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RESEARCH

The influence of resonator outcoupling and losses on the photon statistics of the LASER

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Abstract
An extension to the quantum-mechanical laser master equation for the density operator is derived to incorporate the beam-splitter effect caused by typical dielectric laser output couplers. This effect gives rise to a significant change in the photon statistical distribution of the part of the laser light reflected back into the resonator and, therefore, may have an influence on the total laser output photon statistics. Different cases without and with additional intra-cavity losses were discussed and their influence on the expected laser photon statistics was deduced. As a result, it was found that the well-known Poisson distribution of laser light is in most cases the result of additional losses or absorption, which act uncorrelatedly on single photons. In a laser with negligible additional losses where outcoupling is dominated by the beam-splitter effect, the photon statistics reveal to be mainly non-Poisson. A Poisson distribution would only occur for very low outcoupling rates, i.e., high finesse cavities. It is found that in the limit of strong outcoupling even the distribution of thermal light can result.

Keywords Laser theory · Beam splitter · Photon statistics · Poisson distribution

1 Introduction

Quantum-mechanical theories of lasers exist since the early days of laser research, being more or less complex in terms of how the various laser transition types and interactions with the outside world are taken into account. Therein different approaches are known using different but equivalent formulations either based on a density matrix description or on the Fokker–Planck equation [1, 2]. In all these descriptions an important approximation and assumption have been used since: The outcoupling of photons out of the laser resonator was modeled by a loss term in analogy to what an equivalent intra-cavity absorption of photons would cause to the laser quantum state, thereby incorporating the finite cavity Q-factor into the description of the laser. Although this procedure is consistent in terms of energy and average photon number it has so far not been considered that a typical dielectric laser output mirror is not just a photon extracting device like an absorbing or scattering object. Moreover, such an output coupler has a complex action on the incoming intra-cavity photon state and is in fact nothing else than a beam splitter with a certain splitting ratio. The photon state, therefore, reflected back into the laser resonator is not just shifted to a lower average photon number by the number of lost photons but a rather complex combination of various number states that are binomially distributed. The goal of this paper is to investigate the consequences of these findings on the laser photon statistics of a continuous-wave (cw) laser and to compare the results with the common laser theory used so far.

2 The laser output coupler—a beam splitter

In many cases, lasers are equipped with an output coupler (OC) mirror that physically is simply a beam splitter with a dielectric coating and thus a certain reflectivity R and transmission T as shown in Fig. 1. Assuming a loss-less OC we find $R + T = 1$. This, however, has specific consequences for a quantum state that passes through, or is reflected by, that beam splitter. The goal of this section is to derive the temporal

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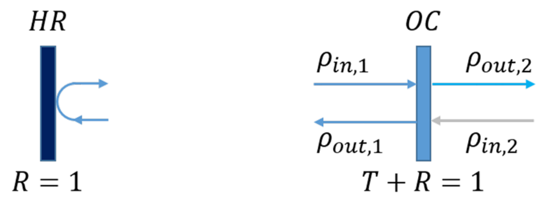

Fig. 1 Schematic view of a laser cavity with a high-reflective mirror *HR* and its output coupler *OC* acting as a beam splitter (laser medium not shown)

evolution of a quantum state in an empty laser resonator subjected only to outcoupling via said beam splitter. The effects caused by the laser medium will be treated in Sect. 3.

To derive the effect of the beam-splitting property of the OC has on the quantum state of a laser in Sect. 3, we first consider a given state

$$\rho_{\mathrm{in},1} = \sum_{n,m} \rho_{n,m} |n\rangle\langle m|, \tag{1}$$

incident onto the OC mirror from inside the laser cavity (beam splitter input channel 1). Therein, $\rho_{n,m}$ are the coefficients of the density operator in number-state representation and n and m are the photon number state indices. The laser output is considered as the beam splitter output channel 2. When there is no back reflection of the outcoupled laser beam into itself, nor an intentional external state to be launched into the lasing mode through the OC from outside the laser cavity (e.g. by injection), we can consider that the input channel 2 of the beam splitter is the vacuum state

$$\rho_{in,2} = |0\rangle\langle 0|. \tag{2}$$

A beam splitter is a quantum mechanically described by the operator [3]

$$B_\theta = e^{i\theta\left(a^\dagger b - ab^\dagger\right)}, \tag{3}$$

with a and b being the annihilation operators acting on the input states 1 and 2, respectively, and θ being a mixing angle. Therein, we have neglected eventual additional phase shifts that may arise from the specific design of the beam splitter but which are of no importance to the findings in this paper. The result of the beam splitting operation in the output channels can be described by the output annihilation operators

$$\begin{aligned} a' &= B_\theta a B_\theta^\dagger = ac - bs, \\ b' &= B_\theta b B_\theta^\dagger = as + bc, \end{aligned} \tag{4}$$

acting on the output states 1 and 2, respectively. Therein $|c| = \sqrt{R} = \cos\theta$ and $|s| = \sqrt{T} = \sin\theta$, such that $|c|^2 + |s|^2 = 1$. We can thus easily derive the generally combined output state generated by the beam splitter interaction from the input state $\rho_{\mathrm{in},1} \otimes \rho_{\mathrm{in},2}$ as

$$\begin{aligned} \rho_{\mathrm{out}} =& \rho_{\mathrm{out},1} \otimes \rho_{\mathrm{out},2} = B_\theta\left(\rho_{\mathrm{in},1} \otimes \rho_{\mathrm{in},2}\right)B_\theta^\dagger \\ =& B_\theta\left(\sum_{n,m} \rho_{n,m}|n\rangle\langle m| \otimes |0\rangle\langle 0|\right)B_\theta^\dagger \\ =& \sum_{n,m} \rho_{n,m} B_\theta\left(\frac{\left(a^\dagger\right)^n}{\sqrt{n!}}|0\rangle\langle 0|\frac{(a)^m}{\sqrt{m!}} \otimes |0\rangle\langle 0|\right)B_\theta^\dagger \\ =& \sum_{n,m} \rho_{n,m}\left(\frac{\left(a'^\dagger\right)^n}{\sqrt{n!}}|0\rangle\langle 0|\frac{\left(a'\right)^m}{\sqrt{m!}} \otimes |0\rangle\langle 0|\right) \\ =& \sum_{n,m} \rho_{n,m}\frac{\left(a^\dagger c - b^\dagger s\right)^n}{\sqrt{n!}}|0\rangle\frac{\left(a^\dagger c^* - b^\dagger s^*\right)^m}{\sqrt{m!}}\langle 0| \otimes |0\rangle\langle 0| \\ =& \sum_{n,m} \rho_{n,m}\frac{1}{\sqrt{n!}}\frac{1}{\sqrt{m!}}\sum_{q=0}^{n}\binom{n}{q}c^{n-q}(-s)^q\sqrt{(n-q)!\,q!} \\ & \times \sum_{r=0}^{m}\binom{m}{r}\left(c^*\right)^{m-r}\left(-s^*\right)^r\sqrt{(m-r)!\,r!} \\ & \times |n-q\rangle\langle m-r| \otimes |q\rangle\langle r| \\ =& \sum_{n,m} \rho_{n,m}\sum_{q=0}^{n}\sum_{r=0}^{m}\binom{n}{q}^{\frac{1}{2}}\binom{m}{r}^{\frac{1}{2}}c^{n-q}(-s)^q\left(c^*\right)^{m-r}\left(-s^*\right)^r \\ & \times |n-q\rangle\langle m-r| \otimes |q\rangle\langle r|. \end{aligned} \tag{5}$$

Therein, $\binom{n}{k} = n!/(k!(n-k)!)$ is the binomial coefficient and q and r are additional numbering indices arising from the sum expansion of the binomial formulas, related to the number states in the laser output beam. Using the identity

$$\sum_{n,m=0}^{\infty}\sum_{q=0}^{n}\sum_{r=0}^{m} \equiv \sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\sum_{n=q}^{\infty}\sum_{m=r}^{\infty}, \tag{6}$$

we obtain

$$\begin{aligned} \rho_{\mathrm{out}} =& \sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\sum_{n=q}^{\infty}\sum_{m=r}^{\infty}\rho_{n,m}\binom{n}{q}^{\frac{1}{2}}\binom{m}{r}^{\frac{1}{2}}c^{n-q}(-s)^q(c^*)^{m-r}(-s^*)^r \\ & \times |n-q\rangle\langle m-r| \otimes |q\rangle\langle r| \\ =& \sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\rho_{n+q,m+r}\binom{n+q}{q}^{\frac{1}{2}}\binom{m+r}{r}^{\frac{1}{2}}c^n(-s)^q(c^*)^m(-s^*)^r \\ & \times |n\rangle\langle m| \otimes |q\rangle\langle r|. \end{aligned} \tag{7}$$

Thus a general superposition of combined output states of the beam splitter results in which each laser output state $|q\rangle\langle r|$ in the output channel 2 is accompanied by a reflected state in the output channel 1 of

$$\begin{aligned} & \rho_{\mathrm{out},1}(|q\rangle\langle r|) = \\ & \sum_{n,m}\rho_{n+q,m+r}\binom{n+q}{q}^{\frac{1}{2}}\binom{m+r}{r}^{\frac{1}{2}}c^n(-s)^q(c^*)^m(-s^*)^r \\ & \times |n\rangle\langle m|. \end{aligned} \tag{8}$$

However, we have to consider that the outcoupled laser beam, i.e., the state of the output channel 2 of the beam splitter, will usually be measured. This measurement quantum mechanically has as result that we have to take the trace over the channel 2 output states, thereby finally defining the state being reflected by the OC into the cavity (i.e., the new intra-cavity state after beam-splitter action,)

$$\rho'_{\mathrm{out},1} = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty} \rho_{\mathrm{out},1}(|q\rangle\langle r|)\delta_{qp}\delta_{rp}, \tag{9}$$

with its coefficients

$$\begin{aligned}\rho'_{n,m} &= \sum_{p=0}^{\infty} \rho_{n+p,m+p} \begin{pmatrix} n+p \\ p \end{pmatrix}^{\frac{1}{2}} \begin{pmatrix} m+p \\ p \end{pmatrix}^{\frac{1}{2}} c^n (c^*)^m |s|^{2p} \\ &= \sum_{p=0}^{\infty} \rho_{n+p,m+p} \begin{pmatrix} n+p \\ n \end{pmatrix}^{\frac{1}{2}} \begin{pmatrix} m+p \\ m \end{pmatrix}^{\frac{1}{2}} c^n (c^*)^m |s|^{2p}.\end{aligned} \tag{10}$$

The δ_{nm} in Eq. 9 are the Kronecker symbols evaluating to unity for identical indices n and m while being zero otherwise. Both versions are equivalent as a result of the symmetry of the binomial coefficient. Itcan be easily shown that the distribution in Eq. 10 is normalized to

$$\sum_{n=0}^{\infty} \rho'_{n,n} = 1, \tag{11}$$

and thus represents a physical distribution, which is not the case for the distribution in Eq. 8. Accordingly, it can be easily shown that the average photon number reflected back into the cavity is

$$\langle n \rangle' = \sum_{n=0}^{\infty} n\rho'_{n,n} = \sum_{k=0}^{\infty} k\rho_{k,k}|c|^2 = |c|^2\langle n \rangle = R\langle n \rangle. \tag{12}$$

Therefore, in each reflection step, i.e., after each round-trip in the empty laser cavity, the originally contained average photon number is reduced according to the reflectivity R of the OC as expected. The average photon number after k round trips thus obtains as

$$\langle n \rangle_k = |c|^{2k}\langle n \rangle_0 = e^{k\ln|c|^2}\langle n \rangle_0. \tag{13}$$

As the photon state is, however, spread over the whole cavity and not localized, outcoupling is happening in a continuous way. One round trip takes the cavity round-trip time $\tau_{\mathrm{RT}} = 2L_o/c$ with L_o being the optical length of the cavity, and thus we can set $k \to t/\tau_{\mathrm{RT}}$ to obtain

$$\langle n \rangle(t) = \langle n \rangle(0) e^{\frac{\ln|c|^2}{\tau_{\mathrm{RT}}}t} = \langle n \rangle(0) e^{-\frac{t}{\tau_c}}. \tag{14}$$

As a consequence, the cavity photon lifetime is given by

$$\tau_c = -\frac{\tau_{\mathrm{RT}}}{\ln|c|^2} = -\frac{\tau_{\mathrm{RT}}}{\ln R} = -\frac{2L_o}{c\ln R}, \tag{15}$$

being identical to the semi-classically derived value [4]. This allows us to derive the equation of motion of an empty resonator with an outcoupling beam splitter according to

$$\dot{\rho}_{n,m}|_{\mathrm{OC}} = \frac{\Delta\rho_{n,m}}{\Delta t} = \frac{\rho'_{n,m} - \rho_{n,m}}{\Delta t} \tag{16}$$

Therein, Δt is an effective time interval over which the transition $\rho_{n,m} \to \rho'_{n,m}$ takes place. It can be determined from the equivalence with the decay corresponding to Eq. 14 by evaluating the decay of the average photon number, for which the derivative of the photon number in time has to be identical to the photon number multiplied by the negative exponential decay rate $\frac{1}{\tau_c}$:

$$\langle \dot{n} \rangle = \sum_{n=0}^{\infty} n\dot{\rho}_{n,n}|_{\mathrm{OC}} = \sum_{n=0}^{\infty} \frac{n\rho'_{n,n} - n\rho_{n,n}}{\Delta t} = \frac{|c|^2 - 1}{\Delta t}\langle n \rangle \overset{!}{=} -\frac{1}{\tau_c}\langle n \rangle \tag{17}$$

$$\Rightarrow \Delta t = \left(1 - |c|^2\right)\tau_c = -\frac{\left(1 - |c|^2\right)}{\ln|c|^2}\tau_{\mathrm{RT}} = -\frac{1-R}{\ln R}\tau_{\mathrm{RT}}. \tag{18}$$

The effective time interval thus gradually changes from very short times for $R \to 0$ to the round-trip time τ_{RT} for $R \to 1$ as can be seen in Fig. 2.

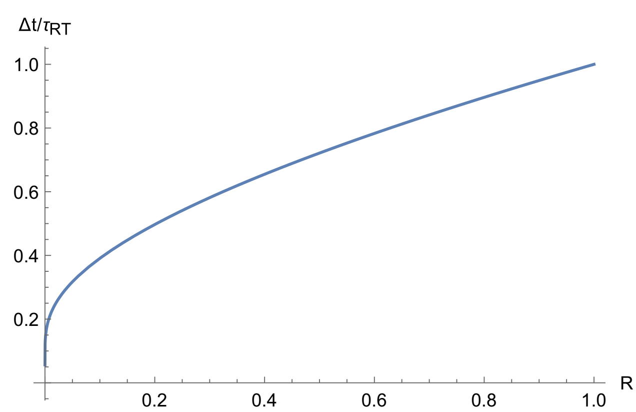

Fig. 2 Effective time interval of the transition $\rho_{n,m} \to \rho'_{n,m}$ in fractions of the round-trip time as a function of the OC reflectivity R

As the result, the equation of motion of the empty resonator with an outcoupling beam splitter is given by

$$\dot{\rho}_{n,m}|_{\mathrm{OC}} = -\frac{\ln|c|^2}{(1-|c|^2)\tau_{\mathrm{RT}}} \times \left(\rho_{n,m} - \sum_{p=0}^{\infty} \rho_{n+p,m+p} \binom{n+p}{n}^{\frac{1}{2}} \binom{m+p}{m}^{\frac{1}{2}} c^n (c^*)^m |s|^{2p}\right). \tag{19}$$

3 The density-operator master equation of the laser

3.1 The original master equation

The quantum-mechanical description of the temporal evolution of the intra-cavity laser state is governed by the well-known master equation for the density-operator coefficients in number-state representation, here in its original form taken from [1]

$$\dot{\rho}_{n,m} = -\frac{N'_{n,m}A}{1+N_{n,m}B/A}\rho_{n,m} + \frac{\sqrt{nm}A}{1+N_{n-1,m-1}B/A}\rho_{n-1,m-1} - \frac{1}{2}\frac{\omega}{Q_{\mathrm{c}}}(n+m)\rho_{n,m} + \frac{\omega}{Q_{\mathrm{c}}}\sqrt{(n+1)(m+1)}\rho_{n+1,m+1}, \tag{20}$$

with ω being the laser angular frequency and

$$A = 2\left(\frac{g}{\gamma}\right)^2 R_p \tag{21}$$

$$B = 4\left(\frac{g}{\gamma}\right)^2 A = 8\left(\frac{g}{\gamma}\right)^4 R_{\mathrm{p}} \tag{22}$$

$$N'_{n,m} = \frac{1}{2}(n+m+2) + \frac{(n-m)^2 B}{8A} \tag{23}$$

$$N_{n,m} = \frac{1}{2}(n+m+2) + \frac{(n-m)^2 B}{16A}. \tag{24}$$

A is the unsaturated linear gain proportional to the pump rate R_{p}, B is a saturation parameter, γ is the decay constant of the upper laser level and g the field coupling constant of the laser transition, proportional to the vacuum Rabi frequency. In this master equation, the first two terms describe the changes to the density operator matrix elements due to gain and gain saturation, while the latter two terms in the equation introduce cavity outcoupling by assuming an absorption of photons inside the cavity by injected atoms in the ground state, such that this absorption yields the corresponding Q-factor of the laser cavity, Q_{c} [1]. It directly follows from this original version of the master equation that changes in $\rho_{n,m}$ arising from outcoupling are only due to a coupling to $\rho_{n,m}$ itself and to $\rho_{n+1,m+1}$. However, from Eq. 19 we immediately see that the random partitioning of the beam splitter behavior of the OC mirror causes much more complicated couplings and therefore, Eq. 20 may not correctly describe laser photon statistics (and laser photon state dynamics) in a general way. As the original master equation gives rise to a Poisson photon statistics, no discrepancy may been so far suspected.

3.2 The new master equation based on beam-splitter effects

In analogy to Eq. 20 we define a new master equation of the laser by replacing the last two terms, emulating outcoupling by absorption, by the temporal change of the matrix elements in Eq. 19. The new master equation thus is given by

$$\dot{\rho}_{n,m} = -\frac{N'_{n,m}A}{1+N_{n,m}\varsigma}\rho_{n,m} + \frac{\sqrt{nm}A}{1+N_{n-1,m-1}\varsigma}\rho_{n-1,m-1} - \frac{\ln|c|^2}{(1-|c|^2)\tau_{RT}} \times \left(\rho_{n,m} - \sum_{p=0}^{\infty} \rho_{n+p,m+p}\binom{n+p}{n}^{\frac{1}{2}}\binom{m+p}{m}^{\frac{1}{2}} c^n (c^*)^m |s|^{2p}\right), \tag{25}$$

and therefore the equation of motion of the diagonal elements, i.e., the evolution of the statistical distribution of the photons, becomes

$$\dot{\rho}_{n,n} = -\frac{(n+1)A}{1+(n+1)\varsigma}\rho_{n,n} + \frac{nA}{1+n\varsigma}\rho_{n-1,n-1} - \frac{\ln|c|^2}{(1-|c|^2)\tau_{\mathrm{RT}}} \times \left(\rho_{n,n} - \sum_{p=0}^{\infty}\rho_{n+p,n+p}\binom{n+p}{n}|c|^{2n}|s|^{2p}\right). \tag{26}$$

Therein,

$$\varsigma = \frac{B}{A} = 4\left(\frac{g}{\gamma}\right)^2, \tag{27}$$

is the pump-rate independent saturation parameter.

3.2.1 Laser rate equation

In this paragraph, we will compare the laser rate equation deduced from the new master equation with the one arising from the original description. Therefore, the rate equation for the average photon number is deduced as

$$
\begin{aligned}
\frac{\mathrm{d}\langle n\rangle}{\mathrm{d}t} &= \sum_{n=0}^{\infty} n\dot{\rho}_{n,n} \\
&= -\sum_{n=0}^{\infty} \frac{n(n+1)A}{(1+(n+1)\varsigma)}\rho_{n,n} + \sum_{n=1}^{\infty}\frac{n^2A}{(1+n\varsigma)}\rho_{n-1,n-1} \\
&\quad -\frac{1}{\Delta t}\left(\sum_{n=0}^{\infty} n\rho_{n,n} - \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} n\rho_{n+p,n+p}\binom{n+p}{p}|c|^{2n}|s|^{2p}\right) \\
&= -\sum_{n=0}^{\infty} \frac{n(n+1)A}{(1+(n+1)\varsigma)}\rho_{n,n} + \sum_{n=0}^{\infty}\frac{(n+1)^2A}{(1+(n+1)\varsigma)}\rho_{n,n} \\
&\quad -\frac{1}{\Delta t}\left(\langle n\rangle - \sum_{p=0}^{\infty}\sum_{k=p}^{\infty}(k-p)\rho_{k,k}\binom{k}{p}|c|^{2(k-p)}|s|^{2p}\right) \\
&= \sum_{n=0}^{\infty}\frac{(n+1)A}{(1+(n+1)\varsigma)}\rho_{n,n} \\
&\quad -\frac{1}{\Delta t}\left(\langle n\rangle - \sum_{k=0}^{\infty}\rho_{k,k}\sum_{p=0}^{k}(k-p)\frac{k!}{p!(k-p)!}|c|^{2(k-p)}|s|^{2p}\right) \\
&= \sum_{n=0}^{\infty}\frac{(n+1)A}{(1+(n+1)\varsigma)}\rho_{n,n} \\
&\quad -\frac{1}{\Delta t}\left(\langle n\rangle - \sum_{k=0}^{\infty}|c|^2k\rho_{k,k}\sum_{p=0}^{k-1}\binom{k-1}{p}|c|^{2(k-1-p)}|s|^{2p}\right) \\
&= \sum_{n=0}^{\infty}\frac{(n+1)A}{(1+(n+1)\varsigma)}\rho_{n,n} - \frac{1-|c|^2}{\Delta t}\langle n\rangle \\
&= \frac{A}{1+\varsigma}\sum_{n=0}^{\infty}\left(1+\sum_{k=1}^{\infty}\frac{(-\varsigma)^{k-1}}{(1+\varsigma)^k}n^k\right)\rho_{n,n} - \frac{\langle n\rangle}{\tau_{\mathrm{c}}} \\
&= \frac{A}{1+\varsigma}\left(1+\sum_{k=1}^{\infty}\frac{(-\varsigma)^{k-1}}{(1+\varsigma)^k}\left\langle n^k\right\rangle\right) - \frac{\langle n\rangle}{\tau_{\mathrm{c}}}.
\end{aligned}
\tag{28}
$$

Therein, we were using the identity

$$
\sum_{p=0}^{\infty}\sum_{k=p}^{\infty} \equiv \sum_{k=0}^{\infty}\sum_{p=0}^{k}. \tag{29}
$$

For comparison, we perform the same operation on Eq. 20 and obtain

$$
\begin{aligned}
\frac{\mathrm{d}\langle n\rangle}{\mathrm{d}t} &= \sum_{n=0}^{\infty} n\dot{\rho}_{n,n} \\
&= -\sum_{n=0}^{\infty} \frac{n(n+1)A}{(1+(n+1)\varsigma)}\rho_{n,n} + \sum_{n=1}^{\infty}\frac{n^2A}{(1+n\varsigma)}\rho_{n-1,n-1} \\
&\quad -\sum_{n=0}^{\infty} n\frac{\omega}{Q_{\mathrm{c}}}n\rho_{n,n} + \sum_{n=-1}^{\infty} n\frac{\omega}{Q_{\mathrm{c}}}(n+1)\rho_{n+1,n+1} \\
&= -\sum_{n=0}^{\infty} \frac{n(n+1)A}{(1+(n+1)\varsigma)}\rho_{n,n} + \sum_{n=0}^{\infty}\frac{(n+1)^2A}{(1+(n+1)\varsigma)}\rho_{n,n} \\
&\quad -\frac{\omega}{Q_c}\left\langle n^2\right\rangle + \sum_{k=0}^{\infty}\frac{\omega}{Q_{\mathrm{c}}}(k-1)k\rho_{k,k} \\
&= \sum_{n=0}^{\infty}\frac{(n+1)A}{(1+(n+1)\varsigma)}\rho_{n,n} - \frac{\omega}{Q_{\mathrm{c}}}\langle n\rangle.
\end{aligned}
\tag{30}
$$

Comparing Eq. 28 with Eq. 30 shows that also the typical laser rate equation does not change its form when assuming a beamsplitter as OC and the correct correspondence between the cavity Q factor Q_{c} and the cavity photon lifetime is given by

$$
\frac{\omega}{Q_c} = \frac{1}{\tau_{\mathrm{c}}}. \tag{31}
$$

As a result we can, therefore, state, first, that in terms of its consequence on the laser rate equation, beam-splitter outcoupling cannot be distinguished from outcoupling emulated by appropriate absorption terms. This means that in both descriptions of a laser we obtain an identical power-to-pump-rate dependence, i.e., especially the same threshold and same slope efficiency. We obtain the semi-classical CW solution to Eq. 28 by assuming a very narrow photon distribution, i.e., in the limit $\rho_{\varphi,\varphi} = 1$ for $n = \varphi$, with φ being the intra-cavity photon number, and $\rho_{n,n} = 0$ for $n \neq \varphi$. This results in

$$
\frac{\mathrm{d}\langle n\rangle}{\mathrm{d}t} \approx \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{(\varphi+1)A}{(1+(\varphi+1)\varsigma)} - \frac{\varphi}{\tau_{\mathrm{c}}} \overset{!}{=} 0, \tag{32}
$$

with the solution

$$
\varphi = \frac{A\tau_{\mathrm{c}} - (1+\varsigma)}{2\varsigma} + \sqrt{\frac{A\tau_c}{\varsigma} + \frac{\left(A\tau_{\mathrm{c}} - (1+\varsigma)\right)^2}{4\varsigma^2}}, \tag{33}
$$

which is depicted for different saturation parameter values in Fig. 3. For operation far above threshold, i.e., by neglecting the first term under the root, we obtain

$$
\varphi \approx \frac{A\tau_{\mathrm{c}}}{\varsigma} - \frac{(1+\varsigma)}{\varsigma}, \tag{34}
$$

from which we can deduce the semi-classical laser slope η_p (wrt. pump rate) and threshold A_{th} as

$$
\eta_p = \frac{\tau_c}{\varsigma}, \tag{35}
$$

$$
A_{th} = \frac{1+\varsigma}{\tau_{\mathrm{c}}}. \tag{36}
$$

Second, whenever additional intra-cavity losses occur that act on single photons, e.g., reabsorption or scattering, this can be well modeled by a term

$$
\dot{\rho}_{n,m}\big|_{\mathrm{abs}} = -\frac{1}{2}\frac{1}{\tau_{\mathrm{loss}}}(n+m)\rho_{n,m} + \frac{1}{\tau_{loss}}\sqrt{(n+1)(m+1)},\,\rho_{n+1,m+1} \tag{37}
$$

analogous to the term of the original master equation, modified to take into account only the loss-related decay. In

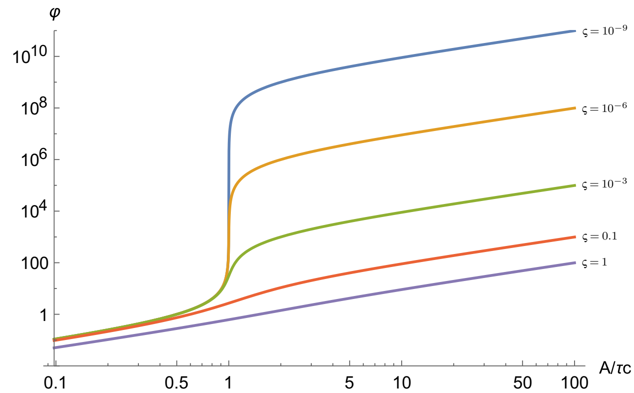

Fig. 3 Semi-classical photon number φ from Eq. 33 as a function of relative pump rate A/τ_c for different saturation parameters $\varsigma = 1, 0.1, 10^{-3}, 10^{-6}, 10^{-9}$ (from bottom to top, respectively)

analogy to Eq. 15 we obtain this loss-related lifetime τ_{loss} for an intra-cavity round-trip loss L_{RT} as

$$\tau_{\text{loss}} = -\frac{2L_o}{c\ln\left(1-L_{\text{RT}}\right)} = -\frac{\tau_{\text{RT}}}{\ln\left(1-L_{\text{RT}}\right)}. \tag{38}$$

3.2.2 Detailed balance in CW operation

To describe the photon statistics of a continuous wave (CW) laser under pure beam-splitter outcoupling, we have to find the steady-state solution of Eq. 26, i.e., $\dot{\rho}_{n,n} = 0\ \forall n$. This is given when the input flow of probability into each $\rho_{n,n}$ equals the output flow of probability from each $\rho_{n,n}$. Fig. 4 depicts the different flows occurring in the diagonal elements of the new master equation given by Eq. 26. In contrast to the detailed balance of the diagonal elements of the original master equation [1] given by

$$\frac{(n+1)A}{1+(n+1)\varsigma}\rho_{n,n} = \frac{\omega}{Q_c}(n+1)\rho_{n+1,n+1}, \tag{39}$$

the beam-splitter outcoupling generates a multitude of couplings between the different diagonal elements. Looking, e.g., at the $\rho_{n,n}$ level in Fig. 4, the upward pointing blue arrow is the only flow of probability that goes to higher levels than n, n. All flows depicted by dark red arrows outgoing from $\rho_{n,n}$ and all corresponding flows to lower lying states depicted by the other red arrows only cause re-distributions of probability within these lower-lying states. All flows depicted by green arrows ending in the levels k, k with $k = 0...n$ therefore constitute the inflow of probability entering into the set of states $k = 0...n$ which, as probability is conserved, has to be compensated by the upward outflow from $\rho_{n,n}$. Therefore, a detailed balance in the diagonal elements of the new master equation is obtained by the condition

$$\frac{(n+1)A}{1+(n+1)\varsigma}\rho_{n,n} = -\frac{\ln|c|^2}{\left(1-|c|^2\right)\tau_{RT}}\sum_{k=0}^{n}\sum_{l=n+1}^{\infty}\rho_{l,l}\binom{l}{k}|c|^{2k}|s|^{2(l-k)}. \tag{40}$$

4 Laser photon statistics

To derive the laser photon statistics under pure beam-splitter outcoupling, i.e., while neglecting additional cavity losses, we rewrite the detailed balance condition in Eq. 40 using $|s|^2 = 1-|c|^2$ as

$$\begin{aligned}\frac{(n+1)A}{1+(n+1)\varsigma}\rho_{n,n} &= -\frac{\ln|c|^2}{\tau_{\text{RT}}}\sum_{l=n+1}^{\infty}\rho_{l,l}\frac{1}{1-|c|^2}\sum_{k=0}^{n}\binom{l}{k}|c|^{2k}|s|^{2(l-k)}\\ &= -\frac{\ln|c|^2}{\tau_{\text{RT}}}\sum_{l=n+1}^{\infty}\rho_{l,l}\frac{1}{1-|c|^2}\sum_{k=0}^{n}\binom{l}{k}|c|^{2k}\left(1-|c|^2\right)^{(l-k)}\\ &= -\frac{\ln|c|^2}{\tau_{\text{RT}}}\sum_{l=n+1}^{\infty}\rho_{l,l}F_n^l\left(|c|^2\right).\end{aligned} \tag{41}$$

Therein, we define the contribution function of the higher photon states by

$$F_n^l\left(|c|^2\right) := \frac{1}{1-|c|^2}\sum_{k=0}^{n}\binom{l}{k}|c|^{2k}\left(1-|c|^2\right)^{(l-k)} \quad \forall l > n. \tag{42}$$

This contribution function defines the precise coupling between the different matrix elements and thus has an important impact on the photon statistical distribution in CW operation.

The contribution function has a specific shape depending on the outcoupling as $R = |c|^2$ changes as can be seen in Fig. 5. For high OC reflectivities, i.e., low outcoupling or a high-finesse cavity, the contribution function mainly allows for a significant contribution of the next higher state $l = n+1$ only while for lower OC reflectivities it gradually allows more and more higher states to contribute. In the limit of very low OC reflectivity, i.e., strong outcoupling, all states higher than n contribute equally to state n in the detailed balance.

A closer look at the form of Eq. 42 shows that $F_n^l\left(|c|^2\right)$ can be seen as a scaled cumulative distribution function

$$F_n^l\left(|c|^2\right) = \frac{1}{1-|c|^2}\sum_{k=0}^{n}B\left(l,|c|^2;k\right), \tag{43}$$

of a binomial probability distribution in k given by

$$B\left(l,|c|^2;k\right) = \binom{l}{k}\left(|c|^2\right)^k\left(1-|c|^2\right)^{(l-k)}. \tag{44}$$

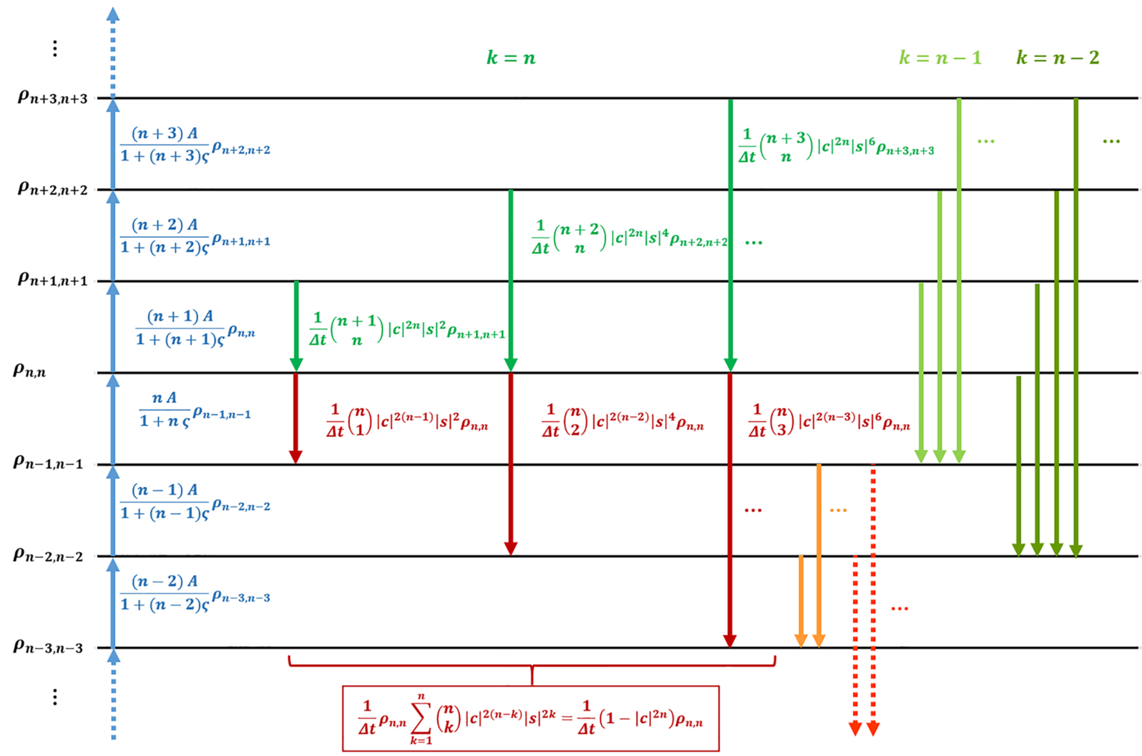

Fig. 4 View of the different flows of probability arising from the different terms of the new master equation. The blue arrows show the effect of the first two terms of Eq. 25 providing the upward flow of probability. The green-scale arrows denote downward flows arising from above the level of $\rho_{n,n}$ while the red-scale arrows indicate redistributions from this level or below of that level

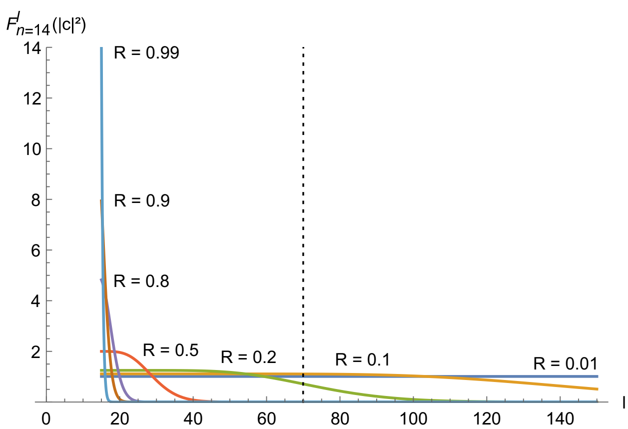

Fig. 5 Contribution function $F_n^l(|c|^2)$ on the example of $n = 14$ and $|c|^2 = R = 0.99, 0.9, 0.8, 0.5, 0.2, 0.1, 0.01$ (from left peak to right flat line at an abscissa of approximately 1). The dashed line shows the point at $l = 70$, where the curve for $|c|^2 = 0.2$ is at about its half maximum, cf. Fig. 6

As such a binomial distribution in k, shown as an example in Fig. 6 for an OC reflectivity of $R = |c|^2 = 0.2$ and different l, has a mean of $\mu = l|c|^2$ and a variance $\sigma^2 = l|c|^2(1-|c|^2)$, we can derive the following important behavior of the contribution function: For $\mu = n$, i.e., $l = l_{\mathrm{HW}} := \frac{n}{|c|^2}$ the cumulative distribution function reaches approximately 1/2 of its maximum value, as in Eq. 43 the summation adds over approximately half of the total distribution. Figure 6 gives an example for this reasoning. Therefore, we can define an effective half width of the contribution function as

$$\Delta n_{\mathrm{HW}} := l_{\mathrm{HW}} - n = \frac{n}{|c|^2} - n = \left(\frac{1}{|c|^2} - 1\right)n. \tag{45}$$

Thus, the contribution function couples approximately all higher states from $l = n+1, ..., l = n/|c|^2$ to contribute to state n in the detailed balance and gradually discards all further higher lying states.

As the contribution function is only defined for $l > n$ and monotonically falling towards higher l, it has its peak at $l = n+1$ with

$$\begin{aligned} F_n^{l=n+1}(|c|^2) &= \frac{1}{1-|c|^2}\sum_{k=0}^{n}\binom{n+1}{k}|c|^{2k}(1-|c|^2)^{(l-k)} \\ &= \frac{1}{1-|c|^2}\left(1 - \binom{n+1}{n+1}|c|^{2(n+1)}\right) \\ &= \frac{1-|c|^{2(n+1)}}{1-|c|^2}. \end{aligned} \tag{46}$$

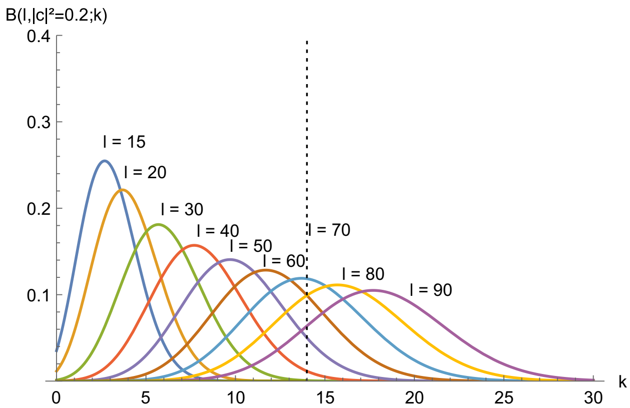

Fig. 6 Binomial distribution function $B(l,|c|^2;k)$ on the example of $|c|^2 = 0.2$ and $l = 15, 20, 30, 40, 50, 60, 70, 80, 90$ (from left peak to right peak). The dashed line shows $k = 14$, i.e., in the contribution function $F^l_{n=14}(|c|^2 = 0.2)$ in Fig. 5 the above binomial distributions are summed up over k until that line $k = 0...n = 14$ for each value of l. As for $n = 14$ and $|c|^2 = 0.2$ we obtain $n/|c|^2 = 70$, the dashed line falls on the mean of the binomial distribution with $l = 70$. Therefore, the corresponding contribution function $F^l_{n=14}(|c|^2 = 0.2)$ reduces to about its half maximum at $l = 70$

4.1 Laser photon statistics at low outcoupling

At low outcoupling, i.e., in the limit $R = |c|^2 \to 1$, the width of the contribution function Δn_{HW} narrows to zero while its peak value approaches

$$\lim_{|c|^2\to 1} F_n^{l=n+1}\left(|c|^2\right) = n+1, \tag{47}$$

as can be easily seen by applying l'Hopital's rule on the final term in Eq. 46. Therefore, the detailed balance in Eq. 41 approaches

$$\begin{aligned}\frac{(n+1)A}{1+(n+1)\varsigma}\rho_{n,n} &= -\frac{\ln|c|^2}{\tau_{\mathrm{RT}}}\sum_{l=n+1}^{\infty}\rho_{l,l}F_n^l\left(|c|^2\right)\\ &\overset{|c|^2\to 1}{\approx} -\frac{\ln|c|^2}{\tau_{\mathrm{RT}}}(n+1)\rho_{n+1,n+1}.\end{aligned} \tag{48}$$

Rearranging the terms result in the recursion relation

$$\begin{aligned}\rho_{n+1,n+1} &\approx -\frac{\tau_{\mathrm{RT}}}{\ln|c|^2}\frac{A}{1+(n+1)\varsigma}\rho_{n,n}\\ &= \rho_{0,0}\frac{\left(\frac{A\tau_{\mathrm{RT}}}{-\ln|c|^2}\right)^{n+1}}{\prod_{k=1}^{n+1}(1+k\varsigma)}\\ &= \rho_{0,0}\frac{\left(\frac{A\tau_{\mathrm{RT}}}{-\varsigma\ln|c|^2}\right)^{n+1}}{\prod_{k=1}^{n+1}\left(\frac{1}{\varsigma}+k\right)},\end{aligned} \tag{49}$$

which for large $n >> 1/\varsigma$, i.e., for laser operation far above threshold, approaches the well-known Poisson distribution

$$\rho_{n,n} \approx \rho_{0,0}\frac{\left(\frac{A\tau_{\mathrm{RT}}}{-\varsigma\ln|c|^2}\right)^n}{\prod_{k=1}^{n}\left(\frac{1}{\varsigma}+k\right)} \to \rho_{0,0}\frac{\langle n\rangle^n}{n!} = e^{-\langle n\rangle}\frac{\langle n\rangle^n}{n!}, \tag{50}$$

with the mean

$$\langle n\rangle = \frac{A\tau_{\mathrm{RT}}}{-\varsigma\ln|c|^2} = \frac{A\tau_c}{\varsigma}. \tag{51}$$

This result coincides with the one originally deduced from the master equation that emulated outcoupling by absorption of intra-cavity photons and is the one well-known from university courses on lasers [1]. It is the reason why we always think of lasers being generators of coherent states of light with a Poisson photon statistics. However, as we will see in the next paragraph, this no longer holds for a laser with strong beamsplitter outcoupling only.

In addition, as we can see in Fig. 7, the coupling to the next higher state only, resulting in the Poisson distribution, is only obtained at extremely low outcoupling. Looking at the values of $F_n^l\left(|c|^2\right)/F_n^{l=n+1}\left(|c|^2\right)$ for $l = n+2, n+3, n+4, n+5$ at only 1% of outcoupling, i.e., at $|c|^2 = R = 0.99$, we see that the state n is only coupled to state $n+1$ with already a significant reduction of the coupling to state $n+2$ for $n << 123$, defined as the point at which the contribution of $l = n+2$ is half that of $l = n+1$. Mathematically, this point is given by

$$\begin{aligned}n_{1/2} = &\frac{1}{\ln\left(|c|^2\right)}W_{-1}\left(\frac{|c|^{\frac{1}{1-|c|^2}}\ln\left(|c|^2\right)}{2\left(1-|c|^2\right)}\right)\\ &-\frac{1}{2\left(1-|c|^2\right)} - 1 \quad \forall\ 0.5 \le |c|^2 \le 1.\end{aligned} \tag{52}$$

Therein, $W_{-1}(x)$ is the lower branch of the Lambert W function, satisfying $W(x)e^{W(x)} = x$. It has to be noted, that no solution for $n_{1/2}$ exists for $|c|^2 < 0.5$, as then $F_n^{l=n+2}(|c|^2)/F_n^{l=n+1}(|c|^2)$ is always greater than 1/2 (see Fig. 8). This shows, that for $|c|^2 < 0.5$ many higher adjacent states contribute to the detailed balance and thus the distribution is non-Poisson for pure beam-splitter outcoupling.

In contrast to this, e.g., at $|c|^2 = R = 0.9$, for a state n with $n > 100$ at least the five higher lying states contribute equally to the detailed balance of state n, thus causing a deviation from a pure Poisson statistics.

This can be defined more rigorously by the condition that $\Delta n_{\mathrm{HW}} < 1$ for at least all n up to and around the average photon number $\langle n\rangle$. Therefore, we obtain a minimum OC reflectivity $R_{\min}$ depending on the average photon number for which the coupling of each state to the next higher state is the only significant contribution in the detailed balance, given by

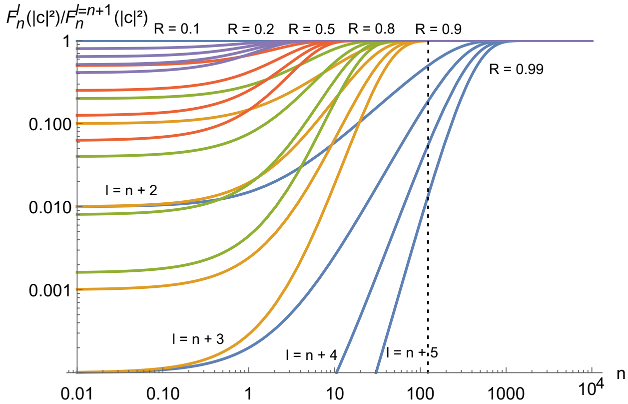

Fig. 7 Contribution function $F_n^l(|c|^2)$ normalized to $F_n^{l=n+1}(|c|^2)$ for $l = n+2, n+3, n+4, n+5$ (grouped with same color, curves from top to bottom within one group) at $|c|^2 = R = 0.99, 0.9, 0.8, 0.5, 0.2, 0.1$ (different colors in blue, yellow, green, red and violet, respectively). The dotted line at $n \approx 123$ shows the point where $F_n^{l=n+2}(|c|^2 = 0.99) = 1/2 \times F_n^{l=n+1}(|c|^2 = 0.99)$

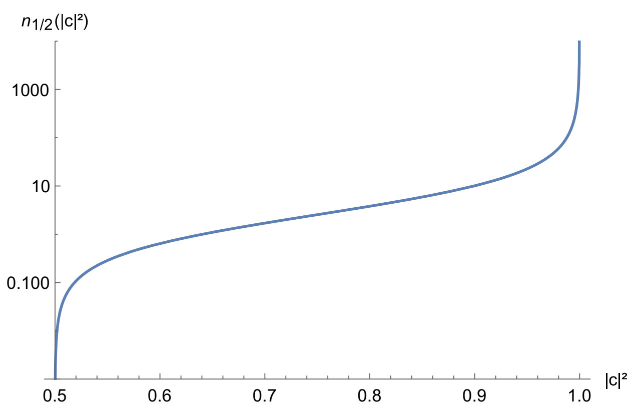

Fig. 8 Photon state number $n_{1/2}$ at which the contribution of $l = n + 2$ is half that of $l = n + 1$ as a function of the OC reflectivity $R = |c|^2$

$$\Delta n_{\mathrm{HW}}(\langle n\rangle) = \left(\frac{1}{|c|^2} - 1\right)\langle n\rangle$$
$$< 1 \Leftrightarrow R_{\min} = |c|^2_{\min} = \frac{\langle n\rangle}{\langle n\rangle + 1}. \tag{53}$$

As in typical lasers, the average photon numbers are often much larger than 10^6, we see that only extremely low outcoupling, which is technically difficult to achieve, creates an inter-state coupling that causes a pure Poisson distribution under beam-splitter outcoupling.

4.2 Laser photon statistics at strong outcoupling

At strong outcoupling, i.e., in the limit $R = |c|^2 \to 0$, the width of the contribution function $\Delta n_{\mathrm{HW}} \to \infty$ such that we can set

$$\lim_{|c|^2\to 0} F_n^l(|c|^2) = 1. \tag{54}$$

Therefore, the detailed balance in Eq. 41 approaches

$$\frac{(n+1)A}{1+(n+1)\varsigma}\rho_{n,n} = -\frac{\ln|c|^2}{\tau_{\mathrm{RT}}}\sum_{l=n+1}^{\infty}\rho_{l,l}F_n^l(|c|^2)$$
$$\overset{|c|^2\to 0}{\approx} -\frac{\ln|c|^2}{\tau_{\mathrm{RT}}}\sum_{l=n+1}^{\infty}\rho_{l,l}. \tag{55}$$

Rearranging the terms and exploiting the normalization condition for the diagonal elements results in the relation

$$\rho_{n,n} \approx -\frac{\ln|c|^2}{\tau_{\mathrm{RT}}}\frac{1+(n+1)\varsigma}{(n+1)A}\sum_{l=n+1}^{\infty}\rho_{l,l}$$
$$= -\frac{\varsigma\ln|c|^2}{A\tau_{\mathrm{RT}}}\left(\frac{1}{(n+1)\varsigma}+1\right)\left(1-\sum_{l=0}^{n}\rho_{l,l}\right), \tag{56}$$

which can easily be solved inductively starting from $\rho_{0,0}$. As a result, we obtain

$$\rho_{0,0} \approx \frac{\frac{-\varsigma\ln|c|^2}{A\tau_{\mathrm{RT}}}\left(\frac{1}{\varsigma}+1\right)}{\frac{-\varsigma\ln|c|^2}{A\tau_{\mathrm{RT}}}\left(\frac{1}{\varsigma}+1\right)+1} \tag{57}$$

$$\rho_{1,1} \approx \frac{\frac{-\varsigma\ln|c|^2}{A\tau_{\mathrm{RT}}}\left(\frac{1}{2\varsigma}+1\right)}{\left(\frac{-\varsigma\ln|c|^2}{A\tau_{\mathrm{RT}}}\left(\frac{1}{\varsigma}+1\right)+1\right)\left(\frac{-\varsigma\ln|c|^2}{A\tau_{\mathrm{RT}}}\left(\frac{1}{2\varsigma}+1\right)+1\right)} \tag{58}$$

$$\vdots$$

$$\rho_{n,n} \approx \frac{\frac{-\varsigma\ln|c|^2}{A\tau_{\mathrm{RT}}}\left(\frac{1}{(n+1)\varsigma}+1\right)}{\prod_{k=0}^{n}\left(\frac{-\varsigma\ln|c|^2}{A\tau_{\mathrm{RT}}}\left(\frac{1}{(k+1)\varsigma}+1\right)+1\right)}. \tag{59}$$

For large $n >> 1/\varsigma$, i.e., for laser operation far above threshold, we can approximate this distribution to

$$\rho_{n,n} \overset{n>>1/\varsigma}{\approx} \frac{\frac{-\varsigma\ln|c|^2}{A\tau_{\mathrm{RT}}}}{\prod_{k=0}^{n}\left(\frac{-\varsigma\ln|c|^2}{A\tau_{\mathrm{RT}}}+1\right)} \approx \frac{\frac{-\varsigma\ln|c|^2}{A\tau_{RT}}}{\left(\frac{-\varsigma\ln|c|^2}{A\tau_{\mathrm{RT}}}+1\right)^{n+1}}$$
$$= \frac{\left(\frac{A\tau_{RT}}{-\varsigma\ln|c|^2}\right)^n}{\left(1+\frac{A\tau_{\mathrm{RT}}}{-\varsigma\ln|c|^2}\right)^{n+1}} = \frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}}, \tag{60}$$

and obtain the well-known Bose–Einstein distribution with identical mean as in Eq. 51. Thus, for strong outcoupling, we find the statistics of thermal light.

Several attempts have been performed by the author to find other analytical solutions to the photon statistics for finite values of outcoupling other than the extremes of

$R \to 0$ and $R \to 1$, but without success. Therefore, it currently seems that in all other cases a numerical calculation will most likely be necessary, which, however, will be very time and memory consuming when real laser states at high photon numbers are to be investigated.

4.3 Laser photon statistics including losses

Nevertheless, there is an important influence that we have so far discarded in the above description: Intra-cavity losses. If we take losses into account which act on single photons and thus can be described by Eq. 37, we introduce an additional coupling in the detailed balance that predominantly couples states n with $n+1$. The detailed balance then reads

$$\frac{(n+1)A}{1+(n+1)\varsigma}\rho_{n,n} = -\frac{\ln|c|^2}{\tau_{\mathrm{RT}}}\sum_{l=n+1}^{\infty}\rho_{l,l}F_n^l\left(|c|^2\right) + \frac{n+1}{\tau_{\mathrm{loss}}}\rho_{n+1,n+1}. \tag{61}$$

Therefore, beam-splitter outcoupling losses and additional losses create fundamentally different couplings and thus affect the photon statistics evolution in a completely different way. Whenever the loss-related coupling dominates, i.e., when

$$\frac{n+1}{\tau_{\mathrm{loss}}} > -\frac{\ln|c|^2}{\tau_{\mathrm{RT}}}F_n^{l=n+1}\left(|c|^2\right) = -\frac{\ln|c|^2}{\tau_{\mathrm{RT}}}\frac{1-|c|^{2(n+1)}}{1-|c|^2}, \tag{62}$$

a Poisson distribution results. The round-trip loss L_{RT} yielding at least the same coupling as for the $l=n+1$ term in the beam-splitter outcoupling therefore results in

$$L_{\mathrm{RT}} = 1 - e^{-\frac{\tau_{RT}}{\tau_{\mathrm{loss}}}}, \tag{63}$$

with

$$\frac{\tau_{\mathrm{RT}}}{\tau_{\mathrm{loss}}} = -\frac{1-|c|^{2(n+1)}}{\left(1-|c|^2\right)(n+1)}\ln|c|^2. \tag{64}$$

The intra-cavity loss needed to obtain this equivalence point is shown in Fig. 9. As can be seen, already at very low loss values we obtain a dominant coupling of the next higher state $l=n+1$ and thus the statistics will develop into a Poisson statistics as soon as n increases significantly. This arises from the fact that the loss term contains the factor $n+1$, while the contribution function $F_n^{l=n+1}\left(|c|^2\right)$ reaches $n+1$ only for $|c|^2 \to 1$, see Eq. 47. For large values of n, additional losses on the order of $1/n$ at low outcoupling already cause a dominance of these losses with respect to the outcoupling losses concerning photon statistics. Therefore, it will be very difficult to experimentally realize a laser that generates pure beam-splitter OC dominated statistics. If we calculate the maximum intra-cavity losses not to surpass to have the chance to observe the beam-splitter effect on photon statistics for a given average photon number $\langle n\rangle$,

$$L_{\mathrm{RT}}^{\max} = 1 - e^{\ln|c|^2\frac{1-|c|^{2(\langle n\rangle+1)}}{\left(1-|c|^2\right)(\langle n\rangle+1)}} = 1 - |c|^{2\frac{1-|c|^{2(\langle n\rangle+1)}}{\left(1-|c|^2\right)(\langle n\rangle+1)}}, \tag{65}$$

we obtain the curves in Fig. 10. At low outcoupling, the loss should be smaller than $1/\langle n\rangle$ while at higher values of outcoupling, the allowable losses can be higher, yet still very low. Therefore, we evaluate the relative maximum loss at large $\langle n\rangle$

$$\lim_{\langle n\rangle\to\infty}\frac{L_{\mathrm{RT}}^{\max}}{\frac{1}{\langle n\rangle}} = \lim_{\langle n\rangle\to\infty}\left(\langle n\rangle - \langle n\rangle|c|^{2\frac{1-|c|^{2(\langle n\rangle+1)}}{\left(1-|c|^2\right)(\langle n\rangle+1)}}\right) = \frac{-\ln|c|^2}{1-|c|^2}. \tag{66}$$

As can be seen in Figs. 10 and 11 the allowable losses can be higher for strong outcoupling. For a typical 1 mW laser at visible or near-infrared wavelengths, i.e., for $\langle n\rangle \approx 10^6$–$10^7$ at around 1–2% outcoupling, the intra-cavity losses need to be lower than 10^{-6}–10^{-7}.

5 Discussion

The results presented before show, based on a simple QM approach to the description of a laser, that the appearance of a Poisson photon statistics is not intrinsically related to the laser process, i.e., stimulated emission, but the result of (uncorrelated) loss mechanisms acting on single photons. It can be stated therefore that so far unavoidable losses in laser resonators are the reason for the Poisson photon statistics that we commonly attribute to lasers. However,

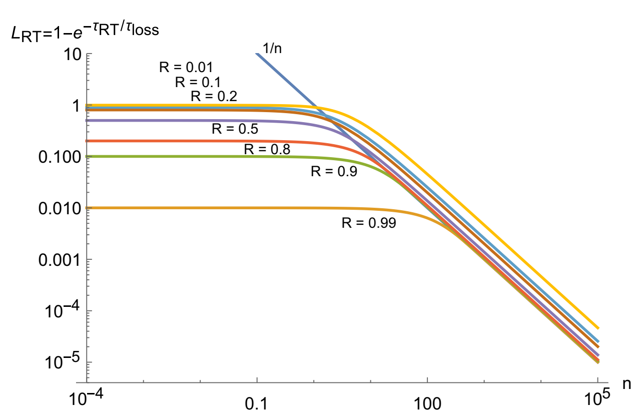

Fig. 9 Necessary loss, L_{RT} from Eq. 63, needed to create loss-dominated coupling in the detailed balance for OC reflectivities of $|c|^2 = R = 0.99, 0.9, 0.8, 0.5, 0.2, 0.1, 0.01$ (from bottom to top, respectively). For large values of n the loss curves show the same power law and fall as $1/n$, shown by the blue line

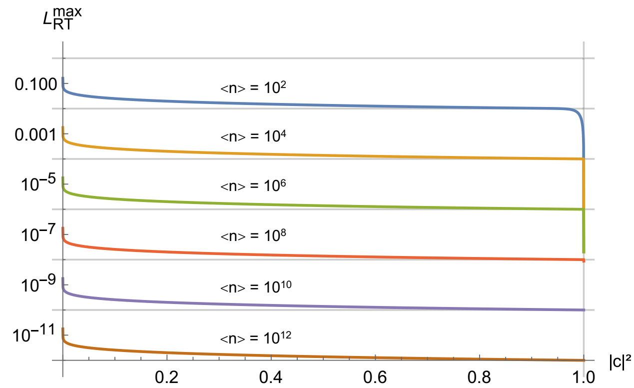

Fig. 10 Maximum loss L_{RT}^{max} allowed from Eq. 65 to create beam-splitter-dominated coupling in the detailed balance for average photon numbers of $\langle n\rangle = 10^2, 10^4, 10^6, 10^8, 10^{10}$ and 10^{12} (from top to bottom, respectively)

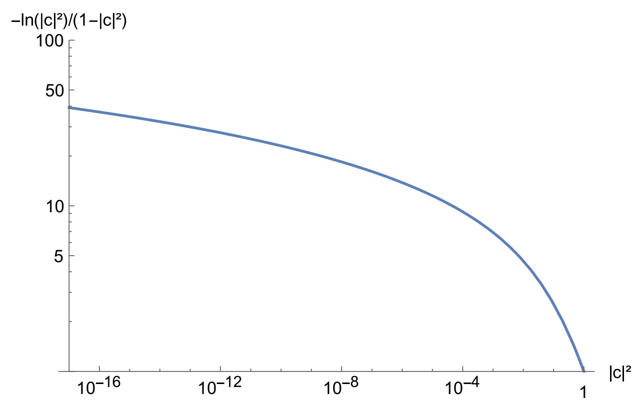

Fig. 11 Maximum loss L_{RT}^{max} allowed from Eq. 65 normalized to the maximum allowed loss at $|c|^2 \to 1$ at high $\langle n\rangle$, i.e., Eq. 66

it could be shown that the correlations arising from the beam-splitting properties of commonly used laser output couplers may in general cause highly interesting couplings that allow for non-Poisson distributions. Unexpectedly to first sight, in the limit of high output coupling and very low losses even the statistics of thermal light may be produced by a laser even when being above threshold. This is a consequence of the strong statistical mixing occurring as a result of the beam-splitter action of the output coupler at low reflectivities. However, as was shown also in the above calculations, extreme care will have to be taken in designing and realizing a laser experiment for this purpose as even small additional passive losses or absorption will cause significant coupling that will force the statistics evolution into a Poisson distribution.

6 Conclusion

In conclusion, the QM laser master equation for the density operator has been extended to incorporate the beam-splitter effect caused by typical dielectric laser output couplers. Different cases without and with additional intra-cavity losses were discussed and their influence on the expected laser photon statistics was deduced. As a result it was found that the well-known Poisson distribution of laser light is in most cases the result of additional losses or absorption, which act uncorrelatedly on single photons. In a theoretical laser with negligible losses where outcoupling is dominated by the beam-splitter effect, a Poisson distribution would only occur for very low outcoupling rates, i.e., high finesse cavities, while more complex distributions are expected for larger outcoupling rates. In the limit of strong outcoupling and low intra-cavity losses even the distribution of thermal light results. The findings of this theory, therefore, shed light on the complex QM interactions and couplings that can be caused by the beam-splitting effect of an outcoupling mirror of a laser resonator, but they also show the limits to be taken into account for future experiments that may try to realize these effects in extremely low-loss laser resonators only subjected to the beam-splitter outcoupling effect. Further research will be performed on the investigation of possible influences of the beam-splitter effect on laser linewidth and coherence.

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Declarations

Conflict of interest The author declares no conflict of interest.

Research involving Human Participants and/or Animals Not applicable.

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