n)$ be an embedding so that $n = p + q - 1$. Then

$$M_\varphi = (M \setminus \mathrm{im}(\varphi)^\circ) \cup_{\varphi|_{S^{p-1}\times S^{q-1}}} (D^p \times S^{q-1})$$

is the manifold obtained by surgery of dimension $(p-1)$, or surgery of codimension q, from M.

For every surgery operation there is a *dual* surgery operation: If $\varphi\colon S^{p-1} \times D^q \hookrightarrow \mathrm{Int}(M)$ is an embedding, then we have an embedding $\tilde{\varphi}\colon S^{q-1} \times D^p \hookrightarrow M_\varphi$ given by the identification of $S^{q-1} \times D^p$ with the second factor in

$$M_\varphi = (M \setminus \mathrm{im}(\varphi)^\circ) \cup_{S^{p-1}\times S^{q-1}} (D^p \times S^{q-1}).$$

It is then clear by construction that

$$(M_\varphi)_{\tilde{\varphi}} \cong M.$$

Hence, every surgery of dimension $(p-1)$ can be undone by its dual surgery, which is of dimension q.

Note that for $q = n$ we have $p = 1$, so

$$S^{p-1} \times D^q = S^0 \times D^n = D^n \sqcup D^n,$$

hence φ is the embedding of two copies of D^n. If these two copies of D^n are mapped to different connected components, we obtain the connected sum of these connected components.

Definition 4.1.2. Let M_1^n and M_2^n be connected manifolds. The *connected sum* of M_1 and M_2, denoted by $M_1 \# M_2$, is defined as follows: Let $\varphi_i\colon D^n \hookrightarrow \mathrm{Int}(M_i)$ be embeddings and assume that precisely one of φ_1, φ_2 is orientation-preserving if both M_1 and M_2 are orientable. Then we define

$$M_1 \# M_2 = (M_1 \setminus \varphi_1(D^n)^\circ) \cup_{\varphi_1\circ\varphi_2^{-1}} (M_2 \setminus \varphi_2(D^n)^\circ).$$

Similarly, if M_1^n and M_2^n are connected with connected and non-empty boundaries, then the *boundary connected sum* of M_1 and M_2, denoted by $M_1 \natural M_2$ is defined as follows: Let $\varphi_i \colon D^{n-1} \hookrightarrow \partial M_i$ be embeddings and assume that precisely one of φ_i is orientation-preserving if both ∂M_1 and ∂M_2 are orientable. Then we define

$$M_1 \natural M_2 = M_1 \cup_{\varphi_1 \circ \varphi_2^{-1}} M_2.$$

For the boundary connected sum, since we only glue along a subset of the boundary, we additionally need to straighten the corners to turn $M_1 \natural M_2$ into a smooth manifold. Then we have by construction that

$$\partial(M_1 \natural M_2) = \partial M_1 \# \partial M_2.$$

By the disc theorem of Palais [89, Theorem 5.5] any two embeddings of D^n into a manifold M^n, that are orientation-preserving if M is orientable, are isotopic. Thus, the connected sum operation and the boundary connected sum orientation are well-defined. Further, we have the following immediate consequences:

- $M_1 \natural M_2 \simeq M_1 \vee M_2$,
- $M^n \# S^n \cong M \cong M \natural D^n$, and
- $M = M_1 \# M_2$ or $M = M_1 \natural M_2$ is orientable if and only if both M_1 and M_2 are orientable, in which case M has an orientation that agrees with that of M_i on $M_i \setminus \operatorname{im} \varphi_i^\circ$.

Lemma 4.1.3. *Let $\varphi \colon D^n \sqcup D^n \cong S^0 \times D^n \hookrightarrow \mathrm{Int}(M^n)$ be an embedding. Let φ_i, $i = 1, 2$, be the restriction of φ to the i-th copy of D^n. Then M_φ is diffeomorphic to*

- *$M \# (S^1 \times S^{n-1})$, if M is connected, and precisely one of φ_i is orientation-preserving if M is orientable,*
- *$M \# (S^1 \tilde{\times} S^{n-1})$, if M is connected, and φ_i are both orientation-preserving or both orientation-reversing if M is orientable,*
- *$M_1 \# M_2$, if $M = M_1 \sqcup M_2$ with M_i connected and φ_i maps to M_i. If M is orientable, the orientation on M_i is chosen so that precisely one of φ_i is orientation-preserving.*

Proof. The last case follows directly from the construction of M_φ. So suppose that M is connected. Any two (orientation-preserving if M is orientable) embeddings of D^n into M are isotopic by [89, Theorem 5.5], hence we can assume that $\mathrm{im}(\varphi_i)$ are both contained in an embedded disc $D^n \subseteq M$. By fixing an orientation on this disc, we can speak of φ_i being orientation-preserving or orientation-reversing, even if M is not orientable.

The disc $D^n \subseteq M$ with the images of each φ_i removed is then diffeomorphic to a sphere S^n with 3 pairwise disjoint discs removed, which in turn is diffeomorphic to $(D^1 \times S^{n-1}) \setminus D^{n\circ}$ for an embedded disc $D^n \subseteq \mathrm{Int}(D^1 \times S^{n-1})$. Hence,

$$\begin{aligned} M_\varphi &\cong (M \setminus D^{n\circ}) \cup_{S^{n-1}} ((D^1 \times S^{n-1}) \setminus D^{n\circ}) \cup_{S^{n-1} \sqcup S^{n-1}} D^1 \times S^{n-1} \\ &\cong (M \setminus D^{n\circ}) \cup_{S^{n-1}} ((D^1 \times S^{n-1}) \cup_{S^{n-1} \sqcup S^{n-1}} (D^1 \times S^{n-1})) \setminus D^{n\circ} \\ &\cong M \# ((D^1 \times S^{n-1}) \cup_{S^{n-1} \sqcup S^{n-1}} (D^1 \times S^{n-1})). \end{aligned}$$

The gluing of the two copies of $D^1 \times S^{n-1}$ is either by the identity or a fixed orientation-reversing isometry α of S^{n-1}, depending on whether the embeddings φ_i preserve or reverse the orientation. If both gluings are via the identity or both via α, which is the case if precisely one of φ_i is orientation-preserving, we obtain $S^1 \times S^{n-1}$, and in the other case the non-trivial bundle $S^1 \tilde{\times} S^{n-1}$. □

If M_1^n and M_2^n are connected with $n \geq 3$, then it follows from van Kampen's theorem that

$$\pi_1(M_1 \# M_2) \cong \pi_1(M_1) * \pi_1(M_2). \tag{4.1.1}$$

We can also calculate the cohomology as follows. For that let $p_i\colon M_1\#M_2 \to M_i$ be the maps given by collapsing $M_{3-i} \setminus \varphi_{3-i}(D^n)^\circ$ to a point.

Lemma 4.1.4. *Let R be a commutative ring and let M_1^n, M_2^n be connected, closed and orientable over R. Then*

$$p_1^* \oplus p_2^*\colon H^*(M_1;R) \oplus H^*(M_2;R) \to H^*(M_1\#M_2;R)$$

is surjective with kernel generated by $1_{H^0(M_1;R)} - 1_{H^0(M_2;R)}$ and $[M_1, \partial M_1; R]^* - [M_2, \partial M_2; R]^*$. Further, if $c \in H^j(\mathrm{BSO}(p), R)$ is a characteristic class, that is stable if $j = n$, then

$$c(M_1\#M_2) = p_1^* c(M_1) + p_2^* c(M_2).$$

Proof. To simplify notation we will write $H^*(-)$ for $H^*(-;R)$. First consider the one point-union $M_1 \vee M_2$. We have maps $\check{p}_i\colon M_1 \vee M_2 \to M_i$ given by collapsing M_{3-i} and $\check{\iota}_i\colon M_i \hookrightarrow M_1 \vee M_2$ given by inclusion. By the Mayer-Vietoris sequence, the induced map

$$\check{p}_1^* \oplus \check{p}_2^*\colon H^*(M_1) \oplus H^*(M_2) \to H^*(M_1 \vee M_2)$$

is surjective with kernel generated by $1_{H^0(M_1)} - 1_{H^0(M_2)}$.

Now consider the long exact sequence in cohomology for the pair $(M_1\#M_2, S^{n-1})$, where we identified the gluing area for the connected sum with S^{n-1}. Then, for $i > 0$ we have

$$H^i(M_1\#M_2, S^{n-1}) \cong H^i(M_1\#M_2/S^{n-1})$$

and $M_1\#M_2/S^{n-1} \cong M_1 \vee M_2$. For $0 < i < n-1$ we have $H^i(S^{n-1}) = 0$, so the collapse map $\rho\colon M_1\#M_2 \to M_1 \vee M_2$ induces isomorphisms on cohomology. This also holds for $i = 0$ since both spaces are connected. Hence, it remains to consider the following part of the long exact sequence:

$$\begin{aligned} 0 \longrightarrow H^{n-1}(M_1 \vee M_2) \xrightarrow{\rho^*} H^{n-1}(M_1\#M_2) \longrightarrow H^{n-1}(S^{n-1}) \\ \longrightarrow H^n(M_1 \vee M_2) \xrightarrow{\rho^*} H^n(M_1\#M_2) \longrightarrow 0. \end{aligned}$$

The groups $H^n(M_1 \vee M_2)$ and $H^n(M_1\#M_2)$ are free and generated by $(\check{p}_1^*[M_1;R]^*, \check{p}_2^*[M_2;R]^*)$ and $[M_1\#M_2;R]^*$, respectively. Since the corresponding dual classes in homology are represented by the manifolds $\iota_1(M_1)$, $\iota_2(M_2)$ and $M_1\#M_2$, respectively, they satisfy $\rho_*[M_1\#M_2;R] = \iota_{1*}[M_1;R] + \iota_{2*}[M_2;R]$. Thus, we obtain in cohomology that

$$\rho^*(\check{p}_i^*[M_i;R]^*) = [M_1\#M_2;R]^*,$$

so ρ^* is surjective in degree n with kernel generated by $\check{p}_i^*([M_1;R]^* - [M_2;R]^*)$. Since $H^{n-1}(S^{n-1}) \cong R$, the map $H^{n-1}(S^{n-1}) \to H^n(M_1 \vee M_2)$ is injective and ρ^* is an isomorphism in degree $(n-1)$. Then the claim follows from the fact that $p_i = \check{p}_i \circ \rho$.

For the characteristic classes first assume that $j \neq n$ and denote by

$$\iota_i\colon M_i \setminus D^{n\circ} \hookrightarrow M_1\#M_2, \quad \iota_i'\colon M_i \setminus D^{n\circ} \hookrightarrow M_i$$

the inclusions. Then, by naturality, we have

$$\iota_i^* c(M_1\#M_2) = c(M_i \setminus D^{n\circ}), \quad {\iota_i'}^* c(M_i) = c(M_i \setminus D^{n\circ}).$$

Since $\iota_i' = p_i \circ \iota_i$ and since ${\iota_j'}^*$ is an isomorphism in degrees $j \neq n$, the map

$$({\iota_1'}^{*-1} \circ \iota_1^*, {\iota_2'}^{*-1} \circ \iota_2^*)\colon H^j(M_1\#M_2) \to H^j(M_1) \oplus H^j(M_2)$$

is the inverse of $p_1^* \oplus p_2^*$. Hence,

$$c(M_1\#M_2) = p_1^* \circ {\iota_1'}^{*-1} \circ \iota_1^* c(M_1\#M_2) + p_2^* \circ {\iota_2'}^{*-1} \circ \iota_2^* c(M_1\#M_2) = p_1^* c(M_1) + p_2^* c(M_2).$$

Finally, assume that $j = n$ and that c is stable. Let W be the boundary connected sum of $[0,1] \times M_1$ and $[0,1] \times M_2$ along the boundary components $\{1\} \times M_1$ and $\{1\} \times M_2$. Then

$$\partial W \cong (M_1 \sqcup M_2) \sqcup (M_1\#M_2).$$

Further, $W \simeq M_1 \vee M_2$ and the maps $\check{\iota}_1$, $\check{\iota}_2$ and ρ can be identified with the inclusions of the boundary components M_1, M_2 and $M_1\#M_2$, respectively. Since c is stable, we have

$$c(M_1\#M_2) = c(T(M_1\#M_2) \oplus \underline{\mathbb{R}}_{M_1\#M_2}) = \rho^* c(TW|_{\rho(M_1\#M_2)}) = \rho^* c(W).$$

Similarly,

$$c(M_i) = \check{\iota}_i^* c(W).$$

Since in degree n, the map $\check{p}_1^* \oplus \check{p}_2^*$ is an isomorphism with inverse $(\check{\iota}_1^*, \check{\iota}_2^*)$, it follows that

$$p_1^* c(M_1) + p_2^* c(M_2) = \rho^*(\check{p}_1^* c(M_1) + \check{p}_2^* c(M_2)) = \rho^* c(W) = c(M_1\#M_2).$$

□

Remark 4.1.5. The condition that c is stable if $j = n$ is necessary. In fact, if M^n is a closed an oriented manifold, then, by [82, Corollary 11.12], we have

$$\chi(M) = \langle e(M), [M] \rangle.$$

Further, we have for closed, connected and orientable manifolds M_1, M_2 that

$$\chi(M_1\#M_2) = \chi(M_1) + \chi(M_2) - (1 + (-1)^n)$$

by Lemma 4.1.4. Thus, if n is even, then the conclusion of Lemma 4.1.4 does not hold for the Euler class.

For higher surgeries there exists in general no simple description of the cohomology ring of M_φ in terms of the cohomology ring of M. We will consider this for the special case of plumbing in Section 4.4.

For a fiber bundle $E \xrightarrow{\pi} B^q$ with fiber F and structure group G we have a preferred embedding $\varphi\colon D^q \times F \hookrightarrow E$ defined as follows: Let $\iota\colon D^q \hookrightarrow B$ be an embedding, that is orientation-preserving if B is orientable. By the disc theorem of Palais [89, Theorem 5.5], the embedding ι is unique up to isotopy. By possibly shrinking the embedding, we can assume that $\iota(D^q)$ is contained in a local trivialization $(U_\alpha, \varphi_\alpha)$, so we obtain a diffeomorphism

$$\varphi_\pi\colon D^q \times F \to \pi^{-1}(\iota(D^q)).$$

For different choices of local trivializations the resulting diffeomorphisms are related to each other via transition functions $D^q \to G$. If G is connected, then, since D^q is contractible, all these transition functions are smoothly homotopic to the constant map 1_G, hence φ_π, considered as an embedding $D^q \times F \hookrightarrow E$, is unique up to isotopy.

Definition 4.1.6. Let $E \xrightarrow{\pi} B^n$ be a fiber bundle with fiber F and structure group G. If G is connected, then the embedding $\varphi_\pi\colon D^q \times F \hookrightarrow E$ is called the *standard embedding* of $D^q \times F$ into E.

From the definition it follows that a standard embedding $\varphi_\pi\colon D^q \times F \to E$ is a bundle map.

For an oriented linear sphere bundle $E \xrightarrow{\pi} B^q$ with fiber S^{p-1}, i.e. a fiber bundle with fiber S^{p-1} and structure group $\mathrm{SO}(p)$, we obtain an embedding $\varphi_\pi\colon D^q \times S^{p-1} \hookrightarrow E$, along which we can perform surgery. Surgeries of this form will play an important role in the subsequent sections.

By using standard embeddings, we can define a generalization of the connected sum. For that, let $\eta\colon \mathrm{Int}(D^q) \setminus \{0\} \to \mathrm{Int}(D^q) \setminus \{0\}$ be the diffeomorphism

$$(x_1, \ldots, x_q) \mapsto \frac{1-|x|}{|x|}(-x_1, x_2, \ldots, x_q).$$

Definition 4.1.7. let $E_1 \xrightarrow{\pi_1} B_1^q$, $E_2 \xrightarrow{\pi_2} B_2^q$ be fiber bundles, both with fiber F and connected structure group G, so that bases, fibers and total space are oriented compatibly. The *fiber connected sum* of π_1 and π_2 is the fiber bundle with fiber F and structure group G, whose total space E is obtained from $(E_1 \setminus \varphi_{\pi_1}(\{0\} \times F)) \cup (E_2 \setminus \varphi_{\pi_2}(\{0\} \times F))$ by identifying $\varphi_{\pi_1}(\mathrm{Int}\, D^q \setminus \{0\} \times F)$ and $\varphi_{\pi_2}(\mathrm{Int}\, D^q \setminus \{0\} \times F)$ via

$$\varphi_{\pi_2} \circ (\eta \times \mathrm{id}_F) \circ \varphi_{\pi_1}^{-1},$$

base space $B = B_1 \# B_2$ and bundle projection π induced from π_1 and π_2. The local trivializations are obtained from those local trivializations (U, φ) of π_i so that $\pi_i(\varphi_{\pi_i}(0, y)) \notin U$ for $y \in F$.

Equivalently, we can construct the space E as the space obtained from

$$(E_1 \setminus \varphi_{\pi_1}(\mathrm{Int}\, D^q \times F)) \cup (E_2 \setminus \varphi_{\pi_2}(\mathrm{Int}\, D^q \times F))$$

by gluing the boundaries along the diffeomorphism

$$\varphi_{\pi_2} \circ (r_{S^{q-1}} \times \mathrm{id}_F) \circ \varphi_{\pi_1}^{-1},$$

where $r_{S^{q-1}}\colon S^{q-1} \times S^{q-1}$ is the diffeomorphism

$$r_{S^{q-1}}(x_1, \ldots, x_q) = (-x_1, x_2, \ldots, x_q).$$

Indeed, the restriction of η to the sphere of radius $\frac{1}{2}$, denoted by $S^{q-1}(\frac{1}{2})$, is the map $r_{S^{q-1}}$ and we obtain this description of E by cutting it along $S^{q-1}(\frac{1}{2}) \times F$.

In fact, by considering a manifold as a fiber bundle with fiber $\{pt\}$ and structure group the trivial group, we recover the definition of the connected sum.

4.2 Plumbing

The plumbing construction was introduced by Milnor [81] to construct manifolds with prescribed intersection form. We also refer to [59], [20, Section 5] and [33, Section 2] for further details on plumbing.

Let $D^p \hookrightarrow \overline{E}_1 \xrightarrow{\pi_1} B_1^q$ and $D^q \hookrightarrow \overline{E}_2 \xrightarrow{\pi_2} B_2^p$ be oriented linear disc bundles over oriented and connected manifolds B_i, such that fibers, base and total space are oriented compatibly. Let $\varphi_{\pi_1}\colon D^q \times D^p \hookrightarrow \overline{E}_1$ and $\varphi_{\pi_2}\colon D^p \times D^q \hookrightarrow \overline{E}_2$ be standard embeddings. Now define the diffeomorphism $I^{\pm}_{p,q}\colon D^p \times D^q \to D^q \times D^p$ by

$$I^{\pm}_{p,q}(x_1, \ldots, x_p, y_1, \ldots, y_q) = (\pm y_1, y_2, \ldots, y_q, \pm x_1, x_2, \ldots, x_p).$$

Definition 4.2.1. The manifold obtained by *plumbing* $\overline{E}_1$ and $\overline{E}_2$, denoted by $\overline{E}_1 \Box \overline{E}_2$, is obtained from $\overline{E}_1 \sqcup \overline{E}_2$ by identifying $\varphi_{\pi_1}(D^q \times D^p)$ and $\varphi_{\pi_2}(D^p \times D^q)$ via $\varphi_{\pi_1} \circ I^{\pm}_{p,q} \circ \varphi_{\pi_2}^{-1}$.

The space $\overline{E}_1 \Box \overline{E}_2$ indeed has a manifold structure after smoothing out the corners which arise at the boundary of the identification area. Since standard embeddings are unique up to isotopy, the diffeomorphism type of $\overline{E}_1 \Box \overline{E}_2$ only depends on the signs we use in the definition of $I^{\pm}_{p,q}$. We then say that $\overline{E}_1 \square \overline{E}_2$ is the manifold obtained by plumbing $\overline{E}_1$ and $\overline{E}_2$ with sign ± 1. Independent of the sign, the map $I^{\pm}_{p,q}$ is orientation-preserving if p or q is even and orientation-reversing if both p and q are odd. Hence, the manifold $\overline{E}_1 \square \overline{E}_2$ is oriented compatibly with $\overline{E}_1$ and $\overline{E}_2$ if p or q is even, and $\overline{E}_1 \square \overline{E}_2$ is oriented compatibly with $\overline{E}_1$ and $-\overline{E}_2$ if both p and q are odd.

We will be interested in the boundaries of plumbings. If $\overline{E}$ is the total space of a disc bundle, we denote by E the total space of its sphere bundle. From the definition we obtain

$$\partial(\overline{E}_1 \square \overline{E}_2) \cong (E_1 \setminus \varphi_{\pi_1}(D^q \times S^{p-1})^\circ) \cup_{\varphi_{\pi_1} \circ I^{\pm}_{p,q} \circ \varphi_{\pi_2}^{-1}} (E_2 \setminus \varphi_{\pi_2}(D^p \times S^{q-1})^\circ). \tag{4.2.1}$$

In particular, if one of the $\overline{E}_i$, say $\overline{E}_2$, is the trivial bundle $S^p \times D^q$, then $\partial(\overline{E}_1 \square \overline{E}_2)$ is obtained by surgery on E_1 along φ_{π_1}. The map

$$I^{\pm}_{p,q}|_{S^{p-1} \times S^{q-1}} \colon S^{p-1} \times S^{q-1} \to S^{q-1} \times S^{p-1}$$

is orientation-preserving if and only if $(p-1)(q-1)$ is even. Hence, the boundary $\partial(\overline{E}_1 \square \overline{E}_2)$ is oriented compatibly with with E_1 and E_2 if and only if $(p-1)(q-1)$ is odd, i.e. if both p and q are even.

We can repeat the process of plumbing by choosing multiple standard embeddings disjoint from each other. A manifold obtained by plumbing multiple disc bundles can then be characterized by a labeled bipartite graph $G = (U, V, E, \pi, \delta)$, where U and V are the sets of vertices, which are assumed to be finite, $E \subseteq U \times V$ is the set of edges, $\delta \colon E \to \{\pm 1\}$ is a function, and π assigns to each vertex an oriented linear disc bundle in the following way: For fixed $p, q \in \mathbb{N}$ we assign to each $u \in U$ an oriented linear disc bundle $D^p \hookrightarrow \overline{E}_u \xrightarrow{\pi_u} B^q_u$ and to each $v \in V$ an oriented linear disc bundle $D^q \hookrightarrow \overline{E}_v \xrightarrow{\pi_v} B^p_v$ with connected and oriented bases, fibers and total spaces that are all oriented compatibly. We denote the standard embeddings of π_u and π_v corresponding to the edge $e = (u, v)$ by $\varphi_{(u,v)}$ and $\varphi_{(v,u)}$, respectively.

Definition 4.2.2. A labeled bipartite graph $G = (U, V, E, \pi, \delta)$ of this form is called a *geometric plumbing graph.*

We define $\overline{M}_G$ as the boundary connected sum of the manifolds obtained by plumbing the bundles π_u and π_v according to each connected component of the graph G, where each edge e corresponds to plumbing with sign $\delta(e)$. We set $M_G = \partial \overline{M}_G$. To define an orientation, if one of p and q is odd, we need to specify the set of vertices, i.e. U or V, from which the orientation should be induced. For that we will always choose V in the following.

Lemma 4.2.3. *If every connected component of G is simply-connected, then the manifold $\overline{M}_G$ is homotopy equivalent to the space*

$$\bigvee_{u \in U} B_u \bigvee_{v \in V} B_v.$$

Proof. First suppose that G is connected. By shrinking the fibers of the disc bundles to their zero-sections, we obtain a homotopy equivalence from $\overline{M}_G$ to the space obtained from $\bigsqcup_{u \in U} B_u \bigsqcup_{v \in V} B_v$ by identifying a point in B_u with one in B_v whenever $(u, v) \in E$. Since G is simply-connected, after picking a root, it is a tree and we homotope all these identification points to a fixed point in the base manifold of the root. In this way we obtain the space $\bigvee_{u \in U} B_u \bigvee_{v \in V} B_v$.

If G is not connected, then the conclusion holds for each boundary component, and hence, since the boundary connected sum of two manifolds is homotopy equivalent to their one-point-union, the claim follows. ☐

The manifolds $\overline{M}_G$ and M_G have been studied in the case where $p = q$ and the base manifolds are spheres, see e.g. [20, Chapter V]. For example, all homotopy spheres that bound parallelizable manifolds can be constructed in this way, see [121, Proposition 1.5]. In the following we will consider the case where p and q may differ, but all bundles π_v, $v \in V$, are trivial with $B_v = S^p$.

4.3 Modifications of Plumbing Graphs

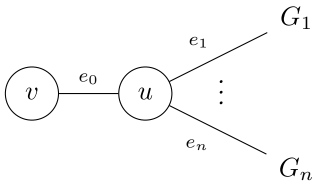
In this section G denotes an arbitrary geometric plumbing graph. Our goal is to simplify the graph data as much as possibly without changing the diffeomorphism type of M_G.

Proposition 4.3.1 (Sign of a Separating Edge). *Suppose G is of the form*

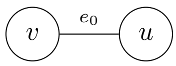
$$G_1 \;\overset{e}{\rule[0.5ex]{2em}{0.4pt}}\; G_2$$

with subgraphs G_1 and G_2 of G. Let G' be the graph obtained from G by reversing the sign of e and by reversing all base and fiber orientations of the bundles associated to vertices in G_2. Then M_G is diffeomorphic to $M_{G'}$.

Proposition 4.3.2 (Vertex of Degree 1). *Suppose G is of the form*

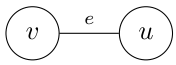


with subgraphs $G_1, \ldots, G_n$ of G. Let G_0 be the graph



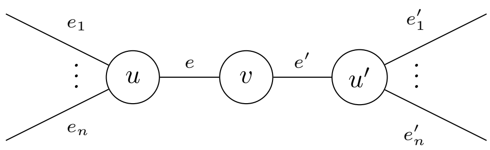
with corresponding restrictions of δ and π. If π_v is trivial with $B_v = S^p$, then M_G is diffeomorphic to $M_{G_0} \# M_{G_1} \# \ldots \# M_{G_n}$, i.e. we can replace G by the disjoint union of the subgraphs $G_0, \ldots, G_n$.

Proposition 4.3.3. *Let G be the graph*

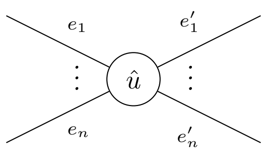


where π_v is trivial with $B_v = S^p$ and $B_u = S^q$. Then $M_G \cong S^{p+q-1}$.

Proposition 4.3.4 (Vertex of Degree 2). *Suppose G is of the following form:*



Suppose π_v is trivial with $B_v = S^p$ and $\delta(e) = 1$, $\delta(e') = -1$. Define $\pi_{\hat{u}}$ as the fiber connected sum of π_u and $\pi_{u'}$. Then we can replace G by the following graph without changing the diffeomorphism type of M_G.



Remark 4.3.5. 1. Proposition 4.3.2 generalizes [33, Proposition 2.6] and [99, Satz 5.9 (A)].

2. By Proposition 4.3.1, we can, after possibly reversing base and fiber orientations in the corresponding subgraphs, apply Proposition 4.3.4 for any possible signs of e and e'.

Proposition 4.3.1 directly follows from the fact, that, by reversing the orientations as assumed, the gluing maps for the plumbings do not change. Proposition 4.3.4 follows from [33, Lemma 2.10]. For convenience we give the proof below.

Proof of Proposition 4.3.4, cf. [33, Lemma 2.10]. Denote the graph

$$\begin{tikzcd} \textcircled{u} \arrow[r, dash, "e"] & \textcircled{v} \arrow[r, dash, "e'"] & \textcircled{u'} \end{tikzcd}$$

by G'. Since $B_v = S^p$ and the bundle π_v is trivial, we have

$$\begin{aligned} E_v \setminus \left(\varphi_{(v,u)}(D^p \times S^{q-1})^\circ \sqcup \varphi_{(v,u')}(D^p \times S^{q-1})^\circ\right) &\cong (S^p \setminus (D^p \sqcup D^p)^\circ) \times S^{q-1} \\ &\cong ([0,1] \times S^{p-1}) \times S^{q-1} \end{aligned}$$

The restrictions of the embeddings $\varphi_{(v,u)}$ and $\varphi_{(v,u')}$ to the boundary $S^{p-1} \times S^{q-1}$ can be identified with the inclusions

$$S^{p-1} \times S^{q-1} \cong \{0\} \times S^{p-1} \times S^{q-1} \xhookrightarrow{\iota_0 \times \mathrm{id}_{S^{p-1}} \times \mathrm{id}_{S^{q-1}}} [0,1] \times S^{p-1} \times S^{q-1}$$

and

$$S^{p-1} \times S^{q-1} \cong \{1\} \times S^{p-1} \times S^{q-1} \xhookrightarrow{\iota_1 \times r_{S^{p-1}} \times \mathrm{id}_{S^{q-1}}} [0,1] \times S^{p-1} \times S^{q-1},$$

where $\iota_i \colon \{i\} \hookrightarrow [0,1]$ are the inclusions.

Hence,

$$\begin{aligned} M_{G'} \cong & E_u \setminus \varphi_{(u,v)}(D^q \times S^{p-1})^\circ \cup_{\varphi_{\pi_u} \circ I^+_{p,q}} [0,1] \times S^{p-1} \times S^{q-1} \\ & \cup_{(\iota_1 \times r_{S^{p-1}} \times \mathrm{id}_{S^{q-1}}) \circ I^-_{q,p} \circ \varphi^{-1}_{\pi_{u'}}} E_{u'} \setminus \varphi_{\pi_{(u',v)}}(D^q \times S^{p-1})^\circ \\ \cong & E_u \setminus \varphi_{(u,v)}(D^q \times S^{p-1})^\circ \cup_{\varphi_{\pi_u} \circ I^+_{p,q} \circ (r_{S^{p-1}} \times \mathrm{id}_{S^{q-1}}) \circ I^-_{q,p} \circ \varphi^{-1}_{\pi_{u'}}} E_{u'} \setminus \varphi_{\pi_{(u',v)}}(D^q \times S^{p-1})^\circ \\ \cong & E_u \setminus \varphi_{(u,v)}(D^q \times S^{p-1})^\circ \cup_{\varphi_{\pi_u} \circ (r_{S^{q-1}} \times \mathrm{id}_{S^{p-1}}) \circ \varphi^{-1}_{\pi_{u'}}} E_{u'} \setminus \varphi_{\pi_{(u',v)}}(D^q \times S^{p-1})^\circ, \end{aligned}$$

which is the fiber connected sum of π_u and $\pi_{u'}$. □

Proof of Proposition 4.3.3. By possibly reversing the orientations on base and fibers of π_v, we can assume that $\delta(e) = 1$. Let $T \colon S^{q-1} \to \mathrm{SO}(p)$ be the clutching function for π_u. Then we have

$$M_G \cong (S^{q-1} \times D^p) \cup_{\tilde{T}} (D^q \times S^{p-1}).$$

Define the diffeomorphism $\bar{T} \colon S^{q-1} \times D^p \to S^{q-1} \times D^p$ by

$$\bar{T}(x, y) = (x, T_x(y)),$$

so $\bar{T}|_{S^{q-1} \times S^{p-1}} = \tilde{T}$. Hence,

$$M_G \cong \bar{T}(S^{q-1} \times D^p) \cup_{\tilde{T}} (D^q \times S^{p-1}) = (S^{q-1} \times D^p) \cup_{\mathrm{id}} (D^q \times S^{p-1}).$$

Now consider the product $D^q \times D^p$, which, after smoothing the corners, is diffeomorphic to D^{p+q}. We have

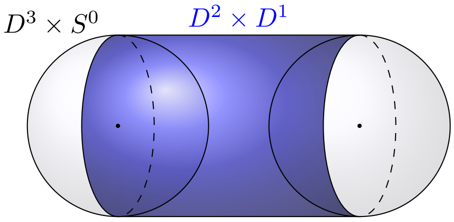
$$\partial(D^q \times D^p) = (S^{q-1} \times D^p) \cup_\partial (D^q \times S^{p-1})$$

and $\partial D^{p+q} = S^{p+q-1}$. Hence,

$$(S^{q-1} \times D^p) \cup_\partial (D^q \times S^{p-1}) \cong S^{p+q-1}.$$

□

Figure 4.1: A trivial embedding of $D^3 \times S^0$



For the proof of Proposition 4.3.2 we need the following notion.

Definition 4.3.6. An embedding $\varphi\colon D^q \times S^{p-1} \hookrightarrow M^{p+q-1}$ into a manifold M is called *trivial*, if there is an embedding $\bar{\varphi}\colon D^{q-1} \times D^p \hookrightarrow M$ so that $\varphi|_{D^{q-1}\times S^{p-1}} = \bar{\varphi}|_{D^{q-1}\times S^{p-1}}$.

Here we consider $D^{q-1} \subseteq D^q$ according to the embedding $\mathbb{R}^{q-1} \cong \mathbb{R}^{q-1} \times \{0\} \subseteq \mathbb{R}^q$.

Lemma 4.3.7. *Let M be connected. Then any two trivial embeddings $\varphi_1, \varphi_2\colon D^q \times S^{p-1} \hookrightarrow M$, that are orientation preserving if M is orientable, are isotopic.*

Proof. First note, that we can assume that both $\bar{\varphi}_1$ and $\bar{\varphi}_2$ are orientation preserving if M is orientable, since otherwise, we can replace $\bar{\varphi}_i$ by the map

$$(x_1, \dots, x_{q-1}, y) \mapsto \bar{\varphi}_i(x_1, \dots, x_{q-2}, -x_{q-1}, y)$$

and φ_i by the map

$$(x_1, \dots, x_q, y) \mapsto \varphi_i(x_1, \dots, x_{q-2}, -x_{q-1}, -x_q, y),$$

which is clearly isotopic to φ_i.

By the disc theorem of Palais [89, Theorem 5.5], any two embeddings of D^p into M are ambient isotopic, where we require them to be orientation-preserving if M is orientable and $p = \dim(M)$. It follows, that, after applying an ambient isotopy, we can assume that $\bar{\varphi}_1|_{\{0\}\times D^p} = \bar{\varphi}_2|_{\{0\}\times D^p}$.

If M is non-orientable, we can introduce a local orientation by enlarging the image of one of $\bar{\varphi}_i$ to a ball D^{p+q-1}, so that φ_i, $\bar{\varphi}_i$ are all contained in this ball. If one of φ_i is orientation-reversing with respect to the orientation of D^{p+q-1}, we can apply an isotopy of this ball that reverses the orientation, which exists by the disc theorem of Palais. If one of $\bar{\varphi}_i$ is orientation-reversing we modify it as in the beginning of the proof. Hence, we can assume that φ_i, $\bar{\varphi}_i$ are all orientation-preserving.

By the uniqueness of tubular neighborhoods, see e.g. [58, Theorem 4.5.3], after applying an isotopy to one of $\bar{\varphi}_i$, there is a smooth map $\phi\colon D^p \to \mathrm{GL}_+(q)$, so that $\bar{\varphi}_1(x, y) = \bar{\varphi}_2(\phi_y(x), y)$. Since D^p is contractible, there exists a smooth homotopy of ϕ to the constant map $\phi \equiv \mathrm{id}_{\mathbb{R}^q}$. This yields an isotopy of $\bar{\varphi}_2$ so that we can assume $\bar{\varphi}_1 = \bar{\varphi}_2$. By the isotopy extension theorem, see e.g. [58, Theorem 8.1.3], this isotopy extends to a diffeotopy of M, in particular we can assume that after the isotopy the assumption $\varphi_i|_{D^{q-1}\times S^{p-1}} = \bar{\varphi}_i|_{D^{q-1}\times S^{p-1}}$ still holds.

Again by the uniqueness of tubular neighborhoods, there is $\phi\colon S^{p-1} \to \mathrm{GL}_+(q)$ so that, after applying an isotopy, $\varphi_1(x, y) = \varphi_2(\phi_y(x), y)$. This isotopy can be chosen so that the condition $\varphi_1|_{D^{q-1}\times S^{p-1}} = \varphi_2|_{D^{q-1}\times S^{p-1}}$ is preserved (cf. [58, Proof of Theorem 4.5.3]), hence ϕ_y fixes $\mathbb{R}^{q-1} \subseteq \mathbb{R}^q$ for all $y \in S^{p-1}$. Then $t \mapsto (1-t)\phi + t\mathrm{id}_{\mathbb{R}^q}$ is a smooth homotopy of ϕ to the constant map $\mathrm{id}_{\mathbb{R}^q}$, so φ_1 is isotopic to φ_2. □

Lemma 4.3.8. *Let $\varphi_1\colon D^q \times S^{p-1} \hookrightarrow M_1^{p+q-1}$ and $\varphi_2\colon D^p \times S^{q-1} \hookrightarrow M_2^{p+q-1}$ be embeddings into connected manifolds and let M be the manifold*

$$M = M_1 \setminus \mathrm{im}(\varphi_1)^\circ \cup_{\varphi_1 \circ I^+_{p,q} \circ \varphi_2^{-1}} M_2 \setminus \mathrm{im}(\varphi_2)^\circ.$$

Suppose that one of φ_1, φ_2 is trivial and that both φ_1 and φ_2 are orientation-preserving if both M_1 and M_2 are orientable. Then M is diffeomorphic to $M_1 \# M_2$ if one of M_1 and M_2 is non-orientable and to $M_1 \# (-1)^{pq-p-q} M_2$ otherwise.

Proof. By possibly interchanging the roles of φ_1 and φ_2 we can assume that φ_1 is trivial. We decompose

$$M_1 \cong M_1 \# S^{p+q-1} \cong M_1 \# ((-1)^q (S^{q-1} \times D^p) \cup_\partial (D^q \times S^{p-1})),$$

the orientation on $S^{q-1} \times D^p$ is chosen so that the induced orientations on the boundary of $S^{q-1} \times D^p$ and $D^q \times S^{p-1}$ are opposite to each other, so that after gluing both pieces together we have a well-defined orientation.

The embedding $\varphi\colon D^q \times S^{p-1} \hookrightarrow (S^{q-1} \times D^p) \cup_\partial (D^q \times S^{p-1})$ of the second factor is trivial: The map $\bar{\varphi}$ is given by

$$D^{q-1} \times D^p \cong (D^{q-1} \times D^p) \cup_{D^{q-1} \times S^{p-1}} (D^q_+ \times S^{p-1}) \hookrightarrow (S^{q-1} \times D^p) \cup_\partial (D^q \times S^{p-1}),$$

where $D^q_+ \subseteq D^q$ denotes the upper half-ball and we embed $D^{q-1} \subseteq S^{q-1}$ as the upper half-sphere into the first factor.

By Lemma 4.3.7, the embeddings φ and φ_1 are isotopic, hence

$$M_1 \setminus \operatorname{im}(\varphi_1)^\circ \cup_{\varphi_1 \circ I^+_{p,q} \circ \varphi_2^{-1}} M_2 \setminus \operatorname{im}(\varphi_2)^\circ \cong M_1 \# ((-1)^q S^{q-1} \times D^p) \cup_{I^+_{p,q} \circ \varphi_2^{-1}} M_2 \setminus \operatorname{im}(\varphi_2)^\circ$$

The map $I^+_{p,q}\colon S^{p-1} \times S^{q-1} \to S^{q-1} \times S^{p-1}$ is orientation-preserving if and only if $(p-1)(q-1)$ is even. Hence, if M_2 is orientable, and if we equip it with the orientation $-(-1)^{(p-1)(q-1)} = (-1)^{pq-p-q}$, we have a well-defined orientation after gluing, so we obtain

$$\begin{aligned} M_1 \setminus \operatorname{im}(\varphi_1)^\circ &\cup_{\varphi_1 \circ I^+_{p,q} \circ \varphi_2^{-1}} M_2 \setminus \operatorname{im}(\varphi_2)^\circ \\ &\cong M_1 \# ((-1)^q S^{q-1} \times D^p \cup_{I^+_{p,q} \circ \varphi_2^{-1}} (-1)^{pq-p-q} M_2 \setminus \operatorname{im}(\varphi_2)^\circ) \\ &\cong M_1 \# (-1)^{pq-p-q} M_2. \end{aligned}$$

□

Proof of Proposition 4.3.2. First note that we can assume that $\delta(e_0) = \delta(e_1) = \cdots = \delta(e_n) = 1$ by Proposition 4.3.1. We then have

$$\begin{aligned} M_{G_0} \cong& E_v \setminus \varphi_{(v,u)}(D^p \times S^{q-1})^\circ \cup_{\varphi_{(v,u)} \circ I^+_{q,p} \circ \varphi^{-1}_{(u,v)}} E_u \setminus \varphi_{(v,u)}(D^q \times S^{p-1})^\circ \\ \cong& D^p \times S^{q-1} \cup_{I^+_{q,p} \circ \varphi^{-1}_{(u,v)}} E_u \setminus \varphi_{(v,u)}(D^q \times S^{p-1})^\circ \\ \cong& D^p \times S^{q-1} \cup_{I^+_{p,q}} ([0,1] \times S^{q-1} \times S^{p-1}) \\ & \cup_{(\iota_1 \times \mathrm{id}_{S^{q-1}} \times \mathrm{id}_{S^{p-1}}) \circ \varphi^{-1}_{(u,v)}} E_u \setminus \varphi_{(v,u)}(D^q \times S^{p-1})^\circ. \end{aligned} \tag{4.3.1}$$

Since we can view the mid-part $[0,1] \times S^{q-1} \times S^{p-1}$ as part of the bundle π_u, the manifold M_G is now obtained from M_{G_0} by cutting out standard embeddings $\varphi_{(u,v_i)}\colon D^q \times S^{p-1} \hookrightarrow (0,1) \times S^{q-1} \times S^{p-1}$, which are of the form $\iota_i \times \mathrm{id}_{S^{p-1}}$ for embeddings $\iota_i\colon D^q \hookrightarrow (0,1) \times S^{q-1}$, and gluing in the corresponding parts of M_{G_i}. We now show that all these embeddings are trivial.

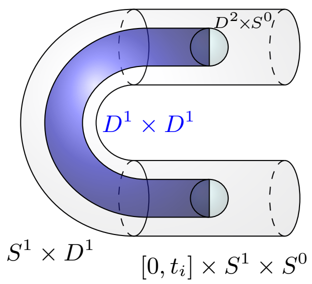
For that we isotope ι_i so that $\iota_i|_{D^{q-1}}\colon D^{q-1} \xrightarrow{\cong} \{t_i\} \times D^{q-1}_i$ for an embedded disc $D^{q-1}_i \subseteq S^{q-1}$. We choose the discs D^{q-1}_i so that they do not intersect. Then we define embeddings $\bar{\varphi}_i\colon D^{q-1} \times D^p \hookrightarrow M_{G_0}$ by identifying

$$D^{q-1} \times D^p \cong (D^{q-1}_i \times D^p) \cup_{D^{q-1}_i \times S^{p-1}} ([0,t_i] \times D^{q-1}_i \times S^{p-1})$$

and mapping it to

$$S^{q-1} \times D^p \cup_{S^{q-1} \times S^{p-1}} ([0,1] \times S^{q-1} \times S^{p-1})$$

FIGURE 4.2: The embeddings $\varphi_{(u,v_i)}$ are trivial



via the obvious inclusions on each part, cf. Figure 4.2. Since the discs D_i^{q-1} do not intersect each other, the maps $\bar{\varphi}_i$ have pairwise disjoint image. Hence, if we equip each M_{G_i} with the orientation induced from each E_v, which equals the orientation induced from $(-1)^{pq-p-q}E_u$, it follows from Lemma 4.3.8, that

$$M_G \cong M_{G_0} \# M_{G_1} \# \dots \# M_{G_n}.$$

□

4.4 Topology of Manifolds Obtained by Plumbing

The goal of this section is to determine the fundamental group and the cohomology ring of M_G, and its characteristic classes.

Lemma 4.4.1. *Suppose $p, q > 2$ and that every connected component of G and every B_u and B_v are simply-connected. Then M_G is simply-connected.*

Proof. By (4.1.1) we can assume that G is connected. The manifold M_G is obtained by gluing the spaces

$$E_u \setminus \left(\bigsqcup_{(u,v)\in E} \varphi_{(u,v)}(D^q \times S^{p-1})^\circ \right) \quad \text{and} \quad E_v \setminus \left(\bigsqcup_{(u,v)\in E} \varphi_{(v,u)}(D^p \times S^{q-1})^\circ \right)$$

according to G. These spaces are fiber bundles with base spaces B_u or B_v with a finite number of discs removed. Since, $p, q > 2$, it follows from the long exact sequence for fiber bundles (Lemma B.1.4), that all these spaces are simply-connected.

The graph is simply-connected, hence, after choosing a root, it is a tree. If M_k is the manifold obtained by gluing according the subgraph of G consisting of all vertices of distance at most k from the root, then it follows inductively from van Kampen's theorem, that M_k is simply-connected. For k large enough we have $M_k = M_G$, and hence M_G is simply-connected. □

Remark 4.4.2. If one does not require that G is simply-connected in Lemma 4.4.1, then, if G is connected, one can show, by using the groupoid version of van Kampen's theorem, that $\pi_1(M_G) \cong \pi_1(G)$.

For the cohomology we fix a ring R and we consider (co-)homology with coefficients in R. If we assume that for every $u \in U$ the Euler class $e(\pi_u)$ vanishes, then, by Proposition B.3.4, see (B.3.3), there exists an element $a_u \in H^{p-1}(B_u)$, so that

$$H^i(E_u) = \pi_u^*(H^i(B_u)) \oplus \theta_{a_u}(H^{i-p+1}(B_u)). \tag{4.4.1}$$

For a geometric plumbing graph G we make the following assumption:

G is simply-connected and either no vertex $v \in V$ is a leaf, or G is of the form $\textcircled{v} \overset{e}{\text{——}} \textcircled{u}$.
(4.4.2)

Theorem 4.4.3. *Let G be a geometric plumbing graph with $p > 1$ satisfying (4.4.2), so that $e(\pi_u) = 0$ and B_u is closed for all $u \in U$, and each bundle π_v is trivial with $B_v \cong S^p$. Then*

$$H^i(M_G) \cong \begin{cases} \left(\sum_{u\in U} 1_{H^0(E_u)}\right) R \cong R, & i = 0; \\ \bigoplus_{u\in U} \pi_u^* H^{p-1}(B_u) & \\ \quad \oplus \left\{ \sum_{u\in U} \lambda_u \cdot a_u \mid \lambda_u \in R, \sum_{(u,v)\in E} \delta(e)\lambda_u = 0 \text{ for all } v \in V \right\}, & i = p-1; \\ \bigoplus_{u\in U} H^q(E_u) \Big/ \bigoplus_{v\in V} \left(\sum_{e=(u,v)\in E} \delta(e)\pi_u^*[B_u]^* \right) R, & i = q; \\ \bigoplus_{u\in U} H^{p+q-1}(E_u) \Big/ \left\{ \sum_{u\in U} \lambda_u [E_u]^* \mid \sum_{u\in U} \lambda_u = 0 \right\} \cong R, & i = p+q-1; \\ \bigoplus_{u\in U} H^i(E_u), & \text{else,} \end{cases}$$

if $q \neq p-1$, and

$$H^i(M_G) \cong \begin{cases} \left(\sum_{u\in U} 1_{H^0(E_u)}\right) R \cong R, & i = 0; \\ \bigoplus_{u\in U} \pi_u^* H^{p-1}(B_u) \Big/ \bigoplus_{v\in V} \left(\sum_{e=(u,v)\in E} \delta(e)\pi_u^*[B_u]^* \right) R & \\ \quad \oplus \left\{ \sum_{u\in U} \lambda_u \cdot a_u \mid \lambda_u \in R, \sum_{(u,v)\in E} \delta(e)\lambda_u = 0 \text{ for all } v \in V \right\}, & i = q; \\ \bigoplus_{u\in U} H^{p+q-1}(E_u) \Big/ \left\{ \sum_{u\in U} \lambda_u [E_u]^* \mid \sum_{u\in U} \lambda_u = 0 \right\} \cong R, & i = 2q; \\ \bigoplus_{u\in U} H^i(E_u), & \text{else,} \end{cases}$$

if $q = p - 1$. The cup product structure is induced from $\bigoplus_{u\in U} H^*(E_u)$.

Remark 4.4.4. If G is any geometric plumbing graph with simply-connected connected components, then we can apply Proposition 4.3.2 to split it into a disjoint union of subgraphs that all either satisfy the assumptions in Theorem 4.4.3, or consist of a single vertex in V. Thus, we can compute the cohomology ring of M_G by applying Theorem 4.4.3 to the corresponding components of G (provided that the assumptions on the bundles as in Theorem 4.4.3 are satisfied).

For the characteristic classes of M_G we have the following result:

Theorem 4.4.5. *Let c be a stable characteristic class. Let G be a geometric plumbing graph satisfying the assumptions of Theorem 4.4.3. If ξ_u denotes the vector bundle corresponding to the bundle π_u, then*

$$c(M_G) = \sum_{u\in U} \pi_u^* c(\xi_u \oplus TB_u),$$

where we identified $H^*(M_G)$ with a quotient of a subring of $\bigoplus_{u\in U} H^*(E_u)$ according to Theorem 4.4.3.

We will apply Theorems 4.4.3 and 4.4.5 in dimension $6k$ with $p = 2k+1$ and $q = 4k$. Recall (cf. Section B.3, or Section 6.1), that the cup product on a closed, oriented $6k$-dimensional manifold M defines a symmetric trilinear form μ_M on $H^{2k}(M;\mathbb{Z})$ and the k-th Pontryagin class $p_k(M)$ defines a linear form on $H^{2k}(M;\mathbb{Z})$. A further important invariant is the k-th power of the second Stiefel-Whitney class $w_2(M)^k \in H^{2k}(M;\mathbb{Z}/2)$.

Corollary 4.4.6. *In the setting of Theorem 4.4.3 let $p = 2k+1$, $q = 4k$, i.e. M_G has dimension $6k$. Let every B_u be simply-connected with torsion-free homology. Then the following assertions hold:*

- *If G is the graph $\textcircled{\scriptsize v}\overset{e}{\text{—}}\textcircled{\scriptsize u}$, then M_G is a simply-connected $6k$-dimensional manifold with torsion-free homology and invariants*

$$(H^{2k}(M_G;\mathbb{Z}), \mu_{M_G}, w_2(M_G)^k, p_k(M_G)) = (H^{2k}(B_u;\mathbb{Z}), 0, (w_2(B_u) + w_2(\pi_u))^k, 0).$$

  If $k = 1$, then $b_3(M_G) = 0$.

- *If G has no vertex in V that is a leaf, then M_G is a simply-connected $6k$-dimensional manifold with torsion-free homology. Further, define $A = \bigoplus_{u\in U} H^{2k}(E_u)$, $\mu = \sum_{u\in U} \mu_{E_u}$ and $p = \sum_{u\in U} p_k(E_u)$ (viewed as a linear form on A). Then $H^{2k}(M_G;\mathbb{Z})$ is given by*

$$\bigoplus_{u\in U} \pi_u^* H^{2k}(B_u;\mathbb{Z}) \oplus \left\{ \sum_{u\in U} \lambda_u \cdot a_u \mid \lambda_u \in \mathbb{Z}, \sum_{e=(u,v)\in E} \delta(e)\lambda_u = 0 \text{ for all } v \in V \right\} \subseteq A,$$

  and we have $w_2(M_G)^k = \sum_{u\in U} \pi_u^*(w_2(B_u) + w_2(\pi_u))^k$ and μ_{M_G} and $p_k(M_G)$ are the restrictions of μ and p to $H^{2k}(M_G;\mathbb{Z})$, respectively. If $k = 1$, then $b_3(M_G) = 0$.

The rest of this section consists of the proofs of Theorems 4.4.3 and 4.4.5.

Lemma 4.4.7. *Let $S^n_k = S^n \setminus (\bigsqcup_k D^n)^\circ$ with $n \geq 2$. Then the inclusion $\iota\colon \bigsqcup_k S^{n-1} \hookrightarrow S^n_k$ of the boundary induces an injective map*

$$\iota^*\colon H^{n-1}(S^n_k) \to H^{n-1}(\textstyle\bigsqcup_k S^{n-1}) \cong \bigoplus_k H^{n-1}(S^{n-1})$$

with image generated by the elements $a_j - a_i$, where a_i is a positively oriented generator of the i-th copy of $H^{n-1}(S^{n-1})$ with respect to the orientation induced by S^n_k.

Proof. We use Lefschetz duality to obtain the following commutative diagram with exact rows and where the vertical arrows are isomorphisms:

$$\begin{tikzcd}
H^{n-1}(S^n_k, \bigsqcup_k S^{n-1}) \arrow[r] \arrow[d, "\cong"] & H^{n-1}(S^n_k) \arrow[r, "\iota^*"] \arrow[d, "\cong"] & H^{n-1}(\bigsqcup_k S^{n-1}) \arrow[r] \arrow[d, "\cong"] & H^n(S^n_k, \bigsqcup_k S^n) \arrow[r] \arrow[d, "\cong"] & 0 \\
H_1(S^n_k) \arrow[r] & H_1(S^n_k, \bigsqcup_k S^{n-1}) \arrow[r] & H_0(\bigsqcup_k S^{n-1}) \arrow[r] & H_0(S^n_k) \arrow[r] & 0
\end{tikzcd}$$

If $n \geq 3$, then S^n_k is simply-connected. If $n = 2$, then the map $H_1(S^2_k) \to H_1(S^2_k, \bigsqcup_k S^1)$ is trivial as any closed loop in S^2_k is homologous to the sum of loops in $\bigsqcup_k S^1$ it encloses. Hence, the map ι^* is injective. To compute the image, it suffices to determine the kernel of the map $H_0(\bigsqcup_k S^{n-1}) \to H_0(S^n_k)$. The claim now follows from the fact that we can join any two boundary components by a path and that the vertical map $H^{n-1}(\bigsqcup_k S^{n-1}) \to H_0(\bigsqcup_k S^{n-1})$ is given by Poincaré duality. $\square$

Define

$$X = \overline{M}_G \setminus \bigcup_{u\in U} \mathrm{Int}(\overline{E}_u) = \bigcup_{v\in V} \overline{E}_v \setminus (\bigcup_{(u,v)\in E} \varphi_{(v,u)}(D^{p\circ} \times D^q)) \cup \bigcup_{u\in U} E_u,$$

where we considered the bundles $\overline{E}_v$ and $\overline{E}_u$ as subspaces of $\overline{M}_G$. Alternatively, the space X is the result of the following pushout:

$$\begin{tikzcd}
\coprod\limits_{e\in E} D^q \times S^{p-1} \arrow[r, hook, "\coprod\limits_{e=(u,v)\in E} \varphi_{(v,u)}\circ I_{q,p}^{\delta(e)}"] \arrow[d, hook, "\coprod\limits_{(u,v)\in E} \varphi_{(u,v)}"] & \coprod\limits_{v\in V} \overline{E}_v \setminus \Big(\bigcup\limits_{(u,v)\in E} \varphi_{(v,u)}(D^{p\circ} \times D^q)\Big) \arrow[d, hook] \\
\coprod\limits_{u\in U} E_u \arrow[r, hook] & X
\end{tikzcd} \tag{4.4.3}$$

For every $v \in V$, by assumption, the manifold $\overline{E}_v \setminus (\bigcup_{(u,v)\in E} \varphi_{(v,u)}(D^{p\circ} \times D^q))$ can be identified with $S^p_{\deg(v)} \times D^q$.

Lemma 4.4.8. *Suppose that G satisfies* (4.4.2). *Then the inclusion $\coprod_{u\in U} E_u \hookrightarrow X$ induces an injective map $H^i(X) \to \bigoplus_{u\in U} H^i(E_u)$ for every i with image given by*

- $\left(\sum_{u\in U} 1_{H^0(E_u)}\right) R \cong R, \quad$ *if* $i = 0$;
- $\bigoplus\limits_{u\in U} \pi_u^* H^{p-1}(B_u) \oplus \left\{ \sum\limits_{u\in U} \lambda_u \cdot a_u \mid \sum\limits_{(u,v)\in E} \delta(e)\lambda_u = 0 \text{ for all } v \in V \right\}, \quad$ *if* $i = p-1$;
- $\bigoplus\limits_{u\in U} H^i(E_u), \quad$ *else.*

We will henceforth identify the cohomology of X with its image in $\bigoplus_{u\in U} H^*(E_u)$.

Proof. The Mayer-Vietoris sequence for the pushout (4.4.3) is given as follows:

$$\cdots \longrightarrow H^i(X) \longrightarrow \bigoplus_{v\in V} H^i(S^p_{\deg(v)}) \bigoplus_{u\in U} H^i(E_u) \longrightarrow \bigoplus_{e\in E} H^i(S^{p-1}) \longrightarrow H^{i+1}(X) \longrightarrow \cdots$$

To simplify notation, we will write u, v or e for a canonical generator of a group $H^i(M^n)$ (e.g. $[M]^*$ if $i = n$ or $1_{H^0(M)}$ if $i = 0$), where M is related to $u \in U$, $v \in V$ or $e \in E$ (e.g. as the fiber, base or total space of π_u or π_v).

For $i = 0$ the middle map of the sequence is given by

$$\bigoplus_{v\in V} Rv \bigoplus_{u\in U} Ru \longrightarrow \bigoplus_{e\in E} Re, \quad v \mapsto \sum_{e=(u,v)\in E} e, \quad u \mapsto -\sum_{e=(u,v)\in E} e.$$

This is the linear map associated to the incidence matrix $Q(G)$, when we view G as a directed graph with edges originating from vertices in V and ending in vertices in U. Since G is simply-connected, it follows from Lemmas C.1 and C.2 that this map is surjective with kernel generated by $\sum_{u\in U} u + \sum_{v\in V} v$. In particular, the image of $H^0(X)$ in $\bigoplus_{u\in U} H^0(E_u)$ is generated by

$$\sum_{u\in U} u = \sum_{u\in U} 1_{H^0(E_u)}.$$

For $1 \leq i < p-1$ or $i \geq p$ the groups $H^i(S^p_k)$ and $H^i(S^{p-1})$ vanish. By exactness, the map $H^i(X) \to \bigoplus_{u\in U} H^i(E_u)$ is an isomorphism for $1 \leq i < p-1$ and for $i > p$ (for $i = 1$ this follows from the surjectivity of the map in degree 0).

Finally, we have to investigate the following part of the sequence:

$$0 \longrightarrow H^{p-1}(X) \longrightarrow \bigoplus_{v\in V} H^{p-1}(S^p_{\deg(v)}) \bigoplus_{u\in U} \pi_u^*(H^{p-1}(B_u)) \oplus Ra_u \\ \longrightarrow \bigoplus_{e\in E} Re \longrightarrow H^p(X) \longrightarrow \bigoplus_{u\in U} H^p(E_u) \longrightarrow 0.$$

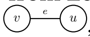
By Lemma 4.4.7, for every $v \in V$ there are generators $a^v_{u_1u_2}$, $u_1, u_2 \in U$ with $(u_1, v), (u_2, v) \in E$, of $H^{p-1}(S^p_{\deg(v)})$ (note that they are not linearly independent), so that the map

$$\bigoplus_{v\in V} H^{p-1}(S^p_{\deg(v)}) \bigoplus_{u\in U} \pi_u^*(H^{p-1}(B_u)) \oplus Ra_u \longrightarrow \bigoplus_{e\in E} Re,$$

which we will denote by ϕ, is then given by

$$\begin{aligned} \phi(a^v_{u_1u_2}) &= \delta(u_1, v)(u_1, v) - \delta(u_2, v)(u_2, v), && \text{for } v \in V, \\ \phi(x) &= 0, && \text{for } x \in \pi_u^*(H^{p-1}(B_u)), u \in U, \\ \phi(a_u) &= \sum_{e=(u,v)\in E} e, && \text{for } u \in U. \end{aligned}$$

The last line follows from Lemma B.3.1.

If G is the graph , then $S^p_{\deg(v)}$ is contractible, so $H^{p-1}(S^p_{\deg(v)})$ is trivial and $\phi(a_u) = e$. Hence, the map ϕ is surjective with kernel $\pi_u^*(H^{p-1}(B_u))$.

Now suppose that no vertex in V is a leaf. Then for any edge $e = (u, v) \in E$, that is connected to a leaf $u \in U$, we have $\phi(a_u) = e$. Hence, for any edge $e' = (u', v) \in E$ connected to v, we have

$$\phi(\delta(u', v)a^v_{u'u} + \delta(u, v)a_u) = e'.$$

Hence, by induction over the distance to the root, we see that the map ϕ is surjective.

We have that $\bigoplus_{u\in U} \pi_u^*(H^{p-1}(B_u))$ is contained in the kernel of ϕ. Further, the image of the map $\bigoplus_{v\in V} H^{p-1}(S^p_{\deg(v)}) \longrightarrow \bigoplus_{e\in E} Re$ is given by

$$\left\{ \sum_{e\in E} \lambda_e e \mid \sum_{e=(u,v)\in E} \delta(e)\lambda_e = 0 \text{ for all } v \in V \right\}.$$

Thus, to determine the kernel of ϕ, we need to determine all elements $\sum_{u\in U} \lambda_u a_u \in \bigoplus_{u\in U} Ra_u$, that get mapped into this set via ϕ. This is precisely the set

$$\left\{ \sum_{u\in U} \lambda_u a_u \mid \sum_{e=(u,v)\in E} \delta(e)\lambda_u = 0 \text{ for all } v \in V \right\},$$

which is the projection of the kernel of ϕ to $\bigoplus_{u\in U} Ra_u$. This finishes the proof. □

Lemma 4.4.9. *Suppose that G satisfies* (4.4.2). *Then the inclusion $M_G \hookrightarrow X$ induces a surjective map $H^i(X) \to H^i(M_G)$ with kernel given by*

$$\ker(H^i(X) \to H^i(M_G)) = \begin{cases} \bigoplus_{v\in V} \left(\sum_{e=(u,v)\in E} \delta(e)\pi_u^*[B_u]^* \right) R, & i = q; \\ \left\{ \sum_{u\in U} \lambda_u [E_u]^* \mid \sum_{u\in U} \lambda_u = 0 \right\}, & i = p + q - 1; \\ 0, & \textit{else}. \end{cases}$$

Proof. By excision, the cohomology of the pair (X, M_G) is isomorphic to the cohomology of the pair

$$\left(\bigsqcup_{v\in V} D^q \times S^p_{\deg(v)}, \bigsqcup_{v\in V} S^{q-1} \times S^p_{\deg(v)}\right).$$

By the long exact sequence, this pair only has non-vanishing cohomology groups in degrees q and $p+q$. We first consider degree q. The inclusion $X \hookrightarrow \overline{M}_G$ then gives the following commutative square:

$$\begin{tikzcd} H^q(\overline{M}_G, M_G) \arrow[r] \arrow[d] & H^q(\overline{M}_G) \arrow[d] \\ H^q(X, M_G) \arrow[r] & H^q(X) \end{tikzcd}$$

For this commutative diagram we will now prove the following claims:

1. $H^q(\overline{M}_G, M_G) \cong \bigoplus_{u\in U} H_p(B_u) \bigoplus_{v\in V} Rv$ and $H^q(\overline{M}_G) \cong \bigoplus_{u\in U} Ru \bigoplus_{v\in V} H^q(S^p)$.
2. The map $H^q(\overline{M}_G, M_G) \to H^q(X, M_G)$ restricted to $\bigoplus_{v\in V} Rv$ is an isomorphism.
3. The map $H^q(\overline{M}_G, M_G) \to H^q(\overline{M}_G)$ restricted to $\bigoplus_{v\in V} Rv$ is injective with image generated by the elements $\sum_{e=(u,v)\in E} \delta(e)u$ for all $v \in V$.
4. The map $H^q(\overline{M}_G) \to H^q(X)$ maps $u \in U$ to $\pi^*[B_u]^*$.

Ad (1). By Lefschetz duality we have $H^q(\overline{M}_G, M_G) \cong H_p(\overline{M}_G)$, so both isomorphisms follow from the fact that $\overline{M}_G$ is homotopy equivalent to $\bigvee_{u\in U} B_u \bigvee_{v\in V} S^p$ by Lemma 4.2.3.

Ad (2). We have the following commutative diagram of inclusions:

$$\begin{tikzcd} \left(\bigsqcup_{v\in V} S^p_{\deg(v)} \times D^q, \bigsqcup_{v\in V} S^p_{\deg(v)} \times S^{q-1}\right) \arrow[r, hook] \arrow[dr, hook] & (X, M_G) \arrow[d, hook] \\ & (\overline{M}_G, M_G) \end{tikzcd} \tag{4.4.4}$$

and Lefschetz duality gives the following commutative square.

$$\begin{tikzcd} H^q(\overline{M}_G, M_G) \arrow[r] \arrow[d, "\cong"] & \bigoplus_{v\in V} H^q(S^p_{\deg(v)} \times D^q, S^p_{\deg(v)} \times S^{q-1}) \arrow[d, "\cong"] \\ H_p(\overline{M}_G) \arrow[r] & \bigoplus_{v\in V} H_p(S^p_{\deg(v)} \times D^q, \bigsqcup_{\deg(v)} S^{p-1} \times D^q) \end{tikzcd}$$

The elements $v \in H^q(\overline{M}_G, M_G)$ in (1) are represented by $B_v \cong S^p \subseteq \overline{M}_G$ in $H_p(\overline{M}_G)$. Each class $[B_v]$ also represents a generator of

$$H_p(S^p_{\deg(v)} \times D^q, \bigsqcup_{\deg(v)} S^{p-1} \times D^q) \cong H_p(S^p, \bigsqcup_{\deg(v)} D^p) \cong H_p(S^p).$$

Hence, when restricted to $\bigoplus_{v\in V} Rv$, the lower horizontal map is an isomorphism. The claim now follows from the commutativity of (4.4.4) and that the map

$$\left(\bigsqcup_{v\in V} S^p_{\deg(v)} \times D^q, \bigsqcup_{v\in V} S^p_{\deg(v)} \times S^{q-1}\right) \hookrightarrow (X, M_G)$$

induces an isomorphism on cohomology by excision.

Ad (3). By Lefschetz duality we have the following commutative diagram.

$$\begin{tikzcd} H^q(\overline{M}_G, M_G) \arrow[r] \arrow[d, "\cong"] & H^q(\overline{M}_G) \arrow[d, "\cong"] \\ H_p(\overline{M}_G) \arrow[r] & H_p(\overline{M}_G, M_G) \end{tikzcd}$$

By (1), for each $v \in V$, the element $v \in H^q(\overline{M}_G, M_G)$ maps to the homology class which is represented by $B_v \cong S^p \subseteq \overline{M}_G$. By isotoping the embedding of the zero-section $B_v \subseteq \overline{E}_v \subseteq \overline{M}_G$ to the boundary of the disc bundle (which is possible as the bundle π_v is trivial), we see that the class of this embedding in $H_p(\overline{M}_G, M_G)$ is represented by the sum of embeddings of fibers of all $\overline{E}_u$ for which $(u, v) \in E$, each class multiplied by the sign $\delta(u, v)$. Each embedding of a fiber of $\overline{E}_u$ represents the dual to the class represented by the embedding of the zero-section $B_u \subseteq \overline{E}_u$. By commutativity of the diagram, it follows that $v \in H^q(\overline{M}_G, M_G)$ gets mapped to $\sum_{e=(u,v)\in E} \delta(e)u \in H^q(\overline{M}_G)$.

Ad (4). The diagram of maps

$$\begin{tikzcd} \bigsqcup_{u\in U} B_u \arrow[r, hook] & \overline{M}_G \\ \bigsqcup_{u\in U} E_u \arrow[r, hook] \arrow[u, "\sqcup_u \pi_u"] & X \arrow[u, hook] \end{tikzcd}$$

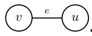
induces the following commutative diagram, where the lower horizontal map is injective by Lemma 4.4.8.

$$\begin{tikzcd} \bigoplus_{u\in U} H^q(B_u) \arrow[d, "\oplus_u \pi_u^*"] & H^q(\overline{M}_G) \arrow[l] \arrow[d] \\ \bigoplus_{u\in U} H^q(E_u) & H^q(X) \arrow[l, hook'] \end{tikzcd}$$

For each $u \in U$, the element $u \in H^q(\overline{M}_G)$ gets mapped to $[B_u]^* \in H^q(B_u)$, which in turn gets mapped to $\pi_u^*[B_u]^* \in H^q(E_u)$. This proves the claim.

Combining claims (1)–(4) it follows that the map $H^q(X, M_G) \longrightarrow H^q(X)$ is given by

$$\begin{aligned} \bigoplus_{v\in V} Rv \longrightarrow H^q(X) &\subseteq \bigoplus_{u\in U} H^q(E_u) \\ v \longmapsto \sum_{e=(u,v)\in E} &\delta(e)\pi_u^*[B_u]^*. \end{aligned}$$

This map is injective; this is clear if G is of the form . If no vertex is a leaf, then this follows from Lemma C.3 (where the arguments work in the same way if some of the entries are -1 instead of 1).

To summarize, we showed for $i < p + q - 1$ that the map $H^i(X) \to H^i(M_G)$ is surjective and only has a non-trivial kernel for $i = q$, which is then generated by the elements $\sum_{e=(u,v)\in E}(\delta(e)\pi_u^*[B_u]^*)$ for $v \in V$.

It remains to consider the case $i = p+q-1$. For each $u \in U$ we have the following commutative diagram of maps of pairs:

$$\begin{tikzcd}
(M_G, \emptyset) \arrow[r, hook] \arrow[d, hook] & (X, \emptyset) \\
\left(M_G, M_G \setminus \left(E_u \setminus \bigsqcup_{(u,v)\in E} \varphi_{(u,v)}(D^q \times S^{p-1})\right)\right) & (E_u, \emptyset) \arrow[u, hook] \arrow[d, hook] \\
\left(E_u \setminus \bigsqcup_{(u,v)\in E} \varphi_{(u,v)}(D^q \times S^{p-1})^\circ, \bigsqcup_{(u,v)\in E} \varphi_{(u,v)}(S^{q-1} \times S^{p-1})\right) \arrow[u, hook] \arrow[r, hook] & \left(E_u, \bigsqcup_{(u,v)\in E} \varphi_{(u,v)}(D^q \times S^{p-1})\right)
\end{tikzcd}$$

The maps not involving X are all orientation-preserving maps that induce isomorphisms on H^{p+q-1} by excision and the long exact sequence. Hence, each $[E_u]^* \in H^{p+q-1}(X)$ gets mapped to the dual of the fundamental class of M_G under the induced map $H^{p+q-1}(X) \longrightarrow H^{p+q-1}(M_G)$. In particular, the kernel of this map is given by

$$\left\{ \sum_{u\in U} \lambda_u [E_u]^* \mid \sum_{u\in U} \lambda_u = 0 \right\}.$$

Further, by the long exact sequence of the pair

$$\left(\bigsqcup_{v\in V} S^p_{\deg(v)} \times D^q, \bigsqcup_{v\in V} S^p_{\deg(v)} \times S^{q-1} \right),$$

the group $H^{p+q-1}(X, M_G)$ is free of rank $|E| - |V|$. Since the kernel of the map

$$H^{p+q-1}(X) \longrightarrow H^{p+q-1}(M_G),$$

which is the image of the map $H^{p+q-1}(X, M_G) \to H^{p+q-1}(X)$, has rank $|U| - 1$, and since in a tree we have $|U| + |V| = |E| + 1$, i.e. $|E| - |V| = |U| - 1$, it follows that $H^{p+q-1}(X, M_G)$ injects into $H^{p+q-1}(X)$, showing that the boundary map $H^{p+q-2}(M_G) \to H^{p+q-1}(X, M_G)$ is trivial. This finishes the proof. ☐

We are now ready to prove Theorems 4.4.3 and 4.4.5, and Corollary 4.4.6.

Proof of Theorem 4.4.3. This directly follows from Lemmas 4.4.8 and 4.4.9. ☐

Proof of Theorem 4.4.5. For $u \in U$ we have the inclusion $\overline{E}_u \hookrightarrow \overline{M}_G$, and, by naturality, the induced map on cohomology maps $c(T\overline{M}_G)$ to $c(T\overline{E}_u)$. If ξ_u denotes the vector bundle corresponding to π_u, then the tangent bundle of $\overline{E}_u$ decomposes as

$$T\overline{E}_u \cong \pi_u^*(\xi_u \oplus TB_u),$$

cf. Lemma B.3.5. Hence,

$$c(T\overline{E}_u) = \pi_u^* c(\xi_u \oplus TB_u).$$

Since $\overline{M}_G \simeq \bigvee_{u\in U} B_u \bigvee_{v\in V} B_v$ by Lemma 4.2.3 and all bundles π_v are trivial, it follows that

$$c(T\overline{M}_G) = \sum_{u\in U} c(T\overline{E}_u) = \sum_{u\in U} \pi_u^* c(\xi_u \oplus TB_u).$$

Now $M_G = \partial \overline{M}_G$, and we denote the inclusion $M_G \hookrightarrow \overline{M}_G$ by ι. Then

$$\iota^* T\overline{M}_G \cong TM_G \oplus \underline{\mathbb{R}}_{M_G},$$

the trivial factor corresponding to the normal bundle of M_G in $\overline{M}_G$. Hence, by the stability of c, we have

$$c(TM_G) = \iota^* \left(\sum_{u \in U} \pi_u^* c(\xi_u \oplus TB_u) \right).$$

To determine this element in $H^*(M_G)$ consider the following commutative diagram:

$$\begin{tikzcd}
M_G \arrow[r, hook] \arrow[d, hook, "\iota"] & X & \bigsqcup_{u \in U} E_u \arrow[l, hook'] \arrow[d, "\bigsqcup_{u \in U} \pi_u"] \\
\overline{M}_G & & \bigsqcup_{u \in U} B_u \arrow[ll, hook']
\end{tikzcd}$$

By Theorem 4.4.3, we need to determine the image of $c(TM_G)$ in the cohomology of $\bigsqcup_{u \in U} E_u$. This is given by

$$\sum_{u \in U} \pi_u^* c(\xi_u \oplus TB_u) \in \bigoplus_{u \in U} H^*(E_u; R).$$

□

Proof of Corollary 4.4.6. In both cases M_G is simply-connected by Lemma 4.4.1. Furthermore, since all B_u are simply-connected $4k$-dimensional manifolds, they have torsion-free homology. Hence, by (4.4.1) and Theorem 4.4.3, also M_G has torsion-free homology. The remaining claims directly follow from Theorems 4.4.3 and 4.4.5. □

Metrics of Positive Ricci Curvature Constructed via Surgery 5

In this chapter we consider surgery on manifolds with a metric of positive Ricci curvature. For that, we first consider the situation for positive scalar curvature in Section 5.1. In Section 5.2 we then introduce the work of Perelman and Burdick on the construction of metrics of positive Ricci curvature on connected sums. Further, in Section 5.3 we consider higher surgeries and state the generalized surgery theorem (Theorem A) with its applications Theorems B and C, whose proofs are also given in this section. The generalized surgery theorem will then be proved in Section 5.4. Finally, in Section 5.5 we compare the generalized surgery theorem to previous results of Sha–Yang and Wraith.

5.1 Surgery and Positive Scalar Curvature

Recall the surgery theorem of Gromov–Lawson and Schoen–Yau.

Theorem 5.1.1 ([101, Corollary 6], [50, Theorem A], Theorem 3.4.6). *Let M^n be a manifold that admits a complete metric of positive scalar curvature and let $\varphi\colon S^{p-1}\times D^q\hookrightarrow M$, $n=p+q-1$, be an embedding. If $q\geq 3$, then M_φ also admits a complete metric of positive scalar curvature.*

Remark 5.1.2. The condition on the codimension cannot be removed. In fact, surgery of codimension 1 is dual to surgery of dimension 0, in particular we obtain $M_1\sqcup M_2$ by surgery of codimension 1 on $M_1\#M_2$. By Corollary 3.4.13, $M\#(-M)$ admits a metric of positive scalar curvature for any closed, simply-connected manifold M^n, $n\geq 5$. Since not every closed, simply-connected manifold of dimension at least 5 admits a metric of positive scalar curvature (cf. Example 3.4.2), Theorem 5.1.1 does not hold for $q=1$.

For codimension 2, note that the 2-torus is obtained from S^2 by surgery of dimension 0, i.e. codimension 2. Since T^2 does not admit a metric of positive scalar curvature, Theorem 5.1.1 does not hold for $q=2$.

For codimension 0, Theorem 5.1.1 in fact holds: Here surgery consists of deleting a connected component that is diffeomorphic to S^n. This result, however, is obvious and is not of much interest for constructions, so this case is not mentioned in Theorem 5.1.1.

Proof idea. By the uniqueness of tubular neighborhoods, see e.g. [58, Theorem 4.5.3], the embedding φ is isotopic to an embedding, where each normal disc $\{x\}\times D^q$ is embedded as the normal disc of radius $\bar r>0$ to $\varphi(S^{p-1}\times\{0\})$ at x. Now consider the product $(S^{p-1}\times D^q)\times\mathbb{R}$ equipped with the product metric of the metric pulled back by φ on $S^{p-1}\times D^q$ and the standard metric on $\mathbb{R}$. Then define the hypersurface

$$M'=\{(x,y,t)\in S^{p-1}\times D^q\times\mathbb{R}\mid (\bar r\|y\|,t)\in \mathrm{im}(\gamma)\},$$

where $\|y\|$ denotes the Euclidean norm on $D^q \subseteq \mathbb{R}^q$ and γ is a smooth curve in the $r-t$ plane that starts as a straight line along the positive r-axis containing $(\bar{r}, 0)$ and ends as a line parallel to the positive t-axis with distance $\varepsilon > 0$. These conditions ensure that M' and $M \setminus \mathrm{im}(\varphi)^\circ$, together with their metrics, can be glued along their common boundary $S^{p-1} \times S^{q-1}$. Further, the metric on the cylindrical part of M' is a product metric.

One can now show that the scalar curvature of M' is given by

$$\mathrm{scal}^{M'} = \mathrm{scal}^{S^{p-1}\times D^q} + O(1)\sin^2(\theta) + (q-1)(q-2)\frac{\sin^2(\theta)}{r^2} - (q-1)\frac{k}{r}\sin(\theta).$$

Here θ denotes the angle between the normal vector to γ and the t-axis, and k is the curvature of γ. The curve γ is then constructed by first making a small bend for which $\mathrm{scal}^{S^{p-1}\times D^q} + O(1)\sin^2(\theta)$ stays positive and continuing it as a straight line until the term $(q-1)(q-2)\frac{\sin^2(\theta)}{r^2}$ strongly dominates the expression (here we see why we need $q \geq 3$). One can then bend γ to a straight line parallel to the t-axis.

Hence, by cutting off the unbounded part of M' we have constructed a metric on

$$M \setminus \mathrm{im}(\varphi)^\circ \cup_{S^{p-1}\times S^{q-1}} [0, t_1] \times (S^{p-1} \times S^{q-1})$$

that coincides with the original metric on $M \setminus \mathrm{im}(\varphi)^\circ$ and which is a product metric on $[t_0, t_1] \times (S^{p-1} \times S^{q-1})$ for some $t_0 \in (0, t_1)$. If ε is sufficiently small one can in fact attach another cylinder so that in a neighborhood of the boundary we have a product metric where the induced metric on $S^{p-1} \times S^{q-1}$ itself is the product of two round spheres. One can now attach $D^p \times S^{q-1}$ with a product metric, where D^p is equipped with a metric that is a product near the boundary and S^{q-1} is equipped with a round metric. Since the boundaries are then isometric and the metrics are products near the boundaries, the metrics glue smoothly to give a metric of positive scalar curvature on M_φ. □

We see that the approach in the proof of Theorem 5.1.1 is *local*, i.e. independently of the metric and the embedding we can leave the metric on M unchanged outside $\mathrm{im}(\varphi)$. Further, the metric in Theorem 5.1.1 gets deformed to a metric that is a product near the boundary of $\varphi(S^{p-1} \times D^q_\varepsilon)$ for some small disc $D^q_\varepsilon = \varepsilon D^q \subseteq D^q$.

This approach cannot work for positive Ricci curvature. First, the proof of Theorem 5.1.1 sketched above does not carry over to positive Ricci curvature as the Ricci curvature in the tangent direction of γ is negative for $\bar{r}$ small whenever γ is curved negatively (with respect to the standard orientation on $\mathbb{R}^2$). Second, we can also show that no other deformation can be successful.

Proposition 5.1.3. *Let M^{p+q-1}, $q > 1$, be a Riemannian manifold and let $N^{p-1} \subseteq M$ be a closed embedded submanifold with trivial normal bundle, hence for $\varepsilon > 0$ small the neighborhood $B_\varepsilon(N)$ is diffeomorphic to $N \times D^q$. Assume that $[t_0, t_1] \times N \times S^{q-1}$ carries a metric g that glues smoothly with the metric on $M \setminus B_\varepsilon N^\circ$ at the boundary component $\{t_0\} \times N \times S^{q-1}$ and is a product metric near the boundary component $\{t_1\} \times N \times S^{q-1}$. Then for ε small, the metric g does not have positive Ricci curvature.*

Proof. If ε is small then the boundary $N \times S^{q-1} \cong \partial B_\varepsilon(N) \subseteq B_\varepsilon(N)$ has positive mean curvature, see e.g. [72, (3.8)]. Hence, the metric g has positive mean curvature at the boundary component $\{t_0\} \times N \times S^{q-1}$. Since it is a product near the boundary component $\{t_1\} \times N \times S^{q-1}$, it has vanishing mean curvature at this boundary component. Assume that g has positive Ricci curvature. Then one can deform it slightly, so that the mean curvature on the boundary component $\{t_1\} \times N \times S^{q-1}$ is in fact strictly positive, see e.g. [21, Proposition 1.2.11]. By Theorem 3.2.4 the boundary of $[t_0, t_1] \times N \times S^{q-1}$ would then be connected, which is a contradiction. □

Proposition 5.1.3 shows that there exists no local surgery technique for positive Ricci curvature that deforms the metric to a product metric near the boundary of $\varphi(S^{p-1} \times D^q_\varepsilon)$. However, a local surgery technique does not necessarily need to have this property; it would be sufficient to deform the metric in any (Ricci-positive) way that allows to attach $D^p \times S^{q-1}$ equipped with a metric of positive Ricci curvature.

Question 5.1.4. Does there exist a local surgery technique for positive Ricci curvature?

If the answer to Question 5.1.4 is affirmative for surgery of a fixed dimension or codimension, it would imply that the existence of metrics of positive Ricci curvature is preserved under any surgery of this dimension or codimension. We can exclude the following cases:

- Codimension at most 2: Here the local surgery approach already fails for positive scalar curvature, cf. Remark 5.1.2,

- Dimension 0: For any two points p_1, p_2 in M^n, any sufficiently small neighborhoods $B_\varepsilon(p_1)$, $B_\varepsilon(p_2)$ of these points have positive mean curvature at their boundaries. Hence, by Theorem 3.2.4, there cannot by a metric of positive Ricci curvature on $[t_0, t_1] \times S^{n-1}$ gluing smoothly with the metric on $M \setminus (B_\varepsilon(p_1)^\circ \sqcup B_\varepsilon(p_2)^\circ)$. Thus, local surgery cannot work in dimension 0. Further, we have
$$\pi_1(M_1 \# M_2) \cong \pi_1(M_1) * \pi_1(M_2)$$
for $n \geq 3$, cf. (4.1.1). Thus, the existence of metrics of positive Ricci curvature cannot be preserved under 0-surgery on closed manifolds by Theorem 3.2.1 and Lemma 4.1.3, unless the embedding of $S^0 \times D^n$ for the surgery maps the two copies of D^n into different connected components and at least one of these connected components is simply-connected.

For all other cases, Question 5.1.4 is open. The existing surgery techniques for positive Ricci curvature, which we will describe in the sections below, are all non-local and take the global structure of the manifold into account.

5.2 Connected Sums

Let M_1^n, M_2^n be manifolds that admit a complete metric of positive Ricci curvature. By Theorem 5.1.1, the connected sum $M_1 \# M_2$ admits a complete metric of positive scalar curvature. As seen in Section 5.1, $M_1 \# M_2$ cannot admit a metric of positive Ricci curvature if both M_1 and M_2 are not simply-connected. On the other hand, if one of M_1, M_2 is simply-connected, we can ask the following question:

Question 5.2.1. Let M_1, M_2 be closed manifolds that admit metrics of positive Ricci curvature and suppose that M_1 is simply-connected. Does $M_1 \# M_2$ admit a metric of positive Ricci curvature?

The techniques we will study in this section go back to Perelman [90], who constructed metrics of positive Ricci curvature on connected sums of copies of $\pm\mathbb{C}P^2$. He made the following observation:

Theorem 5.2.2 ([90], Theorem 3.1.13). *Let M_1^n, M_2^n be manifolds that admit metrics of positive Ricci curvature, so that there exists an isometry $\phi\colon \partial_c M_1 \to \partial_c M_2$ between compact boundary components $\partial_c M_1 \subseteq \partial M_1$ and $\partial_c M_2 \subseteq \partial M_2$. If the second fundamental forms $\mathrm{I\!I}_{\partial_c M_i}$ satisfy*
$$\mathrm{I\!I}_{\partial_c M_1} + \phi^* \mathrm{I\!I}_{\partial_c M_2} \geq 0,$$
then $M_1 \cup_\phi M_2$ admits a metric of positive Ricci curvature that coincides with the original metrics on M_1 and M_2 outside an arbitrarily small neighborhood of the gluing area.

A proof for Theorem 5.2.2 is sketched in [90, Section 4], and detailed proofs were given in [117, Section 2.3] and [11, Section 2.2].

Proof idea. Denote the metric on M_i by g_i. First note, that by a deformation of these metrics as in [21, Proposition 1.2.11], we can assume that the strict inequality

$$\mathrm{I\!I}_{\partial_c M_1} + \phi^* \mathrm{I\!I}_{\partial_c M_2} > 0$$

holds.

A neighborhood of the boundary component $\partial_c M_i$ is diffeomorphic to $[0, \varepsilon) \times \partial_c M_i$ for some $\varepsilon > 0$ and the metric g_i on this neighborhood is of the form $dt^2 + g_{i,t}$, where $g_{i,t}$ is a metric on $\partial_c M_i$.

We define a metric g on $(-\varepsilon, \varepsilon) \times \partial_c M_1$ by $g = dt^2 + g_t$, where, for some $t_0 \in (0, \varepsilon)$,

$$g_t = \begin{cases} g_{1,-t}, & t < -t_0, \\ \phi^* g_{2,t}, & t > t_0, \end{cases}$$

and for $t \in [-t_0, t_0]$, g_t is the unique polynomial of degree 3 in t with coefficients in

$$\mathbb{R} g_{1,t_0} \oplus \mathbb{R} g'_{1,t_0} \oplus \mathbb{R} \phi^* g_{2,t_0} \oplus \mathbb{R} \phi^* g'_{2,t_0},$$

so that the metric g is C^1. The condition on the second fundamental form implies that any sectional curvature involving ∂_t can be made arbitrarily large as $t_0 \to 0$, while all other sectional curvatures remain bounded. Hence, for t_0 small enough, g has positive Ricci curvature.

The metric g is smooth, except for $t = -t_0$ and $t = t_0$, where it is merely C^1. In a similar way as before we smoothen the metric by a polynomial of degree 5, so that the metric becomes C^2. The second t-derivative of this polynomial essentially interpolates linearly between the second t-derivatives of the original metrics, which results in an essentially linear interpolation of the Ricci curvatures. Hence, the new metric, which we denote again by g, has positive Ricci curvature and is C^2.

Finally, we deform g to a smooth metric. This smooth metric can be chosen to be arbitrarily close to g in the C^2-norm. Since the Ricci curvature only depends on the derivatives of g up to order 2, we can achieve that the new metric has positive Ricci curvature. ☐

A special case of Theorem 5.2.2 is where both boundary components have non-negative second fundamental form. This shows that for two manifolds M_1^n, M_2^n, so that each $M_i \setminus D^n$ admits a metric of positive Ricci curvature and so that the boundary has non-negative second fundamental form and the boundary components $\partial_c(M \setminus D^n) \cong S^{n-1}$ are isometric, the connected sum $M_1 \# M_2$ admits a metric of positive Ricci curvature. However, this technique does not iterate, i.e. in this way one cannot construct metrics of positive Ricci curvature on connected sums with more than two summands. For that, Perelman constructed a special metric on $S^n \setminus (\bigsqcup_k D^n)$ with $n = 4$, called *ambient space* in [90], and it was observed by Burdick [22] that this construction generalizes to all $n \geq 4$. In [22] this space is called *docking station.*

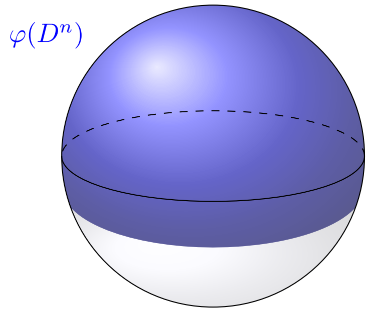
Lemma 5.2.3 ([90, Section 3] and [22, Proposition 1.3]). *For all $n \geq 4$, $k \geq 1$ and $\nu > 0$ there exists a metric of positive Ricci curvature on $S^n \setminus (\bigsqcup_k D^n)$, so that each boundary component is isometric to the round metric of radius 1 on S^{n-1}, and so that all principal curvatures are at least $-\nu$.*

Motivated by this construction, Burdick [22] introduced the following notion.

Definition 5.2.4. Let M^n be a manifold. A metric g on M is a *core metric*, if it has positive Ricci curvature and if there is an embedding $\varphi \colon D^n \hookrightarrow \mathrm{Int}(M)$, such that $g|_{\varphi(S^{n-1})}$ is the round metric of radius 1 and such that the second fundamental form $\mathrm{I\!I}_{\varphi(S^{n-1})}$ is positive semi-definite with respect to the inward pointing normal vector of $S^{n-1} \subseteq D^n$.

Remark 5.2.5. Our definition differs from Burdick's definition, as he requires the second fundamental form to be positive definite. However, the two definitions are equivalent: a metric with positive semi-definite second fundamental form can always be deformed into a metric with positive definite fundamental form while keeping the Ricci curvature positive, e.g. by the deformation [21, Proposition 1.2.11].

FIGURE 5.1: The round metric is a core metric



It is now a consequence of Theorem 5.2.2, Lemma 5.2.3 and the deformation [21, Proposition 1.2.11], that we can take connected sums of manifolds that admit core metrics.

Proposition 5.2.6 ([22, Theorem B]). *Let M_i^n, $1 \leq i \leq k$ be manifolds that admit core metrics. If $n \geq 4$, then $\#_i M_i$ admits a metric of positive Ricci curvature.*

Hence, an approach to answer Question 5.2.1 is to determine which manifolds admit core metrics. This approach can only work in the simply-connected case, since a closed manifold with a core metric is simply-connected. This follows from Theorem 3.2.4, or alternatively from Proposition 5.2.6 above together with Theorem 3.2.1. We note that for non-simply-connected manifolds Burdick [24] has introduced the notion of *socket metrics* and showed that the connected sum of manifolds with core metrics and a manifold with a socket metric admits a metric of positive Ricci curvature. Further, he showed that socket metrics exist on real projective spaces and on certain lens spaces.

The easiest example of a manifold with a core metric is the sphere S^n equipped with the round metric of some radius $r > 0$. Then the requirements of the embedding $D^n \hookrightarrow S^n$ are satisfied by the inclusion of any geodesic ball that contains a hemisphere (for a suitable choice of r), cf. Figure 5.2. The first non-trivial example was given by Perelman [90], who constructed a core metric on $\mathbb{C}P^2$. This construction was generalized by Burdick [22] to all complex and quaternionic projective spaces and to the Caley plane. Further, Burdick constructed core metrics on certain sphere bundles and manifolds obtained by plumbing [23] and on the connected sum of two manifolds with core metrics [24]. All these examples are summarized in the proposition below.

Proposition 5.2.7 ([21], [22], [23], [24],[90]). *The following manifolds admit core metrics:*

1. *S^n, if $n \geq 2$;*
2. *$\mathbb{C}P^n$, $\mathbb{H}P^n$ and $\mathbb{O}P^2$;*
3. *$M_1^n \# M_2^n$ if $n \geq 4$ and M_1, M_2 admit core metrics;*
4. *Total spaces of linear sphere bundles $E \to B$ with fiber and base dimension at least 3 if B is compact and admits a core metric;*
5. *∂W for W obtained by plumbing linear n-disc bundles over n-spheres according to a simply-connected graph with $n \geq 4$, or by plumbing together a p-disc bundle over S^q and a q-disc bundle over S^p with $p \geq 3$, $q \geq 4$.*

Remark 5.2.8. In [23], item 4 of Proposition 5.2.7 is stated with no restriction on the base dimension. However, the proof given in [23] does not work if the base is 2-dimensional, since the metric on M_1 in [23, Definition 3] does not have positive Ricci curvature if $n = 1$. In Theorem C, by using Theorem B, we fix this by giving an alternative proof provided the base dimension is at least 4.

5.3 Generalized Surgery

The first technique to construct metrics of positive Ricci curvature via surgery is due to Sha and Yang [107]. Although their main result (Theorem 3.1.11) concerns connected sums of products of spheres, the technique itself only applies to higher surgeries (see Section 5.5). This technique was later generalized and modified by Wraith [122] and the author [97]. In this section we present the latter, together with its immediate applications. It is the most general version and covers almost all cases of the previous surgery results. We give a discussion of this in Section 5.5.

Recall that $S^{p-1}(\rho)$ denotes the round $(p-1)$-sphere of radius $\rho > 0$. By $D^q_R(N)$ we denote a closed geodesic ball of radius $R > 0$ in $S^q(N)$. We assume that (M^n, g_M) is a Riemannian manifold and that there is an isometric embedding $\iota\colon S^{p-1}(\rho) \times D^q_R(N) \hookrightarrow M$, where $n = p+q-1$.

Theorem A ([97, Theorem A]). *Suppose g_M has positive Ricci curvature and suppose $p, q \geq 3$. Let $r > 0$ and $E \xrightarrow{\pi} B^p$ be a linear S^{q-1}-bundle, so that B is compact and admits a core metric g_B. Then there exists a constant $\kappa = \kappa(p, q, R/N, g_B, r) > 0$, such that if $\frac{\rho}{N} < \kappa$, then the manifold*

$$\hat{M} = (M \setminus \mathrm{im}(\iota)^\circ) \cup_{\iota \circ I^{\pm}_{q,p} \circ \varphi_\pi^{-1}} (E \setminus \mathrm{im}(\varphi_\pi)^\circ)$$

admits a metric of positive Ricci curvature. This metric coincides outside a neighborhood of the gluing area with a submersion metric on E with totally geodesic and round fibers of radius r and with a scalar multiple of the metric g_M on M.

More precisely, the dependence of κ on the metric g_B is completely determined by the smallest eigenvalue of the second fundamental form $\mathrm{I\!I}_{\varphi(S^{p-1})}$.

In Theorem A we have the assumptions that the embedding ι is isometric and that $\frac{\rho}{N} < \kappa$ holds. While the first assumption can be arranged for an arbitrary embedding $S^{p-1} \times D^q \hookrightarrow M$ under certain conditions, see [123], it is not clear in general when the second assumption can be satisfied. In particular, this surgery technique is not local: It can be shown that $\kappa \to 0$ as $R/N \to 0$. Hence, for fixed ρ, i.e. a fixed isometric embedding $S^p(\rho) \hookrightarrow M$, and fixed N, the condition $\frac{\rho}{N} < \kappa$ is not satisfied for small values of R. Thus, in order to achieve this assumption, one needs to extend the embedding $S^p(\rho) \hookrightarrow M$ to a sufficiently large tubular neighborhood. In general it is not clear when this is possible. However, for standard embeddings of linear sphere bundles it is possible to satisfy these assumptions, which leads to the following application:

Theorem B ([97, Theorem B]). *Let G be a simply-connected geometric plumbing graph. Suppose that $p, q \geq 3$. Fix $u_0 \in U$ and suppose that B^q_u, B^p_v are compact and admit core metrics for all other $u \in U$ and all $v \in V$. Then*

1. *If B_{u_0} is compact and admits a metric of positive Ricci curvature, then M_G admits a metric of positive Ricci curvature.*

2. *If B_{u_0} is compact and admits a core metric with $p \geq 4$, then M_G admits a core metric.*

Recall, that, by Proposition 5.2.7, total spaces of linear sphere bundles $E \to B$ admit a core metric if B admits a core metric and fiber and base dimensions are at least 3. By using Theorem B, we can extend this result.

Theorem C ([97, Theorem C]). *Let $E \to B^q$ be a linear S^p-bundle and suppose that*

- *$p = 2$ and $q \geq 4$, or*
- *$q = 2$ and $p \geq 4$.*

If B is closed and admits a core metric, then E admits a core metric.

The proofs of Theorems B and C follow [97, Section 4]. For the proof of Theorem B we need the following result by Burdick, which will allow to construct core metrics.

Proposition 5.3.1 ([22, Theorem 2.5]). *For $q \geq 3$, $p \geq 4$, $R > 1$, and any $\nu > 0$ sufficiently small, there is a core metric on $D^q \times S^{p-1}$ such that the boundary is isometric to $R^2 ds_{q-1}^2 + ds_{p-1}^2$ and the principal curvatures of the boundary are all at least $-\nu$.*

Proof of Theorem B. We denote the total space of the sphere bundle of the disc bundles $\overline{E}_u$ and $\overline{E}_v$ by E_u and E_v, respectively. Since all B_u with $u \neq u_0$ and all B_v have core metrics, the manifold ∂W is obtained by iterated surgeries on the manifold $M = E_{u_0}$ as in Theorem A (cf. (4.2.1)). By a deformation result of Gao and Yau [44] for negative Ricci curvature, that can easily be transferred to positive Ricci curvature (see also [123, Theorem 1.10]) for every $x \in B_{u_0}$ and any open neighborhood U of x we can deform the metric on B_{u_0} to agree with the original metric on $B_{u_0} \setminus U$ and to have constant sectional curvature 1 on a neighborhood of x. Hence, for any $k_1 \in \mathbb{N}$ we can deform the metric on B_{u_0} such that there are positive constants $R_1, \dots, R_{k_1}$ and an isometric embedding

$$D^q_{R_1}(1) \sqcup \dots \sqcup D^q_{R_{k_1}}(1) \hookrightarrow \mathrm{Int}(B_{u_0}).$$

Now we equip E_{u_0} with the metric $g_{\pi_{u_0}}(\rho, \theta)$ according to a connection θ that is flat over each embedded disc as in Lemma B.2.4, so we have an isometric embedding

$$S^{p-1}(\rho) \times D^q_{R_1}(1) \sqcup \dots \sqcup S^{p-1}(\rho) \times D^q_{R_{k_1}}(1) \hookrightarrow E_{u_0}.$$

By Proposition B.2.5 there is a constant $\rho_1 > 0$, so that $g_{E_1}(\rho, \theta)$ has positive Ricci curvature for all $\rho < \rho_1$. By choosing ρ small enough we can satisfy the assumptions of Theorem A. By possibly choosing ρ even smaller we can freely choose the radii of the fibers of the bundles we attach. Hence, by choosing sufficiently small radii for the attached bundles, we can satisfy again the assumptions of Theorem A for the attached bundles.

We repeat this process: Since we glue according to a tree, where we consider u_0 as the root, the manifold M_G is obtained by successively gluing the bundles that correspond to vertices of distance i from the root to the bundles corresponding to vertices of distance $i-1$ from the root. As above we can apply Theorem A for each gluing by possibly decreasing the fiber radii of all the preceding bundles. This finishes the proof of the first part of Theorem B.

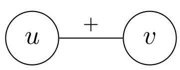
If B_{u_0} admits a core metric, then we can choose the embeddings on which we perform the surgeries to be disjoint from the embedded disc $\varphi(D^q)$. We can also assume that the connection for the bundle π_{u_0} is flat over $\varphi(D^q)$. Hence, if we remove $\varphi(D^q)$ from B_{u_0} and the corresponding part of E_{u_0}, we obtain a boundary component isometric to $ds_{q-1}^2 + \rho ds_{p-1}^2$ with non-negative definite second fundamental form. By [21, Proposition 1.2.11] we can assume that the second fundamental form is positive definite and by possibly choosing ρ smaller and rescaling we can assume that the boundary is isometric to $R^2 ds_{q-1}^2 + ds_{p-1}^2$ for some $R > 1$. Hence, by Theorem 5.2.2, we can glue with the metric from Proposition 5.3.1 (where we assume $p \geq 4$) and obtain a core metric on ∂W. This finishes the proof of Theorem B. □

For the proof of Theorem C we need the following lemma.

Lemma 5.3.2. *Let π_u and π_v denote the trivial bundles*

$$\mathbb{C}P^2 \times D^{n-1} \xrightarrow{\pi_u} \mathbb{C}P^2 \quad \text{and} \quad S^{n-1} \times D^4 \xrightarrow{\pi_v} S^{n-1},$$

respectively. Let W be the manifold obtained by plumbing as follows.



Then ∂W is diffeomorphic to $S^2 \tilde{\times} S^n$.

Proof. According to (4.2.1) we have

$$\partial W \cong (\mathbb{C}P^2 \setminus (D^4)^\circ) \times S^{n-2} \cup_{S^3 \times S^{n-2}} S^3 \times D^{n-1}.$$

The manifold $\mathbb{C}P^2 \setminus (D^4)^\circ$ is diffeomorphic to the disc bundle of the tautological line bundle over $\mathbb{C}P^1 \cong S^2$. Hence, the manifold $(\mathbb{C}P^2 \setminus (D^4)^\circ) \times S^{n-2}$ has the structure of a fiber bundle over S^2 with fiber $D^2 \times S^{n-2}$. On the other hand, the manifold $S^3 \times D^{n-1}$ also has the structure of a fiber bundle over S^2 obtained by the Hopf fibration $S^3 \to S^2$, i.e. the fiber of this bundle is $S^1 \times D^{n-1}$. Since the bundle projection $\partial(\mathbb{C}P^2 \setminus (D^4)^\circ) \cong S^3 \to S^2$ is also given by the Hopf fibration, we glue fibers to fibers, so ∂W has the structure of a fiber bundle over S^2 with fiber

$$D^2 \times S^{n-2} \cup_{S^1 \times S^{n-2}} S^1 \times D^{n-1} \cong S^n.$$

Both bundles have the structure group of the Hopf fibration, which is contained in $\mathrm{SO}(2)$, hence ∂W also has structure group contained in $\mathrm{SO}(2) \subseteq \mathrm{SO}(n+1)$, so it is a linear bundle. It is non-trivial, since under the inclusion $(\mathbb{C}P^2 \setminus (D^4)^\circ) \times S^{n-2} \hookrightarrow \partial W$, the class $w_2(\partial W)$ gets mapped to $w_2((\mathbb{C}P^2 \setminus (D^4)^\circ) \times S^{n-2})$, which is non-trivial as it is the pullback of $w_2(\mathbb{C}P^2 \times S^{n-2})$ under the inclusion $(\mathbb{C}P^2 \setminus (D^4)^\circ) \times S^{n-2} \hookrightarrow \mathbb{C}P^2 \times S^{n-2}$ (which is an isomorphism on H^2). Thus, $w_2(\partial W)$ is non-trivial, so ∂W cannot be diffeomorphic to $S^2 \times S^n$, hence it is the unique non-trivial bundle $S^2 \tilde{\times} S^n$ (cf. Example B.1.6). ☐

Proof of Theorem C. Let $E \xrightarrow{\pi} B^q$ be a linear S^p-bundle, where B is a closed manifold that admits a core metric. First suppose that $p = 2$ and $q \geq 4$. Let $\overline{E} \xrightarrow{\pi_{v_1}} B$ be the disc bundle of π. Let W be the manifold obtained by plumbing as follows:

$$v_1 \overset{+}{\text{———}} u \overset{+}{\text{———}} v_2$$

Here π_u and π_{v_2} denote the trivial bundles $S^3 \times D^q \xrightarrow{\pi_u} S^3$ and $S^q \times D^q \xrightarrow{\pi_{v_2}} S^q$, respectively. By Propositions 4.3.2 and 4.3.3 we have $\partial W \cong E$. By applying Theorem B with $u_0 = u$, we obtain a core metric on ∂W and thus on E.

Now suppose that $q = 2$ and $p \geq 4$. Then B is a 2-dimensional closed manifold with a core metric, hence $B \cong S^2$. There are precisely two isomorphism classes of linear S^p-bundles over S^2, cf. Example B.1.6. If E is the trivial bundle, i.e. $E \cong S^2 \times S^p$, then we can also consider it as a linear S^2-bundle over S^p and apply the first part of Theorem C. If E is the non-trivial bundle then the claim follows from Theorem B and Lemma 5.3.2. ☐

5.4 Proof of the Generalized Surgery Theorem

In this section we prove Theorem A. We follow [97, Section 3]. First we decompose the manifold $\hat{M}$ as follows:

$$\hat{M} \cong (E \setminus \mathrm{im}(\varphi_\pi)^\circ) \cup_{\varphi_\pi \circ I^{\pm}_{p,q}} (I \times S^{p-1} \times S^{q-1}) \cup_{\iota^{-1}} (M \setminus \mathrm{im}(\iota)^\circ), \tag{5.4.1}$$

where I is a closed interval. By possibly reversing the orientations on base and fibers of π, we can assume that we use the map $I^{+}_{p,q}$ in equation (5.4.1). The strategy now is to define a suitable metric of positive Ricci curvature on each part and then glue them together using Theorem 5.2.2.

If the metrics in Theorem 5.2.2 are warped product metrics, then, by observing that the metric we obtain after gluing is again a warped product metric, we obtain the following special case.

Corollary 5.4.1. *Let J be an interval and let $(M_1, g_1), \ldots, (M_k, g_k)$ be Riemannian manifolds. Let $f_1, \ldots, f_k \colon J \to \mathbb{R}_{>0}$ be continuous functions which are smooth on $J \setminus \{x_1, \ldots, x_l\}$, where $x_1, \ldots, x_l \in J$ are interior points. If the metric*

$$g = dt^2 + f_1(t) g_1 + \cdots + f_k(t) g_k$$

on $J \times M_1 \times \cdots \times M_k$ has positive Ricci curvature for all $t \in J \setminus \{x_1, \ldots, x_l\}$ and if

$$f'_{i-}(x_j) \geq f'_{i+}(x_j)$$

for all i, j, then we can smooth the functions $f_1, \ldots, f_k$ on an arbitrarily small neighborhood of each x_j such that the resulting metric has positive Ricci curvature.

We deform the metric on B according to Remark 5.2.5, so that the second fundamental form on $\varphi(S^{p-1})$ is positive definite. We then equip E with the metric $g_\pi(r,\theta)$ constructed in Lemma B.2.4 with respect to the embedding $\varphi\colon D^p \hookrightarrow B$ obtained from the core metric g_B. Then, if we choose the standard embedding φ_π to cover φ (as a bundle map), we have

$$g_\pi(r,\theta)|_{\varphi_\pi(D^p \times S^{q-1})} = g_B|_{\varphi(D^p)} + r^2 \cdot ds^2_{q-1}$$

and over the boundary $\varphi(S^{p-1})$ we have

$$g_\pi(r,\theta)|_{\varphi_\pi(S^{p-1} \times S^{q-1})} = ds^2_{p-1} + r^2 \cdot ds^2_{q-1}.$$

Since the metric is a product over $\varphi(D^p)$, the second fundamental form on $\varphi_\pi(S^{p-1} \times S^{q-1}) \cong S^{p-1} \times S^{q-1}$ with respect to this product structure is given by

$$\mathrm{I\!I}_{\pi^{-1}(\varphi_\pi(S^{p-1} \times S^{q-1}))} = \begin{pmatrix} \mathrm{I\!I}_{\varphi(S^{p-1})} & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.4.2}$$

Since $\mathrm{I\!I}_{\varphi(S^{p-1})} > 0$, the smallest eigenvalue of $\mathrm{I\!I}_{\varphi(S^{p-1})}$, which we denote by λ, is positive. Note that scaling the metric $g_E(r,\theta)$ by a factor $\alpha > 0$ has the effect that λ gets multiplied by α^{-1}, cf. Remark A.6.

By assumption, the metric on $\iota(S^{p-1} \times D^q)$ is the product metric

$$\begin{aligned} g_M|_{\iota(S^{p-1} \times D^q)} &= \rho^2 \cdot ds^2_{p-1} + N^2 \cdot ds^2_q|_{D^q_R(N)} \\ &= \rho^2 \cdot ds^2_{p-1} + (dt^2|_{[0,R]} + N^2 \sin^2\left(\frac{t}{N}\right) ds^2_{q-1}), \end{aligned}$$

cf. Example A.10. Hence, the metric on $\iota(S^{p-1} \times S^{q-1}) \cong S^{p-1} \times S^{q-1}$ is the product metric

$$g_M|_{\iota(S^{p-1} \times S^{q-1})} = \rho^2 \cdot ds^2_{p-1} + N^2 \sin^2\left(\frac{R}{N}\right) ds^2_{q-1}$$

and, again by Example A.10, the second fundamental form with respect to this product structure is given by

$$\mathrm{I\!I}_{\iota(S^{p-1} \times S^{q-1})} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{N}\cot\left(\frac{R}{N}\right) \end{pmatrix}. \tag{5.4.3}$$

In general, the value $\frac{R}{N}$ can be very small, in which case $\mathrm{I\!I}_{\iota(S^{p-1} \times S^{q-1})}$ is negative definite. If it is non-negative, i.e. if $\frac{R}{N} \geq \frac{\pi}{2}$, we decrease R, so that $\mathrm{I\!I}_{\iota(S^{p-1} \times S^{q-1})}$ becomes negative definite.

We will equip the middle part of (5.4.1) with a metric of positive Ricci curvature such that we can glue it to the other parts using Theorem 5.2.2. The metric will be a doubly warped product metric, i.e. it will be given by

$$g_{f,h} = dt^2 + h^2(t) ds^2_{p-1} + f^2(t) ds^2_{q-1},$$

where $f, h\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ are smooth functions. By Corollary A.9, the second fundamental form of $g_{f,h}$ at a slice $t \geq 0$ with respect to ∂_t is given by

$$\mathrm{I\!I}_{\{t\} \times (S^{p-1} \times S^{q-1})} = \begin{pmatrix} \frac{h'(t)}{h(t)} & 0 \\ 0 & \frac{f'(t)}{f(t)} \end{pmatrix}. \tag{5.4.4}$$

Now, in order to glue according to the decomposition (5.4.1) using Theorem 5.2.2, we impose the following boundary conditions:

$$h(0) = \alpha \qquad f(0) = \alpha r \tag{5.4.5}$$
$$h'(0) \leq \lambda \qquad f'(0) \leq 0 \tag{5.4.6}$$
$$h(t_0) = \beta\rho \qquad f(t_0) = \beta N \sin(R/N) \tag{5.4.7}$$
$$h'(t_0) \geq 0 \qquad f'(t_0) \geq \cos(R/N)\,. \tag{5.4.8}$$

Here $\alpha, \beta, t_0 > 0$ can be chosen arbitrarily. Furthermore, in order to use Theorem 5.2.2, the metric $g_{f,h}$ needs to have positive Ricci curvature. For that, let $V \in TS^{p-1}$, $W \in TS^{q-1}$ be unit length vectors (with respect to the metric $g_{f,h}$). Then, by Corollary A.9, the Ricci curvatures are given as follows:

$$\mathrm{Ric}(\partial_t, \partial_t) = -(p-1)\frac{h''}{h} - (q-1)\frac{f''}{f}, \tag{5.4.9}$$
$$\mathrm{Ric}(V, V) = -\frac{h''}{h} + (p-2)\frac{1-(h')^2}{h^2} - (q-1)\frac{h'f'}{hf}, \tag{5.4.10}$$
$$\mathrm{Ric}(W, W) = -\frac{f''}{f} + (q-2)\frac{1-(f')^2}{f^2} - (p-1)\frac{h'f'}{hf}, \tag{5.4.11}$$
$$\mathrm{Ric}(\partial_t, V) = \mathrm{Ric}(\partial_t, W) = \mathrm{Ric}(V, W) = 0.$$

Figure 5.1 contains a sketch of how the graph of such functions h and f would typically look like.

Lemma 5.4.2. *If the functions h and f satisfy* (5.4.5)-(5.4.8) *and the Ricci curvatures* (5.4.9)-(5.4.11) *are positive, then the manifold $\hat{M}$ has a metric of positive Ricci curvature as claimed in Theorem A.*

Proof. We scale the metric g_E by α, so

$$(E \setminus \varphi_\pi(D^p \times S^{q-1})^\circ), \alpha^2 g_E(r, \theta)) \quad \text{and} \quad (I \times S^{p-1} \times S^{q-1}, g_{f,h})$$

have an isometric boundary component by (5.4.5). Scaling by α has the effect that the second fundamental form on $\varphi(S^{p-1})$ becomes bounded from below by $\frac{\lambda}{\alpha}$. Hence, by (5.4.6), (5.4.2) and (5.4.4) the requirements of Theorem 5.2.2 are satisfied for this boundary component (note that we need to reverse the signs in (5.4.4) since ∂_t is the inward normal vector on this boundary component). For the other boundary component we proceed similarly, i.e. we rescale the metric g_M by β so we have isometric boundary components by (5.4.7). Then by (5.4.8), (5.4.3) and (5.4.4) the requirements of Theorem 5.2.2 are satisfied. Now we apply Theorem 5.2.2 to glue according to the decomposition (5.4.1) and rescale the resulting metric by $\frac{1}{\alpha}$. ☐

To construct the functions we need the following existence result for initial value problems.

Lemma 5.4.3 ([113, Lemma 2.3, Theorem 2.13, Theorem 2.14]). *Let $\Phi\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be smooth and let $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$. Then the initial value problem*

$$f'(t) = \Phi(t, f(t)), \quad f(t_0) = x_0$$

has a unique smooth solution on a maximal interval I around t_0. Further, if for every $T > 0$ there exist constants $M(T), L(T)$, so that

$$|\Phi(t, x)| \leq M(T) + L(T)|x|$$

for all $(t, x) \in [-T, T] \times \mathbb{R}^n$, then $I = \mathbb{R}$.

We will now construct functions h and f satisfying the assumptions of Lemma 5.4.2.

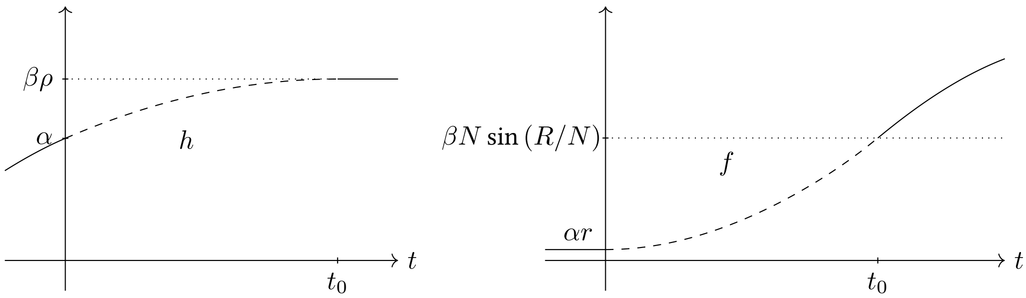


FIGURE 5.1: Sketch of the graph of the functions h and f. The dashed lines correspond to the part on which we have to construct the functions h and f such that they satisfy the boundary conditions (5.4.5)-(5.4.8) and such that the Ricci curvatures (5.4.9)-(5.4.11) are positive.

Definition 5.4.4. Let $h_0 \colon [0,\infty) \to \mathbb{R}$ be the unique smooth function satisfying

$$h_0' = e^{-\frac{1}{2}h_0^2},$$
$$h_0(0) = \sqrt{-2\ln\left(\min\left(\lambda, \frac{1}{2}\right)\right)}.$$

We use the minimum of λ and $\frac{1}{2}$ to cover the case $\lambda \geq 1$, in which $-2\ln(\lambda)$ would be non-positive. The function h_0 is indeed defined on all of $\mathbb{R}$ by Lemma 5.4.3: If we set $n = 1$ and $\Phi(t,x) = e^{-\frac{1}{2}x^2}$, then $h_0'(t) = \Phi(t, h_0(t))$. The condition on Φ in Lemma 5.4.3 is satisfied for $M(T) = 1$ and $L(T) = 0$.

Lemma 5.4.5. *We have*

1. $h_0'(0) = \min\left(\lambda, \frac{1}{2}\right) \leq \lambda$,
2. $h_0, h_0' > 0$,
3. $h_0'' = -h_0 e^{-h_0^2} < 0$,
4. $\lim_{t\to\infty} h_0(t) = \infty$.

Proof. We show that $\lim_{t\to\infty} h_0(t) = \infty$, the remaining statements follow directly from the definition. Since $h_0' > 0$, the function h_0 converges to a limit $L \in (0,\infty]$. If $L < \infty$, then $\lim_{t\to\infty} h_0'(t) = 0$. By the definition of h_0 we have $\lim_{t\to\infty} h_0'(t) = e^{-\frac{1}{2}L^2} > 0$, which is a contradiction. Hence $L = \infty$. □

Definition 5.4.6. For $C \in (0,1)$ let $f_C \colon [0,\infty) \to \mathbb{R}$ be the unique smooth function satisfying

$$f_C'' = Ce^{-h_0^2} f_C,$$
$$f_C(0) = 1,$$
$$f_C'(0) = 0.$$

The function f_C is indeed defined of all of $\mathbb{R}$: If we set $n = 2$ and $\Phi(t,(x,y)) = (y, Ce^{-h_0(t)^2}x)$, then $(f_C, f_C')'(t) = \Phi(t, (f_C, f_C')(t))$. The condition on Φ in Lemma 5.4.3 is satisfied for $M(T) = 0$ and $L(T) = 1$.

Lemma 5.4.7. *We have*

1. $f_C, f_C', f_C'' > 0$ *on* $(0, \infty)$,
2. $\lim_{t\to\infty} f_C(t) = \lim_{t\to\infty} f_C'(t) = \infty$,
3. $\lim_{t\to\infty} f_C(t)h_0'(t) = 0$,
4. $\frac{f_C'}{f_C h_0 h_0'} \in [0, 1]$.

Proof. Ad(1). By definition we have $f_C''(0) > 0$ and hence $f_C'(t) > 0$ for small $t > 0$. Now suppose there is $t > 0$ such that $f_C''(t) = 0$. Let t_0 be the smallest such t, which is positive by the initial conditions. The equation $f_C''(t_0) = 0$ implies $f_C(t_0) = 0$, hence there is $t_1 \in (0, t_0)$ such that $f_C''(t_1) = 0$ which is a contradiction. Hence $f_C'' > 0$ and $f_C, f_C' > 0$ follows from the initial conditions.

Ad(2). We have $\lim_{t\to\infty} f_C(t) = \infty$ since $f_C', f_C'' > 0$. Now set $u = f_C'$. A calculation shows that u satisfies the differential equation

$$u'' + 2h_0 e^{-\frac{1}{2}h_0^2}u' - Ce^{-h_0^2}u = 0.$$

The function h_0 is monotone increasing, hence it has an inverse $h_0^{-1}\colon [s_0, \infty) \to [0, \infty)$, where $s_0 = h_0(0) > 0$. We now define $v\colon [s_0, \infty) \to [0, \infty)$ by

$$v(s) = u(h_0^{-1}(s)).$$

Then v satisfies the differential equation

$$v''(s) + sv'(s) - Cv(s) = 0. \tag{5.4.12}$$

Further, the initial values for v are $v(s_0) = 0$ and $v'(s_0) = Ce^{-\frac{1}{2}s_0^2} > 0$. It follows that $v''(s_0) = -Cs_0e^{-\frac{1}{2}s_0^2} < 0$.

Suppose there is $s > s_0$ such that $v'(s) = 0$ and let s_1 be the smallest such s. Since $v'(s) > 0$ for $s \in [s_0, s_1)$ and $v(s_0) = 0$, it follows that $v(s_1) > 0$ and hence $v''(s_1) = Cv(s_1) > 0$. But $v'(s) > 0$ for $s \in [s_0, s_1)$ and $v'(s_1) = 0$, so $v''(s_1) \le 0$, which is a contradiction. Hence $v' > 0$.

By induction we have

$$v^{(k+2)}(s) + sv^{(k+1)}(s) - (C - k)v^{(k)}(s) = 0$$

for all $k \in \mathbb{N}_0$ and similarly as above we can now show inductively that $v^{(k+1)}$ does not change sign outside a compact set. Indeed, if $v^{(k+1)}(s) = 0$, then $v^{(k+2)}(s) = (C - k)v^{(k)}(s)$, so $v^{(k+2)}$ does not change sign on the zeros of $v^{(k+1)}$ outside a compact set. Hence, the function $v^{(k+1)}$ can change sign at most once outside this compact set. As a consequence, all $v^{(k)}$ converge to a limit. Let $L = \lim_{s\to\infty} v(s) \in (0, \infty]$ and suppose $L < \infty$. Then we have

$$\lim_{s\to\infty} v'(s) = \lim_{s\to\infty} v''(s) = 0$$

and by (5.4.12) it follows that

$$\lim_{s\to\infty} sv'(s) = CL.$$

In particular, there is $s_1 > s_0$, such that $sv'(s) > \frac{CL}{2}$ for all $s > s_1$. Hence, for $s \ge s_1$,

$$v(s) = v(s_1) + \int_{s_1}^{s} v'(r)dr > \int_{s_1}^{s} \frac{CL}{2r}dr = \frac{CL}{2}(\ln(s) - \ln(s_1)) \longrightarrow \infty$$

as $s \to \infty$, which is a contradiction. It follows that $L = \infty$ and therefore also $\lim_{s\to\infty} u(s) = \infty$.

Ad(3). We define the function $y\colon [s_0,\infty) \to \mathbb{R}$ by $y(s) = f_C(h_0^{-1}(s))h_0'(h_0^{-1}(s))$, so

$$y(s) = e^{-\frac{1}{2}s^2} f_C(h_0^{-1}(s))$$

and we need to show that y converges to 0 as $s \to \infty$. A calculation shows that y satisfies

$$y''(s) = -sy'(s) + (C-1)y(s)$$

with $y(s_0) = e^{-\frac{1}{2}s_0^2} > 0$ and $y'(s_0) = -s_0 e^{-\frac{1}{2}s_0^2} < 0$. By definition of y we have $y > 0$ and similarly as before we conclude that $y' < 0$ and y and all its derivatives converge. Since $y > 0$ and $y' < 0$, the limit L of y is finite and non-negative. In particular, both y' and y'' converge to 0, and by a similar argument as above it follows that $L = 0$.

Ad(4). We set $w = \frac{f_C'}{f_C}$. Then the function w satisfies

$$w' = Ce^{-h_0^2} - w^2$$

with $w(0) = 0$. We define $z\colon [s_0,\infty) \to \mathbb{R}$ by

$$z(s) = w(h_0^{-1}(s))\frac{e^{\frac{1}{2}s^2}}{s},$$

so that $z \circ h_0 = \frac{f_C'}{f_C h_0 h_0'}$. We need to show that $z \in [0,1]$. A calculation shows that z satisfies

$$z'(s) = -sz(s)^2 + \frac{s^2-1}{s}z(s) + \frac{C}{s}$$

with $z(s_0) = 0$ and hence $z'(s_0) = \frac{C}{s_0} > 0$, i.e. $z(s) \in (0,1)$ for s near s_0. If $z(s) = 0$ for $s > s_0$, then $z'(s) = \frac{C}{s} > 0$ and if $z(s) = 1$, then $z'(s) = \frac{C-1}{s} < 0$. This shows that z cannot leave the interval $(0,1)$. □

Definition 5.4.8. For $a, b > 0$ define $h_a = a \cdot h_0$ and $f_{b,C} = b \cdot f_C$.

The boundary conditions (5.4.5)-(5.4.8) can easily be satisfied with $h = h_a$ and $f = f_{b,C}$ by suitable choices of a and b, except perhaps the value of f at t_0, but this could for example be achieved by extending f and h by straight lines provided that the value of f at t_0 is less than $\beta N \sin(R/N)$. This is the reason why the constant κ appears in Theorem A.

Now we consider the Ricci curvatures (5.4.9)-(5.4.11).

Lemma 5.4.9. *For a, b, C small enough we have $\mathrm{Ric}(\partial_t, \partial_t) > 0$ and $\mathrm{Ric}(V,V) > 0$ for all t, and $\mathrm{Ric}(W,W) > 0$ for small values of t. Further, let $t_b > 0$ be the smallest value such that $\mathrm{Ric}(W,W)(t_b) = 0$. Then*

1. *$t_b \to \infty$ as $b \to 0$,*
2. *$f_{b,C}'(t_b) \to 1$ as $b \to 0$.*

Proof. We calculate

$$\mathrm{Ric}(\partial_t, \partial_t) = ((p-1) - (q-1)C)e^{-h_0^2},$$

which is positive when $C < \frac{p-1}{q-1}$. We also have

$$\mathrm{Ric}(V,V) = e^{-h_0^2} + (p-2)\frac{a^{-2} - e^{-h_0^2}}{h_0^2} - (q-1)\frac{f_C' h_0'}{f_C h_0}.$$

For $t = 0$ this expression is positive if $a^{-2} > e^{-h_0(0)^2}$. Now suppose there is $t > 0$ such that $\mathrm{Ric}(V, V)(t) = 0$ and let t_a be the minimal such t, which is positive if $a^{-2} > e^{-h_0(0)^2}$. At $t = t_a$ we then have

$$\frac{(p-2)}{a^2} = -h_0(t_a)^2 e^{-h_0(t_a)^2} + (p-2)e^{-h_0(t_a)^2} + (q-1)\frac{f_C'}{f_C h_0 h_0'}(t_a) h_0(t_a)^2 e^{-h_0(t_a)^2}.$$

The left-hand side converges to ∞ as $a \to 0$, while the right-hand side is uniformly bounded by Lemma 5.4.7. Hence, by choosing a sufficiently small, we can achieve that $\mathrm{Ric}(V, V)(t) > 0$ for all t.

Finally, we consider (5.4.11):

$$\mathrm{Ric}(W, W) = -Ce^{-h_0^2} + (q-2)\frac{b^{-2} - (f_C')^2}{f_C^2} - (p-1)\frac{h_0' f_C'}{h_0 f_C}.$$

By choosing C or b small enough, we can achieve that $\mathrm{Ric}(W, W)(t) > 0$ at $t = 0$. At $t = t_b$ we have

$$\begin{aligned}\frac{(q-2)}{b^2} &= f_C(t_b)^2 \left(Ce^{-h_0(t_b)^2} + (q-2)\frac{f_C'(t_b)^2}{f_C(t_b)^2} + (p-1)\frac{f_C' h_0'}{f_C h_0}(t_b) \right) \\ &= Cf_C(t_b)^2 h_0'(t_b)^2 + (q-2) f_C'(t_b)^2 + (p-1)\frac{f_C'}{f_C h_0 h_0'}(t_b) f(t_b)^2 h_0'(t_b)^2.\end{aligned}$$

By Lemma 5.4.7 both the first and third term are uniformly bounded, so $f_C'(t_b) \to \infty$ as $b \to 0$. Hence, $t_b \to \infty$ as $b \to 0$.

Rearranging the terms yields

$$f_{b,C}'(t_b)^2 = b^2 f_C'(t_b)^2 = 1 - \frac{b^2}{(q-2)} f_C(t_b)^2 h_0'(t_b)^2 \left(C + (p-1)\frac{f_C'}{f_C h_0 h_0'}(t_b) \right)$$

and the claim now follows from Lemma 5.4.7. ☐

Proof of Theorem A. By Lemma 5.4.2 it remains to show that there are values of a, b, C for which $h = h_a$ and $f = f_{b,C}$ satisfy the boundary conditions (5.4.5)-(5.4.8) and for which the Ricci curvatures (5.4.9)-(5.4.11) are positive. By Lemma 5.4.9, for a, b, C sufficiently small the Ricci curvatures are positive on $[0, t_b)$. By perhaps choosing a and b even smaller, we can achieve that (5.4.5) and (5.4.6) are satisfied and such that $f'(t_b) > \cos(R/N)$. Now choose $t_1 < t_b$ such that $f'(t_1) > \cos(R/N)$, i.e. the Ricci curvatures on $[0, t_1]$ are strictly positive. We now extend the functions f and h as follows: The function h gets extended by the constant function $h(t_1)$ and f gets extended continuously such that the following holds:

1. $f_-'(t_1) \geq f_+'(t_1)$,
2. f is smooth on (t_1, ∞),
3. $f'(t) \in (\cos(R/N), 1)$ for $t \geq t_1$,
4. $f''(t) < 0$ for $t \geq t_1$.

Then clearly all Ricci curvatures are positive. We now choose $t_0 > t_1$ such that $f(t_0) = N \sin(R/N)\frac{h(t_1)}{\rho}$, which exists if and only if $f(t_1) < N \sin(R/N)\frac{h(t_1)}{\rho}$. This is the case if and only if

$$\frac{\rho}{N} < \frac{h(t_1)}{f(t_1)} \sin\left(\frac{R}{N}\right).$$

The values of f and h at $t = t_1$ only depend on a, b, C and $\cos(R/N)$, which in turn only depend on p, q, λ, r and R/N. The value of λ only depends on the metric g_B. We define the constant κ as the expression on the left-hand side. Smoothing the functions f and h at $t = t_1$ using Corollary 5.4.1 finishes the proof. ☐

5.5 Comparison to Previous Results

The surgery theorem of Sha and Yang is given as follows. As in Theorem A assume that (M^n, g_M) is a Riemannian manifold and that there is an isometric embedding $\iota\colon S^{p-1}(\rho) \times D^q_R(N) \hookrightarrow M$, where $n = p + q - 1$.

Theorem 5.5.1 ([107, Lemma 1]). *Suppose that g_M has non-negative Ricci curvature and suppose $p \geq 2$, $q \geq 3$. Then there exists a constant $\kappa = \kappa(p, q, R/N) > 0$, such that if $\frac{\rho}{N} < \kappa$, then the manifold M_ι admits a metric of positive Ricci curvature.*

In order to modify the trivialization of the normal bundle of the embedded sphere $S^p(\rho)$, and in order to allow to again perform surgery on the part that gets glued in, Wraith proved a modified version of Theorem 5.5.1, which in turn requires a stricter condition on the dimensions involved.

Theorem 5.5.2 ([122, Theorem 0.3]). *Suppose that g_M has positive Ricci curvature and suppose that $p \geq q \geq 3$. Let $T\colon S^{p-1} \to \mathrm{SO}(q)$ be a smooth map, which induces a diffeomorphism $\tilde{T}\colon S^{p-1} \times D^q \to S^{p-1} \times D^q$ defined by $(x, y) \mapsto (x, T_x(y))$. Then there exists a constant $\kappa = \kappa(p, q, R/N, T) > 0$, such that if $\frac{\rho}{N} < \kappa$, then the manifold $M_{\iota\circ\tilde{T}}$ admits a metric of positive Ricci curvature.*

The metric constructed on $M_{\iota\circ\tilde{T}}$ coincides outside a neighborhood of the gluing area with the restriction of g_M to $M \setminus \mathrm{im}(\iota)$ and with $D^p_{R_1}(N_1) \times S^{q-1}(\rho_1)$ on $D^p \times S^{q-1}$. By [122, Proposition 0.4] the quotient $\frac{\rho_1}{N_1}$ can be bounded from above by any constant $\kappa' > 0$, which then gives an additional dependency for κ, while the quotient $\frac{R_1}{N_1}$ is constant independently of M, ι, T and κ'.

Under the assumption $B = S^p$ in Theorem A, the manifold $\hat{M}$ is precisely $M_{\iota\circ\tilde{T}}$, where $T\colon S^{p-1} \to \mathrm{SO}(q)$ is the clutching function of the bundle π. Hence, Theorem 5.5.2 follows from Theorem A. Further, if $T \equiv \mathrm{id}_{\mathbb{R}^q}$, so $\tilde{T}$ is the identity map, we obtain the statement of Theorem 5.5.1, except if $p = 2$. Hence Theorems 5.5.1 and 5.5.2 are both covered by Theorem A, except Theorem 5.5.1 with $p = 2$.

The proof strategies of Theorems 5.5.1 and 5.5.2 are similar to that of Theorem A with different choices of warping functions, and with the difference that we used Theorem 5.2.2 to connect each part, while in the proofs of Theorems 5.5.1 and 5.5.2 the warping functions are modified to connect smoothly to the other parts. We will briefly describe the choices of warping functions.

1. The warping functions in the proof of Theorem 5.5.1. The function f is defined as the unique solution of the equation

$$f'' = \frac{\alpha}{2} f^{-\alpha-1}, \quad f(0) = 1, \quad f'(0) = 0,$$

where $\alpha = \frac{2(q-2)}{p}$. The function h is defined as

$$h = \frac{2}{\alpha} f'.$$

Then $h(0) = 0$, $h'(0) = 1$, so $g_{f,h}$ has a singularity in $t = 0$, so that it descends to a metric on the space obtained from $I \times S^{p-1} \times S^{q-1}$ by collapsing $\{0\} \times S^{p-1} \times \{x\}$ for every $x \in S^{q-1}$, which is $D^p \times S^{q-1}$ (see e.g. [94, Proposition 1.4.7]). One can show that

$$f' = \sqrt{1 - f^{-\alpha}}.$$

As $t \to \infty$ we have

$$f(t) \to \infty, \quad f'(t) \to 1, \quad f''(t) \to 0$$

and that the Ricci curvatures (5.4.9) and (5.4.10) are strictly positive and the Ricci curvature (5.4.11) vanishes. Once f' is sufficiently close to 1, the second derivative f'' then gets modified to become constant 0, so that f becomes a straight line. This bending has the effect that all Ricci curvatures are positive, so that after gluing one can apply the Theorem 3.1.14 to deform the metric to have positive Ricci curvature everywhere.

In contrast to the proof of Theorem A, all Ricci curvatures are non-negative for all t, so rescaling the functions is not necessary. On the other hand, since $f'(t) \to 1$ (instead of $f'(t) \to \infty$ as in the proof of Theorem A), downscaling of f would make it impossible to satisfy the condition (5.4.8), while downscaling of h would destroy the property of $g_{f,h}$ defining a smooth metric on the quotient $D^p \times S^{q-1}$. Thus, we do not have any flexibility for the values of f and h at $t = 0$, and hence we cannot prescribe the metric on $D^p \times S^{q-1}$. On the other hand, this choice of warping functions gives slightly more flexibility for the dimensions involved.

2. The warping functions in the proof of Theorem 5.5.2. The starting point are functions f_0 and h_0, where f_0 is the unique solution of the equation

$$f_0'' = f_0^{-1}, \quad f_0'(0) = 1, \quad f_0'(0) = 0$$

and the function h_0 is defined by

$$h_0 = f_0'.$$

One can then show that $f_0' = \sqrt{2\ln(f_0)}$ and that

$$f_0(t), f_0'(t) \to \infty, \quad f_0''(t) \to 0$$

as $t \to \infty$. The Ricci curvature (5.4.9) is then given by

$$\mathrm{Ric}(\partial_t, \partial_t) = -(p-1)\frac{h_0''}{h_0} - (q-1)\frac{f_0''}{f_0} = ((p-1)-(q-1))\frac{1}{f_0^2} = (p-q)\frac{1}{f_0^2},$$

which is only non-negative if $p \geq q$. The other conditions are similar as in the proof of Theorem A: the Ricci curvature (5.4.10) is positive, while the Ricci curvature (5.4.11) is positive on a bounded interval and becomes negative eventually. The rest of the proof then consists of a sequence of modifications, such as shifting and scaling in order to satisfy the conditions (5.4.5)–(5.4.8) while keeping the Ricci curvatures positive.

Note that h_0 satisfies $h_0 = \sqrt{2\ln({h_0'}^{-1})}$, so

$$h_0' = e^{-\frac{1}{2}h_0^2}.$$

Further, as seen above,

$$\frac{h_0''}{h_0} = -\frac{f_0''}{f_0}$$

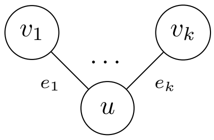
and $h_0'' = e^{-h_0^2}$, so

$$f_0'' = e^{-h_0^2} f_0.$$

Thus, the functions h_0 and f_0 satisfy the same differential equations as the warping functions h_0 and f_C with $C = 1$ in the proof of Theorem A (although with different initial conditions). Modifying the factor C then allows to have positive Ricci curvature for all $p, q \geq 3$.

The applications of Theorems 5.5.1 and 5.5.2 corresponding to Theorem B, which are proven entirely similarly, are given as follows:

Theorem 5.5.3 ([107, Theorem 2] and [121, Theorems 2.2 and 2.3]). *Let G be the geometric plumbing graph*



with compact base manifolds, and

(1) $p \geq 2$, $q \geq 3$, B_u^q a manifold that admits a metric of positive Ricci curvature, and π_{v_i} are trivial bundles with $B_{v_i} = S^p$ and π_u is trivial if $p = 2$, or

(2) $p \geq q \geq 3$, B_u^q a manifold that admits a metric of positive Ricci curvature, and $B_{v_i} = S^p$,

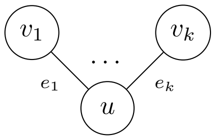
or let G be any simply-connected geometric plumbing graph with compact base manifolds, and

(3) $p = q \geq 3$, $B_{u_0}^q$ admits a metric of positive Ricci curvature for one $u_0 \in U$ and $B_u = B_v = S^q$ for all other $u \in U$ and all $v \in V$.

Then M_G admits a metric of positive Ricci curvature.

Again, note that Theorem B covers all cases in Theorem 5.5.3, except for item (1) with $p = 2$.

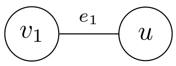
We now discuss applications of item (1), that are stated, but not proven in [107]. For that, let G be the geometric plumbing graph



in Theorem 5.5.3 and, as in item (1), assume that all bundles π_{v_i} are trivial with $B_{v_i} = S^p$. Then, by Proposition 4.3.2, the manifold M_G is diffeomorphic to

$$M_{G_0} \#_{k-1} (S^p \times S^{q-1}),$$

where G_0 is the subgraph



We get the following applications:

1. If $B_u = S^q$, then, by Proposition 4.3.3, $M_G \cong \#_{k-1}(S^p \times S^{q-1})$, which proves [107, Theorem 1], that is, Theorem 3.1.11.

2. If π_u is trivial, then M_G is the manifold $M_k^{p,(q-1)}$ in [107], so we also established [107, Theorem 2].

3. Let $p = 2$, $q = 4$ and let $B_u = \mathbb{C}P^2$ with π_u the trivial bundle. Then, by Lemma 5.3.2, the manifold M_{G_0} is diffeomorphic to $S^2 \tilde{\times} S^3$, so

$$M_G \cong (S^2 \tilde{\times} S^3) \#_{k-1} (S^2 \times S^3) = X_\infty \#_{k-1} M_\infty$$

according to (3.5.1). Together with the manifolds in item (1) with $p = q = 3$ these are precisely all simply-connected 5-manifolds with torsion-free homology, i.e. we showed [107, Theorem 5].

4. It is claimed in [107, (5)] that one can construct the manifold

$$\#_k(S^2 \times S^4)\#_l(S^3 \times S^3)$$

for all $l, k \geq 0$ in this way with π_u trivial. While all of the summands admit core metrics by Proposition 5.2.7 and Theorem C, this technique was not available at the time when [107] was published.

If $p = 3$, $q = 4$, then, by Corollary 4.4.6, M_G can only be spin if B_u is spin, and the only closed, simply-connected spin 4-manifolds that are known to admit a metric of positive Ricci curvature are $\#_m(S^2 \times S^2)$ for all $m \geq 0$ (cf. Proposition 3.5.1). Then, by Corollary 4.4.6, together with the classification of Wall, see Theorem 6.1.2 below, M_{G_0} is diffeomorphic to $\#_{2m}(S^2 \times S^4)$, so M_G is diffeomorphic to

$$\#_{2m}(S^2 \times S^4)\#_{k-1}(S^3 \times S^3).$$

If $p = 4$, $q = 3$, then $B_u = S^3$, so M_{G_0} is diffeomorphic to S^6, so M_G is diffeomorphic to

$$\#_{k-1}(S^3 \times S^3).$$

If $p = 2$, $q = 5$, then B_u is a simply-connected 5-manifold. To construct a spin manifold with torsion-free homology, B_u itself needs to be a spin manifold with torsion-free homology by Theorems 4.4.3 and 4.4.5, so B_u is diffeomorphic to the manifold $\#_m(S^2 \times S^3)$. Then, again by Theorems 4.4.3 and 4.4.5, together with the classification of Wall, M_{G_0} is diffeomorphic to $\#_m(S^2 \times S^4)\#_m(S^3 \times S^3)$. Hence, M_G is diffeomorphic to

$$\#_m(S^2 \times S^4)\#_{m+k-1}(S^3 \times S^3).$$

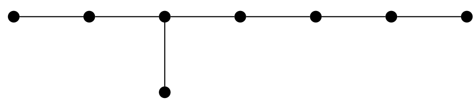
Thus, we cannot construct the manifolds $\#_k(S^2 \times S^4)\#_l(S^3 \times S^3)$ with k odd and $l < k$ in this way.

If we set $B_u = \#_l\mathbb{C}P^2$ and π_u is the unique linear S^2-bundle over B corresponding to $(0, (1, \ldots, 1))$ according to Corollary B.3.9, then, by Corollary 4.4.6 and the classification of Wall, M_{G_0} is diffeomorphic to $\#_l(S^2 \times S^4)$, so M_G is diffeomorphic to

$$\#_l(S^2 \times S^4)\#_{k-1}(S^3 \times S^3).$$

However, in [107] the case where the bundle π_u is non-trivial is not considered, and at the time when [107] was published it was not known whether $\#_l\mathbb{C}P^2$ for $l > 2$ admits a metric of positive Ricci curvature.

Finally, for items (2) and (3) of Theorem 5.5.3, plumbing certain linear D^{4k}-bundles over S^{4k} according to the E8 graph



yields a generator of the group of $(4k-1)$-dimensional homotopy spheres that bound parallelizable manifolds, see [121]. Further, by Proposition 4.3.2, disjoint unions of these graphs can be modified to a connected graph. In dimension $4k+1$, the Kervaire sphere and the standard sphere are the only homotopy spheres that bound parallelizable manifolds. The Kervaire sphere (which is standard in certain dimensions) is the boundary of the manifold obtained by plumbing together two copies of the tangent disc bundle of S^{2k+1}. Hence, by items (2) and (3) of Theorem 5.5.3, all homotopy spheres that bound parallelizable manifolds admit a metric of positive Ricci curvature, see [121]. These include in particular all homotopy spheres in dimension 7.

More generally, it was shown later by Crowley and Wraith [33], that by plumbing linear D^{2k}-bundles over S^{2k} according to a simply-connected graph, one can construct every $(2k-2)$-connected $(4k-1)$-manifolds, whose tangent bundle on its $(2k-1)$-skeleton is trivial, up to connected sum with a homotopy sphere. Hence, every such manifold, after possibly taking a connected sum with a homotopy sphere, admits a metric of positive Ricci curvature. These include all 2-connected 7-manifolds and all 4-connected 11-manifolds. Further, a similar result holds in dimension $4k+1$, if one requires the homology of the manifolds to be torsion-free.

Applications to Manifolds in Dimension $6k$ 6

In this chapter we consider applications of Theorem B to manifolds in dimension $6k$. In Section 6.1 we review the invariants of closed, simply-connected $6k$-dimensional manifolds with torsion-free homology and state the classification results of Wall and Jupp for these manifolds in dimension 6. In Section 6.2 we introduce the notion of algebraic plumbing graphs and prove Theorem D. Further, in Section 6.3 we introduce a reduced form for algebraic plumbing graphs and classify these for low ranks in Section 6.4. We also give an extensive discussion of which closed, simply-connected spin 6-manifolds with torsion-free homology and $b_2 = 2$ can be constructed via an algebraic plumbing graph and thus admits a core metric. Finally, in Section 6.5 we consider further applications.

6.1 Invariants of Closed, Simply-Connected $6k$-Dimensional Manifolds with Torsion-Free Homology

Let M^{6k} be a closed, simply-connected manifold with torsion-free homology. We choose an orientation on M. Then we have the following:

- The cohomology group $H^{2k}(M)$, which is free abelian,
- The trilinear form $\mu_M \colon H^{2k}(M) \times H^{2k}(M) \times H^{2k}(M) \to \mathbb{Z}$ defined by

  $$\mu_M(x, y, z) = \langle x \smile y \smile z, [M] \rangle,$$

  which is symmetric by the graded commutativity of the cup product,
- The k-th power of the second Stiefel-Whitney class

  $$w_2(M)^k \in H^{2k}(M; \mathbb{Z}/2) \cong H^{2k}(M) \otimes \mathbb{Z}/2$$

  by the universal coefficient theorem (see e.g. [34, Theorem 2.33]),
- The Pontryagin class $p_k(M) \in H^{4k}(M)$, which we view as an element of $\mathrm{Hom}(H^{2k}(M), \mathbb{Z})$ via

  $$p_k(M)(x) = \langle p_k(M) \smile x, [M] \rangle.$$

Note that all these invariants are defined on $H^{2k}(M)$. For a finitely generated free abelian group H, a symmetric trilinear form μ on H, an element $w \in H \otimes (\mathbb{Z}/2)$ and a linear form p on H we call (H, μ, w, p) a *system of invariants*. Two systems of invariants (H, μ, w, p) and (H', μ', w', p') are *equivalent*, if there exists an isomorphism $\phi\colon H \to H'$ such that

$$\phi^*\mu' = \mu, \quad \phi(w) = w' \quad \text{and } \phi^*p' = p.$$

A system of invariants (H, μ, w, p) is called *admissible in dimension* $6k$, if it can be realized by a closed, simply-connected manifold M^{6k} with torsion-free homology, i.e. the systems of invariants (H, μ, w, p) and $(H^{2k}(M), \mu_M, w_2^k(M), p_k(M))$ are equivalent.

Note that the equivalence class of $(H^{2k}(M), \mu_M, w_2^k(M), p_k(M))$ is in fact independent of the orientation on M, since $-M$ yields the system of invariants $(H^{2k}(M), -\mu_M, w_2^k(M), -p_k(M))$, which is obtained from the first one via the isomorphism $-\mathrm{id}_{H^{2k}(M)}$.

Since a diffeomorphism between two manifolds induces an isomorphism between the cohomology groups that preserves the characteristic classes and the cup product, it follows that diffeomorphic manifolds have equivalent systems of invariants. The converse does not hold: In general there are additional degrees in which one can have non-trivial cohomology groups and cup products. Further, even if we require that the cohomology groups in all degrees except in degrees 0, $2k$, $4k$ and $6k$ are trivial, our systems of invariants are not sufficient to uniquely determine the diffeomorphism type, as there are exotic spheres Σ^{6k} for some k, see e.g. [68], so that all invariants vanish. This shows that Σ^{6k} and S^{6k} have the same invariants, but are not diffeomorphic.

However, in dimension 6, systems of invariants are almost sufficient to determine the diffeomorphism type. For that, let M^6 be a closed, simply-connected and oriented 6-manifold M with torsion-free homology. Additionally to its system of invariants $(H^2(M), \mu_M, w_2(M), p_1(M))$ we also have the third Betti number $b_3(M) \in \mathbb{N}_0$. We now have the following classification result by Jupp [67].

Theorem 6.1.1 ([67, Theorem 1]). *Orientation-preserving diffeomorphism classes of closed oriented simply-connected 6-manifolds with torsion-free homology are in bijection with equivalence classes of systems of invariants (H, μ, w, p) together with non-negative integers $r \in \mathbb{N}_0$, such that*

$$\mu(W) \equiv p(W) \mod 48 \tag{6.1.1}$$

holds for all $W \in H$ that restrict to w. The bijection assigns to a manifold M the system of invariants $(H^2(M), \mu_M, w_2(M), p_1(M))$ and the integer $b_3(M)/2$.

Theorem 6.1.1 shows that a system of invariants is admissible in dimension 6 if and only if it satisfies (6.1.1).

In the spin case, Theorem 6.1.1 simplifies to the following Theorem by Wall [115].

Theorem 6.1.2 ([115, Theorem 5]). *Orientation-preserving diffeomorphism classes of closed oriented simply-connected spin 6-manifolds with torsion-free homology are in bijection with equivalence classes of systems of invariants $(H, \mu, 0, p)$ together with non-negative integers $r \in \mathbb{N}_0$, such that*

$$4\mu(X) \equiv p(X) \mod 24 \tag{6.1.2}$$

holds for all elements $X \in H$. The bijection assigns to a manifold M the system of invariants $(H^2(M), \mu_M, 0, p_1(M))$ and the integer $b_3(M)/2$.

Remark 6.1.3. 1. In [115, Theorem 5], Wall adds the additional relation

$$\mu(X,X,Y) \equiv \mu(X,Y,Y) \mod 2$$

for all $X,Y \in H$. However, this relation already follows from (6.1.2). In fact, we have

$$\begin{aligned} 3(\mu(X,X,Y)+\mu(X,Y,Y)) &= \mu(X+Y,X+Y,X+Y)-\mu(X,X,X)-\mu(Y,Y,Y) \\ &\equiv \frac{1}{4}(p(X+Y)-p(X)-p(Y)) = 0 \mod 6, \end{aligned}$$

so $\mu(X,X,Y)+\mu(X,Y,Y) \equiv 0 \mod 2$.

2. Theorem 6.1.1 has been generalized by Zhubr [126] to the class of all closed, simply-connected 6-manifolds.

Theorem 6.1.1 shows that the classification of closed, simply-connected 6-manifolds with torsion-free homology is equivalent to the classification of admissible systems of invariants up to equivalence. However, there is no classification known of the latter, except if rk$(H)=1$ (this is obvious) or if rk$(H)=2$, in which case there exists a partial classification by Schmitt [100]. It is given as follows.

For every admissible system of invariants (H,μ,w,p) with rank$(H)=2$, so that p is non-trivial or $D(f_\mu)=0$ (see Appendix D for the definition of D), Schmitt introduces a normal form, i.e. he shows that there exists precisely one choice of basis for H, on which the other invariants fall into one of 36 explicitly given families. It follows that invariants with different normal forms are non-equivalent. Due to the complexity of these families we will focus on the spin case, i.e. admissible systems of invariants (H,μ,w,p) with rank$(H)=2$ and $w=0$. If (u,v) is a basis of H, we call

$$((\mu(u,u,u),\mu(u,u,v),\mu(u,v,v),\mu(v,v,v)),(p(u),p(v)))$$

the *coefficients* of the invariants in the basis (u,v). Note that the coefficients uniquely determine the invariants.

The results of [100, Section 3.4] can be summarized as follows:

Theorem 6.1.4 ([100, Propositions 5–10]). *Let (H,μ,w,p) be an admissible system of invariants with* rank$(H)=2$*, $w=0$ and so that p is non-trivial or $D(f_\mu)=0$. If one of μ and p is non-trivial, then there exists precisely one basis (u,v) of H, so that the coefficients in this basis are contained in one of the following families, where all constants involved are integers:*

$$\begin{aligned} P_1 &= \{((r,0,0,0),(4r+24k,0)) \mid 4r+24k>0\}, \\ Q_1 &= \{((r,2\rho,0,0),(4r+24k,0)) \mid 0 \le r < 6\rho,\ 4r+24k>0\}, \\ R_1 &= \{((r_1,r_2,r_2+2l,0),(4r_1+24k,0)) \mid r_2 \ge 0, l>0 \textit{ or } l<-r_2;\ 4r_1+24k>0\}, \\ R_1' &= \{((r_1,|r_3|,r_3,0),(4r_1+24k,0)) \mid r_3 \ne 0,\ 4r_1+24k>0\}, \\ S_1 &= \{((r_1,r_3+2l,r_3,6\rho_4),(4r_1+24k,0)) \mid 0 \le r_3 < 6\rho_4,\ 4r_1+24k>0\}, \\ K_1 &= \{((6\rho,0,0,0),(0,0)) \mid \rho>0\}, \\ L_1 &= \{((6\rho_1,2\rho_2,0,0),(0,0)) \mid 0 \le \rho_1 < \rho_2\}. \end{aligned}$$

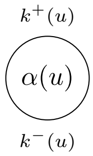
Conversely, any tuple in these families are coefficients of an admissible system of invariants.

Coefficients contained in one of these families are called *normal forms*. Algorithm E.1 brings a given admissible system of invariants into its normal form.

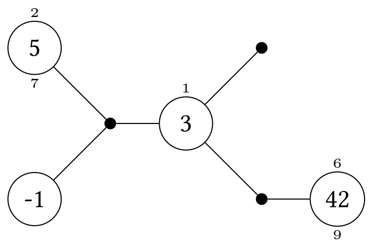
6.2 Algebraic Plumbing Graphs

In this Section we introduce the notion of algebraic plumbing graphs. Let $G = (U, V, E, (\alpha, k^+, k^-))$ be a bipartite graph, which has a labeling $(\alpha, k^+, k^-)\colon U \to \mathbb{Z} \times \mathbb{N}_0^2$ for vertices in U. We call such a graph an *algebraic plumbing graph*.

We will draw vertices $u \in U$ as



If one of $k^+(u)$ and $k^-(u)$ vanishes, then we will omit it. Vertices in V will be drawn as dots (as they do not have any labeling). An example for such a graph is the graph from the introduction:



For every $u \in U$ we introduce the symbols $u^{-k^-(u)}, \ldots, u^{k^+(u)}$ and define the free abelian group

$$A = \bigoplus_{u \in U} \bigoplus_{i=-k^-(u)}^{k^+(u)} \mathbb{Z}u^i.$$

For $k \in \mathbb{N}$ we define the symmetric trilinear form $\mu^k\colon A^3 \to \mathbb{Z}$ by defining it for each $u \in U$ on $\bigoplus_{i=-k^-(u)}^{k^+(u)} \mathbb{Z}u^i$ by

$$\begin{aligned}
\mu^k(u^0, u^0, u^0) &= \frac{\lambda_k}{4}\alpha(u), \\
\mu^k(u^0, u^0, u^l) &= 0, \\
\mu^k(u^0, u^j, u^l) &= \begin{cases} \mathrm{sgn}(j), & j = l, \\ 0, & else, \end{cases} \\
\mu^k(u^i, u^j, u^l) &= 0
\end{aligned}$$

for $i, j, l \in \{-k^-(u), \ldots, k^+(u)\} \setminus \{0\}$, where $\lambda_k \in \mathbb{N}$ is the constant from Lemma B.3.10 (note that it is a multiple of 4). Then extend μ^k to A by setting

$$\mu^k(u_m^i, u_n^j, u_r^k) = 0$$

whenever any two of u_m, u_n, u_r are not equal.

Next we define a linear form $p^k\colon A \to \mathbb{Z}$ by

$$p^k(u^j) = \begin{cases} \lambda_k \alpha(u) + \binom{2k+1}{k}(k^+(u) - k^-(u)), & j = 0, \\ 0, & \text{else}, \end{cases}$$

and we define $w_G \in A \otimes \mathbb{Z}/2$ by

$$w_G = \sum_{\substack{u \in U, \\ i \neq 0}} u^i \mod 2.$$

Finally, we set

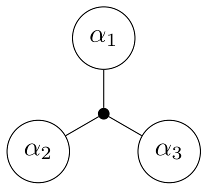
$$H_G = \bigoplus_{\substack{u \in U \\ i \neq 0}} \mathbb{Z}u^i \oplus \left\{ \sum_{u \in U} \lambda_u \cdot u^0 \middle| \lambda_u \in \mathbb{Z}, \sum_{e=(u,v) \in E} \lambda_u = 0 \text{ for all } v \in V \right\} \subseteq A$$

and denote the restrictions of μ^k and p^k to H_G by μ_G^k and p_G^k, respectively (and note, that, by definition, we have $w_G \in H_G \otimes \mathbb{Z}/2$). Further, we set $\mu_G = \mu_G^1$ and $p_G = p_G^1$.

Definition 6.2.1. We call H_G, μ_G^k, w_G and p_G^k the *invariants of* G and we define the *rank of* G by $\operatorname{rank}(H_G)$. We say that G is *spin* if $k^+ = k^- \equiv 0$. Two algebraic plumbing graphs G and G' are k*-equivalent*, denoted $G \sim_k G'$, if the systems of invariants $(H_G, \mu_G^k, w_G, p_G^k)$ and $(H_{G'}, \mu_{G'}^k, w_{G'}, p_{G'}^k)$ are equivalent.

Remark 6.2.2. In the spin case, if two algebraic plumbing graphs are k-equivalent for one k, then they are k-equivalent for all k, since then $\mu_G^k = \frac{\lambda_k}{4}\mu_G$ and $p_G^k = \frac{\lambda_k}{4}p_G$ (recall that $\lambda_1 = 4$). However, in the non-spin case it is not clear if k-equivalence for one k implies k-equivalence for other, or all, k.

Example 6.2.3. Consider the following graph:



Denote by u_i the vertex labeled by α_i. Then

$$A = \mathbb{Z}u_1^0 \oplus \mathbb{Z}u_2^0 \oplus \mathbb{Z}u_3^0$$

and $w_G = 0$. A basis for H_G is given by

$$e_1 = u_1^0 - u_3^0, \quad e_2 = u_2^0 - u_3^0.$$

In this basis we have

$$\begin{aligned}
\mu_G^k(e_1, e_1, e_1) &= \frac{\lambda_k}{4}(\alpha_1 - \alpha_3), \\
\mu_G^k(e_1, e_1, e_2) &= -\frac{\lambda_k}{4}\alpha_3, \\
\mu_G^k(e_1, e_2, e_2) &= -\frac{\lambda_k}{4}\alpha_3, \\
\mu_G^k(e_2, e_2, e_2) &= \frac{\lambda_k}{4}(\alpha_2 - \alpha_3), \\
p_G^k(e_1) &= \lambda_k(\alpha_1 - \alpha_3), \\
p_G^k(e_2) &= \lambda_k(\alpha_2 - \alpha_3).
\end{aligned}$$

We will now define a geometric plumbing graph $\overline{G}^k = (U, V, E, \pi, \delta)$ with the same set of vertices and edges as G. For $u \in U$ set

$$B_u = \#_{k^+(u)} \mathbb{C}P^{2k} \#_{k^-(u)} (-\mathbb{C}P^{2k})$$

(note that the empty connected sum is defined as S^{4k}) and define π_u as the disc bundle of the sphere bundle corresponding to $\alpha(u)$ in Lemma B.3.10. For $v \in V$ we define $B_v = S^{2k+1}$ and π_v as the trivial D^{4k}-bundle over S^{2k+1}, i.e. $\overline{E}_v = S^{2k+1} \times D^{4k}$ and π_v is given by the projection onto the first factor. Finally, we set $\delta(e) = 1$ for all $e \in E$. We set $\overline{G} = \overline{G}^1$. The main result of this chapter is the following:

Theorem D. *Let G be an algebraic plumbing graph.*

- *If $k = 1$, then the system of invariants (H_G, μ_G, w_G, p_G) is admissible in dimension 6.*
- *If every connected component of G is simply-connected, then the system of invariants $(H_G, \mu_G^k, w_G, p_G^k)$ is admissible in dimension $6k$ and realized by the manifold $M_{\overline{G}^k}$. Further, $M_{\overline{G}^k}$ admits a core metric.*
- *If $k = 1$ and every connected component of G is simply-connected, then any closed, simply-connected 6-manifold with torsion-free homology, whose invariants are equivalent to (H_G, μ_G, w_G, p_G), admits a core metric.*

For the proof we first need the following lemma.

Lemma 6.2.4. *Let $G = (U, V, E, (\alpha, k^+, k^-))$ be an algebraic plumbing graph for which all connected components are simply-connected and set $M = M_{\overline{G}^k}$. Then*

1. *M is a closed, simply-connected $6k$-dimensional manifold with torsion-free homology and the systems of invariants $(H^{2k}(M), \mu_M, w_2^k(M), p_k(M))$ and $(H_G, \mu_G^k, w_G, p_G^k)$ are equivalent. In particular, the system $(H_G, \mu_G^k, w_G, p_G^k)$ is admissible in dimension $6k$.*
2. *There exists a k-equivalent subgraph G' of G so that for $M' = M_{\overline{G'}^k}$ the same as in (1) holds and additionally the odd Betti numbers of M' vanish and $M = M' \#_r (S^{2k+1} \times S^{4k-1})$ for some $r \in \mathbb{N}_0$.*
3. *$b_{2k}(M') = b_{2k}(M) = |U'| - |V'| + \sum_{u \in U'} k^+(u) + k^-(u)$.*
4. *M and M' are spin if and only if G and G' are spin.*

Proof. We use Proposition 4.3.2 to split $\overline{G}^k$ into connected components that either satisfy the hypotheses of Theorem 4.4.3 or consist of a single vertex in V as follows: For any $u \in U$ that is connected to a leaf, we remove all edges connected to u except one that connects u to a leaf, this is precisely the modification in Proposition 4.3.2. The corresponding modification of G does not change its invariants, since for such u we always have that u^0 does not appear as a non-zero summand for elements of H_G. We repeat this process until all connected components of $\overline{G}^k$ satisfy the hypotheses of Theorem 4.4.3 and we denote the graph we obtain from G in this way after additionally removing all isolated vertices in V by G'. Then G and G' have the same invariants, since isolated vertices in V do not make any contribution, and the manifolds M and M' then only differ by connected sums of copies of $S^{2k+1} \times S^{4k-1}$.

By Corollary 4.4.6, the cohomology group $H^{2k}(M)$ is given by

$$H^{2k}(M) = \bigoplus_{u\in U} \pi_u^* H^{2k}(B_u) \oplus \left\{ \sum_{u\in U} \lambda_u \cdot a_u \mid \lambda_u \in \mathbb{Z}, \sum_{e=(u,v)\in E'} \delta(e)\lambda_u = 0 \text{ for all } v \in V' \right\}$$

and we define the isomorphism ϕ by mapping a generator of the i-th summand on the left-hand side of

$$H^{2k}(B_u) = \bigoplus_{k^+(u)} H^{2k}(\mathbb{C}P^{2k}) \bigoplus_{k^-(u)} H^{2k}(-\mathbb{C}P^{2k})$$

to u^i and a generator of the i-th summand on the right-hand side to u^{-i}. Further, we define

$$\phi\left(\sum_{u\in U} \lambda_u \cdot a_u\right) = \sum_{u\in U} \lambda_u u^0.$$

It now follows from Theorem 4.4.3 and Corollary B.3.13, that this isomorphism preserves the remaining invariants.

For (3) we need to determine the rank of

$$\left\{ \sum_{u\in U} \lambda_u \cdot u^0 \middle| \sum_{e=(u,v)\in E'} \lambda_u = 0 \text{ for all } v \in V' \right\} \subseteq H_{G'}.$$

The condition for the coefficients λ_u is equivalent to $(\lambda_u)_{u\in U}$ being an element of the kernel of $B(G')^\top$. By Lemma C.3, the matrix $B(G')$ has rank $|V'|$ (as $|U'| \geq |V'|$), hence its kernel has rank $|U'| - |V'|$.

Finally, for (4), note that by (1) w_G (and $w_{G'}$) vanishes if and only if $w_2(M)^k$ (and $w_2(M')^k$) vanishes, which is the case if and only if $w_2(M)$ (and $w_2(M')$) vanishes by the cohomology ring structure of each E_u. □

We can now prove Theorem D.

Proof of Theorem D. First consider an arbitrary algebraic plumbing graph G. By construction, the group H_G is a subgroup of A and the invariants μ_G and p_G are the restrictions of the invariants μ^1 and p^1. Let G^0 be the graph obtained from G be removing all edges. Then the invariants of G^0 are precisely A, μ^1, w_G and p^1 and the system of invariants (A, μ^1, w_G, p^1) is realized by $M_{\overline{G^0}}$ by (1) of Lemma 6.2.4. It follows that (6.1.1) holds for all $W \in A$, and hence also for all $W \in H_G$, that restrict to w_G. Thus, the system (H_G, μ_G, w_G, p_G) is admissible in dimension 6.

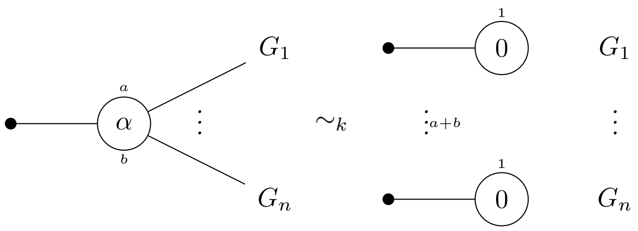
Now assume that every connected component of G is simply-connected. Let $M = M_{\overline{G}^k}$. We apply Lemma 6.2.4 to obtain an equivalent subgraph G' of G and a manifold $M' = M_{\overline{G'}^k}$ with vanishing odd cohomology. Since each connected component of G' is simply-connected, we can apply Theorem B to obtain a core metric on each summand of M', hence M' also admits a core metric; If $k = 1$, then the restrictions on the dimensions in this Theorem require that every connected component contains a vertex in V. This can always be achieved by introducing a new vertex according to Proposition 4.3.4. Since $M = M' \#_r (S^{2k+1} \times S^{4k-1})$ for some $r \in \mathbb{N}_0$, and $S^{2k+1} \times S^{4k-1}$ admits a core metric by Proposition 5.2.7, M admits a core metric.

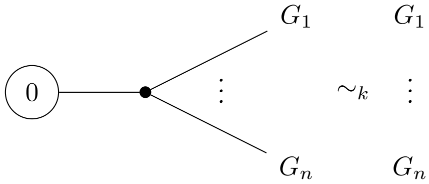
Finally, if $k = 1$ and N is a simply-connected 6-manifold with torsion-free homology, whose invariants are equivalent to (H_G, μ_G, w_G, p_G), then, by Theorem 6.1.1, N is diffeomorphic to $M' \#_r (S^3 \times S^3)$ for some $r \in \mathbb{N}_0$, so N admits a core metric. □

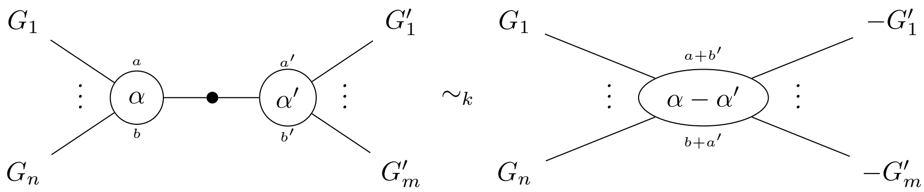
6.3 Reduced Graphs

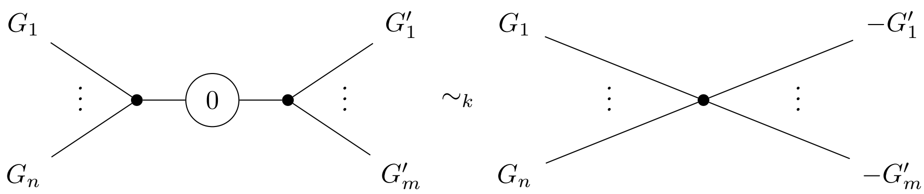
A system of invariants (H, μ^k, w, p^k) can potentially be realized by many different algebraic plumbing graphs. To analyze this, we consider modifications of graphs that do not change the invariants.

Lemma 6.3.1. *We can modify graphs in the following ways without changing their k-equivalence classes.*

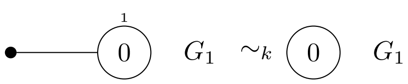
(1) 

(1′) 

(2) 

(2′) 

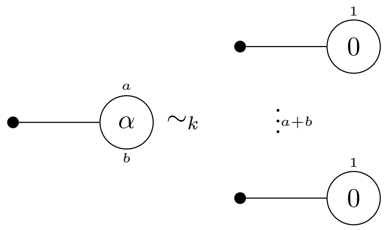
(3) $G_1 \quad \bullet \quad \sim_k \quad G_1$

(3′)  *if G_1 is not spin*

(4) $G_1 \quad G_2 \quad \sim_k \quad -G_1 \quad G_2$

Here G_i, G'_i are (pairwise distinct, and possibly empty) subgraphs, and $-G$ denotes G with α replaced by $-\alpha$ and (k^+, k^-) replaced by (k^-, k^+).

Proof. First note, that $\overline{-G}^k$ is obtained from $\overline{G}^k$ by reversing the orientations of all bases and fibers (but not of the total spaces). Then the equivalences (3) and (4) are clear and the remaining equivalences follow from Propositions 4.3.1, 4.3.2 and 4.3.4, except (1) and $(3')$. For (1) we additionally need to show that



holds. For that, denote the graph on the left-hand side by G and the graph on the right-hand side by G'. Denote the single element of U by u and the elements of U' by $u_1, \dots, u_{a+b}$. Then, by definition, we have

$$H_G = \bigoplus_{\substack{i=-b \\ i\neq 0}}^{a} \mathbb{Z}u^i \quad \text{and} \quad H_{G'} = \bigoplus_{i=1}^{a+b} \mathbb{Z}u_i^1,$$

and an isomorphism $H_G \to H_{G'}$ is given by mapping u^i to u_i^1 for $i > 0$ and u^i to u_{a-i}^1 for $i < 0$. It is now easily verified that μ_G^k, p_G^k, $\mu_{G'}^k$ and $p_{G'}^k$ all vanish, and that

$$w_G = \sum_{\substack{u\in U, \\ i\neq 0}} u^i \mod 2, \quad w_{G'} = \sum_{i=1}^{a+b} u_i^1 \mod 2,$$

hence this isomorphism preserves all invariants.

For $(3')$ denote the graph on the left-hand side by G and the one on the right-hand side by G'. Let $x_0 \in H_{G_1}$ be a primitive element that restricts to w_{G_1} (which exists since w_{G_1} is non-trivial) and extend it to a basis $(x_0, \dots, x_n)$ of H_{G_1}. Let u_1 be the additional vertex in G and u_2 the additional vertex in G'. Then

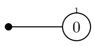
$$w_G = u_1^1 + x_0 \mod 2, \quad w_{G'} = x_0 \mod 2.$$

Hence, by mapping u_1^1 to u_2^0, x_0 to $x_0 - u_2^0$ and x_i to x_i for $i > 0$, we obtain an isomorphism $H_G \to H_{G'}$ that maps w_G to $w_{G'}$. Since the linear and trilinear forms are only non-trivial on elements of H_{G_1}, they are also preserved under this isomorphism.

□

These modifications will be used to bring a given graph into a reduced form.

Definition 6.3.2. Let $G = (U, V, E, (\alpha, k^+, k^-))$ be an algebraic plumbing graph. We call G *reduced*, if it satisfies the following conditions:

- Every connected component of G is simply-connected.
- The graph  only appears as a connected component in G if it is the only non-spin connected component. Every $v \in V$ not contained in this connected component has degree at least 3, and
- Every $u \in U$ with $\alpha(u) = k^+(u) = k^-(u) = 0$ has degree 0 or at least 3.

On every reduced graph the group $(\mathbb{Z}/2)^m$ acts, where m is the number of connected components, by multiplying the i-th connected component by (-1). The orbit of a reduced graph under this action is called its *reduced class*. Two reduced classes are *isomorphic*, if there are two reduced graphs, one contained in each reduced classes, that are isomorphic as labeled bipartite graphs.

Note that, by (4) of Lemma 6.3.1, reduced graphs in the same class are k-equivalent.
The following result is now a direct consequence of Lemmas 6.2.4 and 6.3.1.

Corollary 6.3.3. *Let $G = (U, V, E, (\alpha, k^+, k^-))$ be an algebraic plumbing graph. If every connected component of G is simply-connected, then G is k-equivalent to a reduced graph $G' = (U', V', E', \alpha')$. Further, we have* $\mathrm{rank}(H_G) = |U'| - |V'| + \sum_{u \in U'} k^+(u) + k^-(u)$.

It is now a natural question, how "good" this notion of reduced form is, i.e. if non-isomorphic reduced classes contain non-equivalent graphs.

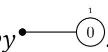
Question 6.3.4. Let G_1, G_2 be reduced graphs that are k-equivalent for some k. Are their reduced classes isomorphic?

This question is open, but we will answer it affirmatively in some special cases in the next section.

6.4 Reduced Graphs of Low Rank

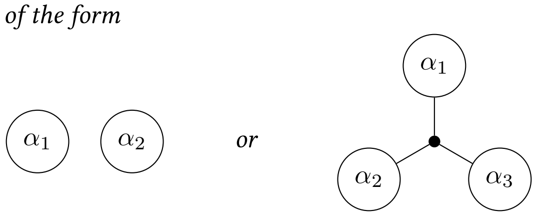
In this section we classify reduced graphs of rank at most 3 up to isomorphism of reduced classes. We also consider the problem of classification up to equivalence and obtain an almost complete result for ranks at most 2. Clearly the only reduced graph of rank 0 is the empty graph, which defines S^{6k}.

Proposition 6.4.1. *Let G be a reduced graph of rank 1. Then the following assertions hold:*

- *If G is spin, then it is of the form* α. *The manifold $M_{\overline{G}^k}$ is the total space of a linear S^{2k}-bundle over S^{4k}. Two graphs* α_1 *and* α_2 *are k-equivalent if and only if $\alpha_1 = \pm\alpha_2$.*
- *If G is not spin, then it is given by* . *The graph G has trivial trilinear form μ_G^k and trivial linear form p_G^k. If $k = 1$, then the manifold $M_{\overline{G}}$ is the unique non-trivial linear S^4-bundle over S^2.*

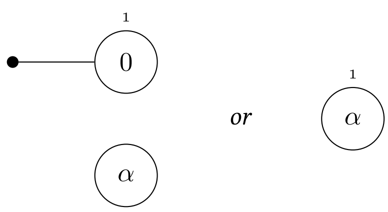
Proposition 6.4.2. *Let G be a reduced graph of rank 2. Then the following assertions hold:*

- *If G is spin, then it is of the form*



  with $\alpha_i \neq 0$ in the second case. For every such graph G there is at most one reduced class that is non-isomorphic, but k-equivalent to that of G.

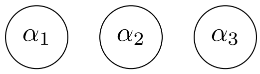
- *If G is not spin, its reduced class contains a graph of the form*

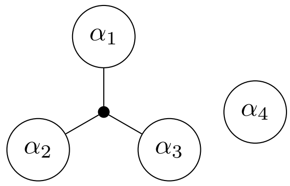


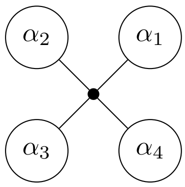
  For every such graph G every reduced class, that is k-equivalent to that of G, is isomorphic to that of G.

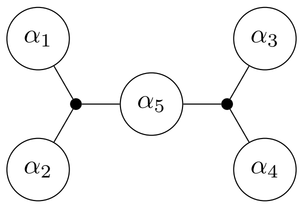
Proposition 6.4.3. *Let G be a reduced graph of rank 3. Then the following assertions hold:*

- *If G is spin, then it is of one of the following forms:*

$(S1)$ 

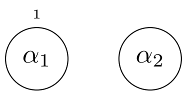
$(S2)$ 

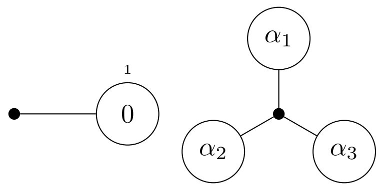
$(S3)$ 

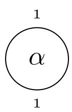
$(S4)$ 

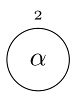
with $\alpha_i \neq 0$ whenever the corresponding vertex has an edge connected to it.

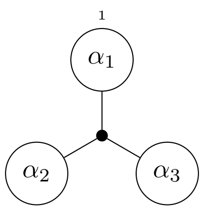
- *If G is not spin, then its reduced class contains a graph of one of the following forms:*

$(N1)$ 

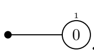
$(N2)$ 

$(N3)$ 

$(N4)$ 

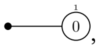
$(N5)$ 

The possibilities for the reduced graphs are a simple cobinatorial consequence of the following lemma.

Lemma 6.4.4. *Let G' be a non-empty connected component of a reduced graph that is not of the form* *. Then*

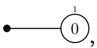
$$2 \cdot |V'| + 1 \leq |U'|.$$

Proof. If V' is empty, then the inequality holds trivially. If V' is non-empty, then, since G' is simply-connected, we can choose a root $v_0 \in V'$ and consider it as a tree. Then, the inequality follows from the fact, that, by definition, v_0 has at least 3 descending vertices in U', while every other $v \in V'$ has at least 2. $\square$

It follows from Lemma 6.4.4 and Corollary 6.3.3 that for a connected component G' of a reduced graph, that is not of the form , we have

$$\operatorname{rank}(H_{G'}) = |U'| - |V'| + \sum_{u \in U'} k^+(u) + k^-(u) \geq 1 + |V'| + \sum_{u \in U'} k^+(u) + k^-(u).$$

Thus, $\operatorname{rank}(H_{G'}) = 1$ implies that V' is empty and $k^+ = k^- \equiv 0$. In a similar way, by going through all possibilities, we obtain all the reduced forms in Propositions 6.4.1–6.4.3. It remains to prove the statements about k-equivalence in Propositions 6.4.1 and 6.4.2.

Proof of Proposition 6.4.1. If G is not of the form , then G is given by , which, by construction, yields the linear S^{2k}-bundle over S^{4k} corresponding to α.

For such a graph G let $u \in U$ be the unique element. Then $H_G = \mathbb{Z}u^0$ and

$$\mu(u^0, u^0, u^0) = \frac{\lambda_k}{4}\alpha = -\mu(-u^0, -u^0, -u^0).$$

Since u^0 and $-u^0$ are the only generators of H_G, this shows that for different absolute values of α we obtain non-equivalent trilinear forms.

In the non-spin case the manifold $M_{\overline{G}}$ is diffeomorphic to the unique non-trivial S^4-bundle over S^2 by Lemma 5.3.2. $\square$

Proof of Proposition 6.4.2. In the non-spin case the linear form p_G^k of the first graph is given by $p_G^k(u_1^1) = 0$, $p_G^k(u_2^0) = \lambda_k \alpha$ (where u_1 denotes the upper vertex and u_2 the lower one), while the linear form on the second graph is given by $p_G^k(u^0) = \lambda_k\alpha + \binom{2k+1}{k}$, $p_G^k(u^1) = 0$, in particular this is always non-zero. Since $\lambda_k > \binom{2k+1}{k}$, see Remark B.3.11, this shows that for different (absolute) values of α we obtain non-equivalent graphs.

In the spin case we first consider the case $k = 1$. We consider the first graph as a graph of the second form by setting $\alpha_3 = 0$. Then, by Example 6.2.3, $e_1 = u_1^0 - u_3^0$ and $e_2 = u_2^0 - u_3^0$ form a basis of H_G and we have

$$\begin{aligned}
\mu_G(e_1, e_1, e_1) &= \alpha_1 - \alpha_3,\\
\mu_G(e_1, e_1, e_2) &= -\alpha_3,\\
\mu_G(e_1, e_2, e_2) &= -\alpha_3,\\
\mu_G(e_2, e_2, e_2) &= \alpha_2 - \alpha_3,\\
p_G(e_1) &= 4(\alpha_1 - \alpha_3),\\
p_G(e_2) &= 4(\alpha_2 - \alpha_3).
\end{aligned}$$

Then the homogeneous polynomials $f(x_1, x_2) = \mu_G(x, x, x)$ (cf. Appendix D), and $p(x_1, x_2) = \frac{1}{4}p_G(x)$, $x = x_1e_1 + x_2e_2$, are given by

$$\begin{aligned}
f(x_1, x_2) &= (\alpha_1 - \alpha_3)x_1^3 - 3\alpha_3x_1^2x_2 - 3\alpha_3x_1x_2^2 + (\alpha_2 - \alpha_3)x_2^3,\\
p(x_1, x_2) &= (\alpha_1 - \alpha_3)x_1 + (\alpha_2 - \alpha_3)x_2.
\end{aligned}$$

By Theorem D.6, the algebra of joint invariants for binary cubic and linear forms are generated by D, R^2, I and J (and there holds a relation between those invariants). We will now show, that for given values of $D(f)$, $R(f,p)^2$, $I(f \cdot p)$ and $J(f \cdot p)$ there exist at most two triples $(\alpha_1, \alpha_2, \alpha_3)$, up to permutation and simultaneous multiplication by (-1), whose invariants are given by these values.

In our case the invariants are given as follows:

$$\begin{aligned}
D(f) =& \alpha_1^2\alpha_2^2 + \alpha_1^2\alpha_3^2 + \alpha_2^2\alpha_3^2 - 2\alpha_1^2\alpha_2\alpha_3 - 2\alpha_1\alpha_2^2\alpha_3 - 2\alpha_1\alpha_2\alpha_3^2, \\
R(f,p) =& (\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3)(\alpha_2 - \alpha_1)(\alpha_1 + \alpha_2 + \alpha_3), \\
I(f \cdot p) =& \alpha_1^2\alpha_2^2 + \alpha_1^2\alpha_3^2 + \alpha_2^2\alpha_3^2 - \alpha_1^2\alpha_2\alpha_3 - \alpha_1\alpha_2^2\alpha_3 - \alpha_1\alpha_2\alpha_3^2, \\
J(f \cdot p) =& -\alpha_1^4\alpha_2^2 - \alpha_1^2\alpha_2^4 - \alpha_1^4\alpha_3^2 - \alpha_1^2\alpha_3^4 - \alpha_2^4\alpha_3^2 - \alpha_2^2\alpha_3^4 + 2\alpha_1^4\alpha_2\alpha_3 + 2\alpha_1\alpha_2^4\alpha_3 + 2\alpha_1\alpha_2\alpha_3^4 \\
&+ \alpha_1^3\alpha_2^2\alpha_3 + \alpha_1^3\alpha_2\alpha_3^2 + \alpha_1^2\alpha_2^3\alpha_3 + \alpha_1\alpha_2^3\alpha_3^2 + \alpha_1^2\alpha_2\alpha_3^3 + \alpha_1\alpha_2^2\alpha_3^3 - 6\alpha_1^2\alpha_2^2\alpha_3^2.
\end{aligned}$$

Note that $D(f)$, $R(f,p)^2$, $I(f \cdot p)$ and $J(f \cdot p)$ are symmetric when viewed as polynomials in $(\alpha_1, \alpha_2, \alpha_3)$, hence we can express them in terms of the elementary symmetric polynomials

$$\begin{aligned}
\sigma_1 &= \alpha_1 + \alpha_2 + \alpha_3, \\
\sigma_2 &= \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3, \\
\sigma_3 &= \alpha_1\alpha_2\alpha_3,
\end{aligned}$$

and we obtain

$$\begin{aligned}
D(f) &= \sigma_2^2 - 4\sigma_1\sigma_3, \\
R(f,p)^2 &= \sigma_1^2(\sigma_1^2\sigma_2^2 - 4\sigma_2^3 - 4\sigma_1^3\sigma_3 + 18\sigma_1\sigma_2\sigma_3 - 27\sigma_3^2), \\
I(f \cdot p) &= \sigma_2^2 - 3\sigma_1\sigma_3, \\
J(f \cdot p) &= -\sigma_1^2\sigma_2^2 + 2\sigma_2^3 + 4\sigma_1^3\sigma_3 - 9\sigma_1\sigma_2\sigma_3.
\end{aligned}$$

Hence, the values of σ_2^2, in particular the value of σ_2 up to sign, and $\sigma_1\sigma_3$ are determined by $D(f)$ and $I(f \cdot p)$. We distinguish two cases.

Case 1. Assume $D(f) \neq 0$, or $D(f) = 0$ and $\sigma_2^2 \neq 0$. From the expression for $J(f \cdot p)$ we obtain

$$(4\sigma_1\sigma_3 - \sigma_2^2)\sigma_1^2 + \sigma_2(2\sigma_2^2 - 9\sigma_1\sigma_3) - J(f \cdot p) = 0.$$

Hence, if $D(f) \neq 0$, then for every choice of sign for σ_2, we obtain, up to sign, at most one solution for σ_1. If $D(f) = 0$ and $\sigma_2^2 \neq 0$ (and thus $\sigma_1\sigma_3 \neq 0$), then from the expression for $R(f,p)^2$ we obtain

$$(2\sigma_1\sigma_2\sigma_3)\sigma_1^2 - (27\sigma_1^2\sigma_3^2 + R(f,p)^2) = 0$$

and as before, every choice of sign for σ_2 uniquely determines σ_1 up to sign. The values for $\alpha_1, \alpha_2, \alpha_3$ are then obtained as the solutions of the equation

$$y^3 - \sigma_1 y^2 + \sigma_2 y - \sigma_3 = 0$$

and choosing a different sign for σ_1 (and thus also for σ_3) results in a simultaneous change of sign for the α_i.

Case 2. Assume $\sigma_2^2 = \sigma_1\sigma_3 = 0$. This implies $\sigma_1 = 0$ or $\sigma_3 = 0$. If $\sigma_1 = 0$, then $\alpha_3 = -\alpha_1 - \alpha_2$, so

$$0 = \sigma_2 = -\alpha_1^2 - \alpha_1\alpha_2 - \alpha_2^2 = -\frac{1}{2}(\alpha_1^2 + \alpha_2^2 + (\alpha_1 + \alpha_2)^2),$$

which implies $\alpha_1 = \alpha_2 = 0$ and hence also $\alpha_3 = 0$.

If $\sigma_3 = 0$, then one α_i, say α_3, vanishes. Then $\sigma_2 = \alpha_1\alpha_2$, hence also one of α_1, α_2 vanishes. Conversely, if two of the α_i vanish, then we have $D(f) = R(f,p)^2 = I(f \cdot p) = J(f \cdot p) = 0$. This shows that the invariants vanish if and only if at least two of the α_i vanish. Then the value of the third one can be determined directly from p (or f).

Thus, we have shown, that for given values $(D, R^2, I, J) \in \mathbb{Z}^4 \setminus \{0\}$ there are at most two triples $(\alpha_1, \alpha_2, \alpha_3)$ (up to permutation and simultaneous multiplication by (-1)), with $D(f) = D$, $R(f,p)^2 = R^2$, $I(f \cdot p) = I$ and $J(f \cdot p) = J$, showing that there are at most two reduced classes of spin graphs of rank two that are 1-equivalent. Since in the spin case $(\mu_G^k, p_G^k) = \frac{\lambda_k}{4}(\mu_G, p_G)$ holds, this result carries over to k-equivalence for all k.

□

Remark 6.4.5. We conjecture that in Proposition 6.4.2 also in the spin case k-equivalent classes are in fact equal. We have actually seen in the proof that this is the case if two of the α_i vanish. We can show that this holds in a few more cases:

1. Suppose $\sigma_1 = \alpha_1 + \alpha_2 + \alpha_3 = 0$. Then $R(f,p)$ vanishes, and it follows that p divides f. The quotient is then given by
$$\frac{f}{p} = x_1^2 + x_1x_2 + x_2^2,$$
which is a homogeneous polynomial whose automorphism group is given by
$$\left\{ \pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \pm\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}, \pm\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \pm\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}, \pm\begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix}, \pm\begin{pmatrix}1 & 0\\ -1 & -1\end{pmatrix} \right\}.$$
Applying these automorphisms to f and p results in a permutation and/or simultaneous multiplication by (-1) of the α_i, so the triple $(\alpha_1, \alpha_2, \alpha_3)$ is uniquely determined, up to permutation and simultaneous multiplication by (-1), among all triples $(\alpha_1', \alpha_2', \alpha_3')$ with $\alpha_1' + \alpha_2' + \alpha_3' = 0$. We now show that this in fact holds for any triple.

   If $\alpha_1+\alpha_2+\alpha_3 = 0$, then we have that $\sigma_1\sigma_3$ (which is an invariant) and R vanish. Conversely, this implies that $\alpha_1 + \alpha_2 + \alpha_3 = 0$ or two of the α_i, say α_1 and α_2 are equal. The latter, under the assumption $\sigma_1\sigma_3 = 0$, implies $\alpha_3 = 0$, or $\alpha_1 = \alpha_2 = 0$. In the case $\alpha_3 = 0$ we obtain the same reduced class of graphs for $\alpha_1 = \alpha_2$ and $\alpha_1 = -\alpha_2$, so we can assume that we are in the case $\alpha_1 + \alpha_2 + \alpha_3 = 0$. In the case where at least two α_i vanish, we have $\sigma_2^2 = 0$ (and note that σ_2^2 is an invariant). However, if $\alpha_1 + \alpha_2 + \alpha_2 = 0$, then σ_2^2 can only vanish if all α_i vanish as seen in Case 2.

   Hence, the values for the α_i are determined uniquely up to permutation and multiplication by (-1) in this case.

2. Suppose that $\sigma_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 = 0$. In the proof of Proposition 6.4.2 we saw that the invariants D, I, R^2 and J determine the value of σ_2^2 and each choice of square root possibly defines a triple $(\alpha_1, \alpha_2, \alpha_3)$. Hence, if $\sigma_2^2 = 0$, then there exists only one choice of square root, so the triple $(\alpha_1, \alpha_2, \alpha_3)$ is determined uniquely by its invariants (up to permutation and simultaneous multiplication by (-1)).

3. Suppose that one of the α_i, say α_1, vanishes (or, equivalently, $\sigma_3 = \alpha_1\alpha_2\alpha_3$ vanishes). If another α_i vanishes, we already saw, that its reduced class is already determined by its k-equivalence class. If α_2 and α_3 are both non-trivial, then each of the invariants D, J, R^2 and I takes the same value for $(0, \alpha_2, \alpha_3)$ and $(0, -\alpha_2, \alpha_3)$. These triples define k-equivalent graphs by $(1')$ and (4) of Lemma 6.3.1. Since these triples are not related to each other via a permutation or simultaneous multiplication by (-1), and we saw in the proof of Proposition 6.4.2 that for fixed values of D, J, R^2 and I there can be at most two triples with these values up to permutation and simultaneous multiplication by (-1), it follows that the reduced class for $(0, \alpha_2, \alpha_3)$ is uniquely determined by its k-equivalence class.

4. Suppose that $D(f) < 0$. Recall that in Case 1 we have that σ_1 satisfies
$$(4\sigma_1\sigma_3 - \sigma_2^2)\sigma_1^2 + \sigma_2(2\sigma_2^2 - 9\sigma_1\sigma_3) - J(f \cdot p) = 0.$$

Choosing the opposite sign for σ_2 gives a possible second solution σ_1' given as the solution of the equation

$$\begin{aligned}(4\sigma_1\sigma_3 - \sigma_2^2){\sigma_1'}^2 &= \sigma_2(2\sigma_2^2 - 9\sigma_1\sigma_3) + J(f \cdot p)\\ &= -\sigma_1^2\sigma_2^2 + 4\sigma_2^3 + 4\sigma_1^3\sigma_3 - 18\sigma_1\sigma_2\sigma_3\\ &= -\frac{R(f,p)^2 + 27\sigma_3^2}{\sigma_1^2} < 0.\end{aligned}$$

Since $4\sigma_1\sigma_3 - \sigma_2^2 = -D(f) > 0$, there exists no solution for σ_1', so the triple $(\alpha_1, \alpha_2, \alpha_3)$ is, up to sign and simultaneous multiplication by (-1), uniquely determined by its invariants.

5. Suppose that $\alpha_1, \alpha_2 \equiv 1 \mod 3$ and $\alpha_3 \equiv 2 \mod 3$. Then $\sigma_1 \equiv 1 \mod 3$ and $\sigma_2, \sigma_3 \equiv 2 \mod 3$. If we multiply the equation in case 1 by σ_3^2, we obtain the following equation for σ_3^2:

$$(\sigma_2(2\sigma_2^2 - 9\sigma_1\sigma_3) - J(f \cdot p))\sigma_3^2 + (4\sigma_1\sigma_3 - \sigma_2^2)\sigma_1^2\sigma_3^2 = 0.$$

Similarly as before the possible second solution σ_3' satisfies

$$(-\sigma_1^2\sigma_2^2 + 4\sigma_2^3 + 4\sigma_1^3\sigma_3 - 18\sigma_1\sigma_2\sigma_3){\sigma_3'}^2 = -(4\sigma_1\sigma_3 - \sigma_2^2)\sigma_1^2\sigma_3^2.$$

In particular, $(4\sigma_1\sigma_3 - \sigma_2^2)\sigma_1^2\sigma_3^2$ is divisible by $-\sigma_1^2\sigma_2^2 + 4\sigma_2^3 + 4\sigma_1^3\sigma_3 - 18\sigma_1\sigma_2\sigma_3$. We have

$$-\sigma_1^2\sigma_2^2 + 4\sigma_2^3 + 4\sigma_1^3\sigma_3 - 18\sigma_1\sigma_2\sigma_3 \equiv 0 \mod 3,$$

hence this term is divisible by 3. However,

$$(4\sigma_1\sigma_3 - \sigma_2^2)\sigma_1^2\sigma_3^2 \equiv 1 \mod 3,$$

so σ_3' cannot be an integer. Thus, the triple $(\alpha_1, \alpha_2, \alpha_3)$ is, up to sign and simultaneous multiplication by (-1), uniquely determined by its invariants.

6. Suppose $\alpha_2 = \alpha_3 = -1$. Then we have $D(f) = 1 + 4\alpha_1$ and $J(f \cdot p) = -2(\alpha_1 + 1)^3$. The second possible solution for σ_1^2 in case 1 is then given by

$$\sigma_1^2 = \frac{27\alpha_1^2}{4\alpha_1 + 1}.$$

The prime divisors of $27\alpha_1^2$ are those of α_1 and 3. Thus, since α_1 and $4\alpha_1 + 1$ are coprime, the only possibilities for α_1 are $\alpha_1 = 0$ or $\alpha_1 = -1$. The first case is covered by item (3), while the second case is covered by item (4). Hence, the triple $(\alpha_1, -1, -1)$ is, up to sign and simultaneous multiplication by (-1), uniquely determined by its invariants.

7. It also holds for triples $(\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_1, \alpha_2, \alpha_3 \in [-1000, 1000]$ as we will see below.

On the other hand, there are triples $(\alpha_1, \alpha_2, \alpha_3)$ and $(\alpha_1', \alpha_2', \alpha_3')$, which are not related via a permutation or simultaneous multiplication by (-1), but whose invariants D, R, I and J are all the same. Below we list all primitive pairs of such triples, where at least one triple is contained in $[-1000, 1000]^3$, together with their invariants and normal forms. These are obtained by applying Algorithm E.2 for every triple $(\alpha_1, \alpha_2, \alpha_3)$ contained in $[-1000, 1000]^3$ and checking whether there exists a second solution.

$(\alpha_1,\alpha_2,\alpha_3)$ $(\alpha_1',\alpha_2',\alpha_3')$	(D,R,I,J) (same for both triples) normal form for $(\alpha_1,\alpha_2,\alpha_3)$ normal form for $(\alpha_1',\alpha_2',\alpha_3')$
$(4,15,30)$ $(-6,-5,60)$	$(44100, 210210, 132300, -105884100)$ $((82201, 112410, 153720, 210210), (4,0))$ ((1, 60, 3570, 210210),(4,0))
$(84,70,15)$ $(-21,20,210)$	$(7452900, -8978970, 22358700, -212862276900)$ $((7174441, 7731570, 8331960, 8978970), (4,0))$ $((1, 210, 43680, 8978970), (4,0))$
$(35,24,-420)$ $(21,60,280)$	$(63680400, 802221420, 191041200, -8298893408400)$ $((308871721, 424570860, 583609320, 802221420), (4,0))$ $((37139281, 103431720, 288054060, 802221420), (4,0))$
$(660,231,70)$ $(-110,-84,1155)$	$(5127992100, -39161432310, 15383976300, -4735808392184100)$ $((253801, 13612830, 730135560, 39161432310), (4,0))$ $((60212161, 521685780, 4519951590, 39161432310), (4,0))$
$(760,184,70)$ $(-80,-65,1456)$	$(2699673600, -45942474240, 12625516800, -3707889653145600)$ $((181103046, 191075080, 201596200, 212696640), (24,0))$ $((109239387, 272815920, 681334160, 1701573120), (12,0))$

In particular, for every such pair the normal forms are different, hence the corresponding systems of invariants are not equivalent. Thus, they do not provide counterexamples to Question 6.3.4.

Remark 6.4.6. The proof of Proposition 6.4.2 in fact provides an algorithm to decide whether a given closed, simply-connected spin 6-manifold M with torsion-free homology and $b_2(M)=2$ can be constructed via an algebraic plumbing graph, see Algorithm E.2. For that, by fixing a basis for $H^2(M)$, recall that the trilinear form μ_M defines a homogeneous polynomial

$$f_{\mu_M} = a_0x_1^3 + 3a_1x_1^2x_2 + 3a_2x_1x_2^2 + a_3x_2^3$$

and a linear form

$$p_M = 4b_0x_1 + 4b_1x_2.$$

The proof of Proposition 6.4.2 then provides at most two algebraic plumbing graphs whose trilinear and linear form have the same invariants. Then we apply Algorithm E.1 to decide whether the systems of invariants are equivalent. This method only works if either $p_{M_1}\neq 0$, or $p_{M_1}=0$ and $D(f_{\mu_{M_1}})=0$. For invariants of graphs G as in the first part of Proposition 6.4.2 we have $p_G=0$ if and only if $\alpha_1=\alpha_2=\alpha_3$, and in this case $D(f)=-3\alpha_1^4$, hence $D(f)=0$ if and only if $\alpha_1=\alpha_2=\alpha_3=0$. Thus, the assumptions in [100, Section 3.4] are not satisfied if $\alpha_1=\alpha_2=\alpha_3\neq 0$. In this case we need a different method.

For that, observe that in this case f_{μ_G} is given by

$$f_{\mu_G} = -3\alpha_3x_1x_2(x_1+x_2).$$

Hence, f_{μ_G} is divisible by $-\alpha_3$. For $\frac{1}{-\alpha_3}f_{\mu_G} = 3x_1x_2(x_1+x_2)$ we calculate the *Hessian* $C(\frac{1}{-\alpha_3}f_{\mu_G})$, see [100, Section 3.3]. It is a binary quadratic form assigned to a binary cubic form with the property that $C(A\cdot f)=A\cdot C(f)$ (a so-called *covariant*). In our case it is given by

$$C\left(\frac{1}{-\alpha_3}f_{\mu_G}\right) = 2x_1^2 + 2x_1x_2 + 2x_2^2.$$

The discriminant of $\frac{1}{2}C(\frac{1}{-\alpha_3}f_{\mu_G})$ equals -3. For binary quadratic forms with discriminant -3 there exists precisely one reduced form, given by $x_1^2+x_1x_2+x_2^2$, to which all such forms are equivalent (see e.g. [30, B.1]). Then [30, Algorithm 5.4.2] provides an algorithm to bring an arbitrary binary quadratic form with discriminant -3 into this reduced form. Further, the automorphism group of this form has 6 elements (cf. Remark 6.4.5).

Hence, for given $((a_0,a_1,a_2,a_3),(b_0,b_1))$ with $b_0 = b_1 = 0$ let $g = \gcd(a_0,a_1,a_2,a_3)$ and compute $C(\frac{1}{g}f_{\mu_M})$. If the Hessian is divisible by 2 and the quotient has discriminant -3, there exist 6 automorphisms of $\mathbb{Z}^2$ that bring the Hessian into reduced form. Then we apply these automorphisms to f_{μ_M} and compare the results to f_G with $\alpha_1 = \alpha_2 = \alpha_3 = -g$.

6.5 Manifolds Obtained from Algebraic Plumbing Graphs

In this section we give examples of manifolds that can or cannot be obtained from an algebraic plumbing graph.

Given $p, q \in \mathbb{N}$ and a set $\mathcal{B}_q$ containing oriented diffeomorphism classes of manifolds of dimension q, we define $\mathcal{T}_{p+q-1}(\mathcal{B}_q)$ as the set containing all diffeomorphism classes of manifolds M_G, where $G = (U, V, E, \pi, \delta)$ is a geometric plumbing graph with simply-connected connected components, so that $B_v \cong S^p$ and π_v is trivial for all $v \in V$, and $B_u \in \mathcal{B}_q$ for all $u \in U$.

We will consider the case where $p = 3$ and $q = 4$, and where $\mathcal{B}_4$ consists of all known examples of 4-manifolds that admit a core metric, i.e.

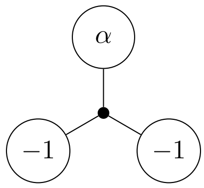
$$\mathcal{B}_4 = \{\#_{i=1}^k \varepsilon_i \mathbb{C}P^2 \mid k \in \mathbb{N}_0, \varepsilon_i \in \{\pm 1\}\}.$$

Then $\mathcal{T}_6(\mathcal{B}_4)$ consists of all manifolds on which we can construct a core metric using Theorem B with all known examples of 4-manifolds that admit a core metric. By construction we have $M_{\overline{G}} \in \mathcal{T}_6(\mathcal{B}_4)$ for any algebraic plumbing graph G with simply-connected connected components. We will now show that also the converse holds.

Proposition 6.5.1. *Let $M \in \mathcal{T}_6(\mathcal{B}_4)$. Then there is an algebraic plumbing graph G so that $M \cong M_{\overline{G}}$.*

For the proof we need the following lemma.

Lemma 6.5.2. *For $\alpha \in \mathbb{Z}$ let G be the reduced graph below.*



Then $M_{\overline{G}}$ is diffeomorphic to the total space of the linear S^2-bundle over $\mathbb{C}P^2$ corresponding to $(\alpha, 1)$ in Corollary B.3.9.

Proof. By Example 6.2.3 the group H_G is generated by the elements $e_1 = u_1^0 - u_2^0$ and $e_2 = u_3^0 - u_2^0$, where we denote by u_1 the vertex labeled by α. Then

$$\begin{aligned}
\mu_G(e_1,e_1,e_1) &= \alpha + 1,\\
\mu_G(e_1,e_1,e_2) &= 1,\\
\mu_G(e_1,e_2,e_2) &= 1,\\
\mu_G(e_2,e_2,e_2) &= 0,\\
p_G(e_1) &= 4(\alpha+1),\\
p_G(e_2) &= 0.
\end{aligned}$$

Now the claim follows from Corollary B.3.12 and Theorem 6.1.2, where we identify e_1 with a and e_2 with b. □

Proof of Proposition 6.5.1. Let G' be a geometric plumbing graph so that $M = M_{G'}$. First we apply Proposition 4.3.4 and modify G' so that all B_u are given by S^4 or $\pm\mathbb{C}P^2$. Further, by applying Proposition 4.3.1, we can achieve that $\delta(e) = 1$ for all $e \in E$. Now we turn G' into an algebraic plumbing graph G by reversing the construction in Section 6.2, except for vertices u for which π_u is an S^2-bundle over $\pm\mathbb{C}P^2$ with spin total space. Every such vertex u gets replaced by a piece of the form as in Lemma 6.5.2, multiplied by ± 1, where all edges connected to u get connected to the vertex labeled by α. By using Corollary 4.4.6 we now obtain that $M_{\overline{G}}$ has the same invariants as M, hence they are diffeomorphic. $\square$

Remark 6.5.3. The bundles in Lemma 6.5.2, together with connected sums of S^2-bundles over S^4, are the only known infinite families of closed, simply-connected spin 6-manifolds M with $b_2(M) = 2$ that admit a metric of positive Ricci curvature, see Section 3.5. Hence, Proposition 6.4.2 and item (6) of Remark 6.4.5 yield an infinite number of new examples of such manifolds and therefore an infinite number of new examples of 6-manifolds with a metric of positive Ricci curvature and $b_2 = 2$. To the best of our knowledge, the corresponding manifolds in dimension $6k$, which are $(2k-1)$-connected, are also new examples of manifolds with a metric of positive Ricci curvature.

Next, we consider how large the class of 6-manifolds we obtain from algebraic plumbing graphs is within the class of all closed, simply-connected 6-manifolds with torsion-free homology.

For manifolds $b_2(M) = 1$, by Proposition 6.4.1, we obtain only one non-spin manifold. In the spin case it follows from Theorem 6.1.2 that for an admissible system of invariants (H, μ, w, p) with $\mathrm{rank}(H) = 1$ and $w = 0$ there exist unique $\alpha, m \in \mathbb{Z}$ and a generator $x \in H$ so that

$$\mu(x, x, x) = \alpha, \quad p(x) = 4\alpha + 24m$$

with $4\alpha + 24m \geq 0$. By Proposition 6.4.1 the systems of invariants obtained from algebraic plumbing graphs of rank one are precisely those with $m = 0$.

For graphs of rank 2 we use Theorem 6.1.4.

Proposition 6.5.4. *The families $P_1, Q_1, R_1, R_1', S_1, K_1, L_1$ in Theorem 6.1.4 are covered as below by coefficients of invariants obtained from simply-connected algebraic plumbing graphs.*

Family	*Subfamily obtained from algebraic plumbing graphs*
P_1	*All elements with* $k = 0$
Q_1	*empty*
R_1/R_1'	*Every given values of* $4r_1 + 24k$ *and* $a_2 = r_2 + 2l$ (R_1) *or* $a_2 = r_3$ (R_1') *can be realized if* $a_2 = 3^\nu p_1^{\nu_1} \dots p_m^{\nu_m}$ *with* $\nu \in \{0, 1\}$, $p_i \equiv 1 \mod 6$, *and* $\nu = 1$ *if* $r_1 + 6k$ *is not divisible by* 3. *Once fixed there are only finitely many possibilities. Additionally all elements with* $k = 0$ *for* R_1'.
S_1	*Every value of* ρ_4 *and* $4r_1 + 24k$ *can be realized, once fixed there only remain finitely many possibilities.*
K_1	*empty*
L_1	*empty*

Remark 6.5.5. The classification in Theorem 6.1.4 has an extension to the non-spin case in [100, Section 3.4], with 36 families in total. By Proposition 6.5.4 we partially cover 4 of the 7 families in the spin case. By considering the non-spin graphs in Proposition 6.4.2, one can show that here we only partially cover 2 of the families in the non-spin case. These are the family P_1 with $k = 0$ and the family R_2 with $\rho_2 = 0$, $r_3 = 1$ and $k = 0$.

For the proof of Proposition 6.5.4 we need the lemma below. This result is well-known. For completeness we also provide a proof.

Lemma 6.5.6. *For a given integer n there exist coprime $(a,b) \in \mathbb{Z}^2$ with $n = a^2 - ab + b^2$ if and only if n is of the form*

$$n = 3^\nu p_1^{\nu_1} \dots p_m^{\nu_m}$$

with $\nu \in \{0,1\}$ and $p_i \equiv 1 \mod 6$ and there are only finitely many such solutions.

Proof. Let ω denote a primitive third root of unity. Then consider the ring $\mathbb{Z}[\omega]$ of *Eisenstein integers*. The map $N\colon \mathbb{Z}[\omega] \to \mathbb{N}_0$,

$$N(a + b\omega) = (a + b\omega)\overline{(a + b\omega)} = a^2 - ab + b^2,$$

is a norm and turns $\mathbb{Z}[\omega]$ into a Euclidean domain. The set of units is given by $\{\pm 1, \pm\omega, \pm\omega^2\}$ and every prime element in this ring is associated to an element in one of the following 3 categories, see e.g. [32, Proposition 4.7]:

- $1 - \omega$ (and we have $N(1 - \omega) = 3$),
- Elements z, so that $N(z) \equiv 1 \mod 6$ and $N(z)$ is prime in $\mathbb{Z}$,
- Elements $p \in \mathbb{Z}$, so that $p \equiv 5 \mod 6$ and p is prime in $\mathbb{Z}$.

Now given

$$n = 3^\nu p_1^{\nu_1} \dots p_m^{\nu_m}$$

with $\nu \in \{0,1\}$ and $p_i \equiv 1 \mod 6$, for every p_i there exists a prime $a_i + b_i\omega \in \mathbb{Z}[\omega]$ with $N(a_i + b_i\omega) = p_i$. Then define

$$a + b\omega = (1 - \omega)^\nu (a_1 + b_1\omega)^{\nu_1} \dots (a_m + b_m\omega)^{\nu_m}$$

and by the multiplicativity of the norm we have

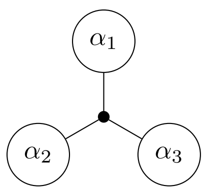
$$a^2 - ab + b^2 = N(a + b\omega) = n.$$

Suppose there exists a prime $p \in \mathbb{Z}$ dividing a and b. Then p^2 divides $N(a + b\omega) = n$, so $p = p_i$ for some i. It follows that $p = (a_i + b_i\omega)\overline{(a_i + b_i\omega)}$ and both factors divide $a + b\omega$, which is not possible by construction of $a + b\omega$.

Conversely, let $(a,b) \in \mathbb{Z}^2$ be coprime and let $p \in \mathbb{Z}$ be a prime dividing $N(a + b\omega)$. If $p \equiv 5 \mod 6$, then p is prime in $\mathbb{Z}[\omega]$, so p divides either $a + b\omega$ or $\overline{a + b\omega} = a - b - b\omega$. In both cases it follows that p divides a and b, which is a contradiction. If $p = 3$ and p^2 divides $N(a + b\omega)$, then either $(1 - \omega)\overline{(1 - \omega)} = 3$, or $(1 - \omega)(1 - \omega) = -3\omega$ and thus 3 divides $a + b\omega$, which is a contradiction.

Finally, since the Eisenstein integers form a lattice in $\mathbb{C}$, there are only finitely many with norm bounded by n, hence there exist only finitely many solutions of $N(a + b\omega) = n$. ☐

Proof of Proposition 6.5.4. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}$ and consider the graph G given by



Let u_i denote the vertex labeled by α_i. By Example 6.2.3 a basis for H_G is given by $e_1 = u_1^0 - u_3^0$ and $e_2 = u_2^0 - u_3^0$ and we have

$$\begin{aligned}
\mu_G(e_1, e_1, e_1) &= \alpha_1 - \alpha_3, \\
\mu_G(e_1, e_1, e_2) &= -\alpha_3, \\
\mu_G(e_1, e_2, e_2) &= -\alpha_3, \\
\mu_G(e_2, e_2, e_2) &= \alpha_2 - \alpha_3, \\
p_G(e_1) &= 4(\alpha_1 - \alpha_3), \\
p_G(e_2) &= 4(\alpha_2 - \alpha_3).
\end{aligned}$$

We assume that the α_i are not all equal, so that p_G is non-trivial and we can apply Theorem 6.1.4. Set $g = \gcd(\alpha_1 - \alpha_3, \alpha_2 - \alpha_3)$. Let $\mu_1, \mu_2 \in \mathbb{Z}$ so that $\mu_1(\alpha_1 - \alpha_3) + \mu_2(\alpha_2 - \alpha_3) = g$. Then for

$$(\lambda_1, \lambda_2) = \frac{1}{g}(\alpha_3 - \alpha_2, \alpha_1 - \alpha_3)$$

we define a new basis (e_1', e_2') of H_G by

$$e_1' = \mu_1 e_1 + \mu_2 e_2, \quad e_2' = \lambda_1 e_1 + \lambda_2 e_2.$$

Then we have

$$\begin{aligned}
\mu_G(e_1', e_1', e_1') &= \mu_1^3 \alpha_1 + \mu_2^3 \alpha_2 - (\mu_1 + \mu_2)^3 \alpha_3, \\
\mu_G(e_1', e_1', e_2') &= \frac{1}{g}(\mu_1^2 \alpha_2(\alpha_3 - \alpha_1) + 2\mu_1\mu_2\alpha_3(\alpha_2 - \alpha_1) + \mu_2^2 \alpha_1(\alpha_2 - \alpha_3)), \\
\mu_G(e_1', e_2', e_2') &= \frac{1}{g^2}(\mu_1(\alpha_1 - \alpha_3)(\alpha_2^2 - \alpha_1\alpha_3) + \mu_2(\alpha_2 - \alpha_3)(\alpha_1^2 - \alpha_2\alpha_3)), \\
\mu_G(e_2', e_2', e_2') &= \frac{1}{g^3}(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3)(\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2 + \alpha_3), \\
p_G(e_1') &= 4g, \\
p_G(e_2') &= 0.
\end{aligned}$$

Let $((a_0, a_1, a_2, a_3), (b_0, b_1))$ be the coefficients of (μ_G, p_G) in this basis. Then, by Theorem 6.1.4, there exists precisely one choice of μ_1, μ_2, so that $((a_0, a_1, a_2, a_3), (b_0, b_1))$ is an element of one of the families P_1, Q_1, R_1, R_1', S_1. We distinguish several cases:

Case 1: $a_3 \neq 0$. If $a_3 \neq 0$, then we are in case S_1. For arbitrary integers g and α_3 set $\alpha_1 = 2g + \alpha_3$ and $\alpha_2 = g + \alpha_3$. Then

$$\alpha_1 - \alpha_3 = 2g, \quad \alpha_2 - \alpha_3 = g,$$

showing that then in fact $g = \gcd(\alpha_1 - \alpha_3, \alpha_2 - \alpha_3)$. The value for a_3 is then given by

$$a_3 = 6(g + \alpha_3).$$

Hence, any value for ρ_4 and $4r_1 + 24k$ in family S_1 can be realized in this way.

On the other hand, if we fix values for g and a_3, then there are only finitely many possibilities for $\alpha_1, \alpha_2, \alpha_3$ to realize these values as they are uniquely determined from the values of $\alpha_1 - \alpha_3$, $\alpha_2 - \alpha_3$ and $\alpha_1 + \alpha_2 + \alpha_3$.

Summarizing, any value of ρ_4 and $4r_1 + 24k$ in family S_1 can be realized. Once these values are fixed, there are only finitely many possibilities.

Case 2: $a_3 = 0$ **and** $a_2 \neq 0$. In this case we have either that two of the α_i coincide, or that $\alpha_1 + \alpha_2 + \alpha_3 = 0$. In the first case, for symmetry reasons we can assume $\alpha_2 = \alpha_3$. Then $g = \alpha_1 - \alpha_3$ and, if we choose $\mu_1 = 1, \mu_2 = 0$ the values for the a_i and b_i are given by

$$((\alpha_1 - \alpha_3, -\alpha_3, -\alpha_3, 0), (4(\alpha_1 - \alpha_3), 0)).$$

Since by assumption $a_2 \neq 0$, so $\alpha_3 \neq 0$ and by possibly replacing e_2' by $-e_2'$, i.e. by replacing a_1 by $-a_1$, we obtain precisely all elements of R_1' with $k = 0$.

Now assume that $\alpha_1 + \alpha_2 + \alpha_3 = 0$. Then the values for the a_i and b_i are given by

$$\begin{aligned} a_0 &= g(\mu_1^2 + \mu_1\mu_2 + \mu_2^2), \\ a_1 &= \alpha_1\mu_2 - \alpha_2\mu_1, \\ a_2 &= \frac{1}{g}(\alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2), \\ a_3 &= 0, \\ b_0 &= 4g, \\ b_1 &= 0. \end{aligned}$$

If we set $x = \frac{\alpha_1 - \alpha_3}{g}$, $y = \frac{\alpha_2 - \alpha_3}{g}$, then we have

$$a_2 = \frac{g}{3}(x^2 - xy + y^2).$$

Note that, since $\alpha_1 + \alpha_2 + \alpha_3 = 0$ we have $g(x+y) = -3\alpha_3$, showing that either g or $x+y$, and thus $(x+y)^2 - 3xy = x^2 - xy + y^2$, is divisible by 3. In particular, $x + y$ is divisible by 3 if and only if $x^2 - xy + y^2$ is divisible by 3.

For a given integer n, by Lemma 6.5.6, there exist coprime $(x, y) \in \mathbb{Z}^2$ with $n = x^2 - xy + y^2$ if and only if n is of the form

$$n = 3^\nu p_1^{\nu_1} \dots p_m^{\nu_m}$$

with $\nu \in \{0, 1\}$ and $p_i \equiv 1 \mod 6$ and there are only finitely many such solutions. Thus, we have seen that for given values of $4r_1 + 24k$, and $a_2 = r_2 + 2l$ or $a_2 = r_3$ in R_1 or R_1', respectively, if $r_1 + 6k$ is divisible by 3, we can realize the value for a_2 if and only if it is of the form $3^\nu p_1^{\nu_1} \dots p_m^{\nu_m}$ with $\nu \in \{0, 1\}$ and $p_i \equiv 1 \mod 6$. If $r_1 + 6k$ is not divisible by 3, we can realize the value for a_2 if and only if it is of the form $3p_1^{\nu_1} \dots p_m^{\nu_m}$ with $p_i \equiv 1 \mod 6$. In both cases there are only finitely many possibilities.

Case 3: $a_2 = a_3 = 0$. Here, as in case 2, either two of the α_i, say α_2 and α_3 are equal, or $\alpha_1 + \alpha_2 + \alpha_3 = 0$. Since $a_2 = 0$ we have in the first case that $\alpha_3 = 0$, implying $a_1 = 0$. In the second case, since $\alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2 = 0$ if and only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$, we have $a_0 = a_1 = b_0 = 0$, which contradicts the assumption that p_G is non-trivial. Thus, we can assume that $\alpha_2 = \alpha_3 = 0$, so the values for the a_i and b_i are given by

$$((\alpha_1, 0, 0, 0), (4\alpha_1, 0)).$$

Hence, we obtain precisely all elements of P_1 with $k = 0$.

□

For higher rank we have no analogue of Theorem 6.1.4. However, we can show that the following class of manifolds cannot be constructed via algebraic plumbing graphs.

Proposition 6.5.7. *Let $E \xrightarrow{\pi} S^{2k} \times S^{2k}$ be a linear S^{2k}-bundle. Then there exists no algebraic plumbing graph G with simply-connected connected components so that $E \cong M_{\overline{G}^k}$.*

Proof. We have $H^{2k}(S^{2k} \times S^{2k}) = H^{2k}(S^{2k}) \oplus H^{2k}(S^{2k})$ and we denote a positively oriented generator of the i-th summand by b_i. Let ξ be the vector bundle corresponding to π and let $\beta_i \in \{0, 1\}$ so that $w_{2k}(\xi) = \beta_1 b_1 + \beta_2 b_2 \mod 2$. Since ξ has rank $2k + 1$ and all cohomology groups of $S^{2k} \times S^{2k}$ in degrees $1 \leq 2k + 1$ except degree $2k$ vanish, we have $w_i(\xi) = 0$ for all $i \neq 0, 2k$. By the Wu formulas, see e.g. [114, Theorem C], it follows that

$$p_k(\xi) \equiv w_{2k}(\xi)^2 = 2\beta_1\beta_2 b_1 b_2 \mod 4.$$

Hence, there exists $\alpha \in \mathbb{Z}$ so that $p_k(\xi) = (4\alpha + 2\beta_1\beta_2)b_1 b_2$. We set $W = \beta_1 b_1 + \beta_2 b_2$.

By Corollary B.3.6 there is $a \in H^{2k}(E)$ so that, if we set $e_1 = \pi^* b_1$, $e_2 = \pi^* b_2$, $e_3 = a$, then (e_1, e_2, e_3) is a basis of $H^{2k}(E)$ with

$$\begin{aligned}
\mu_E(e_1, e_2, e_3) &= 1,\\
\mu_E(e_1, e_3, e_3) &= \beta_2,\\
\mu_E(e_2, e_3, e_3) &= \beta_1,\\
\mu_E(e_3, e_3, e_3) &= \alpha + 2\beta_1\beta_2,\\
\mu_E(e_1, e_1, e_2) &= \mu_E(e_1, e_2, e_2) = \mu_E(e_1, e_1, e_3) = \mu_E(e_2, e_2, e_3) = 0,\\
p_k(E)(e_1) &= p_k(E)(e_2) = 0,\\
p_k(E)(e_3) &= 4\alpha + 2\beta_1\beta_2,\\
w_{2k}(E) &= \beta_1 e_1 + \beta_2 e_2 \mod 2.
\end{aligned}$$

The corresponding homogeneous polynomials $f_{\mu_E}(x_1, x_2, x_3) = \mu_E(x, x, x)$ and $p_E(x_1, x_2, x_3) = p_k(E)(x)$, $x = x_1 e_1 + x_2 e_2 + x_3 e_3$, are then given by

$$\begin{aligned}
f_{\mu_E}(x_1, x_2, x_3) &= 6x_1 x_2 x_3 + 3\beta_1 x_2 x_3^2 + 3\beta_2 x_1 x_3^2 + (\alpha + 2\beta_1\beta_2)x_3^3,\\
p_E(x_1, x_2, x_3) &= (4\alpha + 2\beta_1\beta_2)x_3.
\end{aligned}$$

The ring of invariants for homogeneous polynomials in 3 variables of degree 3 is generated by the two invariants S and T by Theorem D.7. A calculation shows that their values for f_{μ_E} are given by

$$S(f_{\mu_E}) = 1, \quad T(f_{\mu_E}) = -8.$$

An invariant for linear forms is the divisibility d, which is the greatest common divisor of the coefficients. For p_E we obtain

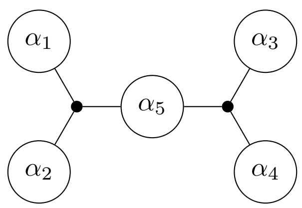
$$d(p_E) = 4\alpha + 2\beta_1\beta_2.$$

Note that p_E divides f_{μ_E} over $\mathbb{Q}$ and we have

$$\frac{f_{\mu_E}}{\frac{1}{d(p_E)}p_E} = 6x_1x_2 + 3\beta_1 x_2 x_3 + 3\beta_2 x_1 x_3 + (\alpha + 2\beta_1\beta_2)x_3^2.$$

We will now go through all the possibilities in Proposition 6.4.3 and show that the invariants are not equivalent to those of E.

The spin case. We start with the spin case. Then, by applying $(1')$ of Lemma 6.3.1, we see that every graph in Proposition 6.4.3, that is spin, is equivalent to a graph G of the form



where some of the α_i may vanish. Denote by u_i the vertex labeled by α_i. Then a basis of H_G is given by $e_1 = u_1^0 - u_2^0$, $e_2 = u_3^0 - u_4^0$ and $e_3 = u_1^0 + u_3^0 - u_5^0$. The corresponding homogeneous polynomials are given by

$$\frac{4}{\lambda_k} f_{\mu_G^k} = (\alpha_1-\alpha_2)x_1^3+(\alpha_3-\alpha_4)x_2^3+(\alpha_1+\alpha_3-\alpha_5)x_3^3+3\alpha_1 x_1^2 x_3+3\alpha_1 x_1 x_3^2+3\alpha_3 x_2^2 x_3+3\alpha_3 x_2 x_3^2$$

and

$$\frac{4}{\lambda_k} p_G^k = 4(\alpha_1 - \alpha_2)x_1 + 4(\alpha_3 - \alpha_4)x_2 + 4(\alpha_1 + \alpha_3 - \alpha_5)x_3.$$

Calculating the invariants S yields $S(f^k_{\mu_G}) = \frac{\lambda_k^4}{4^4}\alpha_1\alpha_2\alpha_3\alpha_4$. Hence, if $S(f_{\mu^k_G}) = 1$, then $\lambda_k = 4$, i.e. $k = 1$ and $|\alpha_1| = |\alpha_2| = |\alpha_3| = |\alpha_4| = 1$. Then, by calculating the invariant $T(f_{\mu_G})$ for all remaining possibilities for $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, we obtain that $S(f_{\mu_G}) = 1$ and $T(f_{\mu_G}) = -8$ if and only if $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = \alpha_4 = -1$, or $\alpha_1 = \alpha_2 = -1$ and $\alpha_3 = \alpha_4 = 1$. For symmetry reasons we can assume that the first case holds.

Then we have

$$f_{\mu_G} = -\alpha_5 x_3^3 + 3x_1^2x_3 + 3x_1x_3^2 - 3x_2^2x_3 - 3x_2x_3^2$$

Now E is spin if and only if $\beta_1 = \beta_2 = 0$, in which case we have

$$f_{\mu_E} = 6x_1x_2x_3,$$

which implies that, for any change of basis, the coefficients of f_{μ_E} are divisible by 6. This is not the case for f_{μ_G}, hence f_{μ_G} and f_{μ_E} cannot be equivalent.

In the non-spin case, note that E can only be non-spin for $k = 1$, since otherwise $H^2(E) = 0$. Hence, we can assume that $k = 1$.

Case (N1). Let u_i be the vertex labeled by α_i. Then a basis for H_G is given by $e_1 = u_1^0$, $e_2 = u_1^1$ and $e_3 = u_2^0$. Then

$$f_{\mu_G} = \alpha_1x_1^3 + \alpha_2x_3^3 + 3x_1x_2^2$$

and $S(f_{\mu_G}) = 0$, hence f_{μ_G} and f_{μ_E} cannot be equivalent.

Case (N2). Let u_i, $1 \le i \le 3$, be the vertex labeled by α_i and let u_4 be the vertex labeled by 0. Then a basis for H_G is given by $e_1 = u_1^0 - u_3^0$, $e_2 = u_2^0 - u_3^0$, and $e_3 = u_4^1$. Then

$$f_{\mu_G} = (\alpha_1 - \alpha_3)x_1^3 + (\alpha_2 - \alpha_3)x_2^3 - 3\alpha_3x_1^2x_2 - 3\alpha_3x_1x_2^2$$

and $S(f_{\mu_G}) = 0$, hence f_{μ_G} and f_{μ_E} cannot be equivalent.

Case (N3). Let u be the vertex labeled by α. Then a basis for H_G is given by $e_1 = u^0$, $e_2 = u^1$ and $e_3 = u^{-1}$. Then

$$f_{\mu_G} = \alpha x_1^3 + 3x_1x_2^2 - 3x_1x_3^2, \quad p_G = \alpha x_1.$$

We have $S(f_{\mu_G}) = 1$, $T(f_{\mu_G}) = -8$ and

$$\frac{f_{\mu_G}}{\frac{1}{d(p_G)}p_G} = \alpha x_1^2 + 3x_2^2 - 3x_3^2.$$

For any change of basis $y_i = a_ix_1 + b_ix_2 + c_ix_3$, we have that the coefficient of y_1y_2 is given by

$$2a_1b_1\alpha + 6a_2b_2 - 6a_3b_3,$$

which is even, in particular not equal to 3. Similarly, the coefficient of y_2y_3 does not equal 3. Hence, we obtain for E that $\beta_1 = \beta_2 = 0$. But then E is spin, while G is not spin.

Case (N4). Let u be the vertex labeled by α. Then a basis for H_G is given by $e_1 = u^0$, $e_2 = u^1$ and $e_3 = u^2$. Then

$$f_{\mu_G} = \alpha x_1^3 + 3x_1x_2^2 + 3x_1x_3^2$$

and $T(f_{\mu_G}) = 8$, hence f_{μ_G} and f_{μ_E} cannot be equivalent.

Case (N5). Let u_i be the vertex labeled by α_i. Then a basis for H_G is given by $e_1 = u_1^0 - u_2^0$, $e_2 = u_3^0 - u_2^0$ and $e_3 = u_1^1$. Then

$$p_G = (4(\alpha_1 - \alpha_2) + 3)x_1 + 4(\alpha_3 - \alpha_2)x_2,$$

so $d(p_G)$ is odd. For p_E we have $d(p_E) \equiv 0$ or $2 \mod 4$, hence p_G and p_E cannot be equivalent. □

Remark 6.5.8. The manifold E admits a Riemannian metric of positive Ricci curvature by Theorem 3.1.7 and a core metric if $k \geq 2$ by Proposition 5.2.7. However, by Proposition 6.5.7, it remains open if E admits a core metric if $k = 1$.

Next we consider homotopy $\mathbb{C}P^3$'s. A closed manifold is a *homotopy* $\mathbb{C}P^3$, if it is homotopy equivalent to $\mathbb{C}P^3$. In terms of invariants, a closed manifold M is a homotopy $\mathbb{C}P^3$ if and only if

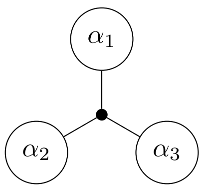
- M is a simply-connected 6-manifold with torsion-free homology,
- $b_3(M) = 0$ and $b_2(M) = 1$,
- $\mu_M(x, x, x) = 1$ for a generator x of $H^2(M)$, and
- $w_2(M) = 0$.

This follows for example from [88, Example 2] and is based on the homotopy classification for simply-connected 6-manifolds by Zhubr [126]. We also note that the homotopy classification given by Wall [115] and Jupp [67] is erroneous, cf. [126, 5.14] or [88, Remark 2].

From Theorem 6.1.2 it follows that there is an infinite family of homotopy $\mathbb{C}P^3$s whose diffeomorphism types are distinguished by p_1, which can take any value congruent $4 \mod 24$ on the generator x. By Proposition 6.4.1, the only homotopy $\mathbb{C}P^3$, that can be constructed via an algebraic plumbing graph, is the standard $\mathbb{C}P^3$. However, we have the following result:

Proposition 6.5.9. *There is an infinite family M_i, $i \in \mathbb{N}_0$, of pairwise non-diffeomorphic homotopy $\mathbb{C}P^3$'s, such that for each $i \in \mathbb{N}_0$ there is a closed, simply-connected 6-manifold N_i so that $M_i \# N_i \in \mathcal{T}_6(\mathcal{B}_4)$. In particular, $M_i \# N_i$ admits a core metric.*

Proof. Let G be the graph



with

$$\alpha_1 = (2i+1)(i+1), \quad \alpha_2 = (2i+1)i, \quad \alpha_3 = \frac{i(i+1)}{2}.$$

Denote the vertices labeled by α_i by u_i. Then a basis for H_G is given by $e_1 = u_1^0 + u_2^0 - 2u_3^0$, $e_2 = iu_1^0 + (i+1)u_2^0 - (2i+1)u_3^0$, and we have

$$\begin{aligned}
\mu_G(x_1, x_1, x_1) &= 1,\\
\mu_G(x_1, x_1, x_2) &= 0,\\
\mu_G(x_1, x_2, x_2) &= 0,\\
\mu_G(x_2, x_2, x_2) &= \frac{i(i+1)(2i+1)}{2},\\
p_G(x_1) &= 4 + 24\frac{i(i+1)}{2},\\
p_G(x_2) &= 4\frac{i(i+1)(2i+1)}{2} + 24\frac{i(i+1)(2i+1)}{6}.
\end{aligned}$$

Hence, $M_{\overline{G}}$ is diffeomorphic to $M_i \# N_i$ if we define M_i and N_i as the unique closed, oriented simply-connected spin 6-manifolds with

- $b_3(M_i) = b_3(N_i) = 0$,
- $b_2(M_i) = b_2(N_i) = 1$,
- $\mu_{M_i}(x, x, x) = 1$ for a generator x of $H^2(M_i)$,
- $p_1(M_i)(x) = 4 + 24\frac{i(i+1)}{2}$,
- $\mu_{N_i}(y, y, y) = \frac{i(i+1)(2i+1)}{2}$ for generator y of $H^2(N_i)$, and
- $p_1(N_i)(y) = 4\frac{i(i+1)(2i+1)}{2} + 24\frac{i(i+1)(2i+1)}{6}$.

The manifold M_i is a homotopy $\mathbb{C}P^3$, and for different values of i we obtain different values for the divisibility of the first Pontryagin class, hence all M_i are pairwise non-diffeomorphic. $\square$

Finally, we consider the question, if a given manifold M can be decomposed into a connected sum $M = M_1 \# M_2$, where $M_1, M_2 \ncong \Sigma^{6k}$ for any homotopy sphere Σ^{6k}. To analyze this on the level of cohomology let H be a finitely generated free abelian group with a symmetric trilinear form $\mu\colon H \times H \times H \to \mathbb{Z}$. Given a subspace $Y \subseteq H$, we say that Y is a direct summand in (H, μ), if there is another subspace $Z \subseteq H$ such that $H = Y \oplus Z$ and

$$\mu(y_1, z_1, z_2) = \mu(z_1, y_1, y_2) = 0$$

for all $y_1, y_2 \in Y$, $z_1, z_2 \in Z$.

Lemma 6.5.10. *Let m be the rank of Y and let $(y_1, \ldots, y_m)$ be a basis of Y. If Y is a direct summand in (H, μ), then for any basis $(x_1, \ldots, x_n)$ of H the matrix*

$$\begin{pmatrix} \mu(x_1,x_1,y_1) & \cdots & \mu(x_1,x_n,y_1) & \cdots & \cdots & \mu(x_1,x_1,y_m) & \cdots & \mu(x_1,x_n,y_m) \\ \vdots & \ddots & \vdots & \cdots & \cdots & \vdots & \ddots & \vdots \\ \mu(x_n,x_1,y_1) & \cdots & \mu(x_n,x_n,y_1) & \cdots & \cdots & \mu(x_n,x_1,y_m) & \cdots & \mu(x_n,x_n,y_m) \end{pmatrix}$$

has rank at most m.

Proof. For each y_i we have a symmetric bilinear form $\mu(\cdot, \cdot, y_i)$. Let A_i denote its matrix in the basis $(x_1, \ldots, x_n)$, i.e. the matrix we consider is given by

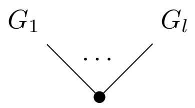
$$A = \left(A_1 | \ldots | A_m\right).$$

Since Y is a direct summand, there exists a subspace $Z \subseteq H$ so that $H = Y \oplus Z$ and products between elements of Y and Z vanish. Let $(z_1, \ldots, z_{n-m})$ be a basis of Z. Then a basis for H is given by $(y_1, \ldots, y_m, z_1, \ldots, z_{n-m})$. Let S denote the change of basis matrix from this matrix to $(x_1, \ldots, x_n)$. Then define

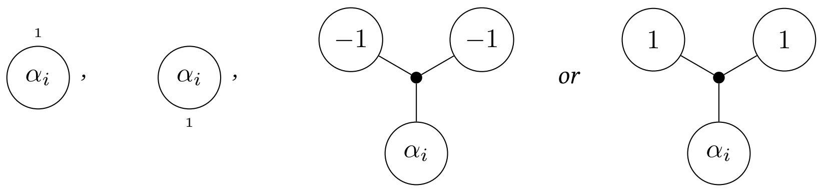
$$A' = S^\top \left(A_1 | \ldots | A_m\right) \begin{pmatrix} S & & 0 \\ & \ddots & \\ 0 & & S \end{pmatrix} = \left(A'_1 | \ldots | A'_m\right),$$

where A'_i is the matrix of $\mu(\cdot, \cdot, y_i)$ in the basis $(y_1, \ldots, y_m, z_1, \ldots, z_{n-m})$. The matrix A' has rank at most m, since it has $(n - m)$ lines that vanish. Since A' and A are obtained from each other via invertible matrices, also A has rank at most m. $\square$

Proposition 6.5.11. *Let G be the graph*



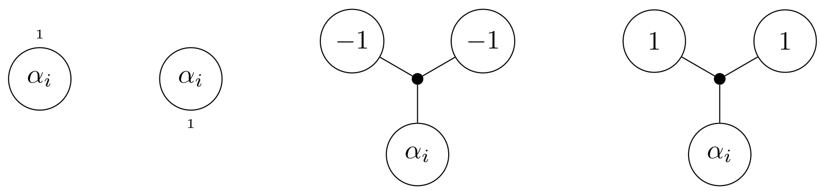
where each G_i is one of



and the edge is connected to the vertex labeled by α_i. If $l \geq 2$, then $M_{\overline{G}^k} \ncong M_1 \# M_2$ for all $6k$-dimensional manifolds M_1, M_2 that are not a homotopy sphere.

Proof. Set $M = M_{\overline{G}^k}$ and let M_1, M_2 be $6k$-dimensional manifolds so that $M_1 \# M_2 \cong M$. Then both $H^{2k}(M_1)$ and $H^{2k}(M_2)$ are direct summands in $(H^{2k}(M), \mu_M)$ by Lemma 4.1.4. By choosing the subspace of smaller rank, we obtain a direct summand Y in $(H^{2k}(M), \mu_M) \cong (H_G, \mu_G^k)$ of rank $m \leq \lfloor b_{2k}(M)/2 \rfloor = \lfloor (2l-1)/2 \rfloor = l-1$.

Fix G_i and let u_1 be the vertex labeled by α_i. If G_i consists of more than one vertex, denote them by u_2 and u_3. Then either $x_i = u_1^0$, $x_i' = u_1^{\pm 1}$, or $x_i = u_1^0 - u_2^0$, $x_i' = u_3^0 - u_2^0$ is a basis of H_{G_i}. We define $\gamma_i = \mu_{G_i}^k(x_i, x_i', x_i')$ and $\beta_i = \mu_{G_i}^k(x_i, x_i, x_i')/\gamma_i$, i.e. we have the following possibilities:



$\beta_i = 0$	$\beta_i = 0$	$\beta_i = 1$	$\beta_i = 1$
$\gamma_i = 1$	$\gamma_i = -1$	$\gamma_i = \frac{\lambda_k}{4}$	$\gamma_i = -\frac{\lambda_k}{4}$

A basis for H_G is now given by $(x_1 - x_l, \ldots, x_{l-1} - x_l, x_1', \ldots, x_l')$. Let $(y_1, \ldots, y_m)$ be a basis of Y. Then there exist $\lambda_{i,j}$ and $\mu_{i,j}$, so that

$$y_i = \sum_{j=1}^{l-1} \lambda_{i,j}(x_j - x_l) + \sum_{j=1}^{l} \mu_{i,j} x_j'.$$

Let $\lambda_{i,l} = -\sum_{j=1}^{l-1} \lambda_{i,j}$. Then the matrix A_i in Lemma 6.5.10 is given as follows:

$$\begin{pmatrix}
* & \cdots & \cdots & * & \gamma_1(\mu_{i,1}+\lambda_{i,1}\beta_1) & 0 & \cdots & 0 & -\gamma_l(\mu_{i,l}+\beta_i\lambda_{i,l}) \\
\vdots & \ddots & & \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\
\vdots & & \ddots & \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\
* & \cdots & \cdots & * & 0 & \cdots & 0 & \gamma_{l-1}(\mu_{i,l-1}+\lambda_{i,l-1}\beta_{l-1}) & -\gamma_l(\mu_{i,l}+\beta_i\lambda_{i,l}) \\
\gamma_1(\mu_{i,1}+\lambda_{i,1}\beta_1) & 0 & \cdots & 0 & \gamma_1\lambda_{i,1} & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \vdots & 0 & \ddots & & & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & & \ddots & & \vdots \\
0 & \cdots & 0 & \gamma_{l-1}(\mu_{i,l-1}+\lambda_{i,l-1}\beta_{l-1}) & \vdots & & & \ddots & 0 \\
-\gamma_l(\mu_{i,l}+\beta_i\lambda_{i,l}) & \cdots & \cdots & -\gamma_l(\mu_{i,l}+\beta_i\lambda_{i,l}) & 0 & \cdots & \cdots & 0 & \gamma_l\lambda_{i,l}
\end{pmatrix}$$

By Lemma 6.5.10, the rank of A is at most $m \leq l-1$. Let A_i^j denote the j-th column of A_i. By considering $A_i^l, \ldots, A_i^{2l-1}$, it follows that either all $\lambda_{i,j}$ vanish, or there is $j_0 \in \{l, \ldots, 2l-1\}$, so that $A_i^{j_0} = 0$ for all i. We first show that the second case implies the first one.

By symmetry reasons we can assume that $j_0 = 2l-1$, i.e. $\lambda_{i,l} = \mu_{i,l} = 0$ for all i. In particular, $\sum_{j=1}^{l-1} \lambda_{i,j} = 0$. Now consider the matrix

$$\left(A_1^l | \ldots | A_1^{2l-2} | \ldots | A_m^l | \ldots | A_m^{2l-2}\right)$$

which then also has rank at most m. By elementary row and column operations we bring it into the following form:

$$A' = \left(\begin{array}{cccc|cccc|c|cccc}
\mu_{1,1} & 0 & \cdots & 0 & \mu_{2,1} & 0 & \cdots & 0 & & \mu_{m,1} & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots & & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 & & \vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \mu_{1,l-1} & 0 & \cdots & 0 & \mu_{2,l-1} & & 0 & \cdots & 0 & \mu_{m,l-1} \\
\lambda_{1,1} & 0 & \cdots & 0 & \lambda_{2,1} & 0 & \cdots & 0 & \cdots & \lambda_{m,1} & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots & & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 & & \vdots & \ddots & \ddots & 0 \\
\vdots & & \ddots & \lambda_{1,l-1} & \vdots & & \ddots & \lambda_{2,l-1} & & \vdots & & \ddots & \lambda_{m,l-1} \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & & 0 & \cdots & \cdots & 0
\end{array}\right)$$

Then the vectors

$$w_i = \begin{pmatrix} \mu_{i,1} \\ \vdots \\ \mu_{i,l-1} \\ \lambda_{i,1} \\ \vdots \\ \lambda_{i,l-1} \\ 0 \end{pmatrix}$$

lie in the space generated by the columns of A'. Since $\mu_{i,l} = \lambda_{i,l} = 0$, and since $(y_1, \ldots, y_m)$ is a basis of Y, it follows that $(w_1, \ldots, w_m)$ is linearly independent. Hence, since A' has rank at most m, $(w_1, \ldots, w_m)$ is a basis (over $\mathbb{Q}$) of the space generated by the columns of A'. Thus, since $\sum_{j=1}^{l-1} \lambda_{i,j} = 0$, every $v = (v_1, \ldots, v_{2l-1})$ in the space generated by the columns of A' satisfies

$$\sum_{i=l}^{2l-2} v_i = 0.$$

In particular, $\lambda_{i,j} = 0$ for all i, j.

Hence we can assume that all $\lambda_{i,j}$ vanish. Then define

$$w_i' = \sum_{j=1}^{l-1} \frac{1}{\gamma_j} A_i^j = \begin{pmatrix} * \\ \vdots \\ * \\ \mu_{i,1} \\ \vdots \\ \mu_{i,l-1} \\ -(l-1)\mu_{i,l} \end{pmatrix}.$$

Since $(y_1, \ldots, y_m)$ is a basis of Y, the elements $w_1', \ldots, w_m'$ are linearly independent and hence a basis over $\mathbb{Q}$ of the space generated by the columns of A. But they are also linearly independent to the elements w_i (which also lie in the space generated by the columns of A since $\lambda_{i,j} = 0$), which is a contradiction unless $m = 0$.

Thus, one of the M_i, say M_1, has vanishing cohomology in degree $2k$, so $H^{2k}(M) = H^{2k}(M_2)$. By Corollary B.3.13 and Theorem 4.4.3 any $x \in H^2(M)$ is non-trivial if any only if $x^k \in H^{2k}(M)$ is non-trivial and all other elements in $H^*(M)$ are obtained from powers of elements in $H^2(M)$ and multiplication by $a \in H^{2k}(M)$. Thus, $H^*(M_2) = H^*(M)$ and M_1 has non-trivial cohomology groups only in degrees 0 and $6k$. Hence, M_1 is a homology sphere and since it is simply-connected it is therefore a homotopy sphere. $\square$

Remark 6.5.12. Since there do not exist exotic spheres in dimensions 6 and 12, we actually obtain for $k = 1, 2$ that $M_{\overline{G}^k}$ does not split as a connected sum $M_1 \# M_2$ for any M_1, M_2 which are not standard spheres.

We obtain the following result:

Theorem E. *For every $k \in \mathbb{N}$ and for every odd $l \in \mathbb{N}$ sufficiently large there exists an infinite family M_j^{6k} of pairwise non-diffeomorphic closed $6k$-dimensional manifolds with torsion-free homology with the following properties:*

- *M_j is $(2k-1)$-connected with $b_{2k}(M_j) = l$,*
- *M_j does not split non-trivially as a connected sum,*
- *M_j is not diffeomorphic to the total space of a linear sphere bundle, a homogeneous space, a biquotient, a cohomogeneity one manifold or a Fano variety,*
- *M_j admits a core metric.*

Further, if $k = 1$ or k is even, then we can replace the conclusion that M_j is $(2k-1)$-connected by M_i being simply-connected and non-spin.

Proof. Let $M = M_{\overline{G}^k}$ with G as in Proposition 6.5.11, where we choose each subgraph G_i to be spin, i.e. one of the third or fourth option. Then, by Proposition 6.5.11, M does not split non-trivially as a connected sum and we have that M is $(2k-1)$-connected with $b_{2k}(M) = 2l - 1$. Further, M admits a core metric by Theorem D and for l sufficiently large it is not diffeomorphic to a homogeneous space, a biquotient, a cohomogeneity one manifold or a Fano variety by Propositions 3.2.6 and 3.2.7.

To show that it is not diffeomorphic to the total space of a linear sphere bundle, note that, for every $\lambda_1, \lambda_2 \in \mathbb{Z}$ we have $\lambda_1 x_1 + \lambda_2 x_2 - (\lambda_1 + \lambda_2) x_3 \in H_G$ (where we use the notation x_i, x_i' as in the proof of Proposition 6.5.11). Hence, there exist $\lambda_1, \lambda_2 \in \mathbb{Z}$, with one of $\lambda_i \neq 0$, so that

$$p_G^k(\lambda_1 x_1 + \lambda_2 x_2 - (\lambda_1 + \lambda_2) x_3) = 0.$$

Then, for i with $\lambda_i \neq 0$, we have

$$\mu_G^k(\lambda_1 x_1 + \lambda_2 x_2 - (\lambda_1 + \lambda_2) x_3, x_i', x_i') = \lambda_i \gamma_i \neq 0.$$

Hence, if $p_G^k \neq 0$, then M is not diffeomorphic to the total space of a linear sphere bundle by Lemma B.3.7.

To obtain a non-spin manifold, we can alternatively choose some of the G_i, say m, to be non-spin, i.e. one of the first and second options. Then M is merely simply-connected, otherwise the same conclusions hold as in the spin case, except that the fact that μ_G^k is non-trivial on $\ker p_G^k$ does not necessarily imply, that M is not diffeomorphic to the total space of a linear sphere bundles, as in this case this follows only for linear S^{p-1}-bundles with $p \geq 2k+1$. Hence, suppose that M is diffeomorphic to the total space E of a linear S^{p-1}-bundle with $1 < p \leq 2k$. We consider the Euler characteristic, which, by Corollary B.3.6 and Theorem 4.4.3 is given by

$$\chi(M) = 2 + 2m(k-1) + (l-1+m)(k+1).$$

Since $\chi(M) > 0$, we have that p is odd by Lemma B.3.7. If $k = 1$, this is a contradiction. If k is even, then $\chi(M)$ is odd for $m \equiv l \mod 2$, which is a contradiction by Lemma B.3.7.

Finally, we need to determine when p_G^k vanishes. We have

$$\begin{aligned} p_G^k(x_i - x_l) &= \lambda_k(\alpha_i - \alpha_l) + \binom{2k+1}{k}((1-\beta_i)\gamma_i - (1-\beta_l)\gamma_l) + 4(\beta_i\gamma_i - \beta_l\gamma_l) \\ &\equiv \binom{2k+1}{k}(\gamma_i(1-\beta_i) - \gamma_l(1-\beta_l)) \mod \lambda_k \end{aligned}$$

and $p_G^k(x_i') = 0$. Hence, $p_G^k \equiv 0 \mod \lambda_k$ if and only if $\beta_i = 0$ and $\gamma_i = \gamma_l$ for all i, or $\beta_i = 1$ for all i (note that $2\binom{2k+1}{k} < \lambda_k$ for $k \geq 3$ and $\lambda_1 = 4$ and $12 \mid \lambda_2$ by Remark B.3.11). It follows that $p_G^k = 0$ if and only if $\beta_i = 0$, $\gamma_i = \gamma_l$ and $\alpha_i = \alpha_l$ for all i, or $\beta_i = 1$ and $\alpha_i + \gamma_i = \alpha_l + \gamma_l$ for all i.

Thus, we can, for example, set $\alpha_i = 0$ for all $i > 1$ and $\alpha_1 = j$ to define the manifold M_j for all $j > 1$. □

Background in Riemannian Geometry A

In this section we review basic notions in Riemannian geometry. We refer to [94] for details and we use the notation introduced therein.

Let (M^n, g) be a Riemannian manifold. By ∇ we denote the Levi-Civita connection of g. Then the *Riemann curvature tensor* R is the $(1,3)$-tensor on M defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for vector fields X, Y, Z on M. From the Riemann curvature tensor we can derive the sectional curvature, the Ricci curvature and the scalar curvature.

Definition A.1. Let $p \in M$ and $u, v \in T_pM$ be linearly independent. The *sectional curvature* of (u, v), denoted by $\sec(u, v)$, is defined by

$$\sec(u,v) = \frac{g(R(u,v)v,u)}{g(u,u)g(v,v) - g(u,v)^2}.$$

The value $\sec(u, v)$ only depends on the 2-plane spanned by u and v. We say that g has positive sectional curvature, if $\sec(u, v) > 0$ for all $p \in M$ and all linearly independent $u, v \in T_pM$.

Definition A.2. Let $p \in M$ and $u, v \in T_pM$. The *Ricci curvature* of (u, v), denoted by $\mathrm{Ric}(u, v)$, is the trace of $R(\cdot, u)v$, i.e. for an orthonormal basis $e_1, \dots, e_n$ of T_pM we have

$$\mathrm{Ric}(u,v) = \sum_{i=1}^n g(R(e_i,u)v,e_i).$$

The Ricci curvature is a symmetric $(0, 2)$-tensor on M. For a non-zero vector $v \in T_pM$ we can extend $\frac{v}{\|v\|}$ to an orthonormal basis $(\frac{v}{\|v\|}, e_2, \dots, e_n)$ of T_pM. Then we obtain the Ricci curvature from the sectional curvature by

$$\mathrm{Ric}(v,v) = \|v\|^2 \sum_{i=2}^n \sec(v,e_i).$$

We say that g has positive Ricci curvature, if $\mathrm{Ric}(v, v) > 0$ for all non-zero $v \in TM$.

Definition A.3. Let $p \in M$. The *scalar curvature*, denoted by scal, is the trace of the Ricci tensor, i.e. for an orthonormal basis $e_1, \dots, e_n$ of T_pM we have

$$\mathrm{scal} = \sum_{i=1}^n \mathrm{Ric}(e_i,e_i).$$

The scalar curvature is a real function on M. We say that g has positive scalar curvature, if $\mathrm{scal} > 0$.

Remark A.4. Let (M_1^n, g_1), (M_2^n, g_2) be Riemannian manifolds. Then we can calculate the curvatures of the product $(M_1 \times M_2,)$ with $g = g_1 + g_2$ from the curvatures of each factor. In fact, we have
$$\nabla^g = \nabla^{g_1} + \nabla^{g_2},$$
hence, for $u, v, w \in T_{p_1}M_1 \cup T_{p_2}M_2$,
$$g(R^g(u,v)v,w) = \begin{cases} g_i(R^{g_i}(u,v)v,w), & u,v,w \in T_{p_i}M_i, \\ 0, & \text{else.} \end{cases}$$
Hence, we obtain for $u_1, v_1 \in T_{p_1}M_1 \setminus \{0\}$ and $u_2, v_2 \in T_{p_2}M_2 \setminus \{0\}$:
$$\begin{aligned} \sec^g(u_i, v_j) &= \sec^{g_i}(u_i, v_j)\delta_{ij}, \\ \mathrm{Ric}^g &= \mathrm{Ric}^{g_1} + \mathrm{Ric}^{g_2}, \\ \mathrm{scal}^g &= \mathrm{scal}^{g_1} + \mathrm{scal}^{g_2}. \end{aligned}$$

An important tool to calculate curvature is the second fundamental form.

Definition A.5. Let (M^n, g) be a Riemannian manifold and let $Y^{n-1} \subseteq M^n$ be an embedded submanifold. Suppose the normal bundle $T^\perp Y$ of Y is trivial and let $N \colon Y \to T^\perp Y$ be a section of unit length vectors. The *shape operator* S is the $(1,1)$-tensor on Y defined by
$$S(v) = \nabla_v N$$
for $v \in TY$. Then the *second fundamental form* $\mathrm{I\!I}$ is the $(0,2)$-tensor on Y defined by
$$\mathrm{I\!I}(u,v) = g(S(u),v) = g(\nabla_u N, v)$$
for $u, v \in T_pS$, $p \in S$. The *mean curvature* of Y, denoted by H, is the trace of $\mathrm{I\!I}$.

Since the normal bundle $T^\perp Y$ has rank 1, there are precisely two possible choices for N. If $Y = \partial_c M \subseteq M$ is a boundary component of M, then, if not stated otherwise, we will consider S and $\mathrm{I\!I}$ with N being the outward pointing normal vector field on $\partial_c M$.

Remark A.6. It will be important to know how the notions we introduced behave under scaling of the metric. For that let $C > 0$ and define $g_C = C^2 g$.

1. For the Levi-Civita connection we have $\nabla^{g_C} = \nabla^g$, since ∇^g is a compatible and torsion-free connection for g_C, and therefore it is the Levi-Civita connection of g_C. Thus, for the Riemann curvature operator we obtain $R^g = R^{g_C}$.
2. For the sectional curvature we have $\sec^{g_C} = \frac{1}{C^2}\sec^g$ by (1).
3. If $(e_1, \dots, e_n)$ is an orthonormal basis of T_pM with respect to g, then $(\frac{1}{C}e_1, \dots, \frac{1}{C}e_n)$ is an orthonormal basis of T_pM with respect to g_C. Hence, it follows that $\mathrm{Ric}^{g_C} = \mathrm{Ric}^g$ and $\mathrm{scal}^{g_C} = \frac{1}{C^2}\mathrm{scal}^g$.
4. If N is a normal vector field with respect to g, then $\frac{1}{C}N$ is a normal vector field with respect to g_C. Therefore $S^{g_C} = \frac{1}{C}S^g$ and $\mathrm{I\!I}^{g_C} = \frac{1}{C}\mathrm{I\!I}^g$.

The shape operator is self-adjoint and hence the second fundamental form is symmetric. Thus, for each $p \in Y$, the second fundamental form on T_pY is diagonalizable and we call its eigenvalues the *principal curvatures*.

Let g_Y denote the metric induced from g on Y and let R^Y denote the Riemann curvature tensor of g_Y. Then one can calculate R from R^Y and $\mathrm{I\!I}$.

Proposition A.7 ([94, Theorems 3.2.2, 3.2.4 and 3.2.5]). *Let $p \in N$ and $u, v, w, z \in T_pY$. Then the following equations hold:*

1. *Tangential curvature equation:*

$$g(R(u,v)w,z) = g_Y(R^Y(u,v)w,z) - \mathbb{I}(u,z)\mathbb{I}(v,w) + \mathbb{I}(u,w)\mathbb{I}(v,z),$$

2. *Normal curvature equation:*

$$g(R(u,v)w,N) = -(\nabla_u \mathbb{I})(v,w) + (\nabla_v \mathbb{I})(u,w),$$

3. *Radial curvature equation:*

$$R(u,N)N = -S^2(u) - (\nabla_N S)(u).$$

We will be interested in manifolds of the form $M = I \times Y$ for a manifold Y, where I is an interval, with metric $g = dt^2 + g_t$ for metrics g_t on Y. Then, for each $t \in I$, we can consider the submanifold $\{t\} \times Y$, whose induced metric is given by g_t and for which $N = \partial_t$ is a unit normal vector field. We denote by g_t' and g_t'' the $(0,2)$-tensors on Y defined by $g_t'(u,v) = \frac{\partial}{\partial t} g_t(u,v)$ and $g_t''(u,v) = \frac{\partial^2}{\partial t^2} g_t(u,v)$, respectively. Further, we write R^t for the Riemann curvature operator of g_t and $\sec^t$, Ric^t and scal^t, for the sectional, Ricci and scalar curvature of g_t, respectively.

Proposition A.8. *For the submanifold $\{t\} \times Y$ of M with unit normal field $N = \partial_t$ we have for all $p \in Y$ and $u, v \in T_pY \setminus \{0\}$, where $\{e_i\}$ is an orthonormal basis of T_pY with respect to g_t:*

1. *Second fundamental form and mean curvature:*

$$\mathbb{I}(u,v) = \frac{1}{2} g_t'(u,v) \text{ and } H = \frac{1}{2} \mathrm{tr}_{g_t} g_t',$$

2. *Sectional curvature:*

$$\begin{aligned} \sec(v,\partial_t) &= -\frac{1}{2} \frac{g_t''(v,v)}{g_t(v,v)} + \frac{1}{4 g_t(v,v)} \sum_i g_t'(v,e_i)^2, \\ \sec(u,v) &= \sec^t(u,v) - \frac{1}{4} \frac{g_t'(u,u) g_t'(v,v) - g_t'(u,v)^2}{g_t(u,u) g_t(v,v) - g_t(u,v)^2}, \end{aligned}$$

3. *Ricci curvature:*

$$\begin{aligned} \mathrm{Ric}(\partial_t,\partial_t) &= -\frac{1}{2} \mathrm{tr}_{g_t} g_t'' + \frac{1}{4} \|g_t'\|_{g_t}^2, \\ \mathrm{Ric}(v,\partial_t) &= 0, \\ \mathrm{Ric}(u,v) &= \mathrm{Ric}^t(u,v) - \frac{1}{2} g''(u,v) + \frac{1}{2} \sum_i g_t'(u,e_i) g_t'(v,e_i) - \frac{1}{4} g_t'(u,v) \mathrm{tr}_{g_t} g_t', \end{aligned}$$

4. *Scalar curvature:*

$$\mathrm{scal} = \mathrm{scal}^t + \frac{3}{4} \|g_t'\|_{g_t}^2 - \frac{1}{4} (\mathrm{tr}_{g_t} g_t')^2 - \mathrm{tr}_{g_t} g_t''.$$

Proof. We extend u and v to local vector fields U and V on Y around p. Then

$$[U, N] = [V, N] = 0 \tag{A.1}$$

by the product structure of M. Hence,

$$\begin{aligned}
\mathbb{I}(U, V) &= \frac{1}{2}(\mathbb{I}(U, V) + \mathbb{I}(V, U)) \\
&= \frac{1}{2}(g_t(\nabla_U N, V) + g_t(U, \nabla_V N)) \\
&= \frac{1}{2}(g_t(\nabla_N U, V) + g_t(U, \nabla_N V)) \\
&= \frac{1}{2}(\frac{\partial}{\partial t} g_t(U, V) - g_t(U, \nabla_N V) + g_t(U, \nabla_N V)) \\
&= \frac{1}{2} g_t'(U, V)
\end{aligned}$$

and

$$H = \mathrm{tr}_{g_t} \mathbb{I} = \frac{1}{2} \mathrm{tr}_{g_t} g_t'.$$

Now we apply the equations in Proposition A.7. The tangential curvature equation yields (where $w \in T_Y M$)

$$g(R(u, v)v, w) = g_t(R^t(u, v)v, w) - \frac{1}{4}(g'(u, w)g'(v, v) - g'(u, v)g'(w, v)). \tag{A.2}$$

For the normal curvature equation first note, that, by (A.1) and since the metric g_t is torsion-free, we have

$$\nabla_u g_t' = \frac{\partial}{\partial t}(\nabla_u g_t) = 0$$

and similarly for $\nabla_v g_t'$. Hence, by the normal curvature equation it follows that

$$g(R(u, v)v, N) = -(\nabla_u \mathbb{I})(v, v) + (\nabla_v \mathbb{I})(u, v) = 0. \tag{A.3}$$

Finally, by the radial curvature equation, we have

$$\begin{aligned}
g(R(u, N)N, v) &= -g_t(S^2(u), v) - g_t((\nabla_N S)(u), v) \\
&= -g_t(S^2(u), v) - g_t(\nabla_N(S(U)), v) + g_t(S(\nabla_N U), v) \\
&= -g_t(S^2(u), v) - \frac{\partial}{\partial t} g_t(S(u), v) + g_t(S(u), \nabla_N U) + g_t(S(\nabla_u N), v) \\
&= -\frac{\partial}{\partial t}\mathbb{I}(u, v) + g_t(S(u), S(v)) \\
&= -\frac{1}{2} g_t''(u, v) + g_t(\sum_i g_t(S(u), e_i)e_i, \sum_j g_t(S(v), v_j)v_j) \\
&= -\frac{1}{2} g_t''(u, v) + \frac{1}{4}\sum_i g_t'(u, e_i) g_t'(v, e_i). \qquad (A.4)
\end{aligned}$$

Further, we have

$$g(R(u, N)N, N) = 0 \tag{A.5}$$

by the skew symmetry property of the Riemann curvature tensor.

The formulas for sec, Ric and scal now directly follow from (A.2)–(A.5). ☐

In case g is a multiply warped product metric, we obtain more explicit expressions.

Corollary A.9. *Suppose $Y = Y_1 \times \cdots \times Y_k$ and suppose*

$$g_t = f_1(t)^2 h_1 + \cdots + f_k(t)^2 h_k$$

for smooth functions $f_i\colon I \to (0,\infty)$ and metrics h_i on Y_i. Then we have for all $p_i \in Y_i$ and $u_i, v_i \in T_{p_i} Y_i$:

1. *Second fundamental form and mean curvature:*

$$\mathrm{I\!I}(u_i, v_j) = f_i' f_i h_i(u_i, v_i)\delta_{ij} \text{ and } H = \sum_{i=1}^{k} \dim(Y_i)\frac{f_i'}{f_i},$$

2. *Sectional curvature:*

$$\begin{aligned} \sec(v_i, \partial_t) &= -\frac{f_i''}{f_i}, \\ \sec(u_i, v_j) &= \begin{cases} \frac{\sec^{h_i}(u_i,v_i) - f_i'^2}{f_i^2}, & i = j, \\ -\frac{f_i' f_j'}{f_i f_j}, & i \neq j, \end{cases} \end{aligned}$$

3. *Ricci curvature:*

$$\begin{aligned} \mathrm{Ric}(\partial_t, \partial_t) &= -\sum_{l=1}^{k} \dim(Y_l)\frac{f_l''}{f_l}, \\ \mathrm{Ric}(v_i, \partial_t) &= 0, \\ \mathrm{Ric}(u_i, v_j) &= \begin{cases} \mathrm{Ric}^{h_i}(u_i, v_i) - h_i(u_i, v_i)(f_i f_i'' + f_i'^2(\dim(Y_i) - 1) & \\ \qquad + f_i f_i' \sum_{l\neq i} \frac{f_l'}{f_l}\dim(Y_l)), & i = j, \\ 0, & i \neq j, \end{cases} \end{aligned}$$

4. *Scalar curvature:*

$$\mathrm{scal} = \sum_l \frac{1}{h_l^2}\mathrm{scal}^{h_l} + \sum_l \dim(Y_l)\frac{f_l'^2}{f_l^2} - \left(\sum_l \dim(Y_l)\frac{f_l'}{f_l}\right)^2 - 2\sum_l \dim(Y_l)\frac{f_l''}{f_l}.$$

Proof. We have

$$\begin{aligned} g_t' &= 2(f_1 f_1' h_1 + \cdots + f_k f_k' h_k), \\ g_t'' &= 2((f_1'^2 + f_1 f_1'')h_1 + \cdots + (f_k'^2 + f_k f_k'')h_k). \end{aligned}$$

Let $(e_1^l, \ldots, e_{\dim(Y_l)}^l)$ be an orthonormal basis of $T_{p_l} Y_l$ with respect to h_l so that $e_1^i = \frac{1}{\|v_i\|} v_i$. Then

$$\left(\frac{1}{f_l} e_m^l\right)_{l,m}$$

is an orthonormal basis of $T_p Y$ with respect to g_t. We then have

$$\begin{aligned} \mathrm{tr}_{g_t} g_t' &= \sum_{l,m} g_t'\left(\frac{1}{f_l}e_m^l, \frac{1}{f_l}e_m^l\right) = \sum_{l,m} 2 f_l f_l' h_l\left(\frac{1}{f_l}e_m^l, \frac{1}{f_l}e_m^l\right) = 2\sum_l \dim(Y_l)\frac{f_l'}{f_l}, \\ \mathrm{tr}_{g_t} g_t'' &= 2\sum_l \dim(Y_l)\left(\frac{f_l'^2}{f_l^2} + \frac{f_l''}{f_l}\right), \\ \|g_t'\|_{g_t}^2 &= \sum_{l_1,l_2,m_1,m_2} g_t'\left(\frac{1}{f_{l_1}}e_{m_1}^{l_1}, \frac{1}{f_{l_2}}e_{m_2}^{l_2}\right)^2 = 2\sum_{l_1,l_2,m_1,m_2} \frac{f_{l_1}' f_{l_2}'}{f_{l_1} f_{l_2}} g_t\left(e_{m_1}^{l_1}, e_{m_2}^{l_2}\right)^2 \\ &= 2\sum_l \dim(Y_l)\frac{f_l'^2}{f_l^2}, \end{aligned}$$

and

$$\sum_{l,m} g_t'\left(v_i, \frac{1}{f_l}e_m^l\right)^2 = 2f_i'^2 h_i(v_i, v_i),$$

$$\sum_{l,m} g_t'\left(u_i, \frac{1}{f_l}e_m^l\right) g_t'\left(v_i, \frac{1}{f_l}e_m^l\right) = 4f_i'^2 \sum_m h_i(u_i, e_m^i)h_i(v_i, e_m^i) = 4f_i'^2 h_i(u_i, v_i).$$

The formulas for the curvatures are now direct consequences of these equations, Proposition A.8 and Remarks A.4 and A.6. ☐

Example A.10. Let $R > 0$. Consider the warped product metric

$$g = dt^2 + f^2(t)ds_{n-1}^2$$

with

$$f(t) = R\sin\left(\frac{t}{R}\right)$$

on the space obtained from $[0, R\pi] \times S^{n-1}$ by collapsing $\{0\} \times S^{n-1}$ and $\{R\pi\} \times S^{n-1}$ each to a point, which is diffeomorphic to S^n. Let $p \in S^{n-1}$ and $u, v \in T_pS^{n-1}$ be orthogonal unit tangent vectors with respect to ds_{n-1}^2. Then $\frac{1}{f(t)}u$ and $\frac{1}{f(t)}v$ are orthogonal unit tangent vectors with respect to g. Hence, by Corollary A.9, we obtain

$$\begin{aligned}
\mathrm{I\!I}\left(\frac{1}{f(t)}u, \frac{1}{f(t)}v\right) &= 0,\\
\mathrm{I\!I}\left(\frac{1}{f(t)}u, \frac{1}{f(t)}u\right) &= \frac{f'(t)}{f(t)} = \frac{\cos\left(\frac{t}{R}\right)}{R\sin\left(\frac{t}{R}\right)} = \frac{1}{R}\cot\left(\frac{t}{R}\right),\\
\sec(u, \partial_t) &= \frac{-f''(t)}{f(t)} = \frac{1}{R^2},\\
\sec(u, v) &= \frac{\sec^{ds_{n-1}^2}(u, v) - f'(t)^2}{f(t)^2} = \frac{1 - \cos^2\left(\frac{t}{R}\right)}{R^2\sin^2\left(\frac{t}{R}\right)} = \frac{1}{R^2}.
\end{aligned}$$

In particular, g has constant curvature $\frac{1}{R^2}$ and hence g is isometric to $R^2 ds_n^2$, see e.g. [76, Corollary 12.5]. Of course this could also be verified directly by defining an isometric embedding of g into $\mathbb{R}^{n+1}$ whose image is $R^2 \cdot S^n$.

Fiber Bundles B

In this chapter we give an introduction to fiber bundles and analyze geometric and topological properties of linear sphere bundles.

B.1 Fiber Bundles and Principal Bundles

In this section we introduce basic concepts of fiber bundles. For further details we refer to [108].

Definition B.1.1. Let F be a manifold with an effective left-action of a Lie group G. A *fiber bundle with fiber F and structure group G* is a smooth map $E \xrightarrow{\pi} B$ together with a maximal set of local trivializations $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$, where $\mathcal{A}$ is some index set, that is,

1. $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ is an open covering of B,
2. $\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a diffeomorphism for all $\alpha \in \mathcal{A}$,
3. For every $\alpha \in \mathcal{A}$ the diagram

$$\begin{tikzcd} U_\alpha \times F \arrow[rr, "\varphi"] \arrow[dr, "\mathrm{pr}_1"] & & \pi^{-1}(U_\alpha) \arrow[dl, "\pi"] \\ & U_\alpha & \end{tikzcd}$$

   commutes,
4. For every $\alpha, \beta \in \mathcal{A}$ there are maps $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$, called the *transition functions*, so that $\varphi_\beta^{-1} \circ \varphi_\alpha(p, x) = (p, g_{\alpha\beta}(p)x)$ for all $(p, x) \in (U_\alpha \cap U_\beta) \times F$, and
5. For every pair (U, φ) for which $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}} \cup \{(U, \varphi)\}$ satisfies (1)–(4), there is $\alpha \in \mathcal{A}$, so that $(U, \varphi) = (U_\alpha, \varphi_\alpha)$.

For every $x \in B$, the space $\pi^{-1}(x)$ is called the *fiber over x* and is denoted by E_x.

A *principal G-bundle* is a fiber bundle with fiber G and structure group G, where the action is given by left-multiplication.

By dropping any assumptions on smoothness (the spaces then do not need to be manifolds) and merely require all maps to be continuous, we obtain the definition of a *topological fiber bundle with fiber F and structure group G* and of a *topological principal G-bundle*.

We will often consider a fiber bundle just as a map $E \xrightarrow{\pi} B$ without explicitly stating the local trivialization, but it will always be assumed that there is a fixed maximal set of local trivializations.

If $P \xrightarrow{\pi} B$ is a principal G-bundle, then we have a right action of G on P defined by right multiplication on G in local trivializations. This action is well-defined, i.e. it does not depend on the choice of local trivialization, since left and right multiplication on G commute. By definition, the action of G on P is free and transitive.

Definition B.1.2. A *bundle map* between fiber bundles $E \xrightarrow{\pi} B$, $E' \xrightarrow{\pi'} B'$ with fiber F and structure group G is a map $f\colon E \to E'$, so that

1. For each $x \in B$ there is $x' \in B'$, so that f maps E_x to $E_{x'}$ and $f|_{E_x}\colon E_x \to E_{x'}$ is a diffeomorphism, hence f descends to a map $\check{f}\colon B \to B'$, and
2. For all $\alpha \in \mathcal{A}$, $\alpha' \in \mathcal{A}'$ and all $x \in U_\alpha \cap \check{f}^{-1}(U_{\alpha'})$ the map

$$\varphi_{\alpha'}^{-1} \circ f \circ \varphi_\alpha|_{\{x\}\times F}\colon \{x\} \times F \to \{x'\} \times F$$

is given by the action of an element $g_{\alpha\beta}(x) \in G$, so that the map $g\colon U_\alpha \cap \check{f}^{-1}(U_{\alpha'}) \to G$ is smooth.

An invertible bundle map is called a *bundle isomorphism* and two fiber bundles, for which there exists a bundle isomorphism between them, are called *isomorphic*.

Given a fiber bundle $E \xrightarrow{\pi} B$ with fiber F and structure group G and a map $f\colon B' \to B$, define

$$E' = \{(x, e) \in B' \times E \mid f(x) = \pi(e)\}.$$

Then $\pi'\colon E' \to B'$, $(x, e) \mapsto x$ is a fiber bundle with fiber F and structure group G, whose local trivializations are given by $(f^{-1}(U_\alpha), \varphi'_\alpha)$ with $\varphi'_\alpha(x, y) \mapsto (x, \varphi_\alpha(y))$, and $E' \to E$, $(x, e) \mapsto e$ is a bundle map between π' and π. The bundle $f^*\pi = \pi'$ is called the *pull-back of* π *along* f.

Given a fiber bundle $E \xrightarrow{\pi} B$ with fiber F and structure group G, we can construct a principal G-bundle by setting

$$P = \bigsqcup\nolimits_\alpha U_\alpha \times G \,/\!\sim,$$

where $(p_\alpha, x_\alpha) \in U_\alpha \times G$ and $(p_\beta, x_\beta) \in U_\beta \times G$ are defined to be equivalent if $p_\alpha = p_\beta$ and $x_\beta = g_{\alpha\beta}(p_\alpha)x_\alpha$. The bundle projection $P \to B$ is then induced from the projection onto the first factor on $U_\alpha \times G$. This bundle is called the *principal bundle associated to* π, cf. [108, Chapter 8].

Conversely, if $P \xrightarrow{\pi} B$ is a principal G-bundle, we define a fiber bundle with fiber F and structure group G by setting

$$E = P \times_G F = {(P \times F)}/{G},$$

where we consider the diagonal right-action $g(p, x) = (pg, g^{-1}x)$ of G on $P \times F$. This bundle is called the *fiber bundle with fiber* F *associated to* π.

Any fiber bundle $E \xrightarrow{\pi} B$ with fiber F and structure group G is isomorphic to the associated bundle $P \times_G F \to B$, where $P \to B$ is the principal bundle associated to π, see e.g. [64, Theorem 5.3.2]. Hence, every fiber bundle can be recovered from its associated principal bundle.

Given a subgroup $H \subseteq G$ we say that the structure group of $E \xrightarrow{\pi} B$ *reduces* to H, if there is a subset $\mathcal{A}' \subseteq \mathcal{A}$, so that (1)–(4) in Definition B.1.1 holds for $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}'}$ and so that for all $\alpha, \beta \in \mathcal{A}'$ the transition function $g_{\alpha\beta}$ takes values in H. We then obtain a fiber bundle with fiber F and structure group H by adding all $\beta \in \mathcal{A}$ to $\mathcal{A}'$ for which $g_{\alpha\beta}$ takes values in H for all $\alpha \in \mathcal{A}'$.

Example B.1.3.

1. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $G = \mathrm{GL}(p, \mathbb{K})$ act linearly on $\mathbb{K}^p$. Then Definition B.1.1 turns into the definition of a vector bundle.

   By using a partition of unity on the base B, we can construct a Riemannian metric on a vector bundle $E \xrightarrow{\pi} B$. For any $\alpha \in \mathcal{A}$ we can pull the metric on $\pi^{-1}(U_\alpha)$ back along φ_α to $U_\alpha \times \mathbb{K}^p$, so that φ_α is an isometry of vector bundles. Applying the Gram-Schmidt process to the standard basis on $\mathbb{K}^p$ for every $x \in U_\alpha$ yields a map $A\colon U_\alpha \to \mathrm{GL}(p, \mathbb{K})$, so that $U_\alpha \times \mathbb{K}^p \to \pi^{-1}(U_\alpha)$,
   $$(x, v) \mapsto \varphi_\alpha(x, A_x(v))$$
   is a local trivialization, and transition functions between all these local trivializations take values in $\mathrm{O}(p)$, if $\mathbb{K} = \mathbb{R}$, and $\mathrm{U}(p)$, if $\mathbb{K} = \mathbb{C}$. This shows that we can reduce the structure group from $\mathrm{GL}(p, \mathbb{K})$ to $\mathrm{O}(p)$, if $\mathbb{K} = \mathbb{R}$, and to $\mathrm{U}(p)$, if $\mathbb{K} = \mathbb{C}$.

2. The orthogonal group $\mathrm{O}(p)$ acts linearly on D^p and on S^{p-1}. A fiber bundle $E \xrightarrow{\pi} B$ with fiber D^p, or S^{p-1}, and structure group $\mathrm{O}(p)$ is called a *linear disc bundle*, or *linear sphere bundle*, respectively. In this case, the bundle π is the unit disc bundle, or unit sphere bundle of the vector bundle constructed as the associated bundle $P \times_{\mathrm{O}(p)} \mathbb{R}^p \to B$, where $P \to B$ is the principal $\mathrm{O}(p)$-bundle associated to π.

3. If G is a Lie group and $H \subseteq G$ a closed subgroup, then the projection $G \xrightarrow{\pi} G/H$ is a principal H-bundle, where the action is given by left-multiplication, see e.g. [108, 7.4 and 7.5].

   An important example is the homogeneous space $\mathrm{SO}(n+1)/\mathrm{SO}(n)$, where we consider $\mathrm{SO}(n)$ as a subgroup of $\mathrm{SO}(n+1)$ via
   $$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$
   The quotient $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ is then diffeomorphic to S^n via the map
   $$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n+1} \\ \vdots & \ddots & \vdots \\ a_{n+1,1} & \cdots & a_{n+1,n+1} \end{pmatrix} \mapsto \begin{pmatrix} a_{n+1,1} \\ \vdots \\ a_{n+1,n+1} \end{pmatrix}.$$
   Hence, the projection $\mathrm{SO}(n+1) \to S^n$ has the structure of a principal $\mathrm{SO}(n)$-bundle.

The homotopy groups of total space, base and fiber of a fiber bundle are related via a long exact sequence.

Lemma B.1.4 ([55, Theorem 4.41 and Proposition 4.48]). *Let $E \xrightarrow{\pi} B$ be a fiber bundle with fiber F and structure group G, such that B is path-connected. Let $b_0 \in B$, $x_0 \in F$ and let $i\colon F \hookrightarrow E$ be the map given by identifying F with E_{b_0} via a local trivialization followed by the inclusion map. Then there is a long exact sequence*

$$\cdots \longrightarrow \pi_i(F, x_0) \xrightarrow{i_*} (E, i(x_0)) \xrightarrow{\pi_*} \pi_i(B, b_0) \longrightarrow \pi_{i-1}(F, x_0) \longrightarrow \cdots \longrightarrow \pi_0(E, i(x_0)) \longrightarrow 0.$$

The following result is used to classify principal bundles.

Proposition B.1.5 ([108, 19.3]). *For every Lie group G there is a topological principal G-bundle $\mathrm{E}G \xrightarrow{\pi_G} \mathrm{B}G$, called the* universal principal G-bundle, *with $\mathrm{E}G$ contractible, so that for any principal G-bundle $P \xrightarrow{\pi} B$ there exists a map $f\colon B \to \mathrm{B}G$, unique up to homotopy, such that π is isomorphic to the pull-back of π_G along f.*

The space BG is called the *classifying space* of G and the map f is called the *classifying map* of the bundle π.

Note that, since EG is contractible, it follows from the long exact sequence in Lemma B.1.4, that $\pi_i(\mathrm{B}G) \cong \pi_{i-1}(G)$ for all i.

Example B.1.6.

1. By Proposition B.1.5, isomorphism classes of principal $\mathrm{O}(q)$-bundles over S^p are in bijection with elements of $[S^p, \mathrm{BO}(q)] = \pi_p(\mathrm{BO}(q)) \cong \pi_{p-1}(\mathrm{O}(q))$. This fact can also be established directly: For every linear S^{q-1}-bundle $E \xrightarrow{\pi} S^p$ there exists a smooth map $T\colon S^{p-1} \to \mathrm{O}(q)$, called *clutching function*, so that π is isomorphic to the bundle
$$(D^p \times S^{q-1}) \cup_{\tilde{T}} (D^p \times S^{q-1}) \to D^p \cup_\partial D^p \cong S^p, \quad (x,y) \mapsto x,$$
where $\tilde{T}\colon S^{p-1} \times S^{q-1} \to S^{p-1} \times S^{q-1}$ is the diffeomorphism defined by $\tilde{T}(x,y) = (x, T_x(y))$. Further, any two bundles are isomorphic if and only if their clutching functions are homotopic, see e.g. [64, Section 10.7]

2. We have
$$\pi_0(\mathrm{O}(q)) \cong \mathbb{Z}/2$$
and
$$\pi_1(\mathrm{O}(q)) \cong \begin{cases} 1, & q = 1, \\ \mathbb{Z}, & q = 2, \\ \mathbb{Z}/2, & q \geq 3. \end{cases}$$
Hence, there exists a unique non-trivial vector bundle of rank q over S^1 and if $q \geq 3$ there exists a unique non-trivial vector bundle of rank q over S^2. The total spaces of the corresponding linear sphere bundles will be denoted by $S^1 \tilde{\times} S^{q-1}$ and $S^2 \tilde{\times} S^{q-1}$, respectively.

The (co-)homology of BG is harder to determine and is used to characterize bundles in the following sense:

Definition B.1.7. let R be a ring and let $c \in H^i(\mathrm{B}G; R)$. Then c is called a *characteristic class*. For any fiber bundle $E \xrightarrow{\pi} B$ with fiber F and structure group G let $f\colon B \to \mathrm{B}G$ be the classifying map of its associated principal bundle. We then set $c(\pi) = f^*c$. For a manifold M^n and $G = \mathrm{O}(n)$ we set $c(M) = c(TM)$.

Examples for characteristic classes are the Stiefel-Whitney classes $w_i \in H^i(\mathrm{BO}(p); \mathbb{Z}/2)$, the Chern classes $c_i \in H^{2i}(\mathrm{BU}(p); \mathbb{Z})$ and the Pontryagin classes $p_i \in H^{4i}(\mathrm{BO}(p), \mathbb{Z})$ (see e.g. [82] for their definitions).

For a map $f\colon B' \to B$ and a fiber bundle $E \xrightarrow{\pi} B$ with fiber F and structure group G the classifying map for $f^*\pi$ is given by the composition of f with the classifying map of π. Therefore, $c(f^*\pi) = f^*c(\pi)$.

We have inclusions $\iota\colon \mathrm{O}(p-1) \hookrightarrow \mathrm{O}(p)$. For a system of characteristic classes $c \in H^i(\mathrm{O}(p); R)$ defined for all $p \geq 0$ (and all denoted by c), we say that c is *stable*, if it is preserved under ι^*. In this case, for any vector bundle $\xi\colon E \to B$, we have $c(\xi \oplus \underline{\mathbb{R}}_B) = c(\xi)$, where $\underline{\mathbb{R}}_B$ denotes the trivial line bundle over B. For example, the Stiefel-Whitney classes and the Pontryagin classes are stable.

A fiber bundle $E \xrightarrow{\pi} B$ with fiber F and structure group G is *fiber orientable*, if F is orientable and the action of G is orientation-preserving. A *fiber orientation* is then a choice of orientation on the manifold F.

Given an orientation on B, we have the product orientation on $U_\alpha \times F$ for every local trivialization $(U_\alpha, \varphi_\alpha)$, which defines an orientation on $\pi^{-1}(U_\alpha)$ via φ_α. Since the action of G on F is orientation-preserving, this defines a global orientation on E. If we orient E in this way, we say that fiber, base and total space of π are *oriented compatibly*.

Example B.1.8. If $E \xrightarrow{\pi} B$ is a vector bundle, a linear disc bundle, or a linear sphere bundle, then π is fiber orientable if and only if the structure group $\mathrm{O}(p)$ of π reduces to $\mathrm{SO}(p)$. This is the case if and only if the first Stiefel-Whitney class $w_1(\pi) \in H^1(B; \mathbb{Z}/2)$ vanishes, see e.g. [73, Theorem II.1.2].

The group $\mathrm{SO}(p)$ satisfies $\pi_1(\mathrm{SO}(p)) \cong \mathbb{Z}/2$ for $p \geq 3$, cf. Example B.1.6, and its universal cover is called the *Spin group* and denoted by $\mathrm{Spin}(n)$. Similarly as defining a reduction of the structure group, we can define lifts of structure groups, and π is said to admit a *spin structure*, if its structure groups lifts from $\mathrm{SO}(n)$ to $\mathrm{Spin}(n)$. This is the case if and only if both $w_1(\pi)$ and $w_2(\pi)$ vanish, see e.g. [73, Theorem II.1.7]. A manifold, whose tangent bundle admits a spin structure, is called a *spin manifold*.

If F has non-empty boundary, then also ∂E has non-empty boundary (provided B is non-empty) and we have the following result.

Lemma B.1.9. *Let $E \xrightarrow{\pi} B^q$ be a fiber bundle with fiber F and structure group G. If B has empty boundary, then for the boundary ∂E we have the following:*

1. *$\partial E \xrightarrow{\pi|_{\partial E}} B$ is a fiber bundle with fiber ∂F and structure group G.*
2. *If π is fiber oriented, then the induced orientation on ∂F defines a fiber orientation on $\pi|_{\partial E}$.*
3. *If B and E are oriented, so that fiber, base and total space of π are oriented compatibly, then fiber, base and total space of $\pi|_{\partial E}$ are oriented compatibly with respect to the orientations B and $(-1)^q \partial E$.*

Proof. The map $\pi|_{\partial E}$ obtains the structure of a fiber bundle with fiber ∂F and structure group G by restricting the local trivializations $(U_\alpha, \varphi_\alpha)$ to $(U_\alpha, \varphi_\alpha|_{U_\alpha \times \partial F})$. Further, if π is fiber oriented, i.e. the action of G on F is orientation-preserving, then the induced action of G on ∂F is also orientation-preserving, hence $\pi|_{\partial E}$ is fiber-oriented.

Finally, the last claim follows from the fact that the product orientation on $U_\alpha \times \partial F$ and the induced orientation $U_\alpha \times \partial F \subseteq \partial(U_\alpha \times F) \subseteq U_\alpha \times F$ differ by the factor $(-1)^q$. □

B.2 Riemannian Submersions

Let $E \xrightarrow{\pi} B$ be a linear sphere bundle. We will equip the total space E with a Riemannian metric so that π is a Riemannian submersion.

Definition B.2.1. A map $f\colon (M_1, g_1) \to (M_2, g_2)$ between Riemannian manifolds (M_1, g_1) and (M_2, g_2) is a *Riemannian submersion*, if the differential $D_p f\colon T_p M_1 \to T_{f(p)} M_2$ is surjective for all $p \in M$, and the restriction $Df\colon \ker(Df)^\perp \to TN$ is an isometry.

For a Riemannian submersion $f\colon (M_1, g_1) \to (M_2, g_2)$ we set $\mathcal{V} = \ker(Df)$ and $\mathcal{H} = \mathcal{V}^\perp$. Then $TM_1 = \mathcal{V} \oplus \mathcal{H}$. The subbundles $\mathcal{V}$ and $\mathcal{H}$ are called the vertical and horizontal distribution, respectively.

Definition B.2.2. A *principal connection* on a principal G-bundle $P \to B$ is a one-form θ on P with values in the Lie algebra $\mathfrak{g}$, so that

- $\mathrm{Ad}_g(R_g^* \theta) = \theta$ for all $g \in G$, where $R_g\colon P \to P$ denotes the diffeomorphism induced by the action of g (i.e. θ is G-equivariant), and
- For all $V \in \mathfrak{g}$, if $\bar{V}$ denotes the vector field on P defined by $\bar{V}_p = D_1 A_p(V)$ with $A_p(g) = pg$, then $\theta(\bar{V}) = V$.

For example, on the trivial principal G-bundle $P = B \times G \to B$ we can define the trivial principal connection as follows: We have $TP = TB \oplus TG = TB \times (G \times \mathfrak{g})$ and we set

$$\theta(V) = \mathrm{pr}_{\mathfrak{g}}(V)$$

for $V \in TP$.

Now let $P \to B$ be a principal bundle with a principal connection θ and let

$$E = P \times_G F \xrightarrow{\pi} B$$

be a fiber bundle with fiber F associated to $P \to B$. We define a distribution $\mathcal{H}_\theta \subseteq TE$ complementary to $\mathcal{V} = \ker(D\pi)$ as the image of $\ker(\theta) \times TF$ under the projection $P \times F \to P \times_G F$.

Proposition B.2.3 ([8, Theorem 9.59]). *Let $E \xrightarrow{\pi} B$ be a fiber bundle with fiber F and structure group G. Suppose we have the following:*

- *A metric g_B on B,*
- *A principal connection θ on the associated principal bundle $P \to B$,*
- *A G-invariant metric $\hat{g}$ on F.*

Then there exists precisely one metric g on E so that $(E, g) \to (B, g_B)$ is a Riemannian submersion with totally geodesic fibers isometric to $(F, \hat{g})$ and horizontal distribution given by $\mathcal{H}_\theta$.

For example, if $P \to B$ is trivial and θ is the trivial principal bundle, then $\ker(\theta) = TB$, where we identified $TE = TB \times (G \times \mathfrak{g})$. Then the product metric $g_B + \hat{g}$ is a metric with the required properties, hence, by uniqueness, we obtain this metric in Proposition B.2.3.

In case $E \xrightarrow{\pi} B$ is a linear sphere bundle, i.e. a fiber bundle with fiber $F = S^{p-1}$ and structure group $G = \mathrm{O}(p)$, the round metric of radius $r > 0$ on S^{p-1} is invariant under the action of G. For a principal connected θ we denote the metric we obtain in Proposition B.2.3 by $g_\pi(r, \theta)$.

As a consequence, we obtain the following Lemma, which will be useful for gluing constructions.

Lemma B.2.4. *Let $E \xrightarrow{\pi} B^q$ be a linear sphere bundle, i.e. a fiber bundle with fiber $F = S^{p-1}$ and structure group $G = \mathrm{O}(p)$, and let $D_i^q \subseteq B$ be a finite number of pairwise disjoint embedded discs. Then for any $r > 0$ and any metric g_B on B there is a principal connection θ, so that for $g = g_\pi(r, \theta)$ the restriction $g|_{\pi^{-1}(D_i^q)}$ on $\pi^{-1}(D_i^q) \cong D_i^q \times S^{q-1}$ is the product metric*

$$g|_{\pi^{-1}(D_i^q)} = g_B|_{D_i^q} + r^2 ds_{p-1}^2.$$

Proof. We apply Proposition B.2.3 to E, where we equip $F = S^{p-1}$ with the round metric of radius r, which is invariant under the action of $G = \mathrm{O}(p)$. For the principal connection we start with an arbitrary principal connection θ_0. Over each D_i^q we have the trivial principal connection θ_i on $\pi^{-1}(U_i) \cong U_i \times G$, where U_i is a contractible open neighborhood of D_i^q. By using a partition of unity, we can now take convex combinations of the connection θ_0 and the connections θ_i to obtain a principal connection θ on P that is trivial over each D_i^q. Hence the metric $g_\pi(r, \theta)$ is a product metric over each D_i^q. □

The Ricci curvatures of the metric $g_\pi(r, \theta)$ are given in [8, Theorem 9.70]. From the formulas given there one sees that the Ricci curvatures are close to those of B and the fiber S^{p-1} if the radius r of the fibers is small. Hence, we obtain the following.

Proposition B.2.5 ([8, Theorem 9.70]). *Let $E \xrightarrow{\pi} B$ be a linear sphere bundle over a Riemannian manifold (B, g_B) with a connection θ on the corresponding principal $\mathrm{O}(p)$-bundle. If B is compact and the metric g_B has positive Ricci curvature, then there is a constant $r_0 > 0$ such that the metric $g_\pi(r, \theta)$ has positive Ricci curvature for all $r \in (0, r_0)$.*

B.3 Topology of Linear Sphere Bundles

In this section we consider topological properties of the total space of a linear sphere bundle $E \xrightarrow{\pi} B$. The main tool will be the Gysin sequence. By R we will denote an arbitrary commutative ring.

Lemma B.3.1. *Let $E \xrightarrow{\pi} B^q$ be an oriented linear sphere bundle with fiber S^{p-1}. Then there is an element $e_R(\pi) \in H^p(B; R)$, called the* Euler class *of π, and a homomorphism*

$$\psi\colon H^*(B; R) \to H^{*-p+1}(B; R),$$

so that the following sequence, called the Gysin sequence*, is exact:*

$$\cdots \xrightarrow{\cdot\smile e_R(\pi)} H^i(B;R) \xrightarrow{\pi^*} H^i(E;R) \xrightarrow{\psi} H^{i-p+1}(B;R) \xrightarrow{\cdot\smile e_R(\pi)} H^{i+1}(B;R) \xrightarrow{\pi^*} \cdots \quad \text{(B.3.1)}$$

The map ψ has the following properties:

1. *Let $\iota\colon S^{p-1} \hookrightarrow E$ denote the (orientation-preserving) inclusion of a fiber. Then, for any element $x \in H^{p-1}(E; R)$ with $\psi(x) = 1 \in H^0(B; R)$, we have $\iota^*(x) = [S^{p-1}; R]^*$.*
2. *If base, fibers and total space are oriented compatibly, then $\psi([E; R]^*) = (-1)^q [B; R]^*$.*
3. *For $x \in H^i(B; R)$ and $y \in H^j(E; R)$ we have*

$$\psi(\pi^*(x) \smile y) = (-1)^i x \smile \psi(y).$$

We will also write $e(\pi)$ for $e_{\mathbb{Z}}(\pi)$.

Proof. Let $\overline{E} \xrightarrow{\pi} B$ be the corresponding disc bundle. The inclusion $B \hookrightarrow \overline{E}$ of B as the zero-section is a homotopy equivalence, so we will identify $H^*(B; R)$ and $H^*(E; R)$ via the induced map of this inclusion in the following. By the Thom isomorphism theorem, see e.g. [82, Theorem 10.4], there exists a class $u_R \in H^p(\overline{E}, E; R)$, so that

$$\Phi\colon H^i(\overline{E}; R) \to H^{i+p}(\overline{E}, E; R), \quad \Phi(x) = x \smile u_R$$

is an isomorphism for all i. We define $e_R(\pi) \in H^p(B; R)$ as the image of u_R under the induced map of the inclusion $(B, \emptyset) \hookrightarrow (\overline{E}, E)$.

Now consider the following commutative diagram, where unlabeled arrows denote maps induced by the obvious inclusions.

$$\begin{tikzcd}
\cdots \arrow[r, "\cdot\smile e_R(\pi)"] & H^i(B,R) \arrow[r, "\pi^*"] \arrow[d, dash, "="] & H^i(E;R) \arrow[r, "\psi"] \arrow[d, dash, "="] & H^{i-p+1}(B;R) \arrow[r, "\cdot\smile e_R(\pi)"] \arrow[d, "\Phi"] & H^{i+1}(B;R) \arrow[r, "\pi^*"] \arrow[d, dash, "="] & \cdots \\
\cdots \arrow[r] & H^i(\overline{E};R) \arrow[r] & H^i(E;R) \arrow[r, "\delta"] \arrow[d, "\iota^*"] & H^{i+1}(\overline{E},E;R) \arrow[r] \arrow[d, "\iota^*"] & H^{i+1}(\overline{E};R) \arrow[r] & \cdots \\
& 0 \arrow[r] & H^i(S^{p-1};R) \arrow[r, "\delta"] & H^{i+1}(D^p,S^{p-1};R) \arrow[r] & 0 &
\end{tikzcd}$$

The second and third line are exact by the long exact sequences for the pairs $(\overline{E}, E)$ and (D^p, S^{p-1}), and ψ is defined as the unique map so that the diagram commutes, i.e. $\psi(x) = \Phi^{-1}(\delta(x))$. By exactness of the second line, the first line then also is exact.

The first property follows from the fact, that $\iota^*\Phi(1) = \iota^* u_R = [D^p, S^{p-1}; R]^*$.

For the second property we apply the Thom isomorphism to both sides, i.e.

$$\Phi(\psi([E; R]^*)) = \delta([E; R]^*) = (-1)^q [\overline{E}, E; R]^*$$

by Lemma B.1.9, and
$$\Phi([B;R]^*) = [B;R]^* \smile u_R.$$
Note that, since the Thom class is the unique class restricting to an oriented generator on each fiber, the class u_R is the image of the Thom class $u_\mathbb{Z}$ under the change of coefficients along $\mathbb{Z} \to R$, $z \mapsto z \cdot 1_R$. Since Φ is an isomorphism, we have $[B]^* \smile u_\mathbb{Z} = \varepsilon[\overline{E}, E]^*$ with $\varepsilon = \pm 1$. By changing the coefficients from $\mathbb{Z}$ to $R = \mathbb{R}$, it follows from [10, Proposition 6.24], that $\varepsilon = 1$, and hence, by changing coefficients from $\mathbb{Z}$ to arbitrary R, we have $[B;R]^* \smile u_R = [\overline{E}, E; R]^*$.

The third property of ψ follows from the corresponding properties of the coboundary map δ, see e.g. [79, Lemma 1]. $\square$

Remark B.3.2. The Euler class is a characteristic class. This can be seen by defining
$$e_R = e_R(\pi_{\mathrm{SO}(p)}) \in H^p(\mathrm{BSO}(p); R)$$
and then using that the Euler class is natural, which follows from the naturality of the Thom class. Indeed, if $E \xrightarrow{\pi} B$ is a fiber bundle with fiber F and structure group $\mathrm{SO}(p)$ with classifying map $f\colon B \to \mathrm{BSO}(p)$, we have
$$e_R(\pi) = e_R(f^*\pi_{\mathrm{SO}(p)}) = f^*e_R(\pi_{\mathrm{SO}(p)}).$$
However, the class e_R is not stable. In fact, the Euler class vanishes for every vector bundle that has a trivial factor, see e.g. [82, Property 9.7].

Recall that for graded-commutative R-algebras A and B the tensor product $A \otimes_R B$ is the graded-commutative R-algebra with grading
$$(A \otimes B)_k = \bigoplus_{i+j=k} (A_i \otimes A_j)$$
and multiplication
$$(a \otimes b)(c \otimes d) = (-1)^{\deg(b)\deg(c)}(ac \otimes bd).$$
We denote by $\rho_R\colon H^*(-) \to H^*(-;R)$ the map induced by the ring homomorphism $\mathbb{Z} \to R$, $z \mapsto z \cdot 1_R$.

Lemma B.3.3. *Let $E \xrightarrow{\pi} B$ be a vector bundle of rank $2k+1$ and assume that B is simply-connected. Then, for any $W \in H^{2k}(B)$ so that $\rho_{\mathbb{Z}/2}W = w_{2k}(\pi)$ we have*
$$\rho_{\mathbb{Z}/4}W^2 = \rho_{\mathbb{Z}/4}p_k(\pi).$$

Proof. The Wu formula for the bundle π is given by
$$\mathcal{P}_2(w_{2k}(\pi)) = \rho_{\mathbb{Z}/4}p_k(\pi) + \theta_2\left(w_1(\pi)\mathrm{Sq}^{2k-1}w_{2k}(\pi) + \sum_{j=0}^{k-1} w_{2j}(\pi)w_{4k-2j}(\pi)\right),$$
see e.g. [114, Theorem C]. Here $\theta_2\colon H^*(-;\mathbb{Z}/2) \to H^*(-;\mathbb{Z}/4)$ denotes the map induced by the module homomorphism $\mathbb{Z}/2 \to \mathbb{Z}/4$, $1 \mapsto 2$, $\mathrm{Sq}^{2k-1}\colon H^{2k}(B;\mathbb{Z}/2) \to H^{4k-1}(B;\mathbb{Z}/2)$ is the Steenrod square operation and $\mathcal{P}_2\colon H^{2k}(B;\mathbb{Z}/2) \to H^{4k}(B;\mathbb{Z}/4)$ is the Pontryagin square operation. For the Pontryagin square we have
$$\mathcal{P}_2(w_{2k}(\pi)) = \mathcal{P}_2(\rho_{\mathbb{Z}/2}W) = \rho_{\mathbb{Z}/4}W^2.$$
Since $w_i(\pi) = 0$ for all $i \geq 2k+2$ and $w_1(\pi) = 0$ since B is simply-connected, it follows that
$$\rho_{\mathbb{Z}/4}W^2 = \mathcal{P}_2(w_{2k}(\pi)) = \rho_{\mathbb{Z}/4}p_k(\pi).$$
$\square$

Proposition B.3.4. *Let $E \xrightarrow{\pi} B$ be an oriented linear sphere bundle with fiber S^{p-1}, whose Euler class $e_R(\pi) \in H^p(B;R)$ vanishes and suppose that $H^{2p-2}(B)$ has no element of order 2. If p is odd, let $W \in H^{p-1}(B)$ so that $\rho_{\mathbb{Z}/2} W = w_{p-1}(\pi)$ and define (using Lemma B.3.3) $P = \frac{1}{4}(p_i(\pi) - W^2)$ for $i = \frac{2p-2}{4}$. Then there is an element $a \in H^{p-1}(E;R)$, so that*

$$H^*(E;R) \cong \begin{cases} H^*(B;R) \otimes_R R[a] \Big/ \langle 1 \otimes a^2 - \rho_R W \otimes a - \rho_R P \otimes 1 \rangle, & p \text{ odd,} \\ H^*(B;R) \otimes_R \Lambda_R[a], & p \text{ even.} \end{cases}$$

The isomorphism is given by $x \otimes a^i \mapsto \pi^*(x) \smile a^i$.

Proof. Since, the Euler class vanishes, the Gysin sequence (B.3.1) splits into short exact sequences of the form

$$0 \longrightarrow H^i(B;R) \xrightarrow{\pi^*} H^i(E;R) \xrightarrow{\psi} H^{i-p+1}(B;R) \longrightarrow 0. \tag{B.3.2}$$

Following [79, Section 8], let $a \in H^{p-1}(E;R)$, so that $\psi(a) = 1 \in H^0(B;R)$. Define $\theta_a \colon H^*(B;R) \to H^{*+p-1}(E;R)$ by

$$\theta_a(x) = (-1)^{ip} a \smile \pi^*(x)$$

for $x \in H^i(B;R)$. Then, by (3) of Lemma B.3.1, the map θ_a defines a splitting of (B.3.2), hence

$$H^i(E;R) = \pi^*(H^i(B;R)) \oplus \theta_a(H^{i-p+1}(B;R)). \tag{B.3.3}$$

Hence, every $y \in H^i(E;R)$ can be written as $y = \pi^*(x_1) + \pi^*(x_2) \smile a$ for unique elements $x_1 \in H^i(B;R)$, $x_2 \in H^{i-p+1}(B;R)$. In particular, there are $\alpha \in H^{2p-2}(B;R)$ and $\beta \in H^{p-1}(B;R)$ so that

$$a \smile a = \pi^*(\alpha) + \pi^*(\beta) \smile a.$$

Thus, $H^*(E;R)$ is isomorphic to

$$H^*(B;R) \otimes_R R[a] \Big/ \langle 1 \otimes a^2 - \beta \otimes a - \alpha \otimes 1 \rangle \cdot$$

For the values of α and β first consider the case $R = \mathbb{Z}$. If p is even, then

$$a \smile a = (-1)^{(p-1)^2} a \smile a = -a \smile a$$

and since $H^{2p-2}(B)$ has no element of order 2, we have $a \smile a = 0$, which shows that

$$H^*(E) \cong H^*(B;R) \otimes_R R[a] \Big/ \langle 1 \otimes a^2 \rangle \cong H^*(B;R) \otimes_R \Lambda_R[a].$$

If p is odd, then, by [79, Theorem III], we can choose a so that $\beta = W$. Further, by [79, Theorem IV], we have

$$p_i(\pi) = 4\alpha + \beta^2.$$

Since $H^{2p-2}(B)$ has no element of order 2, this equation determines α uniquely, and we can write

$$\alpha = \frac{1}{4}(p_i(\pi) - \beta^2) = \frac{1}{4}(p_i(\pi) - W^2) = P.$$

For an arbitrary commutative ring R first note that $\psi \circ \rho_R = \rho_R \circ \psi$. This follows from the fact that $\rho_R u_{\mathbb{Z}} = u_R$, cf. [82, Remark on p. 111]. Thus, for any $a \in H^{p-1}(E)$ with $\psi(a) = 1$, we have $\psi(\rho_R a) = \rho_R \psi(a) = 1$ and we denote $\rho_R a \in H^{p-1}(E;R)$ again by a. Hence, if p is even, then $a \smile a = \rho_R 0 = 0$, and if p is odd, then $\alpha = \rho_R P$ and $\beta = \rho_R W$. □

Lemma B.3.5. *Let $E \xrightarrow{\pi} B$ be a linear sphere bundle and let $c \in H^i(\mathrm{BO}(p);R)$ be a stable characteristic class. Denote the vector bundle corresponding to π by ξ. Then*

$$c(E) = \pi^* c(TB \oplus \xi).$$

Proof. Denote by $\overline{E}$ the total space of the disc bundle corresponding to E and let $\iota\colon E \hookrightarrow \overline{E}$ be the inclusion. Then $\iota^* T\overline{E} \cong TE \oplus \mathbb{R}_E$, the trivial factor corresponds to the normal bundle of $E = \partial\overline{E}$. Further, $T\overline{E} \cong \pi^* TB \oplus \pi^*\xi$, which can be verified by turning the bundle into a Riemannian submersion, since then the horizontal distribution is isomorphic to $\pi^* TB$ and the vertical distribution is isomorphic to $\pi^*\xi$.

It follows that

$$c(E) = c(TE \oplus \mathbb{R}_E) = c(\iota^* T\overline{E}) = \iota^* c(\pi^*(TB \oplus \xi)) = \iota^*\pi^* c(TB \oplus \xi) = \pi^* c(TB \oplus \xi).$$

□

Let M^{6k} be a closed, oriented manifold. Then the cup product on M defines a trilinear form $\mu_M\colon H^{2k}(M) \times H^{2k}(M) \times H^{2k}(M) \to \mathbb{Z}$ via

$$\mu_M(x_1, x_2, x_3) = \langle x_1 \smile x_2 \smile x_3, [M]\rangle.$$

By the graded-commutativity of the cup product, the trilinear form μ_M is symmetric. Further, we have the k-th Pontryagin class $p_k(M) \in H^{4k}(M)$, which can be seen as a linear form on $H^{2k}(M)$ via

$$p_1(M)(x) = \langle p_1(M) \smile x, [M]\rangle$$

for $x \in H^{2k}(M)$. These invariants are important invariants for M, see Section 6.1. If M has the structure of a linear S^{2k}-bundle over a $4k$-dimensional manifold, then it is $6k$-dimensional and we can calculate its invariants.

Corollary B.3.6. *Let $E \xrightarrow{\pi} B^{4k}$ be an oriented linear S^{2k}-bundle, where B is closed, connected and oriented and π is oriented compatibly.*

- *If $e(\pi) \neq 0$, then $H^{2k}(E) = \pi^* H^{2k}(B)$ and μ_E and $p_k(E)$ vanish.*
- *If $e(\pi) = 0$, then for any $W \in H^{2k}(B)$ with $\rho_{\mathbb{Z}/2} W = w_{2k}(\pi)$ there exists $a \in H^{2k}(E)$ so that*

$$H^{2k}(E) = \pi^*(H^{2k}(B)) \oplus \mathbb{Z}a$$

and for $x_1, x_2, x_3 \in H^{2k}(B)$ we have

$$\begin{aligned} \mu_E(\pi^* x_1, \pi^* x_2, \pi^* x_3) &= 0, \\ \mu_E(\pi^* x_1, \pi^* x_2, a) &= \langle x_1 \smile x_2, [B]\rangle, \\ \mu_E(\pi^* x_1, a, a) &= \langle x_1 \smile W, [B]\rangle, \\ \mu_E(a, a, a) &= \frac{1}{4}\langle 3W^2 + p_k(\pi), [B]\rangle. \end{aligned}$$

Further, for $x \in H^{2k}(B)$ we have

$$\begin{aligned} p_k(E)(\pi^* x) &= 0, \\ p_k(E)(a) &= \langle p_k(TB \oplus \xi), [B]\rangle. \end{aligned}$$

In particular we have in both cases that if $p_k(E) \neq 0$, then μ_E is trivial on $\ker(p_k(E))$.

Proof. First suppose that $e(\pi)$ is non-trivial. Then the Gysin sequence in degree $2k$ is given by

$$0 \longrightarrow H^{2k}(B) \xrightarrow{\pi^*} H^{2k}(E) \xrightarrow{\psi} H^0(B) \xrightarrow{\cdot\smile e(\pi)} H^{2k+1}(B) \longrightarrow \cdots.$$

Since $e(\pi)$ is non-trivial, the map $\cdot \smile e(\pi)\colon H^0(B) \to H^{2k+1}(B)$ is invective, hence the map π^* is an isomorphism and for $x_1, x_2, x_3 \in H^{2k}(B)$ we have

$$\pi^* x_1 \smile \pi^* x_2 \smile \pi^* x_3 = \pi^*(x_1 \smile x_2 \smile x_3) = 0.$$

Further, by Lemma B.3.5, $p_k(E) = \pi^* p_k(TB \oplus \xi)$, so

$$p_k(E) \smile \pi^* x_1 = \pi^*(p_k(TB \oplus \xi) \smile x_1) = 0.$$

Now suppose that $e(\pi)$ vanishes. Then, by Proposition B.3.4, we have

$$H^{2k}(E) = \pi^* H^{2k}(B) \oplus \mathbb{Z}a$$

and

$$a \smile a = \pi^* W \smile a + \pi^* P.$$

Further, we have

$$\psi(a \smile \pi^*[B]^*) = \psi(\theta_a([B]^*)) = [B]^*,$$

hence, by Lemma B.3.1, $a \smile \pi^*[B]^* = [E]^*$, so

$$[B]^* \frown (a \frown [E]) = [E]^* \frown [E] = 1.$$

It follows that $a \frown [E] = [B]$ and hence

$$\langle \pi^* y \smile a, [E] \rangle = \langle y, [B] \rangle$$

for all $y \in H^{4k}(B)$. Hence, we can calculate

$$\begin{aligned}
\mu_E(\pi^* x_1, \pi^* x_2, \pi^* x_3) &= \langle \pi^*(x_1 \smile x_2 \smile x_3), [E] \rangle = 0, \\
\mu_E(\pi^* x_1, \pi^* x_2, a) &= \langle \pi^*(x_1 \smile x_2) \smile a, [E] \rangle = \langle x_1 \smile x_2, [B] \rangle, \\
\mu_E(\pi^* x_1, a, a) &= \langle \pi^*(x_1 \smile W) \smile a + \pi^*(x_1 \smile P), [E] \rangle = \langle x_1 \smile W, [B] \rangle, \\
\mu_E(a, a, a) &= \langle \pi^* W \smile a \smile a + \pi^* P \smile a, [E] \rangle \\
&= \langle \pi^* W \smile (\pi^* W \smile a + \pi^* P) + \pi^* P \smile a, [E] \rangle \\
&= \langle \pi^* W^2 \smile a + \pi^* P \smile a, [E] \rangle \\
&= \frac{1}{4} \langle 3W^2 + p_k(\pi), [B] \rangle.
\end{aligned}$$

Finally, the expression for $p_k(E)$ follows from Lemma B.3.5. $\square$

The following lemma is a consequence of Lemma B.3.1 and Corollary B.3.6 and provides topological obstructions for the existence of sphere bundle structures on a given topological space.

Lemma B.3.7. *Let $E \xrightarrow{\pi} B^q$ be a linear S^{p-1}-bundle with B closed, oriented and connected.*

1. *For the Euler characteristic we have $\chi(E) = \chi(B)\chi(S^{p-1})$. In particular, $\chi(E)$ vanishes if p is even and $\chi(E)$ is even if p is odd.*

2. *If all cohomology groups of E in odd degrees vanish, then $b_j(E)$ is even for $j = \frac{p+q-1}{2}$.*

3. *If $p + q - 1 = 6k$ and $p_k(E) \neq 0$ and*

   - $2k + 1 \leq p \leq 6k$, *or*
   - $1 < p \leq 2k$ *and E is $(2k-1)$-connected,*

   then μ_E is trivial on $\ker(p_k(E))$.

Proof. The first claim is well-known and holds more generally for fiber bundles [106]. For linear sphere bundles it follows from the Gysin sequence (Lemma B.3.1) that

$$\begin{aligned}
0 &= \sum_{i=0}^{p+q-1} (-1)^i (b_i(B) - b_i(E) + b_{i-p+1}(B)) \\
&= \sum_{i=0}^{q} (-1)^i b_i(B) - \sum_{i=0}^{p+q-1} (-1)^i b_i(E) + \sum_{i=p-1}^{p+q-1} (-1)^i b_i(B) \\
&= \chi(B) - \chi(E) + (-1)^{p-1}\chi(B) \\
&= \chi(S^{p-1})\chi(B) - \chi(E).
\end{aligned}$$

For the second claim, first note, that, since E only has cohomology in even degrees, its Euler characteristic is positive. Hence, by item (1), we have that p is odd, so $\chi(E)$ is even. Then it follows by Poincaré duality, that

$$\chi(E) = 2\sum_{i=0}^{j-1} b_i(E) + b_j(E),$$

and hence $b_j(E)$ is even.

Finally, suppose that $p + q - 1 = 6k$ and $p_k(E) \neq 0$. If $p = 2k + 1$, then, by Corollary B.3.13, the trilinear form is trivial on $\ker(p_k(E))$. Further, if $q < 4k$, then, by Lemma B.3.5, we have that $p_k(E) \in H^{4k}(E)$ vanishes. Hence, we can assume that $4k < q < 6k$ and that E is $(2k - 1)$-connected. Then for $0 < i + p < 2k$, by the Gysin sequence (Lemma B.3.1), we have an isomorphism

$$H^i(B) \xrightarrow{\cdot \smile e(\pi)} H^{i+p}(B).$$

Hence, we have for $0 < i < 2k$

$$H^i(B) \cong \begin{cases} \mathbb{Z}, & p \mid i, \\ 0, & \text{else.} \end{cases}$$

Again from the Gysin sequence we obtain the following exact sequence.

$$H^{2k}(B) \xrightarrow{\pi^*} H^{2k}(E) \xrightarrow{\psi} H^{2k-p+1}(B).$$

We have $H^{2k-p+1}(B) \cong \mathbb{Z}$ or 0. Let $y \in H^{4k}(B)$ so that $\pi^* y = p_k(E)$ (which exists by Lemma B.3.5). Then for every $x \in H^{2k}(B)$ we have

$$\pi^*(x) \smile p_k(E) = \pi^*(x \smile y) = 0.$$

Hence, if $p_k(E) \neq 0$, we have $H^{2k-p+1}(B) \cong \mathbb{Z}$ and ψ is non-trivial, otherwise $p_k(E)$ would be trivial on $H^{2k}(E)$, which would imply $p_k(E) = 0$ by Poincaré duality. Hence, if $a \in H^{2k}(E)$ is a preimage of a generator of $\operatorname{im}\psi$, every $y \in H^{2k}(E)$ can be written as $y = \pi^* x + \lambda a$, where $x \in H^{2k}(B)$. Since $p_k(E)$ is trivial on $\pi^* H^{2k}(B)$, it follows that $\ker(p_k(E)) = \pi^* H^{2k}(B)$, on which μ_E is trivial. $\square$

For the existence of bundles as in Corollary B.3.6 we first consider the case $k = 1$.

Lemma B.3.8. *Let B be a closed, simply-connected 4-manifold. Then isomorphism classes of linear S^2-bundles over B are in bijection with pairs $(x, Y) \in H^2(B; \mathbb{Z}/2) \times H^4(B)$ such that*

$$\rho_{\mathbb{Z}/4} X^2 = \rho_{\mathbb{Z}/4} Y$$

for some $X \in H^2(B)$ with $\rho_{\mathbb{Z}/2} X = x$. The bijection is given by assigning the pair $(w_2(\pi), p_1(\pi))$ to a linear S^2-bundle $E \xrightarrow{\pi} B$. The Euler class of all such bundles vanishes.

Proof. This is precisely the classification of Dold and Whitney [39], except that the condition on (x, Y) is given by

$$\mathcal{P}_2(x) \equiv Y \mod 4,$$

where $\mathcal{P}_2 \colon H^2(B; \mathbb{Z}/2) \to H^4(B; \mathbb{Z}/4)$ is the Pontryagin square operation. Since $H^2(B)$ is free abelian, every class $x \in H^2(B; \mathbb{Z}/2)$ has a preimage $X \in H^2(B)$. Hence,

$$\mathcal{P}_2(x) = \mathcal{P}_2(\rho_{\mathbb{Z}/2}X) = \rho_{\mathbb{Z}/4}X^2,$$

which shows that the condition $\mathcal{P}_2(x) = \rho_{\mathbb{Z}/4}Y$ is equivalent to $\rho_{\mathbb{Z}/4}X^2 = \rho_{\mathbb{Z}/4}Y$ for one, and thus for all $X \in H^2(B)$ with $\rho_{\mathbb{Z}/4}X = x$.

Finally, the Euler class vanishes, since B is simply-connected, hence orientable, and we have $0 = H_1(M) \cong H^3(M)$. □

Let $B = B^{4k}_{\overline{\gamma}} = \#_{i=1}^m \gamma_i \mathbb{C}P^{2k}$ for $\overline{\gamma} = (\gamma_1, \ldots, \gamma_m) \in \{\pm 1\}^m$. We denote by $b_i \in H^2(\gamma_i \mathbb{C}P^{2k})$ a generator of the cohomology ring of the i-th summand of B. Then $b_i^{2k} = \gamma_i [B]^*$. For the case where $k = 1$ we then obtain from Lemma B.3.8 the following corollary.

Corollary B.3.9. *Isomorphism classes of linear S^2-bundles over $B = B^4_{\overline{\gamma}}$ are in bijection with elements $(\alpha, \overline{\beta}) \in \mathbb{Z} \times \{0,1\}^m$. The bijection assigns to α and $\overline{\beta} = (\beta_1, \ldots, \beta_m)$ the linear S^2-bundle $E \xrightarrow{\pi} B$ with $p_1(\pi) = (4\alpha + \sum_i \gamma_i \beta_i)[B]^*$ and $w_2(\pi) = \sum_i \beta_i \rho_{\mathbb{Z}/2} b_i$.*

In higher dimension we have the following result:

Lemma B.3.10. *For all $k \in \mathbb{N}$ there is $\lambda_k \in \mathbb{N}$, so that for every $\alpha \in \mathbb{Z}$ there exists a linear S^{2k}-bundle $E \xrightarrow{\pi} B^{4k}_{\overline{\gamma}}$ with*

$$w_i(\pi) = \begin{cases} 1, & i = 0, \\ 0, & \textit{else}, \end{cases}$$
$$p_i(\pi) = \begin{cases} 1, & i = 0, \\ \alpha \lambda_k [B^{4k}_{\overline{\gamma}}]^*, & i = k, \\ 0, & \textit{else}, \end{cases}$$
$$e(\pi) = 0.$$

Proof. First we consider vector bundles of rank $4k$ over S^{4k}. These bundles are classified by elements in $\pi_{4k}(\mathrm{BSO}(4k)) \cong \pi_{4k-1}(\mathrm{SO}(4k))$, cf. Proposition B.1.5. To determine this group consider the principal $\mathrm{SO}(4k)$-bundle $\mathrm{SO}(4k+1) \to S^{4k}$ from Example B.1.3. The long exact sequence of homotopy groups (Lemma B.1.4) yields

$$\cdots \longrightarrow \pi_{4k}(S^{4k}) \longrightarrow \pi_{4k-1}(\mathrm{SO}(4k)) \longrightarrow \pi_{4k-1}(\mathrm{SO}(4k+1)) \longrightarrow \pi_{4k-1}(S^{4k}) \longrightarrow \cdots.$$

We have $\pi_{4k}(S^{4k}) \cong \mathbb{Z}$ and that $\pi_{4k-1}(S^{4k})$ is trivial. Further, by Bott periodicity, we have $\pi_{4k-1}(\mathrm{SO}(4k+1)) \cong \mathbb{Z}$. It follows that $\pi_{4k}(\mathrm{BSO}(4k)) \cong \pi_{4k-1}(\mathrm{SO}(4k))$ is either isomorphic to $\mathbb{Z}$ or to $\mathbb{Z} \oplus \mathbb{Z}$ and the map $\pi_{4k}(\mathrm{BSO}(4k)) \to \pi_{4k}(\mathrm{BSO}(4k+1))$ is surjective.

Define a homomorphism

$$\phi \colon \pi_{4k}(\mathrm{BSO}(4k)) \to \mathbb{Z} \oplus \mathbb{Z}, \quad [\alpha] \mapsto (e \frown H[\alpha], p_k \frown H[\alpha]),$$

where $H \colon \pi_{4k}(\mathrm{BSO}(4k)) \to H_{4k}(\mathrm{BSO}(4k))$ is the Hurewicz homomorphism.

The tangent bundle of S^{4k} defines an element $[\alpha] \in \pi_{4k}(\mathrm{BSO}(4k))$ with $\phi([\alpha]) = (2, 0)$. Further, there is a vector bundle of rank $(4k+1)$ over S^{4k} with non-vanishing k-th Pontryagin class, see [77, Theorem 3.8]. Since the map $\pi_{4k}(\mathrm{BSO}(4k)) \to \pi_{4k}(\mathrm{BSO}(4k+1))$ is surjective, there is a class $[\beta] \in \pi_{4k}(\mathrm{BSO}(4k))$ with $\phi([\beta]) = (x, y)$ with $y \neq 0$. Hence, the image of ϕ has rank at least two, which shows that $\pi_{4k}(\mathrm{BSO}(4k)) \cong \mathbb{Z} \oplus \mathbb{Z}$ and the map ϕ is injective.

Now we consider vector bundles of rank $2k+1$ over S^{4k}. For that, consider again the long exact sequence

$$\cdots \longrightarrow \pi_{4k}(S^i) \longrightarrow \pi_{4k-1}(\mathrm{SO}(i)) \longrightarrow \pi_{4k-1}(\mathrm{SO}(i+1)) \longrightarrow \pi_{4k-1}(S^i) \longrightarrow \cdots .$$

By Serre's finiteness theorem [106], the groups $\pi_j(S^i)$ with $j > i$ are finite unless $j = 4m-1$ and $i = 2m$ for some $m \in \mathbb{N}$. Hence, the map $\pi_{4k-1}\mathrm{SO}(i) \to \pi_{4k-1}\mathrm{SO}(i+1)$ is rationally an isomorphism for $2k+1 \leq i < 4k-2$ and injective for $i = 4k-1$. Hence,

$$\pi_{4k}(\mathrm{BSO}(2k+1)) \cong \pi_{4k-1}(\mathrm{SO}(2k+1)) \cong \mathbb{Z} \oplus F_k$$

for a finite group F_k. Moreover, the map $\pi_{4k}(\mathrm{BSO}(2k+1)) \to \pi_{4k}(\mathrm{BSO}(4k))$ maps a primitive element $[\alpha]$ of infinite order to a non-zero element $[\beta]$ of $\pi_{4k}(\mathrm{BSO}(4k)) \cong \mathbb{Z} \oplus \mathbb{Z}$. Since the bundle defined by $[\beta]$ has a trivial factor, we have $\phi([\beta]) = (0, \lambda_k)$ for some $\lambda_k \in \mathbb{Z}$. Since ϕ is injective and $[\beta]$ non-trivial, it follows that $\lambda_k \neq 0$. By possibly multiplying with (-1) we can assume that $\lambda_k \in \mathbb{N}$.

Let π_0 be the linear sphere bundle defined by $[\alpha]$. Then $p_k(\pi_0) = \lambda_k [S^{4k}]^*$. Since the bundle defined by $[\beta]$ has vanishing Euler class, its $4k$-th Stiefel-Whitney class also vanishes, see e.g. [82, Property 9.5]. Thus, $w_{4k}(\pi_0) = 0$. Further, $w_i(\pi_0)$ for $0 < i < 4k$, $p_i(\pi_0)$ for $0 < i < k$ and $e(\pi_0)$ vanish as the corresponding cohomology groups of S^{4k} vanish.

Now for given $\alpha \in \mathbb{Z}$ let $f_\alpha \colon S^{4k} \to S^{4k}$ be a map of degree α, i.e. $f_\alpha^* \colon H^{4k}(S^{4k}) \to H^{4k}(S^{4k})$ is multiplication by α. Further, let $h \colon B \to S^{4k}$ be an orientation-preserving collapse map, i.e. for an orientation-preserving embedding $D^{4k} \hookrightarrow B$, $h \colon B \to S^{4k} \cong D^{4k}/\partial D^{4k}$ is the identity on ${D^{4k}}^\circ$, and maps all the other points to ∂D^{4k}. Then $h^* \colon H^{4k}(S^{4k}) \to H^{4k}(B)$ is an isomorphism that maps $[S^{4k}]^*$ to $[B]^*$. We define

$$\pi = h^* f_\alpha^* \pi_0.$$

The claim now follows from the naturality properties of w_i, p_i and e. $\square$

Remark B.3.11. It follows from Corollary B.3.9, that $\lambda_1 = 4$ and we obtain precisely those bundles in Lemma B.3.10, that correspond to $(\alpha, (0, \ldots, 0))$ in Corollary B.3.9. In general, λ_k is a multiple of $\frac{3-(-1)^k}{2}(2k-1)!$, see [77, Theorem 3.8] and for $k > 4$ we have in fact equality by [4, Theorem 1*].

Further, by Lemma B.3.3, since the Stiefel-Whitney classes of every bundle π in Lemma B.3.10 vanish, it follows that $p_k(\pi)$, and thus λ_k is divisible by 4. We have already seen this for $\lambda_1 = 4$ and it also holds for λ_k if $k \geq 3$, since then $(2k-1)!$ is divisible by 4. However, for $k = 2$, we have $(2k-1)! = 6$, showing that λ_2 is divisible by 12.

Note that for $k \geq 3$ we have

$$\binom{2k+1}{k} = \frac{(2k+1)!}{k!(k+1)!} = (2k-1)!\frac{2k(2k+1)}{k!(k+1)!}$$

and

$$\frac{2k(2k+1)}{k!(k+1)!} \leq \frac{2k(2k+2)}{k!(k+1)!} = \frac{4}{(k-1)!k!} < 1,$$

showing that $\binom{2k+1}{k} < (2k-1)! \leq \lambda_k$. Since $\lambda_1 = 4$ and $\lambda_2 \geq 12$, the inequality $\binom{2k+1}{k} < \lambda_k$ holds for all $k \in \mathbb{N}$. This fact is useful in Chapter 6.

For the manifold $B = B^{4k}_{\overline{\gamma}}$ we have

$$H^*(B;R) \cong R[b_1,\dots,b_m] \Big/ \langle b_ib_j, \gamma_i b_i^{2k} - \gamma_j b_j^{2k}, b_i^{2k+1} \mid i \neq j \rangle$$

cf. Lemma 4.1.4. If $E \xrightarrow{\pi} B$ is a linear sphere bundle we will denote the images $\pi^* b_i$ in $H^*(E;R)$ again by b_i.

Further,

$$w_j(B) = \begin{cases} \binom{2k+1}{j/2} \sum_i \rho_{\mathbb{Z}/2} b_i^{j/2}, & j \text{ even}, \\ 0, & \text{else}, \end{cases}$$

and

$$p_j(B) = \binom{2k+1}{j} \sum_i b_i^{2j},$$

see [82, Corollary 11.15 and Example 15.6]. For $k = 1$ we obtain

$$w_2(B) = \sum_i \rho_{\mathbb{Z}/2} b_i$$

and

$$p_1(B) = 3 \sum_i b_i^2 = 3 \sum_i \gamma_i [B]^*.$$

Thus, combining Proposition B.3.4 Lemma B.3.5 and Corollary B.3.6 with the existence results in Lemmas B.3.8 and B.3.10 and Corollary B.3.9 yields the following corollaries.

Corollary B.3.12. *Let $E \xrightarrow{\pi} B^4_{\overline{\gamma}}$ be an oriented linear S^2-bundle that corresponds to the tuple $(\alpha, \overline{\beta}) \in \mathbb{Z} \times \{0,1\}^m$ in Corollary B.3.9. Then*

$$H^*(E;R) \cong R[a, b_1,\dots,b_m] \Big/ \langle b_ib_j, \gamma_i b_i^2 - \gamma_j b_j^2, b_i^3, a^2 - a \textstyle\sum_i \beta_i b_i - \alpha\gamma_1 b_1^2 \mid i \neq j \rangle,$$

where $a, b_1, \dots, a_m$ have degree 2. Further,

$$\begin{aligned} w_2(E) &= \sum_i (1-\beta_i) b_i, \\ p_1(E) &= (4\alpha + \sum_i (3+\beta_i)\gamma_i)\gamma_1 b_1^2. \end{aligned}$$

In particular, we have

$$H^2(E) = \bigoplus_i \mathbb{Z} b_i \oplus \mathbb{Z} a$$

and

$$\begin{aligned} \mu_E(b_i \smile b_j \smile b_m) &= 0, \\ \mu_E(b_i \smile b_j \smile a) &= \delta_{ij}\gamma_i, \\ \mu_E(b_i \smile a \smile a) &= \beta_i \gamma_i, \\ \mu_E(a \smile a \smile a) &= \alpha + \sum_i \beta_i\gamma_i, \\ p_1(E)(b_i) &= 0, \\ p_1(E)(a) &= 4\alpha + \sum_i (3+\beta_i)\gamma_i. \end{aligned}$$

Corollary B.3.13. *Let $E \xrightarrow{\pi} B^{4k}_{\gamma}$ be an oriented linear S^{2k}-bundle corresponding to $\alpha \in \mathbb{Z}$ in Lemma B.3.10. Then*

$$H^*(E;R) \cong R[a, b_1, \dots, b_m] \Big/ \langle b_i b_j, \gamma_i b_i^{2k} - \gamma_j b_j^{2k}, b_i^{2k+1}, a^2 - \tfrac{1}{4}\alpha\lambda_k\gamma_1 b_1^{2k} \mid i \neq j \rangle,$$

where a has degree $2k$ and $b_1, \dots, b_m$ have degree 2. Further,

$$w_j(E) = \begin{cases} \binom{2k+1}{j/2} \sum_i b_i^{j/2}, & j \text{ even}, \\ 0, & \text{else}, \end{cases}$$

$$p_j(E) = \begin{cases} \left(\binom{2k+1}{k} \sum_i \gamma_i + \alpha\lambda_k\right) \gamma_1 b_1^{2k}, & j = k, \\ \binom{2k+1}{j} \sum_i b_i^{2j}, & \text{else}. \end{cases}$$

In particular, we have

$$H^{2k}(E) = \bigoplus_i \mathbb{Z} b_i^k \oplus \mathbb{Z} a$$

and

$$\begin{aligned} \mu_E(b_i^k \smile b_j^k \smile b_m^k) &= 0, \\ \mu_E(b_i^k \smile b_j^k \smile a) &= \delta_{ij}\gamma_i, \\ \mu_E(b_i^k \smile a \smile a) &= 0, \\ \mu_E(a \smile a \smile a) &= \frac{\lambda_k}{4}\alpha, \\ w_2(E)^k &= \sum_i b_i^k, \\ p_k(E)(b_i) &= 0, \\ p_k(E)(a) &= \binom{2k+1}{k} \sum_i \gamma_i + \lambda_k \alpha. \end{aligned}$$

Incidence and Adjacency Matrix of a Directed Graph C

In this chapter we recall some well-known facts about the adjacency and incidence matrices of a directed graph. For convenience we also include self-contained proofs.

Let R be a commutative ring and let $G = (V, E)$ be a directed graph, where $V = \{v_1, \ldots, v_n\}$ is the set of vertices and $E = \{e_1, \ldots, e_m\} \subseteq V \times V$ is the set of edges. The *incidence matrix* of G, denoted by $Q(G)$, is the $n \times m$-matrix with entries in R defined by

$$Q(G)_{ij} = \begin{cases} 1, & \text{if there is } k \text{ so that } e_j = (v_i, v_k), \\ -1, & \text{if there is } k \text{ so that } e_j = (v_k, v_i), \\ 0, & \text{else.} \end{cases}$$

Lemma C.1. *Suppose G is connected. Then the matrix $Q(G)^\top$ has kernel generated by $(1, \ldots, 1)^\top$. In particular, $Q(G)$ has rank $n - 1$.*

Proof. Let $x \in R^n$ such that $x^\top Q(G) = 0$. Then, for every $e = (v_i, v_j) \in E$, we have $x_i - x_j = 0$, i.e. $x_i = x_j$. Since G is connected, for any $v_i, v_j \in V$ there is a path between v_i and v_j, so $x_i = x_j$ for all $1 \leq i, j \leq n$. □

Lemma C.2. *Suppose the underlying undirected graph of G is simply-connected, i.e. a tree. Then the matrix $Q(G)^\top$, when considered as a linear map $R^n \to R^m$, is surjective.*

Proof. Let $e_i \in E$. Since G is simply-connected, the edge e_i divides G into two subtrees. Let G' be the subtree from which e_i originates with root the vertex connected to e_i. Then define $x \in R^n$ by $x_j = 1$ whenever v_j is contained in G' and $x_j = 0$ otherwise. Then it follows that $Q(G)^\top x$ is the i-th standard basis vector of R^m and hence $Q(G)^\top$ is surjective. □

Now let $G = (U, V, E)$, be a bipartite graph, where $U = \{u_1, \ldots, u_r\}$ and $V = \{v_1, \ldots, v_s\}$ are the sets of vertices and $E \subseteq U \times V$ is the set of edges. The *biadjacency matrix* of G, denoted by $B(G)$, is the $r \times s$-matrix with entries in R defined by

$$B(G)_{ij} = \begin{cases} 1, & (u_i, v_j) \in E, \\ 0, & \text{else.} \end{cases}$$

Lemma C.3. *Suppose G is a tree so that no $v \in V$ is a leaf. Then $B(G)$ has full rank.*

Proof. We show that $B(G)$, when considered as a linear map $R^s \to R^r$, is injective. Let $x \in R^s$ with $B(G)x = 0$ and let $u_i \in U$ be a leaf. Then there is a unique $v_j \in V$ so that $(u_i, v_j) \in E$, hence $x_j = 0$.

Now we remove all the leaves, which are vertices in U, and all $v \in V$ that turn into leaves by this procedure and get a subgraph G' that is again a tree and has no vertex in V as a leaf. For all those $v_j \in V$ that get removed we already have $x_j = 0$. Hence, by removing these entries from x, we obtain a vector x' with $B(G')x' = 0$.

Hence, by induction, it follows that $x = 0$. $\square$

Invariant Theory of SL($n, \mathbb{C}$) D

In this chapter we introduce the basic notions and result of the invariant theory of GL($n, \mathbb{C}$). We refer to [26] and [111] for further details.

A polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ in n variables is *homogeneous* of degree d, if all non-zero terms of f have degree d. Thus, f can be written as

$$f = \sum_{\substack{i_1,\ldots,i_n \in \mathbb{N}_0 \\ i_1+\cdots+i_n=d}} a_{i_1,\ldots,i_n} x_1^{i_1} \ldots x_n^{i_n}$$

with $a_{i_1,\ldots,i_n} \in \mathbb{C}$. If $(e_1, \ldots, e_n)$ denotes the standard basis of $\mathbb{C}^n$, then the assignment

$$f \mapsto \sum_{i_1,\ldots,i_n} a_{i_1,\ldots,i_n} e_1^{i_1} \ldots e_n^{i_n} = f(e_1, \ldots, e_n)$$

defines an isomorphism between the space of homogeneous polynomials of degree d and the d-th symmetric power $S_d(\mathbb{C}^n)$. The standard action of the group GL($n, \mathbb{C}$) induces an action on $S_d(\mathbb{C}^n)$. The corresponding action on homogeneous polynomials is then given by

$$(A \cdot f)(x_1, \ldots, x_n) = f(A^\top(x_1, \ldots, x_n)).$$

This action in turn induces an action on the polynomial ring $\mathbb{C}[S^d(\mathbb{C}^n)]$. For every subgroup $\Gamma \subseteq$ GL($n, \mathbb{C}$) we obtain an induced action.

Definition D.1. An element $I \in \mathbb{C}[S_d(\mathbb{C}^n)]$ is called an *invariant* for Γ, if it is invariant under the action of Γ. If $d_1, \ldots, d_k \in \mathbb{N}$, then an element $I \in \mathbb{C}[S_{d_1}(\mathbb{C}^n) \oplus \cdots \oplus S_{d_k}(\mathbb{C}^n)]$ is a *joint invariant* for Γ, if it is invariant under the direct sum action of Γ.

If we consider the coefficients $a_{i_1,\ldots,i_n}$ as indeterminates, then an invariant can be viewed as a polynomial in the variables $a_{i_1,\ldots,i_n}$. The set of invariants forms a $\mathbb{C}$-algebra.

From the definition we obtain, that for two homogeneous polynomials f_1, f_2 of degree d, if there exists $A \in \Gamma$ so that $A \cdot f_1 = f_2$, then $I(f_1) = I(f_2)$ for all invariants I. The converse does not hold in general, for example, there exists the notion of *nullforms*, which are homogeneous polynomials on which all invariants vanish.

Given an integral n-ary form of degree d, that is, a symmetric multilinear map

$$\mu \colon \mathbb{Z}^n \times \overset{d}{\cdots} \times \mathbb{Z}^n \to \mathbb{Z},$$

we extend μ linearly to a $\mathbb{C}$-multilinear map on $(\mathbb{C}^n)^d$ and define a homogeneous polynomial f_μ of degree d in n variables via the equation

$$f_\mu(v) = \mu(v, \dots, v)$$

for all $v \in \mathbb{C}^n$. Two integral n-ary forms μ_1, μ_2 of degree d are *equivalent*, if there exists $A \in \mathrm{GL}(n, \mathbb{Z})$ so that $\mu_1 = A^* \mu_2$. This is the case if and only if $f_{\mu_1} = A^\top \cdot f_{\mu_2}$. Thus, if μ_1 and μ_2 are equivalent, then all invariants for Γ on f_{μ_1} and f_{μ_2} coincide if $\mathrm{GL}(n, \mathbb{Z}) \subseteq \Gamma$. Since $\det(A) \in \{\pm 1\}$ for all $A \in \mathrm{GL}(n, \mathbb{Z})$, we will consider the group

$$\Gamma = \widetilde{\mathrm{SL}}(n, \mathbb{C}) = \{A \in \mathrm{GL}(n, \mathbb{C}) \mid \det(A) = \pm 1\}.$$

Theorem D.2 (Hilbert's finiteness theorem, [26, Section 3.1]). *For n and d fixed and $\Gamma = \mathrm{SL}(n, \mathbb{C})$ or $\widetilde{\mathrm{SL}}(n, \mathbb{C})$, the algebra of (joint) invariants for Γ is finitely generated.*

In fact, this result holds for a much larger class of groups, for example for any algebraic group that is linearly reductive (see e.g. [80, Chapter 19] for the definition). Here we use that $\widetilde{\mathrm{SL}}(n, \mathbb{C})^\circ = \mathrm{SL}(n, \mathbb{C})$ and that an algebraic group G is linearly reductive if and only if G° is reductive, see e.g. [80, Corollary 22.43].

To obtain invariants for $\widetilde{\mathrm{SL}}(n, \mathbb{C})$ from those for $\mathrm{SL}(n, \mathbb{C})$ we need the following observation.

Lemma D.3. *Let I be a (joint) invariant for $\mathrm{SL}(n, \mathbb{C})$. Set $a'_{i_1,\dots,i_n} = (-1)^{i_1} a_{i_1,\dots,i_n}$. If*

$$I((a_{i_1,\dots,i_n})_{i_1,\dots,i_n}) = I((a'_{i_1,\dots,i_n})_{i_1,\dots,i_n}),$$

then I is a (joint) invariant for $\widetilde{\mathrm{SL}}(n, \mathbb{C})$.

Proof. This follows from the fact that $\widetilde{\mathrm{SL}}(n, \mathbb{C})$ is generated by $\mathrm{SL}(n, \mathbb{C})$ and the matrix

$$\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

and this matrix transforms $(a_{i_1,\dots,i_n})$ into $(a'_{i_1,\dots,i_n})$. □

An important invariant for binary forms is the *discriminant* D. For a binary form of degree it is defined as follows: Let

$$f = \sum_{i=0}^{n} a_{i,n-i} x_1^i x_2^{n-i}$$

be a homogeneous polynomial of degree n in two variables. To simplify notation we set $a_i = a_{i,n-i}$. By setting $x_2 = 1$ we obtain a polynomial $f(x_1, 1)$ in one variable of degree d. Since $\mathbb{C}$ is algebraically closed, there exist $\alpha_1, \dots, \alpha_d \in \mathbb{C}$ (which are the roots of $f(x_1, 1)$), so that

$$f(x_1, 1) = a_n(x_1 - \alpha_1) \dots (x_1 - \alpha_d).$$

Then define

$$D(f) = a_n^{2d-2} \prod_{i<j} (\alpha_i - \alpha_j)^2.$$

The discriminant is an invariant for $\mathrm{SL}(n, \mathbb{C})$. By Lemma D.3 it is also an invariant for $\widetilde{\mathrm{SL}}(n, \mathbb{C})$, since the transformation $a_i \mapsto a'_i$ has the effect that α_i gets replaced by $-\alpha_i$, which does not change the value of $D(f)$.

In particular, in degree 2 the discriminant is given by

$$D(f) = a_1^2 - 4a_2a_0$$

and in degree 3 it is given by

$$D(f) = a_2^2 a_1^2 - 4a_3 a_1^3 - 4a_2^3 a_0 - 27a_3^2 a_0^2 + 18a_3 a_2 a_1 a_0.$$

In fact, for these degrees the discriminant generates the whole ring of invariants.

Theorem D.4 ([57, XVI, XVII]). *Let $n = 2$ and $\Gamma = \mathrm{SL}(n, \mathbb{C})$ or $\Gamma = \widetilde{\mathrm{SL}}(n, \mathbb{C})$. Then for $d = 2$ and for $d = 3$ the algebra of invariants for Γ is generated by D.*

In degree 4 we have two further invariants I and J defined by

$$I(f) = a_0 a_4 - 4a_1 a_3 + 3a_2^2$$

and

$$J(f) = a_0 a_2 a_4 - a_0 a_3^2 - a_1^2 a_4 + 2a_1 a_2 a_3 - a_2^3.$$

Theorem D.5 ([57, XVIII]). *Let $n = 2$ and $\Gamma = \mathrm{SL}(n, \mathbb{C})$ or $\Gamma = \widetilde{\mathrm{SL}}(n, \mathbb{C})$. Then for $d = 4$ the algebra of invariants for Γ is generated by I and J.*

As before, the case $\Gamma = \widetilde{\mathrm{SL}}(n, \mathbb{C})$ is obtained from the case $\Gamma = \mathrm{SL}(n, \mathbb{C})$ by Lemma D.3 by observing that the transformation in Lemma D.3 transforms a_i into $(-1)^i a_i$.

For the applications in Chapter 6 we consider joint invariants for $n = 2$, $d_1 = 3$ and $d_2 = 1$. For a binary cubic polynomial $f = \sum_{i=0}^{3} a_i x_1^i x_2^{n-i}$ and a binary linear form $p = b_1 x_1 + b_0 x_2$ we define the joint invariant R, called the *resultant*, by

$$R(f, p) = a_3 b_0^3 + a_2 b_0^2 b_1 + a_1 b_0 b_1^2 + a_0 b_1^3.$$

Further, we consider the invariants D, I and J as joint invariants via $D(f)$, $I(f \cdot p)$ and $J(f \cdot p)$.

Theorem D.6 ([100, Theorem 4 and Corollary 1]). *Let $n = 2$, $d_1 = 3$ and $d_2 = 1$. Then*

- *For $\Gamma = \mathrm{SL}(n, \mathbb{C})$ the algebra of joint invariants is generated by D, R, I and J.*
- *For $\Gamma = \widetilde{\mathrm{SL}}(n, \mathbb{C})$ the algebra of joint invariants is generated by D, R^2, I and J.*

Further, the relation

$$27J^2 = \frac{1}{256} DR^2 + I^3$$

holds.

Finally, we consider ternary cubic polynomials, that is, $n = d = 3$ and f is of the form

$$\begin{aligned} f = &a_{300} x_1^3 + 3a_{210} x_1^2 x_2 + 3a_{201} x_1^2 x_3 + 3a_{120} x_1 x_2^2 + 6a_{111} x_1 x_2 x_3 + 3a_{102} x_1 x_3^2 \\ &+ a_{030} x_2^3 + 3a_{021} x_2^2 x_3 + 3a_{012} x_2 x_3^2 + a_{003} x_3^3. \end{aligned}$$

Here we have two invariants S and T defined by

$$\begin{aligned} S(f) = &a_{300} a_{120} a_{021} a_{003} - a_{300} a_{120} a_{012}^2 - a_{300} a_{111} a_{030} a_{003} + a_{300} a_{111} a_{021} a_{012} + a_{300} a_{102} a_{030} a_{012} \\ &- a_{300} a_{102} a_{021}^2 - a_{210}^2 a_{021} a_{003} + a_{210}^2 a_{012}^2 + a_{210} a_{201} a_{030} a_{003} - a_{210} a_{201} a_{021} a_{012} \\ &+ a_{210} a_{120} a_{111} a_{003} - a_{210} a_{120} a_{102} a_{012} - 2a_{210} a_{111}^2 a_{012} + 3a_{210} a_{111} a_{102} a_{021} - a_{210} a_{102}^2 a_{030} \\ &- a_{201}^2 a_{030} a_{012} + a_{201}^2 a_{021}^2 - a_{201} a_{120}^2 a_{003} + 3a_{201} a_{120} a_{111} a_{012} - a_{201} a_{120} a_{102} a_{021} \\ &- 2a_{201} a_{111}^2 a_{021} + a_{201} a_{111} a_{102} a_{030} + a_{120}^2 a_{102}^2 - 2a_{120} a_{111}^2 a_{102} + a_{111}^4 \end{aligned}$$

and

$$\begin{aligned}
T(f) = & a_{300}^2 a_{030}^2 a_{003}^2 - 6a_{300}^2 a_{030} a_{021} a_{012} a_{003} + 4a_{300}^2 a_{030} a_{012}^3 + 4a_{300}^2 a_{021}^3 a_{003} - 3a_{300}^2 a_{021}^2 a_{012}^2 \\
& - 6a_{300} a_{210} a_{120} a_{030} a_{003}^2 + 18a_{300} a_{210} a_{120} a_{021} a_{012} a_{003} - 12a_{300} a_{210} a_{120} a_{012}^3 \\
& + 12a_{300} a_{210} a_{111} a_{030} a_{012} a_{003} - 24a_{300} a_{210} a_{111} a_{021}^2 a_{003} + 12a_{300} a_{210} a_{111} a_{021} a_{012}^2 \\
& + 6a_{300} a_{210} a_{102} a_{030} a_{021} a_{003} - 12a_{300} a_{210} a_{102} a_{030} a_{012}^2 + 6a_{300} a_{210} a_{102} a_{021}^2 a_{012} \\
& + 6a_{300} a_{201} a_{120} a_{030} a_{012} a_{003} - 12a_{300} a_{201} a_{120} a_{021}^2 a_{003} + 6a_{300} a_{201} a_{120} a_{021} a_{012}^2 \\
& + 12a_{300} a_{201} a_{111} a_{030} a_{021} a_{003} - 24a_{300} a_{201} a_{111} a_{030} a_{012}^2 + 12a_{300} a_{201} a_{111} a_{021}^2 a_{012} \\
& - 6a_{300} a_{201} a_{102} a_{030}^2 a_{003} + 18a_{300} a_{201} a_{102} a_{030} a_{021} a_{012} - 12a_{300} a_{201} a_{102} a_{021}^3 \\
& + 4a_{300} a_{120}^3 a_{003}^2 - 24a_{300} a_{120}^2 a_{111} a_{012} a_{003} - 12a_{300} a_{120}^2 a_{102} a_{021} a_{003} \\
& + 24a_{300} a_{120}^2 a_{102} a_{012}^2 + 36a_{300} a_{120} a_{111}^2 a_{021} a_{003} + 12a_{300} a_{120} a_{111}^2 a_{012}^2 \\
& + 12a_{300} a_{120} a_{111} a_{102} a_{030} a_{003} - 60a_{300} a_{120} a_{111} a_{102} a_{021} a_{012} - 12a_{300} a_{120} a_{102}^2 a_{030} a_{012} \\
& + 24a_{300} a_{120} a_{102}^2 a_{021}^2 - 20a_{300} a_{111}^3 a_{030} a_{003} - 12a_{300} a_{111}^3 a_{021} a_{012} \\
& + 36a_{300} a_{111}^2 a_{102} a_{030} a_{012} + 12a_{300} a_{111}^2 a_{102} a_{021}^2 - 24a_{300} a_{111} a_{102}^2 a_{030} a_{021} \\
& + 4a_{300} a_{102}^3 a_{030}^2 + 4a_{210}^3 a_{030} a_{003}^2 - 12a_{210}^3 a_{021} a_{012} a_{003} + 8a_{210}^3 a_{012}^3 \\
& - 12a_{210}^2 a_{201} a_{030} a_{012} a_{003} + 24a_{210}^2 a_{201} a_{021}^2 a_{003} - 12a_{210}^2 a_{201} a_{021} a_{012}^2 \\
& - 3a_{210}^2 a_{120}^2 a_{003}^2 + 12a_{210}^2 a_{120} a_{111} a_{012} a_{003} - 24a_{120}^2 a_{111}^2 a_{102}^2 + 24a_{120} a_{111}^4 a_{102} \\
& + 6a_{210}^2 a_{120} a_{102} a_{021} a_{003} - 12a_{210}^2 a_{120} a_{102} a_{012}^2 + 12a_{210}^2 a_{111}^2 a_{021} a_{003} - 24a_{210}^2 a_{111}^2 a_{012}^2 \\
& - 24a_{210}^2 a_{111} a_{102} a_{030} a_{003} - 27a_{210}^2 a_{102}^2 a_{021}^2 + 36a_{210}^2 a_{111} a_{102} a_{021} a_{012} \\
& + 24a_{210}^2 a_{102}^2 a_{030} a_{012} - 12a_{210} a_{201}^2 a_{030} a_{021} a_{003} + 24a_{210} a_{201}^2 a_{030} a_{012}^2 \\
& - 12a_{210} a_{201}^2 a_{021}^2 a_{012} + 6a_{210} a_{201} a_{120}^2 a_{012} a_{003} - 60a_{210} a_{201} a_{120} a_{111} a_{021} a_{003} \\
& + 36a_{210} a_{201} a_{120} a_{111} a_{012}^2 + 18a_{210} a_{201} a_{120} a_{102} a_{030} a_{003} - 6a_{210} a_{201} a_{120} a_{102} a_{021} a_{012} \\
& + 36a_{210} a_{201} a_{111}^2 a_{030} a_{003} - 12a_{210} a_{201} a_{111}^2 a_{021} a_{012} - 60a_{210} a_{201} a_{111} a_{102} a_{030} a_{012} \\
& + 36a_{210} a_{201} a_{111} a_{102} a_{021}^2 + 6a_{210} a_{201} a_{102}^2 a_{030} a_{021} + 12a_{210} a_{120}^2 a_{111} a_{102} a_{003} \\
& - 12a_{210} a_{120}^2 a_{102}^2 a_{012} - 12a_{210} a_{120} a_{111}^3 a_{003} - 12a_{210} a_{120} a_{111}^2 a_{102} a_{012} \\
& + 36a_{210} a_{120} a_{111} a_{102}^2 a_{021} - 12a_{210} a_{120} a_{102}^3 a_{030} + 24a_{210} a_{111}^4 a_{012} \\
& - 36a_{210} a_{111}^3 a_{102} a_{021} + 12a_{210} a_{111}^2 a_{102}^2 a_{030} + 4a_{201}^3 a_{030}^2 a_{003} - 12a_{201}^3 a_{030} a_{021} a_{012} \\
& + 8a_{201}^3 a_{021}^3 + 24a_{201}^2 a_{120}^2 a_{021} a_{003} - 27a_{201}^2 a_{120}^2 a_{012}^2 \\
& - 24a_{201}^2 a_{120} a_{111} a_{030} a_{003} + 36a_{201}^2 a_{120} a_{111} a_{021} a_{012} + 6a_{201}^2 a_{120} a_{102} a_{030} a_{012} \\
& - 12a_{201}^2 a_{120} a_{102} a_{021}^2 + 12a_{201}^2 a_{111}^2 a_{030} a_{012} - 24a_{201}^2 a_{111}^2 a_{021}^2 \\
& + 12a_{201}^2 a_{111} a_{102} a_{030} a_{021} - 3a_{201}^2 a_{102}^2 a_{030}^2 - 12a_{201} a_{120}^3 a_{102} a_{003} + 12a_{201} a_{120}^2 a_{111}^2 a_{003} \\
& + 36a_{201} a_{120}^2 a_{111} a_{102} a_{012} - 12a_{201} a_{120}^2 a_{102}^2 a_{021} - 36a_{201} a_{120} a_{111}^3 a_{012} \\
& - 12a_{201} a_{120} a_{111}^2 a_{102} a_{021} + 12a_{201} a_{120} a_{111} a_{102}^2 a_{030} + 24a_{201} a_{111}^4 a_{021} \\
& - 12a_{201} a_{111}^3 a_{102} a_{030} + 8a_{120}^3 a_{102}^3 - 8a_{111}^6 .
\end{aligned}$$

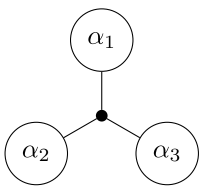
Theorem D.7 ([111, Theorem 4.4.6]). *Let $n = 3$ and $\Gamma = \mathrm{SL}(n, \mathbb{C})$ or $\Gamma = \widetilde{\mathrm{SL}}(n, \mathbb{C})$. Then for $d = 3$ the algebra of invariants for Γ is generated by S and T.*

Algorithms for Systems of Invariants E

This appendix provides explicit algorithms involving systems of invariants.

Algorithm E.1 computes the normal form of an admissible system of invariants (H, μ, w, p) with $\mathrm{rank}(H) = 2$ and $w = 0$ according to the classification of Schmitt [100], see Theorem 6.1.4. For that, the algorithm first determines the kernel of the linear form p. A generator of this kernel is then extended to a basis of H. This extension is then modified, so that the coefficients in this basis are in normal form.

Algorithm E.2 determines, whether a given system of invariants can be realized by an algebraic plumbing graph of the form



For that, the algorithm follows the steps in the proof of Proposition 4.3.4, i.e. it reconstructs possible values for $\alpha_1, \alpha_2, \alpha_3$ from the invariants D, R^2, I, J, cf. Theorem D.6. See Remark 6.4.6 for a detailed description.

Algorithm E.1: Algorithm that computes for an admissible system of invariants (H, μ, w, p) with $\operatorname{rank}(H) = 2$ and $w = 0$ its normal form

Data: $a_0, a_1, a_2, a_3, b_0, b_1 \in \mathbb{Z}$ coefficients of an admissible system of invariants
Result: $r_0, r_1, r_2, r_3, s_0, s_1 \in \mathbb{Z}$ defining a normal form for $((a_0, a_1, a_2, a_3), (b_0, b_1))$

if $b_1 \neq 0$ *or* $b_2 \neq 0$ **then**
  $(\mu_0, \mu_1) \leftarrow$ integers, so that $\mu_0 b_0 + \mu_1 b_1 = \gcd(b_0, b_1)$; `/* Can be computed with Euclidean algorithm */`
  $(\lambda_0, \lambda_1) \leftarrow \frac{1}{\gcd(b_0,b_1)}(b_1, -b_0)$; `/* Coefficients for a generator of` $\ker(p)$, $(u', v') = (\mu_0 u + \mu_1 v, \lambda_0 u + \lambda_1 v)$ `is a basis */`
  $(r_0, r_1, r_2, r_3, s_0, s_1) \leftarrow$ coefficients in the basis (u', v');
  if $r_3 = 0$ **then**
    if $r_2 = 0$ **then**
      if $r_1 = 0$ **then**
        return $(r_0, r_1, r_2, r_3, s_0, s_1)$; `/* Case` P_1 `*/`
      else
        $r_0 \leftarrow r_0 \mod 3r_1$; `/* Change of basis of the form` $(u', v') \mapsto (u' + mv', v')$ `*/`
        return $(r_0, r_1, r_2, r_3, s_0, s_1)$; `/* Case` Q_1 `*/`
    else
      $m \leftarrow$integer, so that $r_1 + 2mr_2 \in (-|r_2|, |r_2|]$;
      $r_0 \leftarrow r_1 + 3mr_1 + 3m^2 r_2$;
      $r_1 \leftarrow |r_1 + 2mr_2|$; `/* Change of basis of the form` $(u', v') \mapsto (u' + mv', \pm v')$ `*/`
      return $(r_0, r_1, r_2, r_3, s_0, s_1)$; `/* Cases` R_1 `and` R_1' `*/`
  else
    if $r_3 < 0$ **then**
      $(r_1, r_3) \leftarrow -(r_1, r_3)$; `/* Change of basis of the form` $(u', v') \mapsto (u', -v')$ `*/`
    $m \leftarrow$integer, so that $r_2 + mr_3 \in [0, r_3)$;
    $r_0 \leftarrow r_0 + 3mr_1 + 3m^2 r_2 + m^3 r_3$;
    $r_1 \leftarrow r_1 + 2mr_2 + m^2 r_3$;
    $r_2 \leftarrow r_2 + mr_3$; `/* Change of basis of the form` $(u', v') \mapsto (u' + mv', v')$ `*/`
    return $(r_0, r_1, r_2, r_3, s_0, s_1)$; `/* Case` S_1 `*/`
else
  $D \leftarrow a_3^2 a_2^2 - 4a_4 a_2^3 - 4a_3^3 a_1 - 27a_4^2 a_1^2 + 18a_4 a_3 a_2 a_1$;
  if $D = 0$ **then**
    $(\mu_0, \mu_1) \leftarrow$coprime integers, so that $\frac{\mu_0}{\mu_1}$ is a root of multiplicity at least 2 of $a_0 y^3 + 3a_1 y^2 + 3a_2 y + a_3$; `/* Exists by assumption on` D, `cf. [100, Proposition 3] */`
    $(\lambda_0, \lambda_1) \leftarrow$integers, so that $\lambda_1 \mu_0 - \lambda_0 \mu_1 = 1$; `/* Can be computed with Euclidean algorithm */`
    $(r_0, r_1, r_2, r_3, s_0, s_1) \leftarrow$ coefficients in the basis (u', v');
    if $r_1 = 0$ **then**
      return $(|r_0|, r_1, r_2, r_3, s_0, s_1)$; `/* Case` K_1 `*/`
    else
      $r_0 \leftarrow r_0 \mod 3r_1$; `/* Change of basis of the form` $(u', v') \mapsto (u' + mv', v')$ `*/`
      return $(r_0, r_1, r_2, r_3, s_0, s_1)$; `/* Case` L_1 `*/`

Algorithm E.2: An algorithm that computes for M with $b_2(M) = 2$ a simply-connected algebraic plumbing graph G with $M \cong M_{\overline{G}}$ if it exists

Data: $a_0, a_1, a_2, a_3, b_0, b_1 \in \mathbb{Z}$ coefficients of the invariants μ_M and $p_1(M)$ in a fixed basis

Result: $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}$ defining a simply-connected algebraic plumbing graph G with $M \cong M_{\overline{G}}$ if it exists

Compute D, R, I, J;
$\sigma_2^2 \leftarrow 4I - 3D$; `/* Candidate for` σ_2^2 `*/`
$\sigma_1\sigma_3 \leftarrow I - D$; `/* Candidate for` $\sigma_1\sigma_3$ `*/`
if $\sigma_2^2 > 0$ **then**
    $\sigma_{2,1} \leftarrow \sqrt{\sigma_2^2}$;
    $\sigma_{2,2} \leftarrow -\sqrt{\sigma_2^2}$; `/* Two candidates for` σ_2 `*/`
if $D \neq 0$ *or* $\sigma_2^2 > 0$ **then**
    for $i \leftarrow 1$ **to** 2 **do**
        if $D \neq 0$ **then**
            $\sigma_{1,i}^2 \leftarrow \frac{\sigma_{2,i}(2\sigma_2^2 - 9\sigma_1\sigma_3) - J}{D}$;
        else
            $\sigma_{1,i}^2 \leftarrow \frac{27(\sigma_1\sigma_3)^2 + R^2}{2\sigma_1\sigma_3 \cdot \sigma_{2,i}}$; `/* Two candidates for` σ_1^2 `*/`
        if $\sigma_{1,i}^2 \geq 0$ **then**
            $\sigma_{1,i} \leftarrow \sqrt{\sigma_{1,i}^2}$;
            $\sigma_{3,i} \leftarrow \frac{\sigma_1\sigma_3}{\sigma_{1,i}}$;
            $(\alpha_1, \alpha_2, \alpha_3) \leftarrow$ zeroes of $y^3 - \sigma_{1,i}y^2 + \sigma_{2,i}y - \sigma_{3,i} = 0$; `/* Roots are possibly complex */`
            if $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}$ *and not* $\alpha_1 = \alpha_2 = \alpha_3$ **then**
                if *the normal forms of* $(\alpha_1 - \alpha_3, -3\alpha_3, -3\alpha_3, \alpha_2 - \alpha_3, 4(\alpha_1 - \alpha_3), 4(\alpha_2 - \alpha_3))$ *and* $(a_0, a_1, a_2, a_3, b_0, b_1)$ *via Algorithm E.1 are equal* **then**
                    return $(\alpha_1, \alpha_2, \alpha_3)$; `/* Success */`
            if $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}$ *and* $\alpha_1 = \alpha_2 = \alpha_3$ **then**
                $g \leftarrow \gcd(a_0, a_1, a_2, a_3)$;
                Compute Hessian $C = C(\frac{1}{g} f_{\mu_M})$;
                if C *is divisible by 2 and* $D(\frac{1}{2}C) = -3$ **then**
                    Bring $\frac{1}{2}C$ into normal form via [30, Algorithm 5.4.2], which yields 6 possible bases (u_i, v_i), $1 \leq i \leq 6$, of $\mathbb{Z}^2$;
                    Compute coefficients $((a_{0,i}, a_{1,i}, a_{2,i}, a_{3,i}), (0,0))$ for f_{μ_M} w.r.t each (u_i, v_i);
                    if $((a_{0,i}, a_{1,i}, a_{2,i}, a_{3,i}), (0,0)) = ((0, -\alpha_3, -\alpha_3, 0), (0,0))$ *for one* i **then**
                        return $(\alpha_1, \alpha_2, \alpha_3)$; `/* Success */`
else if $\sigma_2^2 = D = 0$ **then**
    if *D=R=I=J=0 and the normal forms of* $(\frac{\gcd(b_0,b_1)}{4}, 0, 0, 0, \gcd(b_0, b_1), 0)$ *and* $(a_0, a_1, a_2, a_3, b_0, b_1)$ *via Algorithm E.1 are equal* **then**
        return $(\frac{\gcd(b_0,b_1)}{4}, 0, 0)$; `/* Success */`

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Index

Symbols

$B(G)$	biadjacency matrix of G. 115
$B(a_0, \dots, a_n)$	Brieskorn manifold. 10
$\mathrm{B}G$	classifying space of G. 101
$B^{4k}_{\overline{\gamma}}$	$\#_{i=1}^{k} \gamma_i \mathbb{C}P^{2k}$. 111
$b_i(M)$	i-th Betti number. 8
c_i	i-th Chern class. 102
D	discriminant. 118
D^n	closed n-dimensional disc. 7
$D^q_R(N)$	geodesic ball of radius R in $S^q(N)$. 48
ds^2_n	round metric on S^n. 9
$\overline{E}_1 \square \overline{E}_2$	plumbing of $\overline{E}_1$ and $\overline{E}_2$. 27
$\mathrm{E}G$	total space of universal principal G-bundle. 101
$e(\pi)$	$e_{\mathbb{Z}}(\pi)$. 105
e_R	universal Euler class. 106
$e_R(\pi)$	Euler class of π. 105
E_x	fiber over x. 99
$f^*\pi$	pull-back of π along f. 100
$G /\!\!/ H$	biquotient. 10
$\overline{G}^k$	geometric plumbing graph corresponding to G. 68
$g_\pi(r, \theta)$	submersion metric with totally geodesic and round fibers of radius r and principal connection θ. 104
H	mean curvature. 94
$\mathcal{H}$	horizontal distribution. 103
I	invariant for binary quartic forms. 119
$\mathrm{I\!I}$	second fundamental form. 94
$I^{\pm}_{p,q}$	diffeomorphism between $D^p \times D^q$ and $D^q \times D^p$ used for plumbing. 27
J	invariant for binary quartic forms. 119
$M_1 \# M_2$	connected sum. 23
$M_1 \natural M_2$	boundary connected sum. 24
$\overline{M}_G$	manifold obtained by plumbing along G. 28