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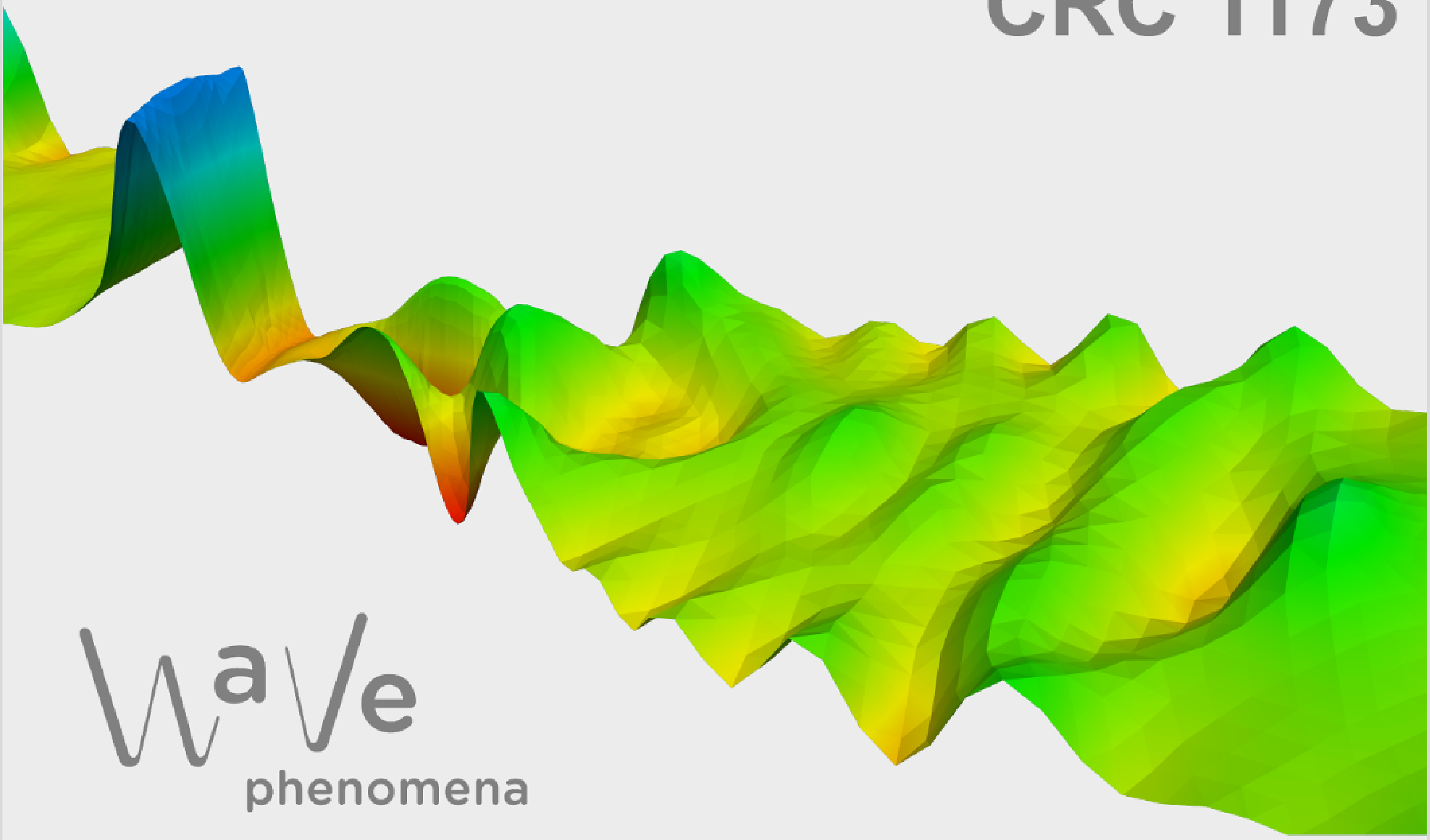

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THE ABSENCE OF THE EFIMOV EFFECT IN SYSTEMS OF ONE- AND TWO-DIMENSIONAL PARTICLES

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Abstract. We study virtual levels of N-particle Schrödinger operators and prove that if the particles are one-dimensional and $N \geq 3$, then virtual levels at the bottom of the essential spectrum correspond to eigenvalues. The same is true for two-dimensional particles if $N \geq 4$. These results are applied to prove the non-existence of the Efimov effect in systems of $N \geq 4$ one-dimensional or $N \geq 5$ two-dimensional particles.

1. Introduction

In recent years the Efimov effect has attracted large interest. This effect is named after the physicist V. Efimov and can be stated as follows: A system of three quantum particles in dimension three, interacting through attractive short-range potentials, has an infinite number of bound states if the subsystems do not have negative spectrum and at least two of them are resonant [10], i.e. any arbitrarily small negative perturbation of the pair potential leads to a negative spectrum. In this situation we also say that the two-body Hamiltonian of the system has a virtual level.
The Efimov effect is a surprising phenomenon, because although the pair interactions are short-range, the system of three particles behaves as a system of two particles with long-range potential. Another interesting feature is its universality. This means that the existence of the effect as well as the distribution of the eigenvalues do not depend on the shape of the potentials of interaction of particles. It is only important that they are short-range and resonant. Indeed, the counting function $N(z)$ of the eigenvalues of the three-body Hamiltonian below $z < 0$ obeys the following asymptotics

$$\lim_{z \to 0-} \frac{N(z)}{|\ln |z||} = \mathcal{A}_0 > 0, \tag{1.1}$$

where the constant $\mathcal{A}_0$ depends on the masses of the particles, but not on the potentials [10].
For a long time the Efimov effect was regarded by many as a theoretical peculiarity. After the theoretical discovery of the Efimov effect great efforts were made to verify it experimentally. However, it took more than 30 years before in 2002 it was found in an ultra-cold gas of caesium atoms [16]. This experiment was a milestone and opened the way to many further experiments in different systems of ultra-cold atoms in many laboratories all over the world [11, 15, 7].

In addition, it lead to a resurgence of interest to the Efimov effect, see for example the review of P. Naidon and S. Endo [17], which contains 400 references. Since then, many predictions of phenomena similar to the Efimov effect have been made [21, 22, 19, 20]. Some of these predictions focus on the question whether an Efimov-type effect can be found in N-particle systems consisting of one- or two-dimensional particles under the assumptions that the $(N-1)$-particle subsystems have a virtual level at the bottom of the essential spectrum. For example in [19] and [21] it is predicted that such an effect occurs for systems of $N = 4$ two-dimensional or $N = 5$ one-dimensional particles if the interactions in subsystems of less than $N-1$ particles are absent. In the table below we give a list of systems of N identical particles where the Efimov effect is expected in the physics literature.

System	**Art of Interactions**	**Resonant subsystems**	**Predicted in**	**Does a mathematical proof exist?**
5 bosons, $d = 1$	four-body no two-body no three-body	four-body	[21]	No
4 bosons, $d = 2$	three-body no two-body	three-body	[19]	No
3 bosons, $d = 3$	two-body	two-body	[10]	Yes, [37]
3 fermions, $d = 2$	two-body	two-body	[20]	Yes, [14]

Table 1. Systems for which an Efimov-type effect is expected by physicists

Besides these systems, effects similar to the Efimov effect are expected for systems with mixed dimensions, i.e. where the particles move in the three-dimensional space, but some of them are confined in a lower-dimensional space. Such an effect is called confinement-induced Efimov effect, see for example [22]. From a mathematical point of view the question of existence or non-existence of Efimov-type effects in systems with mixed dimensions is completely open.

The first mathematically rigorous proof of the Efimov effect for systems of three three-dimensional particles was given by D. R. Yafaev [37] using a system of symmetrized Faddeev equations combined with the low-energy asymptotics of the resolvents of the two-body Hamiltonians. In [39] he also showed that the Hamiltonian has only a finite number of bound states if at most one of the subsystems is resonant. Later, A. V. Sobolev proved the asymptotics (1.1) for the eigenvalue counting function [28]. In the years after the mathematical confirmation of the Efimov effect different techniques were developed and many other mathematical results related to this effect were obtained [35, 23, 31, 30, 6, 33].
In particular, it was proved in [35] that the existence of the Efimov effect depends on the nature of the virtual levels in the subsystems. If the virtual level in the two-body subsystems correspond to eigenvalues, which for example is the case if the three-particle Hamiltonian is considered on certain symmetry subspaces of $L^2(\mathbb{R}^6)$, then the Efimov effect is absent.
For a long time it has been expected that due to the same reason the Efimov effect does not exist for systems of $N \geq 4$ three-dimensional bosons. However, to prove that virtual levels in subsystems of $N-1$ particles correspond to eigenvalues and not to resonances was a very challenging problem, because the sum of the pair potentials does not decay in all directions at infinity, which makes it difficult to use Green's functions. This problem was first solved by D. Gridnev [13] and recently a proof with simpler methods and less restrictions on the potentials was given in [6]. In addition, it was shown in [6] that the Efimov effect can not occur in systems of $N \geq 4$ one- or two-dimensional spinless fermions.

For the case of three two-dimensional spinless fermions, which is not covered by [6], it was predicted in the physics literature that an effect similar to the Efimov effect is present, namely the so-called super Efimov effect [20]. The first mathematical proof of this prediction was given by D. Gridnev [14]. Recently, this result was improved by H. Tamura, where the conditions on the potentials [32] were less restrictive.

Much less is mathematically known about the existence of the Efimov effect in systems of $N \geq 3$ one- or two-dimensional bosons or systems without symmetry restrictions. For such systems consisting of three one- or two-dimensional particles the absence of the Efimov effect was first proved by G. Zhislin and one of the authors of this paper in [34] under very strong restrictions on the potentials. Later, in [36] these restrictions were relaxed, but unfortunately Lemma 1 in [36] contains a mistake. We will correct this mistake in Section 6 at the end of this paper. For systems of $N \geq 4$ one- or two-dimensional bosons mathematical results are unknown. The main goal of our work is to fill this gap, at least partially.

Our main results are the following: For systems of $N \geq 3$ one-dimensional bosons or particles without symmetry restrictions with pair interactions we prove that the existence of virtual levels in the $(N-1)$-particle subsystems does not imply the infiniteness of the number of negative eigenvalues. For systems of N two-dimensional particles we prove the same result except for $N = 4$.

The method of the proof is analogous to the proof in [6]. We study the decay of solutions of the Schrödinger equation corresponding to a virtual level and show that these solutions are eigenfunctions. Then we use arguments similar to [35]. To obtain the decay rate of the solutions we apply a modification of Agmon's method [3], developed in [6]. This method requires estimates on the quadratic form of a multi-particle Schrödinger operator on functions supported far from the origin. In order to obtain these estimates we make a partition of unity of the configuration space according to decompositions of the original system into clusters with careful estimates of the localization error.

On the technical level however this work is very different from [6]. A crucial difference between one- or two-dimensional particles and d-dimensional particles with $d \geq 3$ is that in lower dimensions the common Hardy inequality does not hold. This manifests in particular in the fact that the one-particle Schrödinger operator $h = -\Delta + V$ in dimension one or two with a short-range potential $V \not\equiv 0$ has negative eigenvalues if $\int V(x)\,\mathrm{d}x \leq 0$. Consequently, if we know that h does not have negative spectrum, we can immediately say that $\int V(x)\,\mathrm{d}x > 0$. This simple observation plays an important role in our proof.

On the other hand to localize regions in the configuration space in [6] we used a special type of the cut-off functions constructed in [35]. For this choice of the cut-off functions, due to Hardy's inequality, the localization error can be compensated by a small part of the kinetic energy. Since in one- and two-dimensional cases the Hardy inequalities are different, this construction can not be applied in lower dimensions. To overcome this obstacle, we develop in Section 3 an advanced way to construct the cut-off functions, which is better compatible with one- and two-dimensional variants of Hardy's inequality.

The paper is organized as follows. In Section 2 we discuss virtual levels of one-body Schrödinger operators with short-range potentials in dimension one and two. We prove that virtual levels correspond to resonances and give an estimate for the decay rate of the corresponding solutions. This section is contained for completeness. Readers only interested in the multi-particle case can skip it and go immediately to Section 3, where we extend the study of virtual levels to the multi-particle case. We prove that for systems of $N \geq 3$ one-dimensional or $N \geq 4$ two-dimensional particles virtual levels correspond to eigenvalues. We also derive lower bounds for the decay rates of the zero energy eigenfunctions. In Section 4 we discuss systems of three two-dimensional particles, which is the only case where a virtual level might correspond to a resonance. We show that there exists a solution of the Schrödinger equation in the space $L^2(\mathbb{R}^4, (1+|x|)^{-\delta})$, $\delta > 0$. Section 5 is devoted to the absence of the Efimov effect for multi-particle systems in dimension one and two. In Section 6 we give the proof for the absence of the Efimov effect in systems of three one- or two-dimensional particles.

2. Virtual levels of one-particle Schrödinger operators in dimension one and two

Although the main subject of this paper are virtual levels of multi-particle systems consisting of one- or two-dimensional particles and the non-existence of the Efimov effect in such systems, in this section we discuss virtual levels of one-particle Schrödinger operators in these dimensions. Some of the results of this section will be applied later to study the multi-particle case, others are given for a better understanding of one- and two-dimensional systems.

2.1. **Notation and assumptions.** In this section we consider the one-particle Schrödinger operator

$$h = -\Delta + V \tag{2.1}$$

in $L^2(\mathbb{R}^d)$ with $d = 1$ or $d = 2$. Within the whole section we assume that $V \neq 0$. Furthermore, we assume that V is relatively form bounded with relative bound zero, i.e. for every $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$, such that

$$\langle |V|\psi, \psi\rangle \leq \varepsilon \|\nabla \psi\|^2 + C(\varepsilon)\|\psi\|^2 \tag{2.2}$$

holds for any $\psi \in H^1(\mathbb{R}^d)$. This condition is fulfilled if $V \in L^p_{\text{loc}}(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ with $p = 1$ if $d = 1$ and $p > 1$ if $d = 2$, [9]. Due to the KLMN-theorem [24, Theorem X.17] under this assumption the operator h is self adjoint on $L^2(\mathbb{R}^d)$ with the associated quadratic form

$$Q[\psi] = \|\nabla \psi\|^2 + \langle V\psi, \psi\rangle \tag{2.3}$$

with form domain $H^1(\mathbb{R}^d)$. For any $\varepsilon \in (0,1)$ we define

$$h_\varepsilon = h + \varepsilon\Delta. \tag{2.4}$$

For any self-adjoint operator $\mathcal{A}$ we denote by $\mathcal{S}(\mathcal{A})$, $\mathcal{S}_{\text{ess}}(\mathcal{A})$ and $\mathcal{S}_{\text{disc}}(\mathcal{A})$ the spectrum, the essential spectrum and the discrete spectrum of $\mathcal{A}$, respectively.
Following [8] we introduce function spaces which will be important for our studies in this paper. For dimension $d \geq 3$ the homogenous Sobolev space $\tilde{H}^1(\mathbb{R}^d)$ is defined as the completion of $C_0^\infty(\mathbb{R}^d)$-functions with respect to the norm

$$\|u\|_{\tilde{H}^1} = \left(\int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{2.5}$$

It follows from Hardy's inequality that for $d \geq 3$ a sequence of functions $u_n \in \tilde{H}^1(\mathbb{R}^d)$ with $\|u_n\|_{\tilde{H}^1} \to 0$ converges to zero in $L^2_{\text{loc}}(\mathbb{R}^d)$. It is also known (see, for example [8]) that for $d = 1$ and $d = 2$ the completion of $C_0^\infty(\mathbb{R}^d)$ with respect to (2.5) does not lead to a function space, because constant functions are identified. In order to avoid this problem we add a local L^2 norm to the gradient semi-norm and define for $d = 1$ or $d = 2$

$$\|u\|_{\tilde{H}^1} = \left(\int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x + \int_{\{|x|\leq 1\}} |u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{2.6}$$

Let $\tilde{H}^1(\mathbb{R}^d)$ be the completion of $C_0^\infty(\mathbb{R}^d)$ with respect to the norm (2.6), then

$$\tilde{H}^1(\mathbb{R}^d) = \left\{u \in L^1_{\text{loc}}(\mathbb{R}^d), \nabla u \in L^2(\mathbb{R}^d)\right\}. \tag{2.7}$$

In Appendix A we collect some elementary properties of the space $\tilde{H}^1(\mathbb{R}^d)$, $d=1,2$, which we use in this paper.

2.2. **Properties of virtual levels of one-particle Schrödinger operators with short-range potentials.**

Definition 2.1. Assume that the potential V satisfies (2.2). We say that the operator h, defined in (2.1), has a virtual level at zero if

$$h \geq 0, \qquad \inf \mathcal{S}\,(h_\varepsilon) < 0 \qquad \text{and} \qquad \inf \mathcal{S}_{\text{ess}}\,(h_\varepsilon) = 0 \tag{2.8}$$

holds for any sufficiently small $\varepsilon > 0$.

Remark. Note that the Laplace operator is critical in dimension one and two, i.e. for any $V \in L^1(\mathbb{R}^d)$ satisfying (2.2) and $V \not\equiv 0$ with $\int_{\mathbb{R}^d} V(x)\,\mathrm{d}x \leq 0$ the operator h has at least one negative eigenvalue, see [27]. Consequently, for $d=1,2$ the condition $h \geq 0$ implies $\int_{-\infty}^{\infty} V(x)\,\mathrm{d}x > 0$. On the other hand, the condition $\inf \mathcal{S}(h_\varepsilon) < 0$ yields that $V(x)$ has a non-trivial negative part.

Let us briefly motivate our goals for this section. For the case $d=3$ it was shown that the one-particle Schrödinger operator $h \geq 0$ with short-range potential has a virtual level if and only if $h\psi = 0$ has a solution in $\tilde{H}^1\left(\mathbb{R}^3\right)$. This solution does not belong to $L^2\left(\mathbb{R}^3\right)$ and decays as $|x|^{-1}$ as $|x| \to \infty$, see [38].
Moreover, by applying Hardy's inequality one can see that in case $d \geq 3$ for short-range potentials the operator h has a virtual level at zero if and only if $h \geq 0$ and for any $\varepsilon > 0$ the operator $\tilde{h} = -\Delta + V - \varepsilon(1+|x|)^{-2}$ has a discrete eigenvalue below zero.
Our goal is to generalize these two statements to the cases $d=1,2$. This will be done in the following two theorems.

Theorem 2.2 (Solutions of the Schrödinger equation corresponding to virtual levels)**.** *Assume that $d=1$ or $d=2$ and that the potential V satisfies $V \not\equiv 0$, condition (2.2) and*

$$|V(x)| \leq C\,(1+|x|)^{-2-\nu}\,, \qquad x \in \mathbb{R}^d, \quad |x| \geq A \tag{2.9}$$

for constants $A, C, \nu > 0$. If h has a virtual level at zero, then the following assertions hold:

- **(i)** *There exists a solution $\varphi_0 \in \tilde{H}^1(\mathbb{R}^d)$, $\varphi_0 \not\equiv 0$, of the equation $-\Delta\varphi_0 + V\varphi_0 = 0$, i.e. for all $\psi \in \tilde{H}^1(\mathbb{R}^d)$*
$$\langle \nabla\varphi_0, \nabla\psi \rangle + \langle V\varphi_0, \psi \rangle = 0. \tag{2.10}$$
- **(ii)** *Let $d=1$. Then for the functions φ_0 satisfying (2.10) we have*
$$(1+|x|)^{-\frac{1}{2}-\varepsilon}\varphi_0 \in L^2(\mathbb{R}) \qquad \text{for any } \varepsilon > 0. \tag{2.11}$$
- **(iii)** *Let $d=2$. Then for the functions φ_0 satisfying (2.10) we have*
$$(1+|x|)^{-1}\,(1+|\ln(|x|)|)^{-\frac{1}{2}-\varepsilon}\,\varphi_0 \in L^2(\mathbb{R}^2) \qquad \text{for any } \varepsilon > 0. \tag{2.12}$$
- **(iv)** *If in addition the potential V is relatively $-\Delta$-bounded, i.e. there exists a constant $C > 0$, such that*
$$\|V\psi\|^2 \leq C\left(\|\Delta\psi\|^2 + \|\psi\|^2\right) \tag{2.13}$$
holds for all functions $\psi \in H^2(\mathbb{R}^d)$, then there exists a constant $\delta_0 > 0$, such that for any function $\psi \in H^1(\mathbb{R}^d)$ satisfying $\langle \nabla\psi, \nabla\varphi_0 \rangle = 0$
$$\langle h\psi, \psi \rangle \geq \delta_0 \|\nabla\psi\|^2. \tag{2.14}$$

Remark. **(i)** Note that the left-hand side of (2.10) is well-defined due to condition (2.9) and inequalities (A.4) and (A.5) in Appendix A.

(ii) Similarly to Theorem 2.1 in [6] we use the condition (2.13) on the potential V only to be able to apply the unique continuation theorem. Without this condition we are not able to prove uniqueness of the solution φ_0 of the equation $-\Delta\varphi_0+V\varphi_0 = 0$ in $\tilde{H}^1(\mathbb{R}^d)$. However, similarly to [6] we can show that the subspace $\mathcal{M} \subset \tilde{H}^1(\mathbb{R}^d)$ of functions φ satisfying (2.10) is finite-dimensional and that for each $\psi \in \tilde{H}^1(\mathbb{R}^d)$, satisfying $\langle \nabla\varphi, \nabla\psi\rangle = 0$ for all $\varphi \in \mathcal{M}$ holds (2.14).

(iii) Theorem 2.2 gives a lower bound on the decay rate of solutions of the Schrödinger equation corresponding to virtual levels. It is easy to see that if the potentials are compactly supported and $V(x) = V(|x|)$, then estimates (2.53) and (2.54) are almost sharp. It is also easy to see that the solution can not be an eigenfunction, it is a zero energy resonance.

Theorem 2.3 (Necessary and sufficient condition for a virtual level)**.** *Let $d = 1, 2$. We assume that $V \neq 0$ satisfies (2.2) and (2.9) and that $h \geq 0$. Further, let $\mathcal{U}$ be a continuous, strictly negative potential satisfying for $|x| \geq A$ the condition*

$$|\mathcal{U}(x)| \leq C|x|^{-2} \text{ if } d = 1 \quad \text{and} \quad |\mathcal{U}(x)| \leq C|x|^{-2}\ln^{-2}(|x|) \text{ if } d = 2 \tag{2.15}$$

for some $A, C > 0$. Then h has a virtual level at zero if and only if for any $\varepsilon > 0$ we have

$$\inf \mathcal{S}\,(h + \varepsilon\mathcal{U}) < 0. \tag{2.16}$$

Remark. **(i)** Note that in dimension $d \geq 3$ Hardy's inequality yields that $\inf \mathcal{S}(h+\varepsilon\mathcal{U}) = 0$ for sufficiently small $\varepsilon > 0$ if h does not have a virtual level. For dimension $d = 1$ or $d = 2$ it does not follow from Hardy's inequality. However, Theorem 2.3 shows that it is still true.

(ii) Assume that $V \neq 0$ satisfies (2.2) and (2.9), $h \geq 0$ and that h does not have a virtual level at zero. Then for small $\varepsilon_0 > 0$ the operator $h_{\varepsilon_0} > 0$ also does not have a virtual level and therefore Theorem 2.3 can be applied to the operator h_{ε_0}. Hence, there exists $\varepsilon_1 > 0$, such that

$$(1 - \varepsilon_0)\|\nabla\psi\|^2 + \langle V\psi, \psi\rangle + \varepsilon_1\langle \mathcal{U}\psi, \psi\rangle \geq 0 \tag{2.17}$$

holds for any function $\psi \in H^1(\mathbb{R}^d)$ with $\mathcal{U}$ defined as in Theorem 2.3. This modification of Theorem 2.3 will be used in the case of multi-particle systems in the next sections.

(iii) Assume that $h \geq 0$ does not have a virtual level and that the potential $V \neq 0$ satisfies (2.2) and (2.9). Then by choosing $\mathcal{U}$ according to Theorem 2.3 with $\mathcal{U}(x) = -1$ for $|x| \leq 1$ we obtain from (2.17) that for any $\psi \in \tilde{H}^1(\mathbb{R}^d)$

$$\|\psi\|^2_{\tilde{H}^1} \leq \frac{1 + \varepsilon_1 - \varepsilon_0}{\varepsilon_1}\|\nabla\psi\|^2 + \frac{1}{\varepsilon_1}\langle V\psi, \psi\rangle. \tag{2.18}$$

Theorem 2.3 will be proved in Appendix B, where similar statements for multi-particle Schrödinger operators are established. We turn to the

Proof of Theorem 2.2. Since for any $\varepsilon > 0$ we have $\inf \mathcal{S}_{\mathrm{disc}}(h_\varepsilon) < 0$, we find a sequence of eigenfunctions $\psi_n \in H^1(\mathbb{R}^d)$, corresponding to eigenvalues $E_n < 0$ of the operator $h_{n^{-1}}$, i.e.

$$-\left(1 - n^{-1}\right)\Delta\psi_n + V\psi_n = E_n\psi_n. \tag{2.19}$$

We normalize the functions ψ_n by the condition $\|\psi_n\|_{\tilde{H}^1} = 1$. Then there exists a subsequence, also denoted by $(\psi_n)_{n\in\mathbb{N}}$, which converges weakly in $\tilde{H}^1(\mathbb{R}^d)$ to a function $\varphi_0 \in \tilde{H}^1(\mathbb{R}^d)$. At first, we prove that φ_0 is a solution of the equation $-\Delta\varphi_0 + V\varphi_0 = 0$ in $\tilde{H}^1(\mathbb{R}^d)$ and that $\varphi_0 \neq 0$. Indeed, we have the following

Lemma 2.4. *Assume that h has a virtual level at zero and that V satisfies (2.2) and (2.9). Then the function φ_0 defined above is not zero and for any $\psi \in \tilde{H}^1(\mathbb{R}^d)$*

$$\langle \nabla\varphi_0, \nabla\psi\rangle + \langle V\varphi_0, \psi\rangle = 0. \tag{2.20}$$

Proof of Lemma 2.4. Since φ_0 is the weak limit of the sequence $(\psi_n)_{n\in\mathbb{N}}$ in $\tilde{H}^1(\mathbb{R}^d)$, we have $\varphi_0 \in L^2_{\text{loc}}(\mathbb{R}^d)$ and by Proposition A.1 **(iii)** $\psi_n \to \varphi_0$ strongly in $L^2_{\text{loc}}(\mathbb{R}^d)$. First, we show that

$$\int_{\{|x|\le R\}} V(x)|\psi_n(x)|^2 \,\mathrm{d}x \longrightarrow \int_{\{|x|\le R\}} V(x)|\varphi_0(x)|^2 \,\mathrm{d}x \quad \text{as} \quad n\to\infty \tag{2.21}$$

for any fixed $R > 0$. We write

$$\langle V\psi_n, \psi_n\rangle_{B(R)} - \langle V\varphi_0, \varphi_0\rangle_{B(R)} = \langle V(\psi_n - \varphi_0), \psi_n\rangle_{B(R)} + \langle V\varphi_0, (\psi_n - \varphi_0)\rangle_{B(R)}, \tag{2.22}$$

where $B(R) = \{x \in \mathbb{R}^d : |x| \le R\}$. Let χ be a piecewise differentiable function satisfying $\chi(x) = 1$ for $x \in B(R)$ and $\chi(x) = 0$ if $x \notin B(R+1)$. Then we get by Cauchy Schwarz

$$\begin{aligned}\langle |V||\psi_n - \varphi_0|, |\psi_n|\rangle_{B(R)} &\le \langle |V|^{\frac12}|\psi_n - \varphi_0|\chi, |V|^{\frac12}|\psi_n|\chi\rangle \\ &\le \left(\langle |V||\psi_n - \varphi_0|\chi, |\psi_n - \varphi_0|\chi\rangle\right)^{\frac12} \left(\langle |V||\psi_n|\chi, |\psi_n|\chi\rangle\right)^{\frac12}.\end{aligned} \tag{2.23}$$

We estimate the two factors on the r.h.s. of (2.23) separately. By assumption (2.2) we get

$$\langle |V||\psi_n - \varphi_0|\chi, |\psi_n - \varphi_0|\chi\rangle \le \varepsilon \|\nabla_0 \left(|\psi_n - \varphi_0|\chi\right)\|^2 + C(\varepsilon)\|(\psi_n - \varphi_0)\chi\|^2. \tag{2.24}$$

Due to $\|\nabla_0\psi_n\| \le 1$, $\|\nabla_0\varphi_0\| \le 1$, $0 \le \chi \le 1$ and $|\nabla_0\chi| \le C$ for some $C > 0$, the first term on the r.h.s. of (2.24) is arbitrarily small if $\varepsilon > 0$ is small enough. The second term tends to zero as $n \to \infty$ because $\psi_n \to \varphi_0$ in $L^2_{\text{loc}}(\mathbb{R}^d)$. Similarly, we can show that $\langle |V||\psi_n|\chi, |\psi_n|\chi\rangle$ is bounded and therefore $\langle |V||\psi_n - \varphi_0|, |\psi_n|\rangle_{B(R)}$ tends to zero as $n \to \infty$. Analogously we get $\langle V(\psi_n - \varphi_0), \varphi_0\rangle_{B(R)} \to 0$ as $n \to \infty$. Hence, we get (2.21).
By taking $R > A$, condition (2.9) together with inequality (A.4) for $d = 1$ and (A.5) for $d = 2$, respectively, implies

$$\begin{aligned}\int_{\{|x|>R\}} |V(x)||\psi_n(x)|^2 \,\mathrm{d}x &\le C \int_{\{|x|>R\}} \frac{|\psi_n(x)|^2}{(1+|x|)^{2+\nu}} \,\mathrm{d}x \\ &\le \tilde{C} R^{-\frac{\nu}{2}} \|\psi_n\|^2_{\tilde{H}^1} = \tilde{C} R^{-\frac{\nu}{2}}\end{aligned} \tag{2.25}$$

for some constants $C, \tilde{C} > 0$. Since $\|\varphi_0\|_{\tilde{H}^1} \le 1$, by the same arguments we get

$$\int_{\{|x|>R\}} |V(x)||\varphi_0(x)|^2 \,\mathrm{d}x \le \tilde{C} R^{-\frac{\nu}{2}}, \tag{2.26}$$

which together with (2.21) implies that $\langle V\varphi_0, \varphi_0\rangle$ is well-defined and

$$\langle V\psi_n, \psi_n\rangle \to \langle V\varphi_0, \varphi_0\rangle \qquad \text{as } n \to \infty. \tag{2.27}$$

Recall that

$$\begin{aligned}\langle V\psi_n, \psi_n\rangle &\le -\left(1 - n^{-1}\right)\|\nabla\psi_n\|^2 \\ &= \left(1 - n^{-1}\right)\left(-1 + \int_{\{|x|\le 1\}} |\psi_n|^2 \,\mathrm{d}x\right) \longrightarrow -1 + \int_{\{|x|\le 1\}} |\varphi_0|^2 \,\mathrm{d}x.\end{aligned} \tag{2.28}$$

Sending n to infinity and using (2.27) yields

$$\langle V\varphi_0, \varphi_0\rangle \le -1 + \int_{\{|x|\le 1\}} |\varphi_0|^2 \,\mathrm{d}x = -1 - \|\nabla\varphi_0\|^2 + \|\varphi_0\|^2_{\tilde{H}^1}. \tag{2.29}$$

Since $\|\varphi_0\|_{\tilde{H}^1} \le 1$ and the operator h is non-negative, we get $\|\varphi_0\|_{\tilde{H}^1} = 1$ and

$$\|\nabla\varphi_0\|^2 + \langle V\varphi_0, \varphi_0\rangle = 0. \tag{2.30}$$

Standard arguments show that φ_0 satisfies (2.20) for any $\psi \in \tilde{H}^1(\mathbb{R}^d)$. $\square$

Now we turn to the proof of statements **(ii)** and **(iii)** of Theorem 2.2, i.e. the estimate of the weighted $L^2(\mathbb{R}^d)$ norm of φ_0. At first, we prove a weighted $L^2(\mathbb{R}^d)$-estimate for the functions ψ_n.

Lemma 2.5. *Assume that h has a virtual level at zero and that V satisfies* (2.2) *and* (2.9)*. Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of eigenfunctions corresponding to negative eigenvalues $E_n < 0$ of the operator $h_{n^{-1}}$, normalized as $\|\psi_n\|_{\tilde{H}^1} = 1$. Then the following assertions hold:*

(i) *If $d = 1$, then for any $\alpha_0 < \frac{1}{2}$ there exists a $C > 0$, such that for all $n \in \mathbb{N}$ we have*

$$\|\nabla(|x|^{\alpha_0}\psi_n)\| \le C \quad \text{and} \quad \|(1+|x|)^{\alpha_0-1}\psi_n\| \le C. \tag{2.31}$$

(ii) *If $d = 2$, then for any $\alpha_0 < \frac{1}{2}$ there exists a $C > 0$, such that for all $n \in \mathbb{N}$ we have*

$$\|\nabla(|\ln(|x|)|^{\alpha_0}\psi_n)\| \le C \quad \text{and} \quad \|(1+|x|)^{-1}(1+|\ln(|x|)|)^{\alpha_0-1}\psi_n\| \le C. \tag{2.32}$$

Proof of Lemma 2.5. The proof is a modification of the proof of Lemma 2.4 in [6]. At first, we prove the Lemma for the case $d = 1$. For any $\varepsilon > 0$ and $R > 0$ we define the function

$$G_\varepsilon(x) = \frac{|x|^{\alpha_0}}{1+\varepsilon|x|^{\alpha_0}}\chi_R(x), \tag{2.33}$$

where χ_R is a C^1-cutoff function with

$$\chi_R(x) = \begin{cases} 0, & |x| \le R, \\ 1, & |x| \ge 2R. \end{cases} \tag{2.34}$$

We multiply the eigenvalue equation

$$-(1-n^{-1})\Delta\psi_n + V\psi_n = E_n\psi_n \tag{2.35}$$

by $G_\varepsilon^2\overline{\psi_n}$ and integrate by parts to obtain

$$\left(1-n^{-1}\right)\langle\nabla\psi_n, \nabla\left(G_\varepsilon^2\psi_n\right)\rangle + \langle V\psi_n, G_\varepsilon^2\psi_n\rangle = E_n\|G_\varepsilon\psi_n\|^2 < 0. \tag{2.36}$$

Since

$$\mathrm{Re}\langle V\psi_n, G_\varepsilon^2\psi_n\rangle = \langle V\psi_n, G_\varepsilon^2\psi_n\rangle \qquad \text{and} \qquad \mathrm{Re}\, E_n\|G_\varepsilon\psi_n\|^2 = E_n\|G_\varepsilon\psi_n\|^2, \tag{2.37}$$

we have

$$\mathrm{Re}\langle\nabla\psi_n, \nabla\left(G_\varepsilon^2\psi_n\right)\rangle = \langle\nabla\psi_n, \nabla\left(G_\varepsilon^2\psi_n\right)\rangle. \tag{2.38}$$

Note that

$$\begin{aligned} \mathrm{Re}\langle\nabla\psi_n, \nabla(G_\varepsilon^2\psi_n)\rangle &= \mathrm{Re}\langle\nabla\psi_n, G_\varepsilon\psi_n\nabla G_\varepsilon\rangle + \mathrm{Re}\langle(\nabla\psi_n)G_\varepsilon, \nabla(G_\varepsilon\psi_n)\rangle \\ &= \mathrm{Re}\langle\nabla(\psi_n G_\varepsilon), \psi_n\nabla G_\varepsilon\rangle - \mathrm{Re}\langle\psi_n\nabla G_\varepsilon, \psi_n\nabla G_\varepsilon\rangle \\ &\quad + \mathrm{Re}\langle\nabla(\psi_n G_\varepsilon), \nabla(\psi_n G_\varepsilon)\rangle - \mathrm{Re}\langle\psi_n\nabla G_\varepsilon, \nabla(\psi_n G_\varepsilon)\rangle \\ &= \mathrm{Re}\langle\nabla(\psi_n G_\varepsilon), \nabla(\psi_n G_\varepsilon)\rangle - \mathrm{Re}\langle\psi_n\nabla G_\varepsilon, \psi_n\nabla G_\varepsilon\rangle. \end{aligned} \tag{2.39}$$

Therefore, we obtain

$$\langle\nabla\psi_n, \nabla(G_\varepsilon^2\psi_n)\rangle = \|\nabla(\psi_n G_\varepsilon)\|^2 - \|\psi_n\nabla G_\varepsilon\|^2. \tag{2.40}$$

This together with (2.36) yields

$$\left(1-n^{-1}\right)\left(\|\nabla(\psi_n G_\varepsilon)\|^2 - \int|\psi_n|^2|\nabla G_\varepsilon|^2\,\mathrm{d}x\right) + \int V|\psi_n G_\varepsilon|^2\,\mathrm{d}x < 0. \tag{2.41}$$

Now we estimate the function $|\nabla G_\varepsilon|$. For $|x| > 2R$ we have

$$|\nabla G_\varepsilon| = \frac{\alpha_0|x|^{\alpha_0-1}}{(1+\varepsilon|x|^{\alpha_0})^2} \le \alpha_0|x|^{-1}|G_\varepsilon|. \tag{2.42}$$

For $|x| \in [R, 2R]$ the function $|\nabla G_\varepsilon|$ is uniformly bounded in ε, which implies

$$\int_{\{R\le|x|\le 2R\}} |\nabla G_\varepsilon|^2 |\psi_n|^2 \,\mathrm{d}x \le C_0 \int_{\{R\le|x|\le 2R\}} |\psi_n|^2 \,\mathrm{d}x, \tag{2.43}$$

for some $C_0 > 0$ which depends on R only. Now we use inequality (A.4) to estimate the r.h.s. of (2.43). We get

$$\int_{\{R\le|x|\le 2R\}} |\psi_n|^2 \,\mathrm{d}x \le (1+4R^2) \int_{\{R\le|x|\le 2R\}} \frac{|\psi_n|^2}{1+x^2} \,\mathrm{d}x \le C_H(1+4R^2)\|\psi_n\|^2_{\tilde{H}^1}, \tag{2.44}$$

where C_H is a Hardy-type constant in (A.4). This, together with (2.43) and $\|\psi_n\|_{\tilde{H}^1} = 1$ implies

$$\int_{\{R\le|x|\le 2R\}} |\nabla G_\varepsilon|^2 |\psi_n|^2 \le C_1 \tag{2.45}$$

for some $C_1 > 0$ which is independent of $n \in \mathbb{N}$ and $\varepsilon > 0$. Substituting (2.42) and (2.45) into (2.41) we obtain

$$\left(1 - n^{-1}\right) \|\nabla(\psi_n G_\varepsilon)\|^2 + \langle V G_\varepsilon \psi_n, G_\varepsilon \psi_n\rangle - \alpha_0^2 \int_{\{|x|>2R\}} \frac{|G_\varepsilon \psi_n|^2}{|x|^2} \,\mathrm{d}x \le C_2, \tag{2.46}$$

where $C_2 > 0$ does not depend on $n \in \mathbb{N}$ or $\varepsilon > 0$. The function $G_\varepsilon \psi_n$ is supported outside the ball with radius R. Therefore, choosing $R > A$ we can use (2.9) and apply Hardy's inequality for the half-line, which yields

$$(1-\gamma_0)\|\nabla(G_\varepsilon\psi_n)\|^2 + \langle V G_\varepsilon\psi_n, G_\varepsilon\psi_n\rangle - \alpha_0^2 \langle |x|^{-2} G_\varepsilon\psi_n, G_\varepsilon\psi_n\rangle \ge 0 \tag{2.47}$$

for all $\alpha_0^2 < \frac{1}{4}$ and $\gamma_0 < (1 - 4\alpha_0^2)$. For $n > 2\gamma_0^{-1}$ estimates (2.46) and (2.47) imply

$$\frac{\gamma_0}{2} \|\nabla(G_\varepsilon\psi_n)\|^2 \le C_2. \tag{2.48}$$

Taking the limit $\varepsilon \to 0$ yields $\|\nabla\left(|x|^{\alpha_0}\psi_n\right)\| \le C$ for some $C > 0$.
Applying Hardy's inequality for the half-line to the function $G_\varepsilon\psi_n$ and taking the limit $\varepsilon \to 0$ implies

$$\|(1+|x|)^{\alpha_0 - 1}\psi_n\| \le C. \tag{2.49}$$

This completes the proof of Lemma 2.5 for $d = 1$. Now we assume $d = 2$. For $\varepsilon > 0$ and $0 < \alpha_0 < \frac{1}{2}$ let

$$G_\varepsilon(x) = \frac{|\ln(|x|)|^{\alpha_0}}{1 + \varepsilon|\ln(|x|)|^{\alpha_0}} \chi_R(x), \tag{2.50}$$

where χ_R is a C^1-cutoff function with

$$\chi_R(x) = \begin{cases} 0, & |x| \le R, \\ 1, & |x| \ge 2R. \end{cases} \tag{2.51}$$

Due to (2.9) and Hardy's inequality in dimension two we get similarly to (2.47) that

$$(1-\gamma_0)\|\nabla(G_\varepsilon\psi_n)\|^2 + \langle V G_\varepsilon\psi_n, G_\varepsilon\psi_n\rangle - \alpha_0^2 \langle |x|^{-2} \left(\ln|x|\right)^{-2} G_\varepsilon\psi_n, G_\varepsilon\psi_n\rangle \ge 0 \tag{2.52}$$

for all $\alpha_0^2 < \frac{1}{4}$ and $\gamma_0 < (1 - 4\alpha_0^2)$. Now the proof is a straightforward modification of the one-dimensional case. $\square$

Statements **(ii)** and **(iii)** of Theorem 2.2 follow from the following

Corollary 2.6. *The weak limit φ_0 of the sequence $(\psi_n)_{n\in\mathbb{N}}$ has the following properties.*

(i) *If $d = 1$, then*

$$(1+|x|)^{\alpha_0 - 1}\varphi_0 \in L^2(\mathbb{R}) \qquad \textit{for any } \alpha_0 < \frac{1}{2}. \tag{2.53}$$

(ii) *If $d = 2$, then*

$$(1+|x|)^{-1}\,(1+\ln(|x|))^{\alpha_0-1}\,\varphi_0 \in L^2(\mathbb{R}^2) \qquad \textit{for any } \alpha_0 < \frac{1}{2}. \tag{2.54}$$

Proof of Corollary 2.6. Let $d = 1$. Since $(\psi_n)_{n\in\mathbb{N}}$ converges to φ_0 in $L^2_{\mathrm{loc}}(\mathbb{R})$ and for any $\alpha_0 < \frac{1}{2}$ we have the estimate $\|(1+|x|)^{\alpha_0-1}\psi_n\| \le C$ uniformly in $n \in \mathbb{N}$, for every $\alpha_0 < \frac{1}{2}$ we get

$$(1+|x|)^{\alpha_0-1}\psi_n \to (1+|x|)^{\alpha_0-1}\varphi_0 \quad \text{in } L^2(\mathbb{R}) \quad \text{as } n \to \infty. \tag{2.55}$$

The case $d = 2$ follows analogously. □

To complete the proof of Theorem 2.2 it remains to prove statement **(iv)**. This is a straightforward generalization of Lemma 2.10 in [6], which is based on the the unique continuation theorem [25, Theorem 2.1]. □

3. Virtual levels of systems of N one- or two-dimensional particles

In this section we introduce virtual levels of Schrödinger operators corresponding to systems consisting of N one- or two-dimensional particles. We prove several results on the decay rate of solutions of the Schrödinger equation corresponding to virtual levels of multi-particle systems. The main result of this section is Theorem 3.2, where we give sufficient conditions in terms of a Hardy-type constant, such that virtual levels of multi-particle Schrödinger operators correspond to eigenvalues and prove an estimate for the decay rate of the corresponding eigenfunctions. In Corollaries 3.3 and 3.4 and Theorem 3.5 we discuss applications of Theorem 3.2 to multi-particle systems.

3.1. **Notation and definitions for multi-particle systems.** We consider a system of $N \ge 3$ quantum particles in dimension $d = 1$ or $d = 2$ with masses $m_i > 0$ and position vectors $x_i \in \mathbb{R}^d$, $i = 1, \dots, N$. Such a system is described by the Hamiltonian

$$H_N = -\sum_{i=1}^{N} \frac{1}{m_i}\Delta_{x_i} + \sum_{1\le i<j\le N} V_{ij}(x_{ij}), \quad x_{ij} = x_i - x_j \tag{3.1}$$

acting on $L^2(\mathbb{R}^{dN})$. The potentials V_{ij} describe the particle pair interactions and in the following we assume that they satisfy $V_{ij} \neq 0$ and the conditions (2.2) and (2.9).

Separation of the center of mass of the system. We will consider the operator H_N in the center-of-mass frame. Following [26], we denote by $\langle\cdot,\cdot\rangle_m$ the scalar product on $\mathbb{R}^{dN}$ which is given by

$$\langle x, y\rangle_m = \sum_{i=1}^{N} m_i\langle x_i, y_i\rangle, \qquad |x|_m^2 = \langle x, x\rangle_m, \qquad x, y \in \mathbb{R}^{dN}. \tag{3.2}$$

Here, $\langle\cdot,\cdot\rangle$ is the standard scalar product on $\mathbb{R}^d$. Let X be the space $\mathbb{R}^{dN}$ equipped with the scalar product $\langle\cdot,\cdot\rangle_m$ and let

$$X_0 = \left\{ x = (x_1, \dots, x_N) \in X \;:\; \sum_{i=1}^{N} m_i x_i = 0 \right\} \tag{3.3}$$

be the space of positions of the particles in the center of mass frame and $X_c = X \ominus X_0$ be the space of the center of mass position of the system. We denote by P_0 and P_c the orthogonal projections from X on X_0 and X_c, respectively.
Furthermore, we introduce $-\Delta$, $-\Delta_0$ and $-\Delta_c$ as the Laplace-Beltrami operators on $L^2(X)$,

$L^2(X_0)$ and $L^2(X_c)$, respectively. Then, corresponding to the decomposition $L^2(X) = L^2(X_0) \otimes L^2(X_c)$ we find

$$-\Delta = -\Delta_0 \otimes \mathrm{Id} + \mathrm{Id} \otimes (-\Delta_c). \tag{3.4}$$

Since for every $x \in X$ we have

$$(P_0 x)_i - (P_0 x)_j = x_i - x_j, \tag{3.5}$$

the potential $V(x) = \sum_{1 \le i < j \le N} V_{ij}(x_{ij})$ satisfies

$$V(x) = V(P_0 x). \tag{3.6}$$

Therefore, H_N is unitarily equivalent to the operator

$$H \otimes \mathrm{Id} + \mathrm{Id} \otimes (-\Delta_c), \tag{3.7}$$

where

$$H = -\Delta_0 + V. \tag{3.8}$$

In view of (3.7) the center of mass of the system moves like a free particle and the operator H corresponds to the relative motion of the system.

Clusters and Cluster Hamiltonians. A cluster C of the system is defined as a non-empty subset of $\{1, \dots, N\}$ and we denote by $|C|$ the number of particles contained in C. For $1 < |C| < N$ we define the space of the relative positions of the particles in the cluster C by

$$X_0[C] = \{x \in X_0 : x_i = 0 \text{ if } i \notin C\}. \tag{3.9}$$

Let $-\Delta_0[C]$ be the the Laplace-Beltrami operator on $L^2(X_0[C])$ and

$$V[C] = \sum_{i,j \in C,\ i<j} V_{ij} \tag{3.10}$$

the potential of the interactions between the particles in the cluster C. Then for $1 < |C| < N$ the cluster Hamiltonian with reduced center of mass, acting on $L^2(X_0[C])$, is given by

$$H[C] = -\Delta_0[C] + V[C] \tag{3.11}$$

and describes the internal dynamics of the cluster C. For $C = \{1, \dots, N\}$ we have $X_0[C] = X_0$, so we set $H[C] = H$. For $|C| = 1$ we have $X[C] = \{0\}$ and we set $H[C] = 0$.
Let $P_0[C]$ be the orthogonal projection from X_0 to $X_0[C]$ and for $x \in X_0$ let

$$q[C] = P_0[C]x. \tag{3.12}$$

Partitions of the system. We say that $Z = (C_1, \dots, C_p)$ is a partition or cluster decomposition of the system of order $|Z| = p$ if and only if

$$C_i \neq \emptyset, \qquad C_i \cap C_j = \emptyset, \qquad \bigcup_{j=1}^{p} C_j = \{1, \dots, N\} \tag{3.13}$$

holds for all $i, j = 1, \dots, p$ with $i \neq j$. We refer to $C \subset Z$ as a cluster of the partition $Z = (C_1, \dots, C_p)$ if $C = C_i$ for some $i = 1, \dots, p$. Let

$$X_0(Z) = \bigoplus_{C_k \subset Z} X_0[C_k], \qquad X_c(Z) = X_0 \ominus X_0(Z). \tag{3.14}$$

This gives rise to the decomposition

$$L^2(X_0(Z)) = \bigotimes_{C_k \subset Z} L^2(X_0[C_k]). \tag{3.15}$$

By abuse of notation we denote the operator

$$\mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \otimes (-\Delta_0[C_k]) \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \quad \text{and} \quad \mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \otimes H[C_k] \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id}, \tag{3.16}$$

acting on $L^2(X_0(Z))$, by $-\Delta_0[C_k]$ and $H[C_k]$, respectively. The cluster decomposition Hamiltonian of the partition Z is defined by

$$H(Z) = \sum_{C_k \subset Z} H[C_k] \tag{3.17}$$

and describes the joint internal dynamics of the clusters in Z. Let $-\Delta_0(Z)$ be the Laplace-Beltrami operator on $L^2(X_0(Z))$. Then

$$-\Delta_0(Z) = \sum_{C_k \subset Z} -\Delta_0[C_k]. \tag{3.18}$$

We denote the potential of the inter-cluster interaction by

$$I(Z) = V - \sum_{C_k \subset Z} V[C_k]. \tag{3.19}$$

Then the Hamiltonian of the whole system can be written as

$$H = H(Z) \otimes \mathrm{Id} + \mathrm{Id} \otimes (-\Delta_c(Z)) + I(Z), \tag{3.20}$$

where $-\Delta_c(Z)$ is the Laplace-Beltrami operator on $L^2(X_c(Z))$. We introduce the projections $P_0(Z)$ and $P_c(Z)$ from X_0 on $X_0(Z)$ and $X_c(Z)$, respectively. For $x \in X_0$ let

$$q(Z) = P_0(Z)x, \qquad \xi(Z) = P_c(Z)x. \tag{3.21}$$

To emphasize the dependence on $q(Z)$ and $\xi(Z)$ we will write

$$-\Delta_{q(Z)} = -\Delta_0(Z) \quad \text{and} \quad -\Delta_{\xi(Z)} = -\Delta_c(Z) \tag{3.22}$$

and

$$H = -\Delta_{q(Z)} - \Delta_{\xi(Z)} + V \quad \text{or} \quad H = H(Z) - \Delta_{\xi(Z)} + I(Z). \tag{3.23}$$

Note that the i-th coordinates of $q(Z)$ and $\xi(Z)$ are vectors q_i and ξ_i given by

$$q_i = x_i - x_{C_l}, \qquad \xi_i = x_{C_l} \tag{3.24}$$

where C_l is the cluster which contains the particle i. Here,

$$x_{C_l} = \frac{1}{\sum_{j \in C_l} m_j} \sum_{j \in C_l} m_j x_j \tag{3.25}$$

is the center of mass of the cluster C_l.
For $\kappa > \kappa' > 0$, $R > 0$ and partitions Z with $1 < |Z| < N$ we define the regions

$$\begin{aligned} B(R) &= \{x \in X_0 \ : \ |x|_m \le R\}, \\ K(Z,\kappa) &= \{x \in X_0 \ : \ |q\,(Z)\,|_m \le \kappa |\xi\,(Z)\,|_m\}, \\ K_R(Z,\kappa) &= \{x \in X_0 \ : \ |q\,(Z)\,|_m \le \kappa |\xi\,(Z)\,|_m, \ |x|_m \ge R\}, \\ K_R(Z,\kappa',\kappa) &= K_R(Z,\kappa) \setminus K_R(Z,\kappa'). \end{aligned} \tag{3.26}$$

For the entire system $Z = \{1, \ldots, N\}$ we set

$$K(Z,\kappa) = \{x \in X_0 : |x|_m \le \kappa\}. \tag{3.27}$$

We will use the regions defined in (3.26) to make a partition of unity of X_0 corresponding to different cluster decompositions of the N-particle system. Now we extend Definition 2.1 of a virtual level to the case of multi-particle systems but first we give two remarks which justify our assumptions.

(i) For three-particle systems with the essential spectrum starting at zero, the existence of resonances in two-particle subsystems may lead, to the appearence of an infinite series of negative eigenvalues accumulating logarithmically at zero, the so-called Efimov effect. These eigenvalues and the corresponding eigenfunctions have many interesting properties. One of the goals of our work is to study whether similar effects may occur in systems of N one- or two-dimensional particles. Due to this specific interest we will consider only the case when $\mathcal{S}_{\rm ess}(H) = [0,\infty)$. By the HVZ theorem this yields $H[C] \geq 0$ for any cluster C with $|C| < N$.

(ii) The assumption $H[C] \geq 0$ is a strong restriction on the potentials V_{ij}. Since we consider one-or two-dimensional particles only, this in particular implies $\int V_{ij}\,{\rm d}x > 0$.

Definition 3.1. Assume that the potentials V_{ij} satisfy (2.2) and (2.9). Let $C \subseteq \{1,\dots,N\}$ be a cluster. We say that $H[C]$ has a virtual level at zero if $H[C] \geq 0$ and

(i) there exists a constant $\varepsilon_0 \in (0,1)$, such that
$$\inf \mathcal{S}_{\rm ess}\left(H[C] + \varepsilon_0\Delta_0[C]\right) = 0, \tag{3.28}$$

(ii) for any $\varepsilon \in (0,1)$ we have
$$\inf \mathcal{S}\left(H[C] + \varepsilon\Delta_0[C]\right) < 0. \tag{3.29}$$

Remark. **(i)** Note that if (3.28) is fulfilled for some $\varepsilon_0 > 0$, then it also holds for all $0 < \tilde\varepsilon_0 < \varepsilon_0$.

(ii) Let $H[C] \geq 0$. Then condition (3.28) can not be fulfilled if there exists a subcluster $\tilde C \subset C$ with $1 < |\tilde C| < |C|$ such that $H[\tilde C]$ has a virtual level. Indeed, in this case we have $\inf \mathcal{S}\left(H[\tilde C] + \varepsilon\Delta_0[\tilde C]\right) < 0$ for any $\varepsilon \in (0,1)$ and according to the HVZ theorem (3.28) does not hold.
On the other hand, if (3.28) does not hold for a cluster C and any $\varepsilon_0 \in (0,1)$, then due to the HVZ theorem there exists at least one subcluster $\tilde C$ of the cluster C with $1 < |\tilde C| < |C|$, such that for any $\varepsilon \in (0,1)$ we have
$$\inf \mathcal{S}\left(H[\tilde C] + \varepsilon\Delta_0[\tilde C]\right) < 0. \tag{3.30}$$
Among these subclusters we choose one with the smallest number of particles and denote it by C_0. If C_0 has only two particles, then $\inf \mathcal{S}_{\rm ess}(H[C_0]+\varepsilon_0\Delta_0[C_0]) = 0$ and according to the definition of a virtual level, the operator $H[C_0]$ has a virtual. Let $|C_0| \geq 3$. Then, since C_0 is the smallest cluster for which (3.30) holds for any $\varepsilon \in (0,1)$, for any subcluster $C' \subsetneq C_0$ with $|C'| > 1$ inequality (3.30) can not hold for all $\varepsilon \in (0,1)$, i.e. for some $\varepsilon_1 \in (0,1)$ we have
$$\inf \mathcal{S}\left(H[C'] + \varepsilon_1\Delta_0[C']\right) = 0. \tag{3.31}$$
Obviously, this is also true for all $0 < \tilde\varepsilon_1 < \varepsilon_1$. Since C_0 has only a finite number of subclusters, we can choose $\varepsilon_1 > 0$ in (3.31), such that this inequality holds for all subclusters of C_0. Applying the HVZ theorem yields
$$\inf \mathcal{S}_{\rm ess}\left(H[C_0] + \varepsilon_0\Delta_0[C_0]\right) = 0 \tag{3.32}$$
for any $\varepsilon_0 \in (0,\varepsilon_1)$. At the same time, $\inf \mathcal{S}\left(H[C_0] + \varepsilon\Delta_0[C_0]\right) < 0$ for any $\varepsilon \in (0,1)$. Hence, $H[C_0]$ has a virtual level at zero.

(iii) Similarly to the case of one-particle Schrödinger operators we can give necessary and sufficient conditions for the operator H to have a virtual level at zero in terms of perturbations of the operator with additional potentials. This result can be found in Appendix B, Theorem B.1.

3.2. **Statements of our results on the decay rates of solutions corresponding to virtual levels.** Now we give our main results of this section, namely the existence of solutions of the Schrödinger equation in the presence of a virtual level and estimates of the decay rate of these solutions. For these estimates certain Hardy-type constants play an important role. Let

$$\mathcal{M} = \big\{\psi \in C_0^1(X_0 \setminus B(1)) \,:\, \psi(x) = 0 \text{ for } x_i = x_j\,, 1 \le i, j \le N,\ i \ne j\big\} \tag{3.33}$$

and let

$$\tilde{C}_H(X_0) = \inf_{0 \ne \psi \in \mathcal{M}} \frac{\|\nabla_0 \psi\|}{\||x|_m^{-1}\psi\|}. \tag{3.34}$$

Remark. If the particles are two-dimensional, the sets $\{x_i = x_j\}$ have co-dimension two and the set $\mathcal{M}$ is dense in $H^1(X_0 \setminus B(1))$. In this case the constant $\tilde{C}_H(X_0)$ coincides with the Hardy constant $C_H(X_0) = \frac{d(N-1)-2}{2} = N-2$ of the $d(N-1)$-dimensional space X_0, see for example inequality (2.18) in [8]. However, for one-dimensional particles the sets $\{x_i = x_j\}$ are hyperplanes and the closure of $\mathcal{M}$ with respect to the $H^1(X_0)$ norm includes only functions with trace zero on $\{x_i = x_j\}$. Below we will see that in this case we have $\tilde{C}_H(X_0) \ge \frac{N-1}{2}$.

The main result of this section is the following

Theorem 3.2. *Let H be the Hamiltonian of a system of $N \ge 3$ d-dimensional particles with $d \in \{1, 2\}$, where the potentials $V_{ij} \ne 0$ satisfy (2.2) and (2.9). Assume that H has a virtual level at zero and for the constant $\tilde{C}_H(X_0)$ defined in (3.34) we have $\tilde{C}_H(X_0) > 1$. Then*

(i) *zero is a simple eigenvalue of H and for the corresponding eigenfunction φ_0 we have*

$$\nabla_0 \left(|x|_m^\alpha \varphi_0\right) \in L^2(X_0) \quad \textit{and} \quad (1 + |x|_m)^{\alpha-1}\varphi_0 \in L^2(X_0) \tag{3.35}$$

for any $0 \le \alpha < \tilde{C}_H(X_0)$.

(ii) *There exists a constant $\delta_0 > 0$, such that for any function $\psi \in H^1(X_0)$ satisfying $\langle \nabla_0\varphi_0, \nabla_0\psi\rangle = 0$*

$$(1-\delta_0)\|\nabla_0\psi\|^2 + \langle V\psi, \psi\rangle \ge 0. \tag{3.36}$$

Corollary 3.3. *If $d = 2$ and $N \ge 4$, then we have $\tilde{C}_H(X_0) = C_H(X_0) = N - 2 > 1$. Therefore, Theorem 3.2 can be applied. In particular, it shows that in this case the solution φ_0 of the Schrödinger equation corresponding to the virtual level is a non-degenerate eigenfunction satisfying*

$$(1 + |x|_m)^{\alpha-1}\varphi_0 \in L^2(X_0) \qquad \textit{for any } \alpha < N - 2. \tag{3.37}$$

Corollary 3.4. *If $d = 1$ and $N \ge 4$, each of the hyperplanes $\{x_i = x_j\}$ divides the space X_0 into two half-spaces. Taking one of these hyperplanes and using that the Hardy constant for the half-space is given by $\frac{N-1}{2}$* [18, Proposition 4.1] *we get $\tilde{C}_H(X_0) \ge \frac{N-1}{2} > 1$. Hence, Theorem 3.2 can be applied. This implies that zero is a simple eigenvalue of H and the corresponding eigenfunction φ_0 satisfies*

$$(1 + |x|_m)^{\alpha-1}\varphi_0 \in L^2(X_0) \qquad \textit{for any } \alpha < \frac{N-1}{2}. \tag{3.38}$$

Remark. We can significantly improve the estimate from below for the constant $\tilde{C}_H(X_0)$ given in Corollary 3.4 by taking into account that the traces of functions in $\mathcal{M}$ are zero not only on one of the hyperplanes $\{x_i = x_j\}$, but on all of them. For example, if we have a system of $N = 4$ identical particles, then there are six hyperplanes $\{x_i = x_j\}$ which cut the space X_0 into congruent sectors S_i. One can show that the hyperplanes are the nodal set of a homogeneous harmonic polynomial of degree six. Its restriction to the unit sphere is a spherical harmonic of degree six and an eigenfunction corresponding to the first eigenvalue of the Dirichlet-Laplacian on $S_i \cap \mathbb{S}^2$. This, together with [18, Proposition 4.1] implies that in this case the constant $\tilde{C}_H(X_0)$ is given by $\tilde{C}_H(X_0) = \left(\frac{1}{4} + 42\right)^{\frac{1}{2}} = \frac{13}{2}$.

Note that the constant $\tilde{C}_H(X_0)$, which gives a lower bound on the decay rate of the eigenfunction φ_0, does not depend on the potentials. However, for one-dimensional particles it does depend on the ratios of the masses of the particles. In particular if $d = 1$, $N = 3$ we get the following

Theorem 3.5. *For a system of three one-dimensional particles with masses $m_1, m_2, m_3 > 0$ let*

$$\theta_i = \arccos\left(\frac{\sqrt{m_j m_k}}{\sqrt{m_i + m_j}\sqrt{m_i + m_k}}\right). \tag{3.39}$$

Then we have

$$\tilde{C}_H(X_0) = \frac{\pi}{\theta_0}, \quad \text{where} \quad \theta_0 = \max\{\theta_i,\ i = 1, 2, 3\}. \tag{3.40}$$

Remark. **(i)** It is easy to see that for $d = 1$, $N = 3$ we have $\frac{\pi}{3} \le \theta_0 \le \frac{\pi}{2}$. The constant $\tilde{C}_H(X_0)$ takes its maximal value $\tilde{C}_H(X_0) = 3$ for $\theta_0 = \frac{\pi}{3}$, which corresponds to the case $m_1 = m_2 = m_3$. On the other hand, if one of the masses m_i tends to infinity, then $\theta_0 \to \frac{\pi}{2}$ and therefore $\tilde{C}_H(X_0) \to 2$.

(ii) Corollary 3.3, Corollary 3.4 and Theorem 3.5 show that for all multi-particle systems consisting of one- or two-dimensional particles, except for the case $d = 2$ and $N = 3$, virtual levels correspond to eigenvalues. This fact will be used in Section 5 to prove the absence of the Efimov effect in multi-particle systems in dimension one and two.

(iii) Note that if the dimension of the particles is $d \ge 3$, the eigenfunction φ_0 corresponding to a virtual level decays with the same rate as the fundamental solution of the Laplace operator [6], [5]. Theorem 3.2 shows that for one-dimensional particles the decay rate is higher.

For $d = 2$, $N = 3$ we do not expect that solutions of the Schrödinger equation corresponding to a virtual level are eigenfunctions. We will discuss this case in Section 4.

3.3. **Proof of Theorem 3.2 – Auxiliary results.** To prove Theorem 3.2 we need several auxiliary results. The first one is a generalization of Theorem 2.2 to potentials which do not necessarily decay at infinity.

Theorem 3.6. *Let $h = -\Delta + V$ acting on $L^2(\mathbb{R}^k)$, $k \in \mathbb{N}$, where the potential V satisfies (2.2). Suppose that h has a virtual level at zero and that there exist constants $\alpha_0 > 1$, $b > 0$ and $\gamma_0 \in (0, 1)$, such that for any function $\psi \in H^1(\mathbb{R}^k)$ with $\operatorname{supp}(\psi) \subset \{x \in \mathbb{R}^k : |x| \ge b\}$ we have*

$$\langle h\psi, \psi\rangle - \gamma_0 \|\nabla\psi\|^2 - \alpha_0^2 \langle |x|^{-2}\psi, \psi\rangle \ge 0. \tag{3.41}$$

Then zero is a simple eigenvalue of h and the corresponding eigenfunction φ_0 satisfies

$$\begin{aligned} &(1 + |x|)^{\alpha-1}\varphi_0 \in L^2(\mathbb{R}^k) && \text{if } k \ne 2, \\ \text{and} \quad &(1 + |x|)^{\alpha-1}(1 + |\ln(|x|)|)^{-1}\varphi_0 \in L^2(\mathbb{R}^k) && \text{if } k = 2 \end{aligned} \tag{3.42}$$

for any $\alpha < \alpha_0$. Moreover, there exists a constant $\delta_0 > 0$, such that for any function $\psi \in H^1(\mathbb{R}^k)$ with $\langle \nabla\psi, \nabla\varphi_0\rangle = 0$

$$\langle h\psi, \psi\rangle \ge \delta_0 \|\nabla\psi\|^2. \tag{3.43}$$

Remark. **(i)** By Lemma C.1 condition (3.41) implies $\inf \mathcal{S}_{\text{ess}}(h_\varepsilon) = 0$ for sufficiently small $\varepsilon > 0$.

(ii) Theorem 3.6 is a generalization of Theorem 2.1 in [6] to dimensions $k = 1$ and $k = 2$. Therefore, we only need to prove the theorem for these dimensions.

(iii) Note that Theorem 3.6 does not require that the potential V decays in all directions, which is the case if we consider multi-particle systems where V is the sum of the pair-potentials.

(iv) For dimension $k = 1$ or $k = 2$ Theorem 3.6 considers the case which is in some sense complementary to the one studied in Theorem 2.2. In Theorem 2.2 we assumed that the potential V decays fast at infinity. In Theorem 3.6 we do not require any decay of the potential. Instead of this we need inequality (3.41) for functions ψ which are supported far away from the origin. This condition can not be fulfilled for $k = 1$ and $k = 2$ if V decays fast at infinity. Moreover, under the conditions of Theorem 3.6 virtual levels correspond to eigenvalues of h. In contrast to that, under the conditions of Theorem 2.2 they correspond to resonances.

Proof of Theorem 3.6 for dimensions $k = 1$ and $k = 2$. Since $\inf \mathcal{S}(h_\varepsilon) < 0$ for every $\varepsilon > 0$, we find a sequence of eigenfunctions $\psi_n \in H^1(\mathbb{R}^k)$, corresponding to eigenvalues $E_n < 0$ of the operator $h_{n^{-1}}$, i.e.

$$-\left(1 - n^{-1}\right) \Delta\psi_n + V\psi_n = E_n\psi_n. \tag{3.44}$$

We normalize the functions ψ_n by the condition $\|\psi_n\|_{\tilde{H}^1} = 1$. In the first step we show a uniform bound for the $L^2(\mathbb{R}^k)$ norm of the functions ψ_n.

Lemma 3.7. *Assume that h has a virtual level at zero, where V satisfies (2.2) and suppose that (3.41) holds for some $\alpha_0 > 1$. Then there exists a constant $C > 0$, such that for any eigenfunction $\psi_n \in H^1(\mathbb{R}^k)$ corresponding to a negative eigenvalue of the operator $h_{n^{-1}}$, normalized by $\|\psi_n\|_{\tilde{H}^1} = 1$, we have*

$$\|\nabla(|x|^{\alpha_0}\psi_n)\| \le C \tag{3.45}$$

and

$$\begin{aligned} &\|(1 + |x|)^{\alpha_0 - 1}\psi_n\| \le C && \text{if } k = 1 \\ \text{and} \quad &\|(1 + |x|)^{\alpha_0 - 1}(1 + |\ln(|x|)|)^{-1}\psi_n\| \le C && \text{if } k = 2. \end{aligned} \tag{3.46}$$

The proof of Lemma 3.7 is a straightforward generalization of the proof of Lemma 2.5 in Section 2 and Lemma 2.4 in [6] and we omit it here.
Since we have normalized the sequence $(\psi_n)_n$ as $\|\psi_n\|_{\tilde{H}^1} = 1$, there exists a subsequence, also denoted by $(\psi_n)_{n\in\mathbb{N}}$, which converges weakly in $\tilde{H}^1(\mathbb{R}^k)$ to a function $\varphi_0 \in \tilde{H}^1(\mathbb{R}^k)$.
Now we show that φ_0 is an eigenfunction of h corresponding to the eigenvalue zero. This is done in the following

Lemma 3.8. *Assume that h has a virtual level at zero, where V satisfies (2.2) and suppose that (3.41) holds for some $\alpha_0 > 1$. Then the function φ_0 given above is an eigenfunction of the operator h corresponding to the eigenvalue zero, satisfying $\|\varphi_0\|_{\tilde{H}^1} = 1$.*

Proof of Lemma 3.8. By Lemma 3.7 estimate (3.46) holds for some $\alpha_0 > 1$. Hence, we get convergence of the subsequence $(\psi_n)_{n\in\mathbb{N}}$ in $L^2(\mathbb{R}^k)$ and the limit satisfies $(1 + |x|)^\alpha \varphi_0 \in L^2(\mathbb{R}^k)$ for any $\alpha < \alpha_0 - 1$. In particular, $\varphi_0 \in H^1(\mathbb{R}^k)$.
Due to the semi-continuity of the norm we have $\|\varphi_0\|_{\tilde{H}^1} \le 1$. Since $(\psi_n)_{n\in\mathbb{N}}$ converges to φ_0 in $L^2(\mathbb{R}^k)$ and V satisfies (2.2), we get

$$\langle V\psi_n, \psi_n\rangle \to \langle V\varphi_0, \varphi_0\rangle, \qquad (n \to \infty). \tag{3.47}$$

Now it follows analogously to the proof of Lemma 2.5 that φ_0 satisfies

$$\|\nabla\varphi_0\|^2 + \langle V\varphi_0, \varphi_0\rangle = 0 \tag{3.48}$$

and $\|\varphi_0\|_{\tilde{H}^1} = 1$. Recall that $\varphi_0 \in H^1(\mathbb{R}^k)$ and therefore is an eigenfunction of h corresponding to the eigenvalue zero. This completes the proof of Lemma 3.8. □

It remains to prove (3.43) and the uniqueness of φ_0. This is a straightforward modification of the proof of (2.9) in [6]. The only difference is that we normalize the sequence of eigenfunctions by $\|\psi_n\|_{\tilde{H}^1} = 1$ instead of $\|\nabla\psi_n\| = 1$.

A new estimate of the localization error. To prove Theorem 3.2 we will follow the strategy of the proof of Theorem 4.4 in [6], where so-called geometric methods were used to get lower bounds for the quadratic form of the multi-particle Schrödinger operator. These methods include, for example, a separation of regions $K(Z,\kappa)$ corresponding to different partitions Z. A crucial element of these methods is a partition of unity of the configuration space, which requires an appropriate estimate of the localization error. Note that in fact the localization error is responsible for the existence or non-existence of the Efimov effect.
If the dimension of the particles is $d \geq 3$, one can use a localization error estimate given in [35, Lemma 5.1]. This estimate shows that when we separate a cone $K(Z,\kappa)$, then the localization error can be estimated as $\varepsilon|q(Z)|^{-2}$ with arbitrarily small $\varepsilon > 0$. Using Hardy's inequality this term can be controlled by a small part of the kinetic energy. However, if the particles are one- or two-dimensional, the estimate given in [35] cannot be used, because Hardy's inequality fails in dimension one and two. Therefore, we need a significant improvement of the estimate of the localization error. This is done in the following

Theorem 3.9. *Given $\varepsilon > 0$ and $\kappa > 0$, for each partition Z with $|Z| \geq 2$ one can find a constant $0 < \kappa' < \kappa$ and piecewise continuously differentiable functions $u_Z, v_Z : X_0 \to \mathbb{R}$, such that*

$$u_Z^2 + v_Z^2 = 1, \qquad u_Z(x) = \begin{cases} 1, & x \in K\left(Z,\kappa'\right), \\ 0, & x \notin K\left(Z,\kappa\right), \end{cases} \tag{3.49}$$

and

$$|\nabla_0 u_Z|^2 + |\nabla_0 v_Z|^2 < \varepsilon\left[|v_Z|^2|x|_m^{-2} + |u_Z|^2|q|_m^{-2}\ln^{-2}\left(|q|_m|\xi|_m^{-1}\right)\right] \tag{3.50}$$

for $x \in K\left(Z,\kappa',\kappa\right)$. Here $q = q(Z)$ and $\xi = \xi(Z)$.

To prove Theorem 3.9 we will use an auxiliary result for scalar functions, namely the following

Lemma 3.10. *For any $\varepsilon > 0$ and $0 < \beta < 1$ one can find a constant $0 < \alpha < \beta^2$ and a non-increasing function $u \in H^1(\alpha,\beta) \cap C([\alpha,\beta])$, such that $u(\alpha) = 1$, $u(\beta) = 0$ and*

$$(u'(t))^2 \leq \varepsilon t^{-2}\ln^{-2}(t), \qquad \alpha \leq t \leq \beta. \tag{3.51}$$

Proof of Lemma 3.10. Let $\varepsilon > 0$ and $\beta \in (0,1)$ be fixed. For any $0 < \gamma < 1$ and $\alpha \in (0,\beta^2)$ let $u : [\alpha,\beta] \to \mathbb{R}$ be given by

$$u(t) := \begin{cases} \left|\ln(\alpha\beta^{-1})\right|^{-\gamma}\left|\ln(t\beta^{-1})\right|^{\gamma} & \text{if } \alpha \leq t \leq \beta^2, \\ \left|\ln(\alpha\beta^{-1})\right|^{-\gamma}\left|\ln\beta\right|^{\gamma-1}\left|\ln(t\beta^{-1})\right| & \text{if } \beta^2 \leq t \leq \beta. \end{cases} \tag{3.52}$$

Obviously, $u \in C([\alpha,\beta]) \cap H^1(\alpha,\beta)$ with $u(\beta) = 0$ and $u(\alpha) = 1$.
At first, we prove the claimed estimate for $(u'(t))^2$ for $\alpha \leq t \leq \beta^2$ by choosing the constant $\gamma > 0$ sufficiently small. For $\alpha < t < \beta^2$ we have

$$\left(u'(t)\right)^2 = \gamma^2\left|\ln(\alpha\beta^{-1})\right|^{-2\gamma}\left|\ln(t\beta^{-1})\right|^{2(\gamma-1)}t^{-2}. \tag{3.53}$$

Note that $\alpha\beta^{-1} < 1$ and $t\beta^{-1} < 1$ for $\alpha \leq t \leq \beta^2$ and therefore $\left|\ln(\alpha\beta^{-1})\right| \geq \left|\ln(t\beta^{-1})\right|$, which yields

$$\left(u'(t)\right)^2 \leq \gamma^2\left|\ln(t\beta^{-1})\right|^{-2}t^{-2}, \qquad \alpha < t < \beta^2. \tag{3.54}$$

Furthermore, for $t \leq \beta^2$ we have $\left|\ln(t\beta^{-1})\right| \geq |\ln\sqrt{t}| = \frac{1}{2}\left|\ln t\right|$. This implies

$$\left(u'(t)\right)^2 \leq 4\gamma^2|\ln t|^{-2}t^{-2}, \qquad \alpha < t < \beta^2. \tag{3.55}$$

Choosing $0 < \gamma < \frac{\sqrt{\varepsilon}}{2}$ we get

$$\left(u'(t)\right)^2 \leq \varepsilon|\ln t|^{-2}t^{-2}, \qquad \alpha < t < \beta^2. \tag{3.56}$$

Now we estimate $(u'(t))^2$ for $\beta^2 < t < \beta$. In this case we have

$$\left(u'(t)\right)^2 = \left|\ln(\alpha\beta^{-1})\right|^{-2\gamma} \left|\ln\beta\right|^{2(\gamma-1)} t^{-2}. \tag{3.57}$$

Since $\beta < 1$, we have $|\ln\beta^2| \geq |\ln t|$ for $\beta^2 \leq t \leq \beta$ and therefore

$$\begin{aligned}\left(u'(t)\right)^2 &\leq \left|\ln(\alpha\beta^{-1})\right|^{-2\gamma} \left|\ln\beta\right|^{2(\gamma-1)} |\ln\beta^2|^2 |\ln t|^{-2} t^{-2} \\ &= 4\left|\ln(\alpha\beta^{-1})\right|^{-2\gamma} \left|\ln\beta\right|^{2\gamma} |\ln t|^{-2} t^{-2} \leq \varepsilon |\ln t|^{-2} t^{-2}\end{aligned} \tag{3.58}$$

if α is chosen small enough. This completes the proof of Lemma 3.10 $\square$

Now we turn to the

Proof of Theorem 3.9. Let Z be a partition with $|Z| \geq 2$ and let $\varepsilon > 0$ and $0 < \kappa < 1$ be fixed. We construct functions u_Z, v_Z which satisfy the conditions of Theorem 3.9.
Let $v_1 \in H^1(\mathbb{R}_+)$ be a non-decreasing function with $v_1(t) = 1$ for $t \geq \kappa$ and $0 \leq v_1(t) < 1$ for $t < \kappa$, such that $v_1'(t)(1 - v_1^2)^{-\frac{1}{2}} \to 0$ as $t \nearrow \kappa$. For $x \in X_0,\ x = q + \xi$, let

$$v_Z(x) = v_1\left(\frac{|q|_m}{|\xi|_m}\right), \qquad u_Z(x) = \sqrt{1 - v_Z^2(x)}. \tag{3.59}$$

Then for $x \in K(Z, \kappa)$ we have

$$\begin{aligned}|\nabla_0 u_Z|^2 + |\nabla_0 v_Z|^2 &= |\nabla_0 v_Z|^2 \left(1 - v_Z^2\right)^{-1} \\ &= (v_1'(t))^2 \left(1 - v_1^2(t)\right)^{-1} \left(1 + |q|_m^2 |\xi|_m^{-2}\right) |\xi|_m^{-2},\end{aligned} \tag{3.60}$$

where $t = |q|_m |\xi|_m^{-1}$. For $x \in K(Z, \kappa)$ we have $|\xi|_m^{-2} \leq \left(1 + \kappa^2\right) |x|_m^{-2}$. This implies

$$|\nabla_0 u_Z|^2 + |\nabla_0 v_Z|^2 \leq (v_1'(t))^2 \left(1 - v_1^2(t)\right)^{-1} (1 + \kappa^2)^2 |x|_m^{-2}. \tag{3.61}$$

Since $v_1'(t)(1 - v_1^2(t))^{-\frac{1}{2}} \to 0$ as $t \nearrow \kappa$, we can find $0 < \kappa'' < \kappa$ so close to κ that

$$(v_1'(t))^2 \left(1 - v_1^2(t)\right)^{-1} (1 + \kappa^2)^2 \leq \varepsilon v_1^2(t), \qquad \kappa'' \leq t < \kappa. \tag{3.62}$$

This implies

$$|\nabla_0 u_Z|^2 + |\nabla_0 v_Z|^2 \leq \varepsilon v_Z^2 |x|_m^{-2}, \qquad x \in K(Z, \kappa) \setminus K(Z, \kappa''). \tag{3.63}$$

Now we define v_Z for $x \in K(Z, \kappa'')$. By Lemma 3.10, for given $\tilde{\varepsilon} > 0$ we find a constant $0 < \kappa' < \kappa''$ and a non-decreasing function v_2, such that

$$v_2(\kappa') = 0,\ v_2(\kappa'') = v_1(\kappa'') \quad \text{and} \quad \left(v_2'(t)\right)^2 \leq \tilde{\varepsilon}|t|^{-2} \ln^{-2} t \quad \text{for} \quad \kappa' < t < \kappa''. \tag{3.64}$$

Let v_2 be such a function and for $x \in K(Z, \kappa''),\ x = q + \xi$, let

$$v_Z(x) = v_2\left(\frac{|q|_m}{|\xi|_m}\right), \qquad u_Z(x) = \sqrt{1 - v_Z^2(x)}. \tag{3.65}$$

Then, similar to (3.60) we have

$$\left(|\nabla_0 u_Z|^2 + |\nabla_0 v_Z|^2\right) u_Z^{-2} = (v_2'(t))^2 \left(1 - v_2^2(t)\right)^{-1} u_Z^{-2} \left(1 + |q|_m^2 |\xi|_m^{-2}\right) |\xi|_m^{-2}, \tag{3.66}$$

where $t = |q|_m |\xi|_m^{-1}$. Since v_2 is non-decreasing, we have $(1 - v_2^2(t))^{-1} u_Z^{-2} \leq \left(1 - v_2^2(k'')\right)^{-2}$ for $t \leq \kappa''$. Substituting this estimate into (3.66) we have

$$\left(|\nabla_0 u_Z|^2 + |\nabla_0 v_Z|^2\right) u_Z^{-2} \leq (v_2'(t))^2 \left(1 - v_2(k'')^2\right)^{-2} \left(1 + (\kappa'')^2\right) |\xi|_m^{-2}. \tag{3.67}$$

Recall that $v_2(\kappa'')$ is close to one, but strictly less then one. Due to $\left(v_2'(t)\right)^2 \leq \tilde{\varepsilon}|t|^{-2} \ln^{-2} t$ we get

$$\left(|\nabla_0 u_Z|^2 + |\nabla_0 v_Z|^2\right) u_Z^{-2} \leq \tilde{\varepsilon}|t|^{-2} \ln^{-2} t \left(1 - v_2(k'')^2\right)^{-2} \left(1 + (\kappa'')^2\right) |\xi|_m^{-2}. \tag{3.68}$$

Choosing $\tilde{\varepsilon} > 0$ so small that $\tilde{\varepsilon}\left(1 - v_2(k'')^2\right)^{-2}\left(1 + (\kappa'')^2\right) < \varepsilon$ and using $t = |q|_m|\xi|_m^{-1}$ completes the proof of Theorem 3.9. □

3.4. **Proof of Theorem 3.2.** Now we turn to the proof of Theorem 3.2. It is an application of Theorem 3.6 and we use geometrical methods to prove that all conditions of the latter theorem are fulfilled. Since the pair potentials V_{ij} are relatively form bounded, so is $V = \sum_{1\le i<j\le N} V_{ij}(x_{ij})$. Hence, we only need to show that condition (3.41) is fulfilled for any $0 \le \alpha < \tilde{C}_H(X_0)$. This is done in the following

Lemma 3.11. *Let $d \in \{1, 2\}$ and $N \ge 3$. Assume that the potentials V_{ij} satisfy (2.2) and (2.9). Further, suppose that H has a virtual level at zero. Then for any $0 \le \alpha < \tilde{C}_H(X_0)$ there exist constants γ_0, $R > 0$, such that for any function $\varphi \in H^1(X_0)$ with $\operatorname{supp}(\varphi) \subset \{x \in X_0 : |x|_m \ge R\}$ we have*

$$L[\varphi] := (1 - \gamma_0)\|\nabla_0\varphi\|^2 + \langle V\varphi, \varphi\rangle - \alpha^2\||x|_m^{-1}\varphi\|^2 \ge 0. \tag{3.69}$$

In the proof of Lemma 3.11 we use the following

Lemma 3.12. *Let Z be a partition of the system, such that $\dim(X_0(Z)) = 2$. Furthermore, let $0 < \kappa < 1$. Then there exists $\varepsilon > 0$, such that for any function $\psi \in H^1(R_0)$ with $\operatorname{supp}(\psi) \subset K_R(Z, \kappa)$ and any $0 < \kappa' < \kappa$ we have*

$$\|\nabla_q\psi\|^2 - \varepsilon\left\||q|_m^{-1}\ln^{-1}\left(|q|_m|\xi|_m^{-1}\right)\psi\right\|^2_{K_R(Z,\kappa',\kappa)} \ge 0. \tag{3.70}$$

Proof of Lemma 3.12. We introduce the new variable $y = \frac{q}{|\xi|_m}$. Then we get

$$\begin{aligned}
&\|\nabla_q\psi\|^2 - \varepsilon\left\||q|_m^{-1}\ln^{-1}\left(|q|_m|\xi|_m^{-1}\right)\psi\right\|^2_{K_R(Z,\kappa',\kappa)} \\
&\ge \int\int_{\{\kappa'|\xi|_m\le|q|_m\le\kappa|\xi|_m\}}\left(|\nabla_q\psi|^2 - \varepsilon|q|_m^{-2}\left|\ln^{-2}\left(|q|_m|\xi|_m^{-1}\right)\right||\psi|^2\right)\mathrm{d}q\,\mathrm{d}\xi \\
&= \int\frac{1}{|\xi|_m^2}\int_{\{\kappa'|\le|y|_m\le\kappa\}}\left(|\nabla_y\tilde{\psi}(y,\xi)|^2 - \varepsilon|y|_m^{-2}\left|\ln^{-2}\left(|y|_m\right)\right||\tilde{\psi}(y,\xi)|^2\right)\mathrm{d}y\,\mathrm{d}\xi,
\end{aligned} \tag{3.71}$$

where $\tilde{\psi}(y,\xi) = \psi(y|\xi|_m, \xi)$. Note that $\tilde{\psi}(y,\xi) = 0$ for $|y|_m \ge \kappa$. Due to $\kappa < 1$ we have $(\ln|y|_m)^{-2} \le C(1 + (\ln|y|_m)^2)^{-1}$ for some $C > 0$ and $|y|_m \le \kappa$. Therefore, applying Corollary A.3 to the function $\tilde{\psi}(y,\xi)$ for fixed ξ shows that the r.h.s. of (3.71) is non-negative for sufficiently small $\varepsilon > 0$. This completes the proof of Lemma 3.12. □

Now we turn to the

Proof of Lemma 3.11. The proof follows the idea of the proof of Theorem 4.4 in [6]. We make a partition of unity of the support of φ, separating regions $K(Z, \kappa)$ which correspond to different partitions Z of the system into clusters.

We start by estimating the functional $L[\varphi]$ in regions $K(Z, \kappa)$ corresponding to partitions Z into two clusters. Let $\kappa_2 \in (0, 1)$ be so small that $K_R(Z, \kappa_2)$ and $K_R(Z', \kappa_2)$ for clusters $Z \ne Z'$ with $|Z| = |Z'| = 2$ do not overlap. Such a constant κ_2 exists according to [4] (an English version can be found in [6, Theorem B.2]). By Theorem 3.9 we get

$$L[\varphi] \ge \sum_{Z:|Z|=2} L_2[\varphi u_Z] + L_2'[\mathcal{V}^{(2)}\varphi], \tag{3.72}$$

where $\mathcal{V}^{(2)} = \sqrt{1 - \sum_{Z:|Z|=2} u_Z^2}$ and the functionals $L_2, L_2' : H^1(X_0) \to \mathbb{R}$ are given by

$$\begin{aligned} L_2[\psi] &= (1-\gamma_0)\|\nabla_0\psi\|^2 + \langle V\psi,\psi\rangle - \alpha^2\||x|_m^{-1}\psi\|^2 \\ &\qquad\qquad - \varepsilon\||q(Z)|_m^{-1}\ln^{-1}\left(|q(Z)|_m|\xi(Z)|_m^{-1}\right)\psi\|^2_{K_R(Z,\kappa_2',\kappa_2)}, \\ L_2'[\psi] &= (1-\gamma_0)\|\nabla_0\psi\|^2 + \langle V\psi,\psi\rangle - (\alpha^2+\varepsilon)\||x|_m^{-1}\psi\|^2, \end{aligned} \tag{3.73}$$

where $\varepsilon > 0$ can be chosen arbitrarily small if $\kappa_2' > 0$ is sufficiently small. Recall that the functions u_Z are supported in the region $K(Z,\kappa_2)$, i.e where the two clusters in Z are far away from each other. Note also that the terms $\varepsilon\||x|_m^{-1}\psi\|^2$ and $\varepsilon\||q(Z)|_m^{-1}\ln^{-1}\left(|q(Z)|_m|\xi(Z)|_m^{-1}\right)\psi\|^2_{K_R(Z,\kappa_2',\kappa_2)}$ come from the estimate for the localization error given in Theorem 3.9.

Let Z be an arbitrary partition into two clusters, $q = q(Z),\ \xi = \xi(Z)$ and $\psi = \varphi u_Z$. Our goal is to show that $L_2[\psi] \geq 0$. We have

$$\begin{aligned} L_2[\psi] =& \langle H(Z)\psi,\psi\rangle - \gamma_0\|\nabla_q\psi\|^2 + (1-\gamma_0)\left\|\nabla_\xi\psi\right\|^2 + \langle I(Z)\psi,\psi\rangle \\ &- \alpha^2\left\||x|_m^{-1}\psi\right\|^2 - \varepsilon\left\||q|_m^{-1}\ln^{-1}\left(|q|_m|\xi|_m^{-1}\right)\psi\right\|^2_{K_R(Z,\kappa_2',\kappa_2)}. \end{aligned} \tag{3.74}$$

First, we estimate the inter-cluster potential $I(Z)$ by

$$|I(Z)(x)| \leq C|\xi|_m^{-2-\nu} \leq \varepsilon|\xi|_m^{-2} \tag{3.75}$$

for $x \in \operatorname{supp}(\psi)$ and sufficiently large $R > 0$. Furthermore, on the support of ψ we have $|q|_m \leq \kappa_2|\xi|_m$ and therefore the Poincaré-Friedrich inequality [1, Theorem 6.30] yields

$$\gamma_0\|\nabla_q\psi\|^2 \geq \frac{\gamma_0}{4\kappa_2^2}\||\xi|_m^{-1}\psi\|^2. \tag{3.76}$$

By choosing $\kappa_2 > 0$ small enough this implies

$$\gamma_0\|\nabla_q\psi\|^2 + \langle I(Z)\psi,\psi\rangle - \alpha^2\left\||x|_m^{-1}\psi\right\|^2 \geq 0 \tag{3.77}$$

and therefore

$$L_2[\psi] \geq \langle H(Z)\psi,\psi\rangle - 2\gamma_0\|\nabla_q\psi\|^2 - \varepsilon\left\||q|_m^{-1}\ln^{-1}\left(|q|_m|\xi|_m^{-1}\right)\psi\right\|^2_{K_R(Z,\kappa_2',\kappa_2)}. \tag{3.78}$$

To estimate the r.h.s. of (3.78) we distinguish between several cases.

(i) If $\dim(X_0(Z)) = 1$, we have $d = 1$ and $N = 3$. Assume that $Z = (C_1, C_2)$ with $|C_1| = 2$, then $H[C_2] = 0$ and

$$\langle H(Z)\psi,\psi\rangle = \langle H[C_1]\psi,\psi\rangle \quad \text{and} \quad \|\nabla_{q(Z)}\psi\| = \|\nabla_{q[C_1]}\psi\|. \tag{3.79}$$

We estimate the last term on the r.h.s. of (3.74) by

$$\varepsilon\left\||q|_m^{-1}\ln^{-1}\left(|q|_m|\xi|_m^{-1}\right)\psi\right\|^2_{K_R(Z,\kappa_2',\kappa_2)} \leq \varepsilon\|(1+|q|_m)^{-1}\psi\|^2_{K_R(Z,\kappa_2',\kappa_2)} \tag{3.80}$$

for $\kappa_2 > 0$ small enough and $R > 0$ sufficiently large. This yields

$$L_2[\psi] \geq \langle H[C_1]\psi,\psi\rangle - 2\gamma_0\|\nabla_{q[C_1]}\psi\|^2 - \varepsilon\|(1+|q|_m)^{-1}\psi\|^2_{K_R(Z,\kappa_2',\kappa_2)}. \tag{3.81}$$

Since by the remark after Definition 3.1 the operator $H[C_1]$ does not have a virtual level and $V[C_1] \neq 0$, we can use Theorem 2.3 to conclude that $L_2[\psi] \geq 0$ for $\varepsilon > 0$ and $\gamma_0 > 0$ small enough and $R > 0$ sufficiently large.

(ii) If $\dim(X_0(Z)) \geq 2$, we use again that for clusters C with $1 < |C| < N$ the operator $H[C]$ does not have a virtual level, which implies

$$\langle H(Z)\psi,\psi\rangle - 3\gamma_0\|\nabla_q\psi\|^2 \geq 0 \tag{3.82}$$

for small $\gamma_0 > 0$ and therefore

$$L_2[\psi] \geq \gamma_0\|\nabla_q\psi\|^2 - \varepsilon\left\||q|_m^{-1}\ln^{-1}\left(|q|_m|\xi|_m^{-1}\right)\psi\right\|^2_{K_R(Z,\kappa_2',\kappa_2)}. \tag{3.83}$$

If $\dim(X_0(Z)) = 2$, we apply Lemma 3.12 with $\kappa = \kappa_2$ and $\kappa' = \kappa_2'$ to conclude that $L_2[\psi] \geq 0$. If $\dim(X_0(Z)) \geq 3$, we use (3.80) and Hardy's inequality in the form of (A.21). This implies $L_2[\psi] \geq 0$.

Now we estimate $L_2'[\mathcal{V}^{(2)}\varphi]$. If $N = 3$, the function $\mathcal{V}^{(2)}\varphi$ is supported in the region where all particles are separated, i.e. there exists a constant $c > 0$, such that $|x_{ij}| \geq c|x|_m$ for all $i \neq j$. This implies

$$|V(x)| \leq C|x|_m^{-2-\nu} \leq \varepsilon |x|_m^{-2}, \tag{3.84}$$

where $\varepsilon > 0$ can be chosen arbitrarily small for sufficiently large $R > 0$. Therefore,

$$L_2'[\mathcal{V}^{(2)}\varphi] \geq (1-\gamma_0)\|\nabla_0(\mathcal{V}^{(2)}\varphi)\|^2 - (\alpha^2 + 2\varepsilon)\||x|_m^{-1}\mathcal{V}^{(2)}\varphi\|^2. \tag{3.85}$$

Since $\mathcal{V}^{(2)}\varphi$ can be approximated (in the norm of $H^1(X_0)$) by functions in $\mathcal{M}$, we get

$$\|\nabla_0(\mathcal{V}^{(2)}\varphi)\|^2 \geq \left(\tilde{C}_H(X_0)\right)^2 \||x|_m^{-1}\mathcal{V}^{(2)}\varphi\|^2. \tag{3.86}$$

Due to $\alpha < \tilde{C}_H(X_0)$ we obtain $L_2'[\mathcal{V}^{(2)}\varphi] \geq 0$ for the case $N = 3$ by choosing $\gamma_0, \varepsilon > 0$ small enough.

If $N \geq 4$, we make a partition of unity of the support of $\mathcal{V}^{(2)}\varphi$. Let $\kappa_3 \in (0,1)$ be so small that $K(Z,\kappa_3)$ and $K(\tilde{Z},\kappa_3)$ do not overlap on the support of $\mathcal{V}^{(2)}\varphi$ for partitions $Z \neq \tilde{Z}$ with $|Z| = |\tilde{Z}| = 3$. Such a constant κ_3 exists due to [4] (see [6, Theorem B.2] for an English version). Applying Theorem 3.9 we get

$$L_2'[\mathcal{V}^{(2)}\varphi] \geq \sum_{Z:|Z|=3} L_3[\mathcal{V}^{(2)}\varphi u_Z] + L_3'[\mathcal{V}^{(3)}\varphi], \tag{3.87}$$

where $\mathcal{V}^{(3)} = \mathcal{V}^{(2)}\sqrt{1 - \sum_Z u_Z^2}$ and the functionals $L_3, L_3' : H^1(X_0) \to \mathbb{R}$ are given by

$$\begin{aligned} L_3[\psi] &= (1-\gamma_0)\|\nabla_0\psi\|^2 + \langle V\psi,\psi\rangle - (\alpha^2+\varepsilon)^2\||x|_m^{-1}\psi\|^2 \\ &\qquad\qquad - \varepsilon\||q(Z)|_m^{-1}\ln^{-1}\left(|q(Z)|_m|\xi(Z)|_m^{-1}\right)\psi\|^2_{K_R(Z,\kappa_3',\kappa_3)}, \\ L_3'[\psi] &= (1-\gamma_0)\|\nabla_0\psi\|^2 + \langle V\psi,\psi\rangle - (\alpha^2+\varepsilon)\||x|_m^{-1}\psi\|^2 \end{aligned} \tag{3.88}$$

for some $\varepsilon > 0$ which can be chosen arbitrarily small. Let Z be an arbitrary partition into three clusters. Then by the same arguments as for partitions Z with $|Z| = 2$ we can prove $L_3[\mathcal{V}^{(2)}\varphi u_Z] \geq 0$. If $N \geq 5$, we continue this process for all partitions Z with $|Z| \leq N-1$ and finally arrive at the point where it remains to estimate the functional

$$L'[\tilde{\psi}] := (1-\gamma_0)\|\nabla_0\tilde{\psi}\|^2 + \langle V\tilde{\psi},\tilde{\psi}\rangle - (\alpha^2+\varepsilon)\||x|_m^{-1}\tilde{\psi}\|^2 \geq 0 \tag{3.89}$$

for functions $\tilde{\psi} := \mathcal{V}^{(N-1)}\varphi$ supported in the region where all particles are separated from each other, i.e. there exists a constant $c > 0$, such that $|x_{ij}| \geq c|x|_m$ for $x \in \operatorname{supp}(\mathcal{V}^{(N-1)}\varphi)$. Therefore, we have

$$|V(x)| \leq C(1+|x|_m)^{-2-\nu} \leq \varepsilon(1+|x|_m)^{-2} \tag{3.90}$$

on the support of $\mathcal{V}^{(N-1)}\varphi$ if $R > 0$ is large enough. This implies

$$L'[\mathcal{V}^{(N-1)}\varphi] \geq (1-\gamma_0)\|\nabla_0\big(\mathcal{V}^{(N-1)}\varphi\big)\|^2 - (\alpha^2+2\varepsilon)\||x|_m^{-1}\mathcal{V}^{(N-1)}\varphi\|^2. \tag{3.91}$$

Similarly to (3.86) we have

$$\left\|\nabla_0\left(\mathcal{V}^{(N-1)}\varphi\right)\right\|^2 \geq \left(\tilde{C}_H(X_0)\right)^2 \left\||x|_m^{-1}\mathcal{V}^{(N-1)}\varphi\right\|^2. \tag{3.92}$$

Since $\alpha < (\tilde{C}_H(X_0))$, we can choose γ_0, $\varepsilon > 0$ sufficiently small to obtain $L'[\mathcal{V}^{(N-1)}\varphi] \geq 0$. This completes the proof of Lemma 3.11 and therefore the proof of Theorem 3.2. $\square$

3.5. **Proof of Theorem 3.5.** Now we turn to the proof of Theorem 3.5. Since in the case of three one-dimensional particles the configuration space X_0 is two-dimensional, we are able to improve the geometric methods and therefore to derive the exact value of the constant $\widetilde{C}_H(X_0)$. The proof of the theorem follows from the following lemma, where we collect some geometric properties of the space X_0.

Lemma 3.13. *Let $d = 1$ and $N = 3$. Then the following statements hold.*

- **(i)** *The lines $x_1 = x_2$, $x_1 = x_3$ and $x_2 = x_3$ divide the space X_0 into six sectors $S_1, S_2, \dots, S_6$ with angles $\theta_1 = \theta_4$, $\theta_2 = \theta_5$ and $\theta_3 = \theta_6$. The angles θ_i, $i = 1, 2, 3$ are given by*

$$\theta_i = \arccos\left(\frac{\sqrt{m_j m_k}}{\sqrt{m_i + m_j}\sqrt{m_i + m_k}}\right). \tag{3.93}$$

- **(ii)** *Let $\psi \in H_0^1(S_i)$. Then we have*

$$\|\nabla_0 \psi\| \geq \frac{\pi}{\theta_i} \| |x|_m^{-1} \psi \|. \tag{3.94}$$

Proof of Lemma 3.13. The half lines $x_1 = x_2 \geq 0$, $x_1 = x_3 \geq 0$ and $x_2 = x_3 \leq 0$ in X_0 are spanned by the vectors

$$\begin{aligned} u_{12} &= \left(1, 1, -\frac{m_1 + m_2}{m_3}\right)^\top, \quad u_{13} = \left(1, -\frac{m_1 + m_3}{m_2}, 1\right)^\top \\ \text{and} \quad u_{23} &= \left(\frac{m_2 + m_3}{m_1}, -1, -1\right)^\top, \quad \text{respectively.} \end{aligned} \tag{3.95}$$

Let S_1 be the sector between the half-lines $x_1 = x_2 \leq 0$ and $x_1 = x_3 \geq 0$, S_2 the sector between the half-lines $x_1 = x_2 \geq 0$ and $x_2 = x_3 \leq 0$ and S_3 the sector between the half-lines $x_2 \leq x_3 \geq 0$ and $x_1 = x_3 \geq 0$. Here, we always choose the one sector with angle $0 < \theta_i < \pi$, see Figure 1. To illustrate the situation we choose an orthogonal basis $\{v_1, v_2\}$ of X_0 with $v_1 = u_{12}$ and $v_2 = (m_2, -m_1, 0)^\top$.

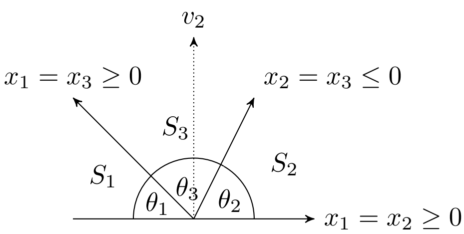

FIGURE 1. The sectors S_1, S_2, S_3

Let S_4, S_5 and S_6 be the sectors which we get by reflecting the sectors S_1, S_2 and S_3 at the origin. Obviously, $\theta_i = \theta_{i+3}$, $i = 1, 2, 3$. Since $\langle -u_{12}, u_{13}\rangle_m > 0$, we have $\theta_1 \in \left(0, \frac{\pi}{2}\right)$ and analogously we see that $\theta_2, \theta_3 \in \left(0, \frac{\pi}{2}\right)$. The angle θ_1 can be computed by the formula

$$\cos(\theta_1) = \frac{\langle -u_{12}, u_{13}\rangle_m}{|u_{12}|_m |u_{13}|_m} = \frac{\sqrt{m_2 m_3}}{\sqrt{m_1 + m_2}\sqrt{m_1 + m_3}}. \tag{3.96}$$

Similarly we can see that the angles θ_2 and θ_3 also satisfy (3.93). This completes the proof of statement **(i)** of Lemma 3.13.

Now we turn to the proof of the Hardy-type inequality (3.94) for the sectors S_i. According to [18, Proposition 4.1] functions $v \in H^1(\mathbb{R}^2)$ supported in a sector $S \subset \mathbb{R}^2$ satisfy

$$\|\nabla v\| \geq (\Lambda(G))^{\frac{1}{2}} \, \||x|^{-1}v\|, \tag{3.97}$$

were $\Lambda(G)$ is the first eigenvalue of the Dirichlet problem for the Laplace-Beltrami operator in $G = S \cap \mathbb{S}^1$. In dimension two G can be identified with the interval $(0, \theta)$ where θ is the angle of S. The Dirichlet eigenvalues of the Laplacian on an interval of length $l > 0$ are given by $\lambda_k = \left(\frac{k\pi}{l}\right)^2$. Therefore, we have $\Lambda(G) = \left(\frac{\pi}{\theta}\right)^2$, which implies that for any function $v \in H^1(\mathbb{R}^2)$ supported in S we have

$$\|\nabla v\| \geq \frac{\pi}{\theta} \||x|^{-1}v\|. \tag{3.98}$$

This completes the proof of Lemma 3.13 and therefore of Theorem 3.5. $\square$

4. Virtual levels of systems of three two-dimensional particles

In this section we consider systems of three two-dimensional particles. This is the only case of multi-particle systems in lower dimensions where we have $\tilde{C}_H(X) = 1$, which leaves a possibility for virtual levels to correspond to resonances and not to eigenvalues. We give the following

Theorem 4.1 (Virtual levels of systems of three two-dimensional particles)**.** *Let H be the Hamiltonian of a system of three two-dimensional particles. Assume that the potentials $V_{ij} \neq 0$ satisfy (2.2) and (2.9) and that H has a virtual level at zero. Then there exists a function $\varphi_0 \in \tilde{H}^1(X_0)$, $\varphi_0 \neq 0$, satisfying*

$$\|\nabla_0 \varphi_0\|^2 + \langle V\varphi_0, \varphi_0\rangle = 0 \tag{4.1}$$

and

$$(1 + |x|_m)^{-\alpha} \varphi_0 \in L^2(X_0) \qquad \text{for any } \alpha > 0. \tag{4.2}$$

Proof. To prove Theorem 4.1 we take a sequence $(\psi_n)_{n\in\mathbb{N}}$ of eigenfunctions corresponding to eigenvalues $E_n < 0$ of the operator $H + n^{-1}\Delta_0$, i.e.

$$-\left(1 - n^{-1}\right)\Delta_0\psi_n + V\psi_n = E_n\psi_n. \tag{4.3}$$

We normalize the functions ψ_n by $\|\nabla_0\psi_n\| = 1$. Then there exists a subsequence of $(\psi_n)_{n\in\mathbb{N}}$, also denoted by $(\psi_n)_{n\in\mathbb{N}}$, which converges weakly in $\tilde{H}^1(X_0)$ to a function $\varphi_0 \in \tilde{H}^1(X_0)$. Due to the Rellich-Kondrachov theorem we have convergence of ψ_n to φ_0 in $L^2_{\text{loc}}(X_0)$.
At first, we show that $\varphi_0 \neq 0$ and establish the decay property (4.2) of the function φ_0. Due to Lemma 3.11 there exist constants $\gamma_0 > 0$ and $R > 0$, such that for every function $\psi \in H^1(X_0)$ supported in the region $\{|x|_m \geq R\}$

$$(1 - \gamma_0)\|\nabla_0\psi\|^2 + \langle V\psi, \psi\rangle \geq 0. \tag{4.4}$$

Applying Lemma 2.3 in [6] we see that the weak limit $\varphi_0 \in \tilde{H}^1(X_0)$ of the sequence $(\psi_n)_{n\in\mathbb{N}}$ of eigenfunctions normalized by $\|\nabla_0\psi_n\| = 1$ is not zero.
In the next step we show that φ_0 satisfies the estimate (4.2) on the decay rate. To do this we first give the following estimate for a weighted L^2 norm of the functions ψ_n.

Lemma 4.2. *Let H be the Hamiltonian of a system of three two-dimensional particles. Assume that the potentials V_{ij} satisfy (2.2) and (2.9) and that H has a virtual level at zero. Then, for any $0 \leq \alpha < 1$ there exists a constant $C > 0$, such that for all $n \in \mathbb{N}$ we have*

$$\|\nabla_0\left(|x|_m^\alpha \psi_n\right)\| \leq C \qquad \text{and} \qquad \|\left(1 + |x|_m\right)^{\alpha-1}\psi_n\| \leq C. \tag{4.5}$$

Proof. The proof is a straightforward modification of the proof of Lemma 2.4 in [6]. $\square$

By Lemma 4.2 we get convergence of $(\psi_n)_{n\in\mathbb{N}}$ to φ_0 in $L^2(X_0,(1+|x|_m)^{-\alpha}\mathrm{d}x)$ for any $\alpha>0$. This shows that the function φ_0 satisfies (4.2).
Our next goal is to prove that

$$\|\nabla_0\varphi_0\|^2+\langle V\varphi_0,\varphi_0\rangle=0. \tag{4.6}$$

Note that first we have to prove that $\langle V\varphi_0,\varphi_0\rangle$ is well defined. Since we do not know whether φ_0 is square-integrable, we can not use the arguments of Lemma 3.8. We prove that $\langle V_{ij}\varphi_0,\varphi_0\rangle$ is well-defined for each pair of particles $\beta=(i,j)$. By Corollary A.2 to do this, it is sufficient to show that

$$\int\int_{\{|q_\beta|_m\le 1\}}|\varphi_0|^2\,\mathrm{d}q_\beta\,\mathrm{d}\xi_\beta<\infty. \tag{4.7}$$

In other words, it is enough to prove that the restriction of the function φ_0 to cylindrical regions $\{|q_\beta|_m\le 1\}$, $\beta\in\{(1,2),(1,3),(2,3)\}$, is square-integrable. Here and in the following for $\beta=(i,j)$ we denote by q_β,ξ_β the variables $q[C]$, $\xi[C]$, where $C=\{i,j\}$.
To prove (4.7) we need to make several steps. Let $(\psi_n)_{n\in\mathbb{N}}$ be the sequence of eigenfunctions of the operator $H+n^{-1}\Delta_0$, normalized by $\|\nabla_0\psi_n\|=1$. Furthermore, let $\chi_1:\mathbb{R}_+\to[0,1]$ be a function with $\chi_1\in C^1(\mathbb{R}_+)$ and $(1-\chi_1^2)^{\frac12}\in C^1(\mathbb{R}_+)$, satisfying

$$\chi_1(t)=0,\ 0\le t\le 1,\qquad \chi_1(t)=1,\ t\ge 2. \tag{4.8}$$

For $b>0$ let $\chi(x)=\chi_1\left(\frac{|x|_m}{b}\right)$. The first step to prove that $\langle V_{ij}\varphi_0,\varphi_0\rangle$ is well-defined is the following

Lemma 4.3. *Let ψ_n and χ be defined as above. Then, for any $\varepsilon>0$ we can find $b>0$ and $n_0\in\mathbb{N}$, such that for all $n>n_0$ we have*

$$\textit{(i)}\quad \|\nabla_0(\chi\psi_n)\|<\varepsilon,\qquad \textit{(ii)}\quad \langle V_{ij}\chi\psi_n,\chi\psi_n\rangle<\varepsilon,\quad i,j\in\{1,2,3\}. \tag{4.9}$$

Proof of Lemma 4.3. For $\psi\in H^1(X_0)$ let

$$L[\psi]=\|\nabla_0\psi\|^2+\langle V\psi,\psi\rangle. \tag{4.10}$$

Then, by definition of the functions ψ_n we have $L[\psi_n]\le\frac1n\|\nabla_0\psi_n\|^2=\frac1n$. On the other hand, by the IMS localisation formula we get

$$L[\psi_n]=L[(1-\chi^2)^{\frac12}\psi_n]+L[\chi\psi_n]-\int_{X_0}\left(|\nabla_0\chi|^2+|\nabla_0(1-\chi^2)^{\frac12}|^2\right)|\psi_n|^2\,\mathrm{d}x. \tag{4.11}$$

We estimate the terms on the r.h.s. of (4.11) separately. Due to $H\ge0$ the first term is non-negative. Since χ is supported in the region $\{|x|_m\ge b\}$, by Lemma 3.11 with $\alpha=0$ we get

$$L[\chi\psi_n]\ge\gamma_0\|\nabla_0(\chi\psi_n)\|^2 \tag{4.12}$$

for some $\gamma_0>0$ if $b>0$ is large enough. Now we estimate the last term on the r.h.s. of (4.11). Note that $\nabla_0\chi$ and $\nabla_0\left(1-\chi^2\right)^{\frac12}$ are supported in the region $\{b\le|x|\le 2b\}$ and satisfy

$$|\nabla_0\chi|^2+|\nabla_0\left(1-\chi^2\right)^{\frac12}|^2\le\frac{C}{b^2} \tag{4.13}$$

for some $C>0$ which does not depend on b. This, together with the estimate (4.5) on the decay rate of ψ_n we get, uniformly in $n\in\mathbb{N}$,

$$\int_{X_0}\left(|\nabla_0\chi|^2+|\nabla_0(1-\chi^2)^{\frac12}|^2\right)|\psi_n|^2\,\mathrm{d}x\le 4C\int_{\{|x|_m\ge b\}}\frac{|\psi_n|^2}{|x|_m^2}\,\mathrm{d}x\le\varepsilon_1(b) \tag{4.14}$$

for some $\varepsilon_1(b)$ with $\varepsilon_1(b)\to0$ as $b\to\infty$. Combining this with (4.11) and (4.12) we obtain

$$L[\psi_n]\ge\gamma_0\|\nabla_0(\chi\psi_n)\|^2-\varepsilon_1(b). \tag{4.15}$$

Since $L[\psi_n] \le \frac{1}{n}$, it follows from (4.15) that for fixed $\varepsilon > 0$ we can choose $n_0 \in \mathbb{N}$ and $b > 0$ large enough, such that $\|\nabla_0 (\chi\psi_n)\|^2 \le \varepsilon$ holds uniformly for $n \ge n_0$. This completes the proof of statement **(i)** of the Lemma.

Now we turn to the proof of assertion **(ii)**. At first, we note that for any pair (i_0, j_0) we have

$$\langle V_{i_0j_0}\chi\psi_n, \chi\psi_n\rangle = L[\chi\psi_n] - \|\nabla_0 (\chi\psi_n)\|^2 - \sum_{(i,j)\neq(i_0,j_0)} \langle V_{ij}\chi\psi_n, \chi\psi_n\rangle, \tag{4.16}$$

i.e. $\langle V_{i_0j_0}\chi\psi_n, \chi\psi_n\rangle$ can be estimated by estimating the r.h.s. of (4.16). For the first term we get by (4.11) and (4.13)

$$L[\chi\psi_n] \le L[\psi_n] + C \int_{\{|x|_m \ge b\}} \frac{|\psi_n|^2}{|x|_m^2}\, \mathrm{d}x. \tag{4.17}$$

Now, by using $L[\psi_n] \le \frac{1}{n}$ and the estimate (4.5) for the functions ψ_n we obtain

$$L[\chi\psi_n] \le \frac{1}{n} + \varepsilon_2(b), \tag{4.18}$$

where $\varepsilon_2(b) \to 0$ as $b \to \infty$. Substituting this in (4.16) we get

$$\langle V_{i_0j_0}\chi\psi_n, \chi\psi_n\rangle \le \frac{1}{n} + \varepsilon_2(b) - \sum_{(i,j)\neq(i_0,j_0)} \langle V_{ij}\chi\psi_n, \chi\psi_n\rangle. \tag{4.19}$$

Now we estimate the last term on the r.h.s. of (4.19). Since the Hamiltonians of the clusters consisting of two particles do not have negative spectrum, we have

$$\langle V_{ij}\chi\psi_n, \chi\psi_n\rangle \ge -\|\nabla_0 (\chi\psi_n)\|^2 \ge -\varepsilon, \tag{4.20}$$

where according to statement **(i)** of the Lemma the constant $\varepsilon > 0$ can be chosen arbitrarily small if $b > 0$ and $n \in \mathbb{N}$ are sufficiently large. Inserting this in (4.19) we get

$$\langle V_{i_0j_0}\chi\psi_n, \chi\psi_n\rangle \le \frac{1}{n} + \varepsilon_2(b) + 2\varepsilon, \tag{4.21}$$

which completes the proof of Lemma 4.3. □

Now we turn to the proof of the well-definedness of $\langle V_{ij}\varphi_0, \varphi_0\rangle$. Recall that we need to show that

$$\int\int_{\{|q_\beta|_m \le 1\}} |\varphi_0|^2\, \mathrm{d}q_\beta\, \mathrm{d}\xi_\beta < \infty. \tag{4.22}$$

Since the cluster Hamiltonians for non-trivial clusters do not have virtual levels and $V_{ij} \neq 0$, by the remark **(iii)** after Theorem 2.3 we get

$$\int\int_{\{|q_\beta|_m \le 1\}} |\chi\psi_n|^2\, \mathrm{d}q_\beta\, \mathrm{d}\xi_\beta \le C_1 \|\nabla_{q_\beta}(\chi\psi_n)\|^2 + C_2 \langle V_{ij}\chi\psi_n, \chi\psi_n\rangle \tag{4.23}$$

for some constants $C_1, C_2 > 0$ and $\beta = (i,j)$. Now by Lemma 4.3 we see that the r.h.s. of (4.23) can be done arbitrarily small if the constant $b > 0$ in the definition of the function χ and $n \in \mathbb{N}$ are sufficiently large. Hence, for any $\varepsilon > 0$ we find $b > 0$, such that

$$\int\int_{\{|q_\beta|_m \le 1\}} |\chi\psi_n|^2\, \mathrm{d}q_\beta\, \mathrm{d}\xi_\beta \le \varepsilon. \tag{4.24}$$

Recall that for $|\xi_\beta|_m > 2b$ we have $\chi(x) = 1$ and therefore

$$\int_{\{|\xi_\beta|_m \ge 2b\}} \int_{\{|q_\beta|_m \le 1\}} |\psi_n(x)|^2\, \mathrm{d}x = \int_{\{|\xi_\beta|_m \ge 2b\}} \int_{\{|q_\beta|_m \le 1\}} |\chi\psi_n(x)|^2\, \mathrm{d}x \le \varepsilon \tag{4.25}$$

for $b > 0$ and $n \in \mathbb{N}$ large enough. Furthermore, we have $\psi_n \to \varphi_0$ in $L^2_{\mathrm{loc}}(X_0)$. Therefore, we get

$$\int_{\{|\xi_\beta|_m \le 2b\}} \int_{\{|q_\beta|_m \le 1\}} |\psi_n|^2 \, \mathrm{d}q_\beta \, \mathrm{d}\xi_\beta \to \int_{\{|\xi_\beta|_m \le 2b\}} \int_{\{|q_\beta|_m \le 1\}} |\varphi_0|^2 \, \mathrm{d}q_\beta \, \mathrm{d}\xi_\beta. \tag{4.26}$$

This, together with (4.25) shows that the integral

$$\int \int_{\{|q_\beta|_m \le 1\}} |\varphi_0|^2 \, \mathrm{d}q_\beta \, \mathrm{d}\xi_\beta \tag{4.27}$$

is bounded and thus $\langle V_{ij}\varphi_0, \varphi_0\rangle$ is well-defined.
Now we show that $\langle V_{ij}\psi_n, \psi_n\rangle \to \langle V_{ij}\varphi_0, \varphi_0\rangle$ as $n \to \infty$. At first, we consider the integral

$$\int_{\{|\xi_\beta|_m \ge 2b\}} \int |V_{ij}||\psi_n|^2 \, \mathrm{d}q_\beta \, \mathrm{d}\xi_\beta \tag{4.28}$$

and prove that it can be done arbitrarily small if $b > 0$ and $n \in \mathbb{N}$ are large enough. By Corollary A.2 we have

$$\begin{aligned} &\int_{\{|\xi_\beta|_m \ge 2b\}} \int |V_{ij}||\psi_n|^2 \, \mathrm{d}q_\beta \, \mathrm{d}\xi_\beta \\ &\qquad \le C \int_{\{|\xi_\beta|_m \ge 2b\}} \left(\int |\nabla_{q_\beta}\psi_n|^2 \, \mathrm{d}q_\beta + \int_{\{|q_\beta|_m \le 1\}} |\psi_n|^2 \, \mathrm{d}q_\beta \right) \mathrm{d}\xi_\beta. \end{aligned} \tag{4.29}$$

Note that by Lemma 4.3 we get for arbitrary $\varepsilon > 0$

$$\int_{\{|\xi_\beta|_m \ge 2b\}} \int |\nabla_{q_\beta}\psi_n|^2 \, \mathrm{d}q_\beta \, \mathrm{d}\xi_\beta = \int_{\{|\xi_\beta|_m \ge 2b\}} \int |\nabla_{q_\beta}(\chi\psi_n)|^2 \, \mathrm{d}q_\beta \, \mathrm{d}\xi_\beta \le \varepsilon \tag{4.30}$$

if $b > 0$ and $n \in \mathbb{N}$ are large enough. Substituting this inequality and inequality (4.25) in (4.29) yields

$$\int_{\{|\xi_\beta|_m \ge 2b\}} \int |V_{ij}||\psi_n|^2 \, \mathrm{d}q_\beta \, \mathrm{d}\xi_\beta \le 2\varepsilon. \tag{4.31}$$

Due to

$$\int |V_{ij}||\varphi_0|^2 \, \mathrm{d}q_\beta \, \mathrm{d}\xi_\beta < \infty \tag{4.32}$$

we also obtain

$$\int_{\{|\xi_\beta|_m \ge 2b\}} \int |V_{ij}||\varphi_0|^2 \, \mathrm{d}q_\beta \, \mathrm{d}\xi_\beta \le \varepsilon \tag{4.33}$$

for $b > 0$ large enough. Now we consider the region $\{|\xi_\beta|_m \le 2b\}$. Due to the decay property (2.9) of the potentials V_{ij} and the estimates (4.5) and (4.2) for the functions ψ_n and φ_0 we get

$$\int_{\{|\xi_\beta|_m \le 2b\}} \int_{\{|q_\beta|_m \ge b_1\}} |V_{ij}||\psi_n|^2 \, \mathrm{d}q_\beta \, \mathrm{d}\xi_\beta < \varepsilon \tag{4.34}$$

and

$$\int_{\{|\xi_\beta|_m \le 2b\}} \int_{\{|q_\beta|_m \ge b_1\}} |V_{ij}||\varphi_0|^2 \, \mathrm{d}q_\beta \, \mathrm{d}\xi_\beta < \varepsilon \tag{4.35}$$

where $\varepsilon > 0$ can be chosen arbitrarily small if $b_1 > 0$ is large enough and estimate (4.34) holds uniformly in $n \in \mathbb{N}$.
Estimates (4.31) - (4.35) show that to prove convergence $\langle V_{ij}\psi_n, \psi_n\rangle \to \langle V_{ij}\varphi_0, \varphi_0\rangle$ it suffices to show that $\langle V_{ij}\psi_n, \psi_n\rangle_\Omega \to \langle V_{ij}\varphi_0, \varphi_0\rangle_\Omega$ for the compact set

$$\Omega := \{x \in X_0 \,:\, |q_\beta|_m \le b_1,\ |\xi_\beta|_m \le 2b\}. \tag{4.36}$$

We write

$$\langle V_{ij}\psi_n, \psi_n\rangle_\Omega - \langle V_{ij}\varphi_0, \varphi_0\rangle_\Omega = \langle V_{ij}(\psi_n - \varphi_0), \psi_n\rangle_\Omega + \langle V_{ij}\varphi_0, (\psi_n - \varphi_0)\rangle_\Omega. \tag{4.37}$$

Since ψ_n converges to φ_0 in $L^2_{\mathrm{loc}}(X_0)$, $\|\nabla_{q_\beta}\psi_n\| \le 1$, $\|\nabla_{q_\beta}\varphi_0\| \le 1$ and the potential V_{ij} satisfies (2.2), both summands on the r.h.s. of (4.37) tend to zero as $n \to \infty$. Combining this with the estimates (4.31) - (4.35) we conclude $\langle V_{ij}\psi_n, \psi_n\rangle \to \langle V_{ij}\varphi_0, \varphi_0\rangle$ for every pair (i,j) of particles and therefore $\langle V\psi_n, \psi_n\rangle \to \langle V\varphi_0, \varphi_0\rangle$ as $n \to \infty$.
Since by definition of the functions ψ_n

$$\langle V\psi_n, \psi_n\rangle \le -\left(1 - n^{-1}\right), \tag{4.38}$$

we get $\langle V\varphi_0, \varphi_0\rangle \le -1$. On the other hand, $H \ge 0$ and $\|\nabla_0\varphi_0\| \le 1$. This shows

$$\|\nabla_0\varphi_0\|^2 + \langle V\varphi_0, \varphi_0\rangle = 0, \tag{4.39}$$

which completes the proof of Theorem 4.1. □

5. Absence of the Efimov effect in multi-particle systems consisting of one- or two-dimensional particles

In this section we prove that the Efimov effect does not occur in systems of $N \ge 4$ one-dimensional or $N \ge 5$ two-dimensional particles. The absence of the Efimov effect for such systems is mainly caused by the fact that in these cases virtual levels of the cluster Hamiltonians $H[C]$ with $|C| = N-1$ correspond to eigenvalues, as we have shown in Section 3. We follow the strategy of the proof of Theorem 5.1 in [6], which itself is based on ideas of [35]. However, on a technical level the proof in this section is slightly different from those in [6] and [35] because Hardy's inequality, which plays an important role in [6] and [35], is different in lower dimensions. The main result of this section is the following

Theorem 5.1. *Let $d = 1$ and $N \ge 4$ or $d = 2$ and $N \ge 5$. Suppose that every pair potential $V_{ij} \ne 0$ satisfies (2.9) and is operator bounded with respect to $-\Delta$ with relative bound zero, i.e. for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$, such that*

$$\|V_{ij}\psi\|^2 \le \varepsilon\|\Delta\psi\|^2 + C(\varepsilon)\|\psi\|^2, \qquad \psi \in H^2(\mathbb{R}^d). \tag{5.1}$$

Furthermore, assume that $H[C] \ge 0$ for all clusters C with $|C| = N-1$ and there exists $\varepsilon \in (0,1)$, such that

$$\mathcal{S}_{\mathrm{ess}}\left(-(1-\varepsilon)\Delta_0[C] + V[C]\right) = [0, \infty). \tag{5.2}$$

Then the discrete spectrum of H is finite.

Remark. We emphasize that in Theorem 5.1 the cluster Hamiltonian $H[C]$ with $|C| = N-1$ may have a virtual level at zero. For clusters C' with $1 < |C'| < N-1$ however, the Hamiltonian $H[C']$ are not allowed to have a virtual level, which is a consequence of (5.2) and the HVZ theorem.

Proof of Theorem 5.1. For $\varepsilon > 0$ we define the functional $L : H^1(X_0) \to \mathbb{R}$ as

$$L[\varphi] := \langle H\varphi, \varphi\rangle - \varepsilon\||x|_m^{-2}\varphi\|^2 \tag{5.3}$$

and prove that $L[\varphi] \ge 0$ for any function $\varphi \in H^1(X_0)$ with $\mathrm{supp}\,(\varphi) \subset \{|x|_m \ge R\}$ if $R > 0$ is large enough and $\varepsilon > 0$ is small enough. This implies finiteness of the discrete spectrum of H, see Lemma C.1 in Appendix C (see also [40]).
We fix a constant $\kappa > 0$, such that $K_R(Z,\kappa) \cap K_R(Z',\kappa) = \emptyset$ for all partitions $Z \ne Z'$ with $|Z| = |Z'| = 2$. By applying Theorem 3.9 we get

$$L[\varphi] \ge \sum_{Z:|Z|=2} L_2[\varphi u_Z] + L_2'[\varphi\mathcal{V}], \tag{5.4}$$

where $\mathcal{V} = \sqrt{1 - \sum_{Z:|Z|=2} u_Z^2}$ and the functionals L_2 and L' are defined by

$$
\begin{aligned}
L_2[\psi] &:= \langle H\psi, \psi\rangle - \varepsilon \| |x|_m^{-2} \psi\|^2 \\
&\quad - \varepsilon_1 \left\| |q(Z)|_m^{-1} |\ln(|q(Z)|_m |\xi(Z)|_m^{-1})|^{-1} \psi \right\|^2_{K_R(Z,\kappa',\kappa)}, \\
L_2'[\psi] &:= \langle H\psi, \psi\rangle - (\varepsilon + \varepsilon_1) \| |x|_m^{-2} \psi\|^2.
\end{aligned}
\tag{5.5}
$$

Here, the constants $\kappa > 0$ and $\varepsilon_1 > 0$ can be chosen arbitrarily small and $\kappa' \in (0, \kappa)$ depends on κ and ε_1 only. For the sake of simplicity we omit the index Z in the following computations and write q and ξ instead of $q(Z)$ and $\xi(Z)$, respectively. At first, we prove $L_2[\varphi u_Z] \geq 0$. We distinguish between the following two types of partitions $Z = (C_1, C_2)$:

(i) $|C_1| < N - 1$ and $|C_2| < N - 1$,
(ii) $|C_1| = N - 1$ or $|C_2| = N - 1$.

In the first case the operators $H[C_1]$ and $H[C_2]$ do not have virtual levels, which implies that there exists a constant $\mu_0 > 0$, such that

$$
\langle H(Z)\psi, \psi\rangle \geq \mu_0 \|\nabla_q \psi\|^2 \tag{5.6}
$$

holds for any function $\psi \in H^1(X_0)$. Repeating the arguments which were used in the proof of Lemma 3.11 we get $L_2[\varphi u_Z] \geq 0$.

We turn to case **(ii)**, where the Hamiltonian $H[C_1]$ or $H[C_2]$ may have a virtual level. Suppose that $|C_1| = N - 1$ and that $H[C_1]$ has a virtual level. According to Theorem 3.2, Corollaries 3.3 and 3.4 and Theorem 3.5 zero is a simple eigenvalue of the operator $H[C_1]$. Let φ_0 be the corresponding eigenfunction normalized by $\|\varphi_0\| = 1$. Let

$$
f(\xi) := \|\nabla_q \varphi_0\|^{-2} \langle \nabla_q (\varphi u_Z(\cdot, \xi)), \nabla_q \varphi_0 \rangle_{L^2(X_0(Z))} \tag{5.7}
$$

and

$$
g(q, \xi) := \varphi u_Z(q, \xi) - f(\xi)\varphi_0(q). \tag{5.8}
$$

Then we have

$$
\varphi u_Z = f\varphi_0 + g \quad \text{and} \quad \langle \nabla_q g(\cdot, \xi), \nabla_q \varphi_0\rangle_{L^2(X_0(Z))} = 0 \tag{5.9}
$$

for almost every ξ. For $|\xi|_m \leq \frac{R}{2}$ we have $f(\xi) = 0$ and $g(q, \xi) = 0$, because $\varphi u_Z = 0$ for $|x| \leq R$. We write

$$
\begin{aligned}
L_2[\varphi u_Z] &= \langle H[C_1]\, g, g\rangle + \langle H[C_1]\varphi_0 f, \varphi_0 f\rangle + 2\,\mathrm{Re}\langle g, H[C_1]\varphi_0 f\rangle \\
&\quad + \|\nabla_\xi (\varphi u_Z)\|^2 + \langle I(Z)\varphi u_Z, \varphi u_Z\rangle - \varepsilon \| |x|_m^{-2} \varphi u_Z\|^2 \\
&\quad - \varepsilon_1 \| |q|_m^{-1} |\ln(|q|_m |\xi|_m^{-1})|^{-1} \varphi u_Z\|^2_{K_R(Z,\kappa',\kappa)}.
\end{aligned}
\tag{5.10}
$$

Due to $H[C_1]\varphi_0 = 0$ the second term and the third term on the r.h.s. of (5.10) are zero. Now we estimate the term $\langle I(Z)\varphi u_Z, \varphi u_Z\rangle$. For fixed $\varepsilon_2 > 0$ we get

$$
|I(Z)(x)| \leq C|\xi|_m^{-2-\nu} \leq \frac{\varepsilon_2}{4} |\ln(|\xi|_m)|^{-2} |\xi|_m^{-2} \tag{5.11}
$$

for $x \in K_R(Z, \kappa)$ if $R > 0$ is large enough. Since $\varphi u_Z(q, \xi) = 0$ for $|\xi|_m \leq \frac{R}{2}$, we can apply the one- or two- dimensional Hardy inequality in the ξ-variable to obtain

$$
|\langle I(Z)\varphi u_Z, \varphi u_Z\rangle| \leq \frac{\varepsilon_2}{4} \| |\ln(|\xi|_m)|^{-1} |\xi|_m^{-1} \varphi u_Z\|^2 \leq \varepsilon_2 \|\nabla_\xi(\varphi u_Z)\|^2. \tag{5.12}
$$

This, together with (5.10) implies

$$
\begin{aligned}
L_2[\varphi u_Z] \geq& \langle H[C_1]g, g\rangle + (1 - \varepsilon_2)\|\nabla_\xi (\varphi u_Z)\|^2 - \varepsilon \| |x|_m^{-2} \varphi u_Z\|^2 \\
&- \varepsilon_1 \| |\ln(|q|_m |\xi|_m^{-1})|^{-1} |q|_m^{-1} \varphi u_Z\|^2_{K_R(Z,\kappa',\kappa)}.
\end{aligned}
\tag{5.13}
$$

Since

$$
\left\| |x|_m^{-2} \varphi u_Z \right\|^2 \leq \left\| |\xi|_m^{-1} (\ln^{-1} |\xi|_m) \varphi u_Z \right\|^2 \tag{5.14}
$$

for $|x|_m > 1$ and we have $|\xi|_m \geq \frac{R}{2}$ on the support of φu_Z, we get

$$4\varepsilon \|\nabla_\xi \left(\varphi u_Z\right)\|^2 - \varepsilon \||x|_m^{-2} \varphi u_Z\|^2 \geq 0. \tag{5.15}$$

Substituting this inequality into (5.13) yields

$$\begin{aligned} L_2[\varphi u_Z] \geq \langle H[C_1]g, g\rangle + (1-\varepsilon_3)\|\nabla_\xi \left(\varphi u_Z\right)\|^2 \\ - \varepsilon_1 \|| \ln(|q|_m |\xi|_m^{-1})|^{-1} |q|_m^{-1} \varphi u_Z\|^2_{K_R(Z,\kappa',\kappa)}, \end{aligned} \tag{5.16}$$

where $\varepsilon_3 = \varepsilon_2 + 4\varepsilon$. Now we estimate the term

$$\langle H[C_1]g, g\rangle - \varepsilon_1 \|| \ln(|q|_m |\xi|_m^{-1})|^{-1} |q|_m^{-1} \varphi u_Z\|^2_{K_R(Z,\kappa',\kappa)}. \tag{5.17}$$

This is done in the following

Lemma 5.2. *Let $1 < \alpha < \tilde{C}_H(X_0)$ and let C_1 be a cluster with $|C_1| = N-1$ and the functions f, g be defined by (5.7) and (5.8). Then for $\varepsilon_1 > 0$ small enough and $R > 0$ sufficiently large*

$$\begin{aligned} \langle H[C_1]g, g\rangle - \varepsilon_1 \|| \ln(|q|_m |\xi|_m^{-1})|^{-1} |q|_m^{-1} \varphi u_Z\|^2_{K_R(Z,\kappa',\kappa)} \\ \geq - \int_{\{|\xi|_m \geq \frac{R}{2}\}} |\xi|_m^{-2\alpha} |f(\xi)|^2 \, \mathrm{d}\xi. \end{aligned} \tag{5.18}$$

Proof of Lemma 5.2. Due to Theorem 3.2 the orthogonality in (5.9) implies

$$\langle H[C_1]g, g\rangle \geq \delta_0 \|\nabla_q g\|^2 \tag{5.19}$$

for some $\delta_0 > 0$. Therefore,

$$\begin{aligned} \langle H[C_1]g, g\rangle - \varepsilon_1 \|| \ln(|q|_m |\xi|_m^{-1})|^{-1} |q|_m^{-1} \varphi u_Z\|^2_{K_R(Z,\kappa',\kappa)} \\ \geq \delta_0 \|\nabla_q g\|^2 - \varepsilon_1 \|| \ln(|q|_m |\xi|_m^{-1})|^{-1} |q|_m^{-1} \varphi u_Z\|^2_{K_R(Z,\kappa',\kappa)}. \end{aligned} \tag{5.20}$$

Since $\varphi u_Z = \varphi_0 f + g$, we have

$$|\nabla_q(\varphi u_Z)|^2 = |\nabla_q(\varphi_0 f + g)|^2 \leq 2|\nabla_q \varphi_0 f|^2 + 2|\nabla_q g|^2, \tag{5.21}$$

which yields

$$\|\nabla_q g\|^2_{K_R(Z,\kappa',\kappa)} \geq \frac{1}{2}\|\nabla_q(\varphi u_Z)\|^2_{K_R(Z,\kappa',\kappa)} - \|\nabla_q \varphi_0 f\|^2_{K_R(Z,\kappa',\kappa)}. \tag{5.22}$$

Since $\varphi u_Z = 0$ for $|q|_m = \kappa|\xi|_m$, we get similarly as in the proof of Lemma 3.11 that

$$\frac{\delta_0}{2}\|\nabla_q(\varphi u_Z)\|^2_{K_R(Z,\kappa',\kappa)} - \varepsilon_1 \|| \ln(|q|_m |\xi|_m^{-1})|^{-1} |q|_m^{-1} \varphi u_Z\|^2_{K_R(Z,\kappa',\kappa)} \geq 0 \tag{5.23}$$

if $\varepsilon_1 > 0$ is small enough. Combining this inequality with (5.22) and (5.20) yields

$$\langle H[C_1]g, g\rangle - \varepsilon_1 \|| \ln(|q|_m |\xi|_m^{-1})|^{-1} |q|_m^{-1} \varphi u_Z\|^2_{K_R(Z,\kappa',\kappa)} \geq -\delta_0 \|\nabla_q \varphi_0 f\|^2_{K_R(Z,\kappa',\kappa)}. \tag{5.24}$$

Now we estimate the term $\|\nabla_q \varphi_0 f\|^2_{K_R(Z,\kappa',\kappa)}$. By Theorem 3.2 we have

$$|\nabla_q \left(|q|_m^\alpha \varphi_0\right)| \in L^2(X_0(Z)) \quad \text{and} \quad (1+|q|_m)^{\alpha-1}\varphi_0 \in L^2(X_0(Z)) \tag{5.25}$$

for any $0 \leq \alpha < \tilde{C}_H(X_0)$. This implies

$$|q|_m^\alpha \, |\nabla_q \varphi_0| \in L^2(X_0(Z)). \tag{5.26}$$

Due to $|\xi|_m \geq \frac{R}{2}$ for $x \in K_R(Z,\kappa)$ we get

$$\begin{aligned}\int_{K_R(Z,\kappa',\kappa)} |\nabla_q \varphi_0 f|^2 \,\mathrm{d}x &= \int_{\{|\xi|_m \geq \frac{R}{2}\}} |f(\xi)|^2 \int_{\kappa'|\xi|_m}^{\kappa|\xi|_m} |\nabla_q \varphi_0|^2 \,\mathrm{d}q \,\mathrm{d}\xi \\ &= \int_{\{|\xi|_m \geq \frac{R}{2}\}} |f(\xi)|^2 \int_{\kappa'|\xi|_m}^{\kappa|\xi|_m} |q|_m^{-2\alpha} |q|_m^{2\alpha} |\nabla_q \varphi_0|^2 \,\mathrm{d}q \,\mathrm{d}\xi \\ &\leq (\kappa')^{-2\alpha} \int_{\{|\xi|_m \geq \frac{R}{2}\}} |\xi|_m^{-2\alpha} |f(\xi)|^2 \int_{\kappa'|\xi|_m}^{\kappa|\xi|_m} |q|_m^{2\alpha} |\nabla_q \varphi_0|^2 \,\mathrm{d}q \,\mathrm{d}\xi,\end{aligned} \tag{5.27}$$

where in the last inequality we used $|q|_m \geq \kappa'|\xi|_m$. Since $|q|_m^\alpha |\nabla_q \varphi_0| \in L^2(X_0(Z))$, we have

$$\int_{\kappa'|\xi|_m}^{\kappa|\xi|_m} |q|_m^{2\alpha} |\nabla_q \varphi_0|^2 \,\mathrm{d}q \leq (\kappa')^{2\alpha} \delta_0^{-1} \tag{5.28}$$

for $|\xi|_m \geq \frac{R}{2}$ if $R > 0$ is sufficiently large. This yields

$$-\delta_0 \|\nabla_q \varphi_0 f\|^2_{K_R(Z,\kappa',\kappa)} \geq -\int_{\{|\xi|_m \geq \frac{R}{2}\}} |\xi|_m^{-2\alpha} |f(\xi)|^2 \,\mathrm{d}\xi, \tag{5.29}$$

which completes the proof of Lemma 5.2. □

We continue to estimate the functional $L_2[\varphi u_Z]$. Combining (5.16) with Lemma 5.2 we get

$$L_2[\varphi u_Z] \geq (1-\varepsilon_3)\|\nabla_\xi(\varphi u_Z)\|^2 - \varepsilon_1 \int_{\{|\xi|_m \geq \frac{R}{2}\}} |\xi|_m^{-2\alpha} |f(\xi)|^2 \,\mathrm{d}\xi. \tag{5.30}$$

In the next step we estimate the term $\|\nabla_\xi(\varphi u_Z)\|^2$. This is done in the following

Lemma 5.3. *Let $\delta > 0$. There exists a constant $\omega > 0$ which depends on $\|\varphi_0\|$, $\|\nabla_q \varphi_0\|$ and $\|\Delta_q \varphi_0\|$ only, such that*

$$\|\nabla_\xi (\varphi u_Z)\|^2 \geq \omega \left(\||\xi|_m^{-1-\delta} \varphi_0 f\|^2 + \||\xi|_m^{-1-\delta} g\|^2\right). \tag{5.31}$$

Remark. For the case $\dim X_c(Z) = 3$, a statement similar to Lemma 5.3 was proved in [35]. In the proof of Lemma 5.3 we follow the ideas of this work.

Proof of Lemma 5.3. Since $\varphi u_Z(q,\xi) = 0$ for $|\xi|_m \leq \frac{R}{2}$, we can apply the one- or two-dimensional Hardy inequality in the space $X_c(Z)$ to the function $\varphi u_Z(q,\cdot)$ for fixed q. This implies

$$\begin{aligned}\|\nabla_\xi (\varphi u_Z)\|^2 &\geq \frac{1}{4}\||\xi|_m^{-1-\delta} \varphi u_Z\|^2 = \frac{1}{4}\||\xi|_m^{-1-\delta}\varphi_0 f + |\xi|_m^{-1-\delta} g\|^2 \\ &\geq \frac{1}{4}\left(\||\xi|_m^{-1-\delta}\varphi_0 f\|^2 + \||\xi|_m^{-1-\delta} g\|^2 - 2|\langle |\xi|_m^{-1-\delta}\varphi_0 f, |\xi|_m^{-1-\delta} g\rangle|\right).\end{aligned} \tag{5.32}$$

Since $\langle \nabla_q \varphi_0, \nabla_q g\rangle_{L^2(X_0(Z))} = 0$, we have

$$\langle \nabla_q |\xi|_m^{-1-\delta}\varphi_0 f, \nabla_q |\xi|_m^{-1-\delta} g\rangle = 0 \tag{5.33}$$

and by Lemma 5.3 in [35] we can find a constant $\omega > 0$ which depends on $\|\varphi_0\|$, $\|\nabla_q \varphi_0\|$ and $\|\Delta_q \varphi_0\|$ only, such that

$$\left|\langle |\xi|_m^{-1-\delta}\varphi_0 f, |\xi|_m^{-1-\delta} g\rangle\right| \leq \frac{1}{2}(1-4\omega)\left(\||\xi|_m^{-1-\delta}\varphi_0 f\|^2 + \||\xi|_m^{-1-\delta} g\|^2\right). \tag{5.34}$$

Substituting this inequality in (5.32) yields

$$\|\nabla_\xi (\varphi u_Z)\|^2 \geq \omega \left(\||\xi|_m^{-1-\delta}\varphi_0 f\|^2 + \||\xi|_m^{-1-\delta} g\|^2\right), \tag{5.35}$$

which completes the proof of Lemma 5.3. □

Combining (5.30) with (5.31) and using $\|\varphi_0\| = 1$ we get

$$L_2[\varphi u_Z] \geq (1-\varepsilon_3)\omega \int_{\{|\xi|_m \geq \frac{R}{2}\}} |\xi|_m^{-2-2\delta} |f(\xi)|^2 \,\mathrm{d}\xi - \varepsilon_1 \int_{\{|\xi|_m \geq \frac{R}{2}\}} |\xi|_m^{-2\alpha} |f(\xi)|^2 \,\mathrm{d}\xi. \tag{5.36}$$

Choosing $\delta < \alpha - 1$ and $\varepsilon_1, \varepsilon_3 > 0$ small enough yields $L_2[\varphi u_Z] \geq 0$.
To complete the proof of Theorem 5.1 it remains to show $L_2'[\varphi \mathcal{V}] \geq 0$ for every $\varphi \in H^1(X_0)$ with $\operatorname{supp}(\varphi) \subset \{x \in X_0 \,:\, |x|_m \geq R\}$, where L_2' is the functional defined in (5.5). Note that for all partitions $Z = (C_1, \dots, C_p)$ with $p = 3, 4, \dots, N-1$ the Hamiltonians $H[C_i]$ do not have a virtual level if $|C_i| > 1$. Hence, we can estimate the functional $L_2'[\mathcal{V}\varphi] \geq 0$ in cones corresponding to partitions Z with $|Z| \geq 3$ in the same way as in the proof of Lemma 3.11. In the region which remains after separation of the cones corresponding to all partitions Z with $|Z| \leq N-1$ we have $|V_{ij}(x_{ij})| \leq |x|_m^{-2-\nu}$ for all $i \neq j$. Applying Hardy's inequality in the space X_0 completes the proof. ☐

6. Absence of the Efimov effect in systems of three one- or two-dimensional particles

Now we prove that the Efimov effect is absent for systems of three one- or two-dimensional particles. This was first proved in [34] under restrictive conditions on the pair potentials. There, the potentials had to be compactly supported or short-range and negative at infinity. Later, in [36] the restrictions on the potentials were relaxed. Unfortunately, Lemma 1 in [36] contains a mistake. Below we follow the ideas of [36] and correct this mistake. We give the proof for both, the one- and the two-dimensional case.

6.1. Systems of three one-dimensional particles.

Theorem 6.1 (Absence of the Efimov effect for systems of three one-dimensional particles)**.** *Let H be the Hamiltonian corresponding to a system of $N = 3$ one-dimensional particles. Suppose that each pair potential V_{ij} satisfies* (2.2) *and* (2.9) *and that $H[C] \geq 0$ for any cluster C with $|C| = 2$. Then the discrete spectrum of H is finite.*

In the proof of Theorem 6.1 we will use the following lemmas.

Lemma 6.2. *Consider the Schrödinger operator $h = -\Delta + V$ in $L^2(\mathbb{R})$, such that $h \geq 0$ and the potential V satisfies* (2.2) *and* (2.9)*. Then there exists a constant $C > 0$, such that for any $b_0 > A$ and any function $\psi \in H^1(\mathbb{R})$*

$$J[\psi, b_0] := \int_{-b_0}^{b_0} \left(|\psi'(t)|^2 + V(t)|\psi(t)|^2\right) \,\mathrm{d}t \geq -C b_0^{-1-\nu} \left(|\psi(b_0)|^2 + |\psi(-b_0)|^2\right). \tag{6.1}$$

Here, ν and A are the constants given by (2.9)*.*

Proof of Lemma 6.2. Let $\psi \in H^1(\mathbb{R})$ and $b_0 > A$. For $n \geq 2$ we define the function ψ_n as $\psi_n(t) = \psi(t)$ for $-b_0 \leq t \leq b_0$, $\psi_n(t) = 0$ for $t < -nb_0$ and for $t > nb_0$, $\psi_n(t) = \psi(-b_0)\frac{nb_0 + t}{b_0(n-1)}$ for $-nb_0 < t < -b_0$ and $\psi_n(t) = \psi(b_0)\frac{nb_0 - t}{b_0(n-1)}$ for $b_0 < t < nb_0$. Since ψ and ψ_n coincide for $-b_0 \leq t \leq b_0$, we have

$$\begin{aligned}\langle h\psi_n, \psi_n\rangle \leq \int_{-b_0}^{b_0} \left(|\psi'(t)|^2 + V(t)|\psi(t)|^2\right) \,\mathrm{d}t + \int_{-nb_0}^{-b_0} \left(|\psi_n'(t)|^2 + |V(t)||\psi_n(t)|^2\right) \mathrm{d}t \\ + \int_{b_0}^{nb_0} \left(|\psi_n'(t)|^2 + |V(t)||\psi_n(t)|^2\right) \,\mathrm{d}t.\end{aligned} \tag{6.2}$$

At first, we estimate the two last integrals of the r.h.s of (6.2). Since $\psi_n'(t) = \frac{\psi(-b_0)}{b_0(n-1)}$ for $t \in (-nb_0, -b_0)$ and $\psi_n'(t) = -\frac{\psi(b_0)}{b_0(n-1)}$ for $t \in (b_0, nb_0)$, we get

$$\int\limits_{-nb_0}^{-b_0} |\psi_n'(t)|^2 \,\mathrm{d}t + \int\limits_{b_0}^{nb_0} |\psi_n'(t)|^2 \,\mathrm{d}t < \varepsilon, \tag{6.3}$$

where $\varepsilon > 0$ can be chosen arbitrarily small if n is large enough. Moreover, $0 \leq \frac{nb_0+t}{b_0(n-1)} \leq 1$ for $t \in (-nb_0, -b_0)$. This implies

$$\int\limits_{-nb_0}^{-b_0} |V(t)||\psi_n(t)|^2 \mathrm{d}t \leq |\psi(-b_0)|^2 \int\limits_{-nb_0}^{-b_0} |V(t)| \mathrm{d}t \tag{6.4}$$

and analogously we get

$$\int\limits_{b_0}^{nb_0} |V(t)||\psi_n(t)|^2 \mathrm{d}t \leq |\psi(b_0)|^2 \int\limits_{b_0}^{nb_0} |V(t)| \mathrm{d}t. \tag{6.5}$$

This, together with (6.2) and (6.3) yields

$$\langle h\psi_n, \psi_n \rangle \leq J[\psi, b_0] + |\psi(-b_0)|^2 \int\limits_{-nb_0}^{-b_0} |V(t)| \mathrm{d}t + |\psi(b_0)|^2 \int\limits_{b_0}^{nb_0} |V(t)| \,\mathrm{d}t + \varepsilon. \tag{6.6}$$

Now we estimate the integrals on the r.h.s of (6.6). For any $0 < \delta < \nu$ we have $-1 - \nu + \delta < -1$. Since $b_0 \geq A$, we get by (2.9)

$$\int_{b_0}^{nb_0} |V(t)| \,\mathrm{d}t \leq c b_0^{-1-\delta} \int_A^\infty |t|^{-1-\nu+\delta} \,\mathrm{d}t \leq c_1 b_0^{-1-\delta} \tag{6.7}$$

for some constants $c, c_1 > 0$. Analogously we have

$$\int_{-nb_0}^{-b_0} |V(t)| \,\mathrm{d}t \leq c_1 b_0^{-1-\delta}. \tag{6.8}$$

Due to $h \geq 0$ we conclude from (6.6), (6.7) and (6.8) that

$$J[\psi, b_0] \geq -c b_0^{-1-\delta} \left(|\psi(b_0)|^2 + |\psi(-b_0)|^2 \right) - \varepsilon.$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, this completes the proof. $\square$

Lemma 6.3. *Let $C_0 > 0$. Then for any sufficiently large $b > 0$ and for any $\psi \in H^1(\mathbb{R})$*

$$\int_b^\infty \left(|\psi'(t)|^2 - C_0 t^{-2-\nu} |\psi(t)|^2 \right) \,\mathrm{d}t \geq -2C_0 b^{-1-\nu} |\psi(b)|^2. \tag{6.9}$$

Proof of Lemma 6.3. Let $\psi \in H^1(\mathbb{R})$ and $\tilde{\psi}(t) = \psi(t) - \psi(b)$. Then $\tilde{\psi}'(t) = \psi'(t)$ and we have

$$\begin{aligned} \int_b^\infty \left(|\psi'(t)|^2 - C_0 t^{-2-\nu} |\psi(t)|^2 \right) \,\mathrm{d}t \geq& \int_b^\infty \left(|\tilde{\psi}'(t)|^2 - 2C_0 t^{-2-\nu} |\tilde{\psi}(t)|^2 \right) \,\mathrm{d}t \\ &- 2C_0 \int_b^\infty t^{-2-\nu} |\psi(b)|^2 \,\mathrm{d}t. \end{aligned} \tag{6.10}$$

Since $\tilde{\psi}(b) = 0$, we can use the one-dimensional Hardy inequality, which for sufficiently large $b > 0$ yields

$$\int_b^\infty \left(|\tilde{\psi}'(t)|^2 - 2C_0 t^{-2-\nu} |\tilde{\psi}(t)|^2 \right) \,\mathrm{d}t \geq 0. \tag{6.11}$$

This, together with (6.10) implies

$$\int_b^\infty \left(|\psi'(t)|^2 - C_0 t^{-2-\nu}|\psi(t)|^2\right) \mathrm{d}t \geq -2C_0|\psi(b)|^2 \int_b^\infty t^{-2-\nu}\,\mathrm{d}t. \tag{6.12}$$

Computing the integral on the r.h.s. of (6.12) completes the proof. □

Lemma 6.4. *Let $b_2 > b_1$. Then for any $\psi \in H^1(\mathbb{R})$*

$$|\psi(b_i)|^2 \leq 2(b_2-b_1)^{-1}\int_{b_1}^{b_2} |\psi(x)|^2\,\mathrm{d}x + 2(b_2-b_1)\int_{b_1}^{b_2} |\psi'(x)|^2\,\mathrm{d}x, \quad i=1,2. \tag{6.13}$$

Remark. Lemma 6.4 is the one-dimensional analogue of Lemma 2 in [36].

Proof of Lemma 6.4. For $x \in (b_1, b_2)$ we write

$$\psi(x) = \int_{b_1}^x \psi'(t)\,\mathrm{d}t + \psi(b_1). \tag{6.14}$$

Therefore, we have

$$|\psi(b_1)|^2 \leq 2|\psi(x)|^2 + 2\left(\int_{b_1}^{b_2} |\psi'(x)|\,\mathrm{d}x\right)^2, \quad x \in (b_1,b_2). \tag{6.15}$$

Applying the Cauchy-Schwarz inequality to the integral on the r.h.s. of (6.15) yields

$$|\psi(b_1)|^2 \leq 2|\psi(x)|^2 + 2(b_2-b_1)\int_{b_1}^{b_2} |\psi'(x)|^2\,\mathrm{d}x, \quad x \in (b_1,b_2). \tag{6.16}$$

Integrating both sides of (6.16) over (b_1, b_2) and dividing by (b_2-b_1) implies

$$|\psi(b_1)|^2 \leq 2(b_2-b_1)^{-1}\int_{b_1}^{b_2} |\psi(x)|^2\,\mathrm{d}x + 2(b_2-b_1)\int_{b_1}^{b_2} |\psi'(x)|^2\,\mathrm{d}x. \tag{6.17}$$

Similarly we can prove the statement for b_2. □

Proof of Theorem 6.1. As in the proof of Theorem 5.1 we show that

$$L[\varphi] := \int \left(|\nabla_0\varphi|^2 + V|\varphi|^2 - \varepsilon|x|_m^{-4}|\varphi|^2\right)\,\mathrm{d}x \geq 0 \tag{6.18}$$

holds for all functions $\varphi \in H^1(X_0)$ with $\operatorname{supp}(\varphi) \subset \{|x|_m \geq R\}$ if $\varepsilon > 0$ is small enough and $R > 0$ is sufficiently large. Let $Z = (C_1, C_2)$ be a partition into two clusters with $|C_1| = 2$. First, we estimate the part of the quadratic form L corresponding to the cone $K(Z,\kappa)$, where $\kappa > 0$ is so small that cones $K(Z,\kappa)$ and $K(Z',\kappa)$ corresponding to different partitions do not overlap. Denote the particles in C_1 by i and j and the third particle by k. In the following we will need subtle geometric arguments and therefore we introduce a basis of X_0 and work with the corresponding coordinates. Recall that $\dim(X_0(Z)) = 1$ and $\dim(X_c(Z)) = 1$. Choosing a vector $u_1 \in X_0(Z)$ and a vector $u_2 \in X_c(Z)$, both normalized with respect to the norm $|u_i|_m = 1$, we get an orthonormal basis of X_0. Denote by $\tilde{q}$ and $\tilde{\xi}$ the coefficients corresponding to the basis $\{u_1, u_2\}$. Then we have $|q|_m = |\tilde{q}|$, $|\xi|_m = |\tilde{\xi}|$ and we can represent $K_R(Z,\kappa)$ as

$$K_R(Z,\kappa) = \left\{(\tilde{q},\tilde{\xi}) \in \mathbb{R}^2 \,:\, |\tilde{q}| \leq \kappa|\tilde{\xi}|,\ |\tilde{q}|^2 + |\tilde{\xi}|^2 \geq R^2\right\} \tag{6.19}$$

and $\varphi = \varphi(\tilde{q}, \tilde{\xi})$ as a function of $\tilde{q}$ and $\tilde{\xi}$. We have

$$\begin{aligned}\int_{K_R(Z,\kappa)} \left(|\nabla_0\varphi|^2 + V|\varphi|^2 - \varepsilon|x|_m^{-4}|\varphi|^2\right) \mathrm{d}x &= \int_{K_R(Z,\kappa)} \left(|\partial_{\tilde{q}}\varphi|^2 + V_{ij}|\varphi|^2\right) \mathrm{d}x \\ &+ \int_{K_R(Z,\kappa)} \left(|\partial_{\tilde{\xi}}\varphi|^2 + (V_{ik}+V_{jk})|\varphi|^2 - \varepsilon|x|_m^{-4}|\varphi|^2\right) \mathrm{d}x\end{aligned} \tag{6.20}$$

and estimate the two integrals on the r.h.s of (6.20) separately. By choosing $\kappa > 0$ small enough we have $|\tilde{\xi}| \geq \frac{R}{2}$ and therefore

$$\int_{K_R(Z,\kappa)} \left(|\partial_{\tilde{q}}\varphi|^2 + V_{ij}|\varphi|^2\right) \mathrm{d}x = \int_{\{|\tilde{\xi}|\geq \frac{R}{2}\}} \int_{\{|\tilde{q}|\leq\kappa|\tilde{\xi}|\}} \left(|\partial_{\tilde{q}}\varphi|^2 + V_{ij}|\varphi|^2\right) \mathrm{d}\tilde{q}\,\mathrm{d}\tilde{\xi}. \tag{6.21}$$

Applying Lemma 6.2 to the integral over $\tilde{q}$ in (6.21) with $b_0 = \kappa|\tilde{\xi}|$ we get

$$\begin{aligned}&\int_{\{|\tilde{\xi}|\geq \frac{R}{2}\}} \int_{\{|\tilde{q}|\leq\kappa|\tilde{\xi}|\}} \left(|\partial_{\tilde{q}}\varphi|^2 + V_{ij}|\varphi|^2\right) \mathrm{d}\tilde{q}\,\mathrm{d}\tilde{\xi} \\ &\quad\geq -C \int_{\{|\tilde{\xi}|\geq \frac{R}{2}\}} |\tilde{\xi}|^{-1-\nu} \left(|\varphi(\kappa\tilde{\xi},\tilde{\xi})|^2 + |\varphi(-\kappa\tilde{\xi},\tilde{\xi})|^2\right) \mathrm{d}\tilde{\xi}.\end{aligned} \tag{6.22}$$

Now we estimate the second integral on the right hand side of (6.20). Note that for $R > 0$ sufficiently large we have

$$|V_{ik}(x_{ik})| + |V_{jk}(x_{jk})| \leq c|\tilde{\xi}|^{-2-\nu} \tag{6.23}$$

for some $c > 0$. This, together with $|x|_m^{-1} \leq |\tilde{\xi}|^{-1}$ implies

$$\begin{aligned}&\int_{K_R(Z,\kappa)} \left(|\partial_{\tilde{\xi}}\varphi|^2 + (V_{ik}+V_{jk})|\varphi|^2 - |x|_m^{-4}|\varphi|^2\right) \mathrm{d}x \\ &\qquad\qquad \geq \int_{K_R(Z,\kappa)} \left(|\partial_{\tilde{\xi}}\varphi|^2 - C|\tilde{\xi}|^{-2-\nu}|\varphi|^2\right) \mathrm{d}x\end{aligned} \tag{6.24}$$

for some $C > 0$, where without loss of generality we assumed that $\nu < 2$. To estimate the integral on the r.h.s. of (6.24) we first integrate over the variable $\tilde{\xi}$ for fixed $\tilde{q}$. Let us describe the domain of integration first. The integral over $\tilde{\xi}$ is from the boundary of $K_R(Z,\kappa)$ to infinity. Note that if $|\tilde{q}|$ is small, then the boundary of $K_R(Z,\kappa)$ is given by an arc with radius R, see Figure 2.

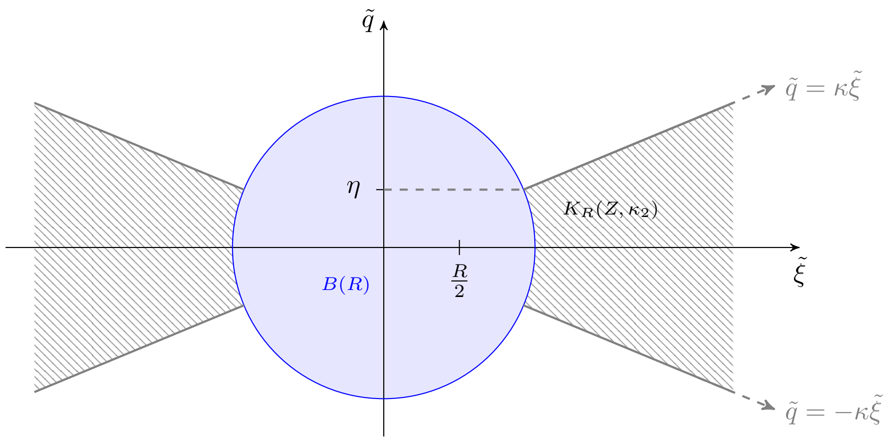

FIGURE 2. The cone $K_R(Z,\kappa)$

By definition, the function φ vanishes on this arc. For large values of $|\tilde{q}|$ the boundary of $K_R(Z,\kappa)$ is given by the straight lines $|\tilde{q}| = \kappa|\tilde{\xi}|$. Let $x = (\tilde{q}, \tilde{\xi})$ be a point of intersection of the ball $B(R)$ with the set $\{|\tilde{q}| = \kappa|\tilde{\xi}|\}$. Then $|\tilde{q}| = \left(1+\kappa^{-2}\right)^{-\frac{1}{2}} R =: \eta$ and we have

$$\begin{aligned}\int_{K_R(Z,\kappa)} &\Big(|\partial_{\tilde{\xi}}\varphi|^2 - c|\tilde{\xi}|^{-2-\nu}|\varphi|^2\Big)\,\mathrm{d}x \\ &= \int_{\{|\tilde{q}|<\eta\}} \int_{\left\{|\tilde{\xi}|\geq\sqrt{R^2-|\tilde{q}|^2}\right\}} \Big(|\partial_{\tilde{\xi}}\varphi|^2 - c|\tilde{\xi}|^{-2-\nu}|\varphi|^2\Big)\,\mathrm{d}\tilde{\xi}\,\mathrm{d}\tilde{q} \\ &\quad + \int_{\{|\tilde{q}|\geq\eta\}} \int_{\left\{|\tilde{\xi}|\geq\kappa^{-1}|\tilde{q}|\right\}} \Big(|\partial_{\tilde{\xi}}\varphi|^2 - c|\tilde{\xi}|^{-2-\nu}|\varphi|^2\Big)\,\mathrm{d}\tilde{\xi}\,\mathrm{d}\tilde{q}.\end{aligned} \tag{6.25}$$

Since $\varphi(\tilde{q},\tilde{\xi}) = 0$ for $|\tilde{\xi}| \leq \sqrt{R^2 - |\tilde{q}|^2}$, the one-dimensional Hardy inequality implies

$$\int_{\{|\tilde{q}|<\eta\}} \int_{\left\{|\tilde{\xi}|\geq\sqrt{R^2-|\tilde{q}|^2}\right\}} \Big(|\partial_{\tilde{\xi}}\varphi|^2 - c|\tilde{\xi}|^{-2-\nu}|\varphi|^2\Big)\,\mathrm{d}\tilde{\xi}\,\mathrm{d}\tilde{q} \geq 0 \tag{6.26}$$

if $R > 0$ is large enough. To estimate the second integral on the r.h.s of (6.25) we apply Lemma 6.3 with $b = \kappa^{-1}|\tilde{q}|$, which yields

$$\begin{aligned}\int_{\{|\tilde{q}|\geq\eta\}} &\int_{\left\{|\tilde{\xi}|\geq\kappa^{-1}|\tilde{q}|\right\}} \Big(|\partial_{\tilde{\xi}}\varphi|^2 - c|\tilde{\xi}|^{-2-\nu}|\varphi(x)|^2\Big)\,\mathrm{d}\tilde{\xi}\,\mathrm{d}\tilde{q} \\ &\geq -C\int_{\{|\tilde{q}|\geq\eta\}} |\tilde{q}|^{-1-\nu}\left(|\varphi(\tilde{q},\kappa^{-1}|\tilde{q}|)|^2 + |\varphi(\tilde{q},-\kappa^{-1}|\tilde{q}|)|^2\right)\mathrm{d}\tilde{q}\end{aligned} \tag{6.27}$$

for some $C > 0$. Combining (6.22) and (6.27) with (6.20) we get

$$\begin{aligned}\int_{K_R(Z,\kappa)} &\left(|\nabla_0\varphi|^2 + V|\varphi|^2 - \varepsilon|x|_m^{-4}|\varphi|^2\right)\mathrm{d}x \\ &\geq -C\int_{\{|\tilde{\xi}|\geq\frac{R}{2}\}} |\tilde{\xi}|^{-1-\nu}\Big(|\varphi(\kappa|\tilde{\xi}|,\tilde{\xi})|^2 + |\varphi(-\kappa|\tilde{\xi}|,\tilde{\xi})|^2\Big)\,\mathrm{d}\tilde{\xi} \\ &\quad - C\int_{\{|\tilde{q}|\geq\eta\}} |\tilde{q}|^{-1-\nu}\left(|\varphi(\tilde{q},\kappa^{-1}|\tilde{q}|)|^2 + |\varphi(\tilde{q},-\kappa^{-1}|\tilde{q}|)|^2\right)\mathrm{d}\tilde{q}.\end{aligned} \tag{6.28}$$

Note that the integrals on the r.h.s of (6.28) are in fact integrals of the function φ over the edges of the cone $K(Z,\kappa)$. We introduce polar coordinates (ρ,θ) in the space X_0. Let $\theta_0 = \arctan(\kappa) \in \left(0, \frac{\pi}{2}\right)$. At first, we consider the integral over $\tilde{\xi}$ in (6.28) and integrate initially over the set where $\tilde{\xi} > 0$. We have

$$\int_{\{\tilde{\xi}\geq\frac{R}{2}\}} \tilde{\xi}^{-1-\nu}|\varphi(\kappa\tilde{\xi},\tilde{\xi})|^2\,\mathrm{d}\tilde{\xi} = c\int_R^\infty \rho^{-1-\nu}|\varphi(\rho,\theta_0)|^2\,\mathrm{d}\rho \tag{6.29}$$

and

$$\int_{\{\tilde{\xi}\geq\frac{R}{2}\}} \tilde{\xi}^{-1-\nu}|\varphi(-\kappa\tilde{\xi},\tilde{\xi})|^2\,\mathrm{d}\tilde{\xi} = c\int_R^\infty \rho^{-1-\nu}|\varphi(\rho,-\theta_0)|^2\,\mathrm{d}\rho, \tag{6.30}$$

where c is a constant depending on θ_0 only. Together with the analogous integrals for $\tilde{\xi} < -\frac{R}{2}$ we get

$$\begin{aligned}\int_{\{|\tilde{\xi}|\geq\frac{R}{2}\}} &|\tilde{\xi}|^{-1-\nu}\Big(|\varphi(\kappa|\tilde{\xi}|,\tilde{\xi})|^2 + |\varphi(-\kappa|\tilde{\xi}|,\tilde{\xi})|^2\Big)\,\mathrm{d}\tilde{\xi} \\ &= c\int_R^\infty \rho^{-1-\nu}\left(|\varphi(\rho,\theta_0)|^2 + |\varphi(\rho,\pi-\theta_0)|^2 + |\varphi(\rho,\pi+\theta_0)|^2 + |\varphi(\rho,-\theta_0)|^2\right)\mathrm{d}\rho.\end{aligned} \tag{6.31}$$

Similarly we can represent the second integral on the r.h.s. of (6.28). Hence, we arrive at

$$\begin{aligned}&\int_{K_R(Z,\kappa)} \left(|\nabla_0\varphi|^2 + V|\varphi|^2 - \varepsilon|x|_m^{-4}|\varphi|^2\right) \mathrm{d}x \\ &\quad\geq -C\int_R^\infty \rho^{-1-\nu}\left(|\varphi(\rho,\theta_0)|^2 + |\varphi(\rho,\pi-\theta_0)|^2 + |\varphi(\rho,\pi+\theta_0)|^2 + |\varphi(\rho,-\theta_0)|^2\right) \mathrm{d}\rho\end{aligned} \tag{6.32}$$

for some $C > 0$. To estimate the r.h.s. of (6.32) let us estimate exemplarily the integral

$$\int_R^\infty \rho^{-1-\nu}|\varphi(\rho,\theta_0)|^2 \,\mathrm{d}\rho. \tag{6.33}$$

We choose $\kappa' > \kappa$ such that $K(Z,\kappa')$ and $K(Z',\kappa')$ do not overlap for any pair of two-cluster partitions $Z \neq Z'$ and denote $\theta_1 = \arctan(\kappa') \in \left(0, \frac{\pi}{2}\right)$. Applying Lemma 6.4 to the function $\varphi(\rho,\cdot)$ for fixed ρ and with $b_1 = \theta_0$ and $b_2 = \theta_1$ we get

$$|\varphi(\rho,\theta_0)|^2 \leq C(\theta_0,\theta_1)\int_{\theta_0}^{\theta_1} \left(|\varphi(\rho,\theta)|^2 + |\partial_\theta\varphi(\rho,\theta)|^2\right) \mathrm{d}\theta \tag{6.34}$$

for some $C(\theta_0,\theta_1) > 0$. Substituting inequality (6.34) into (6.33) we get

$$\begin{aligned}&\int_R^\infty \rho^{-1-\nu}|\varphi(\rho,\theta_0)|^2 \,\mathrm{d}\rho \\ &\quad\leq C(\theta_0,\theta_1)\int_R^\infty\int_{\theta_0}^{\theta_1} \rho^{-1-\nu}\left(|\varphi(\rho,\theta)|^2 + |\partial_\theta\varphi(\rho,\theta)|^2\right) \mathrm{d}\theta\,\mathrm{d}\rho \\ &\quad= C(\theta_0,\theta_1)\int_{\theta_0}^{\theta_1}\int_R^\infty \rho^{-1-\nu}\left(|\varphi(\rho,\theta)|^2 + |\partial_\theta\varphi(\rho,\theta)|^2\right) \mathrm{d}\rho\,\mathrm{d}\theta.\end{aligned} \tag{6.35}$$

Applying inequality (A.17) for fixed θ yields

$$C(\theta_0,\theta_1)\int_R^\infty \rho^{-1-\nu}|\varphi(\rho,\theta)|^2 \,\mathrm{d}\rho \leq \varepsilon\int_R^\infty |\partial_\rho\varphi(\rho,\theta)|^2\rho \,\mathrm{d}\rho \tag{6.36}$$

where $\varepsilon > 0$ can be chosen arbitrarily small if $R > 0$ is large enough. Substituting this inequality into (6.35) and using

$$\left|\frac{\partial\varphi}{\partial\rho}\right|^2 + \frac{1}{\rho^2}\left|\frac{\partial\varphi}{\partial\theta}\right|^2 \leq |\nabla_0\varphi|^2 \tag{6.37}$$

we obtain

$$\int_R^\infty \rho^{-1-\nu}|\varphi(\rho,\theta_0)|^2 \,\mathrm{d}\rho \leq \varepsilon\int_{K_R(Z,\kappa,\kappa')} |\nabla_0\varphi|^2 \,\mathrm{d}x \tag{6.38}$$

for sufficiently large $R > 0$. We can estimate the other integrals on the r.h.s. of (6.32) by the same arguments. Therefore, we obtain

$$\int_{K_R(Z,\kappa)} \left(|\nabla_0\varphi|^2 + V|\varphi|^2 - \varepsilon|x|_m^{-4}|\varphi|^2\right) \mathrm{d}x \geq -4\varepsilon\int_{K_R(Z,\kappa,\kappa')} |\nabla_0\varphi|^2 \,\mathrm{d}x. \tag{6.39}$$

Summing inequality (6.39) over all partitions Z with $|Z| = 2$, inserting the resulting inequality into the definition of L and using that the cones $K_R(Z,\kappa')$ and $K_R(Z',\kappa')$ do not intersect we get

$$L[\varphi] \geq \int_{K_R^c(\kappa)} \left((1-4\varepsilon)|\nabla_0\varphi|^2 + V|\varphi|^2 - \varepsilon|x|_m^{-4}|\varphi|^2\right) \mathrm{d}x, \tag{6.40}$$

where $K_R^c(\kappa) = X_0 \setminus \left(B(R) \cup \bigcup_{Z:|Z|=2} K(Z,\kappa)\right)$. Note that $K_R^c(\kappa)$ is the region in X_0, where all particles are far away from each other. Therefore, we can estimate $|V(x)| \le C|x|_m^{-2-\nu}$. Moreover, we can assume $\nu < 2$ and therefore $|x|_m^{-4} \le |x|_m^{-2-\nu}$. Hence, we get

$$L[\varphi] \ge \int_{K_R^c(\kappa)} \left((1-4\varepsilon)|\nabla_0\varphi|^2 - (C+\varepsilon)|x|_m^{-2-\nu}|\varphi|^2\right) \mathrm{d}x. \tag{6.41}$$

Using polar coordinates (ρ,θ) and $|\nabla_0\varphi| \ge \left|\frac{\partial\varphi}{\partial\rho}\right|$ we find

$$L[\varphi] \ge \int_{\omega\in I}\int_R^\infty \left((1-4\varepsilon)\,|\partial_\rho\varphi|^2 - (C+\varepsilon)\rho^{-2-\nu}|\varphi|^2\right)\rho\,\mathrm{d}\rho\mathrm{d}\theta, \tag{6.42}$$

where $I \subset [0,2\pi]$ is the set of angles corresponding to the region $K_R^c(\kappa)$. Now since $\varphi(\rho,\theta) = 0$ for $\rho \le R$, we can apply inequality (A.17) to the function $u(\rho) = \varphi(\rho,\theta)$ for fixed $\theta \in I$. Choosing R sufficiently large completes the proof of Theorem 6.1. □

6.2. **Systems of three two-dimensional particles.** Now we turn to systems of three two-dimensional bosons or three non-identical two-dimensional particles. For systems of three spinless fermions in dimension two there exists the so-called super Efimov effect, see [14]. We prove that for systems without symmetry restrictions such an effect can not occur. Our result is the following

Theorem 6.5 (Absence of the Efimov effect for systems of three two-dimensional particles)**.** *Let H be the Hamiltonian corresponding to a system of $N=3$ two-dimensional particles. Assume that $H[C] \ge 0$ for any cluster C with $|C|=2$ and that the pair potentials V_{ij} satisfy* (2.2) *and* (2.9) *and they are radially symmetric, i.e. $V_{ij}(x_{ij}) = V_{ij}(|x_{ij}|)$. Then the discrete spectrum of H is finite.*

First, we give some auxiliary Lemmas which are analogous to Lemma 6.2 and Lemma 6.3 for $d=2$.

Lemma 6.6. *Let $d=2$ and consider the operator $h = -\Delta + V$ acting in $L^2(\mathbb{R}^2)$, where $h \ge 0$ and $V(x) = V(|x|)$ satisfies* (2.2) *and* (2.9)*. Then there exists a constant $c>0$, such that for any $b_0 > A$ and for any function $\psi \in H^1(\mathbb{R}^2)$ we have*

$$J[\psi,b_0] := \int_{\{|x|\le b_0\}} \left(|\nabla\psi(x)|^2 + V(x)|\psi(x)|^2\right) \mathrm{d}x \ge -cb_0^{-\nu}\int_0^{2\pi} |\psi(b_0,\theta)|^2\,\mathrm{d}\theta. \tag{6.43}$$

Remark. Lemma 6.6 does not hold if we restrict the operator h to anti-symmetric functions. This is the reason why our proof of Theorem 6.5 does not work for a fermionic system, where the super Efimov effect is known to exist.

Proof of Lemma 6.6. Let $\psi \in H^1(\mathbb{R}^2)$ and $b_0 > A$. We introduce polar coordinates $x = (\rho,\omega)$ and write $\psi(x) = \sum\limits_{n=-\infty}^{\infty}\psi_n(x)$ with $\psi_n(x) = R_n(\rho)\mathrm{e}^{\mathrm{i}n\omega}$. For $k\in\mathbb{N},\ k\ge 2$, let

$$R_n^k(\rho) := \begin{cases} R_n(\rho) & \text{if } \rho \le b_0, \\ R_n(b_0)\ln\left(kb_0\rho^{-1}\right)(\ln k)^{-1} & \text{if } b_0 < \rho \le kb_0, \\ 0 & \text{if } \rho > kb_0. \end{cases} \tag{6.44}$$

We set $\psi_n^k : \mathbb{R}^2 \to \mathbb{C},\ \psi_n^k(x) = R_n^k(|x|)\mathrm{e}^{\mathrm{i}n\omega}$. Then we have $J[\psi_n^k,b_0] = J[\psi_n,b_0]$ and therefore

$$J[\psi,b_0] = \sum_{n=-\infty}^{\infty} J[\psi_n^k,b_0] \quad \text{for any } k\ge 2. \tag{6.45}$$

Now we estimate $J[\psi_n^k, b_0]$ for fixed $k, n \in \mathbb{N}$ with $k \geq 2$. Due to

$$|\nabla\psi|^2 = |\frac{\partial\psi}{\partial\rho}|^2 + \frac{1}{\rho^2}|\frac{\partial\psi}{\partial\omega}|^2 \geq |\frac{\partial\psi}{\partial\rho}|^2 \tag{6.46}$$

and $V(x) = V(|x|)$ we can estimate

$$J[\psi_n^k, b_0] \geq 2\pi \int_0^{b_0} \left(|\partial_\rho R_n^k(\rho)|^2 + V(\rho)|R_n^k(\rho)|^2\right) \rho \mathrm{d}\rho. \tag{6.47}$$

Using $\langle h\tilde{\psi}_n^k, \tilde{\psi}_n^k\rangle \geq 0$ for the radial function $\tilde{\psi}_n^k(x) = R_n^k(|x|)$ and (6.47) yields

$$J[\psi_n^k, b_0] \geq -2\pi \int_{b_0}^{\infty} \left(|\partial_\rho R_n^k(\rho)|^2 + V(\rho)|R_n^k(\rho)|^2\right) \rho \mathrm{d}\rho. \tag{6.48}$$

Easy computation shows that

$$\partial_\rho R_n^k(\rho) = \begin{cases} -R_n(b_0)\,(\ln(k))^{-1}\,\rho^{-1} & \text{if } b_0 < \rho < kb_0, \\ 0 & \text{if } \rho > kb_0. \end{cases} \tag{6.49}$$

This implies

$$\int_{b_0}^{\infty} |\partial_\rho R_n^k(\rho)|^2 \,\rho \mathrm{d}\rho \leq |R_n(b_0)|^2(\ln(k))^{-2} \int_{b_0}^{kb_0} \rho^{-1}\,\mathrm{d}\rho = |R_n(b_0)|^2(\ln(k))^{-1}. \tag{6.50}$$

Since $|V(\rho)| \leq C\,(1+\rho)^{-2-\nu}$ for $\rho > b_0$, we get

$$\begin{aligned} \int_{b_0}^{\infty} |V(\rho)||R_n^k(\rho)|^2 \,\rho \mathrm{d}\rho &\leq C|R_n(b_0)|^2(\ln(k))^{-2} \int_{b_0}^{kb_0} (1+\rho)^{-2-\nu} \left(\ln(kb_0\rho^{-1})\right)^2 \rho \mathrm{d}\rho \\ &\leq C|R_n(b_0)|^2 \int_{b_0}^{kb_0} (1+\rho)^{-2-\nu}\,\rho \mathrm{d}\rho, \end{aligned} \tag{6.51}$$

where for the last inequality we used that $(\ln(k))^{-2}\left(\ln(kb_0\rho^{-1})\right)^2 \leq 1$ for $\rho \in (b_0, kb_0)$. By inserting

$$\int_{b_0}^{kb_0} (1+\rho)^{-2-\nu}\,\rho \mathrm{d}\rho \leq \int_{b_0}^{\infty} \rho^{-1-\nu}\,\mathrm{d}\rho = b_0^{-\nu} \tag{6.52}$$

in inequality (6.51) we find

$$\int_{b_0}^{\infty} |V(\rho)||R_n^k(\rho)|^2 \,\rho \mathrm{d}\rho \leq C|R_n(b_0)|^2 b_0^{-\nu}. \tag{6.53}$$

Combining (6.48) with (6.50) and (6.53) we obtain

$$J[\psi_n^k, b_0] \geq -2\pi C|R_n(b_0)|^2 b_0^{-\nu} - 2\pi|R_n(b_0)|(\ln(k))^{-1}. \tag{6.54}$$

Recall that the left hand side of (6.54) coincides with $J[\psi_n, b_0]$ and in particular does not depend on k. Therefore, sending k to infinity and using

$$2\pi \sum_{n=-\infty}^{\infty} |R_n(b_0)|^2 = \int_0^{2\pi} |\psi(b_0,\omega)|^2\,\mathrm{d}\omega \tag{6.55}$$

completes the proof of Lemma 6.6. $\square$

The following lemma is analogous to Lemma 6.3

Lemma 6.7. *Let $C_0 > 0$. Then for any sufficiently large $b > 0$ and for any $\psi \in H^1(\mathbb{R}^2)$ we have*

$$\int_{\{|x|\geq b\}} \left(|\nabla\psi(x)|^2 - C_0|x|^{-2-\nu}|\psi(x)|^2\right) \mathrm{d}x \geq -2C_0 b^{-\nu}(2\pi)^{-1}\int_0^{2\pi} |\psi(b,\theta)|^2 \,\mathrm{d}\theta. \tag{6.56}$$

Proof of Lemma 6.7. Let $\psi \in H^1(\mathbb{R}^2)$. We write $\psi = \psi_0 + \psi_1$ with $\psi_0 = \mathcal{P}^{m=0}\psi$ and $\psi_1 = \mathcal{P}^{|m|\geq 1}\psi$, where P^m is the projection onto the space of functions with angular momentum m. Then for ψ_1 we have

$$|\nabla\psi_1|^2 = |\partial_\rho\psi_1|^2 + \frac{1}{\rho^2}|\partial_\theta\psi_1|^2 \geq \frac{1}{\rho^2}|\psi_1|^2 \tag{6.57}$$

and therefore

$$\int_{\{|x|\geq b\}} \left(|\nabla\psi_1(x)|^2 - C_0|x|^{-2-\nu}|\psi_1(x)|^2\right) \mathrm{d}x \geq 0 \tag{6.58}$$

if $b > 0$ is sufficiently large. Hence, it suffices to prove inequality (6.56) for the spherically symmetric function ψ_0. For $|x| \geq b$ let $\tilde\psi(|x|) = \psi_0(|x|) - \psi_0(b)$, such that $\tilde\psi(b) = 0$ and we extend $\tilde\psi$ with zero to the region $\{|x| < b\}$. Then, similarly to the one-dimensional case we obtain

$$\begin{aligned}&\int_{\{|x|\geq b\}} \left(|\nabla\psi_0(x)|^2 - C_0|x|^{-2-\nu}|\psi_0(x)|^2\right) \mathrm{d}x \\ &\geq \int_{\{|x|\geq b\}} \left(|\nabla\tilde\psi(x)|^2 - 2C_0|x|^{-2-\nu}|\tilde\psi(x)|^2\right) \mathrm{d}x - \int_{\{|x|\geq b\}} 2C_0|x|^{-2-\nu}|\psi_0(b)|^2 \,\mathrm{d}x.\end{aligned} \tag{6.59}$$

Since $\tilde\psi(|x|) = 0$ for $|x| \leq b$, we can apply the two-dimensional Hardy inequality to the function $\tilde\psi$, which implies that the first integral on the r.h.s of (6.59) is non-negative. Hence, we arrive at

$$\int_{\{|x|\geq b\}} \left(|\nabla\psi_0(x)|^2 - C_0|x|^{-2-\nu}|\psi_0(x)|^2\right) \mathrm{d}x \geq -2C_0\int_{\{|x|\geq b\}} |x|^{-2-\nu}|\psi_0(b)|^2 \,\mathrm{d}x. \tag{6.60}$$

Computing the integral on the r.h.s. of (6.60) completes the proof of Lemma 6.7. $\square$

Proof of Theorem 6.5. In the proof we follow the same strategy as in the proof of Theorem 6.1. Let

$$L[\varphi] := \int \left(|\nabla_0\varphi|^2 + V|\varphi|^2 - \varepsilon|x|_m^{-4}|\varphi|^2\right) \mathrm{d}x. \tag{6.61}$$

We show that $L[\varphi] \geq 0$ for all functions $\varphi \in H^1(X_0)$ with $\mathrm{supp}\,(\varphi) \subset \{|x|_m \geq R\}$ if $\varepsilon > 0$ is small enough and $R > 0$ is sufficiently large. First, we estimate the part of $L[\varphi]$ corresponding to the cone $K(Z,\kappa)$ for an arbitrary partition Z into two clusters. Assume that $Z = (C_1, C_2)$ with $C_1 = \{i,j\}$ and $C_2 = \{k\}$. Note that the spaces $X_0(Z)$ and $X_c(Z)$ are both two-dimensional. We choose orthonormal bases of $X_0(Z)$ and $X_c(Z)$ and denote by $\tilde q_1$, $\tilde q_2$, $\tilde\xi_1$, $\tilde\xi_2$ the corresponding coordinates. We write $\tilde q = (\tilde q_1, \tilde q_2)$, $\tilde\xi = (\tilde\xi_1, \tilde\xi_2)$ and $\varphi = \varphi(\tilde q, \tilde\xi)$. Similarly to (6.20) we write

$$\begin{aligned}\int_{K_R(Z,\kappa)} \left(|\nabla_0\varphi|^2 + V|\varphi|^2 - \varepsilon|x|_m^{-4}|\varphi|^2\right) \mathrm{d}x &= \int_{K_R(Z,\kappa)} \left(|\nabla_{\tilde q}\varphi|^2 + V_{ij}|\varphi|^2\right) \mathrm{d}x \\ &+ \int_{K_R(Z,\kappa)} \left(|\nabla_{\tilde\xi}\varphi|^2 + (V_{ik}+V_{jk})|\varphi|^2 - \varepsilon|x|_m^{-4}|\varphi|^2\right) \mathrm{d}x.\end{aligned} \tag{6.62}$$

To estimate the integrals on the r.h.s of (6.62) we introduce polar coordinates $\tilde{q} = (\rho_1, \beta_1)$ and $\tilde{\xi} = (\rho_2, \beta_2)$ in the planar spaces $X_0(Z)$ and $X_c(Z)$. For the first integral on the r.h.s. of (6.62) we use Lemma 6.6 for fixed $\tilde{\xi}$ with $b_0 = \kappa|\tilde{\xi}|$. Then similarly to (6.22) we get

$$\begin{aligned}\int_{K_R(Z,\kappa)} \left(|\nabla_{\tilde{q}}\varphi|^2 + V_{ij}|\varphi|^2\right) \,\mathrm{d}x &= \int_{\{|\tilde{\xi}|\geq \frac{R}{2}\}} \int_{\{|\tilde{q}|\leq \kappa|\tilde{\xi}|\}} \left(|\nabla_{\tilde{q}}\varphi|^2 + V_{ij}|\varphi|^2\right) \,\mathrm{d}x \\ &\geq -C \int_{\{|\tilde{\xi}|\geq \frac{R}{2}\}} |\tilde{\xi}|^{-\nu} \int_0^{2\pi} |\varphi(\kappa|\tilde{\xi}|, \beta_1, \tilde{\xi})|^2 \,\mathrm{d}\beta_1 \,\mathrm{d}\tilde{\xi}\end{aligned} \tag{6.63}$$

for some $C > 0$. For the second integral on the r.h.s. of (6.62) we use Lemma 6.7 for fixed $\tilde{q}$ and with $b = \kappa^{-1}|\tilde{q}|$, which similarly to (6.27) yields

$$\begin{aligned}&\int_{K_R(Z,\kappa)} \left(|\nabla_{\tilde{\xi}}\varphi|^2 + (V_{ik} + V_{jk})|\varphi|^2 - \varepsilon|x|_m^{-4}|\varphi|^2\right) \,\mathrm{d}x \\ &\qquad\geq -C \int_{\{|\tilde{q}|\geq \eta\}} |\tilde{q}|^{-\nu} \int_0^{2\pi} |\varphi(\tilde{q}, \kappa^{-1}|\tilde{q}|, \beta_2)|^2 \,\mathrm{d}\beta_2 \,\mathrm{d}\tilde{q},\end{aligned} \tag{6.64}$$

where $\eta = \left(1 + \kappa^{-2}\right)^{-1} R$ is analogous to the proof of Theorem 6.1. Combining (6.63) and (6.64) with (6.62) implies

$$\begin{aligned}&\int_{K_R(Z,\kappa)} \left(|\nabla_0\varphi|^2 + V|\varphi|^2 - \varepsilon|x|_m^{-4}|\varphi|^2\right) \,\mathrm{d}x \\ &\quad\geq -C \int_{\{|\tilde{\xi}|\geq \frac{R}{2}\}} |\tilde{\xi}|^{-\nu} \int_0^{2\pi} |\varphi(\kappa|\tilde{\xi}|, \beta_1, \tilde{\xi})|^2 \,\mathrm{d}\beta_1 \,\mathrm{d}\tilde{\xi} \\ &\qquad - C \int_{\{|\tilde{q}|\geq \eta\}} |\tilde{q}|^{-\nu} \int_0^{2\pi} |\varphi(\tilde{q}, \kappa^{-1}|\tilde{q}|, \beta_2)|^2 \,\mathrm{d}\beta_2 \,\mathrm{d}\tilde{q}.\end{aligned} \tag{6.65}$$

In the set $\{(|\tilde{q}|, |\tilde{\xi}|) \in \mathbb{R}_+ \times \mathbb{R}_+\}$ we introduce the polar coordinates (ρ, θ), where $\rho^2 = |\tilde{q}|^2 + |\tilde{\xi}|^2 = |x|_m^2$ and $\theta = \arctan\left(\frac{|\tilde{q}|}{|\tilde{\xi}|}\right) \in [0, \frac{\pi}{2})$. Then $\rho_1 = \rho\sin(\theta)$ and $\rho_2 = \rho\cos(\theta)$. We represent the function $\varphi(x)$ as a function $\tilde{\varphi}(\rho, \theta, \beta_1, \beta_2)$. Note that the integrals on the r.h.s of (6.65) are integrals of the function $|\varphi(x)|^2$ over the set where $|\tilde{q}| = \kappa|\tilde{\xi}|$, i.e. where $\theta_0 = \arctan(\kappa)$. Therefore, for the first integral on the r.h.s of (6.65) we get

$$\begin{aligned}&\int_{\{|\tilde{\xi}|\geq \frac{R}{2}\}} |\tilde{\xi}|^{-\nu} \int_0^{2\pi} |\varphi(\kappa|\tilde{\xi}|, \beta_1, \tilde{\xi})|^2 \,\mathrm{d}\beta_1 \,\mathrm{d}\tilde{\xi} \\ &= \int_{\frac{R}{2}}^{\infty} \int_0^{2\pi} \int_0^{2\pi} \rho_2^{-\nu} |\varphi(\kappa\rho_2, \beta_1, \rho_2, \beta_2)|^2 \,\mathrm{d}\beta_1 \,\mathrm{d}\beta_2 \,\rho_2 \mathrm{d}\rho_2, \\ &= c\int_R^{\infty} \int_0^{2\pi} \int_0^{2\pi} \rho^{-\nu} |\tilde{\varphi}(\rho, \theta_0, \beta_1, \beta_2)|^2 \,\mathrm{d}\beta_1 \,\mathrm{d}\beta_2 \,\rho \mathrm{d}\rho,\end{aligned} \tag{6.66}$$

where $c > 0$ is a constant which comes from the transformation of variables if we represent the function $\rho_2 \mapsto \varphi(\kappa\rho_2, \beta_1, \rho_2, \beta_2)$ as function $\rho \mapsto \tilde{\varphi}(\rho, \theta_0, \beta_1, \beta_2)$, where $\theta_0 = \arctan(\kappa)$. In the first equality in (6.66) we used that $\dim(X_c(Z)) = 2$, which implies that the Jacobian of the transformation to polar coordinates in $X_c(Z)$ gives a factor ρ_2. In the last equality of (6.66) we used that the function $\tilde{\varphi}$ is zero for $\rho < R$. Similarly we get

$$\begin{aligned}&\int_{\{|\tilde{q}|\geq\eta\}} |\tilde{q}|^{-\nu} \int_0^{2\pi} |\varphi(\tilde{q}, \kappa^{-1}|\tilde{q}|, \beta_2)|^2 \,\mathrm{d}\beta_2 \,\mathrm{d}\tilde{q} \\ &= c' \int_R^\infty \int_0^{2\pi} \int_0^{2\pi} \rho^{-\nu} |\tilde{\varphi}(\rho, \theta_0, \beta_1, \beta_2)|^2 \,\mathrm{d}\beta_1 \,\mathrm{d}\beta_2 \,\rho \mathrm{d}\rho\end{aligned} \tag{6.67}$$

for some $c' > 0$. Therefore, by combining (6.66) and (6.67) with (6.65) we obtain

$$\begin{aligned}&\int_{K_R(Z,\kappa)} \left(|\nabla_0\varphi|^2 + V|\varphi|^2 - \varepsilon|x|_m^{-4}|\varphi|^2\right) \,\mathrm{d}x \\ &\qquad \geq -C \int_R^\infty \int_0^{2\pi} \int_0^{2\pi} \rho^{1-\nu} |\tilde{\varphi}(\rho, \theta_0, \beta_1, \beta_2)|^2 \,\mathrm{d}\beta_1 \,\mathrm{d}\beta_2 \,\mathrm{d}\rho\end{aligned} \tag{6.68}$$

for some $C > 0$. Now as in the proof of Theorem 6.1 we estimate the integral on the r.h.s. of (6.68), which is an integral over the edge of $K(Z, \kappa)$ given by $\{|\tilde{q}| = \kappa|\tilde{\xi}|\}$, by an integral over the set $K(Z, \kappa, \kappa')$ for some κ' which is slightly larger than κ. For this purpose let $\kappa' > \kappa$ be so small that the cones $K_R(Z, \kappa')$ and $K_R(Z', \kappa')$ do not overlap for partitions $Z \neq Z'$ with $|Z| = |Z'| = 2$ and let $\theta_1 = \arctan(\kappa')$. We apply Lemma 6.4 to the function $\varphi(\rho, \cdot, \theta_1, \theta_2)$ for fixed ρ, θ_1, θ_2 and with $b_1 = \theta_0,\ b_2 = \theta_1$. Then we get

$$\begin{aligned}&\int_R^\infty \int_0^{2\pi} \int_0^{2\pi} \rho^{1-\nu} |\tilde{\varphi}(\rho, \theta_0, \beta_1, \beta_2)|^2 \,\mathrm{d}\beta_1 \,\mathrm{d}\beta_2 \,\mathrm{d}\rho \\ &\leq C(\theta_0, \theta_1) \int_R^\infty \int_0^{2\pi} \int_0^{2\pi} \int_{\theta_0}^{\theta_1} \rho^{1-\nu} \left(|\tilde{\varphi}(\rho, \theta, \beta_1, \beta_2)|^2 + |\partial_\theta \tilde{\varphi}(\rho, \theta, \beta_1, \beta_2)|^2\right) \,\mathrm{d}\theta \,\mathrm{d}\beta_1 \,\mathrm{d}\beta_2 \,\mathrm{d}\rho,\end{aligned} \tag{6.69}$$

where $C(\theta_0, \theta_1)$ depends on θ_0 and θ_1 only. Using the scalar form of the four-dimensional Hardy inequality [8, eq. (2.15)] we obtain

$$\int_R^\infty \rho^{1-\nu} |\tilde{\varphi}(\rho, \theta, \beta_1, \beta_2)|^2 \,\mathrm{d}\rho \leq \int_R^\infty \rho^{3-\nu} |\partial_\rho \tilde{\varphi}(\rho, \theta, \beta_1, \beta_2)|^2 \,\mathrm{d}\rho. \tag{6.70}$$

Therefore, we get

$$\begin{aligned}&\int_R^\infty \rho^{1-\nu} \left(|\tilde{\varphi}(\rho, \theta, \beta_1, \beta_2)|^2 + |\partial_\theta \tilde{\varphi}(\rho, \theta, \beta_1, \beta_2)|^2\right) \,\mathrm{d}\rho \\ &\qquad \leq R^{-\nu} \int_R^\infty \rho^3 \left(|\partial_\rho \tilde{\varphi}(\rho, \theta, \beta_1, \beta_2)|^2 + \frac{|\partial_\theta \tilde{\varphi}(\rho, \theta, \beta_1, \beta_2)|^2}{\rho^2}\right) \,\mathrm{d}\rho.\end{aligned} \tag{6.71}$$

Recall that (ρ, θ) are the polar coordinates corresponding to $(|\tilde{q}|, |\tilde{\xi}|)$, which implies

$$\left(|\partial_\rho \tilde{\varphi}(\rho, \theta, \beta_1, \beta_2)|^2 + \frac{|\partial_\theta \tilde{\varphi}(\rho, \theta, \beta_1, \beta_2)|^2}{\rho^2}\right) = |\partial_{|\tilde{q}|}\varphi|^2 + |\partial_{|\tilde{\xi}|}\varphi|^2 \leq |\nabla_0\varphi|^2. \tag{6.72}$$

This yields

$$\begin{aligned}&R^{-\nu} \int_R^\infty \int_0^{2\pi} \int_0^{2\pi} \int_{\theta_0}^{\theta_1} \rho^3 \left(|\partial_\rho \tilde{\varphi}(\rho, \theta, \beta_1, \beta_2)|^2 + \frac{|\partial_\theta \tilde{\varphi}(\rho, \theta, \beta_1, \beta_2)|^2}{\rho^2}\right) \,\mathrm{d}\theta \,\mathrm{d}\beta_1 \,\mathrm{d}\beta_2 \,\mathrm{d}\rho \\ &\leq \varepsilon \int_{K_R(Z,\kappa,\kappa')} |\nabla_0\varphi|^2 \,\mathrm{d}x,\end{aligned} \tag{6.73}$$

where $\varepsilon > 0$ can be chosen arbitrarily small if $R > 0$ is sufficiently large. Here we used that the Jacobian of the transformation from the coordinates $x = (\tilde{q}_1, \tilde{q}_2, \tilde{\xi}_1, \tilde{\xi}_2)$ to the variables $(\rho, \theta, \beta_1, \beta_2)$ is given by $\rho^3 \sin(\theta) \cos(\theta)$ and we can estimate

$$0 < \sin(\theta_0) \cos(\theta_0) \leq \sin(\theta) \cos(\theta) \tag{6.74}$$

for any $\theta \in (\theta_0, \theta_1)$ if $0 < \kappa < \kappa' < 1$. Combining (6.73) with (6.71), (6.69) and (6.68) we get

$$\int_{K_R(Z,\kappa)} \left(|\nabla_0 \varphi|^2 + V|\varphi|^2 - \varepsilon |x|_m^{-4} |\varphi|^2\right) \mathrm{d}x \geq -\varepsilon \int_{K_R(Z,\kappa,\kappa')} |\nabla_0 \varphi|^2 \, \mathrm{d}x. \tag{6.75}$$

This inequality is an analogue to (6.38) in the proof of Theorem 6.1. Now we can complete the proof of Theorem 6.5 by repeating the same steps as in the proof of Theorem 6.1 if we replace the scalar form of the two-dimensional Hardy inequality by the scalar form of the four-dimensional one. □

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Appendix A. Properties of the space $\tilde{H}^1(\mathbb{R}^d)$

Here we collect some properties of the space $\tilde{H}^1(\mathbb{R}^d)$ for dimensions $d = 1$ and $d = 2$. These spaces were introduced M. Birman in [8, Section 2] and are intensively discussed in the book [12] by R. Frank, A. Laptev and T. Weidl. For convenience we give the statements and some of these properties below.

Proposition A.1 (Properties of $\tilde{H}^1(\mathbb{R}^d)$, $d = 1, 2$)**.** *The following assertions hold.*

(i) *(Hardy's inequality for the half-line, inequality* (2.17) *in* [8]*) Let $d = 1$ and $u \in \tilde{H}^1(\mathbb{R})$, such that $\liminf_{t\to 0} |u(t)| = 0$. Then*

$$\int_0^\infty \frac{|u(t)|^2}{t^2} \, \mathrm{d}t \leq 4 \int_0^\infty |u'(t)|^2 \, \mathrm{d}t. \tag{A.1}$$

(ii) *(Two-dimensional Hardy inequality, inequality* (2) *in* [29]*) Let $d = 2$ and assume that $u \in \tilde{H}^1(\mathbb{R}^2)$, represented in polar coordinates (r, θ), satisfies*

$$\int_0^{2\pi} u(1, \theta) \, \mathrm{d}\theta = 0, \tag{A.2}$$

where $u(1, \theta)$ is understood in the trace sense. Then

$$\int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2(1 + \ln^2(|x|))} \, \mathrm{d}x \leq 4 \int_{\mathbb{R}^2} |\nabla u|^2 \, \mathrm{d}x. \tag{A.3}$$

(iii) *Let $d = 1$. Then there exists a constant $C > 0$, such that for all functions $u \in \tilde{H}^1(\mathbb{R})$*

$$\int_{-\infty}^{\infty} \frac{|u(x)|^2}{1 + x^2} \, \mathrm{d}x \leq C \|u\|^2_{\tilde{H}^1}. \tag{A.4}$$

(iv) *Let $d = 2$. Then there exists a constant $C > 0$, such that for all functions $u \in \tilde{H}^1(\mathbb{R}^2)$*

$$\int_{\mathbb{R}^2} \frac{|u(x)|^2}{1 + x^2(\ln|x|)^2} \, \mathrm{d}x \leq C \|u\|^2_{\tilde{H}^1}. \tag{A.5}$$

(v) *Let $u \in \tilde{H}^1(\mathbb{R}^d)$ and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\tilde{H}^1(\mathbb{R}^d)$, such that $u_n \rightharpoonup u$ weakly in $\tilde{H}^1(\mathbb{R}^d)$. Then for every measurable bounded set $B \subset \mathbb{R}^d$ we have $\chi_B u_n \to \chi_B u$ in $L^2(\mathbb{R}^d)$.*

Proof. **(i)** We borrow the proof of this inequality from the book [12], which is currently in preparation. We are grateful to R. Frank, A. Laptev and T. Weidl for sharing it with us before publishing.
By applying twice the product rule for weakly differentiable functions we get

$$\left|u' - \frac{1}{2x}u\right|^2 = |u'|^2 + \frac{1}{4x^2}|u|^2 - \frac{1}{2x}\left(|u|^2\right)' = |u'|^2 - \frac{1}{4x^2}|u|^2 - \left(\frac{1}{2x}|u|^2\right)'. \tag{A.6}$$

Hence, for fixed $0 < \varepsilon < M < \infty$ we have

$$\begin{aligned} 0 \le \int_\varepsilon^M \left|u' - \frac{1}{2x}u\right|^2 \mathrm{d}x = \int_\varepsilon^M |u'|^2 \,\mathrm{d}x - \int_\varepsilon^M \frac{1}{4x^2}|u|^2 \,\mathrm{d}x \\ - \frac{1}{2M}\left(|u(M)|^2\right) + \frac{1}{2\varepsilon}\left(|u(\varepsilon)|^2\right). \end{aligned} \tag{A.7}$$

Note that

$$|u(x)|^2 \le x \int_0^x |u'(y)|^2 \,\mathrm{d}y. \tag{A.8}$$

Indeed, we have

$$\begin{aligned} |u(x) - u(x')| &= \left|\int_{x'}^x u'(y)\,\mathrm{d}y\right| \\ &\le \left(\int_{x'}^x |u'(y)|^2 \,\mathrm{d}y\right)^{\frac{1}{2}} \left(\int_{x'}^x 1 \,\mathrm{d}y\right)^{\frac{1}{2}} \le \left(\int_0^x |u'(y)|^2 \,\mathrm{d}y\right)^{\frac{1}{2}} x^{\frac{1}{2}}. \end{aligned} \tag{A.9}$$

Now (A.8) follows from the assumption that $\liminf_{x'\to 0} |u(x')| = 0$.
Now we take inequality (A.7) and let $M \to \infty$. Note that the first integral on the r.h.s. of (A.7) converges as $M \to \infty$, because $u' \in L^2(0,\infty)$. By (A.8) we get $\sup_M |u(M)|M^{-1} < \infty$. Therefore,

$$\int_\varepsilon^\infty |u|^2 x^{-2} \,\mathrm{d}x < \infty. \tag{A.10}$$

The finiteness of this integral and the fact that $x \mapsto x^{-1}$ is not integrable implies

$$\liminf_{x\to\infty} |u(x)|^2 x^{-1} = 0. \tag{A.11}$$

Choosing M in (A.7) along a sequence where this lim inf is realized we obtain

$$0 \le \int_\varepsilon^\infty |u'|^2 \,\mathrm{d}x - \int_\varepsilon^\infty \frac{1}{4x^2}|u|^2 \,\mathrm{d}x + \frac{1}{2\varepsilon}\left(|u(\varepsilon)|^2\right). \tag{A.12}$$

Now we let $\varepsilon \to 0$. Since $u' \in L^2(0,\infty)$, the first integral on the r.h.s. of (A.12) converges. By the same argument, together with (A.8) we get $\lim_{\varepsilon\to 0} |u(\varepsilon)|^2\varepsilon^{-1} = 0$. This shows that the second integral on the r.h.s. of (A.12) also converges and we have

$$0 \le \int_0^\infty |u'|^2 \,\mathrm{d}x - \int_0^\infty \frac{1}{4x^2}|u|^2 \,\mathrm{d}x. \tag{A.13}$$

(ii) The proof can be found in [29].
(iii) We take a smooth function ξ with $0 \le \xi \le 1$ on $\mathbb{R}_+$, $\xi = 0$ on $(0, 1/2]$ and $\xi = 1$ on $[1,\infty)$. Then

$$\int_0^\infty \frac{|u|^2}{1+x^2} \,\mathrm{d}x \le 2\int_0^\infty (1-\xi)^2 \frac{|u|^2}{1+x^2} \,\mathrm{d}x + 2\int_0^\infty \xi^2 \frac{|u|^2}{1+x^2} \,\mathrm{d}x. \tag{A.14}$$

The first term on the r.h.s. of (A.14) can be controlled by

$$2\int_0^\infty (1-\xi)^2 \frac{|u|^2}{1+x^2}\,\mathrm{d}x \le 2 \sup_{0\le s\le 1} \frac{1}{1+s^2}\int_0^1 |u|^2\,\mathrm{d}x = 2\int_0^1 |u|^2\,\mathrm{d}x. \tag{A.15}$$

The second term on the r.h.s. of (A.14) can be estimated by using the Hardy inequality as

$$\begin{aligned} 2\int_0^\infty \xi^2 \frac{|u|^2}{1+x^2}\,\mathrm{d}x &\le 8\int_0^\infty |(\xi u)'|^2\,\mathrm{d}x \le 16\int_0^\infty \left(\xi^2|u'|^2 + (\xi')^2\,|u|^2\right)\,\mathrm{d}x \\ &\le C\left(\int_0^\infty |u'|^2\,\mathrm{d}x + \int_0^1 |u|^2\,\mathrm{d}x\right), \end{aligned} \tag{A.16}$$

where C depends on ξ, but not on u.
(iv) The proof is a simple modification of the proof of assertion **(iii)**.
(v) The proof follows immediately from the Rellich-Kondrachov theorem, see [2, Theorem 6.3] □

Remark. Inequality (A.3) is equivalent to the scalar inequality

$$\int_0^\infty \frac{|u(t)|^2}{t\left(1+\ln^2(t)\right)}\,\mathrm{d}t \le 4\int_0^\infty t(u'(t))^2\,\mathrm{d}t, \qquad u(1)=0. \tag{A.17}$$

Corollary A.2. *It follows from Proposition A.1* **(iv)** *and* **(v)** *that if V satisfies* (2.2) *and* (2.9)*, then there exists a constant $C>0$, such that for all $u\in \tilde H^1(\mathbb{R}^d)$ we have*

$$\int_{\mathbb{R}^d} |V(x)||u(x)|^2\,\mathrm{d}x \le C\|u\|^2_{\tilde H^1}. \tag{A.18}$$

Corollary A.3. *For any function $u\in\tilde H^1(\mathbb{R}^2)$ with $\operatorname{supp}(u)\subset\{x\in\mathbb{R}^2 : |x|<1\}$ and any constant $\nu\in(0,1)$ we have*

$$\int_{\{|x|\ge\nu\}} \frac{|u|^2}{|x|^2(1+\ln^2(|x|))}\,\mathrm{d}x \le 4\int_{\{|x|\ge\nu\}} |\nabla u|^2\,\mathrm{d}x. \tag{A.19}$$

Proof of Corollary A.3. Since the function $v(x)=|x|^{-2}\left(1+(\ln(|x|))^2\right)^{-1}$ is spherically symmetric, due to the rearrangement inequality it suffices to show that the inequality holds for spherically symmetric functions. Let $u\in\tilde H^1(\mathbb{R}^2)$ be spherically symmetric with $\operatorname{supp}(u)\subset\{x:|x|<1\}$. Then the function $\tilde u$ given by

$$\tilde u(x) = \begin{cases} u(x) & \text{if } |x|\ge\nu, \\ u(\nu) & \text{if } |x|<\nu \end{cases} \tag{A.20}$$

is also an element of $\tilde H^1(\mathbb{R}^2)$. Applying the two-dimensional Hardy inequality (A.3) to the function $\tilde u$, using that $\tilde u$ is constant for $|x|\le\nu$ and that $\tilde u$ and u coincide for $|x|\ge\nu$ proves (A.19). □

Remark. Analogously to the proof of Corollary A.3 one can see that if $d\ge 3$, then for any function $u\in\tilde H^1(\mathbb{R}^d)$ we have

$$\int_{\{|x|\ge\nu\}} \frac{|u|^2}{|x|^2}\,\mathrm{d}x \le \frac{4}{(d-2)^2}\int_{\{|x|\ge\nu\}} |\nabla u|^2\,\mathrm{d}x. \tag{A.21}$$

Appendix B. Necessary and sufficient conditions for virtual levels

In this section we prove Theorem 2.3, which is stated for one-particle Schrödinger operators in dimension one or two. Afterwards, we give an analogue of this theorem for the multi-particle case.

Proof of Theorem 2.3. We only need to prove that the absence of a virtual level of h implies that (2.16) does not hold. The proof of the other direction follows from Theorem 2.2 and the variational principle.
Let $d = 1$. Note that we can assume that $\mathcal{U}(x) = -(1 + |x|)^{-2}$. For $\psi \in H^1(\mathbb{R})$ we write

$$\psi_0(x) = \psi(x) - \psi(0). \tag{B.1}$$

Then, $\psi_0(0) = 0$ and we can apply Hardy's inequality on the half line $\mathbb{R}_+$ to obtain

$$\int_0^\infty \frac{|\psi_0(x)|^2}{|x|^2}\,\mathrm{d}x \le 4\int_0^\infty |\psi_0'(x)|^2\,\mathrm{d}x = 4\int_0^\infty |\psi'(x)|^2\,\mathrm{d}x. \tag{B.2}$$

Furthermore, we have

$$\begin{aligned}\langle V\psi,\psi\rangle &= \int_{-\infty}^{\infty} V(x)|\psi(0)|^2\,\mathrm{d}x + \int_{-\infty}^{\infty} V(x)|\psi_0(x)|^2\,\mathrm{d}x + 2\mathrm{Re}\int_{-\infty}^{\infty} V(x)\psi(0)\psi_0(x)\,\mathrm{d}x\\ &\ge \int_{-\infty}^{\infty} V(x)|\psi(0)|^2\,\mathrm{d}x + \int_{-\infty}^{\infty} V(x)|\psi_0(x)|^2\,\mathrm{d}x - 2\int_{-\infty}^{\infty} |V(x)||\psi(0)\psi_0(x)|\,\mathrm{d}x.\end{aligned} \tag{B.3}$$

Note that for any $\delta > 0$

$$2|\psi(0)\psi_0(x)| \le \delta|\psi(0)|^2 + \delta^{-1}|\psi_0(x)|^2, \tag{B.4}$$

which together with (B.3) implies

$$\begin{aligned}\langle V\psi,\psi\rangle &\ge |\psi(0)|^2\int_{-\infty}^{\infty} (V(x) - \delta|V(x)|)\,\mathrm{d}x + \int_{-\infty}^{\infty} |\psi_0(x)|^2\left(V(x) - \delta^{-1}|V(x)|\right)\mathrm{d}x\\ &\ge |\psi(0)|^2\int_{-\infty}^{\infty} (V(x) - \delta|V(x)|)\,\mathrm{d}x - (1+\delta^{-1})\int_{-\infty}^{\infty} |V(x)||\psi_0(x)|^2\,\mathrm{d}x\\ &\ge |\psi(0)|^2\int_{-\infty}^{\infty} (V(x) - \delta|V(x)|)\,\mathrm{d}x - C(1+\delta^{-1})\|\psi_0\|^2_{\tilde{H}^1},\end{aligned} \tag{B.5}$$

where in the last estimate we used Corollary A.2. Since $\psi_0(0) = 0$, we have $\|\psi_0\|^2_{\tilde{H}^1} \le C\|\psi_0'\|^2$. This, together with (B.5) yields

$$\langle V\psi,\psi\rangle \ge |\psi(0)|^2\int_{-\infty}^{\infty} (V(x) - \delta|V(x)|)\,\mathrm{d}x - C(\delta)\int_{-\infty}^\infty |\psi_0'(x)|^2\,\mathrm{d}x. \tag{B.6}$$

Since $\int_{-\infty}^\infty V(x)\,\mathrm{d}x > 0$, we can choose the constant $\delta > 0$ sufficiently small, such that $\int_{-\infty}^\infty (V(x) - \delta|V(x)|)\,\mathrm{d}x \ge \frac{1}{2}\int_{-\infty}^\infty V(x)\,\mathrm{d}x =: C_0$. This, together with $\psi'(x) = \psi_0'(x)$ implies

$$|\psi(0)|^2 \le C_0^{-1}\langle V\psi,\psi\rangle + C_1(\delta)\|\psi'\|^2 \tag{B.7}$$

for some constant $C_1(\delta) > 0$ which depends on V and δ only. This yields

$$\begin{aligned} \varepsilon_1 \int\limits_{-\infty}^{\infty} \frac{|\psi(x)|^2}{1+x^2}\,\mathrm{d}x &\le 2\varepsilon_1|\psi(0)|^2 \int\limits_{-\infty}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x + 2\varepsilon_1 \int\limits_{-\infty}^{\infty} \frac{|\psi_0(x)|^2}{1+x^2}\,\mathrm{d}x \\ &\le \varepsilon_1\left(2\pi|\psi(0)|^2 + 8\|\psi'\|^2\right) \le \varepsilon_1\left(2\pi C_0^{-1}\langle V\psi,\psi\rangle + C_2(\delta)\|\psi'\|^2\right), \end{aligned} \tag{B.8}$$

where $C_2(\delta) = C_1(\delta) + 8$. We distinguish between two cases:
(i) If $2\pi C_0^{-1}\langle V\psi,\psi\rangle < C_2(\delta)\|\psi'\|^2$, then (B.8) yields

$$\varepsilon_1 \int\limits_{-\infty}^{\infty} \frac{|\psi(x)|^2}{1+x^2}\,\mathrm{d}x \le 2\varepsilon_1 C_2(\delta)\|\psi'\|^2. \tag{B.9}$$

Now since h does not have a virtual level, we can choose $\varepsilon_1 > 0$ sufficiently small to conclude that

$$\langle h\psi,\psi\rangle - \varepsilon_1 \int_{-\infty}^{\infty} \frac{|\psi(x)|^2}{1+x^2}\,\mathrm{d}x \ge 0. \tag{B.10}$$

(ii) If $2\pi C_0^{-1}\langle V\psi,\psi\rangle \ge C_2(\delta)\|\psi'\|^2$, we have $\langle V\psi,\psi\rangle > 0$ and

$$\varepsilon_1 \int\limits_{-\infty}^{\infty} \frac{|\psi(x)|^2}{1+x^2}\,\mathrm{d}x \le 4\varepsilon_1\pi C_0^{-1}\langle V\psi,\psi\rangle. \tag{B.11}$$

By choosing $0 < \varepsilon_1 < (4\pi)^{-1}C_0$ we obtain

$$\langle h\psi,\psi\rangle - \varepsilon_1 \int\limits_{-\infty}^{\infty} \frac{|\psi(x)|^2}{1+x^2}\,\mathrm{d}x \ge \|\psi'\|^2 \ge 0. \tag{B.12}$$

This implies (B.10) and therefore the statement of Theorem 2.3 for the case $d = 1$.
Now we assume that $d = 2$. For $\psi \in H^1(\mathbb{R}^2)$ we write $\psi_0(x) = \psi(x) - a_0$, where

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} \psi(1,\theta)\,\mathrm{d}\theta. \tag{B.13}$$

Then $\int_0^{2\pi}\psi_0(1,\theta)\,\mathrm{d}\theta = 0$ and thus we can apply the two-dimensional Hardy inequality (A.3) to the function ψ_0. Proceeding as in the proof of the one-dimensional case yields the statement for $d = 2$ and therefore completes the proof of Theorem 2.3. □

Now we extend Theorem 2.3 to the case of multi-particle Schrödinger operators.

Theorem B.1. *Let H be the Schrödinger operator corresponding to a system of $N \ge 2$ one- or two-dimensional particles, where the potentials $V_{ij} \ne 0$ satisfy (2.2) and (2.9) and let $H \ge 0$. Then H has a virtual level at zero if and only if the following two assertions hold.*

(i) *There exists an $\varepsilon_0 > 0$, such that for any cluster C with $1 < |C| < N$ we have*

$$H[C] - \varepsilon_0\left(1 + |q[C]|_m^2(\ln(|q[C]|_m)^2)\right)^{-1} \ge 0. \tag{B.14}$$

(ii) *For any $\varepsilon > 0$ we have*

$$\inf \mathcal{S}\left(H - \varepsilon\left(1 + |x|_m^2\ln^2(|x|_m)\right)^{-1}\right) < 0. \tag{B.15}$$

Proof of Theorem B.1. For $N = 2$ the statement was proved in Theorem 2.3. Now assume that $N \geq 3$. First we prove that if H has a virtual level at zero, then (B.15) is true. According to remark **(ii)** after Theorem 3.5 we know that if $d = 1$, $N \geq 3$ or $d = 2$, $N \geq 4$ zero is an eigenvalue of H. Taking the corresponding eigenfunction as a trial function shows that (B.15) holds for any $\varepsilon > 0$. For $d = 2$, $N = 3$ we do not know if zero is an eigenvalue. However, by Theorem 4.1 we know that there is a function $\varphi_0 \in \tilde{H}^1(X_0)$ with $\|\nabla_0\varphi_0\|^2 + \langle V\varphi_0, \varphi_0\rangle = 0$, which yields (B.15).

In the rest of the proof we will use induction in the number of particles. Assume that the system has $N \geq 3$ particles and that the theorem holds for all system with the number of particles less or equal to $N - 1$.
If H has a virtual level and condition (B.14) does not hold, then there exists at least one cluster C in this system with $1 < |C| < N$, such that for this cluster C and all $\varepsilon > 0$ condition (B.14) does not hold as well. Among such clusters we choose one with the smallest number of particles and denote it by C_0. Then we have

$$\inf \mathcal{S}\left(H[C_0] - \varepsilon\left(1 + |q[C_0]|_m^2(\ln(|q[C_0]|_m)^2\right)^{-1}\right) < 0 \qquad \text{for any } \varepsilon > 0. \tag{B.16}$$

If C_0 consists of only two particles, then by Theorem 2.3 this condition implies that $H[C_0]$ has a virtual level at zero. However, by the remark after Definition 3.1 Hamiltonians of non-trivial clusters can not have virtual levels at zero if the Hamiltonian of the whole system has a virtual level. Therefore, C_0 must consist of at least three particles. Now since C_0 is the smallest cluster for which (B.14) does not hold for any small $\varepsilon > 0$, for each cluster $\tilde{C} \subsetneq C_0$ with $|\tilde{C}| > 1$ we have

$$H[\tilde{C}] - \varepsilon_0\left(1 + |q[\tilde{C}]|_m^2(\ln(|q[\tilde{C}]|_m))^2\right)^{-1} \geq 0 \tag{B.17}$$

for some $\varepsilon_0 > 0$. Since $|C_0| < N - 1$, (B.16), (B.17) and the induction assumption yield that $H[C_0]$ has a virtual level, which according to the remark after the Definition 3.1 contradicts the assumption that H has a virtual level.
To complete the proof of the theorem we have to show that if condition (B.14) and (B.15) are fulfilled, then H has a virtual level.
At first, we prove that condition **(i)** of Definition 3.1 is fulfilled, namely that there exists a constant $\varepsilon_0 \in (0, 1)$, such that

$$\inf \mathcal{S}_{\text{ess}}\left(H + \varepsilon_0\Delta_0\right) = 0. \tag{B.18}$$

Recall that due to the remark after Definition 3.1 to prove (B.18) it suffices to show that for any C with $1 < |C| < N$ the operator $H[C]$ does not have a virtual level. Assume for contradiction that there exists a cluster C_1 with $1 < |C_1| < N$, such that $H[C_1]$ has a virtual level at zero. Then, due to the induction assumption we have

$$\inf \mathcal{S}\left(H[C_1] - \varepsilon\left(1 + |q[C_1]|_m^2(\ln(|q[C_1]|_m))^2\right)^{-1}\right) < 0. \tag{B.19}$$

This is a contradiction to (B.14). Hence, condition **(i)** of Definition 3.1 is fulfilled.
It remains to prove that if conditions (B.14) and (B.15) of Theorem B.1 hold, then condition **(ii)** of Definition 3.1 is fulfilled, namely

$$\inf \mathcal{S}\left(H + \varepsilon\Delta_0\right) < 0 \qquad \text{for any } \varepsilon \in (0, 1). \tag{B.20}$$

If $\dim(X_0) \geq 3$, we can use Hardy's inequality to conclude that (B.20) holds. If $\dim(X_0) < 3$, i.e. the system consists of three one-dimensional particles, (B.15) implies that for any $n \in \mathbb{N}$ the operator $H - n^{-1}\left(1 + |x|_m^2(\ln(|x|_m))^2\right)^{-1}$ has a negative eigenvalue. We take a sequence of eigenfunctions ψ_n corresponding to these eigenvalues, normalized by $\|\psi_n\|_{\tilde{H}^1} = 1$. Applying the same arguments as in the proof of Theorem 3.5 we see that ψ_n converges in $L^2(X_0)$ to a function

ψ_0 which is an eigenfunction of the operator H corresponding to the eigenvalue zero. For this function we have

$$(1-\varepsilon)\|\nabla_0\psi_0\|^2 + \langle V\psi_0, \psi_0\rangle = -\varepsilon\|\nabla_0\psi_0\|^2 < 0 \tag{B.21}$$

for any $\varepsilon > 0$. This proves that condition **(ii)** of Definition 3.1 is fulfilled and completes the proof of Theorem B.1. □

Appendix C. A sufficient condition for finiteness of the discrete spectrum

In this section we give a criterion for the finiteness of the number of negative eigenvalues, which we used in the proofs of Theorem 5.1, Theorem 6.1 and Theorem 6.5. This criterion, in a slightly different form, is due to G. Zhislin and is a part of the proof of the main result in [40]. For the convenience of the reader we give it here.

Lemma C.1. *Let $h = -\Delta + V$ in $L^2(\mathbb{R}^k)$, $k \in \mathbb{N}$, where V satisfies (2.2). Assume there exist constants $\beta, \varepsilon, b > 0$, such that*

$$\langle h\psi, \psi\rangle - \varepsilon\langle |x|^{-\beta}\psi, \psi\rangle \geq 0 \tag{C.1}$$

holds for any $\psi \in H^1(\mathbb{R}^k)$ with $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^k,\ |x| \geq b\}$. Then the following assertions hold.

(i) $\inf \mathcal{S}_{\text{ess}}(h) \geq 0$.
(ii) *The operator h has at most a finite number of negative eigenvalues.*
(iii) *Zero is not an infinitely degenerate eigenvalue of h.*

To prove Lemma C.1 we use the following

Lemma C.2. *Assume that V satisfies (2.2). Let $\beta > 0$, $\varepsilon > 0$ and $\tilde{b} > b > 0$. Then there exist a constant $C(\varepsilon, \beta)$ and a function $\chi_1 \in C^1(\mathbb{R}^k)$, $0 \leq \chi_1 \leq 1$, with*

$$\chi_1(x) = \begin{cases} 1, & |x| \leq b, \\ 0, & |x| \geq \tilde{b}, \end{cases} \tag{C.2}$$

such that for all $\psi \in H^1(\mathbb{R}^k)$ we have

$$\langle h\psi, \psi\rangle \geq \langle h\psi\chi_1, \psi\chi_1\rangle - C(\varepsilon, \beta)\|\psi\chi_1\|^2 + \langle h\psi\chi_2, \psi\chi_2\rangle - \varepsilon\||x|^{-\beta}\psi\chi_2\|^2_{\{b\leq|x|\leq\tilde{b}\}}, \tag{C.3}$$

where $\chi_2 = \sqrt{1-\chi_1^2}$.

Proof of Lemma C.2. Let $\beta, \varepsilon > 0$ and $b, \tilde{b} > 0$ with $\tilde{b} > b$ be fixed. Furthermore, let $u : \mathbb{R}_+ \to [0,1]$ be a C^1-function, such that $u(t) = 1$, $t \leq b$ and $u(t) = 0$, $t \geq \tilde{b}$. We assume that u is strictly monotonically decreasing on $(b, \tilde{b})$. Let $v = \sqrt{1-u^2}$. We choose u in such a way that $v'(t)(1-v^2(t))^{-\frac{1}{2}} \to 0$ as $t \to \tilde{b}_-$. For $x \in \mathbb{R}^k$ let

$$\chi_1(x) = u(|x|), \qquad \chi_2(x) = v(|x|). \tag{C.4}$$

Then we have

$$|\nabla\chi_1|^2 + |\nabla\chi_2|^2 = \frac{|\nabla\chi_2|^2}{1-\chi_2^2} = \frac{(v'(|x|))^2}{1-v^2(|x|)}. \tag{C.5}$$

Since $v'(|x|)(1-v^2(|x|))^{-\frac{1}{2}} \to 0$ as $|x| \to \tilde{b}_-$ and $v(|x|)$ is close to one in a vicinity of $|x| = \tilde{b}$, we can choose b' so close to $\tilde{b}$ that

$$\frac{(v'(|x|))^2}{1-v^2(|x|)} \leq \varepsilon v^2(|x|)|x|^{-\beta}, \qquad b' \leq |x| \leq \tilde{b}. \tag{C.6}$$

This, together with (C.5) implies

$$|\nabla\chi_1|^2 + |\nabla\chi_2|^2 \leq \varepsilon\chi_2^2(x)|x|^{-\beta}, \qquad b' \leq |x| \leq \tilde{b}. \tag{C.7}$$

Now we estimate $|\nabla\chi_1|^2 + |\nabla\chi_2|^2$ for $b \le |x| \le b'$. Recall that $u(t) > u(b') > 0$ for $b < t < b'$. Hence, we get

$$\frac{(v'(|x|))^2}{1 - v^2(|x|)} \le Cu^2(|x|)|x|^{-\beta}, \qquad b \le |x| \le b' \tag{C.8}$$

for some $C > 0$ which depends on b' (which itself depends on ε and β). Due to the IMS formula we have

$$\langle h\psi, \psi\rangle = \langle h\psi\chi_1, \psi\chi_1\rangle + \langle h\psi\chi_2, \psi\chi_2\rangle - \int \left(|\nabla\chi_1|^2 + |\nabla\chi_2|^2\right)|\psi|^2 \, \mathrm{d}x. \tag{C.9}$$

This, together with (C.7) and (C.8) completes the proof of Lemma C.2. □

Now we turn to the

Proof of Lemma C.1. We construct a finite-dimensional subspace $M \subset L^2(\mathbb{R}^k)$, such that $\langle h\psi, \psi\rangle > 0$ holds for any $\psi \in H^1(\mathbb{R}^k)$, $\psi \neq 0$ which is orthogonal to M. Let $\varepsilon, \beta, b > 0$, such that (C.1) is fulfilled. Let χ_1 and χ_2 be functions according to Lemma C.2. Then by assumption of the lemma for any function $\psi \in H^1(\mathbb{R}^k)$

$$\begin{aligned} \langle h\psi, \psi\rangle &\ge \langle h\psi\chi_1, \psi\chi_1\rangle - C(\varepsilon,\beta)\|\psi\chi_1\|^2 + \langle h\psi\chi_2, \psi\chi_2\rangle - \varepsilon\||x|^{-\beta}\psi\chi_2\|^2_{\{b \le |x| \le \tilde{b}\}} \\ &\ge \langle h\psi\chi_1, \psi\chi_1\rangle - C(\varepsilon,\beta)\|\psi\chi_1\|^2, \end{aligned} \tag{C.10}$$

because $\operatorname{supp}(\chi_2) \subset \{x \in \mathbb{R}^k : |x| \ge b\}$. Thus, to prove statements **(i)-(iii)** it suffices to show that

$$\langle h\psi\chi_1, \psi\chi_1\rangle - C(\varepsilon,\beta)\|\psi\chi_1\|^2 \ge 0 \tag{C.11}$$

holds for any function $\psi \in H^1(\mathbb{R}^k)$ with $\psi \perp M$ (in $L^2(\mathbb{R}^k)$) for some finite-dimensional space $M \subset H^1(\mathbb{R}^k)$. By condition (2.2) we get

$$\langle h\psi\chi_1, \psi\chi_1\rangle - C(\varepsilon,\beta)\|\psi\chi_1\|^2 \ge (1-\varepsilon)\|\nabla(\psi\chi_1)\|^2 - C'(\varepsilon,\beta)\|\psi\chi_1\|^2 \tag{C.12}$$

for some $C'(\varepsilon,\beta) > 0$. For $l \in \mathbb{N}$ let

$$M_l := \{\varphi_1\chi_1, \ldots, \varphi_l\chi_1\}, \tag{C.13}$$

where $\{\varphi_1, \ldots, \varphi_l\}$ is an orthonormal set of eigenfunctions corresponding to the l lowest eigenvalues of the Laplacian, acting on $L^2\left(\{|x| \le \tilde{b}\}\right)$ with Dirichlet boundary conditions. For $\psi \perp M_l$ we have $\psi\chi_1 \perp \varphi_1, \ldots \varphi_l$, which for sufficiently large l implies

$$\|\nabla(\psi\chi_1)\|^2 \ge (1-\varepsilon)^{-1} C'(\varepsilon,\beta)\|\psi\chi_1\|^2. \tag{C.14}$$

Therefore, we conclude $L[\psi\chi_1] > 0$. This proves statements **(i)-(iii)** of Lemma C.1. □

Data availability statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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