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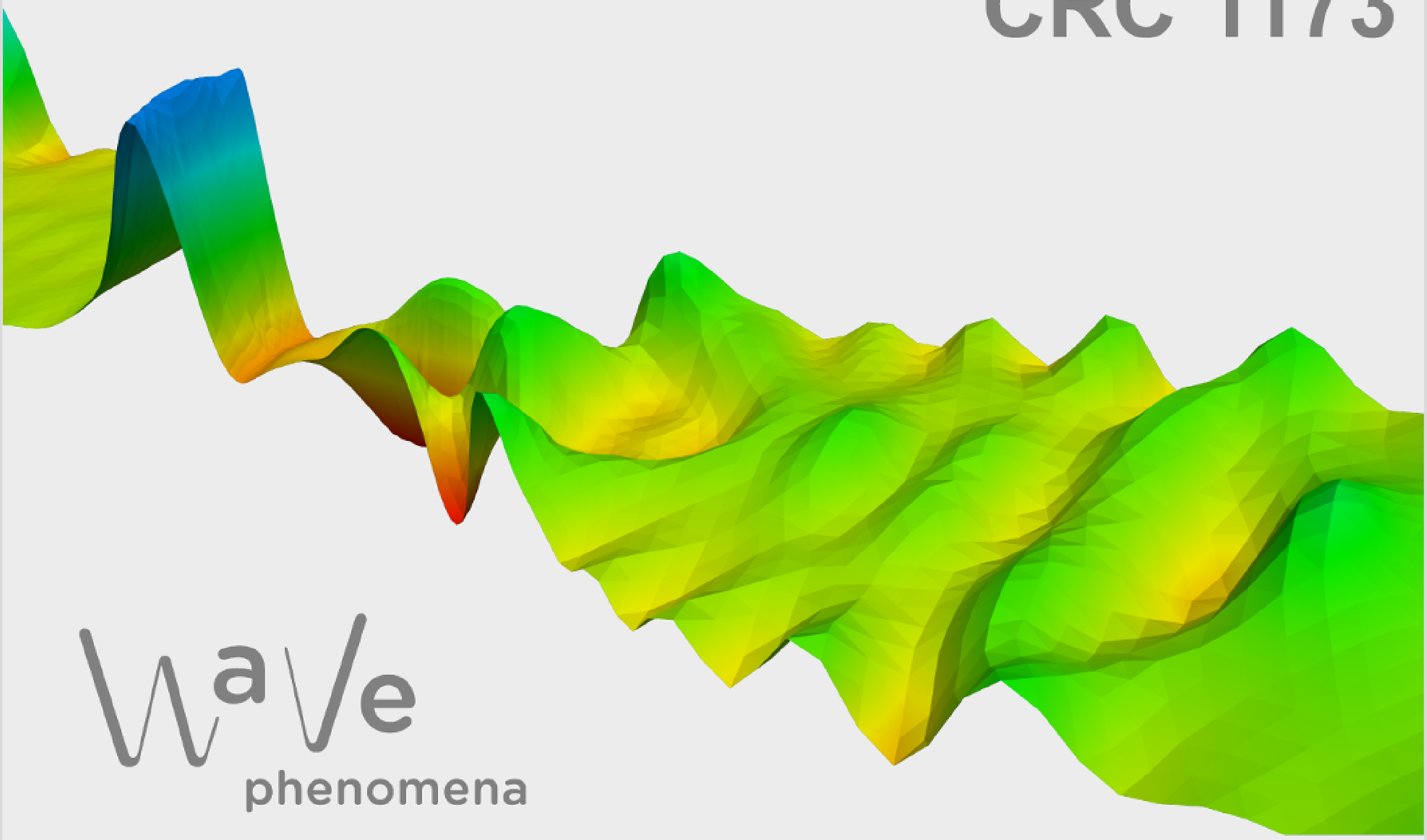

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ABSENCE OF POSITIVE EIGENVALUES OF MAGNETIC SCHRÖDINGER OPERATORS

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Abstract. We study sufficient conditions for the absence of positive eigenvalues of magnetic Schrödinger operators in $\mathbb{R}^d$, $d \geq 2$. In our main result we prove the absence of eigenvalues above certain threshold energy which depends explicitly on the magnetic and electric field. A comparison with the examples of Miller–Simon shows that our result is sharp as far as the decay of the magnetic field is concerned. As applications, we describe several consequences of the main result for two-dimensional Pauli and Dirac operators, and two and three dimensional Aharonov–Bohm operators.

Contents

1. Introduction and description of main results

The question of the absence of positive eigenvalues of Schrödinger operators has a long history. In 1959 Kato proved that the operator $-\Delta + V$ in $L^2(\mathbb{R}^d)$ has no positive eigenvalues if V is continuous and such that

$$V(x) = o(|x|^{-1}) \qquad |x| \to \infty, \tag{1.1}$$

by deriving suitable lower bounds on solutions of the Schrödinger equation. His lower bound showed that for positive energies these solutions decay so slowly at infinity that they are not normalizable, see [20]. It is known that condition (1.1) is essentially optimal since there exist oscillatory potentials of the Wigner-von Neumann type, decaying as $|x|^{-1}$, which produce positive eigenvalues of the associated Schrödinger operator, see [31, 37] or [30, Ex. VIII.13.1].

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Kato's result was generalized by Simon [31], who considered, for $d = 3$, potentials of the class $L^2 + L^\infty$ which are smooth outside a compact set and allow there a decomposition $V = V_1 + V_2$ with $V_1 = o(|x|^{-1})$, $V_2(x) = o(1)$, and

$$\omega_0 = \limsup_{|x|\to\infty} x \cdot \nabla V_2(x) < \infty. \tag{1.2}$$

Under these conditions Simon proved the absence of eigenvalues of $-\Delta + V$ in the interval (ω_0, ∞). Note that $\omega_0 \geq 0$ since $V_2(x) \to 0$ as $|x| \to \infty$. Later it was shown by Agmon [1] that under similar assumptions the operator $-\Delta + V$, in any dimension, has no eigenvalues in the interval $(\omega_0/2, \infty)$.

By exploiting a clever exponentially weighted virial identity, Froese et al. proved the absence of all positive eigenvalues of $-\Delta + V$ under relative compactness conditions on V and $x \cdot \nabla V$, [12, 13]. Their conditions on the regularity and decay on V and $x \cdot \nabla V$ were still global but much more general than the pointwise conditions of Kato, Simon, and Agmon. Yet another approach to the problem is based on Carleman estimates in L^p-spaces. This method allows to further weaken the regularity and decay conditions and to include rough potentials, see the works of Jerison and Kenig [18], Ionescu and Jerison [18], and the article [21] by Koch and Tataru.

Much less is known about the absence of positive eigenvalues for magnetic Schrödinger operators of the form

$$H = (P - A)^2 + V, \qquad P = -i\nabla, \tag{1.3}$$

in particular in dimension two. In the above equation A stands for a magnetic vector potential satisfying $\operatorname{curl} A = B$. The results obtained by Koch and Tataru in [21] cover also Schrödinger operators with magnetic fields. But they impose decay conditions on the vector potential A which are *not gauge invariant* and which imply, in the case of dimension two, that the total flux of the magnetic field must vanish. Therefore they cannot be applied to two-dimensional Schrödinger operators with magnetic fields of non-zero flux.

Certain implicit conditions for the absence of eigenvalues of the operator (1.3) in $\mathbb{R}^2$ were recently found by Fanelli, Krejčiřík and Vega in [10], see also [11]. However, their result guarantees absence of *all* eigenvalues of H, not only of the positive ones. Consequently the hypotheses needed in [10] include some smallness conditions on V and B which are not necessary for the absence of positive eigenvalues only. In [14] Ikebe and Saito proved the limiting absorption principle for H under certai pointwise decay conditions on V and B, see Remark 1.5.

In this work we develop quadratic form methods which are an effective tool to rule out positive eigenvalues for a large class of magnetic Schrödinger operators while at the same time allowing the existence of negative eigenvalues, which one does not want to rule out a priori. In addition, intuition from physics and experience from the rigorous study of Schrödinger operators without magnetic fields clearly show that while eigenvalues depend on global properties of the potential and the magnetic field, energies in the essential spectrum depend only on asymptotic properties. Thus, the nonexistence of eigenvalues embedded in the essential spectrum should depend *only on the asymptotic behavior* of the potential and the magnetic field, as long as the local behavior of the potential and magnetic field is not so singular such that it destroys the self–adjointness of the magnetic Schrödinger operator. Our results make this intuition rigorous: the *local behavior* of the magnetic field and the potential is *largely irrelevant* for the non-existence of positive eigenvalues. Our results also cover cases where the magnetic field decays so slowly that no choice of vector potential satisfies the conditions in [21]. Moreover, the famous Miller–Simon examples show that our results are sharp concerning the decay of the magnetic field at infinity.

In dimension two we identify the magnetic field with a scalar function which, in turn, can be interpreted as a vector field in $\mathbb{R}^3$ perpendicular to the plane $\mathbb{R}^2$. In general dimensions the magnetic field is an antisymmetric two–form, which can be identified with an antisymmetric matrix–valued function on $\mathbb{R}^d$ satisfying, in the sense of distributions, the condition

$$\partial_j B_{k,i} + \partial_k B_{i,j} + \partial_i B_{j,k} \quad \forall d \in \{1, \dots, d\}\,. \tag{1.4}$$

Here $B_{j,k}(x)$ denotes the entries of B at a point $x \in \mathbb{R}^d$. If $d = 3$, then B can be identified with a vector field with components $B_1 = B_{3,2}$, $B_2 = B_{1,3}$, and $B_3 = B_{2,1}$. Equation (1.4) thus coincides with the usual divergence free condition

$$\nabla \cdot B = 0 \quad [d = 3] \tag{1.5}$$

dictated by Maxwell's equations. It is well known that there exists a vector potential A, a one–form, such that $B = \mathrm{curl} A$ or $B = dA$, with the exterior derivative.

1.1. **The method.** Let us briefly describe our method and its most important novel features. As already mentioned above we build upon the technique invented by R. Froese and I. Herbst and M. and Th. Hoffmann–Ostenhof [13] and further developed in [12]. The latter is based on weighted virial identities which require working with dilations and their generator. For non-magnetic Schrödinger operators this is facilitated by the fact that the momentum operator P has very simple commutation relations with dilations. In particular, the domain of P is invariant under dilations. This is not true anymore for the magnetic operators, since the vector potential spoils the dilation invariance of the domain of $P - A$. One of the crucial new features of our approach shows that to overcome this difficulty one has to work with a vector potential A in the Poincaré gauge and exploit its connection with the dilations and the virial theorem. This connection, which enables us to develop a quadratic form version of the magnetic virial theorem, is explained in Section 3. We also show that the rather different conditions of Kato and Agmon–Simon are, in fact, just two sides of the same coin. Kato's condition for the absence of positive eigenvalues can be easily recovered from the quadratic form version of the virial of the potential, see Section 3.3 for details. Moreover, the use of the Poincaré gauge leads to very natural decay conditions on B required for the absence of positive eigenvalues. The well-known example by Miller and Simon then shows that these conditions are sharp, see Section 6.

1.2. **A typical result.** In order to describe a typical result with general and easy to verify conditions on the magnetic field B and the potential V, we need some more notation. We denote by $L^p = L^p(\mathbb{R}^d)$, $1 \le p \le \infty$ the usual scale of Lebesgue spaces. Moreover, we need their locally uniform versions

$$L^p_{\mathrm{loc,unif}} = \Big\{ V : \sup_{x \in \mathbb{R}^d} \int_{|x-y| \le 1} |V(y)|^p \, dy < \infty \Big\} \tag{1.6}$$

with norms

$$\|V\|_{L^p_{\mathrm{loc,unif}}} := \sup_{x \in \mathbb{R}^d} \Big(\int_{|x-y| \le 1} |V(y)|^p \, dy \Big)^{1/p} \tag{1.7}$$

when $1 \le p < \infty$ and the obvious modification for $p = \infty$. Clearly these spaces are nested, that is, $L^q_{\mathrm{loc,unif}} \subset L^p_{\mathrm{loc,unif}}$ when $1 \le p \le q \le \infty$. Moreover, we need

Definition 1.1 (Vanishing at infinity locally uniformly (in L^p))**.** A function $V \in L^p_{\mathrm{loc,unif}}$ with

$$\lim_{R \to \infty} \|\mathbb{1}_{\ge R} V\|_{L^p_{\mathrm{loc,unif}}} = 0 \tag{1.8}$$

vanishes at infinity locally uniformly in $L^p_{\mathrm{loc,unif}}$.

Here $\mathbb{1}_{\ge R}$ is the characteristic function of the set $\{x \in \mathbb{R}^d : |x| \ge R\}$. In fact, we will only need the $p = 1, 2$ versions of vanishing locally uniformly in L^p at infinity.

Given a magnetic field B and a point $w \in \mathbb{R}^d$ let $\widetilde{B}_w(x) \coloneqq B(x+w)[x]$. More precisely, $\widetilde{B}_w$ is a vector–field on $\mathbb{R}^d$ with components

$$(\widetilde{B}_w)_j(x) \coloneqq (B(x+w)[x])_j = \sum_{m=1}^{d} B_{j,m}(x+w)\, x_m\,, \quad j = 1, \dots, d\,. \tag{1.9}$$

Using translations, we will usually assume $w = 0$, in which case we will simply write $\widetilde{B}$. In dimension two, identifying the magnetic field with a scalar, the vector field $\widetilde{B}_w$ is given by $\widetilde{B}_w(x) = B(x+w)(-x_2, x_1)$ and in three dimensions it is given by the cross product $\widetilde{B}_w(x+w) = B(x+w) \wedge x$.

The simplest version of our results is given by

Theorem 1.2 (Simple version)**.** *Given a magnetic field B assume that $\widetilde{B}_w \in L^p_{loc,unif}$ for some $p > d$ and some $z \in \mathbb{R}^d$. Then there exists a vector potential $A \in L^2_{loc}(\mathbb{R}^d, \mathbb{R}^d)$ with $B = dA$. Moreover, let V be a potential with $V \in L^q_{loc,unif}$ for some $q > d/2$ which allows a splitting $V = V_1 + V_2$ such that $xV_1 \in L^{q_1}_{loc,unif}$ for some $q_1 > d$ and $x \cdot \nabla V_2 \in L^{q_2}_{loc,unif}$ for some $q_2 > d/2$ and assume that $\widetilde{B}$ and xV_1 vanish at infinity locally uniformly in L^2 and V, V_1, and $x \cdot \nabla V_2$ vanish at infinity locally uniformly in L^1.*

Then the magnetic Schrödinger operator $(P-A)^2 + V$, defined via quadratic form methods, has essential spectrum equal to $[0, \infty)$ and no positive eigenvalues.

Remarks 1.3. (i) The decay condition on xV_1, respectively $x \cdot V_2$, are generalizations, in terms of local L^p conditions, of the pointwise conditions of Kato [20], respectively Agmon [1] and Simon [31]. For a generalization using only natural quadratic form conditions, see Theorems 1.6 and 4.8 below.

(ii) Even in this simplest version the conditions on B and V allow for strong local singularities and the decay condition at infinity is rather mild: for example, if one splits V in such a way that V_1 is compactly supported. Then xV_1 is zero outside a compact set, so clearly vanishing at infinity. The condition $xV_1 \in L^{q_1}_{\text{loc,unif}}$ for some $q_1 > d$ allows for rather large local singularities. In particular, the virial $x \cdot \nabla V$ has only to exist in a neighborhood of infinity in order to be able to apply Theorem 1.2. One can also include a long range part of V in V_1. Moreover, since $|\widetilde{B}_w(x)| \lesssim |B(x+w)||x|$, the magnetic field can have strong local singularities, in particular at w. The decay of the magnetic field B has to be faster than $\langle x - w \rangle^{-1}$, which is in line of what one expects from the Miller–Simon examples, see Section 6.1.

Let us now briefly describe our main results in full generality.

1.3. Full quadratic form version: absence of all positive eigenvalues.

It turns out that the absence of positive eigenvalues depends, in a sense, *only* on the behavior of $\widetilde{B}$, xV and $x \cdot \nabla V$ *at infinity* with respect to the operator $(P-A)^2$. The latter are to be understood in a weak sense according to the following

Definition 1.4 (Vanishing at infinity)**.** We say that a potential W vanishes at infinity with respect to $(P-A)^2$ if for some $R_0 > 0$ its quadratic form domain $\mathcal{Q}(W)$ contains all $\varphi \in \mathcal{D}(P-A)$ with $\text{supp}(\varphi) \in \mathcal{U}^c_{R_0}$ and for $R \geq R_0$ there exist positive α_R, γ_R with $\alpha_R, \gamma_R \to 0$ as $R \to \infty$ such that

$$|\langle \varphi, W\varphi \rangle| \leq \alpha_R \|(P-A)\varphi\|_2^2 + \gamma_R \|\varphi\|_2 \quad \text{for all } \varphi \in \mathcal{D}(P-A) \text{ with } \text{supp}(\varphi) \subset \mathcal{U}^c_R \tag{1.10}$$

Here $\mathcal{U}_R = \{x \in \mathbb{R}^d : |x| < R\}$ and $\mathcal{U}^c_R = \mathbb{R}^d \setminus \mathcal{U}_R$ is its complement.

By monotonicity we may assume, without loss of generality, that α_R and γ_R are decreasing in $R \geq R_0$.

This definition is inspired by Section 3 in [19]. It allow us to effectively treat magnetic fields and potentials which can have severe singularities even close to infinity.

In order to guarantee that there is a locally square integrable vector potential A with $dA = B$, we need

Lemma 1.5. *Given a magnetic field B and $w \in \mathbb{R}^d$ let $\widetilde{B}_w$ be given by (1.9) and assume that*

$$\int\limits_{|x-w|<R} |x-w|^{2-d}\Big(\log\frac{R}{|x-w|}\Big)^2\, |\widetilde{B}_w(x)|^2\, dx < \infty$$

for all $R > 0$. Then there exists a vector potential $A \in L^2_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d)$ with $B = dA$ in the sense of distributions.

We then have

Theorem 1.6. *Given a magnetic field B assume that it fulfills the condition of Lemma 1.5 for some $w \in \mathbb{R}^d$ and that $\widetilde{B}_w^2$ given by (1.9) is relatively form bounded and vanishes at infinity with respect to $(P-A)^2$. Moreover, assume that the potential V is form small and vanishes at infinity with respect to $(P-A)^2$ and allows for a splitting $V = V_1 + V_2$, such that $|xV_1|^2$ and $x\cdot\nabla V_2$ are also form small and vanish at infinity with respect to $(P-A)^2$.*

Then the magnetic Schrödinger operator $(P-A)^2+V$, defined via quadratic form methods, has essential spectrum $[0,\infty)$ and no positive eigenvalues.

Remark 1.7. Some comments concerning Theorem 1.6: (i) We only need relative form boundedness of $\widetilde{B}_w^2$ with respect to $(P-A)^2$. Its relative form bound does not have to be less than one.

(ii) While the conditions on the potential V and the magnetic field B with respect to $(P-A)^2$ might be difficult to check, the diamagnetic inequality shows that it is enough to check them with respect to the non-magnetic kinetic energy P^2, see [3].

(iii) One can again absorb strong local singularities of the potential in a suitable choice of V_1. Thus the local behavior of the potential V and the magnetic field B is *largely irrelevant* for the non-existence of positive eigenvalues. Moreover, the virial $x\cdot\nabla V_2$ has to exist only in a weak quadratic form sense, see Lemma 3.7 and the discussion in Section 3.3.

(iv) An inspection of the proof shows that in Theorem 1.6 it is enough to assume that $x\cdot\nabla V$ is bounded from above at infinity by zero, see Definition 1.8 below for the precise meaning. Classically the force is given by $F = -\nabla V$. Thus $x\cdot F = -x\cdot\nabla V$ is negative, i.e., the force is *confining*, if $x\cdot\nabla V$ is positive, otherwise the force is *repulsive*, i.e., it pushes the particle further to infinity. Thus in order to prevent localization of a quantum particle only the positive part of $x\cdot\nabla V$ should have to be small at infinity.

(v) We would like to stress that unlike all other results on the absence of positive eigenvalues for magnetic Schrödinger operators that we are aware of, with the exception of [10], we impose only conditions on the magnetic field B and not directly on the vector potential A. Decay and regularity conditions on the vector potential A are not invariant under gauge transformations and thus unphysical. The conditions of [10], on the other hand, are quite restrictive. For example, in [10] the authors need that various *global* quantities related to the magnetic field B and to the potential V are absolute form bounded with respect to $(P-A)^2$, i.e. without allowing for lower order terms in the respective bounds and and they need an explicit smallness condition for the various constants involved in their bounds. This makes their conditions difficult to check and the resulting assumptions turn out to be so strong that they rule out existence of any eigenvalue.

However, for a large class of physically relevant potentials and magnetic fields one expects that the corresponding magnetic Schrödinger operator has negative eigenvalues, while it typically should not have positive eigenvalues, at least when the magnetic field and the potential vanish in a suitable sense at infinity. This is exactly what our Theorem 1.6 and its generalizations below provide.

(vi) In order to prove invariance of the essential spectrum, one usually assume that the potential V is relatively $(P-A)^2$ form compact. We do not assume this! In fact, we show in Theorem 4.8 that if the potential V is form small, i.e., form bounded with relative bound < 1, and vanishes at infinity with respect to $(P-A)^2$, then $\sigma_{\mathrm{ess}}((P-A)^2+V)=\sigma_{\mathrm{ess}}((P-A)^2)$. This shows invariance of the essential spectrum under a large class of perturbations. In particular, it confirms the physical intuition that local singularities, as long as they do not destroy form smallness, cannot influence the essential spectrum, at least as a set. For example, one can have a potential with local Hardy type singularity and even a sequence of suitably decreasing Hardy type singularities moving to infinity. Moreover, using ideas of Combesure and Ginibre [4] and Maz'ya and Verbitzky [26], we can allow perturbations with rather strong oscillations, both locally and at infinity.

(vii) Theorem 1.2 above is the most general formulation of our results, when one considers magnetic fields and potentials vanishing at infinity, in a suitable sense. We can allow for much ore general condition on the potential V and the magnetic field B, see the following section and Section 2.3 below for more general assumptions.

1.4. **Full quadratic form version: absence of eigenvalues above a positive threshold.** If $\widetilde{B}^2, |xV_1|^2$ and $x\cdot\nabla V_2$ do not vanish at infinity with respect to $(P-A)^2$, we can still exclude positive eigenvalues above a certain threshold. For this we need

Definition 1.8 (Bounded at infinity)**.** A potential W is bounded from above at infinity with respect to $(P-A)^2$ if for some $R_0>0$ its quadratic form domain $\mathcal{Q}(W)$ contains all $\varphi\in\mathcal{D}(P-A)$ with $\mathrm{supp}(\varphi)\in\mathcal{U}_{R_0}^c$ and for $R\ge R_0$ there exist positive α_R,γ_R with $\lim_{R\to\infty}\alpha_R=0$ and $\liminf_{\mathbb{R}\to\infty}\gamma_R<\infty$ such that

$$\langle\varphi,W\varphi\rangle\le\alpha_R\|(P-A)\varphi\|_2^2+\gamma_R\|\varphi\|_2\quad\text{for all }\varphi\in\mathcal{D}(P-A)\text{ with }\mathrm{supp}(\varphi)\subset\mathcal{U}_R^c \tag{1.11}$$

By monotonicity we may assume, without loss of generality, that α_R and γ_R are decreasing in $R\ge R_0$ in which case we set $\gamma_\infty^+(W):=\lim_{R\to\infty}\gamma_R=\inf_R\gamma_R$, the asymptotic bound upper bound of W (at infinity).

A potential W is bounded from below at infinity with respect to $(P-A)^2$ if $-W$ is bounded from above at infinity. We set $\gamma_\infty^-(W)=\gamma_\infty^+(-W)$.

A potential W is bounded at infinity with respect to $(P-A)^2$ if $\pm W$ are bounded from above at infinity. We set $\gamma_\infty(W):=\max(\gamma_\infty^+(W),\gamma_\infty^-(W))$, the asymptotic bound of W (at infinity).

Using the diamagnetic inequality, one can replace $(P-A)^2$ by P^2 in the definition of the asymptotic bounds $\gamma_\infty^+(W)$ and $\gamma_\infty(W)$.

We split $V=V_1+V_2$ and set

$$\beta^2:=\gamma_\infty\big(\widetilde{B}^2\big),\quad\omega_1^2:=\gamma_\infty\big((xV_1)^2\big),\quad\omega_2:=\gamma_\infty^+\big(x\cdot\nabla V_2\big) \tag{1.12}$$

Under mild regularity conditions the magnetic Schrödinger operator $(P-A)^2+V$ has $[0,\infty)$ as its essential spectrum and our main result, Theorem 4.8, implies that it has no eigenvalues larger than

$$\Lambda(B,V)=\Lambda:=\frac{1}{4}\left(\beta+\omega_1+\sqrt{(\beta+\omega_1)^2+2\omega_2}\right)^2 \tag{1.13}$$

While the β,ω_1, and ω_2 might be difficult to compute directly from the definition it is easy to see

$$\beta\le\limsup_{|x|\to\infty}|\widetilde{B}(x)|,\qquad\omega_1\le\limsup_{|x|\to\infty}|x|\,|V_1(x)|,\qquad\omega_2\le\limsup_{|x|\to\infty}(x\cdot\nabla V_2(x))_+\,. \tag{1.14}$$

once the limits are well-defined and finite. We would like to point out that Theorem 4.8 can be applied also in situations in which the limits in (1.14) might not be defined. Morally, $\gamma_\infty(W)$ is the bounded part of W at infinity, modulo terms which are small uniformly locally in L^1: If W is in the Kato–class outside a compact set, which, in particular, is the case if $W\in L^p_{\mathrm{loc,unif}}$ outside of a compact set, and if $W-W_b$ vanishes at infinity locally uniformly in L^1 for some bounded function W_b, then

$$\gamma_\infty(W)\le\|W_b\|_\infty\,, \tag{1.15}$$

and a similar bound holds for $\gamma_\infty^+(W)$, see Section 5. This allows to effectively compute upper bounds on β, ω_1, and ω_2 when the limits in (1.14) exist not.

1.5. **Relation to previous works.** If $B = 0$, then by choosing $V_1 = V$ and $V_2 = 0$ we obtain a generalization of the result of Kato [20]. On the other hand, by choosing V_1 such that $V_1(x) = o(|x|^{-1})$, and setting $V_2 = V - V_1$ we get $\Lambda = \omega_0/2$, see equation (1.2), and recover thus the results of Agmon [1] and Simon [31]. Moreover, Theorem 4.8 extends all the above mentioned results to magnetic Schrödinger operators with magnetic fields which decay fast enough so that $\beta = 0$, see Section 5 for more details.

Vice-versa, if $V = 0$, then we have $\Lambda = \beta$ which is in agreement with the well–known example by Miller and Simon [27], cf. Section 6 if one corrects a calculation error in their examples. The Miller–Simon examples show that our condition on the magnetic field for absence of eigenvalues above a threshold is sharp.

It is tempting to split $V = sV + (1-s)V$ and to optimize the resulting expression for the threshold energy (1.13) with respect to $0 \le s \le 1$. This minimization problem can be explicitly done. It turns out that the minimum is always given by the minimum of the two extreme cases $s = 0$ and $s = 1$, see Corollary B.2 in Appendix B.

Ikebe and Saito proved in [14] the limiting absorption principle, and hence also the absence of eigenvalues of H under the condition that V allows the same decomposition as above with $|V_1(x)| \le C|x|^{-1-\delta}$, $|V_2(x)| \le C|x|^{-\delta}$, $|x| \cdot \nabla V_2(x)| \le C|x|^{-\delta}$, and that B is continuous and satisfies $|B(x)| \le C|x|^{-1-\delta}$. Here C and δ are positive constants. These conditions are easily covered by Theorem 4.8. Indeed, if V an B satisfy these upper bounds, then $\beta = \omega_1 = \omega_2 = 0$. Hence there are no positive eigenvalues.

1.6. **Essential spectrum.** In Section 7 we establish new sufficient conditions on B under which

$$\sigma_{\mathrm{ess}}((P-A)^2) = [0, \infty).$$

Roughly speaking we require that $B(x) \to 0$, in a mild way, as $|x| \to \infty$ along only one path connecting to infinity, see Theorem 7.5 and Definition 7.3 for details. As a consequence of this result we show that under the assumptions stated in section 2.3 we have $\sigma_{\mathrm{ess}}((P-A)^2) = [0, \infty)$, cf. Corollary 7.7. We also show that if the potential V is form small and vanishes at infinity w.r.t $(P-A)^2$, then $\sigma_{\mathrm{ess}}((P-A)^2+V) = \sigma_{\mathrm{ess}}((P-A)^2)$, see Theorem 7.8. For this one usually assumes that V is *relative form compact* w.r.t. $(P-A)^2$ which is a considerably stronger assumption, excluding, for example, Hardy–type singularities. Our result proves invariance of the essential spectrum under a conditions which includes all physically relevant examples, even exotic one with strong singularities or oscillations.

2. Magnetic Schrödinger operators and the Poincaré gauge

First let us fix some notation. Given a set M and two functions $f_1,\, f_2 : M \to \mathbb{R}$, we write $f_1(m) \lesssim f_2(m)$ if there exists a numerical constant c such that $f_1(m) \le c\, f_2(m)$ for all $m \in M$. The symbol $f_1(m) \gtrsim f_2(m)$ is defined analogously. Moreover, we use the notation

$$f_1(m) \sim f_2(m) \quad \Leftrightarrow \quad f_1(m) \lesssim f_2(m) \;\wedge\; f_2(m) \lesssim f_1(m),$$

and

$$\lim_{|x|\to\infty} f(x) = L \quad \Leftrightarrow \quad \lim_{r\to\infty} \operatorname*{ess\,sup}_{|x|=r} f(x) = L. \tag{2.1}$$

The quantities $\limsup_{|x|\to\infty} f(x)$ and $\liminf_{|x|\to\infty} f(x)$ are defined in a similar way. We will use $\partial_j = \frac{\partial}{\partial x_j}$ for the usual partial derivatives in the weak sense, i.e., as distributions.

For any $u \in L^r(\mathbb{R}^d)$ with $1 \le r \le \infty$ we will use the shorthand

$$\|u\|_r := \|u\|_{L^r(\mathbb{R}^d)}$$

for the L^r-norm of u and

$$\|T\|_{r\to r} := \|T\|_{L^r(\mathbb{R}^d)\to L^r(\mathbb{R}^d)}$$

for a norm of a bounded linear operator $T : L^r(\mathbb{R}^d) \to L^r(\mathbb{R}^d)$. The space $L_{\rm loc}(\mathbb{R}^d)$ is the space of all complex valued functions f such that $f\mathbb{1}_K \in L^r(\mathbb{R}^d)$ for al compact sets $K \subset \mathbb{R}^d$. Here $\mathbb{1}_K$ stands for the indicator function of K. By $L^r_{\rm loc}(\mathbb{R}^d, \mathbb{R}^d)$ we denote the space of all vector fields v which are locally in L^r, that is, $|v| := (\sum_{j=1}^d v_j^2)^{1/2}\mathbb{1}_K$ is $L^r_{\rm loc}(\mathbb{R}^d)$.

The space $\mathcal{C}_0^\infty = \mathcal{C}_0^\infty(\mathbb{R}^d)$ is the space of all complex valued test–functions f which are infinitely often differentiable and have compact support. Given measurable complex valued functions $f, g \in L^2(\mathbb{R}^d)$ we denote by

$$\langle f, g\rangle = \int_{\mathbb{R}^d} \overline{f(x)} \cdot g(x)\, dx$$

the usual scalar product on $L^2(\mathbb{R}^d)$. By the symbol

$$\mathcal{U}_R(x) = \{y \in \mathbb{R}^d \,:\, |x - y| < R\}$$

we denote the ball of radius R centered at a point $x \in \mathbb{R}^d$. If $x = 0$, we abbreviate

$$\mathcal{U}_R = \mathcal{U}_R(0).$$

2.1. **The magnetic Schrödinger operator.** Given a magnetic vector potential $A \in L^2_{\rm loc}(\mathbb{R}^d, \mathbb{R}^d)$, the magnetic Sobolev space is defined by

$$\mathcal{H}^1(\mathbb{R}^d) := \mathcal{D}(P - A) = \Big\{u \in L^2(\mathbb{R}^d) \,:\, (P - A)\, u \in L^2(\mathbb{R}^d)\Big\}, \tag{2.2}$$

equipped with the graph norm

$$\|u\|_{\mathcal{H}^1} = \Big(\|(P - A)u\|_2^2 + \|u\|_2^2\Big)^{1/2}. \tag{2.3}$$

Here $P = -i\nabla$ is the momentum operator. Note that for $u \in \mathcal{D}(P - A)$ one has $Au \in L^1_{\rm loc}(\mathbb{R}^d, \mathbb{R}^d)$. So we only know that $Pu \in L^1_{\rm loc}(\mathbb{R}^d)$ for a typical $u \in \mathcal{D}(P - A)$, which is one of the sources for technical difficulties of Schrödinger operators with magnetic fields. Nevertheless, Kato's inequality shows $|\varphi| \in \mathcal{D}(P)$ for any $\varphi \in \mathcal{D}(P - A)$ and the diamagnetic inequality [15, 31] yields

$$|((P - A)^2 + \lambda)^{-1}\varphi| \le (P^2 + \lambda)^{-1}|\varphi| \tag{2.4}$$

for all $\lambda > 0$ and $\varphi \in L^2(\mathbb{R}^d)$. It is well-know that

$$q_{A,0}(\varphi) := \langle (P - A)\varphi, (P - A)\varphi\rangle = \|(P - A)\varphi\|_2^2 \tag{2.5}$$

is a closed quadratic form on $\mathcal{H}^1(\mathbb{R}^d)$ for any magnetic vector potential $A \in L^2_{\rm loc}(\mathbb{R}^d, \mathbb{R}^d)$, and that $\mathcal{C}_0^\infty(\mathbb{R}^d)$ is dense in $\mathcal{D}(P - A)$, see [33, Thm. 2.2]. Since every closed positive quadratic form on a Hilbert space corresponds to a unique self-adjoint positive operators, the quadratic form $q_{A,0}$ defines an operator, which we denote by $H_{A,0} = (P - A)^2$.

A potential V is a locally integrable, measurable function $V : \mathbb{R}^d \to \mathbb{R}$. Hence its quadratic form domain $\mathcal{Q}(V) = \mathcal{D}(|V|^{1/2})$ contains $\mathcal{C}_0^\infty(\mathbb{R}^d)$. The quadratic form q_V corresponding to V is given by

$$q_V(\varphi, \varphi) = \big\langle |V|^{1/2}\varphi, \operatorname{sgn}(V)|V|^{1/2}\varphi\big\rangle. \tag{2.6}$$

With a slight abuse of notation, we will often write $q_V(\varphi, \varphi) = \langle\varphi, V\varphi\rangle$.

The potential V is form bounded w.r.t $(P - A)^2$ if its form domain $\mathcal{Q}(V)$ contains $\mathcal{D}(P - A)$ and there exists $\alpha, \beta < \infty$ such that

$$|\langle\varphi, V\varphi\rangle| \le \alpha\|(P - A)\varphi\|_2^2 + \beta\|\varphi\|_2^2 \quad \text{for all} \varphi \in \mathcal{D}(P - A). \tag{2.7}$$

The infimum

$$\alpha_0 = \inf\{\alpha > 0 : \text{ there exists } \beta < \infty \text{ such that (2.7) holds for all } \varphi \in \mathcal{D}(P - A)\}$$

is called the relative form bound of V_- with respect to $(P - A)^2$, and similar for other pairs of positive operators and their quadratic forms.

We say that V is relative form small w.r.t $(P-A)^2$ if the relative bound $\alpha_0 < 1$, i.e., the bound (2.7) hols for some $0 \le \alpha < 1$ and $\beta < \infty$ and if $\alpha_0 = 0$ one says that V is infinitesimally form small w.r.t. $(P - A)^2$.

It is well–known, see [30] and [36], that V is relatively form bounded w.r.t. $(P - A)^2$ if and only if the operator

$$C_\lambda \coloneqq ((P - A)^2 + \lambda)^{-1/2} V ((P - A)^2 + \lambda)^{-1/2} \tag{2.8}$$

is bounded on $L^2(\mathbb{R}^d)$ for some (and then all) $\lambda > 0$. More precisely, C_λ is the operator given by the quadratic form

$$\langle \varphi, C_\lambda \varphi \rangle \coloneqq q_V\big(((P - A)^2 + \lambda)^{-1/2}\varphi, ((P - A)^2 + \lambda)^{-1/2}\varphi\big) \tag{2.9}$$

and one can choose

$$\alpha = \|C_\lambda\|_{2\to 2} \quad \text{ and } \beta = \lambda \|C_\lambda\|_{2\to 2}$$

for any $\lambda > 0$ and

$$\alpha_0 = \lim_{\lambda\to\infty} \|C_\lambda\|_{2\to 2}$$

If the potential V is relatively form small with respect to $(P - A)^2$, the KLMN Theorem, see e.g. [36, Theorem 6.24], [30], shows that

$$q_{A,V}(\varphi, \varphi) \coloneqq \|(P - A)\varphi\|_2^2 + q_V(\varphi, \varphi) = \langle (P - A)\varphi, (P - A)\varphi \rangle + \langle \varphi, V\varphi \rangle \tag{2.10}$$

with domain $\mathcal{D}(q_{A,V}) \coloneqq \mathcal{D}(P-A)$ is a closed quadratic form which is bounded from below. It corresponds to a unique self-adjoint operator, which we denote by $H_{A,V} = (P - A)^2 + V$. We will sometimes drop the dependence of $H_{A,V}$ and simply write H for the full magnetic Schrödinger operator.

The diamagnetic inequality implies that if V is form bounded, respectively form small w.r.t P^2, then it is also form bounded, respectively form small w.r.t $(P - A)^2$ with the same constants, see [3].

One could extend the above setting by allowing a splitting $V = V_+ - V_-$, where the positive and negative parts of V are given by $V_\pm = \max(\pm V, 0)$. The discussion in [33] shows that for arbitrary $V_+ \in L^1_{\mathrm{loc}}$, the quadratic form

$$q_{A,V_+}(\varphi, \varphi) \coloneqq \|(P - A)\varphi\|_2^2 + \langle \varphi, V_+\varphi \rangle = \|(P - A)\varphi\|_2^2 + \|\sqrt{V_+}\varphi\|_2^2 \tag{2.11}$$

is well defined and closed on the form domain $\mathcal{D}(Q_{A,V_+}) = \mathcal{D}((P - A)) \cap \mathcal{Q}(V_+)$, where $\mathcal{Q}(V_+) = \mathcal{D}(\sqrt{V_+})$ and that $\mathcal{C}_0^\infty$ is still dense in $\mathcal{D}(Q_{A,V_+})$ in the graph norm $\|\varphi\|_{A,V_+} = (\mathcal{Q}_{A,V_+}(\varphi) + \|\varphi\|_2^2)^{1/2}$. Again this closed quadratic form corresponds to a unique self–adjoint operator H_{A,V_+} and in order to define a self–adjoint operator $H_{A,V}$ via the KLMN theorem. it is enough to assume that V_- is form small w.r.t. H_{A,V_+}.

More important for us is the observation due to Combescure and Ginibre [4] that rather singular potentials V can be form bounded with respect to P^2, and by the diamagnetic inequality then also with respect to $(P - A)^2$. Assume that $V = \nabla \cdot \Sigma + W$, where Σ is a locally square integrable vector field on $\mathbb{R}^d$ and W is locally integrable. For $\varphi \in \mathcal{C}_0^\infty$, an integration by parts shows

$$\langle \varphi, (\nabla \cdot \Sigma)\varphi \rangle = -\operatorname{Im} \langle \Sigma\varphi, P\varphi \rangle = -2 \operatorname{Im} \langle \Sigma\varphi, (P - A)\varphi \rangle$$

Thus

$$|\langle \varphi, (\nabla \cdot \Sigma)\varphi \rangle| \le \|\Sigma\varphi\| \, \|P\varphi\| \quad \text{ and } |\langle \varphi, (\nabla \cdot \Sigma)\varphi \rangle| \le \|\Sigma\varphi\| \, \|(P - A)\varphi\|$$

and this shows that if Σ^2 and W are form bounded w.r.t. $(P-A)^2$, respectively P^2, then the quadratic form

$$\langle\varphi, V\varphi\rangle \coloneqq -2\,\mathrm{Im}\,\langle\Sigma\varphi,(P-A)\varphi\rangle + \langle\varphi, W\varphi \tag{2.12}$$

is also form bounded w.r.t. $(P-A)^2$, respectively P^2. Using Cauchy–Schwarz with epsilon, one can also relate the coefficients in the quadratic form bounds. This allows to treat singular potentials.

Moreover, at least in the non–magnetic case, the beautiful work of Maz'ya and Verbitsky [27] shows that all potential V which are relatively form bounded w.r.t. P^2 are of this form.

2.2. **The Poincaré gauge.** The magnetic field at the point $x\in\mathbb{R}^d$ is given by an antisymmetric two-form $B(x):\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$, which we identify with a matrix valued function B given by

$$B(x) = (B_{j,m}(x))_{j,m=1}^d,$$

which is antisymmetric, $B_{j,m}(x) = -B_{m,j}(x)$ for all $1\le j,m\le d, x\in\mathbb{R}^d$.

Any vector potential A, or more precisely a one form, generates a magnetic fields via the exterior derivative $B = dA$, in the distributional sense. In matrix notation, $B_{j,m} = \partial_j A_m - \partial_m A_j$. In three space dimensions, one can identify the two form B with a vector valued function $B = \mathrm{curl} A$.

For a given magnetic field B and a point $z\in\mathbb{R}^d$ we define the vector field $\widetilde{B}_w$ by equation (1.9), and put

$$A_w(x) \coloneqq \int_0^1 \widetilde{B}(tx)\,dt = \int_0^1 B(tx)[tx]\,dt\,, \tag{2.13}$$

which is the vector potential in the Poincaré gauge. Using translations, it is no loss of generality to assume $w=0$, in which case we will simply write A for the vector potential given by (2.13). By going to spherical coordinates, one easily checks at least for nice, say continuous or even smooth, magnetic fields B, that the above vector potential is well defined and that $dA = B$ in the sense of distributions.

Since B is antisymmetric the vector $\widetilde{B}(x) = B(x)[x]$ is orthogonal to x. Hence, when $w=0$ the vector potential A given by (2.13) satisfies the transversal, or Poincaré, gauge

$$x\cdot A(x) = 0 \qquad \forall\ x\in\mathbb{R}^d, \tag{2.14}$$

which will be very important in our discussion of dilations and the virial theorem for magnetic Schrödinger operators in Section 3. It is easy to see that for A given by (2.13) one has $A\in L^2_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^d)$ for bounded magnetic fields B and this extends to a large class of singular magnetic fields, see Lemma 2.9 below. Except otherwise noted, we will always use the Poincaré gauge in the following. For a nice discussion of the Poincaré gauge from a physics point of vies see [17] and from a more mathematical point of view, but still for rather regular magnetic fields, see [35].

2.3. **Hypotheses.** We will use the following hypotheses on B and V:

Assumption 2.1. The magnetic field B is such that for some $w\in\mathbb{R}^d$ and with $\widetilde{B}_w(x) \coloneqq B(x+w)[x]$

$$\mathbb{R}^d \ni x \mapsto |x-w|^{2-d}\log_+^2\left(\frac{R}{|x-w|}\right)\widetilde{B}_w(x)^2 \in L^1_{\mathrm{loc}}(\mathbb{R}^d) \tag{2.15}$$

for all $R>0$.

As already remarked, there is no loss of generality assuming $w=0$ by using translations. Together with Lemma 2.9 the above mild integrability condition then assures that the corresponding vector potential in the Poincaré gauge is locally square integrable, which is essential in order to define the magnetic Schrödinger operator. The magnetic field B can have severe local singularities, while Assumption 2.1 still holds.

Assumption 2.2. The scalar field $|\widetilde{B}|^2$ is relatively form bounded w.r.t. $(P-A)^2$, where A is the Poincaré gauge vector potential corresponding to B, That is,

$$\langle \varphi, |\widetilde{B}|^2\varphi\rangle = \|\widetilde{B}\varphi\|_2^2 \lesssim \|(P-A)\varphi\|_2^2 + \|\varphi\|_2^2 \qquad \forall\, \varphi \in \mathcal{D}(P-A). \tag{2.16}$$

Assumption 2.3. The potential V is relatively form small w.r.t. $(P-A)^2$, that is, there exist constants $\alpha < 1$ and $\gamma > 0$ such that

$$|\langle \varphi, V\varphi\rangle\| \leq \alpha\,\|(P-A)\varphi\|_2^2 + \gamma\|\varphi\|_2^2 \qquad \forall\, \varphi \in \mathcal{H}^1(\mathbb{R}^d). \tag{2.17}$$

We also need similar conditions on the virial $x\cdot\nabla V$ of the potential. Since we don't want to impose strong differentiability conditions on V, one has to be a bit careful: The virial $x\cdot\nabla V$ is, at first, a distribution given by the quadratic form

$$\langle \varphi, x\cdot\nabla V\varphi\rangle = -d\langle\varphi, V\varphi\rangle - 2\,\mathrm{Re}\,\langle V\varphi, x\cdot\nabla\varphi\rangle$$

when $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$. We assume that this form extends to all $\varphi \in \mathcal{D}(P-A)$. A careful discussion when this is the case, is given in Lemma 3.7 and in Section 3.3.
For the assumptions which give us control of virial $x\cdot\nabla V$, we decompose the potential $V = V_1 + V_2$ How one splits $V = V_1 + V_2$ is quite arbitrary, as long as the conditions below are met.

Assumption 2.4. If the potential is split as $V = V_1 + V_2$, then $V_1, x^2V_1^2$ and $x\cdot\nabla V_2$ are are relatively form bounded w.r.t. $(P-A)^2$.

It will turn out that under this condition the virial $x\cdot\nabla V$ is also relatively form bounded w.r.t. $(P-A)^2$. See the discussion in Section 3.3.

These above assumptions are all we need to prove a quadratic form version of the virial theorem, which allows us to treat rather singular magnetic fields and potentials V and to avoid having explicit conditions on the magnetic vector potential A which are not gauge invariant.

Behaviour at infinity. We need to quantify the notion that the magnetic field B, the potential V and the virial $x\cdot\nabla V$ are bounded, or even vanish, at infinity.

From physical heuristics, one expect that that 'smallness' should not be measured pointwise, but only *relative to* the kinetic energy $(P-A)^2$. The following conditions make this physical intuition precise.

Assumption 2.5. (Vanishing at infinity) The potential V vanishes at infinity w.r.t. $(P-A)^2$ in the sense of Definition 1.4. Moreover, if we split $V = V_1 + V_2$ as in assumption 2.4, then also V_1 vanishes at infinity w.r.t. $(P-A)^2$ in the sense of Definition 1.4.

The precise notion of being bounded at infinity w.r.t. $(P-A)^2$ is given by

Assumption 2.6. (Boundedness of the magnetic field and the virial at infinity) There exist positive sequences $(\varepsilon_j)_j, (\beta_j)_j$ and $(R_j)_j$ with $\varepsilon_j \to 0$ and $R_j \to \infty$ as $j \to \infty$, such that for all $\varphi \in \mathcal{D}(P-A)$ with $\mathrm{supp}(\varphi) \subset \mathcal{U}_j^c = \{x \in \mathbb{R}^d : |x| \geq R_j\}$

$$\|\widetilde{B}\varphi\|_2^2 \leq \varepsilon_j\,\|(P-A)\varphi\|_2^2 + \beta_j^2\,\|\varphi\|_2^2 \tag{2.18}$$

For the decomposition $V = V_1 + V_2$ of the potential, we also assume that there exist positive sequences $(\omega_{1,j})_j$ and $(\omega_{2,j})_j$ such that for all $\varphi \in \mathcal{D}(P-A)$ with $\mathrm{supp}(\varphi) \subset \mathcal{U}_j^c$

$$\|x\,V_1\varphi\|_2^2 \leq \varepsilon_j\,\|(P-A)\varphi\|_2^2 + \omega_{1,j}^2\,\|\varphi\|_2^2 \tag{2.19}$$

$$\langle\varphi, x\cdot\nabla V_2\varphi\rangle \leq \varepsilon_j\,\|(P-A)\varphi\|_2^2 + \omega_{2,j}\,\|\varphi\|_2^2 \tag{2.20}$$

By monotonicity we may assume, without loss of generality, that the sequences β_j, $\omega_{1,j}$, and $\omega_{2,j}$ in assumption 2.6 are decreasing. We define

$$\beta := \lim_{j\to\infty} \beta_j\,, \qquad \omega_k := \lim_{j\to\infty} \omega_{k,j}\,, \quad k = 1, 2. \tag{2.21}$$

the relative bounds of $\widetilde{B}$, etc., at infinity, which give a precise quantitative notion on how large, relative to $(P-A)^2$, the magnetic field B, respectively the virial $x\cdot\nabla V$, are at infinity w.r.t. $(P-A)^2$. These assumptions above are inspired by Section 3 in [19] and allow us to effectively treat magnetic fields and potentials which can have severe singularities even close to infinity.

Unique continuation at infinity. For a unique continuation type argument at infinity, we also need a quantitative version of relative form bondedness.

Assumption 2.7. If $V = V_1 + V_2$, then we assume

$$\|\widetilde{B}\varphi\|_2^2 + \|xV_1\varphi\|_2^2 \le \frac{\alpha_1^2}{4}\,\|(P-A)\varphi\|_2^2 + C_1\|\varphi\|_2^2, \tag{2.22}$$

$$\big\langle \varphi, x\cdot\nabla V_2\,\varphi\big\rangle \;\le\; \alpha_2\,\|(P-A)\varphi\|_2^2 + C_2\|\varphi\|_2^2, \tag{2.23}$$

$$|\big\langle \varphi, V_1\,\varphi\big\rangle| \;\le\; \alpha_3\,\|(P-A)u\|_2^2 + C_3\|\varphi\|_2^2 \tag{2.24}$$

for some $\alpha_j, C_j > 0$, $j = 1,2,3$, all $\varphi\in\mathcal{D}(P-A)$, and

$$\alpha_1 + \alpha_2 + d\alpha_3 < 2. \tag{2.25}$$

Remarks 2.8. (i) In the conditions above, one can use the diamagnetic inequality in order to replace $P-A$ by the nonmagnetic momentum operator P in all relative form boundedness conditions, see [3].

(ii) In Section 5 we show that the above assumptions are satisfied under some mild and, more importantly, easily verifiable regularity and decay conditions on B and V, see Corollaries 5.5 and 5.6.

2.4. **Regularity of the Poincaré gauge map.** Note that the Poincaré gauge map (2.13) is a–priori only well-defined when the magnetic field B is sufficiently regular, say, continuous. Our first result shows that the map (2.13) can be continuously extended to all magnetic field satisfying Assumption 2.1.

Lemma 2.9. *Let $\mathcal{B}$ be the vector space of vector fields $\widetilde{B}$ satisfying*

$$\int_{\mathcal{U}_R} |x|^{2-d}\left(\log\frac{R}{|x|}\right)^2\,|\widetilde{B}(x)|^2\,dx < \infty$$

for all $R>0$. The continuous vector fields are dense in $\mathcal{B}$ and the map $\widetilde{B}\mapsto A := T(\widetilde{B})$ given by

$$A(x) = T(\widetilde{B})(x) := \int_0^1 \widetilde{B}(tx)\,dt \quad \text{for } x\in\mathbb{R}^d\,,$$

extends to map from $\mathcal{B}$ into $L^2_{\rm loc}(\mathbb{R}^d,\mathbb{R}^d)$. In particular, the Poincaré gauge map given in (2.13) is well defined for all magnetic fields satisfying assumption 2.1. Moreover,

$$\int_{\mathcal{U}_R} |x|^{2-d}\,|A(x)|^2\,dx \;\le\; 4\int_{\mathcal{U}_R} |x|^{2-d}\left(\log\frac{R}{|x|}\right)^2\,|\widetilde{B}(x)|^2\,dx\,, \tag{2.26}$$

for any $R>0$.

Proof. Given $B\in\mathcal{B}$ and $R>0$ let

$$\|\widetilde{B}\|_{\mathcal{B},R} := \left(\int_{\mathcal{U}_R} |x|^{2-d}\left(\log\frac{R}{|x|}\right)^2\,|\widetilde{B}(x)|^2\,dx\right)^{1/2}.$$

Also let $\mathcal{A}$ be the space of vector potentials A for which

$$\||A\|_{\mathcal{A},R} := \left(\int_{\mathcal{U}_R} |x|^{2-d} \, |A(x)|^2 \, dx \right)^{1/2}$$

is finite for all $R > 0$. This makes $\mathcal{A}$ and $\mathcal{B}$ locally convex metric spaces and by construction, $\mathcal{A} \subset L^2_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d)$. The metrics consistent with the topologies on $\mathcal{A}$ and $\mathcal{B}$ are, for example,

$$d_{\mathcal{A}}(A_1, A_2) = \sum_{n=0}^{\infty} 2^{-n} \frac{\||A_1 - A_2\|_{\mathcal{A},2^n}}{1 + \|A_1 - A_2\|_{\mathcal{A},2^n}} \quad \text{and} \quad d_{\mathcal{B}}(\widetilde{B}_1, \widetilde{B}_2) = \sum_{n=0}^{\infty} 2^{-n} \frac{2\|\widetilde{B}_1 - \widetilde{B}_2\|_{\mathcal{B},2^n}}{1 + 2\|\widetilde{B}_1 - \widetilde{B}_2\|_{\mathcal{B},2^n}}$$

The usual arguments show that $\mathcal{A}$ and $\mathcal{B}$ are complete metric spaces. Moreover, the usual cutting and mollifying arguments show that the continuous functions are dense in $\mathcal{B}$. In addition, $T(\widetilde{B})$ is well defined and locally bounded when $\widetilde{B}$ is continuous, so $T(\widetilde{B}) \in \mathcal{A}$, when $\widetilde{B}$ is continuous. The bound (2.26) then gives

$$d_{\mathcal{A}}(T(\widetilde{(B)}_1), T(\widetilde{B}_2)) = \sum_{n=0}^{\infty} 2^{-n} \frac{\||T(\widetilde{B}_1 - \widetilde{B}_2)\|_{\mathcal{A},2^n}}{1 + \|T(\widetilde{B}_1 - \widetilde{B}_2)\|_{\mathcal{A},2^n}} \le d_{\mathcal{B}}(\widetilde{B}_1, \widetilde{B}_2)$$

so T is uniformly continuous, thus it extend to a map from $\mathcal{B}$ into $\mathcal{A}$ which we continue to denote by T. This shows that the Poincaré gauge map (2.13) is well defined for all magnetic fields B satisfying assumption 2.1.

Hence it is enough to prove the bound (2.26) and by density, it is enough to prove it for continuous vector fields $\widetilde{B}$. Let g be a radial function, which is positive and finite for almost all $|x| < R$. Since $A(x) = \int_0^1 \widetilde{B}(tx) \, dt$, we have using symmetry

$$\begin{aligned}
\int_{\mathcal{U}_R} g(|x|)|A(x)|^2 \, dx &= \int_0^1 \int_0^1 \int_{\mathcal{U}_R} g(|x|)\widetilde{B}(t_1 x)\widetilde{B}(t_2 x) \, dx dt_1 dt_2 \\
&= 2 \iint_{0 \le t_1 < t_2 \le 1} \int_{|x| \le R} g(|x|)\widetilde{B}(t_1 x)\widetilde{B}(t_2 x) \, dx dt_1 dt_2 \\
&= 2 \int_0^1 \int_0^1 \int_{\mathcal{U}_{tR}} g(|y|/t) t^{1-d} \widetilde{B}(uy)\widetilde{B}(y) \, dy du dt \\
&= 2 \int_{\mathcal{U}_R} \left(\int_{|y|/R}^1 g(|y|/t) t^{1-d} \, dt \right) A(y)\widetilde{B}(y) \, dy \\
&\le 2 \left(\int_{\mathcal{U}_R} g(|y|)|A(y)|^2 \, dy \right)^{1/2} \left(\int_{\mathcal{U}_R} g(|y|)^{-1} \left(\int_{|y|/R}^1 g(|y|/t) t^{1-d} \, dt \right)^2 |\widetilde{B}(y)|^2 \, dy \right)^{1/2}
\end{aligned}$$

where we also used the substitution $t_1 = ut_2$ and $y = t_2 x$ and then Cauchy-Schwarz inequality. Thus as soon as $\int_{|x| \le R} g(|x|)|A(x)|^2 \, dx$ is finite, we arrive at the a-priori bound

$$\int_{\mathcal{U}_R} g(|x|)|A(x)|^2 \, dx \le 4 \int_{\mathcal{U}_R} g(|x|)^{-1} \left(\int_{|x|/R}^1 g(|x|/t) t^{1-d} \, dt \right)^2 |\widetilde{B}(x)|^2 \, dx \, . \tag{2.27}$$

Choosing $g(s) = s^{2-d}$, one easily calculates

$$g(|x|)^{-1} \left(\int_{|x|/R}^1 g(|x|/t) t^{1-d} \, dt \right)^2 = |x|^{2-d} \Big(\log \frac{R}{|x|} \Big)^2 .$$

Plugging this into (2.27) gives (2.26). We note that $A(x) = \int_0^1 \widetilde{B}(tx)\, dt$ is locally bounded as long as $\widetilde{B}$ is locally bounded. Thus for the above choice of g

$$\int_{\mathcal{U}_R} |x|^{2-d}\, |A(x)|^2\, dx$$

is, as required, finite for all continuous $\widetilde{B}$. Hence the a-priori bound (2.26) holds for all continuous $\widetilde{B}$ and extend by density to all of $\mathcal{B}$. ∎

Together with the quadratic form $Q_{A,V}$ we will also need the associated sesqui-linear form

$$q_{A,V}(u,v) = \big\langle (P-A)\, u, (P-A)\, v\big\rangle + \big\langle u, V v\big\rangle = q_{A,0}(u,v) + \big\langle u, V v\big\rangle, \qquad u,v \in \mathcal{H}^1(\mathbb{R}^d). \tag{2.28}$$

and denote by $H = H_{A,V}$ the self-adjoint operator associated with $Q_{A,V}$.

3. Dilations and the magnetic virial theorem

We will write $H_{A,V} = (P-A)^2 + V$, even though, strictly speaking, the operator is only defined via the sum of the corresponding quadratic forms.

3.1. **Dilations and the Poincaré gauge.**

In this subsection we will study the behavior of the magnetic Schrödinger form $Q_{A,V}$ under the action of the dilation group.

Let D_0 be the operator defined on $C_0^\infty(\mathbb{R}^d)$ by

$$D_0 = \frac{1}{2}\left(P\cdot x + x\cdot P\right), \qquad \mathscr{D}(D_0) = C_0^\infty(\mathbb{R}^d). \tag{3.1}$$

Remark 3.1. Note that $D_0 = \frac{1}{2}\left((P-A)\cdot x + x\cdot(P-A)\right)$, when A is in the Poincaré gauge (2.14), since then $x\cdot A(x) = 0$ for all $x\in\mathbb{R}^d$. This is one of the reasons why dilations and the Poincaré gauge work well together. A deeper reason is the representation (3.14) which connects the Poincaré gauge with dilations.

Lemma 3.2. *D_0 is essentially self-adjoint.*

Proof. For $t\in\mathbb{R}$ define the unitary dilation operator U_t by

$$(U_t f)(x) = e^{td/2} f(e^t x) \quad x\in\mathbb{R}^d. \tag{3.2}$$

It is easy to see that U_t is unitary on $L^2(\mathbb{R}^d)$ and forms a group, $U_tU_s = U(t+s)$, for all $t,s\in\mathbb{R}$. In particular, the adjoint is given by $U_t^* = U_{-t}$. Moreover, each U_t leaves $C_0^\infty(\mathbb{R}^d)$ invariant and a direct calculation shows that $t\mapsto U_t$ is strongly differentiable on $C_0^\infty(\mathbb{R}^d)$ with

$$\Big(\frac{d}{dt} U_t f\Big)\Big|_{t=0} = iD_0 f, \qquad \forall\ f\in C_0^\infty(\mathbb{R}^d). \tag{3.3}$$

The claim now follows from [29, Thm. VIII.10]. ∎

We denote by D the closure of D_0, which is self-adjoint, and by D_t the operator given by

$$iD_t = \frac{U_t - U_{-t}}{2t}\,. \tag{3.4}$$

D_t is bounded and symmetric. We will use it to approximate D in the limit $t\to 0$.

Let $\varphi\in\mathcal{D}(P)$. It is easy to check the commutation formula

$$PU_t = e^t U_t P\,, \tag{3.5}$$

since $(PU_t\varphi)(x) = -i\nabla(e^{td/2}\varphi(e^t x)) = -ie^t e^{td/2}(\nabla\varphi)(e^t x) = e^t(U_t(P\varphi))(x)$. In a similar way, one checks that for a multiplication operator V the commutation formula

$$V(\cdot)U_t = U_t V_{-t}(\cdot) \coloneqq U_t V(e^{-t}\cdot) \tag{3.6}$$

holds on its domain, i.e., for all $\varphi \in \mathcal{D}(V)$ we have $(V(U_t\varphi))(x) = e^{td/2}V(x)\varphi(e^t x) = (U_t(V_{-t}\varphi))(x)$ for almost all $x \in \mathbb{R}^d$. A similar result also holds for vector valued multiplication operators, for example,

$$A(\cdot)U_t = U_t A_{-t}(\cdot) \coloneqq U_t A(e^{-t}\cdot) \tag{3.7}$$

For the virial theorem, we want to define the commutator $[H_{A,V}, iD]$, where D is the generator of dilations. Since the two operators involved are unbounded, this usually leads to involved domain considerations. Even worse, in our case we do not know the domain $\mathcal{D}(H_{A,V})$ exactly, nor do we intend to know it, since we prefer to work only with quadratic forms. This seems to make a usable virial theorem impossible to achieve, however, a quadratic form approach turns out to be feasable.

Assume that $u \in \mathcal{D}(H_{A,V})$ and we approximate the unbounded generator of dilations D by the bounded approximations D_t. A slightly formal calculation, for $u \in \mathcal{D}(H_{A,V}) \cap \mathcal{C}_0^\infty$ which might be empty, however, gives

$$\langle u, [H_{A,V}, iD_t]u\rangle = \langle H_{A,V}u, iD_t u\rangle + \langle iD_t u, H_{A,V}u\rangle = 2\,\mathrm{Re}(\langle H_{A,V}u, iD_t u\rangle) \tag{3.8}$$

since iD_t is antisymmetric. Assume that $\mathcal{D}(P-A)$ is invariant under dilations. Then, since $\mathcal{D}(H_{A,V}) \subset \mathcal{D}(q_{A,V}) \subset \mathcal{D}(P-A)$, we have $iD_t u \in \mathcal{D}(P-A)$. So the right hand side of (3.8) can be written as $2\,\mathrm{Re}(q_{A,V}(u, iD_t u))$ and this extends, since $\mathcal{D}(H_{A,V})$ is dense in $\mathcal{D}(q_{A,V}) = \mathcal{Q}(H_{A,V})$, to all of $\mathcal{Q}(H_{A,V})$, the quadratic form domain of $H_{A,V}$. So we simply *define* the commutator $[H_{A,V}, iD_t]$ as the quadratic form

$$\langle u, [H_{A,V}, iD_t]u\rangle \coloneqq 2\,\mathrm{Re}\left(q_{A,V}(u, iD_t u)\right) \tag{3.9}$$

on $\mathcal{Q}(H_{A,V})$. Moreover, we can define the commutator $[H, iD]$, again in the sense of quadratic forms, by

$$\big\langle u, i\,[H,D]\,u\big\rangle \coloneqq \lim_{t\to 0}\langle u, [H_{A,V}, iD_t]u\rangle \coloneqq \lim_{t\to 0} 2\,\mathrm{Re}\left(q_{A,V}(u, iD_t u)\right), \tag{3.10}$$

provided the limit on the right hand side exists. In the remaining part of this section, we will deal with the calculation of the right hand side of (3.10) and, in particular, the claim that $\mathcal{D}(P-A)$ is invariant under dilations under a natural condition on the magnetic field.

By (3.5), the Sobolev space $\mathcal{D}(P)$ is invariant under dilations. To see how one can also get this for the magnetic Sobolev space $\mathcal{D}(P-A)$ let $\varphi \in \mathcal{D}(P-A)$. Then, as distributions,

$$(P-A)U_t\varphi = e^t U_t P\varphi - U_t A_t \varphi = e^t U_t (P-A)\varphi + U_t(e^t A - A_{-t})\varphi\,. \tag{3.11}$$

Since $U_t : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is unitary and $(P-A)\varphi \in L^2(\mathbb{R}^d)$, we have $e^t U_t(P-A)\varphi \in L^2(\mathbb{R}^d)$ for all $t \in \mathbb{R}$. So in order that $U_t\varphi \in \mathcal{D}(P-A)$ we have to check if $(e^t A - A_{-t})\varphi \in L^2(\mathbb{R}^d)$. This is the content of the next proposition.

Proposition 3.3. *Suppose that the magnetic field B satisfies assumption 2.1, the vector potential A corresponding to B is in the Poincaré gauge, and $\widetilde{B}^2$ is relatively form bounded w.r.t. $(P-A)^2$.*
If $\varphi \in D(P-A) = \mathcal{H}^1(\mathbb{R}^d)$, then $(e^t A - A_{-t})\varphi \in L^2(\mathbb{R}^d)$ for all $t \in \mathbb{R}$ and the map $\mathbb{R} \ni t \mapsto (e^t A - A_{-t})\varphi$ is continuous. In particular, $\mathcal{D}(P-A)$ is invariant under dilations.

The main tool for the proof of Proposition 3.3 is the following

Lemma 3.4. *Under the assumptions of Proposition 3.3, if $\varphi \in D(P-A) = \mathcal{H}^1(\mathbb{R}^d)$, then*

$$\|(e^t A - A_{-t})\varphi\| \;\le\; e^t\big(e^{C_z|t|} - 1\big)\|(P-A)\varphi\| + \frac{z\,C_z}{C_z \pm 1}\big(e^{(C_z\pm 1)|t|} - 1\big)\|\varphi\| \tag{3.12}$$

for all $t \in \mathbb{R}$ and $z > 0$, where the $+$ sign holds for $t \ge 0$ and the $-$ sign for $t < 0$ and the constant C_z is given by

$$C_z = \sqrt{d}\,\big\|\widetilde{B}\big((P-A)^2 + z^2\big)^{-\frac{1}{2}}\big\|_{2\to 2}\,.$$

Remark 3.5. In the above bound we use the convention $\frac{C_z}{C_z - 1}\big(e^{(C_z-1)|t|} - 1\big) = |t|$ when $C_z = 1$.

Given Lemma 3.4, the proof of Proposition 3.3 is simple.

Proof of Proposition 3.3: Given $\varphi \in \mathcal{D}(P-A)$, Lemma 3.4 shows that $(e^t A - A_{-t})\varphi \in L^2(\mathbb{R}^d)$ for all $t \in \mathbb{R}$ and then (3.11) shows that $U_t\varphi \in \mathcal{D}(P-A)$ for all $t \in \mathbb{R}$. Thus $\mathcal{D}(P-A)$ is invariant under dilations.

Moreover, the bound (3.12) shows that the map $t \mapsto (e^t A - A_{-t})\varphi$ is continuous at $t = 0$. Since, for any $s, t \in \mathbb{R}$,

$$e^{t+s}A - A_{-(t+s)} = e^s\big(e^t A - A_{-t}\big) + e^s A_{-t} - (A_{-s})_{-t} = e^s\big(e^t A - A_{-t}\big) + U_t^*\big(e^s A - (A_{-s})\big)U_t \tag{3.13}$$

and $U_t\varphi \in \mathcal{D}(P-A)$ for any $\varphi \in \mathcal{D}(P-A)$, continuity of $t \mapsto \big(e^t A - A_{-t}\big)\varphi$ at $t = 0$ implies continuity at all $t \in \mathbb{R}$. ∎

Proof of Lemma 3.4: First of all, it is enough to prove (3.12) for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, since this is dense in $\mathcal{D}(P-A)$ in the graph norm: If (3.12) holds for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, then given $\varphi \in \mathcal{D}(P-A)$, choose a sequence $\varphi_n \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ such that $(P-A)\varphi_n \to (P-A)\varphi$ and $\varphi_n \to \varphi$. By taking a subsequence, if necessary, we can also assume that $\varphi_n \to \varphi$ almost everywhere, hence $(e^t A - A_{-t})\varphi_n \to (e^t A - A_{-t})\varphi$ almost everywhere, in particular, $|(e^t A - A_{-t})\varphi| = \lim_{n\to\infty} |(e^t A - A_{-t})\varphi_n| = \liminf_{n\to\infty} |(e^t A - A_{-t})\varphi_n|$ almost everywhere. Then Fatou's Lemma and (3.12) imply

$$\begin{aligned}\|(e^t A - A_{-t})\varphi\| &= \|\liminf_{n\to\infty} |(e^t A - A_{-t})\varphi_n|\| \le \liminf_{n\to\infty} \|(e^t A - A_{-t})\varphi_n\| \\ &\le e^t\big(e^{C_z|t|} - 1\big)\|(P-A)\varphi\| + \frac{z\,C_z}{C_z \pm 1}\big(e^{(C_z\pm 1)|t|} - 1\big)\|\varphi\|\end{aligned}$$

for all $\varphi \in \mathcal{D}(P-A)$.

Let $t \in \mathbb{R}$. Since A is in the Poincaré gauge, using the change of variables $t = e^{-s}$, we have

$$A = \int_0^\infty e^{-s}\widetilde{B}(e^{-s}\cdot)\,ds = \int_0^\infty e^{-s}U_s^*\,\widetilde{B}\,U_s\,ds\,. \tag{3.14}$$

From the definition of A_{-t} and (3.14) we get

$$\begin{aligned}e^t\,A - A_{-t} &= e^t\int_0^\infty e^{-s}U_s^*\,\widetilde{B}\,U_s\,ds - \int_0^\infty e^{-s}\,U_t^*U_s^*\,\widetilde{B}\,U_sU_t\,ds \\ &= e^t\int_0^\infty e^{-s}\,U_s^*\,\widetilde{B}\,U_s\,ds - e^t\int_t^\infty e^{t-s}U_s^*\,\widetilde{B}\,U_s\,ds \\ &= e^t\int_0^t e^{-s}\,U_s^*\,\widetilde{B}\,U_s\,ds.\end{aligned} \tag{3.15}$$

Let $\varphi \in C_0^\infty(\mathbb{R}^d)$. The above identity then gives

$$v_t := (e^t A - A_{-t})\varphi = e^t\int_0^t e^{-s}\,U_s^*\,\widetilde{B}\,U_s\,\varphi\,ds. \tag{3.16}$$

Define the operator $R_z : D(P-A) \to D(H_0)$ by

$$R_z := ((P-A)^2 + dz^2)^{-1}\,(P - A - iz). \tag{3.17}$$

Here $((P-A-iz))$ is a vector operator, which maps $\varphi \in \mathcal{D}(P-A)$ to the vector function $(P-A-iz)\varphi = \big((P_j - A_j - iz)\varphi\big)_{j=1,\dots,d}$. Then $R_z(P-A+iz)\varphi = \varphi$, so

$$\begin{aligned}\widetilde{B}\,U_s\,\varphi &= \widetilde{B}\,R_z(P-A+iz)\,U_s\,\varphi = \widetilde{B}\,R_z\,U_s\big[e^s(P-A)\,\varphi + (e^sA - A_{-s})\,\varphi + iz\varphi\big] \\ &= \widetilde{B}\,R_z\,U_s\big[e^s(P-A)\,\varphi + v_s + iz\varphi\big],\end{aligned}$$

which in view of (3.16) implies

$$v_t = \int_0^t e^{t-s}\, U_s^* \widetilde{B}\, R_z\, U_s\big(e^s (P-A)\,\varphi + v_s + iz\varphi\big)\, ds. \tag{3.18}$$

Hence, if $t \geq 0$,

$$\begin{aligned} w(t) := \|v_t\| \;&\leq\; K_z \int_0^t e^{t-s}\, \big(e^s \|(P-A)\,\varphi\|_2 + w(s) + z\|\varphi\|\big)\, ds \\ &= E(t) + K_z \int_0^t e^{t-s}\, w(s)\, ds, \end{aligned}$$

where

$$K_z := \|\widetilde{B}\, R_z\|_{2\to 2}\,, \tag{3.19}$$

and

$$E(t) = K_z \int_0^t e^{t-s}\, \big(e^s \|(P-A)\,\varphi\|_2 + z\|\varphi\|_2\big)\, ds\,.$$

Of course, any upper bound on K_z can be used, we will derive a suitable bound on K_z at the end of this proof. The Gronwall–type Lemma A.1 in the Appendix yields

$$w(t) \;\leq\; E(t) + K_z \int_0^t e^{(1+K_z)(t-s)}\, E(s)\, ds\,. \tag{3.20}$$

Note

$$\begin{aligned} &\int_0^t e^{(1+K_z)(t-s)}\, E(s)\, ds = \\ &\quad = K_z \iint\limits_{0<s<s'<t} e^{(1+K_z)(t-s')} e^{s'}\, ds ds'\, \|(P-A)\,\varphi\|_2 + zK_z \iint\limits_{0<s<s'<t} e^{(1+K_z)(t-s')} e^{s'-s}\, ds ds'\, \|\varphi\|_2 \\ &\quad = \Big(\frac{e^t}{K_z}\big(e^{K_z t}-1\big) - te^t\Big)\|(P-A)\,\varphi\|_2 + z\Big(\frac{1}{K_z+1}\big(e^{(K_z+1)t}-1\big) - \big(e^t-1\big)\Big)\|\varphi\|_2 \end{aligned}$$

and a straightforward calculation gives

$$E(t) = K_z t e^t \|(P-A)\varphi\|_2 + zK_z\big(e^t-1\big)\|\varphi\|_2$$

Inserting this into (3.20) gives

$$\|(e^t A - A_{-t})\varphi\| = w(t) \leq e^t\big(e^{K_z t}-1\big)\|(P-A)\,\varphi\|_2 + \frac{zK_z}{K_z+1}\big(e^{(K_z+1)t}-1\big)\|\varphi\|_2$$

which is , at least for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$.

If $t<0$, then setting $\tau = -t > 0$, we get from (3.18)

$$\begin{aligned} \widetilde{w}(\tau) := \|v_{-\tau}\|_2 \;&\leq\; K_z \int_0^\tau e^{s-\tau}\, \big(e^{-s}\|(P-A)\,\varphi\|_2 + w(s) + z\|\varphi\|_2\big)\, ds \\ &= \widetilde{E}(\tau) + K_z \int_0^\tau e^{s-\tau}\, \widetilde{w}(s)\, ds, \end{aligned}$$

with

$$\widetilde{E}(\tau) := K_z \int_0^\tau e^{s-\tau}\, \big(e^{-s}\|(P-A)\,\varphi\|_2 + z\|\varphi\|_2\big)\, ds$$

and the second Gronwall–type bound from Lemma A.1 now gives

$$\widetilde{w}(\tau) \leq \widetilde{E}(\tau) + K_z \int_0^\tau e^{(K_z-1)(\tau-s)}\, \widetilde{E}(s)\, ds\,. \tag{3.21}$$

Similarly as above one calculates

$$\begin{aligned}
&\int_0^\tau e^{(K_z-1)(\tau-s)}\,\widetilde{E}(s)\,ds = \\
&\quad = K_z \iint\limits_{0<s<s'<\tau} e^{(K_z-1)\tau-K_z s'}\,dsds'\,\|(P-A)\,\varphi\|_2 + zK_z \iint\limits_{0<s<s'<t} e^{(K_z-1)\tau-K_z s'+s}\,dsds'\,\|\varphi\|_2 \\
&\quad = \Big(\frac{e^{-\tau}}{K_z}\big(e^{K_z\tau}-1\big)-\tau e^{-\tau}\Big)\|(P-A)\,\varphi\|_2 + z\Big(\frac{1}{K_z-1}\big(e^{(K_z-1)\tau}-1\big)-\big(1-e^{-\tau}\big)\Big)\|\varphi\|_2
\end{aligned}$$

and

$$\widetilde{E}(\tau) = K_z\tau e^{-\tau}\|(P-A)\,\varphi\|_2 + zK_z(1-e^{-\tau})\|\varphi\|_2\,,$$

and plugging this back into (3.21), using $t=-\tau<0$ we arrive at

$$\|(e^tA-A_{-t})\varphi\|_2 = \widetilde{w}(\tau) \le e^t\big(e^{K_z|t|}-1\big)\,\|(P-A)\,\varphi\|_2 + \frac{zK_z}{K_z-1}\big(e^{(K_z-1)|t|}-1\big)\|\varphi\|_2\,.$$

Recalling that we can replace K_z by any upper bound in the above arguments, this proves (3.12), we only have to bound K_z. Let $\psi\in C_0^\infty(\mathbb{R}^d)$. From the definition (3.19) one easily gets

$$K_z = \|\widetilde{B}\,R_z\|_{2\to2} \le \|\widetilde{B}\,((P-A)^2+dz^2)^{-\frac12}\|_{2\to2}\,\|((P-A)^2+dz^2)^{-\frac12}(P-A-iz)\|_{2\to2}.$$

On the other hand, letting $T=((P-A)^2+dz^2)^{-\frac12}(P-A-iz)$ one sees

$$\begin{aligned}
TT^* &= ((P-A)^2+dz^2)^{-\frac12}(P-A-iz)\cdot(P-A+iz)((P-A)^2+dz^2)^{-\frac12} \\
&= ((P-A)^2+dz^2)^{-\frac12}((P-A)^2+dz^2)((P-A)^2+dz^2)^{-\frac12} = \mathbb{1}\,.
\end{aligned} \tag{3.22}$$

Hence by duality $\|((P-A)^2+dz^2)^{-\frac12}(P-A-iz)\|_{2\to2} = \|T\|_{2\to2} = 1$ and thus

$$K_z \,\le\, \|\widetilde{B}(H_0+z^2)^{-\frac12}\|_{2\to2} \,=:\, C_z\,. \tag{3.23}$$

∎

The next result concerns the calculation of $\frac{d}{dt}\big(e^t\,A-A_{-t}\big)\varphi\Big|_{t=0}$ for $\varphi\in\mathcal{D}(P-A)$. Recall that given a magnetic field B, the vector field $\widetilde{B}$ is given by equation (1.9).

Proposition 3.6. *Suppose that the magnetic field B satisfies assumption 2.1, the vector potential A corresponding to B is in the Poincaré gauge, and $\widetilde{B}^2$ is relatively form bounded w.r.t. $(P-A)^2$. Then for all $\varphi\in\mathcal{D}(P-A)$ the map $\mathbb{R}\ni t\mapsto(e^t\,A-A_{-t})\varphi$ is differentiable and*

$$\frac{d}{dt}\big(e^t\,A-A_{-t}\big)\varphi\Big|_{t=0} = \lim_{t\to0}\frac1t(e^t\,A-A_{-t})\varphi = \widetilde{B}\,\varphi \tag{3.24}$$

where the limit is taken in $L^2(\mathbb{R}^d)$.

Proof. Assume that for $\varphi\in\mathcal{D}(P-A)$ the map $t\mapsto\big(e^tA-A_{-t}\big)\varphi$ is differentiable in $t=0$ with derivative given by (3.24). Then (3.13) shows that it is also differentiable in any point $t\in\mathbb{R}$ with derivative

$$\frac{d}{dt}\big(e^t\,A-A_{-t}\big)\varphi = \big(e^tA-A_{-t}\big)\varphi + U_t^*\widetilde{B}U_t\varphi \tag{3.25}$$

By assumption, $\widetilde{B}^2$ is relatively form bounded with respect to $(P-A)^2$, that is, $\widetilde{B}:\mathcal{D}(P-A)\to L^2(\mathbb{R}^d)$ is bounded. Thus the right hand side of (3.25) is in $L^2(\mathbb{R}^d)$ by Proposition 3.3.
Hence it is enough to show differentiability at $t=0$. We will prove, for all $\varphi\in\mathcal{D}(P-A)$,

$$\lim_{t\to0}\frac{1}{|e^t-1|}\Big\|e^t\,A-A_{-t}-(e^t-1)\,\widetilde{B}\Big\|\,\varphi = 0 \quad\text{in}\quad L^2(\mathbb{R}^d), \tag{3.26}$$

which is equivalent to (3.24). First assume that $\varphi \in C_0^\infty(\mathbb{R}^d)$. Using (3.15) we have

$$
\begin{aligned}
v_t :&= \big(e^t A - A_{-t} - (e^t-1)\,\widetilde{B}\big)\,\varphi = \int_0^t e^{t-s}\, U_s^*\,\widetilde{B}\, U_s\varphi\, ds - (e^t-1)\,\widetilde{B}\,\varphi \\
&= \int_0^t e^{t-s}\,\big(U_s^*\,\widetilde{B}\,U_s - \widetilde{B}\big)\varphi\, ds. \qquad (3.27)
\end{aligned}
$$

Using (3.11) we rewrite the integrand as

$$
\begin{aligned}
&\big(U_s^*\,\widetilde{B}\,U_s - \widetilde{B}\big)\varphi = U_s^*\widetilde{B}(U_s-1)\varphi + (U_s^*-1)\widetilde{B}\varphi \\
&\quad = U_s^*\widetilde{B}R_z\big((P-A+iz)U_s - (P-A+iz)\big)\varphi + (U_s^*-1)\widetilde{B}\varphi \\
&\quad = U_s^*\widetilde{B}R_z\Big[U_s\big(e^s(P-A)+e^sA - A_{-s} + iz\big)\varphi - (P-A+iz)\varphi\Big] + (U_s^*-1)\widetilde{B}\varphi \\
&\quad = U_s^*\widetilde{B}R_z\Big[U_s\big((e^s-1)(P-A)\varphi + (e^s-1)\widetilde{B}\varphi + v_s\big) + (U_s-1)(P-A+iz)\varphi\Big] - U_s^*(U_s-1)\widetilde{B}\varphi\,.
\end{aligned}
$$

Setting $w(s) := \|v_s\|_2$, and recalling $\|\widetilde{B}R_z\|_{2\to 2} \le \sqrt{d}\|\widetilde{B}((P-A)^2+z^2)^{-1/2}\| =: C_z$, see (3.23), we get

$$
\begin{aligned}
\|\big(U_s^*\,\widetilde{B}\,U_s - \widetilde{B}\big)\varphi\|_2 \le C_z\Big[|e^s-1|\big(\|(P-A)\varphi\|_2 + \|\widetilde{B}\varphi\|_2\big) + w(s) + \|(U_s-1)(P-A+iz)\varphi\|_2\Big] \\
+ \|(U_s-1)\widetilde{B}\varphi\|_2
\end{aligned}
$$

This implies the integral inequalities

$$
w(t) \le E(t) + C_z \int_0^t e^{t-s}\, w(s)\, ds \quad \text{for } t \ge 0
$$

and

$$
w(t) \le E(t) + C_z \int_0^{|t|} e^{t+s}\, w(-s)\, ds \quad \text{for } t \le 0,
$$

where now

$$
\begin{aligned}
E(t) = &\int_0^t e^{t-s}\Big[C_z\|(U_s-1)(P-A+iz)\varphi\|_2 + \|(U_s-1)\widetilde{B}\varphi\|_2\Big]\, ds \\
&+ \int_0^t e^{t-s}\,(e^s-1)C_z\big(\|(P-A)\varphi\|_2 + \|\widetilde{B}\varphi\|_2\big)\, ds\,,
\end{aligned}
$$

for $t \ge 0$, and

$$
\begin{aligned}
E(t) = &\int_0^{|t|} e^{t+s}\Big[C_z\|(U_s-1)(P-A+iz)\varphi\|_2 + \|(U_s-1)\widetilde{B}\varphi\|_2\Big]\, ds \\
&+ \int_0^{|t|} e^{t+s}\,(1-e^{-s})C_z\big(\|(P-A)\varphi\|_2 + \|\widetilde{B}\varphi\|_2\big)\, ds\,,
\end{aligned}
$$

for $t \le 0$. Lemma A.1 then yields the upper bounds

$$
w(t) \le E(t) + C_z \int_0^t e^{(1+C_z)(t-s)}\, E(s)\, ds \quad \text{for } t \ge 0 \qquad (3.28)
$$

and

$$
w(t) \le E(t) + C_z \int_0^{|t|} e^{(C_z-1)(t-s)}\, E(-s)\, ds \quad \text{for } t \le 0\,. \qquad (3.29)
$$

To continue it is convenient to use, for $\tau \geq 0$,

$$\kappa(\tau) := \sup_{|s|\leq\tau} \|(U_s - 1)\widetilde{B}\varphi\|_2 + C_z \sup_{|s|\leq\tau} \|(U_s - 1)(P - A + iz)\varphi\|_2,$$

so that for $t \geq 0$

$$\begin{aligned} E(t) &\leq \int_0^t e^{t-s}\,\kappa(s)\,ds + \big(\|(P-A)\varphi\|_2 + \|\widetilde{B}\varphi\|_2\big) \int_0^t e^{t-s}\,(e^s - 1)\,ds \\ &\leq \kappa(t)(e^t - 1) + \big(\|(P-A)\varphi\|_2 + \|\widetilde{B}\varphi\|_2\big)(e^t - 1)^2, \end{aligned}$$

since κ is increasing. Analogously, for $t \leq 0$ we have

$$\begin{aligned} E(t) &= \int_0^{|t|} e^{t+s}\kappa(s)\,ds + \big(\|(P-A)\varphi\|_2 + \|\widetilde{B}\varphi\|_2\big) \int_0^{|t|} e^{t+s}\,(1 - e^{-s})\,ds \\ &\leq \kappa(|t|)(1 - e^t) + \big(\|(P-A)\varphi\|_2 + \|\widetilde{B}\varphi\|_2\big)(1 - e^t)^2\,. \end{aligned}$$

So by monotonicity, for $t \geq 0$,

$$\begin{aligned} \int_0^t e^{(1+C_z)(t-s)}\,E(s)\,ds &\leq \Big(\kappa(t)(e^t - 1) + \big(\|(P-A)\varphi\|_2 + \|\widetilde{B}\varphi\|_2\big)(e^t - 1)^2\Big) \int_0^t e^{(1+C_z)(t-s)}\,ds \\ &= \frac{(e^{(1+C_z)t} - 1)(e^t - 1)}{1 + C_z}\Big[\kappa(t) + (e^t - 1)\big(\|(P-A)\varphi\|_2 + \|\widetilde{B}\varphi\|_2\big)\Big] \end{aligned}$$

and, similarly, for $t \leq 0$ we have

$$\int_0^{|t|} e^{(C_z-1)(t-s)}\,E(-s)\,ds \leq \frac{(1 - e^{(C_z-1)t})(1 - e^t)}{C_z - 1}\Big[\kappa(|t|) + (1 - e^t)\big(\|(P-A)\varphi\|_2 + \|\widetilde{B}\varphi\|_2\big)\Big]$$

which in combination with (3.28) and (3.29) implies

$$\begin{aligned} \frac{w(t)}{|e^t - 1|} &= \Big\|\frac{e^t\,A - A_{-t}}{e^t - 1}\varphi - \widetilde{B}\varphi\Big\| \\ &\leq \Big(1 + \frac{C_z|e^{(C_z\pm1)t} - 1|}{C_z \pm 1}\Big)\Big[\kappa(|t|) + |e^t - 1|\big(\|(P-A)\varphi\|_2 + \|\widetilde{B}\varphi\|_2\big)\Big], \end{aligned} \tag{3.30}$$

where the $+$ sign holds when $t \geq 0$ and the $-$ sign when $t < 0$. Since $P - A : \mathcal{D}(P - A) \to L^2(\mathbb{R}^d)$ and $\widetilde{B} : \mathcal{D}(P - A) \to L^2(\mathbb{R}^d)$ are bounded, (3.30) extends to all $\varphi \in \mathcal{D}(P - A)$, by density. Since $\kappa(t) \to 0$ as $t \to 0$, this proves (3.26). ∎

We will need a similar version for the electric potentials. Recall that $iD_t = (U_t - U_{-t})/(2t)$.

Lemma 3.7. *Let A, B, and $\widetilde{B}$ satisfy the same assumptions as in Proposition 3.6 and let V be any electric potential, with form domain $\mathcal{D}(P - A) \subset \mathcal{Q}(V)$, such that the distribution $x \cdot \nabla V$ extends to a quadratic form which is form bounded with respect to $(P - A)^2$. Then with $V_{-t} = U_t^* V U_t = V(e^{-t}\cdot)$,*

$$\lim_{t\to0} \frac{1}{t}\langle\varphi, (V - V_{-t})\psi\rangle = \langle\varphi, x \cdot \nabla V\,\psi\rangle \tag{3.31}$$

and

$$\lim_{t\to0} 2\,\mathrm{Re}\langle\varphi, V iD_t\varphi\rangle = -\langle\varphi, x \cdot \nabla V\,\varphi\rangle \tag{3.32}$$

for all $\varphi, \psi \in \mathcal{D}(P - A)$.

Remark 3.8. By a slight abuse of notation, we use the symbol $\langle\varphi, V\psi\rangle$ for the sesqui–linear form $\langle|V|^{1/2}\varphi, \mathrm{sgn}\,V|V|^{1/2}\psi\rangle$ with domain $\mathcal{D}(q_V) = \mathcal{D}(|V|^{1/2})$ and $\langle\varphi, x \cdot \nabla V\varphi\rangle$ for the (extension of the) form corresponding to the distribution $x \cdot \nabla V$.

Proof. We always have $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Given $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ we denote by $V\psi$ the distribution

$$\mathcal{C}_0^\infty(\mathbb{R}^d) \ni \varphi \mapsto \langle \varphi, V\psi\rangle := \int_{\mathbb{R}^d} \overline{\varphi(x)} V(x)\psi(x)\, dx.$$

Then the distributional derivative $W_s := \frac{d}{ds} V_{-s}$ is given by

$$\langle \varphi, W_s\psi\rangle = \frac{d}{ds}\langle \varphi, V_{-s}\psi\rangle =: -q(U_s\varphi, U_s\psi) = -\langle U_s\varphi, x\cdot\nabla V\, U_s\psi\rangle\,.$$

and, by assumption, the sesqui–linear form q extends to sesqui–linear form with a domain containing $\mathcal{D}(P-A)$ and which is relatively form bounded with respect to $(P-A)^2$. With a slight abuse of notation, we will write also q for this extension.

We claim that for any $\varphi, \psi \in \mathcal{D}(P-A)$ the map

$$\mathbb{R} \ni s \mapsto q(U_s\varphi, U_s\psi) \quad \text{is continuous.} \tag{3.33}$$

Assuming this for the moment, the fundamental theorem of calculous shows

$$\big\langle \varphi, (V - V_{-t})\psi\big\rangle = \int_0^t \frac{d}{ds}\big\langle \varphi, V_{-s}\psi\big\rangle\, ds = \int_0^t q(U_s\varphi, U_s\psi)\, ds \tag{3.34}$$

for any $\varphi, \psi \in \mathcal{C}_0^\infty(\mathbb{R}^d) \subset \mathcal{D}(P-A)$ and, by density, this extend to all $\varphi, \psi \in \mathcal{D}(P-A)$. But then (3.34) implies

$$\frac{d}{dt}\langle \varphi, V_{-t}\psi\rangle\big|_{t=0} = \lim_{t\to 0}\frac{1}{t}\langle \varphi, (V - V_{-t})\psi\rangle = q(\varphi,\psi)$$

which proves (3.31). For (3.32) we note

$$2t\,\mathrm{Re}\langle \varphi, ViD_t\rangle = \mathrm{Re}\,\big(\langle \varphi, VU_t\varphi\rangle - \langle U_t^*\varphi, V\varphi\rangle\big)$$

and

$$\langle \varphi, VU_t\varphi\rangle - \langle U_t^*\varphi, V\varphi\rangle = \langle U_t^*\,\varphi, (V_{-t} - V)\varphi\rangle = \int_0^t q(U_sU_t^*\varphi, U_s\varphi)\, ds$$

again by (3.34). By a simple continuity argument this shows

$$2\,\mathrm{Re}\langle \varphi, ViD_t\varphi\rangle = \frac{1}{t}\int_0^t \mathrm{Re}\, q(U_sU_t^*\varphi, U_s\varphi)\, ds \to q(\varphi,\varphi) = -\langle \varphi, x\cdot\nabla V\,\varphi\rangle$$

as $t\to 0$, which yields (3.32).

It remains to prove (3.33): The sesqui–linear form q being relatively $(P-A)^2$ form bounded is equivalent to

$$\varphi,\psi \mapsto q(\big((P-A)^2 + dz^2\big)^{-1/2}\varphi, \big((P-A)^2 + dz^2\big)^{-1/2}\psi)$$

extending, for $z>0$, to a bounded sesqui–linear form to all $\varphi,\psi \in L^2(\mathbb{R}^d)$. Recalling the definition (3.17) for R_z and (3.22), this is equivalent to

$$\varphi,\psi \mapsto q(\big(R_z\varphi, R_z\psi) =: \widetilde{q}(\varphi,\psi)$$

being a bounded quadratic form, more precisely, extending to a bounded quadratic form on all of $L^2(\mathbb{R}^d)$, for all $z>0$. Using sesqui–linearity, it is easy to see that for all continuous maps $s\mapsto \varphi_s, s\mapsto \psi_s \in L^2(\mathbb{R}^d)$ the map $s\mapsto \widetilde{q}(\varphi_s,\psi_s)$ is continuous for any bounded sesqui–linear form $\widetilde{q}$ on $L^2(\mathbb{R}^d)$.

For $\varphi,\psi \in \mathcal{D}(P-A)$ we have

$$q(U_s\varphi, U_s\psi) = \widetilde{q}((P-A-iz)U_s\varphi, (P-A-iz)U_s\psi)$$

and

$$U_s\varphi = R_z(P-A-iz)U_s\varphi = R_zU_s\big(e^s(P-A)+(e^sA-A_{-s})-iz\big)\varphi\,.$$

The map $s\mapsto e^s(P-A)\varphi$ is clearly continuous for all $\varphi\in\mathcal{D}(P-A)$ and so is the map $s\to(e^sA-A_s)\varphi$ by Proposition 3.3. Thus $s\mapsto\widetilde{\varphi}_s := \big(e^s(P-A)+(e^sA-A_{-s})-iz\big)\varphi$ is continuous for all $\varphi\in\mathcal{D}(P-A)$. Using $s\mapsto U_s$ being strongly continuous and unitary, and

$$U_t\widetilde{\varphi}_t - U_s\widetilde{\varphi}_s = (U_t-U_s)\widetilde{\varphi}_t + U_s(\widetilde{\varphi}_t-\widetilde{\varphi}_s)$$

one sees that the map $s\mapsto\varphi_s := U_s\widetilde{\varphi}_s$ is continuous. Similarly when φ is replaced by $\psi\in\mathcal{D}(P-A)$. Thus

$$\mathbb{R}\ni s\mapsto q(U_s\varphi,U_s\psi) = \widetilde{q}(\varphi_s,\psi_s)$$

is continuous, since $\widetilde{q}$ is a bounded sesqui–linear form. This proves (3.33) and hence the lemma. ∎

3.2. **The commutator as a quadratic form.** This section deals with one of our main results, the rigorous identification of the right hand side of (3.10).

Theorem 3.9 (Magnetic virial theorem)**.** *Let B and V satisfy assumptions 2.1- 2.3 and A be the vector potential in the Poincaré gauge corresponding to the magnetic field B. Assume also that the distribution $x\cdot\nabla V$ extends to a quadratic form which is form bounded with respect to $(P-A)^2$. Then for all $\varphi\in\mathcal{D}(P-A)$, the limit $\lim_{t\to0}2\,\mathrm{Re}\big(q_{A,V}(\varphi,iD_t\varphi)\big)$ exists. Moreover,*

$$\big\langle\varphi,[H,iD]\,\varphi\big\rangle := \lim_{t\to0}2\,\mathrm{Re}\big(q_{A,V}(\varphi,iD_t\varphi)\big) = 2\|(P-A)\varphi\|_2^2 + 2\ \mathrm{Re}\big\langle\widetilde{B}\varphi,(P-A)\,\varphi\big\rangle - \big\langle\varphi,x\cdot\nabla V\varphi\big\rangle\,. \tag{3.35}$$

Remark 3.10. See Remark 3.8 and the proof of Lemma 3.7 for the precise meaning of the quadratic form $\langle\varphi,x\cdot\nabla V\varphi\rangle$.

Proof. Recall that, as a quadratic form, we defined $\langle\varphi,[H_{A,V},iD_t]\varphi\rangle := 2\,\mathrm{Re}\,q_{A,V}(\varphi,iD_t\varphi)$, using the notation from (2.28). See (3.9) and the discussion before it. We show that the limit exists for all $u\in\mathcal{D}(P-A)$ and is given by the right hand side of (3.35). By (3.11)

$$(P-A)U_t\,u = e^tU_t(P-A)\,u + X_t\,u, \tag{3.36}$$

where

$$X_t\,u = U_t(e^t\,A-A_{-t})\,u\,, \tag{3.37}$$

where we recall $A_{-t}=U_t^*\,A\,U_t=A(e^{-t}\cdot)$. Since

$$2t\,\mathrm{Re}\big(q_{A,V}(\varphi,iD_t\varphi)\big) = \mathrm{Re}\big(q_{A,V}(\varphi,U_t\varphi)-q_{A,V}(U_{-t}\,\varphi,\varphi)\big)\,,$$

and

$$\begin{aligned} q_{A,0}(\varphi,U_t\varphi) &= \langle(P-A)\varphi,U_te^t(P-A)\varphi\rangle + \langle(P-A)\varphi,X_t\varphi\rangle\,,\\ q_{A,0}(U_{-t}\varphi,\varphi) &= \langle(P-A)\varphi,U_te^{-t}(P-A)\varphi\rangle + \langle X_{-t}\varphi,(P-A)\varphi\rangle\,, \end{aligned}$$

we get

$$\begin{aligned} 2\,\mathrm{Re}\,q_{A,0}(\varphi,iD_t\varphi) &= \frac{e^t-e^{-t}}{t}\langle(P-A)\varphi,U_t(P-A)\varphi\rangle + \langle(P-A)\varphi,\frac{1}{t}X_t\varphi\rangle - \langle\frac{1}{t}X_{-t}\varphi,(P-A)\varphi\rangle\\ &\to 2\langle(P-A)\varphi,(P-A)\varphi\rangle + 2\,\mathrm{Re}\langle\widetilde{B}\varphi,(P-A)\varphi\rangle \end{aligned}$$

as $t\to0$, because by Proposition 3.6 we have

$$\lim_{t\to0}\frac{1}{t}X_{\pm t}\,u = \pm\widetilde{B}u \quad\text{in}\quad L^2(\mathbb{R}^d).$$

Lemma 3.7 gives $\lim_{t\to 0}\mathrm{Re}\langle\varphi, ViD_t\varphi\rangle = -\langle\varphi, x\cdot\nabla V\varphi\rangle$ and since

$$q_{A,V}(\varphi, iD_t\varphi) = q_{A,0}(\varphi, iD_t\varphi) + \langle\varphi, ViD_t\varphi\rangle\,,$$

this finishes the proof. ∎

An immediate consequence of our magnetic virial theorem is

Corollary 3.11. *Let the assumptions of the magnetic virial Theorem 3.9 above be satisfied. If $\psi\in\mathcal{D}(P-A)$ is a weak eigenfunction of the magnetic Schrödinger operator $H_{A,V}$ corresponding to the energy $E\in\mathbb{R}$, in the sense that*

$$E\langle\varphi,\psi\rangle = q_{A,V}(\varphi,\psi) \tag{3.38}$$

for all $\varphi\in\mathcal{D}(P-A)$, or all $\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^d)$, then

$$0 = 2E + 2\,\mathrm{Re}\big\langle (P-A)\,\psi, \widetilde{B}\,\psi\big\rangle + d\,\langle\psi, V_1\psi\rangle - 2\,-\big\langle\psi,(2V + x\cdot\nabla V)\psi\big\rangle \tag{3.39}$$

Now, of course, the question is for what class of potentials V one can calculate the virial $x\cdot\nabla V$ in a simple way. If $x\cdot\nabla V$ is given by a function which yields a nice quadratic form, then $\langle\varphi, x\cdot\nabla V\varphi\rangle$ is given by the classical expression. On the other hand, the virial $\langle\varphi, x\cdot\nabla V\varphi\rangle$ exists even if V is not at all classically differentiable. A typical example is given in the next section.

3.3. **The Kato form of the virial.**

Lemma 3.12. *Assume that the magnetic field B satisfies assumptions 2.1 and 2.2, A is the magnetic vector-potential in the Poincaré gauge, and V and $|x|^2V^2$ are relatively form bounded with respect to $(P-A)^2$. Then for all $\varphi\in\mathcal{D}(P-A)$ the virial of V, i.e., the quadratic form corresponding to the distribution $x\cdot\nabla V$, is given by*

$$-\big\langle\varphi, x\cdot\nabla V\varphi\big\rangle = -2\,\mathrm{Im}\big\langle xV\varphi,(P-A)\varphi\big\rangle + d\big\langle\varphi, V\varphi\big\rangle \tag{3.40}$$

for all $\varphi\in\mathcal{D}(P-A)$.

Remark 3.13. We call (3.40) the Kato form of the virial. Kato did not consider magnetic fields and used the pointwise conditions V bounded and $\lim_{x\to\infty}|x|V(x)=0$ to conclude absence of positive eigenvalues for non-magnetic Schrödinger operators. Lemma 3.12 allows us not only to extend this to magnetic Schrödinger operators but to replace Kato's pointwise condition by a rather weak and natural smallness condition on the quadratic form $\langle\varphi,|x|^2V^2\varphi\rangle$ at infinity.

Of course, since the vector potential is in the Poincaré gauge $x\cdot A(x)=0$, so $\big\langle xV\varphi,(P-A)\varphi\big\rangle = \big\langle V\varphi, x\cdot P\varphi\big\rangle$, hence the right hand side of (3.40) does not depend on vector potential A. This form is useful, however, see the proof of Lemma 4.4, in particular, the proof of (4.16).

Proof. By definition, the virial is given by $-\langle\varphi, x\cdot\nabla V\varphi\rangle = \lim_{t\to 0}\mathrm{Re}\langle\varphi, ViD_t\varphi\rangle$. We will calculate this limit slightly differently than in Lemma 3.7. As distributions

$$2itD_t\varphi = \int_{-t}^{t} U_s iD\varphi\,ds = \int_{-t}^{t} U_s ix\cdot P\varphi\,ds + \frac{d}{2}\int_{-t}^{t} U_s\varphi\,ds$$

and

$$\frac{1}{|x|}\int_{-t}^{t} U_s ix\cdot P\varphi\,ds = i\int_{-t}^{t} e^s U_s\big(\tfrac{x}{|x|}\cdot P\varphi\big)\,ds = i\int_{-t}^{t} e^s U_s\big(\tfrac{x}{|x|}\cdot(P-A)\varphi\big)\,ds$$

since any vector potential in the Poincaré gauge is transversal, that is, $x \cdot A(x) = 0$ for all $x \in \mathbb{R}^d$. Altogether, we have

$$iD_t\varphi = \frac{i}{2t}|x| \int_{-t}^{t} e^s U_s\big(\tfrac{x}{|x|} \cdot (P-A)\varphi\big)\, ds \; + \frac{d}{4t}\int_{-t}^{t} U_s\varphi\, ds$$

at least when $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$. Thus, in this case,

$$\langle \varphi, V iD_t\varphi\rangle = i\big\langle |x|V\varphi, \frac{1}{2t}\int_{-t}^{t} e^s U_s\big(\tfrac{x}{|x|}\cdot(P-A)\varphi\big)\, ds\big\rangle + \frac{d}{2}\big\langle V\varphi, \frac{1}{2t}\int_{-t}^{t} U_s\varphi\, ds\big\rangle\,. \tag{3.41}$$

Since $\frac{x}{|x|}\cdot(P-A)\varphi \in L^2(\mathbb{R}^d)$ for all $\varphi \in \mathcal{D}(P-A)$, the maps $s \mapsto U_s(\frac{x}{|x|}(P-A)\varphi)$ and $s \mapsto U_s\varphi$ are continuous. Moreover, the map $s \mapsto U_s\varphi$ is continuous in the graph norm corresponding to $P-A$ for any $\varphi \in \mathcal{D}(P-A)$ by a similar argument as in the proof of Lemma 3.7. Also $|x|V\varphi \in L^2(\mathbb{R}^d)$ for any $\varphi \in \mathcal{D}(P-A)$, since xV is relatively $P-A$ bounded, that is, $|x|^2V^2$ is relatively $(P-A)^2$ form bounded, by assumption. But then (3.41) also extends to all $\varphi \in \mathcal{D}(P-A)$ by continuity.

Since for $\varphi \in \mathcal{D}(P-A)$ the map $s \mapsto U_s$ is continuous in the graph norm of $P-A$, we also have $\frac{1}{2t}\int_{-t}^{t} U_s\varphi\, ds \to \varphi$ in the graph norm. In addition, $\frac{1}{2t}\int_{-t}^{t} e^s U_s\big(\frac{x}{|x|}\cdot(P-A)\varphi\big)\, ds \to \frac{x}{|x|}\cdot(P-A)\varphi$ in $L^2(\mathbb{R}^d)$ as $t \to 0$. Then (3.41) yields

$$\lim_{t\to 0}\langle \varphi, V iD_t\varphi\rangle = i\big\langle |x|V\varphi, \tfrac{x}{|x|}\cdot(P-A)\varphi\big\rangle + \frac{d}{2}\langle V\varphi,\varphi\rangle \; = i\langle xV\varphi, (P-A)\varphi\rangle + \frac{d}{2}\langle V\varphi,\varphi\rangle$$

which, taking real parts, finishes the proof of Lemma 3.12. ∎

Remark 3.14. Slightly informally, an alternatively way to derive (3.40) is as follows: For $u, w \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, which is dense in the domain of $P-A$, the quadratic form $\big\langle u, x\cdot\nabla V w\big\rangle$ is given as a distribution by

$$\begin{aligned}\big\langle u, x\cdot\nabla V w\big\rangle &= \big\langle u, x\cdot\nabla(Vw) - Vx\cdot\nabla w\big\rangle = -\big\langle \nabla\cdot(xu), Vw\big\rangle - \big\langle Vu, x\cdot\nabla w\big\rangle\\ &= -d\big\langle u, Vw\big\rangle - \big\langle Vu, x\cdot\nabla w\big\rangle - \big\langle x\cdot\nabla u, Vw\big\rangle\\ &= -d\big\langle u, Vw\big\rangle - i\big(\big\langle xVu, (P-A)w\big\rangle - \big\langle (P-A)u, xVw\big\rangle\big)\end{aligned} \tag{3.42}$$

since the vector potential A is in the Poincaré gauge and $P = -i\nabla$. Under the conditions on V this extends to all $\varphi \in \mathcal{D}(P-A)$.

Corollary 3.15. *Assume that the magnetic field B satisfies assumptions 2.1 and 2.2, A is the magnetic vector-potential in the Poincaré gauge, and the potential V splits as $V = V_1 + V_2$ where V_1 and $|x|^2V_1^2$ are relatively form bounded with respect to $(P-A)^2$ and the distribution $x\cdot\nabla V_2$ extend to a quadratic form which is form bounded with respect to $(P-A)^2$. Then the virial of V is given by*

$$-\big\langle \varphi, x\cdot\nabla V\varphi\big\rangle = -2\,\mathrm{Im}\,\big\langle xV_1\varphi, (P-A)\varphi\big\rangle + d\big\langle \varphi, V_1\varphi\big\rangle - \big\langle \varphi, x\cdot\nabla V_2\,\varphi\big\rangle \tag{3.43}$$

for all $\varphi \in \mathcal{D}(P-A)$.

Proof. Simply combine Lemma 3.7 and Lemma 3.12. ∎

3.4. **The exponentially weighted magnetic virial.** The proof of our main result, see Theorem 4.8 below, is based on finding two different expressions for the commutator $\langle e^F\psi, [H,D]e^F\psi\rangle$, when F is a suitable weight function and ψ is a weak eigenfunction. This is done in

Lemma 3.16. *Assume that the magnetic field B and the electric potential V satisfy assumptions 2.1, 2.2, and 2.3, and A is the vector potential corresponding to B in the Poincaré gauge. Moreover assume that the distribution $x\cdot\nabla V$ extend to a quadratic form, which is fom bounded with respect to $(P-A)^2$. Let $F : \mathbb{R}^d \to \mathbb{R}$ be a smooth and bounded radial function, such that $\nabla F(x) = g(x)x$, and assume that $g \geq 0$ and that the functions $\nabla(|\nabla F|^2)$, $(1+|\cdot|^2)g$, $x\cdot\nabla g$ and $(x\cdot\nabla)^2 g$ are bounded. Let $\psi \in \mathcal{D}(P-A)$ be a weak* *eigenfunction of the magnetic Schrödinger operator $H_{A,V}$, i.e., $E\langle\varphi,\psi\rangle = q_{A,V}(\varphi,\psi)$ for some $E\in\mathbb{R}$ and all $\varphi\in\mathcal{D}(P-A)$, where $q_{A,V}$ is the sesqui–linear form corresponding to the magnetic Schrödiner operator $H_{A,V}$ and set $\psi_F = e^F\,\psi$. Then*

$$\begin{aligned}\big\langle\psi_F, i\,[H,D]\,\psi_F\big\rangle &= \big\langle\psi_F, \big(E+|\nabla F|^2\big)\,\psi_F\big\rangle + 2\,\mathrm{Re}\,\big\langle(P-A)\,\psi_F, \widetilde{B}\,\psi_F\big\rangle - 2\,\mathrm{Im}\,\big\langle(P-A)\,\psi_F, xV_1\,\psi_F\big\rangle\\ &\quad + \|(P-A)\psi_F\|_2^2 + \big\langle\psi_F,(d\,V_1-V)\psi_F\big\rangle - \big\langle\psi_F, x\cdot\nabla V_2\,\psi_F\big\rangle,\end{aligned}\tag{3.44}$$

and

$$\big\langle\psi_F, i\,[H,D]\,\psi_F\big\rangle = -4\,\|\sqrt{g}\,D\,\psi_F\|_2^2 + \big\langle\psi_F, \big((x\cdot\nabla)^2 g - x\cdot\nabla|\nabla F|^2\big)\psi_F\big\rangle\,.\tag{3.45}$$

Remark 3.17. This is a quadratic form version of the bounds of [13], who considered only the nonmagnetic case. Also note that the conditions in [13] are stronger, since they work with operators and not with forms. To get an idea why the bounds from Lemma 3.16 are useful for excluding eigenfunctions for positive energies $E>0$, think of $\big\langle\psi_F, \big(E+|\nabla F|^2\big)\,\psi_F\big\rangle$, respectively $-4\,\|\sqrt{g}\,D\,\psi_F\|$, as the main terms in (3.44) and (3.45), and the other terms as lower order. Then (3.44) and (3.45) contradict each other when $E>0$ unless $\psi=0$.

Before we prove Lemma 3.16 we first collect some auxiliary results, to simplify the calculations. First note that as distributions,

$$(P-A)\psi_F = e^F(P-A)\psi - ie^F\nabla F\psi\,.\tag{3.46}$$

Hence since F and ∇F are bounded we have $\psi_F\in\mathcal{D}(P-A)$ for any $\psi\in\mathcal{D}(P-A)$, so $\big\langle\psi_F, i\,[H,D]\,\psi_F\big\rangle$ is well-defined.

Secondly, note that the operators $\nabla F\cdot P$ and $P\cdot\nabla F$ are well defined on $\mathcal{D}(P-A)$. Indeed, since F is radial we have $\nabla F = gx$ for some function g depending only on $|x|$. This implies $\nabla F\cdot A = 0$, see also (2.14). Hence, as distributions,

$$\nabla F\cdot Pu = gx\cdot Pu = gx\cdot(P-A)u \in L^2(\mathbb{R}^d)\tag{3.47}$$

for all $u\in\mathcal{D}(P-A)$. Similarly,

$$\begin{aligned}P\cdot\nabla F\,u &= P\cdot(gx)u = gP\cdot xu - i(x\cdot\nabla g)u = gx\cdot(P-A)u - igdu - i(x\cdot\nabla g)u\in L^2(\mathbb{R}^d)\,,\\ \langle x\rangle^{-1}D\,u &= \frac{1}{2\langle x\rangle}\big(x\cdot P + P\cdot x\big)u = \frac{x}{\langle x\rangle}\cdot P\,u - \frac{i}{2\langle x\rangle}u = \langle x\rangle^{-1}x\cdot(P-A)\,u - \frac{i}{2\langle x\rangle}u\in L^2(\mathbb{R}^d),\\ gD\,u &= \frac{g}{2}\big(x\cdot P + P\cdot x\big)u = gx\cdot P\,u - \frac{ig}{2}u = gx\cdot(P-A)\,u - \frac{ig}{2}u\in L^2(\mathbb{R}^d),\\ \sqrt{g}D\,u &= \sqrt{g}x\cdot(P-A)\,u - \frac{i\sqrt{g}}{2}u\in L^2(\mathbb{R}^d),\\ \langle x\rangle gD\,u &= \langle x\rangle gx\cdot(P-A)\,u - \frac{i\langle x\rangle g}{2}u\in L^2(\mathbb{R}^d),\end{aligned}\tag{3.48}$$

and

$$D_{\nabla F}\,u \coloneqq \frac{1}{2}\big(\nabla F\cdot P + P\cdot\nabla F\big)u = gDu - \frac{i}{2}(x\cdot\nabla g)u\in L^2(\mathbb{R}^d)\tag{3.49}$$

for all $u\in\mathcal{D}(P-A)$, by the assumptions on g. Note also that $D_{\nabla F}$ is symmetric.

The next result is needed also later, so we single it out.

Lemma 3.18. *Under the conditions of Lemma 3.16 we have*

$$q_{A,V}(u,v) = q_{A,V}(e^{-F}u, e^F v) + 2i\langle D_{\nabla F}\,u, v\rangle + \big\langle\nabla F\,u, \nabla F\,v\big\rangle\tag{3.50}$$

for all $u,v\in\mathcal{D}(P-A)$. In particular, if ψ is a weak eigenfunction corresponding to the energy E of the magnetic Schrödinger operator $H_{A,V}$, then

$$q_{A,V}(\psi_F,\psi_F) = \big\langle\psi_F, (E+|\nabla F|^2)\psi_F\big\rangle\tag{3.51}$$

Proof. A straightforward calculation using the above equations and (3.46) yields

$$\begin{aligned} q_{A,0}(e^{-F}u, e^{F}v) &= \big\langle (P-A+i\nabla F)u, (P-A-i\nabla F)v\big\rangle \\ &= q_{A,0}(u,v) - i\big(\big\langle \nabla F u, (P-A)v\big\rangle + \big\langle (P-A)u, \nabla F v\big\rangle\big) - \big\langle \nabla F u, \nabla F v\big\rangle \\ &= q_{A,0}(u,v) - 2i\big\langle D_{\nabla F}\, u, v\big\rangle - \big\langle \nabla F u, \nabla F v\big\rangle\,. \end{aligned} \tag{3.52}$$

In particular, since $\big\langle e^{-F}\, u, V e^{F}\, v\big\rangle = \big\langle u, Vv\big\rangle$ and $q_{A,0}(u,v) = q_{A,0}(u,v) + \big\langle u, Vv\big\rangle$ this gives (3.50).

If ψ is a weak eigenfunction of $H_{A,V}$ then $q_{A,V}(\psi, v) = E\big\langle \psi, v\big\rangle$ for all $v \in \mathcal{D}(P-A)$. Since $D_{\nabla F}$ is symmetric, $\big\langle D_{\nabla F}\psi_F, \psi_F\big\rangle$ s real and (3.50) implies

$$\begin{aligned} q_{A,V}(\psi_F,\psi_F) &= \mathrm{Re}\, q_{A,V}(\psi_F,\psi_F) = \mathrm{Re}\, q_{A,V}(\psi, e^{F}\psi_F) + \mathrm{Re}\,\big\langle \nabla F\psi_F, \nabla F\psi_F\big\rangle \\ &= \mathrm{Re}\, E\big\langle \psi, e^{F}\psi_F\big\rangle + \mathrm{Re}\,\big\langle \nabla F\psi_F, \nabla F\psi_F\big\rangle = \mathrm{Re}\,\big\langle \psi_F, (E+|\nabla F|^2)\psi_F\big\rangle \end{aligned}$$

∎

Proof of Lemma 3.16. From (3.46) we know that $\psi_F \in \mathcal{D}(P-A) = \mathcal{Q}(H_{A,V})$. Thus for any $\psi \in \mathcal{Q}(H_{A,V})$ our magnetic virial Theorem 3.9 shows

$$\big\langle \psi_F, i\,[H,D]\,\psi_F\big\rangle = 2q_{A,0}(\psi_F,\psi_F) + 2\ \mathrm{Re}\,\big\langle \widetilde{B}\psi_F, (P-A)\,u\big\rangle - \big\langle \psi_F, x\cdot\nabla V\psi_F\big\rangle\,.$$

with $q_{A,0}(\psi_F,\psi_F) = \big\langle (P-A)\psi_F, (P-A)\psi_F\big\rangle$. If ψ is a weak eigenfunction of $H_{A,V}$ with energy E, then

$$\begin{aligned} \big\langle \psi_F, i\,[H,D]\,\psi_F\big\rangle &= 2q_{A,V}(\psi_F,\psi_F) - 2\big\langle \psi_F, V\psi_F\big\rangle + 2\ \mathrm{Re}\,\big\langle \widetilde{B}\psi_F i_F, (P-A)\,u\big\rangle - \big\langle \psi_F, x\cdot\nabla V\psi_F\big\rangle \\ &= 2\big\langle \psi_F, (E+|\nabla F|^2)\psi_F\big) - 2\big\langle \psi_F, V\psi_F\big\rangle + 2\ \mathrm{Re}\,\big\langle \widetilde{B}\psi_F i_F, (P-A)\,u\big\rangle - \big\langle \psi_F, x\cdot\nabla V\psi_F\big\rangle \end{aligned}$$

by (3.51). This proves the first claim of Lemma 3.16.

Applying (3.50) with $u = \psi_F$ and $v = iD_t\psi_F$ one sees

$$\begin{aligned} q(\psi_F, iD_t\psi_F) &= q(\psi, e^{F} iD_t\psi_F) + 2i\langle D_{\nabla F}\psi_F, iD_t\psi_F\rangle + \big\langle \nabla F\,\psi_F, \nabla F\, iD_t\psi_F\big\rangle \\ &= E\,\big\langle \psi_F,\, iD_t\psi\big\rangle - 2\langle D_{\nabla F}\psi_F, D_t\psi_F\rangle + \big\langle\, \psi_F, |\nabla F|^2 iD_t\psi_F\big\rangle\,, \end{aligned}$$

where we again used $q_{A,V}(\psi,v) = E\langle\psi,v\rangle$ for all $v\in\mathcal{D}(P-A)$ and any weak eigenfunction ψ with energy E. Notice that $\langle \psi_F,\, iD_t\psi_F\rangle = i\langle \psi_F, D_t\psi_F\rangle$ is purely imaginary since D_t is symmetric, so taking the real part above shows

$$2\,\mathrm{Re}\, q(\psi_F, iD_t\psi_F) = -4\,\mathrm{Re}\langle D_{\nabla F}\,\psi_F, D_t\psi_F\rangle + 2\,\mathrm{Re}\langle \psi_F, |\nabla F|^2 iD_t\psi_F\rangle. \tag{3.53}$$

Lemma 3.7 gives $2\,\mathrm{Re}\langle \psi_F, |\nabla F|^2 iD_t\psi_F\rangle \to -\langle \psi_F, x\cdot\nabla(|\nabla F|^2)\psi_F\rangle$ as $t\to 0$. Hence (3.53) implies (3.45) as long as

$$\lim_{t\to 0} \mathrm{Re}\langle D_{\nabla F}\,\psi_F, D_t\psi_F\rangle = \|\sqrt{g}D\psi_F\|_2^2 - \frac{1}{4}\langle \psi_F, ((x\cdot\nabla)^2 g)\psi_F\rangle. \tag{3.54}$$

Using $D_{\nabla F}u = gDu - \frac{i}{2}(x\cdot\nabla g)u$ for all $u\in\mathcal{D}(P-A)$, we get

$$\langle D_{\nabla F}\,u, D_t u\rangle = \big\langle gD\,u, D_t u\big\rangle + \frac{1}{2}\big\langle (x\cdot\nabla g)u, iD_t u\big\rangle$$

and we already know from Lemma 3.7 that $\frac{1}{2}\,\mathrm{Re}\,\big\langle (x\cdot\nabla g)u, iD_t u\big\rangle \to -\frac{1}{4}\big\langle u, ((x\cdot\nabla)^2 g)u\big\rangle$ as $t\to 0$. Moreover,

$$\langle x\rangle^{-1} D_t u = \frac{1}{2t}\int_{-t}^{t} \langle x\rangle^{-1} U_s\big(Du\big)\,ds = \frac{1}{2t}\int_{-t}^{t} \frac{\langle e^s x\rangle}{\langle x\rangle} U_s\big(\langle x\rangle^{-1} Du\big)\,ds$$

initially for $u\in\mathcal{C}_0^\infty(\mathbb{R}^d)$, but by density and since $\langle x\rangle^{-1}D : \mathcal{D}(P-A)\to L^2(\mathbb{R}^d)$ is bounded, this extends to all $u\in\mathcal{D}(P-A)$. Thus, by continuity, $\langle x\rangle^{-1}D_t u \to \langle x\rangle^{-1}Du$ in $L^2(\mathbb{R}^d)$ as $t\to 0$ and

$$\langle gD\,u, D_t u\rangle = \langle \langle x\rangle gD\,u, \langle x\rangle^{-1}D_t u\rangle \to \langle \langle x\rangle gD\,u, \langle x\rangle^{-1}Du\rangle = \|\sqrt{g}Du\|_2^2$$

as $t\to 0$ for all $u\in\mathcal{D}(P-A)$. This completes the proof of (3.54) and of the Lemma. ∎

For a type of unique continuation at infinity argument, we will also need the follwing

Lemma 3.19. *Let B and V satisfy assumptions 2.1, 2.3, and 2.7. Assume that ψ and F satisfy conditions of Lemma 3.16. Then there exists $\kappa>0$ and $c_\kappa>0$ such that*

$$\big\langle \psi_F, i\,[H,D]\,\psi_F\big\rangle \,\geq\, \kappa\,\big\langle \psi_F, |\nabla F|^2\,\psi_F\big\rangle - c_\kappa \|\psi_F\|_2^2\,. \tag{3.55}$$

Proof. In what follows the value of a constant c might change from line to line. Since $\psi_F\in\mathcal{H}^1(\mathbb{R}^d)$, Lemma 3.16, the Cauchy-Schwarz inequality and assumption 2.4give

$$\begin{aligned}\big\langle \psi_F, i\,[H,D]\,\psi_F\big\rangle &\geq 2\|(P-A)\psi_F\|_2^2 - 2\|(P-A)\psi_F\|_2\big(\|\widetilde{B}\psi_F\|_2+\|xV_1\psi_F\|_2\big)\\ &\quad-(\alpha_2+d\,\alpha_3)\|(P-A)\psi_F\|_2^2 - c\|\psi_F\|_2^2\,.\end{aligned}$$

Therefore using (3.46) and assumption 2.3 we find that for any $\kappa>0$

$$\begin{aligned}\big\langle \psi_F, i\,[H,D]\,\psi_F\big\rangle &\geq (2-\kappa)\|(P-A)\psi_F\|_2^2+\kappa\,\big\langle \psi_F, |\nabla F|^2\,\psi_F\big\rangle-(\alpha_2+\kappa\alpha_0)\,\|(P-A)\psi_F\|_2^2\\ &\quad-2\|(P-A)\psi_F\|_2\big(\|\widetilde{B}\psi_F\|_2+\|xV_1\psi_F\|_2\big)-c\|\psi_F\|_2^2.\end{aligned}$$

On the other hand assumption 2.4implies that

$$\begin{aligned}2\|(P-A)\psi_F\|_2\big(\|\widetilde{B}\psi_F\|_2+\|xV_1\psi_F\|_2\big) \,&\leq\, \alpha_1\|(P-A)\psi_F\|_2^2+2\,c_1\|(P-A)\psi_F\|_2\,\|\psi_F\|_2\\ &\leq\,(\alpha_1+\kappa)\,\|(P-A)\psi_F\|_2^2+\frac{c_1}{\kappa}\,\|\psi_F\|_2^2.\end{aligned}$$

Hence

$$\begin{aligned}\big\langle \psi_F, i\,[H,D]\,\psi_F\big\rangle &\geq (2-2\kappa-\kappa\alpha_0-\alpha_1-\alpha_2-d\,\alpha_3)\,\|(P-A)\psi_F\|_2^2+\kappa\,\big\langle \psi_F, |\nabla F|^2\,\psi_F\big\rangle\\ &\quad-(c+\kappa^{-1}c_1)\,\|\psi_F\|_2^2,\end{aligned}$$

and the result follows upon setting

$$\kappa=\frac{2-\alpha_1-\alpha_2-d\,\alpha_3}{2+\alpha_0}>0,$$

see (2.25). ∎

4. Absence of positive eigenvalues

We will give the prove of absence of positive eigenvalues in two steps. The first is that putative eigenfunctions corresponding to positive energies have to decay faster than exponentially. In a second step, we prove that any such eigenfunction has to be zero.

4.1. Ridiculously fast decay.

We set $\langle x\rangle_\lambda := \sqrt{\lambda+|x|^2}$ for $x\in\mathbb{R}^d$, $\lambda>0$. For $\lambda=1$, we write simply $\langle x\rangle_1=\langle x\rangle$. We have

Proposition 4.1 (Fast decay)**.** *Assume that B and V satisfy assumptions 2.1- 2.6 and that the magnetic field A corresponding to B is in the Poincaré gauge. Furthermore, assume that ψ is a weak eigenfunction of the magnetic Schrödinger operator $H_{A,V}$ corresponding to the energy $E\in\mathbb{R}$, and that there exist $\overline{\mu}\geq 0$ and $\lambda>0$ such that $x\mapsto e^{\overline{\mu}\,\langle x\rangle_\lambda}\,\psi(x)\in L^2(\mathbb{R}^d)$. If $E+\overline{\mu}^2>\Lambda$ with Λ given by (1.13), then*

$$x\mapsto e^{\mu\langle x\rangle_\lambda}\,\psi(x)\in L^2(\mathbb{R}^d)\qquad \forall\,\mu>0,\quad \forall\,\lambda>0. \tag{4.1}$$

Before we start with the proof, we make some preparations. Obviously it suffices to prove the statement for $\lambda=1$. We will first consider the case $\overline{\mu}=0$, i.e., we only know that $\psi\in\mathcal{D}(P-A)\subset L^2(\mathbb{R}^d)$. The choice

$$F_{\mu,\varepsilon}(x)=\frac{\mu}{\varepsilon}\left(1-e^{-\varepsilon\,\langle x\rangle}\right)\,, \tag{4.2}$$

for the weight function, for some $\mu \ge 0$ and $\varepsilon > 0$, will be convenient. We have $F_{\mu,\varepsilon}(x) \to \mu\langle x\rangle$ as $\varepsilon \to 0$. Also, since

$$\nabla F_{\mu,\varepsilon} = \mu\langle x\rangle^{-1} e^{-\varepsilon\langle x\rangle} x \tag{4.3}$$

we have

$$g_{\mu,\varepsilon}(x) = \mu\langle x\rangle^{-1} e^{-\varepsilon\langle x\rangle}\,. \tag{4.4}$$

Moreover, let

$$\mu_* = \sup\left\{\mu \ge 0 \,:\, e^{\mu\langle x\rangle}\psi \in L^2(\mathbb{R}^d)\right\},$$

the maximal exponential decay rate of the weak eigenfunction ψ. The bound (4.1) is equivalent to $\mu_* = \infty$, so we have to exclude $0 \le \mu_* < \infty$. If $0 \le \mu_* < \infty$, then there exist sequences $\mu_n \searrow \mu_*$, $\varepsilon_n \searrow 0$ as $n \to \infty$, i.e., both sequences are decreasing and $\mu_n \to \mu_*$, $\varepsilon_n \to 0$, as $n \to \infty$, with

$$a_n := \|e^{F_n}\,\psi\|_2 \;\to\; \infty \quad \text{as} \quad n \to \infty, \tag{4.5}$$

where we put $F_n := F_{\mu_n,\varepsilon_n}$. Moreover, we let $g_n(x) := g_{\mu_n,\varepsilon_n}$ and define φ_n by

$$\varphi_n = \frac{e^{F_n}\,\psi}{\|e^{F_n}\,\psi\|}\,. \tag{4.6}$$

Since

$$F_n(x) \;\le\; \mu_n\langle x\rangle\,, \tag{4.7}$$

the function e^{F_n} is bounded uniformly in $n \in \mathbb{N}$ on compact subsets of $\mathbb{R}^d$. So for each compact set $K \subset \mathbb{R}^d$ we have

$$\left\langle \varphi_n, \mathbb{1}_K\,\varphi_n\right\rangle \to 0 \quad \text{as } n \to \infty\,. \tag{4.8}$$

This also implies that for any bounded function W with $W(x) \to 0$ as $x \to \infty$ one has

$$\left\langle \varphi_n, W\varphi_n\right\rangle \to 0 \quad \text{as } n \to \infty. \tag{4.9}$$

In order to use Lemma 3.16 to derive a contradiction, the following is useful.

Lemma 4.2. *Let F_n, g_n, ψ, and φ_n be given as above. If $0 < \mu_* < \infty$, then*

$$\lim_{n\to\infty}\langle e^{F_n}\psi, \varepsilon_n\langle x\rangle e^{F_n}\psi\rangle = 0\,. \tag{4.10}$$

Moreover, if $0 \le \mu_* < \infty$, then

$$\lim_{n\to\infty}\left\langle \nabla F_n\varphi_n, \nabla F_n\varphi_n\right\rangle = \mu_*^2 \tag{4.11}$$

and

$$\lim_{n\to\infty}\left\langle \varphi_n, x, \left((x\cdot\nabla)^2 g_n - x\cdot\nabla|\nabla F|^2\right)\varphi_n\right\rangle = 0 \tag{4.12}$$

Remark 4.3. If $\mu_* > 0$, then ψ decays exponentially and since F_n is bounded for fixed $n \in \mathbb{N}$ we have $\langle e^{F_n}\psi, \langle x\rangle e^{F_n}\psi\rangle < \infty$ for all n.

Lemma 4.4. *Let $0 \le \mu_* < \infty$ and F_n, g_n, and φ_n be given as above. If the potential V is relative form small and vanishes at infinity w.r.t $(P-A)^2$, i.e satisfies assumptions 2.3 and 2.5, then*

$$\lim_{n\to\infty}\langle \varphi_n, V\varphi_n\rangle = 0 \tag{4.13}$$

$$\lim_{n\to\infty}\langle (P-A)\varphi_n, (P-A)\varphi_n\rangle = E + \mu_*^2\,. \tag{4.14}$$

Moreover, if the magnetic field B satisfy assumptions 2.2, and 2.6, then

$$\limsup_{n\to\infty}|\langle \widetilde{B}\,\varphi_n, (P-A)\,\varphi_n\rangle| \le \beta(E+\mu_*^2)^{1/2}\,. \tag{4.15}$$

and if one splits $V = V_1 + V_2$, with V_1 and V_2 satisfying assumptions 2.4 and 2.6 then

$$\limsup_{n\to\infty} \langle \varphi_n, x\cdot\nabla V\varphi_n\rangle \le 2\omega_1(E+\mu_*^2)^{1/2} + \omega_2 \tag{4.16}$$

Remark 4.5. For the proof of similar results in [13], the assumption that V and $x\cdot\nabla V$ are relatively form compact with respect to P^2 is made. Thus they only deal with potentials which are relatively form bounded with relative bound zero. They also do not consider conditions on the Kato form of the virial $x\cdot\nabla V$.

We will prove these two Lemmata later in this section.

Proof of Proposition 4.1. Assume that $0 \le \mu_*^2 < \infty$. It is easy to check that F_n and g_n satisfy the assumptions of the exponentially weighted magnetic virial Lemma 3.16. Thus Lemma 3.16 and Lemma 4.2 show

$$\limsup_{n\to\infty} \langle \varphi_n, i\,[H,D]\,\varphi_n\rangle \le 0\,. \tag{4.17}$$

On the other hand the first equality from Lemma 3.16 together with Lemma 4.4 shows

$$\begin{aligned}\liminf_{n\to\infty} \langle \varphi_n, i\,[H,D]\,\varphi_n\rangle &\ge 2(E+\mu_*^2) - 2(\beta+\omega_1)(E+\mu_*^2)^{1/2} - \omega_2 \\ &= 2\left[\left(\sqrt{E+\mu_*^2} - \frac{\beta+\omega_1}{2}\right)^2 - \left(\frac{\beta+\omega_1}{2}\right)^2 - \frac{\omega_2}{2}\right] > 0\end{aligned} \tag{4.18}$$

if $\sqrt{E+\mu_*^2} > \frac{1}{2}\big(\beta+\omega_1+\sqrt{(\beta+\omega_1)^2+2\omega_2}\big) = \sqrt{\Lambda}$. Clearly, (4.17) and (4.18) contradict each other. Thus $\mu_* = \infty$, which is equivalent to (4.1). ∎

It remains to prove Lemma 4.2 and 4.4.

Proof of Lemma 4.2. Clearly, for any $\delta > 0$

$$\langle \varphi_n, \varepsilon_n\langle x\rangle\varphi_n\rangle = \langle \varphi_n, \mathbb{1}_{\{\varepsilon_n\langle x\rangle<\delta\}}\,\varphi_n\rangle + \langle \varphi_n, \mathbb{1}_{\{\varepsilon_n\langle x\rangle\ge\delta\}}\,\varepsilon_n\langle x\rangle\varphi_n\rangle \le \delta + \langle \varphi_n, \mathbb{1}_{\{\varepsilon_n\langle x\rangle>\delta\}}\,\varepsilon_n\langle x\rangle\varphi_n\rangle$$

One easily checks that $0 \le t \mapsto \frac{1-e^{-t}}{t}$ is decreasing. Thus

$$\gamma_\delta \coloneqq \sup_{t\ge\delta}\frac{1-e^{-t}}{t} = \frac{1-e^{-\delta}}{\delta} < 1 \tag{4.19}$$

which shows

$$F_n = \frac{\mu_n\langle x\rangle}{\varepsilon_n\langle x\rangle}(1-e^{-\varepsilon\langle x\rangle}) \le \mu_n\gamma_\delta\langle x\rangle \quad \text{for all } x \text{ with } \varepsilon_n\langle x\rangle \ge \delta\,.$$

Choose any $\gamma_\delta < \kappa < 1$. If $0 < \mu_* < \infty$ then ψ decays exponentially with rate $\kappa\mu_* < \mu_*$, by the definition of μ_*. Thus

$$\limsup_{n\to\infty} \langle e^{F_n}\psi, \mathbb{1}_{\{\varepsilon_n\langle x\rangle>\delta\}}\,\langle x\rangle e^{F_n}\psi\rangle \le \limsup_{n\to\infty} \langle e^{\mu_n\gamma_\delta\langle x\rangle}\psi, \langle x\rangle e^{\mu_n\gamma_\delta\langle x\rangle}\psi\rangle < \infty$$

since, $\mu_n\gamma_n \to \gamma_\delta\mu_* < \kappa\mu_*$ as $n\to\infty$. This implies (4.10).

For the proof of the remaining part of Lemma 4.2, we note that from (4.3) one gets

$$|\nabla F_n|^2 = \mu_n^2(1-\langle x\rangle^{-2})e^{-2\varepsilon_n\langle x\rangle}\,. \tag{4.20}$$

Since φ_n is normalized this gives

$$\begin{aligned}\mu_n^2 - \langle \nabla F_n\varphi_n, \nabla F_n\varphi_n\rangle &= \langle \varphi_n, (\mu_n^2 - |\nabla F_n|^2)\varphi_n\rangle \\ &= \mu_n^2\left(\langle \varphi_n, (1-e^{-2\varepsilon_n\langle x\rangle})\varphi_n\rangle + \langle \varphi_n, \langle x\rangle^{-2}e^{-2\varepsilon_n\langle x\rangle}\varphi_n\rangle\right)\end{aligned} \tag{4.21}$$

Recall that $\mu_n \searrow \mu_*$. If $\mu_* = 0$, then (4.21) shows

$$\left|\mu_n^2 - \langle \nabla F_n\varphi_n, \nabla F_n\varphi_n\rangle\right| \le 2\mu_n^2 \to 0 \quad \text{as } n \to \infty\,.$$

If $0 < \mu_* < \infty$, then using $0 \le 1 - e^{-2\varepsilon_n\langle x\rangle} \le 2\varepsilon_n\langle x\rangle$ in (4.21) gives

$$\left|\mu_n^2 - \langle \nabla F_n\varphi_n, \nabla F_n\varphi_n\rangle\right| \le \mu_n^2\left(2\langle\varphi_n, \varepsilon_n\langle x\rangle\varphi_n\rangle + \langle\varphi_n, \langle x\rangle^{-2}\varphi_n\rangle\right) \to 0 \quad \text{as } n \to \infty$$

due to (4.10) and (4.9). This proves (4.11).

Using the definitions of F_n and g_n a relatively short calculation shows

$$\left|(x\cdot\nabla)^2 g_n - x\cdot\nabla|\nabla F|^2\right| \lesssim \mu_n(\mu_n+1)\left[\langle x\rangle^{-2} + \langle x\rangle^{-1} + \varepsilon_n\langle x\rangle + \varepsilon_n^2\langle x\rangle\right] e^{-\varepsilon_n\langle x\rangle} \tag{4.22}$$

Since $0 \le t \mapsto te^{-t}$ is bounded, (4.22) implies, if $\mu_* = 0$,

$$\left|\langle\varphi_n, \left((x\cdot\nabla)^2 g_n - x\cdot\nabla|\nabla F|^2\right)\varphi_n\rangle\right| \lesssim \mu_n(\mu_n+1) \to 0 \quad \text{as } n \to \infty$$

If $0 < \mu_* < \infty$, then (4.22) shows

$$\left|\langle\varphi_n, \left((x\cdot\nabla)^2 g_n - x\cdot\nabla|\nabla F|^2\right)\varphi_n\rangle\right| \lesssim \left\langle\varphi_n, \left(\langle x\rangle^{-2} + \langle x\rangle^{-1}\right)\varphi_n\right\rangle + \langle\varphi_n, \varepsilon_n\langle x\rangle\varphi_n\rangle \to 0 \quad \text{as } n \to \infty$$

using again (4.10) and (4.9). This proves (4.12). ∎

In the proof of Lemma 4.4 we need the following auxiliary tool.

Lemma 4.6. *Assume that the potential V is relatively form bounded w.r.t $(P-A)^2$. Then for any family of real-valued bounded function $\xi_j \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $j \in I$, for which $\sup_{j\in I}\|\xi_j\|_\infty$ and $\sup_{j\in I}\|\nabla\xi_j\|_\infty$ are finite, we have*

$$\sup_{j\in I}\sup_{n\in\mathbb{N}}\|(P-A)\xi_j\varphi_n\| < \infty\,. \tag{4.23}$$

where φ_n is the sequence defined in (4.6). Moreover, if $\xi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ is a real-valued function with compact support, then

$$\limsup_{n\to\infty}\|(P-A)\xi\varphi_n\| = 0\,. \tag{4.24}$$

We give the proof of this Lemma after the

Proof of Lemma 4.4. One easily checks that if ξ is an infinitely often differentiable cut–off function with bounded derivative, then $\xi\varphi \in \mathcal{D}(P-A)$ for any $\varphi \in \mathcal{D}(P-A)$.

Let $\chi_l : [0,\infty) \to \mathbb{R}_+$, $l = 1,2$, be infinitely often differentiable on $(0,\infty)$ with $\chi_1(r) = 1$ for $0 \le r \le 1$, $\chi_1(r) > 0$ for $r \le 3/2$, $\chi_1(r) = 0$ for $r \ge 7/4$, and $\chi_2(r) = 0$ for $r \le 5/4$, $\chi_2(r) > 0$ for $r \ge 3/2$, $\chi_2(r) = 1$ for $r \ge 2$. Then $\inf_{r\ge 0}(\chi_1^2(r) + \chi_2^2(r)) > 0$ and thus

$$\xi_1 := \frac{\chi_1}{\sqrt{\chi_1^2+\chi_2^2}}\,, \quad \xi_2 := \frac{\chi_2}{\sqrt{\chi_1^2+\chi_2^2}}$$

are infinitely often differentiable with bounded derivatives and $\xi_1^2 + \xi_2^2 = 1$. Given $R \ge 1$ we set

$$\xi_{<R}(x) := \xi_1(|x|/R), \quad \xi_{\ge R}(x) := \xi_2(|x|/R)$$

which yields a family of infinitely often differentiable real-valued localization functions on $\mathbb{R}^d$ with bounded derivatives. Note that $\xi_{<R}$ has compact support and $\mathrm{supp}(\xi_{\ge R}) \subset \mathcal{U}_j^c = \{x \in \mathbb{R}^d : |x| \ge R\}$

By construction, we have

$$\langle\varphi_n, V\varphi_n\rangle = \langle\xi_{<R}^2\varphi_n, V\varphi_n\rangle + \langle\xi_{\ge R}^2\varphi_n, V\varphi_n\rangle$$

and, recalling that V is form bounded with respect to $(P-A)^2$, we have for fixed $R \ge 1$

$$|\langle\xi_{<R}^2\varphi_n, V\varphi\rangle| = |\langle\xi_{<R}\varphi_n, V\xi_{<R}\varphi_n\rangle| \lesssim \|(P-A)\xi_{<R}\varphi_n\|_2^2 + \|\xi_{<R}\varphi_n\|_2^2 \to 0\,, \text{ as } n \to \infty$$

by Lemma 4.6 and (4.9), since $\xi_{<R}$ has compact support.

Moreover, since V vanishes at infinity w.r.t. $(P-A)^2$, there exist α_R, γ_R with $\varepsilon_R, \beta_R \to 0$ as $R \to \infty$ such that

$$|\langle \xi_{\geq R}^2 \varphi_n, V\varphi_n \rangle| = |\langle \xi_{\geq R}\varphi_n, V\xi_{\geq R}\varphi_n \rangle| \leq \alpha_R \|(P-A)\xi_{\geq R}\varphi_n\|_2^2 + \gamma_R \|\xi_{\geq R}\varphi_n\|_2^2 .$$

Lemma 4.6 then shows

$$\limsup_{n\to\infty} |\langle \xi_{\geq R}^2 \varphi_n, V\varphi_n \rangle| \lesssim \alpha_R + \gamma_R \to 0 , \text{ as } R \to \infty ,$$

which proves (4.13).

Moreover, from Lemma 3.18, we get

$$\begin{aligned} \langle (P-A)\varphi_n, (P-A)\varphi_n \rangle &= E + \langle \nabla F_n \varphi_n, \nabla F_n \varphi_n \rangle - \langle \varphi_n, V\varphi_n \rangle \\ &\to E + \mu_*^2 \quad \text{as } n \to \infty \end{aligned}$$

using also (4.13) and (4.10). This proves (4.14).

For $\widetilde{B}^2$ one can argue exactly the same way as above for V to see that for fixed $j \in \mathbb{N}$

$$\limsup_{n\to\infty} \langle \varphi_n, |\widetilde{B}|^2 \varphi_n \rangle \leq \limsup_{n\to\infty} \langle \xi_{2,j}\varphi_n |\widetilde{B}|^2 \xi_{2,j}\varphi_n \rangle \leq C\varepsilon_j + \beta_j^2$$

where we also used assumption 2.6 and put $C = \sup_{j\in\mathbb{N}} \limsup_{n\to\infty} \|(P-A)\xi_j\varphi_n\|_2^2$, which due to Lemma 4.6 is finite. Since $\varepsilon_j \to 0$ and $\beta_j \to \beta$, as $n \to \infty$, we get

$$\limsup_{n\to\infty} \|\widetilde{B}\varphi_n\| \leq \beta ,$$

Because of $|\langle \widetilde{B}\,\varphi_n, (P-A)\,\varphi_n \rangle| \leq \|\widetilde{B}\,\varphi_n\| \|(P-A)\,\varphi_n\|$ and (4.14) this proves (4.15).

If the potential splits as $V = V_1 + V_2$ with V_1, V_2 satisfying assumptions 2.4 and 2.6, then one can argue exactly as above to see

$$\limsup_{n\to\infty} |\langle xV_1\varphi_n, (P-A)\varphi_n \rangle| \leq \ \omega_1$$

and

$$\limsup_{n\to\infty} |\langle \varphi_n, x\cdot\nabla V_2\varphi_n \rangle| \leq \ \omega_2 .$$

Moreover, if V_1 and $(xV_1)^2$ are form bounded w.r.t. $(P-A)^2$ and $\varphi \in \mathcal{D}(P-A)$ with $\operatorname{supp}(\varphi) \subset \{|x| \geq R\}$, then

$$|\langle \varphi, V_1\,\varphi \rangle| = |\langle |x|^{-1}\varphi, |x|V_1\varphi \rangle| \leq \||x|^{-1}\varphi\| \||x|V_1\varphi\| \lesssim R^{-1}\|\varphi\| \left(\|(P-A)\varphi\|_2^2 + \|\varphi\|_2^2\right)^{1/2} ,$$

so V_1 vanishes at infinity w.r.t. $(P-A)^2$. Thus $\lim_{n\to\infty} \langle \varphi_n, V_1\,\varphi_n \rangle = 0$ and using the mixed form of the virial from Corollary 3.15 yields

$$\limsup_{n\to\infty} \langle \varphi, x\cdot\nabla V\varphi \rangle \leq 2\omega_1(E+\mu_*^2)^{1/2} + \omega_2 . \qquad \blacksquare$$

Remark 4.7. Note that $\Lambda < \beta + \omega$ as soon as $\omega > 0$.

Now we give the

Proof of Lemma 4.6. Let $\psi \in \mathcal{D}(P-A)$ be a weak eigenfunction of the magnetic Schrödinger operator $H_{A,V}$ with eigenvalue E and F_n, $\psi_n = e^{F_n}\psi$ and $\varphi_n = \psi_n/\|\psi_n\|$ as in (4.6). In particular, we have $\sup_n \|\nabla F_n\| \leq \sup_n \mu_n < \infty$. Since V is relatively form bounded with respect to $(P-A)^2$

$$\|(P-A)\varphi\|_2^2 = q_{A,V}(\varphi,\varphi) - \langle \varphi, V\varphi \rangle \leq q_{A,V}(\varphi,\varphi) + \alpha_0\|(P-A)\varphi\|_2^2 + C\|\varphi\|_2^2$$

for some $0 \le \alpha_0 < 1$, $C > 0$, and all $\varphi \in \mathcal{D}(P-A)$. Thus

$$\|(P-A)\varphi\|_2^2 \le (1-\alpha_0)^{-1} \left(q_{A,V}(\varphi,\varphi) + C\|\varphi\|_2^2\right)$$

From the IMS localization formula (C.1) we get

$$\begin{aligned} q_{A,V}(\xi\psi_n, \xi\psi_n) &= \operatorname{Re} q_{A,V}(\xi^2 e^{2F_n}\psi, \psi) + \langle \psi, |\nabla(\xi e^{F_n})|^2\psi\rangle \\ &\le E\|\xi\psi_n\|_2^2 + 2\|(\nabla\xi)\psi_n\|_2^2 + 2\|(\nabla F_n)\xi\psi_n\|_2^2 \end{aligned}$$

since ψ is a weak eigenfunction with energy E. Thus

$$\|(P-A)\xi_j\varphi_n\|_2^2 \lesssim \|\xi_j\varphi_n\|_2^2 + \|(\nabla\xi_j)\varphi_n\|_2^2$$

where the implicit constant is independent of $j \in I$ and $n \in \mathbb{N}$. Since φ_n is normalized, this proves the first claim.

On the other hand, if ξ has compact support then so does $\nabla\xi$. Thus, from (4.9) we get $\|\xi\varphi_n\| \to 0$ and $\|(\nabla\xi)\varphi_n\| \to 0$, as $n \to \infty$. Hence,

$$\|(P-A)\xi\varphi_n\|_2^2 \lesssim \|\xi\varphi_n\|_2^2 + \|(\nabla\xi)\varphi_n\|_2^2 \to 0\,,$$

as $n \to \infty$. ∎

4.2. **Absence of positive eigenvalues.** Now we can prove our main result.

Theorem 4.8. *Let B and V satisfy assumptions 2.1- 2.6. Then the magnetic Schrödinger operator $H_{A,V}$ has no eigenvalues in the interval (Λ, ∞), where Λ is given by (1.13).*

Moreover, if $E \le \Lambda$ is an eigenvalue of $H_{A,V}$ then any weak eigenfunction ψ with energy E cannot decay faster than $e^{e\sqrt{\lambda - E}|x|}$, in the sense that if $x \mapsto e^{\overline{\mu}|x|}\psi(x) \in L^2(\mathbb{R}^d)$ for some $\overline{\mu} > \sqrt{\Lambda - E}$, then ψ is the zero function.

Proof. Let $q_{A,V}$ be the quadratic from corresponding to $H_{A,V}$ and assume that $E\langle\varphi,\psi\rangle = q_{A,V}(\varphi,\psi)$ for all $\varphi \in \mathcal{D}(q_{A,V}) = \mathcal{D}(P-A)$. Furthermore, assume that either $E > \Lambda$ or $E + \overline{\mu}^2 > \Lambda$ for some $\overline{\mu} > 0$ and $x \mapsto e^{\overline{\mu}|x|}\psi(x) \in L^2(\mathbb{R}^d)$. Then from Proposition 4.1 we know that

$$x \mapsto e^{\mu\langle x\rangle_\lambda}\, \psi(x) \in L^2(\mathbb{R}^d) \qquad \forall\, \mu > 0, \quad \forall\, \lambda > 0.$$

where $\langle x\rangle_\lambda = (\lambda + x^2)^{1/2}$.

Let $\mu > 0, \varepsilon > 0, \lambda > 0$, and define

$$F(x) = F_{\mu,\varepsilon,\lambda}(x) = \frac{\mu}{\varepsilon}\left(1 - e^{-\varepsilon\,\langle x\rangle_\lambda}\right)\,,$$

so that

$$\nabla F_{\mu,\varepsilon,\lambda}(x) = x g_{\mu,\varepsilon,\lambda}(x), \quad g_{\mu,\varepsilon,\lambda}(x) = \frac{\mu\, e^{-\varepsilon\,\langle x\rangle_\lambda}}{\sqrt{\lambda + |x|^2}}\,.$$

Denote $\psi_{\mu,\varepsilon,\lambda} = e^{F_{\mu,\varepsilon,\lambda}}\,\psi$. Lemma 3.19 and equation (3.45) then give

$$\kappa\,\langle \psi_{\mu,\varepsilon,\lambda}, |\nabla F_{\mu,\varepsilon,\lambda}|^2\,\psi_{\mu,\varepsilon,\lambda}\rangle \le \langle \psi_{\mu,\varepsilon,\lambda}, \left((x\cdot\nabla)^2 g_{\mu,\varepsilon,\lambda} - x\cdot\nabla|\nabla F_{\mu,\varepsilon,\lambda}|^2\right)\psi_{\mu,\varepsilon,\lambda}\rangle + C\,\|\psi_{\mu,\varepsilon,\lambda}\|_2^2 \tag{4.25}$$

for all $\mu, \varepsilon, \lambda > 0$ and some constant C independent of μ, λ and ε. Moreover, a direct calculation shows

$$\lim_{\varepsilon\to 0}\, x\cdot\nabla|\nabla F_{\mu,\varepsilon,\lambda}(x)|^2 = 2\lambda\mu^2\langle x\rangle_\lambda^{-1}(1 - \langle x\rangle_\lambda^{-2}) > 0 \tag{4.26}$$

and

$$\lim_{\varepsilon\to 0}\, (x\cdot\nabla)^2 g_{\mu,\varepsilon,\lambda}(x) = -2\lambda\mu\langle x\rangle_\lambda^{-3}|x|^2 < 0\,. \tag{4.27}$$

Since

$$\lim_{\varepsilon\to 0} F_{\mu,\varepsilon,\lambda}(x) := F_{\mu,\lambda}(x) = \mu\langle x\rangle_\lambda\,,$$

in view of Proposition 4.1 we can pass to limit $\varepsilon \to 0$ in (4.25) to obtain

$$\kappa\, \mu^2 \left\langle \psi_{\mu,\lambda}, \frac{|x|^2}{\lambda+|x|^2}\, \psi_{\mu,\lambda}\right\rangle \leq C\, \|\psi_{\mu,\lambda}\|_2^2 \qquad \forall\, \mu, \lambda > 0, \tag{4.28}$$

where

$$\psi_{\mu,\lambda}(x) := e^{\mu\langle x\rangle_\lambda}\, \psi(x)\,.$$

Using Proposition 4.1 again and the monotone convergence theorem we finally obtain, by letting $\lambda \to 0$,

$$\kappa\, \mu^2\, \|\psi_\mu\|_2^2 \leq C\, \|\psi_\mu\|_2^2 \qquad \forall\, \mu > 0, \tag{4.29}$$

where $\psi_\mu(x) = e^{\mu|x|}\, \psi(x)$. This is of course impossible for μ large enough. Hence $\psi_\mu = 0$ and the claim follows. ∎

Remark 4.9. Notice that in view of Corollary 7.7 we have $(\Lambda, \infty) \subseteq \sigma_{\mathrm{ess}}(H)$. Hence Theorem 4.8 excludes the presence of all embedded eigenvalues of H strictly larger than Λ.

On the other hand, the possibility of Λ being an eigenvalue of H cannot be in general excluded. Indeed, if B is continuous and compactly supported with $|\int_{\mathbb{R}^2} B| > 2\pi$, and if $V = -B$, then by the Aharonov-Casher theorem, see e.g. [7, Sec. 6.4], $\Lambda = 0$ is an eigenvalue of $H = (P-A)^2 - B$. Sufficient conditions for the absence of positive eigenvalues of the Pauli operator are proved in section 6.4 , see 6.5.

Remark 4.10. In [5] it was proved that if the magnetic fields has the form

$$B(x) = \frac{b(\theta)}{r}\,, \quad x = (r\cos\theta,\, r\sin\theta), \quad b \in L^\infty(\mathbb{S}^1),$$

then the operator H_A, has no eigenvalues above $\|b\|^2_{L^\infty(\mathbb{S}^1)}$. Note that in this particular setting $\Lambda = \|b\|^2_{L^\infty(\mathbb{S}^1)}$.

Remark 4.11. One of the authors of the present paper established in [22] dispersive estimates for the propagator e^{-itH} in weighted L^2-spaces under the condition that H has no positive eigenvalues, see [22, Assumption 2.2]. Theorem 4.8 implies that the latter assumption can be omitted. This was, in fact, one of the main motivations for the present work.

5. Kato–class, local L^p, and pointwise conditions

Below we show that Assumptions 2.1-2.6 are satisfied under mild explicit regularity and decay conditions on the magnetic field B and the potential V. In particular, we give local L^p conditions, which in a natural way extend the pointwise bounds on the potential from in [1, 31].

5.1. Potentials vanishing at infinity.

Recall the Definitions 1.4, respectively 1.8, for a potential V to vanish, respectively being bounded, at infinity w.r.t. $(P-A)^2$. We have

Proposition 5.1. a) *If $V = W_1 + W_2$ and W_1 and W_2 vanish at infinity w.r.t. $(P-A)^2$, then V vanishes at infinity w.r.t. $(P-A)^2$.*

b) *If $V = W_1 + W_2$ and W_1 and W_2 are bounded from above at infinity w.r.t. $(P-A)^2$, then V is bounded from above at infinity w.r.t. $(P-A)^2$. Moreover, $\gamma^+_\infty(V) \leq \gamma^+_\infty(W_1) + \gamma^+_\infty(W_2)$.*

c) *If $V = \nabla\cdot\Sigma + W$ for some real-valued vector field Σ and some potential W and Σ^2 and W are form bounded with respect to $(P-A)^2$, then V is form bounded with respect to $(P-A)^2$.*

d) *If $V = \nabla\cdot\Sigma + W$ for some real-valued vector field Σ and some potential W and Σ^2 and W vanish at infinity w.r.t. $(P-A)^2$ then V vanishes at infinity w.r.t. $(P-A)^2$.*

Remarks 5.2. (i) Again, the diamagnetic inequality implies that one only has to check form boundedness and vanishing w.r.t. P^2.

(ii) The second result is already observed in the work of Combescure and Ginibre [4]. The beautiful work of Maz'ya and Verbitsky [26] shows that V is form bounded w.r.t P^2 if and only if one can split $V = \nabla \cdot \Sigma + W$ such that Σ^2 and W are form bounded w.r.t P^2.

(iii) It is not true, in general, that Σ^2 bounded at infinity implies that $\nabla \cdot \Sigma$ is bounded at infinity w.r.t $(P - A)^2$.

(iv) The choice $\Sigma(x) = x\langle x\rangle^{-\varepsilon} \sin(e^{1/|x|}) = O(\langle x\rangle)^{-\varepsilon}$, for some $\varepsilon > 0$, yields a potential $V = \nabla \cdot \Sigma$ with

$$V(x) = -|x|^{-1} e^{1/|x|} \langle x\rangle^{-\varepsilon} \cos(e^{1/|x|}) + O(\langle x\rangle^{-\varepsilon}) \tag{5.1}$$

which has a severe singularity at zero. Since Σ^2 is infinitesimally for bound and vanishing at infinity w.r.t P^2, the above result shows that so does V. That V vanishes at infinity w.r.t. P^2, which might not be too surprising, since the singularity is local.

(v) The choice $\Sigma(x) = x\langle x\rangle^{-\varepsilon} \sin(e^{|x|}) = O(\langle x\rangle)^{-\varepsilon}$, for some $\varepsilon > 0$, yields a potential $V = \nabla \cdot \Sigma$ with

$$V(x) = |x| e^{|x|} \langle x\rangle^{-\varepsilon} \cos(e^{|x|}) + O(\langle x\rangle^{-\varepsilon}) \tag{5.2}$$

which has again severe oscillations, now at infinity. Nevertheless, it is infinitesimally form bounded and vanishes at infinity w.r.t. P^2 since Σ^2 does. In particular, despite the severe oscillations of V at infinity, our Theorem 7.8 below shows that the perturbation V does not change the essential spectrum.

Proof. The first two claims follows directly from Definitions 1.4 and 1.8.

For the third claim let $\varphi \in \mathcal{C}_0^\infty$, which is dense in $\mathcal{D}(P - A)$ for any real–valued locally square integrable vector potential A and note that by an integration by parts the distribution $\nabla \cdot \Sigma$ is given by

$$\langle \varphi, \nabla \cdot \Sigma\varphi \rangle = -2 \operatorname{Im} \langle \Sigma\varphi, P\varphi \rangle = -2 \operatorname{Im} \langle \Sigma\varphi, (P - A)\varphi \rangle$$

since $\langle \Sigma\varphi, A\varphi \rangle$ is real. Thus the right hans side above extend to all $\varphi \in \mathcal{D}(P - A)$ if Σ^2 is form bounded w.r.t. $(P - A)^2$ and $|\langle \varphi, \nabla \cdot \Sigma\varphi \rangle| \le \|\Sigma\varphi\| \|(P - A)\varphi\|$. So if $\|\Sigma\varphi\|_2^2 \le \alpha\|(P - A)\varphi\|_2^2 + \gamma\|\varphi\|_2^2$, then

$$\begin{aligned} |\langle \varphi, \nabla \cdot \Sigma\varphi \rangle| &\le 2(\alpha\|(P - A)\varphi\|_2^2 + \gamma\|\varphi\|_2^2)^{1/2} \|(P - A)\varphi\| \\ &\le (\varepsilon^{-1}\alpha + \varepsilon)\|(P - A)\varphi\|_2^2 + \varepsilon^{-1}\gamma\|\varphi\|_2^2 \end{aligned} \tag{5.3}$$

for all $\varepsilon > 0$, which proves that $\nabla \cdot \Sigma$ is form bounded w.r.t. $(P - A)^2$. If W is also form bounded w.r.t $(P - A)^2$, then so is their sum $V = \nabla \cdot \Sigma + W$.

Lastly, because of the first part, we only have to show that $\nabla \cdot \Sigma$ vanishes at infinity as soons as Σ^2 vanishes at infinity w.r.t $(P - A)^2$. So assume that there exist α_R and γ_R decreasing with $\alpha_R, \gamma_R \to 0$ as $R \to \infty$ and

$$\|\Sigma\varphi\|_2^2 \le \alpha_R\|(P - A)\varphi\|_2^2 + \gamma_R\|\varphi\|_2^2$$

for all $\varphi \in \mathcal{D}(P - A)$ with $\operatorname{supp}(\varphi) \in \mathcal{U}_R^c$. Setting $\varepsilon = \max(\alpha_R, \gamma_R)^{1/2}$ in (5.3) yields

$$|\langle \varphi, \nabla \cdot \Sigma\varphi \rangle| \le \max(\alpha_R, \gamma_R)^{1/2} \Big(2\|(P - A)\varphi\|_2^2 + \|\varphi\|_2^2 \Big)$$

for all $\varphi \in \mathcal{D}(P - A)$ with $\operatorname{supp}(\varphi) \subset \mathcal{U}_R^c$ and large enough R. This shows that $\nabla \cdot \Sigma$ vanishes at infinity w.r.t. $(P - A)^2$. ∎

5.2. **Local L^p conditions for vanishing at infinity.** An efficient route to local L^p conditions is via the so–called Kato–class, which we recall.

Definition 5.3 (Kato–class)**.** A real-valued and measurable function V on $\mathbb{R}^d$ is in the Kato–class K_d if

$$\lim_{\alpha\to 0}\sup_{x\in\mathbb{R}^d}\int_{|x-y|\le\alpha} g_d(x-y)|V(y)|\,dy = 0 \tag{5.4}$$

where

$$g_d(x) := \begin{cases} |x|^{2-d} & \text{if} \quad d\ge 3 \\ |\ln|x|| & \text{if} \quad d=2 \end{cases} \tag{5.5}$$

One also defines the Kato–norm

$$\|V\|_{K_d} := \begin{cases} \sup_{x\in\mathbb{R}^d}\int_{|x-y|\le 1}|x-y|^{d-2}|V(y)|\,dy\,, & \text{if} \quad d\ge 3 \\ \sup_{x\in\mathbb{R}^2}\int_{|x-y|\le 1/2}|\ln(|x-y|)||V(y)|\,dy\,, & \text{if} \quad d=2 \end{cases}\,. \tag{5.6}$$

There is also a definition for the Kato–class in dimension one, but we do not need it. It is well-known that any Kato–class potential is infinitesimally form bounded with respect to P^2, thus also with respect to $(P-A)^2$ for any vector potential $A\in L^2_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^d)$. It is also clear that $K_d\subset L^1_{\mathrm{loc,unif}}(\mathbb{R}^d)$ and using Hölder's inequality one easily sees $L^p_{\mathrm{loc,unif}}(\mathbb{R}^d)\subset K_d$ for all $p>d/2$.

Lastly, we say that a potential V is in the Kato–class outside a compact set, if there exists a compact set $K\subset\mathbb{R}^d$ such that $\mathbb{1}_{K^c}V\in K_d$. Here $\mathbb{1}_{K^c}$ is the characteristic function of the complement of K.

For potentials which are in the Kato–class outside of a compact set, we have a simple criterium for vanishing.

Proposition 5.4. *Let $\mathbb{1}_{\ge R}$ be the characteristic function of $\{x\in\mathbb{R}^d : |x|\ge R\}$. Given a potential W assume that it is in the Kato–class outside a compact set and that it vanishes at infinity locally uniformly in L^1, that is,*

$$\lim_{R\to\infty}\|\mathbb{1}_{\ge R}W\|_{L^1_{loc,unif}} = 0\,. \tag{5.7}$$

Then W vanishes at infinity w.r.t. P^2 in the sense of Definition 1.4.

Moreover, if $V=\nabla\cdot\Sigma+W$ for some vector field $\Sigma\in L^2_{loc}$ and a potential $W\in L^1_{loc}$ and Σ^2 and W satisfy the above assumptions, then V also vanishes at infinity w.r.t. P^2 in the sense of Definition 1.4.

We prove it later in this section. Useful corollaries are

Corollary 5.5 (Pointwise asymptotic bounds)**.** *Given a magnetic field B and potential $V=V_1+V_2$ assume that $\widetilde{B}^2$, V, V_1, xV_1, and $x\cdot\nabla V_2$ are bounded outside of a compact set and $\lim_{|x|\to\infty}V(x)=\lim_{|x|\to\infty}V_1(x)=0$. Then assumption 2.5 and 2.6 are satisfied and we can take*

$$\beta\le\limsup_{|x|\to\infty}|\widetilde{B}(x)|,\quad \omega_1\le\limsup_{|x|\to\infty}|x\,V_1(x)|,\quad \textit{and}\ \omega_2\le\limsup_{|x|\to\infty}(x\cdot\nabla V_2(x))_+. \tag{5.8}$$

Of course, the above pointwise conditions are way too strong, in general. Assuming $V\in L^1_{\mathrm{loc,unif}}$ let

$$Z(V) := \{W\in L^\infty : V-W \text{ vanishes at infinity locally uniformly in } L^1\} \tag{5.9}$$

$$Z_+(V) := \{W\in L^\infty : (U-W)_+ \text{ vanishes at infinity locally uniformly in } L^1\} \tag{5.10}$$

$$\widetilde{\gamma}_R(V) := \inf\{\|\mathbb{1}_{\ge R}W\|_\infty : W\in Z(V)\}\,,\quad \widetilde{\gamma}^+_R(U) := \inf\{\|\mathbb{1}_{\ge R}W_+\|_\infty : W\in Z_+(V)\} \tag{5.11}$$

$$\widetilde{\gamma}_\infty((V) := \lim_{R\to\infty}\widetilde{\gamma}_R(V) = \inf_{R\gtrsim 1}\widetilde{\gamma}_R(V)\,,\quad \widetilde{\gamma}^+_\infty(V) := \lim_{R\to\infty}\widetilde{\gamma}^+_R(v) = \inf_{R\gtrsim 1}\widetilde{\gamma}^+_R(V) \tag{5.12}$$

Corollary 5.6. *Given a magnetic field B and a potential $V = V_1 + V_2$ assume that $\widetilde{B}^2, (xV_1)^2$, and $(x\cdot\nabla V_2)_+ \in K_d$ are in the Kato–class outside a compact set. Then the asymptotic bounds from assumption 2.6 are bounded by*

$$\beta^2 \le \widetilde{\gamma}_\infty\big(\widetilde{B}^2\big)\,, \quad \omega_1^2 \le \widetilde{\gamma}_\infty\big((xV_1)^2\big)\,, \text{ and } \ \omega_2 \le \widetilde{\gamma}_\infty^+\big(x\cdot\nabla V_2\big)\,. \tag{5.13}$$

Proofs of Corollaries 5.5 and 5.6: If W is bounded, then $\gamma_\infty(W) \le \lim_{R\to\infty}\|\mathbb{1}_{\ge R}W_b\|_\infty = \inf_{R\gtrsim 1}\|\mathbb{1}_{\ge R}W_bW_b\|_\infty$ and $\gamma_\infty^+(W) \le \inf_{R\gtrsim 1}\|\mathbb{1}_{\ge R}W_b^+\|_\infty$. Thus, for any $W_b\in Z(W)$ Propositions 5.1 and 5.4 show

$$\gamma_\infty(W) \le \gamma_\infty(W-W_b) + \gamma_\infty(W_b) = \gamma_\infty(W_b) \le \inf_{R\gtrsim 1}\|W_b\|_\infty$$

Since $W - W_b$ vanishes at infinity w.r.t. $(P-A)^2$, i.e., $\gamma_\infty(W-W_b) = 0$. Thus

$$\gamma_\infty(W) \le \inf_{W_b\in Z(V)}\inf_{R\gtrsim 1}\|W_b\|_\infty = \inf_{R\gtrsim 1}\widetilde{\gamma}_R(W_b) = \widetilde{\gamma}_\infty(W)$$

and similarly for $\gamma_\infty^+(W)$, which proves Corollary 5.6.

For Corollary 5.5, note that if $C = \limsup_{|x|\to\infty} W(x) < \infty$, respectively $C = \limsup_{|x|\to\infty} W_+(x) < \infty$, then for any $\varepsilon > 0$ the functions $(|W| - C - \varepsilon)_+$, respectively $(W - C - \varepsilon)_+$, have compact support, so $\gamma_\infty(W) \le \gamma_\infty(|W|) \le +C + \varepsilon$, respectively $\gamma_\infty^+(W) \le C + \varepsilon$ for any $\varepsilon > 0$, which proves Corollary 5.5. ∎

In the proof of Proposition 5.4 we need

Lemma 5.7. *Given a potential W assume that there exist $R_0 > 0$ and $\alpha_{R,\lambda}, \gamma_{R,\lambda} \ge 0$ for $R_0 > 0$ and $R \ge R_0, \lambda > 0$ such that*

$$\langle\varphi, W\varphi\rangle \le \alpha_{R,\lambda}\|(P-A)\varphi\|_2^2 + \gamma_{R,\lambda}\|\varphi\|_2^2 \tag{5.14}$$

for all $\varphi\in\mathcal{D}(P-A)$ with $\operatorname{supp}(\varphi)\in\mathcal{U}_R^c$. Moreover, assume that $R_0 \le R \mapsto \alpha_{R,\lambda}, \gamma_{R,\lambda}$ are decreasing for fixed $\lambda > 0$ and $\lim_{\lambda\to\infty}\alpha_{R,\lambda} = 0$ for fixed $R \ge R_0$.

Then W is bounded from above at infinity w.r.t $(P-A)^2$ with asymptotic bound

$$\gamma_\infty^+(W) \le \liminf_{\lambda\to\infty}\lim_{R\to\infty}\gamma_{R,\lambda}\,. \tag{5.15}$$

Remark 5.8. The order of the limits in (5.15) is important, since typically one has $\liminf_{\lambda\to\infty}\gamma_{R,\lambda} = \infty$ for any fixed R.

Given any $\alpha_{R,\lambda}, \gamma_{R,\lambda}$ for which (5.14) holds, one can, by a simple monotonicity argument, replace them with $\alpha'_{R,\lambda} \coloneqq \inf_{R_0\le L\le R}\alpha_{L,\lambda}$ and $\gamma'_{R,\lambda} \coloneqq \inf_{R_0\le L\le R},\gamma_{L,\lambda}$, i.e., the required monotonicity in R in Lemma 5.7 is not a restriction.

Proof. Let $\widetilde{\gamma}_\lambda = \lim_{R\to\infty}\gamma_{R,\lambda}$. Pick any $\lambda_0 > 0$ and given R_n, λ_n for $n\in\mathbb{N}_0$ choose inductively $\lambda_{n+1} \ge \lambda_n + 1$ with $\alpha_{R_n,\lambda_{n+1}} \le \frac{1}{n+1}$ and then $R_{n+1} \ge R_n + 1$ with $\gamma_{R_{n+1},\lambda_{n+1}} \le \frac{1}{n+1} + \widetilde{\gamma}_{\lambda_{n+1}}$.

Take a subsequence n_j with $\widetilde{\gamma}_j \coloneqq \widetilde{\gamma}_{n_j} \to \liminf_{n\to\infty}\widetilde{\gamma}_\lambda$ as $j\to\infty$ and set $\alpha_R \coloneqq \frac{1}{n_j+1}$ and $\gamma_R \coloneqq \frac{1}{n_j+1} + \widetilde{\gamma}_j$ for $R\in[R_{n_j}, R_{n_{j+1}})$. With this choice Definition 1.8 is satisfied, so W is asymptotically bounded at infinity w.r.t. $(P-A)^2$ and $\gamma_\infty(W) = \lim_{R\to\infty}\gamma_R = \lim_{j\to\infty}\widetilde{\gamma}_j = \liminf_{\lambda\to\infty}\lim_{R\to\infty}\gamma_{R,\lambda}$. ∎

Proof of Proposition 5.4. Given a locally square integrable magnetic vector potential A we abbreviate $H_0 = (P-A)^2$ for the free magnetic Schrödinger operator defined by quadratic form methods. Given a potential W in the Kato–class, $\varphi\in\mathcal{D}(P-A) = \mathcal{Q}(H_0)$, and $\lambda > 0$ let $f = (H_0+\lambda)^{1/2}\varphi \in L^2$. Then

$$\begin{aligned}|\langle\varphi, W\varphi\rangle| &\le \langle\varphi, |W|\varphi\rangle = \langle f, (H_0+\lambda)^{-1/2}|W|(H_0+\lambda)^{-1/2}f\rangle \le \|(H_0+\lambda)^{-1/2}|W|(H_0+\lambda)^{-1/2}\|_{2\to 2}\|f\|_2^2 \\ &= \|(H_0+\lambda)^{-1/2}|W|(H_0+\lambda)^{-1/2}\|_{2\to 2}\left(\|((P-A)^2\varphi\|_2^2 + \lambda\|\varphi\|_2^2\right)\end{aligned}$$

By duality, $\|(H_0+\lambda)^{-1/2}|W|(H_0+\lambda)^{-1/2}\|_{2\to 2} = \||W|^{1/2}(H_0+\lambda)^{-1}|W|^{1/2}\|_{2\to 2}$. Assume that $|W|$ is bounded, then for $0 \le \mathrm{Re}(z) \le 1$ the operator family $T_z = |W|^z(H_0+\lambda)^{-1}|W|^{1-z}$ is analytic and bounded.

Using the diamagnetic inequality and duality we have

$$\||W|(H_0+\lambda)^{-1}\|_{1\to 1} = \|(H_0+\lambda)^{-1}|W|\|_{\infty\to\infty} = \|(H_0+\lambda)^{-1}|W|\|_\infty \le \|(P^2+\lambda)^{-1}|W|\|_\infty\,,$$

which is finite for any $\lambda>0$ and bounded W. Thus T_z is bounded from $L^1\to L^1$ for $\mathrm{Re}(z)(z)=0$ and from $L^\infty\to L^\infty$ for $\mathrm{Re}(z)=1$ and as in [7] one can use the Stein interpolation theorem [29] to see

$$\|(H_0+\lambda)^{-1/2}|W|(H_0+\lambda)^{-1/2}\|_{2\to 2} \le \|(P^2+\lambda)^{-1}|W|\|_\infty\,.$$

at least for bounded W. If $\mathrm{supp}(\varphi)\subset\mathcal{U}_R^c$, one can replace W by $W_R = \mathbb{1}_{\ge R}W$. Thus

$$|\langle\varphi, W\varphi\rangle| = |\langle\varphi, W_R\varphi\rangle| \le \alpha_{R,\lambda}\|(P-A)\varphi\|_2^2 + \gamma_{R,\lambda}\|\varphi\|_2^2$$

for all $\varphi\in\mathcal{D}(P-A)$ with $\mathrm{supp}(\varphi)\subset\mathcal{U}_R^c$, choosing

$$\begin{aligned}\alpha_{R,\lambda} &= \|(P^2+\lambda)^{-1}|W_R|\|_\infty\,,\\ \gamma_{R,\lambda} &= \lambda\|(P^2+\lambda)^{-1}|W_R|\|_\infty\end{aligned}\tag{5.16}$$

If W_R is unbounded, replace W_R by $\min(|W_R|,n)$ and take the limit $n\to\infty$ to see that the above bounds work also for unbounded W, as long as the right hand side of (5.16) is finite.

Clearly, $\alpha_{R,\lambda}$ and $\gamma_{R,\lambda}$ are decreasing in R for fixed $\lambda>0$. One even has $\lim_{\lambda\to\infty}\|(P^2+\lambda)^{-1}|W|\|_\infty = 0$ if and only if W is in the Kato–class, which is well–known, see [7, 34]. However, we also clearly have $\lim_{\lambda\to\infty}\gamma_{R,\lambda} = \|W_R\|_\infty$, which is finite, if and only if W_R is bounded. Nevertheless, if W_R is in the Kato class for some, hence all, large enough R and $\lim_{R\to\infty}\|W_R\|_{L^1_{\mathrm{loc,unif}}} = 0$ then

$$\lim_{R\to\infty}\|(P^2+\lambda)^{-1}|W_R|\|_\infty = 0\,,\tag{5.17}$$

which together with Lemma 5.7 shows $\gamma_\infty(W)=0$. This proves the first part of Proposition 5.4. The other claim of Proposition 5.4 follows from the above since by Proposition 5.1 $W=\nabla\cdot\Sigma$ vanishes w.r.t $(P-A)^2$ as soon as Σ^2 does.

For the proof of 5.17, we claim that for any potential W and any $0<\alpha\le 1$

$$\|(P^2+\lambda)^{-1}|W|\|_\infty \lesssim \sup_{x\in\mathbb{R}^d}\int_{|x-y|\le\alpha} g_d(x-y)|W(y)|\,dy + \frac{e^{-\sqrt{\lambda}\alpha/4}}{\sqrt{\lambda}\alpha}\|W\|_{L^1_{\mathrm{loc,unif}}}\tag{5.18}$$

where the implicit constant depend only on d. This clearly proves (5.17), since replacing W by $W_R=\mathbb{1}_{\ge R}W$ it yields

$$\limsup_{R\to\infty}\|(P^2+\lambda)^{-1}|W_R|\|_\infty \le C_{\lambda,d}\sup_{x\in\mathbb{R}^d}\int_{|x-y|\le\alpha} g_d(x-y)|W_{R_0}(y)|$$

for any fixed $R_0,\lambda>0$ and all $0\le\alpha\le 1$ as soon as $\lim_{R\to\infty}\|W_R\|_{L^1_{\mathrm{loc,unif}}}=0$. Since W_{R_0} is in the Kato–class, we can then take the limit $\alpha\to 0$ to get (5.17).

It remains to prove (5.18). Note

$$\|(P^2+\lambda)^{-1}|W|\|_\infty = \sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d} G(x,y,\lambda)|W(y)|\,dy$$

where $G(x,y),\lambda = (P^2+\lambda)^{-1}(x,y)$ is the Green's function, i.e., the kernel of $(P^2+\lambda)^{-1}$. We split the integral above in the two regions $|x-y|\le\alpha$ and $|x-y|>\alpha$. The bounds

$$G(x,y,\lambda)\lesssim \lambda^{-1}|x-y|^{-d}e^{-\sqrt{\lambda}|x-y|/2}\tag{5.19}$$

and for $|x-y|\le 1/2$ and $\lambda\ge 1$

$$G(x,y,\lambda)\lesssim\begin{cases}|x-y|^{2-d} & \text{if}\quad d\ge 3\\ |\ln|x-y|| & \text{if}\quad d=2\end{cases}\tag{5.20}$$

are well-know. The second bound immediately gives

$$\sup_{x\in\mathbb{R}^d}\int_{|x-y|\le\alpha} G(x,y,\lambda)|W(y)|\,dy \lesssim \sup_{x\in\mathbb{R}^d}\int_{|x-y|\le\alpha} g_d(x-y)|W(y)|\,dy$$

at least for all $0<\alpha\le 1/2$ and the first one shows

$$\int_{|x-y|>\alpha} G(x,y,\lambda)|W(y)|\,dy \lesssim \lambda^{-1}\int_{|x-y|\ge\alpha} |x-y|^{-d}e^{-\sqrt{\lambda}|x-y|/2}|W(y)|\,dy\,.$$

Integrating over shells $\alpha n\le|x-y|<\alpha(n+1)$ leads to

$$\begin{aligned}\sup_{x\in\mathbb{R}^d}\int_{|x-y|>\alpha} G(x,y,\lambda)|W(y)|\,dy &\lesssim \lambda^{-1}\sum_{n=1}^{\infty} e^{-\sqrt{\lambda}\alpha n/2}\frac{(\alpha(n+1))^d-(\alpha n)^d}{(\alpha n)^d}\|W\|_{L^1_{\mathrm{loc,unif}}}\\ &\lesssim \lambda^{-1}\sum_{n=1}^{\infty} e^{-\sqrt{\lambda}\alpha n/2}\|W\|_{L^1_{\mathrm{loc,unif}}} = \frac{e^{-\sqrt{\lambda}\alpha/2}}{\lambda(1-e^{-\sqrt{\lambda}\alpha/2})}\|W\|_{L^1_{\mathrm{loc,unif}}} \lesssim \frac{e^{-\sqrt{\lambda}\alpha/4}}{\sqrt{\lambda}\alpha}\,\|W\|_{L^1_{\mathrm{loc,unif}}}\end{aligned}$$

since $0<t\mapsto \frac{te^{-t/2}}{1-e^{-t}}$ is bounded. This proves (5.18).

We sketch the proof of the bounds (5.19) and (5.20), for the convenience of the reader: The kernel of the heat semigroup is $e^{-P^2t}(x,y)=(4\pi t)^{-d/2}e^{-\frac{|x-y|^2}{4t}}$. Since $(P^2+\lambda)^{-1}=\int_0^\infty e^{-P^2s-\lambda s}\,ds$ we have

$$G(x,y,\lambda)=\int_0^\infty (4\pi s)^{-d/2}e^{-\frac{|x-y|^2}{4s}}e^{-\lambda s}\,ds \;=|x-y|^{2-d}\int_0^\infty (4\pi u)^{-d/2}e^{-\frac{1}{4u}}e^{-\lambda|x-y|^2u}\,du$$

Moreover, $\frac{1}{4u}+\lambda|x-y|^2u\ge\sqrt{\lambda}|x-y|$ for all $u>0$, so

$$\begin{aligned}G(x,y,\lambda)&\le|x-y|^{2-d}e^{-\sqrt{\lambda}|x-y|/2}\int_0^\infty(4\pi u)^{-d/2}e^{-\frac{1}{8u}}e^{-\lambda|x-y|^2u/2}\,du\\ &=\frac{|x-y|^{-d}e^{-\sqrt{\lambda}|x-y|/2}}{\lambda}\int_0^\infty(4\pi u)^{-d/2}e^{-\frac{1}{8u}}\lambda|x-y|^2ue^{-\lambda|x-y|^2u/2}\,\frac{du}{u}\lesssim\frac{|x-y|^{-d}e^{-\sqrt{\lambda}|x-y|/2}}{\lambda}\end{aligned}$$

since $0<t\mapsto te^{-t}$ is bounded and $c_d=\int_0^\infty(4\pi u)^{-d/2}e^{-\frac{1}{4u}}\,\frac{du}{u}<\infty$ for all $d\ge1$. This proves (5.19).

On the other hand,

$$G(x,y,\lambda)=|x-y|^{2-d}\int_0^\infty(4\pi u)^{-d/2}e^{-\frac{1}{4u}}e^{-\lambda|x-y|^2u}\,du\le\widetilde{c}_d|x-y|^{2-d}$$

where $\widetilde{c}_d=\int_0^\infty(4\pi u)^{-d/2}e^{-\frac{1}{4u}}\,du<\infty$ if $d\ge3$, which proves (5.20) when $d\ge3$.

If $d=2$, then for $0<|x-y|\le1/2$, one has

$$G(x,y,\lambda)=(4\pi)^{-1}\int_0^\infty e^{\frac{1}{4u}}e^{-\lambda|x-y|^2u}\,\frac{du}{u}\lesssim\int_0^1 e^{\frac{1}{4u}}\,\frac{du}{u}+\int_1^{|x-y|^{-2}}\frac{du}{u}+\int_{|x-y|^{-2}}^\infty e^{-\lambda|x-y|^2u}\,\frac{du}{u}$$

Since $\int_0^1 e^{\frac{1}{4u}}\,\frac{du}{u}\lesssim1$ and $\int_{|x-y|^{-2}}^\infty e^{-\lambda|x-y|^2u}\,\frac{du}{u}=\int_1^\infty e^{-\lambda u}\,\frac{du}{u}\le1$ for $\lambda\ge1$, this proves (5.20) when $d=2$. ∎

6. Examples

We recall a couple of examples which show that the decay assumptions on B and V stated in Theorems 1.2, 1.6, and 4.8, and Corollary 5.5 cannot be improved.

6.1. **Miller-Simon revisited.** In [27] Miller and Simon considered, in dimension two, the case $V = 0$ and radial magnetic field $B(x) = b(r)$, $r = |x|$. They proved that

1) If $b(r) = r^{-\alpha} + O(r^{-1-\varepsilon})$ with $0 < \alpha < 1$ and $\varepsilon > 0$ then the spectrum of H is dense pure point,
2) If $b(r) = b_0\, r^{-1} + O(r^{-1-\varepsilon})$ for some $\varepsilon > 0$ then the spectrum of H is dense pure point in $[0, b_0^2)$ and absolutely continuous in $[b_0^2, \infty)$,
3) If $b(r) = \mathcal{O}(r^{-\alpha})$ with $\alpha > 1$ then the spectrum of H is purely absolutely continuous in $(0, \infty)$.

Remark 6.1. Note that $\beta = \limsup_{|x|\to\infty} |\widetilde{B}(x)| + \infty$ in the case (1). On the other hand, Theorem 4.8 guarantees the absence of eigenvalues in the interval (β^2, ∞) for the case (2), and in the interval $(0, \infty)$ for the case (3), even for non–radial magnetic fields. In particular, the Miller–Simon examples show that our result on absence of eigenvalues is *sharp*. These examples even have dense point spectrum in $[0, \beta^2]$.

Since there is a calculation error in the original Miller-Simon paper and also in the book [7], we sketch their argument: Assume that the radial magnetic field b is reasonable, e.g., bounded and use x, y as coordinates in $\mathbb{R}^2$ and $r = (x^2 + y^2)^{1/2}$.

The first observation of Miller and Simon is that if the magnetic field, radial or not, B goes pointwise to zero at infinity, then $\sigma_{\text{ess}}((P - A)^2) = [0, \infty)$ (this is sharpened in Theorem 7.5).

For radial magnetic fields $\widetilde{B}(x) = (-y, x)b(r)$, so the Poincaré gauge the magnetic vector potential is

$$A(x, y) = (-y, x)\int_0^1 b(tr)t\,dt = \frac{(-y, x)}{r}h(r)$$

with $h(r) = r^{-1}\int_0^r b(s)s\,ds$. Expanding $(P - A)^2$ one sees

$$(P - A)^2 = (P_x - A_x)^2 + (P_y - A_y)^2 = P^2 + h(r)^2 - 2\frac{h(r)}{r}L$$

where $L = xP_y - yP_x$ is the angular momentum in the plane. It is well-known that L has eigenvalues $(0, \pm 1, \pm 2, \ldots)$ and it commutes with P^2 and with the radial potential $h(r)^2$. So restricted to the angular momentum channel $\{L = m\}$, the operator $(P - A)^2$ is given by

$$H_m \coloneqq (P - A)^2\big|_{\{L=m\}} = \big(P^2 + V_m\big)|_{\{L=m\}} \quad \text{with } V_m(r) = h(r)^2 - \frac{2mh(r)}{r}$$

Due to the angular momentum barrier the divergence of V_m for small r when $m \neq 0$ is irrelevant.

If $b_0 = \lim_{r\to\infty} r^{-1}b(r) = \infty$, then $h(r) \to \infty$, so V_m is trapping and all operators H_m have discrete spectrum. But if also $b(r) \to 0$ as $r \to \infty$, then $\sigma_{\text{ess}}(H_A) = [0, \infty)$, so $(P - A)^2$ has necessarily dense point spectrum in $[0, \infty)$, proving the first claim (1) above.

If $b_0 = \lim_{r\to\infty} r^{-1}b(r) < \infty$, then $h(r) \to b_0$ and $V_m(r) \to b_0^2$ as $r \to \infty$, so H_m has only discrete spectrum below b_0^2 for any $m \in \mathbb{Z}$. Since $b(r) \to 0$ for $r \to \infty$, the operator has essential spectrum $[0, \infty]$, which must be dense point spectrum in $[0, b_0^2]$. (or any reasonable choice of radial magnetic field b, the effective potential V_m is smooth with decaying derivatives for large r, so the spectrum of H_m above b_0^2 is absolutely continuous for all $m \in \mathbb{Z}$. Thus $(P - A)^2$ has absolutely continuous spectrum in (b_0^2, ∞), which proves the last two claims.

Remark 6.2. In [27] the choice of the vector potential contains a wrong factor of $1/2$ and in the example in [7] there is a mistake in the calculation of the magnetic field. Thus in their examples they concluded incorrectly that the effective potential has the asymptotic $V_m(r) \to b_0^2/4$ for large r.

6.2. **Wigner-von Neumann potential.** Suppose that $B = 0$. Wigner and von Neumann showed that the operator $-\Delta + V$ in $L^2(\mathbb{R}^3)$ with the radial potential

$$V(r) = -\frac{32 \sin r\big[g(r)^3 \cos r - 3g^2(r) \sin^3 r + g(r)\cos r + \sin^3(r)\big]}{(1+g(r)^2)^2}, \qquad g(r) = 2r - \sin(2r), \tag{6.1}$$

has eigenvalue $+1$, see [31, 37] and [30, Ex. VIII.13.1]. As pointed out in [31] for large r

$$V(r) = -\frac{8\sin(2r)}{r} + \mathcal{O}(r^{-2}). \tag{6.2}$$

Theorem 4.8 implies that $-\Delta + V = -\Delta + V_1 + V_2$ has no eigenvalues larger than

$$\Lambda = \frac{1}{2}\Big(\omega_1 + \omega_2 + \sqrt{\omega_1^2 + 2\omega_1\omega_2}\,\Big),$$

with ω_1 and ω_2 defined by equation (1.14). We can thus optimize the splitting $V = V_1 + V_2$ in order to minimize Λ. A quick calculation using (6.2) shows that the optimal choices are $V_1 = V$, $V_2 = 0$ and $V_1 = 0$, $V_2 = V$. In both cases we get $\Lambda = 8$ which coincides with [1, Thm. 4]. Note that [31, Thm. 2] implies absence of eigenvalues in the interval $(16, \infty)$.

For each $|k| > 2$ Arai and Uchiyama constructed in [2] bounded radial potentials which are asymptotically of the form

$$V(x) = \frac{k\sin(2|x|)}{|x|} + O(|x|^{-1-\varepsilon}) \quad \text{as } |x| \to \infty \tag{6.3}$$

for some $\varepsilon > 0$ such that $P^2 + V$ has eigenvalue 1. In these examples also $x \cdot \nabla V$ is bounded and $\omega_1 = \limsup_{|x|\to\infty}(x \cdot \nabla V(x))_+ = 2|k|$. Thus we can conclude that $P^2 + V$ has no eivenvalues $E > |k|^2/2$.

6.3. **Aharonov Bohm vector potentials.** In two dimensions the prototypical Aharonov Bohm magnetic vector potential is given by

$$A^{ab}(x,y) = \frac{(-y,x)}{r^2}B_0\,, \quad r = -(x^2+y^2)^{1/2} \tag{6.4}$$

for some $B_0 \in \mathbb{R}$. This yields a locally square integrable on $R^2 \setminus \{0\}$, it corresponds to a singular magnetic field, which is concentrated in zero, i.e., $B\partial_x A_y^{ab} - \partial_y A_x^{ab} = 0$ outside zero, but for any smooth curve S circling once around zero, the line integral along S is given by

$$\int_S (A_x dx + A_y dy) = 2\pi B_0$$

that is, the 'magnetic field' corresponding to A has total flux $2\pi B_0$. The corresponding magnetic Schrödinger operator H_0^{ab} is now defined as the closure of the quadratic form $q_{ab,0}$ defined first on $\mathcal{C}_0^\infty(\mathbb{R}^2 \setminus \{0\})$ as

$$q_{ab,0}(\varphi,\varphi) = \big\langle (P - A^{ab})\varphi, (P - A^{ab})\varphi)\big\rangle$$

and for any potential V which is form small w.r.t. H_0^{ab}, the operator H_V^{ab} is defined as the form sum

$$q_{ab,V}(\varphi,\varphi) \coloneqq q_{ab,0}(\varphi,\varphi) + \big\langle \varphi, V\varphi\big\rangle\,.$$

For such type of singular magnetic Schrödinger operators we still have a virial theorem and a result on absence of positive eigenvalues for the follwing simple reasons:

For dilation, it makes no difference of one works on $\mathbb{R}^2$ or on $\mathbb{R}^2 \setminus \{0\}$. Thus we can still use dilations to derive a virial theorem. In fact, this is easy.

The first thing one has to check if $\mathcal{D}(P - A^{ab})$ is invariant under dilations. Recall equation (3.11), which for the Aharonov Bohm vector potential reads

$$(P - A^{ab})U_t\varphi = e^t U_t P\varphi - U_t A_t^{ab}\varphi = e^t U_t(P - A^{ab})\varphi + U_t(e^t A^{ab} - A_{-t}^{ab})\varphi = e^t U_t(P - A^{ab})\varphi \tag{6.5}$$

since, the Aharonov Bohm vector potential is homogeneous of degree -1, we have $e^t A^{ab} - A^{ab}_{-t} = 0$ for all $t > 0$. That is, the Aharonov Bohm magnetic momentum operator $P - A^{ab}$ has the same commutation properties with dilations as the free momentum P, which drastically simplifies the analysis!

Theorem 6.3 (Aharonov Bohm magnetic virial theorem)**.** *Let A^{ab} be the Aharonov Bohm vector potential and V satisfy assumptions 2.3. Assume also that the distribution $x \cdot \nabla V$ extends to a quadratic form which is form bounded with respect to $(P - A^{ab})^2$. Then for all $\varphi \in \mathcal{D}(P - A^{ab})$, the limit $\lim_{t\to 0} 2\,\mathrm{Re}\,\big(q_{ab}(\varphi, iD_t\varphi)\big)$ exists. Moreover,*

$$\big\langle \varphi, [H^{ab}_V, iD]\,\varphi \big\rangle := \lim_{t\to 0} 2\,\mathrm{Re}\,\big(q_{ab,V}(\varphi, iD_t\varphi)\big) = 2\|(P - A^{ab})\varphi\|_2^2 - \big\langle \varphi, x\cdot\nabla V\varphi\big\rangle. \tag{6.6}$$

This is proven exactly as Theorem 3.9, the extra term from the magnetic field disappears because of the scaling of the Aharonov Bohm vector potential.

Of course, this theorem then also implies absence of positive eigenvalues under the same conditions on the potential V as in Theorem 4.8, now with $\beta = 0$. For the Aharonov–Bohm Hamiltonian H^{ab}_V no eigenvalues E with

$$E > \frac{1}{4}\left(\omega_1 + \sqrt{\omega_1^2 + 2\omega_2}\right)^2 \tag{6.7}$$

exist.

Remarks 6.4. (i) One can also allow for an angular dependence in the Aharonov–Bohm type potential as in [24].

(ii) In addition to the Aharonov–Bohm potential, one can also allow for an additional regular magnetic field B satisfying assumptions 2.1 and 2.2. One has to modify the right hand sides of (6.6) and of (6.7) accordingly.

(iii) On can also consider the Aharonov–Bohm effect in $\mathbb{R}^3$ where the magnetic field is singular along a line l through the origin.

We leave the straightforward modifications of the technical details to the interested reader.

6.4. **Pauli and magnetic Dirac operators.** In this section we state two consequences of Theorem 4.8 and Corollary 5.5. Let $B : \mathbb{R}^2 \to \mathbb{R}$ be given and consider the Pauli operator

$$P(A) = \begin{pmatrix} (i\nabla + A)^2 + B & 0 \\ 0 & (i\nabla + A)^2 - B \end{pmatrix}$$

in $L^2(\mathbb{R}^2, \mathbb{C}^2)$. It is well-known that the operator $P(B)$ is non-negative, and that if $|\int_{\mathbb{R}^2} B| > 2\pi$, then zero is an eigenvalue of $P(B)$, see also Remark 4.9.

Corollary 6.5. *Assume that $B \in L^p_{\mathrm{loc}}(\mathbb{R}^2)$ for some $p > 2$ and that $B(x) = \mathcal{O}(|x|^{-1})$ as $|x| \to \infty$. Let $A \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ be such that* curl $A = B$*. Then the operator $P(A)$ has no eigenvalues in the interval $(4\beta^2, \infty)$, with β given by (1.14).*

If moreover there exists a compact set $K \subset \mathbb{R}^2$ such that $B \in C^1(\mathbb{R}^2 \setminus K)$, then the operator $P(A)$ has no eigenvalues in the interval $\big(\Lambda_P, \infty\big)$, with

$$\Lambda_P := \min\{4\beta^2, \frac{1}{4}(\beta + \omega + \sqrt{(\beta+\omega)^2 + 2\omega})^2\} \tag{6.8}$$

and

$$\omega = \max\left\{ \limsup_{|x|\to\infty} x\cdot\nabla B(x),\ -\liminf_{|x|\to\infty} x\cdot\nabla B(x)\right\}.$$

Proof. The first part of the statement follows from Theorem 4.8 and Proposition 5.5, respectively 5.6 applied to the components of the Pauli operator with the splitting $V_1 = \pm B$, $V_2 = 0$. The second part follows from the first part and from the application of Theorem 4.8 and Corollary 5.5 with the splitting $V_1 = 0$, $V_2 = \pm B$. ∎

Remark 6.6. A couple of comments are in order: (i) The example of Miller and Simon [27], see Section 6.1, applies to two-dimensional Pauli operators as well. In particular, a quick inspection shows that if $B(r) = b_0\, r^{-1} + \mathcal{O}(r^{-2})$, then the spectrum of $P(A)$ is dense pure point in $[0, b_0^2)$ and absolutely continuous in $[b_0^2, \infty)$. Note that in this case Corollary 6.5 guarantees the absence of eigenvalues for $P(A)$ in the interval (b_0^2, ∞), so this result is *sharp*.

(ii) Under the hypotheses of Corollary 6.5 the essential spectrum of $P(A)$ coincides with $[0, \infty)$, see Corollary 7.7 below.

(iii) Absence of positive eigenvalues of the Pauli operator in $\mathbb{R}^3$ will be treated elsewhere.

The second application of Theorem 4.8 concerns magnetic Dirac operators in $L^2(\mathbb{R}^2, \mathbb{C}^2)$ which in the standard representation have the form

$$\mathbb{D} = \begin{pmatrix} m & \mathrm{Q} \\ \mathrm{Q}^* & -m \end{pmatrix}, \qquad \mathrm{Q} = (P_1 - A_1) + i(P_2 - A_2)\,, \tag{6.9}$$

where m is the mass of the particle. We have

Corollary 6.7. *Let B satisfy the assumptions of Corollary 6.5 and let $A \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ be such that* $\operatorname{curl} A = B$. *Then the Dirac operator $\mathbb{D}$ defined on $\mathcal{D}(P - A)$ has no eigenvalues in*

$$\big(-\infty, -\sqrt{\Lambda_P + m^2}\ \big) \cup \big(\sqrt{\Lambda_P + m^2},\, \infty \big).$$

Proof. Note that

$$\mathbb{D}^2 = P(A) + m^2 \mathbb{1} \tag{6.10}$$

in the sense of sesqui-linear forms on $\mathcal{D}(P - A) \oplus \mathcal{D}(P - A)$. Hence if $\mathbb{D}\psi = E\psi$ for some $\psi \in \mathcal{D}(P - A) \oplus \mathcal{D}(P - A)$, then ψ is a weak eigenfunction of $P(A)$ relative to eigenvalue $E^2 - m^2$. In view of Corollary 6.5 we thus have $E^2 - m^2 \leq \Lambda_P$. ∎

Remark 6.8. Sufficient conditions for the absence of the entire point spectrum of Pauli and Dirac operators with electromagnetic fields where recently found in [6], see also Remark 1.7.(v).

7. The essential spectrum

We have the following dichotomy.

Lemma 7.1 (Dichotomy)**.** *Let $A \in L^2_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d)$. Then either* $\inf \sigma((P - A)^2) > 0$ *or* $\sigma((P - A)^2) = [0, \infty)$.

Remark 7.2. The Landau Hamiltonian, where the vector potential A corresponds to a constant magnetic field, provides an example where $\inf \sigma((P - A)^2) > 0$, see [23].

Proof. Write $H_0 = (P - A)^2$. It suffices to prove the implication

$$0 \in \sigma(H_0) \quad \Rightarrow \quad \sigma(H_0) = [0, \infty). \tag{7.1}$$

Let $D(H_0)$ denote the domain of H_0. To prove (7.1) suppose that $0 \in \sigma(H_0)$. Hence there exits a sequence $\{\widetilde{\varphi}\}_{n\in\mathbb{N}} \subset D(H_0)$ such that $\|\widetilde{\varphi}_n\|_2 = 1$ for all $n \in \mathbb{N}$ and

$$\|H_0\, \widetilde{\varphi}_n\|_2 \ \to\ 0 \qquad n \to \infty. \tag{7.2}$$

Now we define

$$\phi_n(x) = e^{ik\cdot x}\, \widetilde{\varphi}_n(x), \tag{7.3}$$

where $k \in \mathbb{R}^d$ is arbitrary. Then in the sense of distributions

$$(P - A)\, \phi_n(x) = e^{ik\cdot x}\, (P - A + k)\, \widetilde{\varphi}_n(x),$$

and

$$H_0\, \phi_n(x) = (P - A)^2\, \phi_n(x) = e^{ik\cdot x}\, H_0\, \widetilde{\varphi}_n(x) + 2e^{ik\cdot x} k \cdot (P - A)\widetilde{\varphi}_n(x) + |k|^2 \phi_n(x).$$

Since $\|\widetilde{\varphi}_n\|_2 = 1$, it follows that $H_0\, \phi_n \in L^2(\mathbb{R}^d)$. Hence $\phi_n \in D(H_0)$. Moreover the above calculations and the Cauchy-Schwarz inequality show that

$$\begin{aligned} \|(H_0 - |k|^2)\, \phi_n\|_2 &\leq \|H_0\, \widetilde{\varphi}_n\|_2 + 2|k|\, \|(P - A)\widetilde{\varphi}_n\|_2 \;=\; \|H_0\, \widetilde{\varphi}_n\|_2 + 2|k| \sqrt{\langle \widetilde{\varphi}_n, H_0\, \widetilde{\varphi}_n \rangle} \\ &\leq\; \|H_0\, \widetilde{\varphi}_n\|_2 + 2|k|\, \|H_0\, \widetilde{\varphi}_n\|_2^{1/2}. \end{aligned}$$

By (7.2) we thus have $\|(H_0 - |k|^2)\, \phi_n\|_2 \to 0$ for any $k \in \mathbb{R}^d$. Hence $[0, \infty) \subseteq \sigma_{\mathrm{ess}}(H_0)$ and since $H_0 \geq 0$, we conclude that $\sigma(H_0) = \sigma_{\mathrm{ess}}(H_0) = [0, \infty)$. ∎

Next we formulate a condition on B under which $\sigma((P - A)^2) = [0, \infty)$ for any locally square integrable vector potential A with $B = dA$.

Definition 7.3 (Vanishing somewhere at infinity)**.** We say that that the magnetic field B *vanishes somewhere at infinity* if there exist sequences $\{R_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ such that $R_n \to \infty$, $|x_n| \to \infty$ as $n \to \infty$, and

$$\lim_{n\to\infty} R_n^{-d} \int_{|y - x_n| < R_n} \left(\frac{|y|}{R_n}\right)^{2-d} \left(\log \frac{R_n}{|y|}\right)^2 \left|B(x_n + y)[y]\right|^2 dy = 0. \tag{7.4}$$

Remark 7.4. This vanishing condition is quite weak. For example, if one has the pointwise bound $|B(x)| \leq d_n$ for $|x - x_n| < R_n$ and $d_n/R_n \to 0$ as $n \to \infty$, then B vanishes somewhere at infinity, since

$$R_n^{-d} \int_{|y - x_n| < R_n} \left(\frac{|y|}{R_n}\right)^{2-d} \left(\log \frac{R_n}{|y|}\right)^2 \left|B(x_n + y)[y]\right|^2 dy \leq \frac{d_n^2}{R_n^2} \int_{|y|<1} |y|^{4-d} (\log |y|)^2\, dy \to 0$$

as $n \to \infty$. Also, we do not require that the magnetic field $B = dA$ exists as a classical vector field outside the sequence of balls $\mathcal{U}_{R_n}(x_n)$.

Theorem 7.5. *Suppose that A is a locally square integrable magnetic vector potential such that the magnetic field $B = dA$ vanishes somewhere at infinity in the sense of Definition 7.3. Then*

$$\sigma((P - A)^2) = \sigma_{\mathrm{ess}}((P - A)^2) = [0, \infty).$$

Remark 7.6. In case that the magnetic field goes to zero pointwise at infinity, the above result was already shown by Miller and Simon, [27, 7]. As pointed out in [27] the invariance of the essential spectrum is quite remarkable, since the the vector potential A corresponding to the magnetic field B might not have any decay at infinity, i.e., the magnetic kinetic energy $(P - A)^2$ is not a small perturbation of the non–magnetic kinetic energy P^2, in general.

Proof. Let R_n and x_n be the sequences defined in Definition 7.3 and let

$$A_n(x) = \int_0^1 B(x_n + t(x - x_n))\, [t(x - x_n)]\, dt \tag{7.5}$$

be the vector potential related to B via the Poincaré gauge centred at x_n. Then $\operatorname{curl} A_n = \operatorname{curl} A = B$ for all $n \in \mathbb{N}$, at least on $\mathcal{U}_n = \mathcal{U}_{R_n}(x_n)$, and therefore there exits, at least locally on $\mathcal{U}_n$, a scalar gauge field $\chi_n : \mathcal{U}_n \to \mathbb{R}$ with $\nabla\chi_n \in L^2(\mathcal{U}_n)$ such that

$$A_n = A - \nabla\chi_n \quad \text{on } \mathcal{U}_n \tag{7.6}$$

and for all φ with $\operatorname{supp}(\varphi) \subset \mathcal{U}_n$ and $(P - A_N)\varphi \in L^2$ we have $e^{i\chi}\varphi \in \mathcal{D}(P - A)$ and $(P_A)e^{i\chi}\varphi = e^{i\chi}(P - A_n)\varphi$, see [25].

Due to the Dichotomy Lemma 7.1 we only have to show that $0 \in \sigma((P - A)^2)$. To this end we will construct a sequence $\{\phi_n\}_n \subset \mathcal{D}(P - A)$ with $\operatorname{supp}(\phi_n \in \mathcal{U}_n)$ and $\|\phi_n\|_2 = 1$ such that

$$\|(P - A)\,\phi_n\|_2^2 \;\to\; 0 \qquad n \to \infty. \tag{7.7}$$

We choose $\phi_n = e^{i\chi_n}\,\varphi_n$, where

$$\varphi_n(x) = C_d\,R_n^{-\frac{d}{2}}\left(1 - \frac{|x - x_n|}{R_n}\right)_+ ,$$

where the constant C_d depends only on d and is chosen such that $\|\phi_n\| = \|\varphi_n\| = 1$. Then by the above gauge invariance

$$\|(P - A)\,\phi_n\|_2^2 = \|(P - A_n)\,\varphi_n\|_2^2 \le \big(\|P\varphi_n\| + \|A_n\varphi_n\|\big)^2. \tag{7.8}$$

We have

$$\|Pv_n\|_2^2 \;\lesssim\; R_n^{-2} \;\to\; 0 \qquad n \to \infty.$$

Moreover, using the fact that $(1-t)^2 \le 1 \le t^{2-d}$ for all $0 < t < 1$, inequality (2.26) and equations (1.9) and (7.5), we obtain

$$\begin{aligned}
\|A_n v_n\|_2^2 &\lesssim R_n^{-d}\int_{\mathcal{U}_{R_n}(x_n)}\left(1 - \frac{|x - x_n|}{R_n}\right)^2 |A_n(x)|^2\,dx \\
&= R_n^{-d}\int_{\mathcal{U}_{R_n}}\left(1 - \frac{|y|}{R_n}\right)^2 |A_n(x_n + y)|^2\,dy \lesssim R_n^{-d}\int_{\mathcal{U}_{R_n}}\left(\frac{|y|}{R_n}\right)^{2-d} |A_n(x_n + y)|^2\,dy \\
&\le 4R_n^{-d}\int_{\mathcal{U}_{R_n}}\left(\frac{|y|}{R_n}\right)^{2-d}\log^2(R_n/|y|)\,\big|B(x_n + y)[y]\big|^2\,dy.
\end{aligned} \tag{7.9}$$

Thus the assumption that B vanishes somewhere at infinity implies $\|A_n v_n\|_2 \to 0$ as $n \to \infty$. By (7.8) this shows

$$\|(P - A)\,\phi_n\|_2^2 \to\; 0$$

as $n \to \infty$, which proves (7.7). Since $\|\phi_n\|_2 = \|v_n\|_2 = 1$ for all $n \in \mathbb{N}$, it follows that $0 \in \sigma(H_0)$ and applying Lemma 7.1 then shows $\sigma_{\text{ess}}((P - A)^2) = [0, \infty)$. ∎

Corollary 7.7. *Suppose that the magnetic field satisfies assumptions 2.1 and 2.2. Then for any locall;y square integrable magnetic vector potential A with $dA = B$ we have*

$$\sigma_{ess}((P - A)^2) = [0, \infty)\,. \tag{7.10}$$

Proof. We apply Theorem 7.5. Let $R_n \to \infty$ as $n \to \infty$, and let $\{x^n\}_n \subset \mathbb{R}^d$ be a sequence with coordinates given by

$$x_l^n = (n+1)\,R_n \qquad \forall\, l = 1, \dots d, \quad \forall\, n \in \mathbb{N} \tag{7.11}$$

and let

$$C_n := R_n^{-d}\int_{\mathcal{U}_n}\left(\frac{|y|}{R_n}\right)^{2-d}\left(\log\frac{R_n}{|y|}\right)^2 \big|B(x^n + y)[y]\big|^2\,dy.$$

We have to show that $C_n \to 0$ as $n \to \infty$. If we write $x = y + x^n$, then for every $y \in \mathcal{U}_{R_n}$ we get, in view of (1.9) and the Cauchy-Schwarz inequality,

$$\big(B(x^n+y)[y]\big)_j^2 = \left(\sum_{m=1}^d B_{j,m}(x)\, x_m\, \frac{y_m}{x_m}\right)^2 \leq \max_{1\leq m\leq d} \frac{1}{x_m^2}\, \big(\widetilde{B}(x)[x]\big)\big)_j^2\, |y|^2 = \frac{1}{n^2 R_n^2}\, \big(\widetilde{B}(x)[x]\big)\big)_j^2\, |y|^2 .$$

Hence

$$C_n \;\leq\; \frac{1}{n^2 R_n^4}\, \big\langle u_n, |\widetilde{B}|^2\, u_n\big\rangle, \tag{7.12}$$

where

$$u_n(x) := |x-x_n|^{2-\frac{d}{2}} \left(\log \frac{R_n}{|x-x_n|}\right)_+ .$$

A quick calculation now shows that $\|u_n\|_2^2 + \|\nabla u_n\|_2^2 \lesssim R_n^4$, which together with (2.16) and (7.12) further implies

$$C_n \;\lesssim \frac{1}{n^2} + \frac{1}{n^2 R_n^4}\, \big\langle u_n, |A|^2\, u_n\big\rangle . \tag{7.13}$$

We now estimate the last term on the right hand side as follows

$$\begin{aligned}
\big\langle u_n, |A|^2\, u_n\big\rangle \;&= \int_{\mathcal{U}_{R_n}(x_n)} |x-x_n|^{4-d} \left(\log \frac{R_n}{|x-x_n|}\right)^2 |A(x)|^2\, dx \\
&= R_n^{4-d} \int_{\mathcal{U}_{R_n}(x_n)} \left(\frac{|x-x_n|}{R_n}\right)^{4-d} \left(\log \frac{R_n}{|x-x_n|}\right)^2 |A(x)|^2\, dx \\
&\lesssim\; R_n^{4-d} \int_{\mathcal{U}_{R_n}} \left(\frac{|y|}{R_n}\right)^{2-d} |A(y+x_n)|^2\, dy \;\lesssim\; R_n^4\, C_n,
\end{aligned}$$

where in the last step we have used again inequality (2.26) and equation (1.9). Plugging this estimate into (7.13) gives $C_n \to 0$ as $n \to \mathbb{N}$, and therefore, by Theorem 7.5, $\sigma_{\rm ess}((P-A)^2) = [0,\infty)$. ∎

Theorem 7.8. *Suppose that A is a locally square integrable magnetic vector potential and the potential V is form small and vanishes at infinity w.r.t $(P-A)^2$. Then*

$$\sigma_{\rm ess}(H_{A,V}) = \sigma_{\rm ess}((P-A)^2). \tag{7.14}$$

Remark 7.9. We do not assume that V is form compact w.r.t $(P-A)^2$!

Proof. Since V is form small with respect to $(P-A)^2$, the quadratic form $q_{A,V}$ is closed and bounded from below on the form domain $\mathcal{D}(P-A)$. Hence there exists $s \geq 1$ such that the operators $H_{A,0}+s$ and $H_{A,V}+s$ are invertible in $L^2(\mathbb{R}^d)$. We are going to prove that the resolvent difference

$$(H_{A,0}+s)^{-1} - (H_{A,V}+s)^{-1} \quad \text{is compact in} \;\; L^2(\mathbb{R}^d). \tag{7.15}$$

for some large enough $s \geq 1$, which by Weyls theorem implies that the essential spectra of $H_{A,V}$ and $(P-A)^2$ coincide.

In the following, we will abbreviate $H_0 = H_{A,0}$. Let $C_s := (H_0+s)^{-1/2} V (H_0+s)$, more precisely, C_s is the bounded operator associated with the bounded form

$$\big\langle \varphi, C_s \varphi\big\rangle := \big\langle (H_0+s)^{-1/2}\varphi, V(H_0+s)^{-1/2}\varphi\big\rangle = q_V((H_0+d\lambda)^{-1/2}\varphi, (H_0+s)^{-1/2}\varphi)\,,$$

and the relative form bound of V w.r.t $(P-A)^2$ is given by $\lim_{s\to\infty}\|Cs\|_{2\to 2}<1$, see [36, Theorem 6.24], [30]. Choose λ large enough, such that $\|C_\lambda\|<1$. Then Tiktopoulos' formula, [32, Chapter II.3], [36, Theorem 6.25] shows

$$(H_{A,V}+s)^{-1}=(H_0+s)^{-1/2}(1-C_s)^{-1}(H_0+s)^{-1/2}\,.$$

Hence

$$(H_0+s)^{-1}-(H_{A,V}+s)^{-1}=(H_0+s)^{-1/2}(1-C_s)^{-1}C_s(H_0+s)^{-1/2}$$

so we only have to show that

$$C_s(H_0+s)^{-1/2}=(H_0+s)^{-1/2}V(H_0+s)^{-1}$$

is a compact operator. For this let $\xi_{<R},\xi_{\geq R}$ the smooth partition from the proof of Lemma 4.4 with $\xi_{<R}^2+\xi_{\geq R}^2=1$, $\mathrm{supp}(\xi_{<R})\subset\mathcal{U}_{2R}$, $\mathrm{supp}(\xi_{\geq R})\subset\mathcal{U}_R^c$, and $\|\nabla\xi_{<R}\|_\infty,\|\nabla\xi_{\geq R}\|_\infty\lesssim R^{-1}$. With

$$J_{<R}\coloneqq(H_0+s)^{-1/2}\xi_{<R}^2V(H_0+s)^{-1} \tag{7.16}$$

$$J_{\geq R}\coloneqq(H_0+s)^{-1/2}\xi_{\geq R}^2V(H_0+s)^{-1} \tag{7.17}$$

we obviously have $(H_0+s)^{-1/2}V(H_0+s)^{-1}=J_{<R}+J_{\geq R}\,.$

We will show that $\lim_{R\to\infty}\|J_{\geq R}\|_{2\to 2}=0$. So $(H_0+d\lambda)^{-1/2}V(H_0+d\lambda)^{-1}$ is the norm limit of $J_{<R}$ as $R\to\infty$, in particular, it is a compact operators if $J_{<R}$ is compact for all large R. Since

$$\|J_{\geq R}\|_{2\to 2}=\sup_{\|f\|=1}|\langle f,J_{\geq R}f\rangle| \tag{7.18}$$

and with $\varphi=(H_0+s)^{-1/2}f$

$$\begin{aligned}|\langle f,J_{\geq R}f\rangle|&=|\langle\xi_{\geq R}\varphi,V\xi_{\geq R}\varphi\rangle|\leq\alpha_R\|(P-A)\xi_{\geq R}\varphi\|_2^2+\gamma_R\|\xi_{\geq R}\varphi\|_2^2\\&\leq\alpha_R\Big(\|(P-A)\varphi\|+\|\nabla\xi_{\geq R}\|\|\varphi\|\Big)^2+\gamma_R\|\varphi\|_2^2\lesssim\big(\alpha_R(1+R^{-1})^2+\gamma_R\big)\|f\|_2^2\end{aligned}$$

since $(P-A)\xi_{\geq R}\varphi=\xi_{\geq R}(P-A)\varphi-i(\nabla\xi_{\geq R})\varphi$, $\|(P-A)\varphi\|\leq\|f\|$ and $\|\varphi\|\leq s^{-1}\|f\|$. From this and (7.18) one immediately gets $\|J_{\geq R}\|_{2\to 2}\lesssim\alpha_R(1+R^{-1})^2+\gamma_R\to 0$ for $R\to\infty$.

To prove that $J_{<R}$ is compact, we first note that the domain of $H_0=(P-A)^2$ is given by all $\varphi\in\mathcal{D}(P-A)$ for which with $\psi=(P-A)$ the distribution $(P-A)\psi$ is also in L^2. Thus for all $\varphi\in\mathcal{D}((P-A)^2)$ we have

$$(H_0+s)^{-1}(P-A+i\lambda)\cdot(P-A-i\lambda)\varphi=(H_0+s)^{-1}(H_0+d\lambda^2)\varphi=\varphi$$

when $s=d\lambda^2$. Moreover, when $\varphi\in\mathcal{D}((P-A)^2)$ and χ is a bounded $\mathcal{C}^2$ function such that $\nabla\chi$ and $\Delta\chi$ are bounded, then

$$\begin{aligned}(P-A-i\lambda)\chi\varphi&=\chi(P-A-i\lambda)\varphi-i(\nabla\chi)\varphi\in L^2\,,\\(P-A+i\lambda)\cdot(P-A-i\lambda)\chi\varphi&=\chi(P-A+i\lambda)\cdot(P-A-i\lambda)\varphi-2i(\nabla\chi)\cdot(P-A)\varphi-(\Delta\chi)\varphi\\&=\chi(H_0+d\lambda^2)\varphi-2i(\nabla\chi)\cdot(P-A)\varphi-(\Delta\chi)\varphi\in L^2\end{aligned}$$

so also $\chi\varphi\in\mathcal{D}((P-A)^2)$.

Use $\varphi=(H_0+s)^{-1}f$ with $f\in L^2$ and choose $d\lambda^2=s$. Then the last equality yields

$$\begin{aligned}\chi(H_0+s)^{-1}f=\chi\varphi&=(H_0+s)^{-1}(P-A+i\lambda)\cdot(P-A-i\lambda)\chi\varphi\\&=(H_0+s)^{-1}\chi f-2i(H_0+s)^{-1}(\nabla\chi)\cdot(P-A)(H_0+s)^{-1}f-(H_0+s)^{-1}(\Delta\chi)(H_0+s)^{-1}f\,.\end{aligned}$$

Setting $\chi=\xi_{<R}^2$ one sees that $J_{<R}$ can be written as

$$J_{<R}=C_s\Big(J_1-2iJ_2\cdot(P-A)(H_0+s)^{-1}-J_3(H_0+s)^{-1}\Big)\,. \tag{7.19}$$

where we abbreviated $J_1 = (H_0+s)^{-1/2}\chi$, $J_2 = (H_0+s)^{-1/2}(\nabla\chi)$, and $J_3 = (H_0+s)^{-1/2}(\Delta\chi)$.

Note that C_s is bounded and so are $(P-A)(H_0+s)^{-1}$ and $(H_0+s)^{-1}$. Moreover, since $\chi = \xi_{<R}^2$ has compact support, it is well–know that the operators $\chi(P^2+s)^{-1/2}$, $(\nabla\chi)(P^2+s)^{-1/2}$, and $(\Delta\chi)(P^2+s)^{-1/2}$ are compact operators on L^2, see [8, Thm. 5.7.3], for example. The diamagnetic inequality and the Dodds–Fremlin–Pitt theorem [9, 28] then imply that the operators $\chi(H_0+s)^{-1/2}$, $(\nabla\chi)(H_0+s)^{-1/2}$, and $(\Delta\chi)(H_0+s)^{-1/2}$ are also compact, and by duality so are J_1, J_2, and J_3. Thus by (7.19) the operator $J_{<R}$ is a compact operator for all $R>0$. ∎

Corollary 7.10. *Suppose that B satisfies assumptions 2.1, 2.2, and that V is satisfies assumptions 2.3 and 2.6. Then*

$$\sigma_{\mathrm{ess}}(H_{A,V}) = [0,\infty). \tag{7.20}$$

Proof. Combine Theorem 7.5 and Corollary 7.7. ∎

Appendix A. Gronwall type bounds

Lemma A.1. *Let $T>0$ and let $w, E : [0,T] \to [0,\infty)$. If for some $c>0$*

$$w(t) \le E(t) + c\int_0^t e^{t-s}\, w(s)\, ds, \tag{A.1}$$

for all $t\in[0,T]$, then

$$w(t) \le E(t) + c\int_0^t e^{(1+c)(t-s)}\, E(s)\, ds \qquad \forall\, t\in[0,T]. \tag{A.2}$$

Moreover, if

$$w(t) \le E(t) + c\int_0^t e^{s-t}\, w(s)\, ds, \tag{A.3}$$

for all $t\in[0,T]$, then

$$w(t) \le E(t) + c\int_0^t e^{(c-1)(t-s)}\, E(s)\, ds \qquad \forall\, t\in[0,T]. \tag{A.4}$$

Proof. Put $v(t) := \int_0^t e^{t-s}\, w(s)\, ds$. Then $v(0)=0$ and, assuming (A.1),

$$v'(t) = v(t) + w(t) \le E(t) + (1+c_0)v(t)$$

Hence

$$\frac{d}{dt}\Big(e^{-(1+c_0)t}\, v(t)\Big) = e^{-(1+c_0)t}(v'(t) - (1+c_0)v(t)) \le e^{-(1+c_0)t}\, E(t).$$

It follows that

$$e^{-(1+c_0)t}\, v(t) = \int_0^t \frac{d}{ds}\Big(e^{-(1+c_0)s}\, v(s)\Big)\, ds \le \int_0^t e^{-(1+c_0)s}\, E(s)\, ds\,.$$

This implies

$$v(t) \le \int_0^t e^{(1+c_0)(t-s)}\, E(s)\, ds,$$

and (A.2) follows. ∎

Appendix B. Optimizing the threshold

It is tempting to split the potential $V = V_1 + V_2$ at infinity in order to optimize the threshold above which one can exclude existence of eigenvalues. Using $V_1 = sV$ and $V_2 = (1-s)V$, Theorem 4.8 shows the non–existence of eigenvalues with

$$E > \frac{1}{4}\Big(\beta + \omega_1 s + \sqrt{(\beta+\omega_1 s)^2 + 2\omega_2(1-s)}\Big)^2 = \frac{\omega_1^2}{4}(g(s))^2$$

where for $0 \le s \le 1$ we set

$$g(s) \coloneqq b + s + \sqrt{(b+s)^2 + 2c(1-s)} \tag{B.1}$$

with $b = \beta/\omega_1$ and $c = \omega_2/\omega_1^2$. The goal is to minimize g over $s \in [0,1]$.

Lemma B.1 (Bang–Bang type Lemma)**.** *For g given in (B.1) we have $\min_{0\le s\le 1} g(s) \ge \min(g(0), g(1))$. More precisely,*

$$\min_{0\le s\le 1} g(s) = \begin{cases} g(0) & \text{if} \quad c < 2b+2 \\ g(1) & \text{if} \quad c > 2b+2 \end{cases} \tag{B.2}$$

and g is constant if $c = 2b+2$.

Proof. Write $c = 2b+2+r$. Then $(b+s)^2 + 2c(1-s) = (b+2-s)^2 + 2r(1-s)$, hence

$$g(s) = b + s + \sqrt{(b+2-s)^2 + 2r(1-s)}$$

for all $0 \le s \le 1$. Note that g is clerly constant on $[0,1]$ if $r = 0$. On $[0,1]$ the derivative of g is given by

$$g'(s) = 1 + \left((b+2-s)^2 + 2r(1-s)\right)^{-1/2}\big(s - (b+2+r)\big)\,.$$

Fix $0 \le s \le 1$. A calculation shows

$$\Big(\left((b+2-s)^2 + 2r(1-s)\right)^{-1/2}\big(s - (b+2+r)\big)\Big)^2 > 1$$

if and only if $0 < r(r+2b+2) = rc$. Since $c \ge 0$, this implies that if $r < 0$, i.e., $c < 2b+2$, we have $g' > 0$ on $[0,1]$, i.e., g is strictly increasing on $[0,1]$.

On the other hand, if $c > 2b+2$, then also $c - b > b+2 \ge 2$ and $r < 0$, so $g' < 0$ on $[0,1]$, i.e., g is strictly decreasing on $[0,1]$. This proves the lemma. ∎

Corollary B.2. *Setting*

$$\beta^2 \coloneqq \gamma_\infty\big(\widetilde{B}^2\big), \quad \omega_1^2 \coloneqq \gamma_\infty\big((xV)^2\big), \quad \omega_2 \coloneqq \gamma_\infty^+\big(x\cdot\nabla V\big) \tag{B.3}$$

the threshold $\Lambda(B,V)$ defined in (1.13) optimized for splitting the potential as $V = V_1 + V_2$ with $V_1 = sV$, $V_2 = (1-s)V$ and $0 \le s \le 1$ is given by

$$\widetilde{\Lambda}(B,V) = \begin{cases} \frac{1}{2}\left(\beta^2 + \omega_2 + \beta\sqrt{\beta + 2\omega_2}\right) & \text{if} \quad \omega_2 \le 2\omega_1(\beta+\omega_1) \\ (\beta+\omega_1)^2 & \text{if} \quad \omega_2 > 2\omega_1(\beta+\omega_1) \end{cases} \tag{B.4}$$

Proof. Given Lemma B.1 this is just a simple calculation. ∎

Appendix C. IMS localization formula

In one step in the proof of Lemma 4.6 we need a quadratic form version of the well-known IMS localization formula under minimal assumptions on the quadratic form of the magnetic Schrödinger operator. This should be well–known, but we could not find the version we need in the literature.

Theorem C.1 (IMS localization formula). *Let A be a locally square integrable magnetic vector potential and V form small w.r.t. $(P-A)^2$. Then for all bounded real–valued $\xi \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that $\nabla\xi$ is also bounded and all $\varphi \in \mathcal{D}(P-A)$, also $\xi\varphi$ and $\xi^2\varphi \in \mathcal{D}(P-A)$ and*

$$\operatorname{Re} q_{A,V}(\xi^2\varphi,\varphi) = q_{A,V}(\xi\varphi,\xi\varphi) - \big\langle \varphi, |\nabla\xi|^2\varphi\big\rangle \tag{C.1}$$

Proof. As before, one easily checks that $\xi\varphi$ and $\xi^2\varphi$ are in the domain of $P-A$ when φ is. Moreover, the potential V commutes with the multiplication operator ξ, so as quadratic forms $\big\langle \xi^2\varphi, V\varphi\big\rangle = \big\langle \xi\varphi, V\xi\varphi\big\rangle$ and we only have to check the kinetic energy term. Since $(P-A)(\xi^2\varphi) = \xi(P-A)(\xi\varphi) + (P\xi)\xi\varphi$ a short calculation reveals

$$\begin{aligned}\big\langle (P-A)(\xi^2\varphi),(P-A)\varphi\big\rangle &= \big\langle (P-A)(\xi\varphi),(P-A)(\xi\varphi)\big\rangle + \big\langle (P\xi)\varphi,(P-A)(\xi\varphi)\big\rangle \\ &\quad - \big\langle (P-A)(\xi\varphi),(P\xi)\varphi)\big\rangle - \big\langle \varphi, |\nabla\xi|^2\varphi\big\rangle,\end{aligned}$$

so

$$\begin{aligned}\operatorname{Re} q_{A,0}(\xi^2\ \varphi,\varphi) &= \operatorname{Re}\big\langle (P-A)(\xi^2\varphi),(P-A)\varphi\big\rangle \\ &= \big\langle (P-A)(\xi\varphi),(P-A)(\xi\varphi)\big\rangle + \big\langle \varphi, |\nabla\xi|^2\varphi\big\rangle\end{aligned}$$

which proves (C.1). ∎

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