# COLLINEAR AND SOFT DYNAMICS IN PERTURBATIVE QUANTUM CHROMODYNAMICS

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| Referent: | Prof. Dr. Kirill Melnikov |
| Korreferent: | Prof. Dr. Matthias Steinhauser |

# ABSTRACT

In this thesis we compute several important ingredients for to the next-to-next-to-next-to leading order (N3LO) zero-jettiness slicing scheme that is used for providing fully differential cross-sections at N3LO in perturbative quantum chromodynamics.

In the first part of the thesis we calculate next-to-next-to leading order (NNLO) beam functions through higher orders in the dimensional regularization parameter $\epsilon$ as required for the renormalization of the N3LO beam function. We build upon these results to calculate the so-called real-virtual (RV) contributions to the N3LO beam function.

In the second part of the thesis we concentrate on developing new methods to calculate the zero-jettiness soft function. This computation is complicated because of Heaviside functions that are needed to define jettiness. In order to enable the calculation, we develop two different approaches to allow for the application of integration-by-parts identities to integrals with Heaviside functions. We use these two approaches to compute the NNLO soft function to higher orders in $\epsilon$ as required for the renormalization of the N3LO soft function. Finally, we apply one of these approaches to study the N3LO zero-jettiness function and derive the first N3LO contribution to the zero-jettiness soft function, the triple gluon same-hemisphere emission.

# ZUSAMMENFASSUNG

In dieser Dissertation berechnen wir mehrere wichtige Beiträge für ein next-to-next-to-next-to leading order (N3LO) zero-jettiness Slicing-Schema das verwendet wird um vollständig differentielle Streuquerschnitte auf N3LO in störungstheoretischer Quantenchromodynamik zu berechnen.

Im ersten Teil dieser Dissertation berechnen wir next-to-next-to leading order (NNLO) Beamfunktionen zu höheren Ordnungen in dem dimensionalen Regularisierungsparameter $\epsilon$, da diese für die Renormierung der N3LO Beamfunktion benötigt werden. Aufbauend auf diesem Ergebnis berechnen wir sogenannte real-virtual (RV) Beiträge zur N3LO Beamfunktion.

Im zweiten Teil dieser Dissertation konzentrieren wir uns auf die Entwicklung von Methoden zur Berechnung der Softfunktion, diese wird durch Heavisidefunktionen, die benötigt werden um jettiness zu definieren, erschwert. Um die Berechnung zu ermöglichen, entwickeln wir zwei verschiedene Methoden welche die Anwendung von integration-by-parts Identitäten auf Integrale mit Heaviside-Funktionen ermöglichen. Wir verwenden beide Methoden um die NNLO Softfunktion zu höheren Ordnungen in $\epsilon$ zu berechnen, da dieses Ergebnis für die Renormierung der N3LO Softfunktion benötigt wird. Schließlich verwenden wir eine der

beiden Methoden um erste Teil-Ergebnisse der N3LO Softfunktion zu bestimmen, nämlich die Emission dreier Gluonen in dieselbe Hemisphäre.

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# CONTENTS

# LIST OF FIGURES



# LIST OF TABLES

# INTRODUCTION

# 1 INTRODUCTION

## 1.1 OVERVIEW

The fundamental physical interactions are currently best described within the framework of quantum field theory as encapsulated in the Standard Model of Particle Physics (SM). While the SM keeps being confirmed at particle colliders and else-where it is understood that it can not be fully correct up to arbitrary high energy scales. Indeed, the SM is not able to describe several physical effects observed and verified. Besides not including the fundamental force of gravity, it also fails to describe cosmological observations that require the existence of dark matter, the observed imbalance between matter and antimatter in the universe and a few other things.

To remedy these shortcomings many theories that go beyond the Standard Model (BSM) have been proposed over the years. However, none of these theories has so far been confirmed by a direct measurement at a particle collider or in other laboratory experiments. A lead to what a correct BSM theory might look like is expected to come from a disagreement between experimental measurement and theory prediction at the Large Hadron Collider (LHC).

Searching for deviations between theory and experiment has led to great efforts in both communities to increase the precision of experimental measurements and theoretical predictions. Theoretical efforts are complicated by the fact that the LHC is a proton-proton collider and protons are composite particles which cannot be described in perturbative quantum chromodynamics (pQCD). However, theoretical predictions within pQCD are made possible by the collinear factorization framework [1, 2]

$$\sigma^{\text{had}}_{pp\to X} = \sum_{i,j} \int_0^1 \mathrm{d}x_i \int_0^1 \mathrm{d}x_j f_{i/P}(x_i) f_{j/P}(x_j) \sigma_{ij\to X}(p_i, p_j) \left[1 + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{Q}\right)\right]. \tag{1.1.1}$$

In Eq. (1.1.1) $\sigma^{\text{had}}_{pp\to X}$ is the hadronic cross section for the production of the final state $X$ in a proton-proton collision. The $f_{i/P}(x_i)$ are the parton distribution functions (PDFs) that describe the probability to find a parton $i$ with momentum fraction $x_i$ in the proton. Furthermore, $\sigma_{ij\to X}(p_i, p_j)$ is the partonic cross section for the production of the final state $X$ in the collision of partons $i$ and $j$. Finally, $\Lambda_{\text{QCD}} \sim 0.3$ GeV is the scale of non-perturbative QCD and $Q$ is the momentum transfer of the hard scattering process. Thus, at a momentum transfer of $Q \sim 30$ GeV non-perturbative effects are at the percent level so that neglecting non-perturbative contributions in Eq. (1.1.1) provides a good description of proton-proton collisions. A more precise theoretical prediction for proton-proton scattering requires better knowledge of both PDFs and the partonic cross section.

While the PDFs are non-perturbative quantities that need to be extracted from experiment, the partonic cross section $\sigma_{ij\to X}(p_i, p_j)$ can be calculated in pQCD. A better theoretical description of the latter quantity requires the calculation of virtual and real emission contributions to higher orders in the strong coupling constant $\alpha_s$.

These computations are encumbered by an interplay of singularities that appear in both real and virtual emission corrections. Virtual corrections contain so-called loop integrals which may introduce ultra-violet (UV) and infra-red (IR) divergences. The UV divergences are re-absorbed into physical parameters by the process of renormalization. The IR poles stemming from virtual corrections cancel against similar poles from real emissions and collinear counter terms required for the PDF renormalization.

Experimental measurements introduce selection requirements for final states (so-called fiducial cuts) to account for detector limitations and discrimination of backgrounds. Therefore, to compare theory predictions to experimental data, computations of cross sections need to be performed differentially in kinematic observables.

One such comparison for the production and decay of a Higgs boson in proton-proton collisions at a center of mass energy of 13 TeV and in the diphoton decay channel $pp \to H \to \gamma\gamma$ was reported in Ref. [3]. Kinematic cuts were imposed on the final state photons such that the squared four momentum of the diphoton pair is in the range 105 GeV $< p^2_{\gamma\gamma} <$ 160 GeV and the inclusive cross section as well as the Higgs boson $p_T$ distribution were measured for this fiducial region. The reference found good agreement between the measured inclusive cross section $\sigma_{\text{exp}} = [132 \pm 10 \text{ stat } \pm 8 \text{ sys }]$ [fb] and the SM prediction $\sigma_{\text{theo}} = [126 \pm 7\ ]$ [fb]. The comparison for the Higgs boson transverse momentum distribution is shown in Fig. 1.1 and generally also shows good agreement.

Theoretical predictions used for the analysis in Ref. [3] are fully differential with respect to any infra-red safe kinematic feature of the Higgs boson production process at next-to-next-to leading order (NNLO) in QCD. Since the statistical and systematic accuracy of the experimental measurement will be improved with the collection of more data, theoretical predictions for the fully differential cross section will need to be extended to next-to-next-to-next-to leading order (N3LO).

In the context of such computations, the phase-space integrations need to be carried out numerically as fiducial cuts may be complicated functions of the final state momenta. To allow for such a numeric integration a systematical approach to extract and cancel IR divergences is required. This is the focus of current efforts for fully-differential cross sections at N3LO in QCD. We explain this situation in more detail, in the next section.

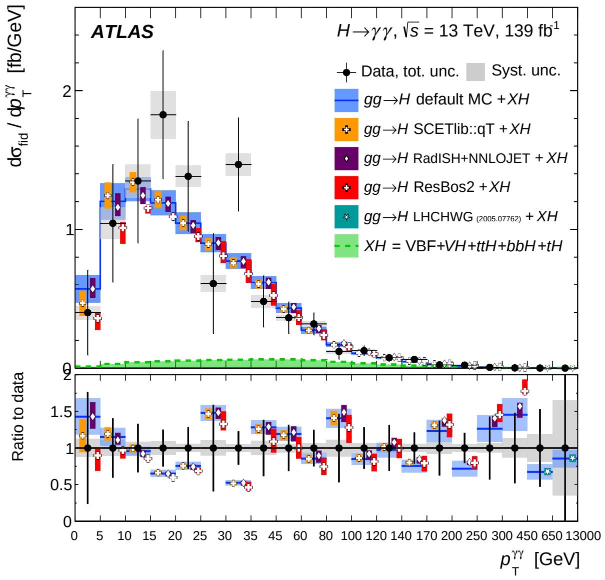


**Figure 1.1:** Differential cross section for $pp \to H \to \gamma\gamma$ extracted from Ref. [3]. The measured cross section is compared to different SM predictions obtained with different programs that provide a fully differential cross section for $pp \to H$. "Total uncertainties are indicated by the error bars on the data points, while the systematic uncertainties are indicated by the boxes. The uncertainties in the predictions are indicated with shaded bands. The bottom panel shows the predicted values from the top panel divided by data."(Ref. [3],page 29,figure 8.)

## 1.2 SOFT AND COLLINEAR DYNAMICS

To understand how IR divergences appear in real-emission processes consider the example of a massless incoming quark with momentum $p$ emitting a gluon with momentum $k$, as illustrated in Fig. 1.2. The associated propagator with the intermediate quark in Fig. 1.2 reads

$$\frac{1}{(k-p)^2} = \frac{1}{-2k\cdot p} = \frac{1}{-2E_g E_q\left[1-\cos\theta_{g,q}\right]}, \tag{1.2.1}$$

where $E_g$ ($E_q$) is the energy of the gluon (quark) and $\theta_{g,q}$ is the relative angle between the initial state quark and the final state gluon. From Eq. (1.2.1) we can infer that the integration over the gluon final state phase space $d^3k_g$ becomes singular in the two cases for which the

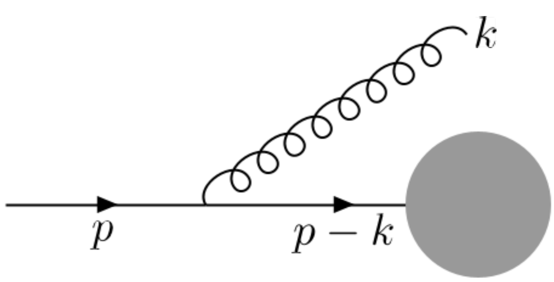


**Figure 1.2:** Massless incoming quark emitting a final state gluon. The gray circle represents an arbitrary process.

propagator $1/(p-k)^2$ becomes singular. The first case appears when the gluon is emitted with zero energy $E_g = 0$, the so-called soft limit. The second case is given by $\theta_{g,q} = 0$, when the gluon is emitted in the same direction as the quark; this is the so-called collinear limit.

While both of these singularities are regularized in dimensional regularization by integrating in $d = 4 - 2\epsilon$ dimensions $\mathrm{d}^3 k_g \to \mathrm{d}^{d-1} k_g$, this makes a numerical evaluation required for complicated observables impossible. The situation can be best illustrate with a toy example. Imagine that we want to calculate the following integral

$$I = \lim_{\epsilon\to 0}\left[\int_0^1 \mathrm{d}x\, \frac{f(x)}{x^{1-\epsilon}} - \frac{f(0)}{\epsilon}\right], \tag{1.2.2}$$

where $f(x)$ is a complicated function but finite everywhere in the integration domain and the dependence of the integrand on $\epsilon$ is similar to the one of phase-space integrals in dimensional regularization. To integrate the first term in Eq. (1.2.2) numerically we first need to set $\epsilon = 0$. However, in this case the integrand behaves like $\sim x^{-1}$ around $x = 0$ and we obtain a logarithmic divergence. Furthermore the $\epsilon \to 0$ limit is directly prohibited in the second term of Eq. (1.2.2), preventing any numeric integration. To solve this problem, the divergence at $x = 0$ needs to be extracted and cancelled against the second term, prior to any integration. This can be achieved by two different methods, subtraction and slicing.

### *Subtraction Methods*

Since we know that the integrand in Eq. (1.2.2) is only singular at $x = 0$, we can simply subtract and add back the function at $f(x)$ at $x = 0$. We obtain

$$\begin{aligned} I &= \lim_{\epsilon\to 0}\left[\int_0^1 \mathrm{d}x\, \frac{f(x) - f(0) + f(0)}{x^{1-\epsilon}} - \frac{f(0)}{\epsilon}\right] \\ &= \lim_{\epsilon\to 0}\left[\int_0^1 \mathrm{d}x\, \frac{f(x) - f(0)}{x^{1-\epsilon}}\right] + \lim_{\epsilon\to 0}\left[f(0)\int_0^1 \mathrm{d}x\, \frac{1}{x^{1-\epsilon}} - \frac{f(0)}{\epsilon}\right] \\ &= \int_0^1 \mathrm{d}x\, \frac{f(x) - f(0)}{x} \end{aligned} \tag{1.2.3}$$

The last line in Eq. (1.2.3) is now finite in the $x = 0$ limit and can thus be evaluated numerically, while the explicit $1/\epsilon$ pole of the third term in the second line in Eq. (1.2.3) cancelled against the second term, the so-called subtraction term, after integration.

For NNLO QCD processes different subtraction schemes exist. Two examples are given by the antenna subtraction [4–16] and the nested soft and collinear subtractions [17–26]. While being numerically more stable than their slicing counterparts, subtraction schemes are also more involved. A subtraction scheme regulates singularities locally at the integrand level and requires precise knowledge of all singularities as well as the integration of the associated subtraction terms. For this reason slicing schemes seem more feasible at N3LO.

*Slicing Methods*

We again want to calculate the integral in Eq. (1.2.2) numerically. To isolate the divergence at $x = 0$, we split the integration into two regions around a small cut-off parameter $\delta \ll 1$. We obtain

$$I = \lim_{\epsilon\to 0}\left[\int_0^\delta \mathrm{d}x\,\frac{f(x)}{x^{1-\epsilon}} + \int_\delta^1 \mathrm{d}x\,\frac{f(x)}{x^{1-\epsilon}} - \frac{f(0)}{\epsilon}\right]. \tag{1.2.4}$$

The second term on the r.h.s. of Eq. (1.2.4) is again finite and can be evaluated numerically, the first term on the other hand can now be simplified in the $\delta \to 0$ limit. We find

$$\begin{aligned} I &= \lim_{\epsilon\to 0}\left[\left(\frac{1}{\epsilon} + \log(\delta)\right) f(0) - \frac{f(0)}{\epsilon} + \mathcal{O}(\delta,\epsilon)\right] + \int_\delta^1 \mathrm{d}x\,\frac{f(x)}{x} \\ &= \log(\delta)\, f(0) + \int_\delta^1 \mathrm{d}x\,\frac{f(x)}{x} + \mathcal{O}(\delta)\,. \end{aligned} \tag{1.2.5}$$

While the subtraction in Eq. (1.2.3) provided an exact result, the slicing in Eq. (1.2.5) usually only provides an approximation for small $\delta$. Additionally, the first term in the r.h.s. of the last line in Eq. (1.2.5) has an explicit factor of $\log\delta$ that needs to cancel against the second term, *after* integration. In practice this leads to large numerical cancellations that need to be controlled.

Despite these shortcomings, as compared to the subtraction method, slicing methods are easier to construct and thus seem more feasible at N3LO. Currently, different slicing schemes are readily available at NNLO in QCD with $N$-jettiness [27–30] and $p_T$ [31–38] being the most prominent.

To understand how slicing works in a physical context consider the production of a vector boson in quark anti-quark collisions $q\bar{q} \to V + X$ at NLO QCD. Two contributions need to be considered. Virtual $\mathcal{O}(\alpha_s)$ corrections to $q\bar{q} \to V$ and real emission corrections described by the process $q\bar{q} \to V + g$. We calculate the $p_T$ distribution of the Vector boson and introduce a cut-off parameter $p_0 \ll 1$, see Fig. 1.3. To fulfill momentum conservation any radiated gluon below this cut-off parameter needs to be soft or collinear while any gluon radiation above

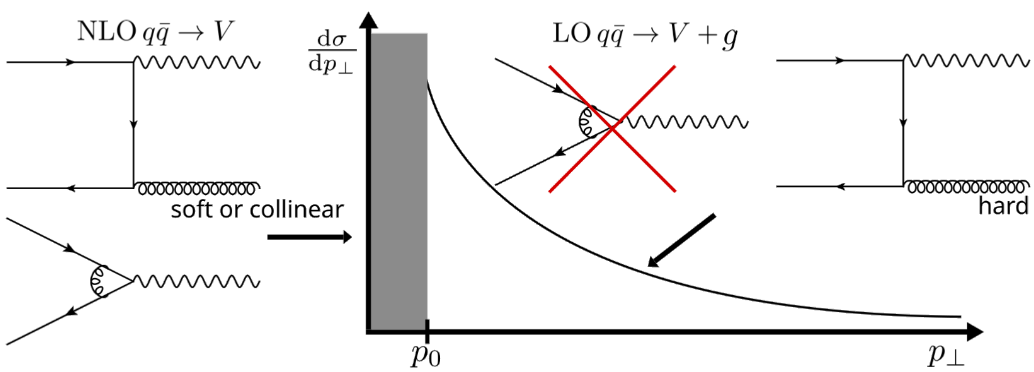


**Figure 1.3:** Visualization of $q_T$ slicing at NLO for the process $q\bar{q} \to V + X$.

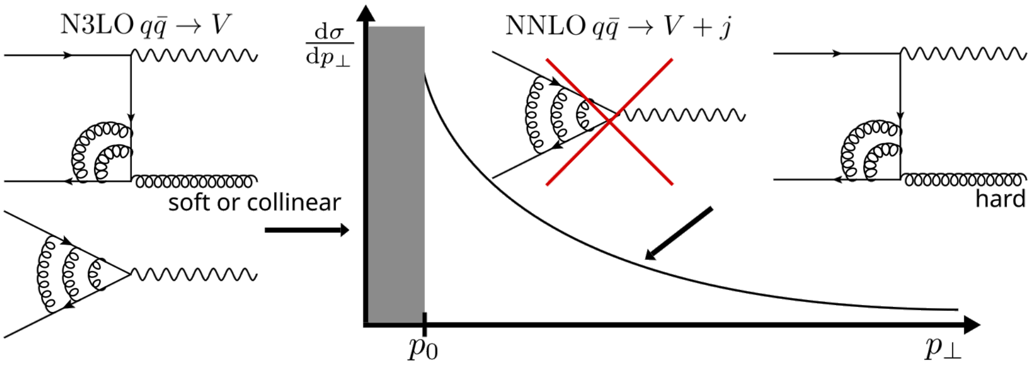


**Figure 1.4:** Visualization of $q_T$ slicing at N3LO for the process $q\bar{q} \to V + X$.

the cut-off parameter needs to be hard. From Fig. 1.3 it becomes clear that we can view the emission in the latter region as the leading Order (LO) process of $q\bar{q} \to V + g$. This process is free of singularities and therefore its cross section can be computed numerically in four dimensions. On the other hand, the process in the first region is $q\bar{q} \to V$ at NLO, for which all emissions need to be soft, collinear or virtual. However, just like in the toy example, we can simplify the integrand in these soft and collinear limits to facilitate the computation.

The extension of this discussion to the N3LO case is illustrated in Fig. 1.4. The first region is now simply $q\bar{q} \to V$ at N3LO with all emissions being either soft collinear or virtual. Conversely, the second region still needs to have *at least* one resolved final state jet such that the most offending singularities are given by the process $q\bar{q} \to V + j$ at NNLO. While the second region now also exhibits divergences from real and virtual emissions, they can be treated with existing NNLO subtraction schemes.

Although the $p_T$ slicing scheme has already been used in fully differential N3LO calculations of color singlet production [39], this scheme can not be extended to final states with additional jets. For this reason we study the use of jettiness as a slicing parameter.

Specifically, we will use the zero-jettiness observable as a slicing parameter for color singlet production $pp \to V + X$ at N3LO.

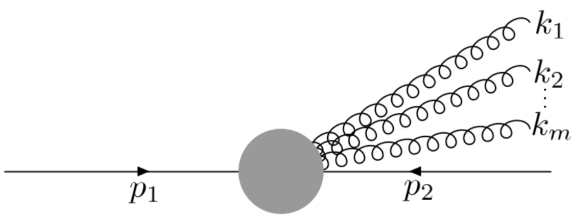


**Figure 1.5:** Kinematic configuration of the process $pp \to V + X$, $p_1$ and $p_2$ are the momenta of the incoming partons, while $k_m$ are the momenta of the final state partons. The final state vector boson $V$ is omitted.

*zero-jetiness slicing*

The zero-jettiness variable is defined as [40, 41]

$$\tau = \sum_m \min_{i\in\{1,2\}} \left[\frac{2p_i \cdot k_m}{Q_i}\right] , \tag{1.2.6}$$

where $p_i$ are the four momenta of the initial state partons, $k_m$ are the momenta of final state QCD partons and $Q_i$ are hardness variables (cf. Fig. 1.5). From the definition in Eq. (1.2.6) it becomes clear that in the limit $\tau \to 0$ all emitted momenta $k_m$ need to become either soft or collinear to the initial state momenta $p_i$ as required for a slicing parameter. In this limit of small $\tau$, the cross section factorizes [40, 41]

$$\lim_{\tau_0\to 0} \mathrm{d}\sigma^{\mathrm{N^3LO}}_{pp\to V+X}\ (\tau \ll \tau_0) = B \otimes B \otimes S \otimes H \otimes \mathrm{d}\sigma^{\mathrm{LO}}_{pp\to V}\ . \tag{1.2.7}$$

In Eq. (1.2.7), $\otimes$ denotes convolutions of the transverse virtuality and longitudinal momenta fractions of particles entering the hard process (cf. Eq. (14) in Ref. [41]). The hard function $H$ describes virtual corrections to the leading order cross section and is process-dependent. It is already known through N3LO QCD for such processes as single vector boson production and Higgs boson production [42, 43]. The beam function $B$ is a process-independent function that describes initial state collinear radiation and is already known through N3LO [44–47]. We note that the beam function is independent of the number of jets in the final state since it describes radiation collinear to the incoming partons. The soft function $S$ is also process-independent. It describes soft radiation off the external partons. It is currently only known through NNLO [48, 49].

Application of the zero-jettiness slicing scheme at N3LO requires the knowledge of the corresponding soft and beam function. In this thesis we contribute towards this goal by calculating varies contributions required for the two functions. These computations require the use of standard multi-loop techniques such as reverse-unitarity [50] and integration-by-part identities (IBPs) [51] (cf. Appendix A.1).

## 1.3 STRUCTURE OF THIS THESIS

This thesis contains two independent parts, that deal with zero-jettiness beam and soft functions. In the first part, we focus on the computation of the beam function.

Specifically, in Chapter 2 we calculate all NNLO QCD beam functions through second order in $\epsilon$, as required for the renormalization of the N3LO beam functions. In Chapter 3 we compute contributions to all N3LO beam functions due to real-virtual emissions. To facilitate these calculations, we employ the by-now standard techniques of collinear projection operators [52], integration-by-parts identities (IBPs) and reverse unitary.

The calculation of the soft function described in Part II can not be performed with the help of the standard methods as Heaviside functions, that appear in the definition of the zero-jettiness observable, complicate the calculation. Therefore, the goal of Chapters 4 and 5 is twofold. First we want to calculate the soft function at NNLO through second order in $\epsilon$ as required for the renormalization of the N3LO soft function. Second, we want to develop new computational techniques which will enable application of integration-by-parts identities to phase-space integrals with Heaviside functions.

In Chapter 4 we achieve this by rewriting Heaviside functions as integrals of auxiliary parameters over delta functions. After writing delta functions as the imaginary part of a propagator-like structure, integration-by-parts identities can be applied. At variance with this, in Chapter 5 we derive modified integration-by-parts identities that can be directly applied to integrals containing Heaviside functions. Both of these approaches allow to express the NNLO soft function through a sum of master integrals, and lead to huge simplifications compared to the original calculations in Refs [48, 49].

In Chapter 6 we apply the modified IBP approach of Chapter 5 to the computation of the N3LO soft function. We focus on the contribution where three gluons are emitted into the same hemisphere. While the calculation turns out to be complicated because of an additional analytic regulator that needs to be introduced, the approach ultimately proves successful allowing us to derive first contributions to the zero-jettiness soft function at N3LO.

We summarize this thesis Chapter 7.

Part I

# BEAM FUNCTION

2

# NNLO BEAM FUNCTION

*This chapter is a revised version of Reference [53].*

In this chapter we discuss the calculation of the NNLO beam function through higher orders in the dimensional regularization parameter $\epsilon$. It is structured as follows. In Section 2.1 we explain the general setup of the calculation, relating collinear limits of QCD cross-sections to beam functions. We use reverse unitarity and IBP-relations to express the bare partonic beam function through master integrals. The calculation of master integrals is discussed in detail in Section 2.2. We proceed by renormalizing the beam function in Section 2.3. Finally, we discuss the results and conclude in Section 2.4.

We note that parts of this chapter were already discussed in the authors master's thesis Ref. [54] in which the quark-to-quark beam function was calculated. However, the computation of the new results in this chapter, the gluon-to-gluon, gluon-to-quark, anti-quark-to-quark and quark-to-gluon beam functions require the recapitulation of concepts and results already discussed in Ref. [54].

## 2.1 GENERAL SETUP

The beam function is a quantity defined in soft-collinear effective theory (SCET) [55–58]. However, it was pointed out in Ref. [59] that the beam function can be related to QCD splitting functions. More precisely, the bare partonic beam function $B_{ij}^b$ is related to the spin-averaged and color-summed collinear splitting function $\left\langle P_{j\to i^*\{m\}}\right\rangle$ which describes the transition of a parton $j$ to a parton $i$ under collinear emission of $m$ other partons. It can be written as

$$B_{ij}^b \sim \sum_{\{m\}} \int \mathrm{dPS}^{(\mathrm{m})} \left\langle P_{j\to i^*\{m\}}\right\rangle . \tag{2.1.1}$$

The phase-space measure is defined as [44, 45]

$$\mathrm{dPS}^{(\mathrm{m})} = \left(\prod_n^m \frac{\mathrm{d}^d k_n}{(2\pi)^{d-1}}\,\delta^+\left(k_n^2\right)\right)\delta\left(2\sum_n^m k_n\cdot p - \frac{t}{z}\right)\delta\left(2\sum_n^m \frac{k_n\cdot\bar{p}}{s} - (1-z)\right), \tag{2.1.2}$$

where

$$\delta^+\left(k_n^2\right) = \delta\left(k_n^2\right)\,\theta\left(k_0\right). \tag{2.1.3}$$

In Eq. (2.1.2) $p$ is the momentum of the parton $j$, $\bar{p}$ is the light cone momentum complementary to $p$, and $k_m$ are the momenta of final state partons. Furthermore, $t$ is the so-called transverse virtuality of the off-shell parton $i$, $z\cdot p$ is its longitudinal momentum and $s = 2p\cdot\bar{p}$. Thus, the last two delta functions in Eq. (2.1.2) introduce the dependence on the transverse virtuality $t$ and the longitudinal momentum fraction $z$ to the beam function as required for the convolutions of the factorization theorem in Ref. [41].

A general formula for the calculation of any splitting function $\left\langle P_{j\to i^*\{m\}}\right\rangle$ is given in Ref [52]. It relies on the use of a projection operator $\mathcal{P}$ acting on matrix elements $M_{j\to i^*\{m\}}$ for the process of a parton $j$ splitting into on-shell partons $\{m\}$ and an off-shell parton $i^*$. This projection operator, together with an axial gauge for gluons, decouples the collinear emissions from hard matrix elements. We obtain

$$\left\langle P_{j\to i^*\{m\}}\right\rangle = \mathcal{P}|M_{j\to i^*\{m\}}|^2, \tag{2.1.4}$$

$$\mathcal{P}|M_{j\to i^*\{m\}}|^2 = \begin{cases} \sum \mathrm{Tr}\left[M_{j\to i^*\{m\}}\frac{\hat{\bar{p}}}{4\bar{p}\cdot p_s}M^\dagger_{j\to i^*\{m\}}\right], & \text{if } i\in\{q,\bar{q}\} \\ -\frac{1}{2(1-\epsilon)}\sum d^\rho_\mu\left(p_s\right)d_{\nu\rho}\left(p_s\right)M^\mu_{j\to i^*\{m\}}M^{\nu\dagger}_{j\to i^*\{m\}}, & \text{if } i\in\{g\} \end{cases} \tag{2.1.5}$$

where

$$d_{\mu\nu}(k) = -g_{\mu\nu} + \frac{k_\mu\bar{p}_\nu + \bar{p}_\mu k_\nu}{k\cdot\bar{p}}, \qquad p_s = p - \sum_m k_m, \tag{2.1.6}$$

and the sum in Eq. (2.1.4) runs over all color and spin degrees of freedom of all external partons.

Putting everything together, we write

$$B^b_{ij} = \mathcal{N}_{ij}\sum_{\{m\}}\frac{1}{\phi_{\{m\}}}\int \mathrm{dPS}^{(\mathrm{m})}\mathcal{P}|M_{j\to i^*\{m\}}|^2, \tag{2.1.7}$$

where $\phi_{\{m\}}$ is a symmetry factor for identical particles and $\mathcal{N}_{ij}$ is an averaging factor, dependent on the color $\langle C_{i/j}\rangle$ and spin $\langle S_{i/j}\rangle$ sums of partons $i$ and $j$

$$\mathcal{N}_{ij} = \frac{1}{\langle C_j\rangle}\frac{\langle S_i\rangle}{\langle S_j\rangle}. \tag{2.1.8}$$

This factor simply introduces the spin and color average for the initial state $j$ and removes the spin average for the final state $i$, that was performed in Ref. [52]. All required averaging factors $\mathcal{N}_{ij}$ are shown in Table 2.1

It is apparent from the definition Eq. (2.1.1) that, just like for the splitting functions, it is sufficient to consider the following set of $i$'s and $j$'s : $(i,j)\in\{(q_l,q_m),(q_l,g),(q_l,\bar{q}_m),(g,g),(g,q_m)\}$[60, 61], where the indices $l$ and $m$ denote quark flavors.

Any other set is obtained by charge conjugation. For example the beam function $B^b_{\bar{q}_l\bar{q}_m}$ is simply equal to $B^b_{q_lq_m}$. Furthermore a flavor-preserving transition in $B^b_{q_lq_m}$ is obtained by setting $l=m$.

The representation of the bare partonic beam function $B^b_{ij}$ given in Eq. (2.1.7), together with a special Feynman rule for the projection operator $\mathcal{P}$, allows us to perform a calculation in a standardized way. We start by drawing all relevant diagrams for the process $|M_{j\to i^*\{m\}}|^2$, then we apply Feynman rules to obtain an expression for the projection of $|M_{j\to i^*\{m\}}|^2$, and finally,

| $(i,j)$ | $(q_l,q_m)$ | $(q_l,g)$ | $(q_l,\bar{q}_m)$ | $(g,g)$ | $(g,q_m)$ |
|---|---|---|---|---|---|
| $\mathcal{N}_{ij}$ | $\frac{1}{N_c}$ | $\frac{1}{N_c^2-1}\ \frac{1}{1-\epsilon}$ | $\frac{1}{N_c}$ | $\frac{1}{N_c^2-1}$ | $\frac{1-\epsilon}{N_c}$ |

**Table 2.1:** Averaging factors $\mathcal{N}_{ij}$ required for the different bare partonic beam functions $B_{ij}^b$

we use integration-by-parts identities to obtain the beam function expressed through master integrals.

For example the calculation of $B_{gg}^b$ at NNLO requires the consideration of the two real-real emission (RR) processes $|M_{g\to g^*\{gg\}}|^2$ and $|M_{g\to g^*\{q\bar{q}\}}|^2$ at tree level, as well as the real-virtual (RV) contribution $|M_{g\to g^*\{g\}}|^2$ at one loop. It is convenient to distinguish amplitudes for the process $g \to g^*\{g\}$ that involve a loop from the ones at tree level. We will refer to the former as $M^l_{g\to g^*\{g\}}$ and to the latter as $M^t_{g\to g^*\{g\}}$. Squaring the amplitude we obtain

$$
\begin{aligned}
|M_{g\to g^*\{g\}}|^2 &= |M^l_{g\to g^*\{g\}} + M^t_{g\to g^*\{g\}}|^2 \\
&= |M^l_{g\to g^*\{g\}}|^2 + 2\,\mathrm{Re}\Big\{M^l_{g\to g^*\{g\}} M^{t\dagger}_{g\to g^*\{g\}}\Big\} + |M^t_{g\to g^*\{g\}}|^2.
\end{aligned}
\tag{2.1.9}
$$

In Eq. (2.1.9) we only keep $2\,\mathrm{Re}\Big\{M^l_{g\to g^*\{g\}} M^{t\dagger}_{g\to g^*\{g\}}\Big\}$, as it is the only terms which is of order $\alpha_s^2$.

The required diagrams for $|M_{g\to g^*\{gg\}}|^2$, $|M_{g\to g^*\{q\bar{q}\}}|^2$ and $\mathrm{Re}\Big\{M^l_{g\to g^*\{g\}} M^{t\dagger}_{g\to g^*\{g\}}\Big\}$ are shown in Fig. 2.1. Diagrams for all other beam functions are shown in Appendix B. We implement Feynman rules in Form [62] to obtain mathematical expressions for all diagrams and employ reverse unitarity[50] to rewrite delta functions appearing in Eq. (2.1.2) as the difference of two propagators

$$
\delta(X) = \frac{\mathrm{i}}{2\pi}\left(\frac{1}{X+\mathrm{i}\epsilon} - \frac{1}{X-\mathrm{i}\epsilon}\right). \tag{2.1.10}
$$

Eq. (2.1.10) effectively maps phase-space integrals in Eq. (2.1.7) onto loop integrals, allowing us to use integration-by-parts (IBP) identities [51] to express the beam function through master integrals. The IBP reduction is performed using FIRE [63].
We find that all five beam functions can be expressed through just twelve master integrals. They include nine double-real master integrals

$$
\begin{aligned}
I_1 &= [1]_{(2)}, & I_2 &= \left[\frac{1}{\bar{p}\cdot(p-k_1)}\right]_{(2)}, \\
I_3 &= \left[\frac{1}{(p-k_{12})^2}\right]_{(2)}, & I_4 &= \left[\frac{1}{(p-k_1)^2\, k_{12}^2\, \bar{p}\cdot k_2}\right]_{(2)}, \\
I_5 &= \left[\frac{1}{(p-k_1)^2\,(p-k_{12})^2\,\bar{p}\cdot k_1}\right]_{(2)}, & I_6 &= \left[\frac{1}{(p-k_1)^2\,(p-k_{12})^2\,\bar{p}\cdot k_2}\right]_{(2)}, \\
I_7 &= \left[\frac{1}{(p-k_{12})^2\,[\bar{p}\cdot(p-k_1)]}\right]_{(2)}, & I_8 &= \left[\frac{1}{k_{12}^2\,[\bar{p}\cdot(p-k_1)]\,(p-k_1)^2}\right]_{(2)},
\end{aligned}
\tag{2.1.11}
$$

$$I_9 = \left[ \frac{1}{(p-k_{12})^2 \, (p-k_2)^2 \, (p-k_1)\cdot\bar{p}} \right]_{(2)},$$

and three real-virtual master integrals

$$I_{10} = \left[ \frac{1}{l^2 \, l\cdot\bar{p} \, (p-l)^2 \, (p-k-l)^2} \right]_{(1)}, \qquad I_{11} = \left[ \frac{1}{l^2 \, (p-k-l)^2} \right]_{(1)}, \qquad (2.1.12)$$
$$I_{12} = \left[ \frac{1}{l^2 \, (p-l)^2 \, (l-k)^2 \, (l-k)\cdot\bar{p}} \right]_{(1)}.$$

In Eqs. (2.1.11) and (2.1.12) we used the notation $k_{12}$ to define the sum of $k_1$ and $k_2$. We labelled all the integrals with the subscripts $[\,]_{(1)}$ and $[\,]_{(2)}$. These labels refer to the phase space element that we use to integrate

$$[f]_{(2)} = \int \mathrm{dPS}^{(2)} \, f, \qquad [f]_{(1)} = \int \mathrm{dPS}^{(1)} \int \frac{\mathrm{d}^d l}{(2\pi)^d} \, f. \qquad (2.1.13)$$

We note that phase-space measures $\mathrm{dPS}^{(2,1)}$ are defined in Eq. (2.1.2). We further note that all integrals but $I_9$ are required for the calculation of $B^b_{q_l q_m}$ and have thus already been determined in Ref. [54]. However, we discuss the calculation of some already known integrals in detail to introduce computational techniques that will be required later.

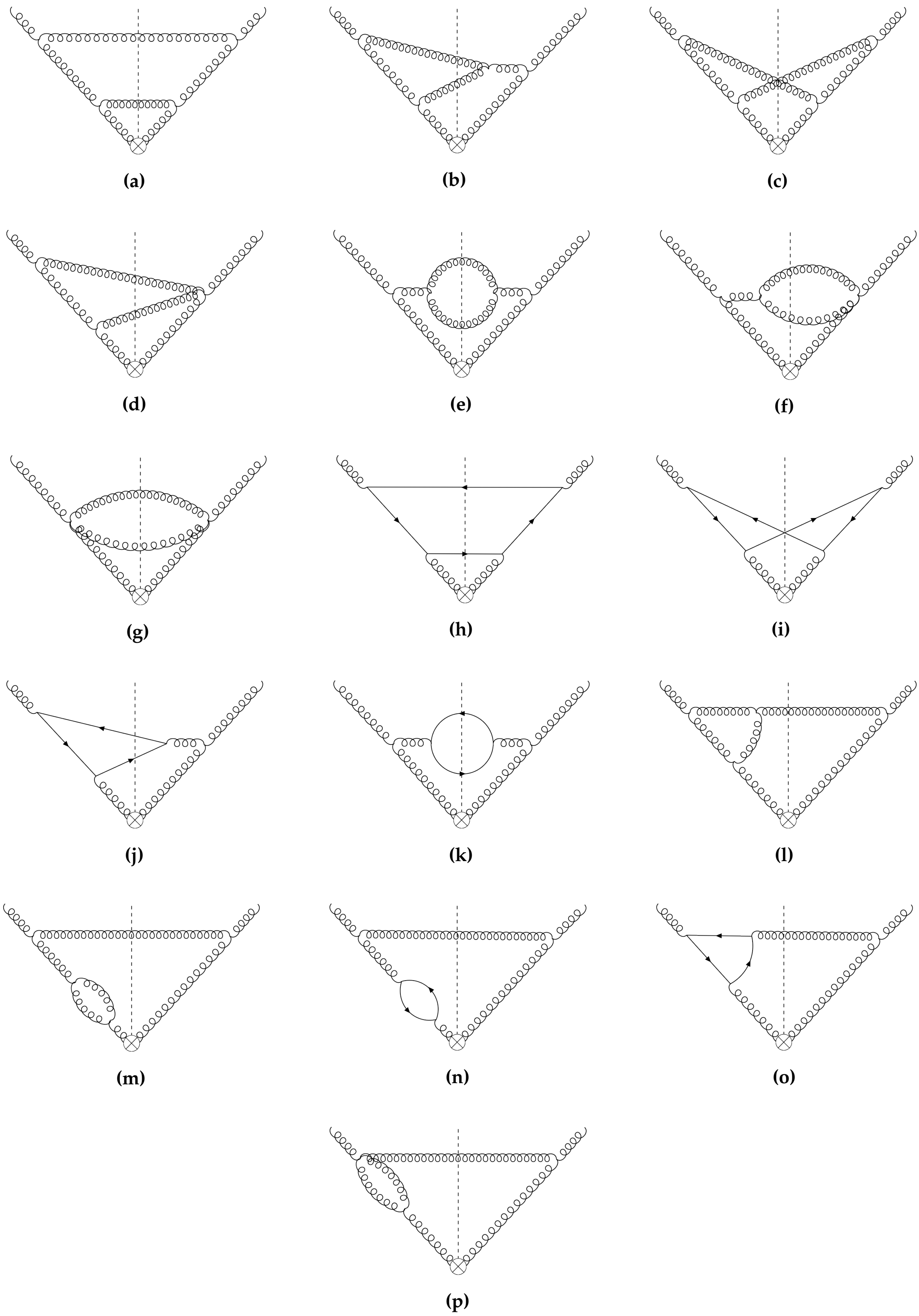


**Figure 2.1:** Diagrams contributing to the $B_{gg}^{(b)}$ beam function. Diagrams for which the fermion flow is reversed and left-right mirror diagrams are not shown. The dashed line represents a "cut" so that all particles crossing it are on the mass-shell. The vertex $\otimes$ denotes the insertion of the projection operator defined in Eq. (2.1.5). Diagrams (a)-(g) are double real gluon emission diagrams, (h)-(k) are quark anti-quark emission diagrams, and (l)-(p) are real-virtual emission diagrams.

## 2.2 CALCULATION OF MASTER INTEGRALS

### 2.2.1 *Master integral $I_1$*

We begin by discussing the simplest integral, the two particle phase-space $I_1$. It reads

$$I_1 = \int \frac{\mathrm{d}^d k_1}{(2\pi)^{d-1}} \int \frac{\mathrm{d}^d k_2}{(2\pi)^{d-1}}\, \delta^+\left(k_1^2\right)\, \delta^+\left(k_2^2\right)\, \delta\left(2k_{12}\cdot p - \tfrac{t}{z}\right) \delta\left(\tfrac{2k_{12}\cdot\bar{p}}{s} - (1-z)\right). \tag{2.2.1}$$

We start by rescaling the external momenta $k_1$, $k_2$, $p$, and $\bar{p}$ such that the dependence on $s$ and $t$ factors out of the integral in Eq. (2.2.1). We write

$$\bar{p} = \tilde{\bar{p}}\frac{s}{\sqrt{t}}, \qquad p = \tilde{p}\sqrt{t}, \qquad k_i = \tilde{k}_i\sqrt{t}, \tag{2.2.2}$$

and obtain

$$I_1 = t^{d-3}\int \frac{\mathrm{d}^d k_1}{(2\pi)^{d-1}} \int \frac{\mathrm{d}^d k_2}{(2\pi)^{d-1}}\, \delta^+\left(k_1^2\right)\, \delta^+\left(k_2^2\right)\, \delta\left(2k_{12}\cdot p - \tfrac{1}{z}\right) \times \delta\left(2k_{12}\cdot\bar{p} - (1-z)\right), \tag{2.2.3}$$

where we dropped the tilde to simplify notations. For integrals such as the one shown in Eq. (2.2.3) it is useful to introduce a Sudakov decomposition for the momenta $k_1$ and $k_2$

$$k_i^\mu = \alpha_i p^\mu + \beta_i \bar{p}^\mu + k_{i\perp}^\mu, \qquad i = 1,2\,, \tag{2.2.4}$$

where $p^2 = \bar{p}^2 = p\cdot k_{i\perp} = \bar{p}\cdot k_{i\perp} = 0$. With this parametrization, the phase-space integration measure becomes

$$\int \frac{\mathrm{d}^d k_i}{(2\pi)^{d-1}}\delta^+\left(k_i^2\right) = \frac{1}{2(2\pi)^{d-1}}\int_0^\infty \mathrm{d}\alpha_i \int_0^\infty \mathrm{d}\beta_i \int \mathrm{d}^{d-2}k_{i\perp}\delta(\alpha_i\beta_i - k_{i\perp}^2), \tag{2.2.5}$$

where the lower boundaries in the first two integrals enforce the constraint $k_i\cdot p > 0$, $k_i\cdot\bar{p} > 0$ is encoded in $\delta^+(k_i^2)$.

We further simplify the expression Eq. (2.2.5) by introducing spherical coordinates for $k_{i\perp}^\mu$ and, since the integrand in Eq. (2.2.3) is independent of any angles associated with $k_{i\perp}^\mu$, we can directly integrate over the $(d-2)$-dimensional solid angle $\Omega_{d-2}$

$$\begin{aligned}\int \frac{\mathrm{d}^d k_i}{(2\pi)^{d-1}}\delta^+\left(k_i^2\right) &= \frac{\Omega_{d-2}}{2(2\pi)^{d-1}}\int_0^\infty \mathrm{d}\alpha_i \int_0^\infty \mathrm{d}\beta_i \int \mathrm{d}|k_{i\perp}|\;|k_{i\perp}|^{d-3}\delta(\alpha_i\beta_i - k_{i\perp}^2)\\ &= \frac{\Omega_{d-2}}{4(2\pi)^{d-1}}\int_0^\infty \mathrm{d}\alpha_i \int_0^\infty \mathrm{d}\beta_i \int \mathrm{d}k_{i\perp}^2\;(k_{i\perp}^2)^{-\epsilon}\;\delta(\alpha_i\beta_i - k_{i\perp}^2).\end{aligned} \tag{2.2.6}$$

We now insert Eq. (2.2.6) into Eq. (2.2.3) and integrate over $k_{i\perp}^2$ to remove delta functions $\delta(\alpha_i\beta_i - k_{i\perp}^2)$. We find

$$\begin{aligned} I_1 =\ & t^{d-3}\frac{1}{16}\frac{(\Omega_{d-2})^2}{(2\pi)^{2d-2}}\int_0^\infty \mathrm{d}\alpha_1 \int_0^\infty \mathrm{d}\beta_1 \int \mathrm{d}k_{1\perp}^2 (k_{1\perp}^2)^{-\epsilon}\delta(\alpha_1\beta_1 - k_{1\perp}^2) \\ & \times \int_0^\infty \mathrm{d}\alpha_2 \int_0^\infty \mathrm{d}\beta_2 \int \mathrm{d}k_{2\perp}^2 (k_{2\perp}^2)^{-\epsilon}\delta(\alpha_2\beta_2 - k_{2\perp}^2)\delta(\alpha_{12}-(1-z))\delta\left(\beta_{12}-\tfrac{1}{z}\right) \\ =\ & t^{d-3}\frac{1}{16}\frac{(\Omega_{d-2})^2}{(2\pi)^{2d-2}}\int_0^\infty \mathrm{d}\alpha_1 \int_0^\infty \mathrm{d}\alpha_2 (\alpha_1\alpha_2)^{-\epsilon}\delta(\alpha_{12}-(1-z)) \\ & \times \int_0^\infty \mathrm{d}\beta_1 \int_0^\infty \mathrm{d}\beta_2 (\beta_1\beta_2)^{-\epsilon}\delta\left(\beta_{12}-\tfrac{1}{z}\right), \end{aligned} \tag{2.2.7}$$

where the shorthand notation $\alpha_{12} = \alpha_1 + \alpha_2$ and $\beta_{12} = \beta_1 + \beta_2$ is used. The remaining two integrals are of the form

$$\int_0^\infty \mathrm{d}\alpha_1 \int_0^\infty \mathrm{d}\alpha_2\ \alpha_1^a\ \alpha_2^b\ \delta(\alpha_1+\alpha_2-x). \tag{2.2.8}$$

They can be computed using the integral definition of the beta function

$$\int_0^1 u^{x-1}(1-u)^{y-1}\mathrm{d}u = \mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{2.2.9}$$

Indeed,

$$\begin{aligned} & \int_0^\infty \mathrm{d}\alpha_1 \int_0^\infty \mathrm{d}\alpha_2\ \alpha_1^a\ \alpha_2^b\ \delta(\alpha_1+\alpha_2-x) = \int_0^x \mathrm{d}\alpha_1\ \alpha_1^a\ \left(1-\frac{\alpha_1}{x}\right)^b\ x^b, \\ & \overset{y=\frac{\alpha_1}{x}}{=} x^{1+a+b}\int_0^1 \mathrm{d}y\ y^a\ \ (1-y)^b \overset{2.2.9}{=} \frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(2+a+b)}x^{1+a+b}. \end{aligned} \tag{2.2.10}$$

Inserting Eq. (2.2.10) into Eq. (2.2.7) we obtain

$$I_1 = t^{d-3}\frac{1}{16}\frac{(\Omega_{d-2})^2}{(2\pi)^{2d-2}}\frac{\Gamma(1-\epsilon)^4}{\Gamma(2-2\epsilon)^2}\left(\frac{1-z}{z}\right)^{1-2\epsilon}. \tag{2.2.11}$$

### 2.2.2 *Master integral $I_2$*

The next-to-easiest integral is the master integral $I_2$. It reads

$$I_2 = \int \frac{\mathrm{d}^d k_1}{(2\pi)^{d-1}} \int \frac{\mathrm{d}^d k_2}{(2\pi)^{d-1}}\ \delta^+\left(k_1^2\right)\ \delta^+\left(k_2^2\right)\ \delta(2k_{12}\cdot p - \tfrac{t}{z})\frac{\delta\left(\frac{2k_{12}\cdot\bar{p}}{s} - (1-z)\right)}{\bar{p}\cdot(p-k_1)}, \tag{2.2.12}$$

and it can be calculated along the lines of $I_1$. Rescaling momenta Eq. (2.2.2) and employing the parametrization Eq. (2.2.6) we obtain

$$I_2 = \frac{t^{d-3}}{s}\frac{1}{8}\frac{(\Omega_{d-2})^2}{(2\pi)^{2d-2}}\int_0^\infty \mathrm{d}\alpha_1 \int_0^\infty \mathrm{d}\alpha_2 \frac{(\alpha_1\alpha_2)^{-\epsilon}}{1-\alpha_1}\delta(\alpha_{12}-(1-z)) \\ \times \int_0^\infty \mathrm{d}\beta_1 \int_0^\infty \mathrm{d}\beta_2 (\beta_1\beta_2)^{-\epsilon}\delta(\beta_{12}-\tfrac{1}{z}). \tag{2.2.13}$$

The second integral in Eq. (2.2.13) can again be evaluated using Eq. (2.2.10), while the first integral can be related to the integral representation of the Gauss hypergeometric function

$${}_2\mathrm{F}_1(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}\,(1-t)^{c-b-1}\,(1-tz)^{-a}\mathrm{d}t. \tag{2.2.14}$$

Indeed

$$\int_0^\infty \mathrm{d}\alpha_1 \int_0^\infty \mathrm{d}\alpha_2\, \alpha_1^a\, \alpha_2^b\, (1-\alpha_1)^{-c}\, \delta(\alpha_1+\alpha_2-x) \\ = \int_0^x \mathrm{d}\alpha_1\, \alpha_1^a \left(1-\frac{\alpha_1}{x}\right)^b (1-\alpha_1)^{-c}\, x^b, \\ \overset{y=\frac{\alpha_1}{x}}{=} x^{1+a+b}\int_0^1 \mathrm{d}y\, y^a\, (1-y)^b\,(1-x\,y)^{-c} \\ \overset{2.2.14}{=} \frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(2+a+b)}x^{1+a+b}\,{}_2\mathrm{F}_1(1+a,c,2+a+b,x). \tag{2.2.15}$$

Combining these results we finally find

$$I_2 = \frac{t^{d-3}}{s}\frac{1}{8}\frac{(\Omega_{d-2})^2}{(2\pi)^{2d-2}}\left(\frac{1-z}{z}\right)^{1-2\epsilon}\frac{\Gamma(1-\epsilon)^4}{\Gamma(2-2\epsilon)^2}\,{}_2\mathrm{F}_1(1,1-\epsilon,2-2\epsilon,1-z). \tag{2.2.16}$$

### 2.2.3 *Master integral $I_3$*

We turn to the calculation of the integral $I_3$. It reads

$$I_3 = \int \frac{\mathrm{d}^d k_1}{(2\pi)^{d-1}}\int \frac{\mathrm{d}^d k_2}{(2\pi)^{d-1}}\,\delta^+\left(k_1^2\right)\,\delta^+\left(k_2^2\right)\,\delta\left(2k_{12}\cdot p-\tfrac{t}{z}\right)\frac{\delta\left(\frac{2k_{12}\cdot\bar{p}}{s}-(1-z)\right)}{(p-k_{12})^2}. \tag{2.2.17}$$

Unlike the previous two integrals, this integral can not be evaluated using the phase-space parametrization Eq. (2.2.6). This happens because the propagator in Eq. (2.2.17) introduces the scalar product $k_{1\perp}\cdot k_{2\perp}$ complicating the integration over the solid angel. To circumvent this problem we insert $1=\int \mathrm{d}^dQ\delta^d(k_1+k_2-Q)$ into the integrand and change the order of integration. The integral can now be split into two integrals that can be evaluated one after another. We find

$$I_3 = t^{d-4}\int \mathrm{d}^dQ\,\delta\left(2Q\cdot p-\tfrac{1}{z}\right)\delta\left(2Q\cdot\bar{p}-(1-z)\right)\frac{\mathrm{F}(Q^2)}{(p-Q)^2}, \tag{2.2.18}$$

$$F(Q^2) = \int \frac{d^d k_1}{(2\pi)^{d-1}} \int \frac{d^d k_2}{(2\pi)^{d-1}} \delta^+(k_1^2)\delta^+(k_2^2)\delta^d(k_{12} - Q), \tag{2.2.19}$$

where we rescaled the momenta to factor out $t$ again. We further simply the calculation of $F(Q^2)$ by exploiting Lorentz invariance and choosing a reference frame such that $Q = (Q_0, 0, 0, 0)$. The integral simplifies to

$$F(Q_0) = \int \frac{d^d k_1}{(2\pi)^{d-1}} \int \frac{d^d k_2}{(2\pi)^{d-1}} \delta^+(k_1^2)\delta^+(k_2^2)\ \delta^{d-1}(\vec{k}_1 + \vec{k}_2)\ \delta(k_{10} + k_{20} - Q_0), \tag{2.2.20}$$

where $k_{i0}$ denotes the time-like component of $k_i$. We first perform the $k_2$ integration removing the delta functions and integrate over $k_1$ after that. We find

$$\begin{aligned}
F(Q_0) &= \int \frac{d^d k_1}{(2\pi)^{2d-2}} \delta^+(k_{10}^2 - \vec{k}_1^{\,2})\delta^+(Q_0^2 - 2Q_0 k_{10}), \\
&= \int dk_{10}\ \Omega_{d-1} \int \frac{d|\vec{k}_1|}{(2\pi)^{2d-2}} |\vec{k}_1|^{d-2}\ \delta^+(k_{10}^2 - \vec{k}_1^{\,2})\delta^+(Q_0^2 - 2Q_0 k_{10}), \\
&\overset{k_s = |\vec{k}_1|^2}{=} \int dk_{10}\ \Omega_{d-1} \int \frac{dk_s}{2(2\pi)^{2d-2}}\ k_s^{\frac{d-3}{2}}\ \delta^+(k_{10}^2 - k_s)\delta^+(Q_0^2 - 2Q_0 k_{10}), \\
&= \int dk_{10}\ \Omega_{d-1} \frac{1}{2(2\pi)^{2d-2}}\ k_{10}^{d-3}\delta^+(Q_0^2 - 2Q_0 k_{10}), \\
&= \frac{\Omega_{d-1}}{(2\pi)^{2d-2}} \frac{Q_0^{d-4}}{2^{d-1}}.
\end{aligned} \tag{2.2.21}$$

The Lorentz invariance is restored by identifying $Q_0^2 = Q^2$, and we obtain

$$F(Q^2) = \frac{\Omega_{d-1}}{(2\pi)^{2d-2}} \frac{(Q^2)^{-\epsilon}}{2^{d-1}}, \tag{2.2.22}$$

$$I_3 = t^{d-4} \frac{\Omega_{d-1}}{(2\pi)^{2d-2}} \frac{1}{2^{d-1}} \int d^d Q\ \delta\left(2Q \cdot p - \tfrac{1}{z}\right) \delta\left(2Q \cdot \bar{p} - (1-z)\right) \frac{(Q^2)^{-\epsilon}}{(p-Q)^2}. \tag{2.2.23}$$

We perform the remaining $Q$ integration in Eq. (2.2.23) by introducing the Sudakov decomposition $Q = \alpha p^\mu + \beta \bar{p}^\mu + Q_\perp^\mu$ and write

$$\begin{aligned}
I_3 = t^{d-4} &\frac{\Omega_{d-1}}{8(2\pi)^{2d-2}} \frac{\Omega_{d-2}}{2^{d-1}} \int_0^\infty d\alpha \int_0^\infty d\beta \int dQ_\perp^2\, (Q_\perp^2)^{-\epsilon} \theta(\alpha\beta - Q_\perp^2) \\
\times &\frac{(\alpha\beta - Q_\perp^2)^{-\epsilon}}{-\beta + \alpha\beta - Q_\perp^2} \delta(\beta - \tfrac{1}{z})\delta(\alpha - (1-z)).
\end{aligned} \tag{2.2.24}$$

The integration over $\alpha$ and $\beta$ is performed by removing the delta functions, while the Heaviside function is removed by changing the integration variable to $Q_\perp^2 = \frac{1-z}{z} y$. We finally obtain,

$$\begin{aligned}
I_3 &= -t^{d-4} \frac{\Omega_{d-1}}{4(2\pi)^{2d-2}} \frac{\Omega_{d-2}}{2^{d-1}} \left(\frac{1-z}{z}\right)^{1-2\epsilon} \int_0^1 dy \frac{(1-y)^{-\epsilon} y^{-\epsilon}}{(1 + \frac{1-z}{z} y)}, \\
&= -t^{d-4} \frac{1}{16} \frac{(\Omega_{d-2})^2}{(2\pi)^{2d-2}} \left(\frac{1-z}{z}\right)^{1-2\epsilon} \frac{\Gamma(1-\epsilon)^4}{\Gamma(2-2\epsilon)^2}\ {}_2F_1\left(1, 1-\epsilon, 2-2\epsilon, \tfrac{z-1}{z}\right),
\end{aligned} \tag{2.2.25}$$

where we used the integral representation of the Gauss hypergeometric function Eq. (2.2.14) again.

### 2.2.4 *Master integral* $I_4$

We next consider the master integral $I_4$

$$I_4 = \int \frac{\mathrm{d}^d k_1}{(2\pi)^{d-1}} \int \frac{\mathrm{d}^d k_2}{(2\pi)^{d-1}} \, \delta^+\left(k_1^2\right) \delta^+\left(k_2^2\right) \delta\left(2k_{12}\cdot p - \tfrac{t}{z}\right) \frac{\delta\left(\frac{2k_{12}\cdot\bar{p}}{s} - (1-z)\right)}{(p-k_1)^2\, k_{12}^2\, \bar{p}\cdot k_2}. \tag{2.2.26}$$

It is calculated along the same lines as $I_3$, and we again split the integral into two parts

$$I_4 = t^{d-5} s^{-1} \int \mathrm{d}^d Q \, \delta\left(2Q\cdot p - \tfrac{1}{z}\right) \delta\left(2Q\cdot\bar{p} - (1-z)\right) \frac{F(Q^2, p\cdot Q, \bar{p}\cdot Q)}{Q^2}, \tag{2.2.27}$$

$$F(Q^2, p\cdot Q, \bar{p}\cdot Q) = \int \frac{\mathrm{d}^d k_1}{(2\pi)^{d-1}} \int \frac{\mathrm{d}^d k_2}{(2\pi)^{d-1}} \frac{\delta^+\left(k_1^2\right)\delta^+\left(k_2^2\right)}{(p-k_1)^2\,\bar{p}\cdot k_2} \delta^d(Q-k_1-k_2). \tag{2.2.28}$$

Due to the presence of the vectors $p$ and $\bar{p}$ the function $F$ will now be a function not only of $Q^2$ but also of $p\cdot Q$ and $\bar{p}\cdot Q$. Nonetheless, our approach remains the same. We exploit Lorentz invariance by setting $Q = (Q_0, 0, 0, 0)$, and find

$$\begin{aligned} F &= -\frac{1}{8p_0\bar{p}_0} \int \frac{\mathrm{d}^{d-1}\vec{k}_1}{(2\pi)^{2d-2}|\vec{k}_1|^4} \delta\left(Q_0 - 2|\vec{k}_1|\right) \frac{1}{1-\vec{n}_p\cdot\vec{n}_k} \frac{1}{1+\vec{n}_{\bar{p}}\cdot\vec{n}_k} \\ &= -\frac{1}{(2p_0Q_0)(2\bar{p}_0Q_0)} \left(\frac{Q_0}{2}\right)^{d-4} \int \frac{\mathrm{d}\Omega_k^{(d-1)}}{(2\pi)^{2d-2}} \frac{1}{(k_n\cdot p_1)\,(k_n\cdot p_2)}. \end{aligned} \tag{2.2.29}$$

While the angular integral in Eq. (2.2.29) is not straightforward to compute, it is discussed in detail in Refs. [64, 65]. The result reads

$$\int \frac{\mathrm{d}\Omega_k^{(d-1)}}{(k_n\cdot p_1)\,(k_n\cdot p_2)} = -\Omega^{(d-2)} \frac{2^{-2\epsilon}}{\epsilon} \frac{\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)} \, {}_2F_1\left(1, 1, 1-\epsilon, 1 - \frac{\rho_{12}}{2}\right), \tag{2.2.30}$$

where $\rho_{12} = (1 - \vec{n}_{p_1}\cdot\vec{n}_{p_2})$. Finally, we write $\rho_{12}$ in the Lorentz invariant way

$$1 - \frac{\rho_{12}}{2} = \frac{1}{2}\left(1 - \vec{n}_p\,\vec{n}_{\bar{p}}\right) = \frac{1}{2}\frac{p\cdot\bar{p}}{p_0\bar{p}_0} = \frac{1}{4p_0\bar{p}_0} = \frac{Q_0^2}{2Q_0p_0\; 2Q_0\bar{p}_0} = \frac{Q^2}{(2Q\cdot p)(2Q\cdot\bar{p})}, \tag{2.2.31}$$

and the function F in Eq. (2.2.29) becomes

$$\begin{aligned} F(Q^2, p\cdot Q, \bar{p}\cdot Q) = &\frac{\Omega^{(d-2)}}{(2\pi)^{2d-2}} \frac{(Q^2)^{-\epsilon}}{(2p\cdot Q)(2\bar{p}\cdot Q)} \frac{\Gamma(1-\epsilon)^2}{\epsilon\,\Gamma(1-2\epsilon)} \\ &\times {}_2F_1\left(1, 1, 1-\epsilon, \frac{Q^2}{(2Q\cdot p)(2Q\cdot\bar{p})}\right). \end{aligned} \tag{2.2.32}$$

We substitute Eq. (2.2.32) into Eq. (2.2.27) and find

$$I_4 = t^{d-5}s^{-1}\int \mathrm{d}^dQ\,\delta\left(2Q\cdot p-\tfrac{1}{z}\right)\delta\left(2Q\cdot\bar{p}-(1-z)\right)\frac{\Omega^{(d-2)}}{(2\pi)^{2d-2}}\frac{(Q^2)^{-1-\epsilon}}{(2p\cdot Q)(2\bar{p}\cdot Q)} \\ \times\frac{\Gamma(1-\epsilon)^2}{\epsilon\,\Gamma(1-2\epsilon)}\,{}_2F_1\left(1,1,1-\epsilon,\frac{Q^2}{(2Q\cdot p)(2Q\cdot\bar{p})}\right). \tag{2.2.33}$$

To facilitate the $Q$ integration, we introduce a Sudakov decomposition $Q^\mu = \alpha p^\mu + \beta\bar{p}^\mu + Q_\perp^\mu$ and integrate over $\alpha$ and $\beta$ removing the delta functions $\delta(\beta - 1/z)$ and $\delta(\alpha-(1-z))$. We obtain

$$I_4 = t^{d-5}s^{-1}\,\frac{z}{(1-z)}\frac{\left[\Omega^{(d-2)}\right]^2}{4(2\pi)^{2d-2}}\frac{\Gamma(1-\epsilon)^2}{\epsilon\,\Gamma(1-2\epsilon)} \\ \times\int_0^{\frac{1-z}{z}}\mathrm{d}Q_\perp^2\,(Q_\perp^2)^{-\epsilon}\left(\tfrac{1-z}{z}-Q_\perp^2\right)^{-(1+\epsilon)}\,{}_2F_1\left(1,1,1-\epsilon,1-\tfrac{Q_\perp^2\,z}{1-z}\right). \tag{2.2.34}$$

Finally, we substitute $Q_\perp^2 = (1-z)(1-u)/z$, integrate over $u$ and find

$$I_4 = -t^{d-5}s^{-1}\frac{\left[\Omega^{(d-2)}\right]^2}{4(2\pi)^{2d-2}}\left(\frac{1-z}{z}\right)^{-1-2\epsilon}\frac{\Gamma(1-\epsilon)^4}{\epsilon^2\,\Gamma(1-2\epsilon)^2} \\ \times\,{}_3F_2\left(1,1,-\epsilon;1-2\epsilon,1-\epsilon,1\right), \tag{2.2.35}$$

where ${}_3F_2$ is the generalized hypergeometric function [66]. The remaining RR master integrals $I_5$ through $I_8$ can be calculated following the above discussion. For this reason we move to the discussion of the master integral $I_9$.

### 2.2.5 *Master integral $I_9$*

We consider the master integral $I_9$, it reads

$$I_9 = \int\frac{\mathrm{d}^dk_1}{(2\pi)^{d-1}}\int\frac{\mathrm{d}^dk_2}{(2\pi)^{d-1}}\,\delta^+\left(k_1^2\right)\delta^+\left(k_2^2\right)\frac{\delta\left(2k_{12}\cdot p-\frac{t}{z}\right)\delta\left(\frac{2k_{12}\cdot\bar{p}}{s}-(1-z)\right)}{(p-k_{12})^2\;(p-k_2)^2\,(p-k_1)\cdot\bar{p}}. \tag{2.2.36}$$

To proceed, we insert $1=\int\mathrm{d}^dQ\delta^d(k_1+k_2-Q)$ into the integral and obtain

$$I_9 = t^{d-5}s^{-1}\int\mathrm{d}^dQ\,\delta\left(2Q\cdot p-\tfrac{1}{z}\right)\delta\left(2Q\cdot\bar{p}-(1-z)\right)\frac{F_9(Q^2,p\cdot Q,\bar{p}\cdot Q)}{(p-Q)^2}, \tag{2.2.37}$$

where

$$F_9(Q^2,p\cdot Q,\bar{p}\cdot Q) = \int\frac{\mathrm{d}^dk_1}{(2\pi)^{d-1}}\int\frac{\mathrm{d}^dk_2}{(2\pi)^{d-1}}\,\frac{\delta^+\left(k_1^2\right)\delta^+\left(k_2^2\right)\delta^d(Q-k_1-k_2)}{\bar{p}\cdot(p-k_1)\,(p-k_2)^2}. \tag{2.2.38}$$

We choose the rest frame of $Q$, use the phase-space parametrization shown in Eq. (2.2.20), remove delta functions by integrating over $k_1$ and introduce spherical coordinates for $k_2$. We further remove the remaining delta functions by integrating over the absolute value of $\vec{k}_2$ and obtain the angular integral

$$F_9 = -\left(\frac{Q_0}{2}\right)^{d-2} \frac{1}{Q_0^2\, Q_0 p_0\, Q_0 \bar{p}_0} \frac{1}{\lambda} \int \frac{d\Omega_k^{(d-1)}}{(2\pi)^{2d-2}} \frac{1}{1-\frac{1}{\lambda}\vec{n}_{\bar{p}}\cdot\vec{n}_k} \frac{1}{1-\vec{n}_p\cdot\vec{n}_k}$$
$$= -\left(\frac{Q_0}{2}\right)^{d-2} \frac{1}{Q_0^2\, Q_0 p_0\, Q_0 \bar{p}_0} \frac{1}{\lambda} \int \frac{d\Omega_k^{(d-1)}}{(2\pi)^{2d-2}} \frac{1}{(k_n\cdot p_1)\,(k_n\cdot p_2)}, \tag{2.2.39}$$

where we introduced the notation $p_1 = (1, \frac{1}{\lambda}\vec{n}_{\bar{p}})$, with $\lambda = 1/(Q_0\bar{p}_0) - 1$, $p_2 = (1, \vec{n}_p)$ and $k_n = (1, \vec{n}_k)$. While the angular integration in Eq. (2.2.39) looks similar to the one in Eq. (2.2.29) it is in fact more difficult as the momentum $p_1$ is not massless anymore. Indeed, this case was also discussed in Ref. [65] and the result is given by the Appell hypergeometric function $F_1$ (see e.g. Ref. [67])

$$\int \frac{d\Omega_k^{(d-1)}}{(k_n\cdot p_1)\,(k_n\cdot p_2)} = -\frac{1}{\epsilon} \frac{2^{1-2\epsilon}\pi^{1-\epsilon}\,\lambda\,\Gamma(1-\epsilon)}{(\lambda - \vec{n}_p\cdot\vec{n}_{\bar{p}})\,\Gamma(1-2\epsilon)}$$
$$\times F_1\left(1, -\epsilon, -\epsilon, 1-2\epsilon, -\frac{1+\vec{n}_p\cdot\vec{n}_{\bar{p}}}{\lambda-\vec{n}_p\cdot\vec{n}_{\bar{p}}}, \frac{-1+\vec{n}_p\cdot\vec{n}_{\bar{p}}}{-\lambda+\vec{n}_p\cdot\vec{n}_{\bar{p}}}\right). \tag{2.2.40}$$

Substituting Eq. (2.2.40) into Eq. (2.2.39) and restoring Lorentz invariance, we obtain

$$F(Q^2, p\cdot Q, \bar{p}\cdot Q) = \frac{1}{(2\pi)^{2d-2}} \frac{1}{\epsilon} \frac{\pi^{1-\epsilon}\,(Q^2)^{-\epsilon}\,\Gamma(1-\epsilon)}{(Q^2 + 2p\cdot Q(1-2\bar{p}\cdot Q))\Gamma(1-2\epsilon)}$$
$$\times F_1\left(1, -\epsilon, -\epsilon, 1-2\epsilon, \frac{Q^2 - 4p\cdot Q\,\bar{p}\cdot Q}{Q^2 + 2p\cdot Q(1-2\bar{p}\cdot Q)}, \frac{Q^2}{Q^2 + 2p\cdot Q(1-2\bar{p}\cdot Q)}\right). \tag{2.2.41}$$

We insert Eq. (2.2.41) into Eq. (2.2.37), introduce the Sudakov decomposition $Q^\mu = \alpha p^\mu + \beta\bar{p}^\mu + Q_\perp^\mu$, remove delta functions by integrating over $\alpha$ and $\beta$ and substitute $Q_\perp^2 = l(1-z)/z$. We find

$$I_9 = -\frac{t^{d-5}}{s} \frac{\Omega^{(d-2)}}{4(2\pi)^{2d-2}} \frac{1}{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}$$
$$\times \int_0^1 dl \frac{\pi^{1-\epsilon}\, z\,(1-z)}{[1-l\,(1-z)]\,[l\,(1-z)+z]} \left(\frac{(1-l)\,l\,(1-z)^2}{z^2}\right)^{-\epsilon}$$
$$\times F_1\left(1, -\epsilon, -\epsilon, 1-2\epsilon, \frac{l\,(1-z)}{l(1-z)-1}, \frac{(l-1)\,(1-z)}{l(1-z)-1}\right). \tag{2.2.42}$$

While the remaining $l$-integration is too complicated to be performed in closed form in $\epsilon$, it is sufficient to know the result expanded $\epsilon$. Since there are no readily available software packages to perform an expansion for the Appell function, we replace Appell function by its integral representation [66]

$$F_1\,(a, b_1, b_2, c, z_1, z_2) = \int_0^1 du \frac{\Gamma(c)\, u^{a-1}\,(1-u)^{c-a-1}}{\Gamma(a)\Gamma(c-a)}\,(1-u\,z_1)^{-b_1}(1-u\,z_2)^{-b_2}. \tag{2.2.43}$$

We find

$$
\begin{aligned}
I_9 = {} & \frac{t^{d-5}}{s} \int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}l \, \frac{\left[\Omega^{(d-2)}\right]^2}{(2\pi)^{2d-2}} \frac{(z-1)\, z\, (1-u)^{-1-2\epsilon}\, \Gamma(1-\epsilon)^2}{4\,[1+l\,(z-1)]\,[l\,(z-1)-z]\,\Gamma(1-2\epsilon)} \\
& \times \left[\frac{(1-l)\, l\, (1-z)^2}{z}\right]^{-\epsilon} \left[\frac{z\,[1+l\,(-1+u+z-u\,z)]}{1+l\,(z-1)}\right]^{\epsilon} \\
& \times \left[\frac{1+u\,(z-1)+l\,(-1+u+z-uz)}{1+l\,(z-1)}\right]^{\epsilon}.
\end{aligned}
\tag{2.2.44}
$$

While the integral over $l$ in Eq. (2.2.44) converges if we Taylor expand around $\epsilon = 0$, the integral over $u$ is proportional to $(1-u)^{-1-2\epsilon}$ and therefore it diverges at $u = 1$ for $\epsilon = 0$. We subtract the divergence at $u = 1$, splitting the integral into two pieces. To this end, we define two functions

$$
\begin{aligned}
M(u,l) = {} & \left[\frac{z\,[1+l\,(-1+u+z-u\,z)]}{1+l\,(z-1)}\right]^{\epsilon} \\
& \times \left[\frac{1+u\,(z-1)+l\,(-1+u+z-uz)}{1+l\,(z-1)}\right]^{\epsilon},
\end{aligned}
\tag{2.2.45}
$$

$$
\begin{aligned}
G(l) = {} & \frac{t^{d-5}}{s} \frac{\left[\Omega^{(d-2)}\right]^2}{(2\pi)^{2d-2}} \frac{(z-1)\, z\, \Gamma(1-\epsilon)^2}{4\,[1+l\,(z-1)]\,[l\,(z-1)-z]\,\Gamma(1-2\epsilon)} \\
& \times \left[\frac{(1-l)\, l\, (1-z)^2}{z}\right]^{-\epsilon},
\end{aligned}
\tag{2.2.46}
$$

and re-write Eq. (2.2.44) as

$$
\begin{aligned}
I_9 &= \int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}l \, (1-u)^{-1-2\epsilon}\, G(l)\, M(u,l) \\
&= \int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}l \, (1-u)^{-1-2\epsilon}\, G(l)\, \left[M(u,l) - M(1,l)\right] \\
&\quad + \int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}l \, (1-u)^{-1-2\epsilon}\, G(l)\, M(1,l).
\end{aligned}
\tag{2.2.47}
$$

The $u = 1$ singularity in the first term on the right-hand side of Eq. (2.2.47) is now regulated, while the last term in Eq. (2.2.47) can be easily integrated over $u$. We find

$$
I_9 = \int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}l \, (1-u)^{-1-2\epsilon} G(l)\, [M(u,l) - M(1,l)] - \frac{1}{2\epsilon} \int_0^1 \mathrm{d}l G(l)\, M(1,l). \tag{2.2.48}
$$

The integrand in Eq. (2.2.48) is now expanded up to order $\epsilon^4$ and integrated using the HyperInt package [68]. The final result is expressed through harmonic polylogarithms (HPLs) [69] $H(\vec{m}_w, z)$. It reads

$$
\begin{aligned}
I_9 = \frac{t^{d-5}}{s} \frac{\left[\Omega^{(d-2)}\right]^2}{(2\pi)^{2d-2}} (1-z)^{-2\epsilon} \Bigg[ \frac{1}{\epsilon} \frac{z}{4\,(1+z)} H(0,z) \\
- \frac{z}{8\,(1+z)} \left(\pi^2 + 4\,H(-1,0,z) - 8\,H(0,0,z) + 4\,H(1,0,z)\right) \Bigg] + \mathcal{O}(\epsilon),
\end{aligned}
\tag{2.2.49}
$$

where we do not show higher orders in $\epsilon$ for the sake of brevity.

### 2.2.6 *Master integral $I_{10}$*

We finally turn to the calculation of the RV master integral $I_{10}$. While this integral has also already been discussed in Ref. [54], it is important to re-examine it for two reasons. First, it complements our discussion of the computation of master integrals and second, the calculation of the N3LO RV beam function in the next chapter will re-use the result for integrals $I_{10}$, $I_{11}$ and $I_{12}$.

The integral $I_{10}$ reads

$$
I_{10} = \int \frac{\mathrm{d}^d k}{(2\pi)^{d-1}} \int \frac{\mathrm{d}^d l}{(2\pi)^d}\, \delta^+\left(k^2\right) \frac{\delta\left(2k\cdot p - \frac{t}{z}\right)\, \delta\left(\frac{2k\cdot \bar{p}}{s} - (1-z)\right)}{l^2\,(l\cdot\bar{p})\,(p-l)^2\,(p-k-l)^2}. \tag{2.2.50}
$$

We rescale $k$ and the loop momenta using Eq. (2.2.2) as well as $l \to \tilde{l}\sqrt{t}$ and obtain

$$
I_{10} = \frac{t^{d-5}}{s} \int \frac{\mathrm{d}^d k}{(2\pi)^{d-1}} \int \frac{\mathrm{d}^d l}{(2\pi)^d}\, \delta^+\left(k^2\right) \frac{\delta\left(2k\cdot p - \frac{1}{z}\right)\, \delta\left(2k\cdot\bar{p} - (1-z)\right)}{l^2\,(l\cdot\bar{p})\,(p-l)^2\,(p-k-l)^2}, \tag{2.2.51}
$$

where we dropped tildes after the rescaling. We first integrate over $l$ and then over $k$. The loop integral in Eq. (2.2.51) is standard except for a linear propagator

$$
I = \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{l^2\;\;(l\cdot\bar{p})\;\;(p-l)^2\;\;(p-k-l)^2}. \tag{2.2.52}
$$

We address this problem by first combining the propagators $1/l^2$ and $1/l\cdot\bar{p}$ . We find

$$
\frac{1}{l^2}\frac{1}{(2l\cdot\bar{p})} = \int_0^\infty \frac{\mathrm{d}y}{(l^2 + 2\,l\cdot\bar{p}\,y)^2} = \int_0^\infty \frac{\mathrm{d}y}{[(l+y\,\bar{p})^2]^2}. \tag{2.2.53}
$$

We then combine $1/(p-l)^2$ and $1/(p-l-k)^2$ and obtain

$$\begin{aligned}\frac{1}{(p-l)^2}\frac{1}{(p-k-l)^2} &= \int_0^1 \frac{\mathrm{d}x}{\left[(1-x)(p-l)^2+x(p-k-l)^2\right]^2} \\ &= \int_0^1 \frac{\mathrm{d}x}{\left[(p-l-xk)^2\right]^2}.\end{aligned} \tag{2.2.54}$$

Putting everything together, we find

$$I = \int_0^\infty \mathrm{d}y \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{[(l+y\,\bar{p})^2]^2\ \left[(p-l-xk)^2\right]^2}. \tag{2.2.55}$$

The integral over $l$ is standard. We obtain

$$\begin{aligned}I =& \frac{12\,\mathrm{i}}{(4\pi)^{d/2}}\frac{\Gamma(4-\frac{d}{2})}{\Gamma(4)} \int_0^\infty \mathrm{d}y \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}u\ u^{-1-\epsilon}\ (1-u)^{-1-\epsilon} \\ &\times\ (2\,x\,k\,p - 2\,y\,\bar{p}(p-x\,k))^{-2-\epsilon}.\end{aligned} \tag{2.2.56}$$

After integrating Eq. (2.2.56) over $y$, the remaining $x$ and $u$ integrations are straightforward and we obtain

$$\begin{aligned}\int \frac{\mathrm{d}^d l}{(2\pi)^d}\frac{1}{l^2\ (l\cdot\bar{p})\ (p-l)^2\ (p-k-l)^2} &= -i\,2^{-2+2\epsilon}\pi^{-2+\epsilon}\frac{\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{\epsilon^2\,\Gamma(1-2\epsilon)} \\ &\times (2p\cdot k)^{-1-\epsilon}\ {}_2F_1\left(1,-\epsilon,1-\epsilon,2\bar{p}\cdot k\right).\end{aligned} \tag{2.2.57}$$

The remaining integration over the on-shell momentum $k$ is performed by introducing the Sudakov decomposition $k^\mu = \alpha p^\mu + \beta\bar{p}^\mu + k_\perp^\mu$. We find

$$\begin{aligned}I_{10} = -i\ &\frac{t^{d-5}}{s}\frac{\left[\Omega^{(d-2)}\right]^2}{4(2\pi)^{2d-2}}\ (1-z)^{-\epsilon}z^{1+2\epsilon}\frac{\Gamma(1-\epsilon)^3\Gamma(1+\epsilon)}{\epsilon^2\,\Gamma(1-2\epsilon)} \\ &\times\ {}_2F_1(1,-\epsilon,1-\epsilon,1-z).\end{aligned} \tag{2.2.58}$$

This concludes the discussion of the evaluation of master integrals. All master integrals are substituted into the expressions for the beam functions and expanded in $\epsilon$. Note that factors of $(1-z)^{-1+n\epsilon}$ and $t^{-1+n\epsilon}$ are expanded in $\epsilon$ using the so-called plus distribution (cf. Appendix A.3). The expansion to the required order in $\epsilon^2$ is performed with the help of the `HypExp` package [70]. The quantity obtained as the result of this procedure is the so-called bare partonic beam function. In the next chapter we renormalize the bare partonic beam function to obtain the so-called matching coefficients.

## 2.3 RENORMALIZATION

In this section we discuss the renormalization of the bare partonic beam functions $B_{ij}^{\text{bare}}$ obtained in the previous sections. This includes the standard ultra-violet (UV) renormalization in the $\overline{\text{MS}}$ scheme, which in this case only requires the renormalization of $\alpha_s$. However, we also discuss the renormalization[1] of infrared poles of $B_{ij}^{\text{bare}}$. The physical quantities obtained after this renormalization procedure are the so-called matching coefficients $I_{ij}$. Thus the goal of this section is to derive a formula that expresses the matching coefficients $I_{ij}$ through the bare partonic beam function $B_{ij}^{\text{bare}}$.

We begin by applying the standard renormalization in the $\overline{\text{MS}}$ scheme

$$\alpha_s^0 = Z(\alpha_s)\left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon}\alpha_s, \tag{2.3.1}$$

$$Z(\alpha_s) = 1 - \frac{\alpha_s}{\pi}\frac{\beta_0}{\epsilon} + \mathcal{O}(\alpha_s^2), \tag{2.3.2}$$

$$\beta_0 = \frac{11\, C_A - 4T_F\, n_f}{3}. \tag{2.3.3}$$

While the replacement in Eq. (2.3.1) is sufficient to remove all ultra-violet poles of the beam function, there are still infrared (IR) poles that need to be taken care off. These IR poles cancel against other IR poles of the soft and hard functions as well as collinear counter terms required for the PDF renormalization. To renormalize the beam function we follow the discussion in Ref. [71].

We start with the relation between the physical and the bare beam function

$$B_{ij}^{\text{bare}}(t,z,\mu) = \int_0^t \mathrm{d}t' Z_i\left(t-t',\mu\right) B_{ij}\left(t',z,\mu\right), \tag{2.3.4}$$

where the renormalization constants $Z_i\left(t-t',\mu\right)$ are known to the three-loop order [60]. The physical beam function in Eq. (2.3.4) is defined as

$$B_i\left(t',z,\mu\right) = \sum_k \int_0^z \frac{\mathrm{d}z'}{z'} I_{ik}\left(t',z',\mu\right) f_{kj}\left(\frac{z}{z'}\right) = \sum_k I_{ik}\left(t',z,\mu\right) \otimes f_{kj}\left(z\right), \tag{2.3.5}$$

where we introduced the shorthand notation for the $z$-convolution in the second line.

We now derive a relation between the matching coefficients $I_{ij}$ and the bare partonic beam functions $B_{ij}^{\text{bare}}$ with the help of Eqs. (2.3.4) and (2.3.5). We begin by multiplying both sides of Eq. (2.3.5) with $Z_i\left(t-t',\mu\right)$ and integrating over $t'$. We obtain

$$B_{ij}^{\text{bare}}(t,z,\mu) = \sum_k \int \mathrm{d}t'\; Z_i\left(t-t',\mu\right) I_{ik}\left(t',z',\mu\right) \otimes f_{kj}, \tag{2.3.6}$$

where we omitted the integration boundaries.

Since we want to compute the matching coefficient, we can use any external states. Thus, we will consider partons $j$ as initial states. We write the "bare" and the "physical" parton PDFs as

$$f_{kj}^{\text{phys}} = \hat{\Gamma}_{kl} \otimes f_{lj}^{\text{bare}}, \tag{2.3.7}$$

$$f_{lj}^{\text{bare}} = \delta_{lj}\delta(1-z), \tag{2.3.8}$$

1 To be consistent with the literature Ref. [60, 61] we call this procedure renormalization, however, one might also call this procedure subtraction.

and insert them into Eq. (2.3.6) to obtain

$$B_{ij}^{\text{bare}}(t,z,\mu) = \sum_k \int \mathrm{d}t'\, Z_i(t-t',\mu)\, I_{ik}(t',z,\mu) \otimes \hat{\Gamma}_{kj}(z). \tag{2.3.9}$$

The quantities $\hat{\Gamma}_{kj}$ follow from the Altarelli-Parisi equation. Up to $\mathcal{O}(\alpha_s^2)$ they read

$$\begin{aligned}\Gamma_{ik} &= \delta_{ik}\delta(1-z) - \frac{\alpha_s(\mu)}{2\pi}\frac{P_{ik}^{(0)}}{\epsilon} \\ &+ \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2 \left[\frac{1}{2\epsilon^2}\left(\sum_m P_{im}^{(0)} \otimes P_{mk}^{(0)} + \frac{\beta_0}{2}P_{ik}^{(0)}\right) - \frac{1}{2\epsilon}P_{ik}^{(1)}\right].\end{aligned} \tag{2.3.10}$$

The quantities $P_{ij}^{(n)}$, $n = 1,2$, are known and can be found in the Appendix of Ref. [60]. After expanding all quantities that appear in Eq. (2.3.9) in powers of $\alpha_s$

$$\Gamma_{jk} = \left(\frac{\alpha_s}{2\pi}\right)^{(n)} \Gamma_{jk}^{(n)}, \tag{2.3.11}$$

$$Z_q(t,\mu) = \delta(t) + \sum_{n=1}\left(\frac{\alpha_s}{4\pi}\right)^n Z_q^{(n)}, \tag{2.3.12}$$

$$I_{ik}(t,z,\mu) = \delta_{ik}\delta(t)\delta(1-z) + \sum_{n=1}\left(\frac{\alpha_s}{4\pi}\right)^n I_{ik}^{(n)}, \tag{2.3.13}$$

$$B_{qq}^{ik}(t,z,\mu) = \delta_{ik}\delta(t)\delta(1-z) + \sum_{n=1}\left(\frac{\alpha_s}{4\pi}\right)^n B_{ik}^{bare\ (n)}, \tag{2.3.14}$$

we obtain

$$I_{ij}^{(1)}(t,z,\mu) = B_{ij}^{bare\ (1)}(t,z) - Z_i^{(1)}(t,\mu)\,\delta_{ij}\,\delta(1-z) + 2\frac{P_{ij}^{(0)}}{\epsilon}\delta(t), \tag{2.3.15}$$

$$\begin{aligned}I_{ij}^{(2)}(t,z,\mu) &= B_{ij}^{bare\ (2)}(t,z) - \int \mathrm{d}t' Z_i^{(1)}(t-t',\mu)\,\left(I_{ij}^{(1)}(t',z,\mu) + 2\Gamma_{ij}^{(1)}\delta(t')\right) \\ &- 4\Gamma_{ij}^{(2)}\delta(t) - 2\sum_k I_{ik}^{(1)}(t,z,\mu)\otimes\Gamma_{kj}^{(1)}(z,\mu) - Z_i^{(2)}(t,\mu)\,\delta_{ij}\,\delta(1-z).\end{aligned} \tag{2.3.16}$$

Eqs. (2.3.15) and (2.3.16) relates the matching coefficients to the bare beam function $B_{ij}^{\text{bare}}$. However, convolutions over $z'$ and $t'$ which appear in Eqs. (2.3.15) and (2.3.16) still need to be computed.

The z convolution $\otimes$ is of the form

$$\int_0^x \frac{\mathrm{d}x_1}{x_1} f(x_1) g\left(\frac{x}{x_1}\right) = \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 f(x_1) g(x_2) \delta(x - x_1 x_2). \tag{2.3.17}$$

These convolutions have been extensively studied and we simply use the Mathematica package MT[72] to evaluate them.
The $t$ convolutions give rise to integrals of the form

$$\int_0^t \mathrm{d}t' L_n(t-t')\, L_{n'}(t), \tag{2.3.18}$$

where

$$L_n(t) = \left[\frac{\ln^n(t)}{t}\right]_+. \tag{2.3.19}$$

To calculate these contributions we consider the integral

$$\int_0^t \mathrm{d}t'(t-t')^{-1-\beta\epsilon}\, t'^{\,-1-\alpha\epsilon} = \frac{\Gamma(-\alpha\epsilon)\Gamma(-\beta\epsilon)}{\Gamma(-(\alpha+\beta)\epsilon)} t^{-1-(\alpha+\beta)\epsilon}. \tag{2.3.20}$$

and expand both sides of the equation in $\epsilon$. Since expansion of $t'^{\,-1-\alpha\epsilon}$ is performed using the plus distribution, Eq. (2.3.20) plays the role of a generating function for the convolutions. Thus, all required convolutions can be found by matching coefficients in $\alpha$ and $\beta$. All convolutions for the computation of the beam function can be found in Appendix A.5.

Combining Eq. (2.3.16) with the above convolutions we obtain all five matching coefficients $I_{q_l q_m}$, $I_{q_l,g}$, $I_{q_l \bar{q}_m}$, $I_{g,g}$ and $I_{g,q_m}$. This concludes our discussion on the renormalization. We discuss the results in the next section.

## 2.4 RESULTS AND CONCLUSION

We successfully calculated all five matching coefficients $I_{q_l q_m}$, $I_{q_l,g}$, $I_{q_l \bar{q}_m}$, $I_{g,g}$ and $I_{g,q_m}$ through second order in the dimensional regularization parameter $\epsilon$. The expressions for these functions are quite lengthy and we decide to only discuss some features of the most complicated coefficient $I_{gg}$. However, we note that expressions for all matching coefficients can be found in digital form in the ancillary file of Ref. [53].

We write the matching coefficient in the following form

$$I_{gg}^{(2)} = \left(\frac{\alpha_s}{4\pi}\right)^2 \sum_{k=0}^{5} \frac{1}{\mu^2} L_k\left(\frac{t}{\mu^2}\right) F_+^{(k)}(z) + \delta(t) F_\delta(z), \tag{2.4.1}$$

$$F_\delta(z) = C_{-1}\delta(1-z) + \sum_{k=0}^{5} C_k L_k(1-z) + F_{\delta,h}(z), \tag{2.4.2}$$

where $L_K$ is the plus distribution defined in Eq. (2.3.19). For breavity reasons we only show the coefficient $C_{-1}$ and the function $F_{\delta,h}(z)$ for $n_f = 0$. The coefficient $C_{-1}$ reads

$$\begin{aligned}C_{-1} = \; & C_A^2\left(-\frac{110\zeta(3)}{9}+\frac{2428}{81}-\frac{67\pi^2}{18}+\frac{11\pi^4}{90}\right)+C_A n_f T_F\left(\frac{40\zeta(3)}{9}-\frac{656}{81}+\frac{10\pi^2}{9}\right)\\ & +\epsilon\Bigg[C_A^2\left(-\frac{938\zeta(3)}{27}+\frac{65\pi^2\zeta(3)}{3}-150\zeta(5)+\frac{14576}{243}-\frac{202\pi^2}{27}+\frac{77\pi^4}{540}\right)\\ & +C_A n_f T_F\left(\frac{280\zeta(3)}{27}-\frac{3904}{243}+\frac{56\pi^2}{27}-\frac{7\pi^4}{135}\right)\Bigg]\\ & +\epsilon^2\Bigg[C_A^2\Bigg(-\frac{5656\zeta(3)}{81}+\frac{220\pi^2\zeta(3)}{27}+\frac{1142\zeta(3)^2}{9}-\frac{638\zeta(5)}{15}+\frac{87472}{729}\\ & -\frac{1214\pi^2}{81}+\frac{67\pi^4}{216}-\frac{593\pi^6}{11340}\Bigg)+C_A n_f T_F\Bigg(\frac{1568\zeta(3)}{81}-\frac{80\pi^2\zeta(3)}{27}+\frac{232\zeta(5)}{15}\\ & -\frac{23360}{729}+\frac{328\pi^2}{81}-\frac{5\pi^4}{54}\Bigg)\Bigg].\end{aligned} \tag{2.4.3}$$

To present the function $F_{\delta,h}(z)$ we write

$$F_{\delta,h}(z)|_{n_f=0} = C_A^2\left(F_0(z)+\epsilon\, F_1(z)+\epsilon^2\, F_2(z)\right), \tag{2.4.4}$$

and introduce the short-hand notation $H_{\vec{a}} = H(\vec{a},z)$. For brevity reasons we only present $F_0$, the function reads

$$\begin{aligned}F_0 = \; & 48\left(z^2-z-\frac{1}{z}+2\right)H_{1,1,1}+\frac{4\left(55z^3-47z^2+58z-55\right)H_{1,1}}{3z}\\ & +\frac{2\left(286z^4-365z^3+342z^2-307z+66\right)H_{0,0}}{3(z-1)z}+\frac{4\left(55z^4-102z^3+105z^2-102z+55\right)H_{1,0}}{3(z-1)z}\\ & +\frac{32\left(z^4-3z^3+3z^2-z+1\right)H_{2,0}}{(z-1)z}+\frac{8\left(7z^4-18z^3+21z^2-10z+7\right)H_{2,1}}{(z-1)z}\\ & +\frac{8\left(3z^4-10z^3-7z^2+10z+7\right)H_{0,0,0}}{(z-1)(z+1)}+\frac{8\left(6z^4-12z^3+18z^2-11z+6\right)H_{1,1,0}}{(z-1)z}\\ & +\frac{\left(z^2+z+1\right)^2}{z(z+1)}\left(-16H_{-2,0}-16H_{-1,2}+16H_{-1,-1,0}-32H_{-1,0,0}+4\pi^2H_{-1}\right)\\ & +\frac{\left(z^2-z+1\right)^2}{(z-1)z}\left(56H_{1,2}+56H_{1,0,0}\right)+\frac{16H_3\left(4z^5-7z^4+7z^2+3\right)}{z\left(z^2-1\right)}\\ & +H_1\left(\frac{2\left(134z^4+102z^3+131z^2+163z-134\right)}{9z(z+1)}-\frac{4\pi^2\left(7z^4+7z^2+13z-7\right)}{3z(z+1)}\right)\\ & +\frac{4H_2\left(99z^4-133z^3+123z^2-111z+33\right)}{3(z-1)z}+H_0\Bigg(\frac{-268z^4-563z^3+462z^2-167z+804}{9(z-1)z}\\ & -\frac{\pi^2\left(44z^5-60z^4+12z^3+64z^2-8z+28\right)}{3(z-1)z(z+1)}\Bigg)-\frac{2\pi^2\left(99z^4+65z^3+55z^2+67z-33\right)}{9z(z+1)}\\ & +\frac{2\left(2460z^4+553z^3+350z^2+255z-2406\right)}{27z(z+1)}-\frac{\left(120z^5-112z^4+88z^3+120z^2-200z+80\right)\zeta(3)}{(z-1)z(z+1)}.\end{aligned} \tag{2.4.5}$$

3

# N3LO REAL-VIRTUAL EMISSION BEAM FUNCTIONS

In this chapter we discuss the calculation of contributions to the N3LO beam function due to real-virtual emissions. This contribution is required for the ongoing computation of the N3LO beam function in Refs. [44–46]. However, unlike the contributions discussed in these references, the complexity of the RV emission contribution is comparable to the NNLO contributions discussed in the previous chapter. Indeed, the computation of the N3LO RV beam function is very similar to the one of the NNLO RV beam function because the Feynman diagrams required for these two quantities are directly related to each other. For this reason the two beam functions share intermediate results and the calculation of the N3LO RV beam function is straightforward after the discussion of the previous chapter.

The chapter is structured as follows. In Section 3.1 we explain the general setup of the calculation and illustrate how the N3LO RV emission contribution can be obtained using results of the previous chapter. In Section 3.2 we explain how to obtain the RV master integrals as the direct product of NNLO RV master integrals. Finally, we discuss the results and conclude in Section 3.3.

## 3.1 GENERAL SETUP

In this section we discuss the calculation of the RV beam function at N3LO. Similar to the NNLO case, the beam function is obtained from the following equation (cf. Eq. (2.1.7))

$$B^b_{ij} = \mathcal{N}_{ij} \sum_{\{m\}} \frac{1}{\phi_{\{m\}}} \int \mathrm{dPS}^{(\mathrm{m})} \mathcal{P} |M_{j\to i^*\{m\}}|^2, \tag{3.1.1}$$

where we need to sum over all possible emissions $m$ under which a parton $j$ changes into a parton $i$.

However, in this chapter we are only interested in the case where $\{m\}$ is a single parton. For example for $B^b_{gg}$ we only consider the matrix element $|M_{g\to g^*\{g\}}|^2$. We distinguish the two loop $M^{ll}_{g\to g^*\{g\}}$, the one loop $M^{l}_{g\to g^*\{g\}}$ and the tree level $M^{t}_{g\to g^*\{g\}}$ amplitudes. The squared amplitude is then written as

$$\begin{aligned} |M_{g\to g^*\{g\}}|^2 &= |M^{ll}_{g\to g^*\{g\}} + M^{l}_{g\to g^*\{g\}} + M^{t}_{g\to g^*\{g\}}|^2 \\ &= M^{ll}_{g\to g^*\{g\}} M^{t\dagger}_{g\to g^*\{g\}} + M^{t}_{g\to g^*\{g\}} M^{ll\dagger}_{g\to g^*\{g\}} + M^{l}_{g\to g^*\{g\}} M^{l\dagger}_{g\to g^*\{g\}} \\ &= 2\,\mathrm{Re}\Big\{ M^{ll}_{g\to g^*\{g\}} M^{t\dagger}_{g\to g^*\{g\}} \Big\} + M^{l}_{g\to g^*\{g\}} M^{l\dagger}_{g\to g^*\{g\}}, \end{aligned} \tag{3.1.2}$$

where we dropped all terms that do not scale as $\alpha_s^3$.

The first term in the last line of Eq. (3.1.2) is the real-virtual-virtual (RVV) contribution while the second term is the real-virtual contribution that we consider in this chapter. The required diagrams are obtained by interfering the seven RV diagrams in Fig. 2.1 with themselves, yielding the diagrams shown in Fig. 3.1. Diagrams for all other RV emission beam functions are given in Appendix B.

We proceed by implementing Feynman rules in Form [62] to obtain mathematical expressions for all diagrams. In doing so we denote the loop momentum appearing in $M^{l}_{g\to g^*\{g\}}$ as $l$ and the one appearing in $M^{l\dagger}_{g\to g^*\{g\}}$ as $u$. Since these amplitudes are the same as for the NNLO RV computation, we can reuse the previous partial fraction and topology definitions. If, for example, a RV amplitude requires the following topologies for the NNLO calculation

$$I^{A}_{1,1,1,a,b,c,e} = \left[\left[\frac{1}{l^2}\right]^a \left[\frac{1}{(p-k-l)^2}\right]^b \left[\frac{1}{(p-l)^2}\right]^c \left[\frac{1}{l\cdot\bar{p}}\right]^e\right]_{(1)}, \tag{3.1.3}$$

$$I^{B}_{1,1,1,a,b,c,e} = \left[\left[\frac{1}{l^2}\right]^a \left[\frac{1}{(p-l)^2}\right]^b \left[\frac{1}{(l-k)^2}\right]^c \left[\frac{1}{l\cdot\bar{p}}\right]^e\right]_{(1)}, \tag{3.1.4}$$

then the topologies needed for the N3LO computation read

$$\begin{aligned} I^{AlAu}_{1,1,1,a,b,c,e,a',b',c',e',f} = \Bigg[ &\left[\frac{1}{l^2+\mathrm{i}0}\right]^a \left[\frac{1}{(p-k-l)^2+\mathrm{i}0}\right]^b \left[\frac{1}{(p-l)^2+\mathrm{i}0}\right]^c \\ &\times \left[\frac{1}{l\cdot\bar{p}+\mathrm{i}0}\right]^e \left[\frac{1}{u^2-\mathrm{i}0}\right]^{a'} \left[\frac{1}{(p-k-u)^2-\mathrm{i}0}\right]^{b'} \\ &\times \left[\frac{1}{(p-u)^2-\mathrm{i}0}\right]^{c'} \left[\frac{1}{u\cdot\bar{p}-\mathrm{i}0}\right]^{e'} \left[\frac{1}{(l+u)^2}\right]^{f} \Bigg]_{(l,u)}, \end{aligned} \tag{3.1.5}$$

$$\begin{aligned} I^{AlBu}_{1,1,1,a,b,c,e,a',b',c',e',f} = \Bigg[ &\left[\frac{1}{l^2+\mathrm{i}0}\right]^a \left[\frac{1}{(p-k-l)^2+\mathrm{i}0}\right]^b \left[\frac{1}{(p-l)^2+\mathrm{i}0}\right]^c \\ &\times \left[\frac{1}{l\cdot\bar{p}+\mathrm{i}0}\right]^e \left[\frac{1}{u^2-\mathrm{i}0}\right]^{a'} \left[\frac{1}{(p-u)^2-\mathrm{i}0}\right]^{b'} \\ &\times \left[\frac{1}{(u-k)^2-\mathrm{i}0}\right]^{c'} \left[\frac{1}{u\cdot\bar{p}-\mathrm{i}0}\right]^{e'} \left[\frac{1}{(l+u)^2}\right]^{f} \Bigg]_{(l,u)}, \end{aligned} \tag{3.1.6}$$

$$\vdots$$

where ellipses stand for all other combinations of the families defined in Eqs. (3.1.3) and (3.1.4). We note that in Eqs. (3.1.3) - (3.1.6) we used the shorthand notations

$$[f]_{(1)} = \int \mathrm{dPS}^{(1)} \int \frac{\mathrm{d}^d l}{(2\pi)^d}\, f, \tag{3.1.7}$$

$$[f]_{(l,u)} = \int \mathrm{dPS}^{(1)} \int \frac{\mathrm{d}^d l}{(2\pi)^d} \int \frac{\mathrm{d}^d u}{(2\pi)^d}\, f, \tag{3.1.8}$$

$$\mathrm{dPS}^{(1)} = \frac{\mathrm{d}^d k}{(2\pi)^{d-1}} \, \delta^+ \left(k^2\right) \, \delta\left(2k \cdot p - \tfrac{t}{z}\right) \, \delta\left(\tfrac{2k\cdot\bar{p}}{s} - (1-z)\right). \tag{3.1.9}$$

We further note that the last propagators in Eqs. (3.1.5) and (3.1.6) are required because the collinear projection operator connects the otherwise disconnected loop momenta. Indeed, the contractions in the projection operator (cf. Eq. (2.1.5))

$$\left\langle P_{j\to i^*\{m\}}\right\rangle = \mathcal{P}|M_{j\to i^*\{m\}}|^2, \tag{3.1.10}$$

$$\mathcal{P}|M_{j\to i^*\{m\}}|^2 = \begin{cases} \sum \mathrm{Tr}\left[M_{j\to i^*\{m\}} \frac{\hat{\bar{p}}}{4\bar{p}\cdot p_s} M^\dagger_{j\to i^*\{m\}}\right], & \text{if } i \in \{q,\bar{q}\} \\ -\frac{1}{2(1-\epsilon)} \sum d^\rho_\mu\left(p_s\right) d_{\nu\rho}\left(p_s\right) M^\mu_{j\to i^*\{m\}} M^{\nu\dagger}_{j\to i^*\{m\}}, & \text{if } i \in \{g\} \end{cases} \tag{3.1.11}$$

may lead to a factor $l \cdot u$, which solely appears in the numerator. This factor is mapped onto integral families Eqs. (3.1.5) and (3.1.6) by means of the identity

$$l \cdot u = \frac{1}{2}\left((l+u)^2 - l^2 - u^2\right). \tag{3.1.12}$$

We further note, that since the "propagator" $1/(l+u)^2$ only appears in the numerator we do not need to assign a Feynman prescription to it.
After mapping all integrals onto families we perform a reduction using `KIRA` [73]. Since `KIRA` automatically looks for shifts of loop momenta, that map different integrals onto each other and since `KIRA` does not keep track of the Feynman prescription, such shifts include for example $l \to u$ and $u \to l$. If applied in our case, this shift of loop momenta would effectively perform a complex conjugation of some integrals. In the context of IBP programs, such relations that map different integrals onto each other are commonly referred to as inter-family relations. We disable all inter-family relations when performing the reduction with `KIRA`, and only apply shifts of loop momenta that preserve the Feynman prescription after the reduction. We further discard all equations that contain an integral with a $1/(l+u)^2$ propagator when generating IBP relations. Proceeding like this we express the beam function in terms of master integrals. We explain their evaluation in the next section.

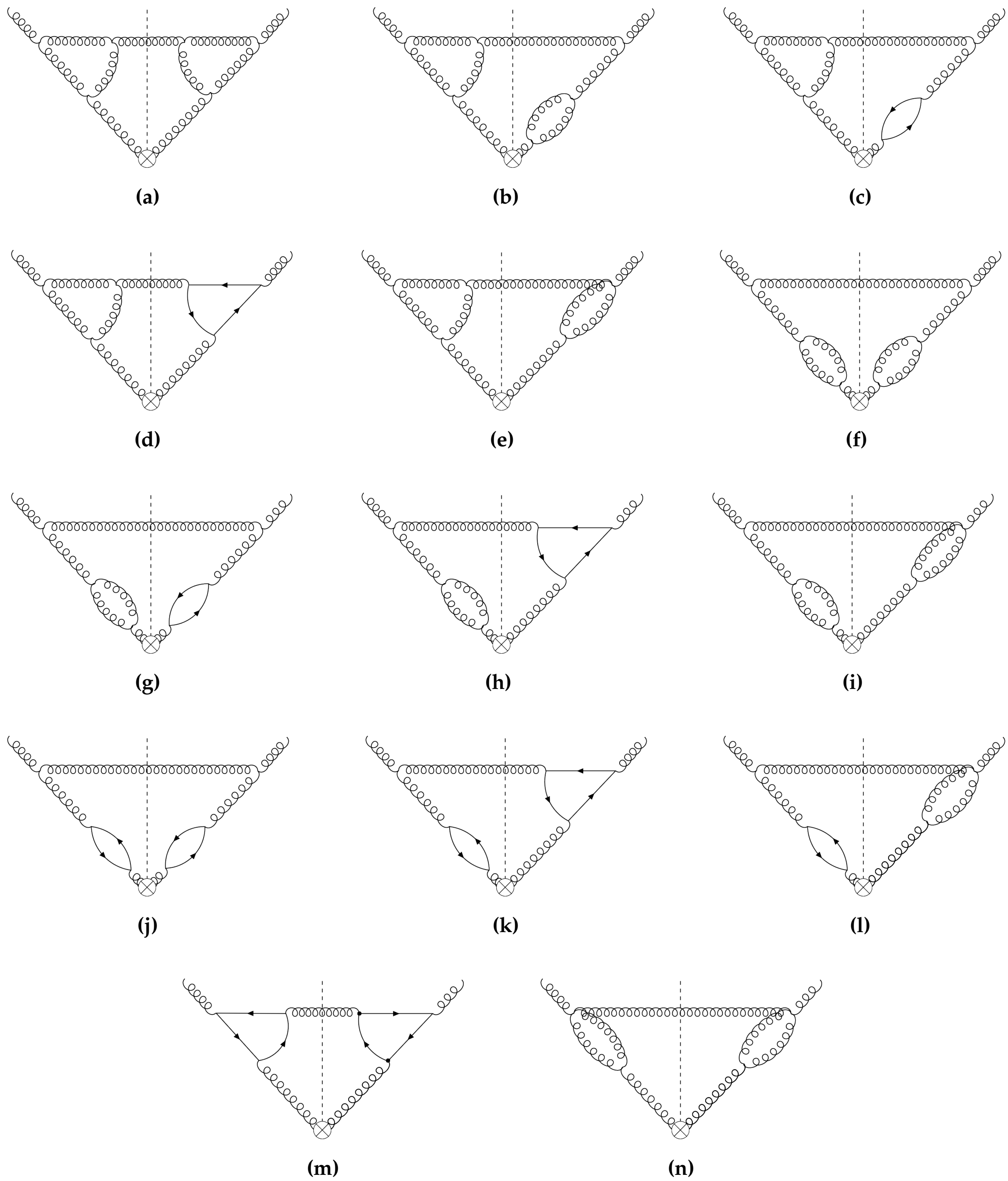


**Figure 3.1:** Diagrams for the RV emission contribution to the N3LO $B_{gg}^{(b)}$ beam function. The dashed line represents a "cut" so that all particles crossing it are on the mass-shell. The vertex $\otimes$ denotes the insertion of the projection operator defined in Eq. (2.1.5). Diagrams which can be obtained by reversing any of the fermion flows and/or mirroring the diagram at the cut are not shown.

## 3.2 CALCULATION OF MASTER INTEGRALS

After shifts of the loop momenta were performed by hand we obtain nine master integrals. Notably the "propagator" $1/(l+u)^2$ linking the two loop momenta together is gone after the reduction. For this reason, all master integrals can be written as a direct product of already known RV master integrals $I_{10}$, $I_{11}$ and $I_{12}$ in Eq. (2.1.12) up to a prefactor. They read

$$I_{10} = -i\,\frac{t^{d-5}}{s}\frac{\left[\Omega^{(d-2)}\right]^2}{4(2\pi)^{2d-2}}\,(1-z)^{-\epsilon}z^{1+2\epsilon}\frac{\Gamma(1-\epsilon)^3\Gamma(1+\epsilon)}{\epsilon^2\,\Gamma(1-2\epsilon)}\;{}_2F_1(1,-\epsilon,1-\epsilon,1-z), \tag{3.2.1}$$

$$I_{11} = i\,t^{d-4}\frac{\left[\Omega^{(d-2)}\right]^2}{16(2\pi)^{2d-2}}(1-z)^{-\epsilon}z^{2\epsilon}\frac{\Gamma(1-\epsilon)^3\Gamma(1+\epsilon)}{\epsilon(1-2\epsilon)\Gamma(1-2\epsilon)}, \tag{3.2.2}$$

$$\begin{aligned} I_{12} = i\,&\frac{t^{d-5}}{s}\frac{\left[\Omega^{(d-2)}\right]^2}{4(2\pi)^{2d-2}}\frac{\Gamma(1-\epsilon)^4\Gamma(2+\epsilon)}{\epsilon^2(1+\epsilon)\Gamma(1-2\epsilon)} \\ &\times\left[\left(\frac{1-z}{z}\right)^{-1-2\epsilon}\Gamma(1+\epsilon)(\cos(\pi\epsilon)+i\sin(\pi\epsilon))\right. \\ &\left.\quad-\;\epsilon\,\frac{z^{1+2\epsilon}(1-z)^{-1-\epsilon}\,{}_2F_1(1,\epsilon+1,\epsilon+2,1-z)}{(1+\epsilon)\Gamma(1-\epsilon)}\right]. \end{aligned} \tag{3.2.3}$$

For example, we find the following master integral

$$\begin{aligned} I_1^{RVRV} = &\int \mathrm{dPS}^{(1)}\int\frac{\mathrm{d}^d l}{(2\pi)^d}\frac{1}{l^2+i0}\frac{1}{(p-k-l)^2+i0}\,\frac{1}{(p-l)^2+i0}\frac{1}{l\cdot\bar{p}+i0} \\ &\times\int\frac{\mathrm{d}^d u}{(2\pi)^d}\frac{1}{u^2-i0}\frac{1}{(p-k-u)^2-i0}\frac{1}{(p-u)^2-i0}\frac{1}{u\cdot\bar{p}-i0}. \end{aligned} \tag{3.2.4}$$

While the individual loop integrals are just the loop integral encountered in $I_{10}$ and its complex conjugate, there is only one phase-space integration. However, the phase-space integration effectively amounts to making the replacements $2k\cdot p \to t/z$ and $2k\cdot\bar{p} \to s(1-z)$ and to multiplying the integrand with the following prefactor $C$

$$C = \frac{\left[\Omega^{(d-2)}\right]}{4(2\pi)^{d-1}}\left(\frac{t(1-z)}{z}\right)^{-\epsilon}. \tag{3.2.5}$$

Exploiting the simplicity of the phase-space integration we re-write the last line in Eq. (3.2.4) as the product of NNLO RV solutions. We find

$$\begin{aligned} I_1^{RVRV} &= \left(\int \mathrm{dPS}^{(1)}\int\frac{\mathrm{d}^d l}{(2\pi)^d}\frac{1}{l^2+i0}\frac{1}{(p-k-l)^2+i0}\,\frac{1}{(p-l)^2+i0}\frac{1}{l\cdot\bar{p}+i0}\right) \\ &\times\left(\int\frac{\mathrm{dPS}^{(1)}}{C}\int\frac{\mathrm{d}^d u}{(2\pi)^d}\frac{1}{u^2-i0}\frac{1}{(p-k-u)^2-i0}\frac{1}{(p-u)^2-i0}\frac{1}{u\cdot\bar{p}-i0}\right), \\ &= \frac{I_{10}I_{10}^*}{C}, \end{aligned} \tag{3.2.6}$$

$$= \frac{t^{-2-3\epsilon}}{s^2} \frac{\left[\Omega^{(d-2)}\right]^3}{4(2\pi)^{3d-3}} (1-z)^{-\epsilon} z^{2+3\epsilon} \frac{\Gamma(1-\epsilon)^6 \Gamma(2+\epsilon)^2}{\epsilon^4 (1+\epsilon)^2 \Gamma(1-2\epsilon)^2}$$
$$\times\ {}_2F_1(1,-\epsilon,1-\epsilon,1-z)^2.$$

Proceeding similarly, we find the results for all master integrals. They are defined as

$$I_1^{RVRV} = [I_{10}, I_{10}]_{RVRV}\,, \tag{3.2.7}$$
$$I_2^{RVRV} = [I_{10}, I_{11}]_{RVRV}\,, \tag{3.2.8}$$
$$I_3^{RVRV} = [I_{10}, I_{12}]_{RVRV}\,, \tag{3.2.9}$$
$$I_4^{RVRV} = [I_{11}, I_{10}]_{RVRV}\,, \tag{3.2.10}$$
$$I_5^{RVRV} = [I_{11}, I_{11}]_{RVRV}\,, \tag{3.2.11}$$
$$I_6^{RVRV} = [I_{11}, I_{12}]_{RVRV}\,, \tag{3.2.12}$$
$$I_7^{RVRV} = [I_{12}, I_{10}]_{RVRV}\,, \tag{3.2.13}$$
$$I_8^{RVRV} = [I_{12}, I_{11}]_{RVRV}\,, \tag{3.2.14}$$
$$I_9^{RVRV} = [I_{12}, I_{12}]_{RVRV}\,, \tag{3.2.15}$$

where we used the shorthand notation

$$[a, b]_{RVRV} = \frac{ab^*}{C}. \tag{3.2.16}$$

They read

$$I_2^{RVRV} = -\frac{t^{-1-3\epsilon}}{s} \frac{\left[\Omega^{(d-2)}\right]^3}{16(2\pi)^{3d-3}} (1-z)^{-\epsilon} z^{1+3\epsilon} \frac{\Gamma(1-\epsilon)^6 \Gamma(1+\epsilon) \Gamma(2+\epsilon)}{\epsilon^3 (1+\epsilon)(1-2\epsilon) \Gamma(1-2\epsilon)^2} \tag{3.2.17}$$
$$\times\ {}_2F_1(1,-\epsilon,1-\epsilon,1-z),$$

$$I_3^{RVRV} = -\frac{t^{-2-3\epsilon}}{s^2} \frac{\left[\Omega^{(d-2)}\right]^3}{16(2\pi)^{3d-3}}\, {}_2F_1(1,-\epsilon,1-\epsilon,1-z) \frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)^6 \Gamma(2+\epsilon)^2}{(1+\epsilon)^2 \Gamma(-2\epsilon)^2}$$
$$\times z^{2+3\epsilon} \Bigg[ -\frac{(1-z)^{-1-\epsilon} \Gamma(-1-\epsilon)}{\Gamma(-\epsilon)^2}\, {}_2F_1(1,1+\epsilon,2+\epsilon,1-z) \tag{3.2.18}$$
$$+ (1-z)^{-1-2\epsilon} \Gamma(1+\epsilon) (\cos(\pi\epsilon) - i \sin(\pi\epsilon)) \Bigg],$$

$$I_4^{RVRV} = I_2^{RVRV}, \tag{3.2.19}$$

$$I_5^{RVRV} = t^{-3\epsilon} \frac{\left[\Omega^{(d-2)}\right]^3}{64(2\pi)^{3d-3}} (1-z)^{-\epsilon} z^{3\epsilon} \frac{\Gamma(1-\epsilon)^6 \Gamma(1+\epsilon)^2}{(1-2\epsilon)^2 \epsilon^2 \Gamma(1-2\epsilon)^2}, \tag{3.2.20}$$

$$I_6^{RVRV} = \frac{t^{-1-3\epsilon}}{s} \frac{\left[\Omega^{(d-2)}\right]^3}{16(2\pi)^{3d-3}} (1-z)^{-1-2\epsilon} z^{1+3\epsilon} \frac{\Gamma(1-\epsilon)^6 \Gamma(1+\epsilon)\Gamma(2+\epsilon)}{\epsilon^3(1+\epsilon)^2(1-2\epsilon)\Gamma(1-2\epsilon)^2} \times \Bigg[ -\epsilon(1-z)^\epsilon \, {}_2F_1(1,1+\epsilon,2+\epsilon,1-z) + (1+\epsilon)\Gamma(1-\epsilon)\Gamma(1+\epsilon)(\cos(\pi\epsilon) - i\sin(\pi\epsilon)) \Bigg], \tag{3.2.21}$$

$$I_7^{RVRV} = \left(I_3^{RVRV}\right)^*, \tag{3.2.22}$$

$$I_8^{RVRV} = \left(I_6^{RVRV}\right)^*, \tag{3.2.23}$$

$$I_9^{RVRV} = \frac{t^{-2-3\epsilon}}{s^2} \frac{\left[\Omega^{(d-2)}\right]^3}{4(2\pi)^{3d-3}} (1-z)^{-2-3\epsilon} z^{2+3\epsilon} \frac{\Gamma(1-\epsilon)^6 \Gamma(2+\epsilon)^2}{\epsilon^4(1+\epsilon)^4\Gamma(1-2\epsilon)^2} \times \Bigg[ \epsilon^2(1-z)^{2\epsilon} \, {}_2F_1(1,1+\epsilon,2+\epsilon,1-z)^2 - 2\epsilon(1+\epsilon)(1-z)^\epsilon \cos(\pi\epsilon)\Gamma(1-\epsilon)\Gamma(1+\epsilon) \, {}_2F_1(1,1+\epsilon,2+\epsilon,1-z) + (1+\epsilon)^2\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)^2 \Bigg]. \tag{3.2.24}$$

We note that in Eqs. (3.2.18) and (3.2.21) - (3.2.23) it is sufficient to only take the real part of the master integrals. However, it serves as a useful cross-check to keep the imaginary part for all integrals and let them cancel between each other in the full expression for the bare beam function.

This concludes our discussion of master integrals for RV contributions to the N3LO beam function. All master integrals are substituted into the expression for the beam function and any factors of $(1-z)^{-1+n\epsilon}$ and $t^{-1+n\epsilon}$ are written as a plus distribution (cf. Appendix A.3), the resulting expression is expanded to the zeroth order in $\epsilon$. The quantity obtained by this procedure is the RV contribution to the bare partonic beam function. We discuss the results and conclude in the next section.

## 3.3 RESULTS AND CONCLUSION

The individual contributions to the RV bare partonic beam functions read

$$\left(\frac{4\pi}{\alpha_s}\right)^3 B_{gg}^{(b)} = \frac{C_A^3}{4} \Bigg\{ \frac{1}{\epsilon^6} \frac{16}{9} \delta(1-z)\delta(t) + \frac{1}{\epsilon^5} \Bigg[ \delta(t) \Bigg( -\frac{16}{3} L_0(1-z) - \frac{16(1-z)^3}{3z} + \frac{32(1-z)^2}{3z} - \frac{16(1-z)}{z} + \frac{16}{3z} \Bigg) - \frac{16\delta(1-z) L_0\left(\frac{t}{\mu^2}\right)}{3\mu^2} \Bigg] \Bigg\} + \mathcal{O}\left(\epsilon^{-4}\right), \tag{3.3.1}$$

$$\begin{aligned}\left(\frac{4\pi}{\alpha_s}\right)^3 B_{gq}^{(b)} &= \frac{1}{4}\frac{1}{\epsilon^5}\left\{C_A^2 C_F\left(-\frac{8(1-z)^2}{3z}-\frac{8}{3z}\right)\delta(t)\right.\\ &\quad\left.+C_A C_F^2\left(\frac{32(1-z)^2}{3z}+\frac{32}{3z}\right)\delta(t)+C_F^3\left(-\frac{32(1-z)^2}{3z}-\frac{32}{3z}\right)\delta(t)\right\}\\ &\quad+\mathcal{O}\left(\epsilon^{-4}\right),\end{aligned} \tag{3.3.2}$$

$$\begin{aligned}\left(\frac{4\pi}{\alpha_s}\right)^3 B_{qg}^{(b)} &= \frac{1}{4}\frac{1}{\epsilon^5}\left\{-\frac{8}{3}C_A^2 T_F\delta(t)-\frac{1}{3}16C_A^2 T_F(1-z)^2\delta(t)\right.\\ &\quad\left.+\frac{16}{3}C_A^2 T_F(1-z)\delta(t)\right\}+\mathcal{O}\left(\epsilon^{-4}\right),\end{aligned} \tag{3.3.3}$$

$$\begin{aligned}\left(\frac{4\pi}{\alpha_s}\right)^3 B_{qq}^{(b)} &= \frac{C_A^2 C_F}{4}\left\{\frac{1}{\epsilon^6}\frac{16}{9}\delta(t)\delta(1-z)+\frac{1}{\epsilon^5}\left[\delta(t)\left(-\frac{16}{3}L_0(1-z)-\frac{1}{3}8(1-z)\right.\right.\right.\\ &\quad\left.\left.\left.+\frac{16}{3}\right)-\frac{16\delta(1-z)L_0\left(\frac{t}{\mu^2}\right)}{3\mu^2}\right]\right\}+\mathcal{O}\left(\epsilon^{-4}\right).\end{aligned} \tag{3.3.4}$$

For brevity reason we only displayed the first orders in $\epsilon$, however all contributions are calculated through zeroth order.

To summarize, in this chapter we computed the simplest contribution to the N3LO beam functions, the real-virtual emission contributions. The calculation was straightforward, as intermediate and final results of the RV emission contributions to the NNLO beam functions discussed in the previous chapter could be used. We discuss the computation of the soft function in the next chapter.

Part II

# SOFT FUNCTION

4

# NNLO SOFT FUNCTION AND AUXILIARY PARAMETERS

*This chapter is based on Reference [53].*

In this chapter we discuss the calculation of the NNLO zero-jettiness soft function. We begin by explaining the general setup in Section 4.1, relating the soft function to soft limits of QCD cross-sections that are given by the so-called eikonal functions. We re-write Heaviside functions that appear in the definition of the zero-jettiness observable as integrals of delta functions over auxiliary parameters. We then use reverse unitarity to express the soft function in terms of master integrals. We discuss the calculation of these master integrals in Section 4.2. Afterwards we explain how to perform the remaining integrations over auxiliary parameters in Section 4.3. Finally, we present the results for the NNLO soft-function in Section 4.4.

## 4.1 GENERAL SETUP

Similar to the beam function, the zero-jettiness soft function is a quantity originally defined in SCET. However, in full analogy to Eq. (2.1.1) it can be calculated by considering soft limits of squares of scattering amplitudes in QCD, which are given by the so-called eikonal functions $\xi$ [52]. The soft-function is then obtained by integrating the eikonal function over an appropriate phase-space. To be precise, the bare soft function $S$ is related to the eikonal function $\xi_{\{m\}}$ which describes the soft emission of $m$ partons

$$S = \sum_{\{m\}} \frac{1}{\phi_{\{m\}}} \int \mathrm{dPS}_{\mathrm{S}}^{(\mathrm{m})} M_m \xi_{\{m\}}, \tag{4.1.1}$$

where $\phi_{\{m\}}$ is an identical particle factor,

$$\mathrm{dPS}_S^{(m)} = \prod_{n=1}^{m} \frac{\mathrm{d}^d k_n}{(2\pi)^{d-1}} \delta^+ \left(k_n^2\right) = \prod_{n=1}^{m} [dk_n], \tag{4.1.2}$$

is the unresolved phase space for $m$ emissions and

$$M_m = \delta\left(\tau - \sum_{i=1}^{m} \min[k_i \cdot n, k_i \cdot \bar{n}]\right), \tag{4.1.3}$$

is the $m$-particle measurement function. Note that, to be consistent with the literature [48, 49], we have changed the definition of the zero-jettiness variable in Eq. (4.1.3) compared to the one already introduced in chapters Chapters 2 and 3. That is, we rescaled the light-like momenta $p$ and $\bar{p}$ as $n = 2p$ and $\bar{n} = 2\bar{p}$ and set $n \cdot \bar{n} = 2$.

The value of the sum $\sum_{i=1}^{m} \min[k_i \cdot n, k_i \cdot \bar{n}]$ in Eq. (4.1.3) depends on the relative orientation of $k_i, n$ and $\bar{n}$ where $i \in \mathbb{N}$. To express it in a unique way we use Heaviside functions. Explicitly, for up to two emissions the observable reads

$$M_0 = \delta(\tau), \tag{4.1.4}$$

$$M_1 = \delta(\tau - n \cdot k_1)\,\theta(\bar{n} \cdot k_1 - n \cdot k_1) + \delta(\tau - \bar{n} \cdot k_1)\,\theta(n \cdot k_1 - \bar{n} \cdot k_1), \tag{4.1.5}$$

$$\begin{aligned} M_2 = [&\delta(\tau - n \cdot k_1 - n \cdot k_2)\,\theta(\bar{n} \cdot k_1 - n \cdot k_1)\,\theta(\bar{n} \cdot k_2 - n \cdot k_2) + (n^\mu \leftrightarrow \bar{n}^\mu) \\ &+ \delta(\tau - \bar{n} \cdot k_1 - n \cdot k_2)\,\theta(n \cdot k_1 - \bar{n} \cdot k_1)\,\theta(\bar{n} \cdot k_2 - n \cdot k_2) + (n^\mu \leftrightarrow \bar{n}^\mu)]. \end{aligned} \tag{4.1.6}$$

We refer to different sets of delta functions and Heaviside-functions in Eqs. (4.1.4) - (4.1.6) as "configurations". Since the NNLO eikonal function $\xi$ is invariant under exchange of $n$ and $\bar{n}$, it is sufficient to only consider two configurations, which we refer to as $A$ and $B$. Hence, we write

$$M_1 = M_A^{(1)} + M_B^{(1)}, \tag{4.1.7}$$

$$M_2 = 2\,M_A + 2\,M_B, \tag{4.1.8}$$

where

$$M_A^{(1)}(k_1) = \delta(\tau - n \cdot k_1)\,\theta(\bar{n} \cdot k_1 - n \cdot k_1), \tag{4.1.9}$$

$$M_B^{(1)}(k_1) = \delta(\tau - \bar{n} \cdot k_1)\,\theta(n \cdot k_1 - \bar{n} \cdot k_1), \tag{4.1.10}$$

$$M_A(k_1, k_2) = \delta(\tau - n \cdot k_1 - n \cdot k_2)\,\theta(\bar{n} \cdot k_1 - n \cdot k_1)\,\theta(\bar{n} \cdot k_2 - n \cdot k_2), \tag{4.1.11}$$

$$M_B(k_1, k_2) = \delta(\tau - n \cdot k_1 - \bar{n} \cdot k_2)\,\theta(\bar{n} \cdot k_1 - n \cdot k_1)\,\theta(n \cdot k_2 - \bar{n} \cdot k_2). \tag{4.1.12}$$

To further facilitate the calculation we expand the bare soft function in a series in the bare strong coupling constant

$$S = \sum_{i=0}^{n} \left[g_{b,s}^2\right]^i S^{(i)}. \tag{4.1.13}$$

The zeroth order is simply defined as [48]

$$S^{(0)} = \delta(\tau). \tag{4.1.14}$$

To compute the NLO contribution to the zero-jettiness soft function we need to consider corrections due to the emission of a soft gluon. This contribution reads

$$S^{(1)} = S_A^{(1)} + S_B^{(1)} = \int \mathrm{dPS}_S^{(1)}\, M_A^{(1)}\, \xi_g^{(1)} + \int \mathrm{dPS}_S^{(1)}\, M_B^{(1)}\, \xi_g^{(1)}, \tag{4.1.15}$$

where [52]

$$\xi_g^{(1)} = C_a\, 2\, \frac{n \cdot \bar{n}}{(n \cdot k_1)(\bar{n} \cdot k_1)}, \tag{4.1.16}$$

and $C_a = C_F(C_A)$ if the incoming particles are quarks (gluons), respectively. Since $\xi_g^{(1)}$ is symmetric under exchange of $n \leftrightarrow \bar{n}$ it follows that $S_B^{(1)} = S_A^{(1)}$.

To compute $S_A^{(1)}$ we introduce the Sudakov decomposition

$$k_1^\mu = \alpha_1 \frac{n^\mu}{2} + \beta_1 \frac{\bar{n}^\mu}{2} + k_{1\perp}^\mu, \tag{4.1.17}$$

use spherical coordinates for $k_{i\perp}^\mu$ and remove the delta function $\delta(\alpha_1\beta_1 - k_{1\perp}^2)$ by integrating over $k_{1\perp}^2$. The phase space becomes

$$[\mathrm{d}k_1] = \frac{\mathrm{d}\Omega_1^{(d-2)}}{4(2\pi)^{d-1}} \,\mathrm{d}\alpha_1 \,\mathrm{d}\beta_1 \,(\alpha_1\beta_1)^{-\epsilon}, \qquad \alpha_1\,, \beta_1 \in [0, \infty)\,. \tag{4.1.18}$$

Using this parametrization in Eq. (4.1.18) the computation of $S_A^{(1)}$ is straightforward, and we obtain

$$\begin{aligned} S_A^{(1)}(\tau) = &\; C_a \int \mathrm{dPS}_S^{(1)} \,\delta\left(\tau - k_1 \cdot n\right) \theta\left(k_1 \cdot \bar{n} - k_1 \cdot n\right) \,2\, \frac{n \cdot \bar{n}}{\left(k_1 \cdot n\right)\left(k_1 \cdot \bar{n}\right)} \\ = &\; C_a \left[\frac{\Omega^{(d-2)}}{4(2\pi)^{d-1}}\right] 4 \frac{\tau^{-1-2\epsilon}}{\epsilon}. \end{aligned} \tag{4.1.19}$$

At NNLO we need to consider three different contributions to the soft function

$$\begin{aligned} S^{(2)} &= \int \mathrm{dPS}_S^{(1)} \,M_1 \,\xi_g^{(2)} + \frac{1}{2!} \int \mathrm{dPS}_S^{(2)} \,M_2 \,\xi_{gg}^{(2)} + \int \mathrm{dPS}_S^{(2)} \,M_2 \,\xi_{q\bar{q}}^{(2)} \\ &= S_g^{(2)} + S_{gg}^{(2)} + S_{q\bar{q}}^{(2)}. \end{aligned} \tag{4.1.20}$$

The single gluon emission contribution $S_g^{(2)}$ in Eq. (4.1.20) comes from loop corrections to the NLO result. It has already been calculated to arbitrary order in $\epsilon$ in Ref. [48]. It reads

$$S_g^{(2)} = -\frac{\left[\Omega_{d-2}\right]^2}{2(2\pi)^{2d-2}} C_a C_A \frac{\Gamma(1-\epsilon)^5 \Gamma(1+\epsilon)^3}{\Gamma(1-2\epsilon)^2 \Gamma(1+2\epsilon)} \frac{\tau^{-1-4\epsilon}}{\epsilon^3}. \tag{4.1.21}$$

We continue with the discussion of the double-real emission pieces $S_{gg}^{(2)}$ and $S_{q\bar{q}}^{(2)}$. The required eikonal functions $\xi_{gg}^{(2)}$ and $\xi_{q\bar{q}}^{(2)}$ can be found in Eq. (A1) and Eq. (A3) of Ref. [52]. They read

$$\xi_{q\bar{q}}^{(2)} = T_F \,C_a \left(\mathcal{T}_{11} + \mathcal{T}_{22} - 2\mathcal{T}_{12}\right), \tag{4.1.22}$$

$$\xi_{gg}^{(2)} = C_a \left[4\, C_a \,\xi_{12}(k_1)\, \xi_{12}(k_2) + C_A \left(2\xi_{12} - \xi_{11} - \xi_{22}\right)\right], \tag{4.1.23}$$

where

$$\mathcal{T}_{ij} = -\frac{2(p_i \cdot p_j)(k_1 \cdot k_2) + [p_i \cdot (k_1 - k_2)][p_j \cdot (k_1 - k_2)]}{2(k_1 \cdot k_2)^2 [p_i \cdot (k_1 + k_2)][p_j \cdot (k_1 + k_2)]}, \tag{4.1.24}$$

$$\begin{aligned}
\xi_{ij} &= \frac{(1-\epsilon)}{(k_1\cdot k_2)^2}\frac{p_i\cdot k_1\, p_j\cdot k_2 + p_j\cdot k_1\, p_i\cdot k_2}{p_i\cdot(k_1+k_2)\; p_j\cdot(k_1+k_2)} \\
&- \frac{(p_i\cdot p_j)^2}{2p_i\cdot k_1\, p_j\cdot k_2\, p_i\cdot k_2\, p_j\cdot k_1}\left[2 - \frac{p_i\cdot k_1\, p_j\cdot k_2 + p_i\cdot k_2\, p_j\cdot k_1}{p_i\cdot(k_1+k_2)\; p_j\cdot(k_1+k_2)}\right] \\
&+ \frac{p_i\cdot p_j}{2k_1\cdot k_2}\left[\frac{2}{p_i\cdot k_1\, p_j\cdot k_2} + \frac{2}{p_j\cdot k_1\, p_i\cdot k_2}\right. \\
&\left. - \frac{1}{p_i\cdot(k_1+k_2)p_j\cdot(k_1+k_2)}\left(4 + \frac{(p_i\cdot k_1\, p_j\cdot k_2 + p_i\cdot k_2\, p_j\cdot k_1)^2}{p_i\cdot k_1\, p_j\cdot k_2\, p_i\cdot k_2\, p_j\cdot k_1}\right)\right],
\end{aligned} \tag{4.1.25}$$

$$\xi_{ij}(k_1) = \frac{p_i\cdot p_j}{(p_i\cdot k_1)(p_j\cdot k_1)}, \tag{4.1.26}$$

with $p_1 = n$, $p_2 = \bar{n}$.

We note that the previous results for $S^{(2)}_{q\bar{q}}$ and $S^{(2)}_{gg}$ in Refs. [48, 49] were obtained by directly integrating $\mathcal{T}_{ij}$ and $\xi_{ij}$ over the relevant phase space. However this approach is not extendable to N3LO as the N3LO eikonal functions is quite involved (cf. Eqs. (C.7)-(C.10) of Ref. [74]) and the associated integration is highly non-trivial. For this reason the goal of this chapter is twofold. First, we want to re-calculate $S^{(2)}_{q\bar{q}}$ and $S^{(2)}_{gg}$ to higher order in $\epsilon$. Second, we want to develop a method that can be extended to N3LO.

A standard way to reduce the number and complexity of integrals in loop calculations are the IBP relations, cf. Appendix A.1. We already saw in Chapters 2 and 3 that this technique can be adapted to phase-space integrals by re-writing delta functions as the sum of two "propagators"

$$\delta(X) = \frac{\mathrm{i}}{2\pi}\left(\frac{1}{X+\mathrm{i}\epsilon} - \frac{1}{X-\mathrm{i}\epsilon}\right). \tag{4.1.27}$$

This method allows us to use publicly-available reduction tools like `FIRE` [63] or KIRA [73], that are programmed to work with linear or quadratic propagators. While the identity Eq. (4.1.27) was sufficient in the case of the beam function, this is not the case for the soft function, as the definition of the zero-jettiness observable in Eqs. (4.1.11) and (4.1.12) contains Heaviside functions.

To remedy this problem, we map Heaviside functions onto delta functions, using the following identity [1]

$$\theta(b-a) = \int_0^1 \mathrm{d}z\, \delta\left(z\, b - a\right)\, b, \tag{4.1.28}$$

which holds for $a, b \in [0,\infty)$. Since, for real emissions, $k_{1,2}\cdot n$, $k_{1,2}\cdot\bar{n} \in [0,\infty)$, Eq. (4.1.28) can be used to remove Heaviside functions in Eqs. (4.1.11) and (4.1.12) at the expense of introducing an auxiliary integral. We find

$$M_A = \delta\left(\tau - n\cdot k_1 - n\cdot k_2\right)\theta\left(\bar{n}\cdot k_1 - n\cdot k_1\right)\theta\left(\bar{n}\cdot k_2 - n\cdot k_2\right)$$

1 See also Section 4.2.2 in Ref. [75] and Appendix B in the journal version of Ref. [76].

$$
\begin{aligned}
&= \int_0^1 \mathrm{d}z_1 \int_0^1 \mathrm{d}z_2\, \delta\left(\tau - n\cdot k_1 - n\cdot k_2\right) \delta\left(z_1 \bar{n}\cdot k_1 - n\cdot k_1\right) \bar{n}\cdot k_1 \\
&\quad \times \delta\left(z_2 \bar{n}\cdot k_2 - n\cdot k_2\right) \bar{n}\cdot k_2 = \int_0^1 \mathrm{d}z_1 \int_0^1 \mathrm{d}z_2\, M_{Az},
\end{aligned} \tag{4.1.29}
$$

$$
\begin{aligned}
M_B &= \delta\left(\tau - n\cdot k_1 - \bar{n}\cdot k_2\right) \theta\left(\bar{n}\cdot k_1 - n\cdot k_1\right) \theta\left(n\cdot k_2 - \bar{n}\cdot k_2\right) \\
&= \int_0^1 \mathrm{d}z_1 \int_0^1 \mathrm{d}z_2\, \delta\left(\tau - n\cdot k_1 - \bar{n}\cdot k_2\right) \delta\left(z_1 \bar{n}\cdot k_1 - n\cdot k_1\right) \bar{n}\cdot k_1 \\
&\quad \times \delta\left(z_2 n\cdot k_2 - \bar{n}\cdot k_2\right) n\cdot k_2 = \int_0^1 \mathrm{d}z_1 \int_0^1 \mathrm{d}z_2\, M_{Bz}.
\end{aligned} \tag{4.1.30}
$$

We use the representations Eqs. (4.1.29) and (4.1.30) in Eq. (4.1.20), postpone the $z_i$ integrations until the very end and perform phase-space integrations first. We then use reverse unitarity and IBP relations to express the soft function in terms of master integrals. We continue by computing the master integrals as functions of $z_1$ and $z_2$. Finally, we integrate over these auxiliary parameters.

We illustrate these steps by discussing the computation of $S^{(2)}_{q\bar{q}}$ in detail; the computation of $S^{(2)}_{gg}$ is completely analogous. According to the earlier discussion, contributions to the soft functions due to the emission of a $q\bar{q}$ pairs read

$$
\begin{aligned}
S^{(2)}_{q\bar{q}} &= 2 \int_0^1 \mathrm{d}z_1 \int_0^1 \mathrm{d}z_2 \int \mathrm{dPS}^{(2)}_S\, M_{Az}\, \xi^{(2)}_{q\bar{q}} + 2 \int_0^1 \mathrm{d}z_1 \int_0^1 \mathrm{d}z_2 \int \mathrm{dPS}^{(2)}_S\, M_{Bz}\, \xi^{(2)}_{q\bar{q}} \\
&= T_F\, C_a\, n_f \left(2\, S^{(2)}_{q\bar{q},A} + 2\, S^{(2)}_{q\bar{q},B}\right),
\end{aligned} \tag{4.1.31}
$$

where we separated the contributions of configurations $A$ and $B$. They read

$$
S^{(2)}_{q\bar{q},A,B} = \int_0^1 \mathrm{d}z_1 \int_0^1 \mathrm{d}z_2 \int \mathrm{dPS}^{(2)}_S\, M_{Az,Bz} \left(\mathcal{T}_{11} + \mathcal{T}_{22} - 2\mathcal{T}_{12}\right). \tag{4.1.32}
$$

After writing all delta functions in Eqs. (4.1.29) and (4.1.30) as linear combinations of the corresponding "propagators" and doing partial fractioning, we perform a reduction to master integrals using `FIRE` [63]. In configuration $A$ we find four master integrals

$$
\begin{aligned}
I_{A1} &= \langle 1 \rangle_{(A)}, & I_{A2} &= \left\langle \left(\frac{n\cdot k_1}{2} - \frac{\tau z_1}{2(z_1 - z_2)}\right)^{-1} \right\rangle_{(A)}, \\
I_{A3} &= \left\langle (k_1\cdot k_2)^{-1} \right\rangle_{(A)}, & I_{A4} &= \left\langle \left(\frac{n\cdot k_1}{2} - \frac{\tau z_1}{2(z_1 - z_2)}\right)^{-1} (k_1\cdot k_2)^{-1} \right\rangle_{(A)},
\end{aligned} \tag{4.1.33}
$$

where we defined

$$
\langle f \rangle_{(A)} = \int \mathrm{dPS}^{(2)}_S\, \delta\left(\tau - n\cdot k_1 - n\cdot k_2\right)\, \delta(n\cdot k_1 - z_1 \bar{n}\cdot k_1)\, \delta(n\cdot k_2 - z_2 \bar{n}\cdot k_2)\, f. \tag{4.1.34}
$$

For configuration $B$, we obtain six master integrals

$$
I_{1B} = \langle 1 \rangle_{(B)},
$$

$$
\begin{aligned}
I_{2B} &= \left\langle \left( \frac{n\cdot k_1}{2} + \frac{\tau z_1}{2(1-z_1)} \right)^{-1} \right\rangle_{(B)}, \\
I_{3B} &= \left\langle (k_1\cdot k_2)^{-1} \right\rangle_{(B)}, \\
I_{4B} &= \left\langle \left( \frac{n\cdot k_1}{2} + \frac{\tau z_1}{2(1-z_1)} \right)^{-1} (k_1\cdot k_2)^{-1} \right\rangle_{(B)}, \\
I_{5B} &= \left\langle \left( \frac{n\cdot k_1}{2} - \frac{\tau}{2(1-z_2)} \right)^{-1} \right\rangle_{(B)}, \\
I_{6B} &= \left\langle \left( \frac{n\cdot k_1}{2} - \frac{\tau}{2(1-z_2)} \right)^{-1} (k_1\cdot k_2)^{-1} \right\rangle_{(B)},
\end{aligned}
\tag{4.1.35}
$$

where we used

$$
\langle f \rangle_{(B)} = \int \mathrm{dPS}_S^{(2)}\ \delta(\tau - n\cdot k_1 - \bar{n}\cdot k_2)\ \delta(n\cdot k_1 - z_1 \bar{n}\cdot k_1)\ \delta(\bar{n}\cdot k_2 - z_2 n\cdot k_2)\ f. \tag{4.1.36}
$$

We discuss the calculation of the master integrals in detail in the next section.

## 4.2 CALCULATION OF MASTER INTEGRALS

We evaluate the master integrals of Eqs. (4.1.33) and (4.1.35) by integrating over gluon momenta $k_{1,2}$. Since all contributions to the soft function are symmetric with respect to $z_1 \leftrightarrow z_2$ permutation, for the calculations below we will assume that $z_1 > z_2$.

### 4.2.1 *Master integral $I_{A1}$*

We begin by discussing the configuration $A$. The simplest integral reads

$$
I_{A1} = \int \mathrm{dPS}_S^{(2)}\ \delta\left(\tau - n\cdot k_1 - n\cdot k_2\right)\ \delta(n\cdot k_1 - z_1\bar{n}\cdot k_1)\ \delta(n\cdot k_2 - z_2 \bar{n}\cdot k_2). \tag{4.2.1}
$$

To compute it, we follow the discussion in Section 2.2.1. We introduce the Sudakov decomposition

$$
k_i^\mu = \alpha_i \frac{n^\mu}{2} + \beta_i \frac{\bar{n}^\mu}{2} + k_{i\perp}^\mu, \tag{4.2.2}
$$

use spherical coordinates for $k_{i\perp}^\mu$ and remove the delta function $\delta(\alpha_i\beta_i - k_{i\perp}^2)$ by integrating over $k_{i\perp}^2$. The phase space becomes

$$
[\mathrm{d}k_i] = \frac{\mathrm{d}\Omega_i^{(d-2)}}{4(2\pi)^{d-1}}\, \mathrm{d}\alpha_i\, \mathrm{d}\beta_i\, (\alpha_i\beta_i)^{-\epsilon}, \qquad \alpha_i\,, \beta_i \in [0,\infty)\,. \tag{4.2.3}
$$

Combining Eq. (4.2.3) with Eq. (4.2.1) we obtain

$$I_{A1} = [N]^2 \int \prod_{i=1}^{2} \mathrm{d}\alpha_i \mathrm{d}\beta_i \, [\alpha_i\beta_i]^{-\epsilon} \, \delta(\tau - \beta_1 - \beta_2)\delta(\beta_1 - z_1\alpha_1)\delta(\beta_2 - z_2\alpha_2), \tag{4.2.4}$$

where we defined

$$[N] = \frac{\Omega^{(d-2)}}{4(2\pi)^{d-1}}. \tag{4.2.5}$$

We integrate over $\alpha_1$ and $\alpha_2$ and use (cf. Eq. (2.2.10))

$$\int_0^\infty \mathrm{d}\beta_1 \int_0^\infty \mathrm{d}\beta_2 \; \beta_1^a \; \beta_2^b \; \delta(\beta_1 + \beta_2 - x) = \frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(2+a+b)} x^{1+a+b}, \tag{4.2.6}$$

to perform the remaining integrations. The result reads

$$\begin{aligned} I_{A1} &= [N]^2 \frac{1}{(z_1 z_2)^{1-\epsilon}} \int_0^\infty \mathrm{d}\beta_1 \mathrm{d}\beta_2 \; (\beta_1\beta_2)^{-2\epsilon} \;\; \delta(\tau - \beta_1 - \beta_2) \\ &= [N]^2 \frac{\tau^{1-4\epsilon}}{(z_1 z_2)^{1-\epsilon}} \frac{\Gamma^2(1-2\epsilon)}{\Gamma(2-4\epsilon)}. \end{aligned} \tag{4.2.7}$$

### 4.2.2 *Master integral $I_{A2}$*

The next-to-easiest integral is $I_{A2}$. It reads

$$I_{A2} = \int \mathrm{dPS}_S^{(2)} \; \delta\left(\tau - n\cdot k_1 - n \cdot k_2\right) \; \delta(n\cdot k_1 - z_1 \bar{n}\cdot k_1) \; \delta(n \cdot k_2 - z_2 \bar{n}\cdot k_2) \tag{4.2.8}$$

$$\times \left( \frac{n\cdot k_1}{2} - \frac{\tau z_1}{2(z_1 - z_2)} \right)^{-1}. \tag{4.2.9}$$

Since the additional propagator only depends on $n \cdot k_1 \sim \beta_1$ the integration is identical to the one of $I_{A1}$ up to the last integration. We find

$$I_{A2} = [N]^2 \frac{1}{(z_1 z_2)^{1-\epsilon}} \int_0^\tau \mathrm{d}\beta_1 \frac{(\beta_1(1-\beta_1))^{-2\epsilon}}{\frac{\beta_1}{2} - \frac{\tau z_1}{2(z_1-z_2)}}. \tag{4.2.10}$$

We scale out $\tau$, integrate over $\beta_1$ and obtain

$$\begin{aligned} I_{A2} &= [N]^2 \frac{\tau^{-4\epsilon}}{(z_1 z_2)^{1-\epsilon}} \int_0^1 \mathrm{d}\beta_1 \frac{(\beta_1(1-\beta_1))^{-2\epsilon}}{\frac{\beta_1}{2} - \frac{z_1}{2(z_1-z_2)}} \\ &= [N]^2 \frac{(-2)\tau^{-4\epsilon}}{(z_1 z_2)^{1-\epsilon}} \frac{z_1 - z_2}{z_1} \frac{\Gamma^2(1-2\epsilon)}{\Gamma(2-4\epsilon)} \; {}_2F_1\left(1, 1-2\epsilon, 2-4\epsilon, \frac{z_1 - z_2}{z_1}\right). \end{aligned} \tag{4.2.11}$$

### 4.2.3 *Master integral* $I_{A3}$

We now discuss the integral $I_{A3}$, it reads

$$I_{A3} = \int \mathrm{dPS}_S^{(2)}\ \delta\left(\tau - n\cdot k_1 - n\cdot k_2\right)\ \delta(n\cdot k_1 - z_1 \bar{n}\cdot k_1)\ \delta(n\cdot k_2 - z_2\bar{n}\cdot k_2) \times \frac{2}{2k_1\cdot k_2}. \tag{4.2.12}$$

This is the first integral that contains a scalar product $k_1 \cdot k_2$. We use the Sudakov decomposition Eq. (4.2.2) to write it as

$$2k_1\cdot k_2 = \alpha_1\beta_2 + \beta_1\alpha_2 - 2\sqrt{\alpha_1\beta_2\alpha_2\beta_1}\cos\phi_{12} = \alpha_1\alpha_2\left(z_1 + z_2 - 2\sqrt{z_2 z_1}\cos\phi_{12}\right). \tag{4.2.13}$$

We note that in the second line of Eq. (4.2.13), we used the relation $\beta_i = \alpha_i z_i$, that arises from the delta function in Eq. (4.2.12). The result in Eq. (4.2.13) is quite remarkable, as the factorization of integration over $\phi_{12}$ and integrations over $\alpha_{1,2}$ simplifies the computation significantly. Consider the angular integration. We find

$$\begin{aligned}
\int \frac{\mathrm{d}\Omega_1^{(d-2)}\mathrm{d}\Omega_2^{(d-2)}}{2k_1\cdot k_2} &= \frac{1}{\alpha_1\alpha_2}\int \frac{\mathrm{d}\Omega_1^{(d-2)}\mathrm{d}\Omega_2^{(d-2)}}{z_1+z_2-2\sqrt{z_2 z_1}\cos\phi_{12}} \\
&= \frac{\Omega^{(d-2)}\Omega^{(d-3)}}{\alpha_1\alpha_2}\int_0^\pi \frac{\mathrm{d}\theta\left(\sin^2\theta\right)^{-\epsilon}}{z_1+z_2-2\sqrt{z_2z_1}\cos\phi_{12}} \\
&\overset{x=\cos\phi_{12}}{=} \frac{\Omega^{(d-2)}\Omega^{(d-3)}}{\alpha_1\alpha_2}\int_{-1}^1 \frac{\mathrm{d}x\left(1-x^2\right)^{-\epsilon-\frac{1}{2}}}{z_1+z_2-2\sqrt{z_2z_1}\ x} \\
&\overset{y=\frac{1+x}{2}}{=} \frac{\Omega^{(d-2)}\Omega^{(d-3)}}{\alpha_1\alpha_2}4^{-\epsilon}\int_0^1 \frac{\mathrm{d}y\left[y\left(1-y\right)\right]^{-\epsilon-\frac{1}{2}}}{\left(\sqrt{z_1}+\sqrt{z_2}\right)^2 - 4y\sqrt{z_1\ z_2}} \\
&= \frac{\Omega^{(d-2)}\Omega^{(d-3)}}{\alpha_1\alpha_2}4^{-\epsilon}\frac{\Gamma\left(\frac{1}{2}-\epsilon\right)^2}{\Gamma(1-2\epsilon)}\frac{{}_2F_1\left(1,\frac{1}{2}-\epsilon,1-2\epsilon,\frac{4\sqrt{z_1}\sqrt{z_2}}{\left(\sqrt{z_1}+\sqrt{z_2}\right)^2}\right)}{\left(\sqrt{z_1}+\sqrt{z_2}\right)^2} \\
&= \frac{\left[\Omega^{(d-2)}\right]^2}{\alpha_1\alpha_2\ z_1\left(1+\sqrt{\frac{z_2}{z_1}}\right)^2}\,{}_2F_1\left(1,\frac{1}{2}-\epsilon,1-2\epsilon,\frac{4\sqrt{z_2/z_1}}{\left(1+\sqrt{z_2/z_1}\right)^2}\right).
\end{aligned} \tag{4.2.14}$$

The hypergeometric function in Eq. (4.2.14) can be simplified using the following identity

$${}_2F_1\left(1,\frac{1}{2}-\epsilon,1-2\epsilon,\frac{4z}{(1+z)^2}\right) = (1+z)^2\,{}_2F_1\left(1,1+\epsilon,1-\epsilon,z^2\right), \tag{4.2.15}$$

which is valid for $|z| < 1$. Since we work in the region where $z_2 < z_1$, we can immediately use Eq. (4.2.15) to simplify Eq. (4.2.14) and we obtain

$$\int \frac{\mathrm{d}\Omega_1^{(d-2)}\mathrm{d}\Omega_2^{(d-2)}}{2k_1\cdot k_2}\bigg|_{\alpha_i\to\frac{\beta_i}{z_i}} = \frac{\left[\Omega^{(d-2)}\right]^2 z_2}{\beta_1\beta_2}\,{}_2F_1\left(1,1+\epsilon,1-\epsilon,\frac{z_2}{z_1}\right). \tag{4.2.16}$$

We substitute Eq. (4.2.16) back into Eq. (4.2.12), and use the representation Eq. (4.2.7) divided by the solid angels $\left[\Omega^{(d-2)}\right]^2$ to integrate over $\alpha_1$, $\alpha_2$, $\beta_2$. We find

$$I_{A3} = [N]^2 \frac{2\, z_2}{(z_1 z_2)^{1-\epsilon}}\, {}_2F_1\left(1, 1+\epsilon, 1-\epsilon, \frac{z_2}{z_1}\right) \int_0^\tau \mathrm{d}\beta_1\, \beta_1^{-1-2\epsilon} (\tau-\beta_1)^{-1-2\epsilon}. \tag{4.2.17}$$

We scale out $\tau$, integrate over $\beta_1$ and obtain

$$I_{A3} = [N]^2 \frac{2\tau^{-1-4\epsilon}}{(z_1 z_2)^{1-\epsilon}} \frac{\Gamma^2(-2\epsilon)}{\Gamma(-4\epsilon)}\, z_2\, {}_2F_1\left(1, 1+\epsilon, 1-\epsilon, \frac{z_2}{z_1}\right). \tag{4.2.18}$$

### 4.2.4 *Master integral $I_{A4}$*

We now calculate the last master integral in configuration $A$. It reads

$$\begin{aligned} I_{A4} = &\int \mathrm{dPS}_S^{(2)}\, \delta\left(\tau - n\cdot k_1 - n\cdot k_2\right)\, \delta(n\cdot k_1 - z_1 \bar{n}\cdot k_1)\, \delta(n\cdot k_2 - z_2\bar{n}\cdot k_2) \\ &\times \left(\frac{n\cdot k_1}{2} - \frac{\tau z_1}{2(z_1-z_2)}\right)^{-1} (k_1\cdot k_2)^{-1}. \end{aligned} \tag{4.2.19}$$

The integration is straightforward as we can use the integral representations Eq. (4.2.16) and Eq. (4.2.7) to directly obtain

$$\begin{aligned} I_{A4} = [N]^2 &\frac{(-4)\tau^{-2-4\epsilon}}{(z_1 z_2)^{1-\epsilon}} z_2\, {}_2F_1\left(1, 1+\epsilon, 1-\epsilon, \frac{z_2}{z_1}\right) \\ &\times \frac{(z_1-z_2)}{z_1} \int_0^1 \mathrm{d}\beta_1 \frac{(\beta_1(1-\beta_1))^{-2\epsilon-1}}{1-\frac{z_1-z_2}{z_1}\beta}. \end{aligned} \tag{4.2.20}$$

Integrating over $\beta_1$ we obtain the final result

$$\begin{aligned} I_{A4} = [N]^2 &\frac{(-4)\tau^{-2-4\epsilon}}{(z_1 z_2)^{1-\epsilon}} \frac{\Gamma^2(-2\epsilon)}{\Gamma(-4\epsilon)} \\ \times z_2 &\frac{z_1-z_2}{z_1}\, {}_2F_1\left(1, 1+\epsilon, 1-\epsilon, \frac{z_2}{z_1}\right)\, {}_2F_1\left(1, -2\epsilon, -4\epsilon, \frac{z_1-z_2}{z_1}\right). \end{aligned} \tag{4.2.21}$$

The remaining master integrals are all part of configuration $B$. Integrals in this configuration can be computed in complete analogy to the integrals contributing to configuration $A$. Explicit calculations for these integrals are given in Appendix C.1

This concludes our discussion of the calculation of master integrals. We explain how to perform the remaining integrations over auxiliary variables $z_{1,2}$ in the next section.

## 4.3 Integration over auxiliary parameters

After substituting the solutions for the master integrals into the expression for the soft function, we still need to integrate over the auxiliary parameters $z_1$ and $z_2$. For example, consider the contribution to the soft-function $S^{(2)}_{q\bar{q},A}$ defined in Eq. (4.1.32). Written in terms of master integrals, it reads

$$\begin{aligned} S^{(2)}_{q\bar{q},A} = 2\int_0^1 \mathrm{d}z_1 \int_0^{z_1} \mathrm{d}z_2 \Bigg[ &\frac{32\epsilon(2\epsilon-1)z_1 z_2}{\tau^2(z_1-z_2)^2}\, I_{A1} + \frac{8\epsilon(2\epsilon-1)z_1 z_2(z_1+z_2)}{\tau(z_1-z_2)^3}\, I_{A2} \\ &+ \frac{8(z_1+z_2)\left(16\epsilon^3 z_1 z_2 - \epsilon^2(z_1+z_2)^2 + \epsilon\left(z_1^2 - 6z_1z_2 + z_2^2\right) + z_1 z_2\right)}{(4\epsilon-1)(z_1-z_2)^4}\, I_{A3} \\ &+ \frac{8\tau z_1 z_2\left(\epsilon^2(z_1+z_2)^2 - z_1 z_2\right)}{(z_1-z_2)^5}\, I_{A4} \Bigg]. \end{aligned} \tag{4.3.1}$$

It appears that after inserting our solutions for the master integrals into Eq. (4.3.1) we will have to perform a non-trivial integration over $z_1$ and $z_2$. However, after changing variables $z_2 = t\, z_1$ the $z_1$ integration factors out. We find

$$\begin{aligned} S^{(2)}_{q\bar{q},A} = 2\left[\frac{\Omega^{(d-2)}}{4(2\pi)^{d-1}}\right]^2 \tau^{-1-4\epsilon} \int_0^1 \mathrm{d}z_1\, z_1^{-1+\epsilon} \int_0^1 \mathrm{d}t \Bigg[ &{}_2F_1(1, 1+\epsilon, 1-\epsilon, t) \\ \times \Bigg( &\frac{16(1+t)t^\epsilon\left(16\epsilon^3 t - \epsilon^2(1+t)^2 + \epsilon\left(t^2 - 61 + t\right) + t\right)\Gamma(-2\epsilon)^2}{(4\epsilon-1)(1-t)^4\Gamma(-4\epsilon)} \\ &- \frac{32t^{\epsilon+1}\left(\epsilon^2(1+t)^2 - t\right)\Gamma(-2\epsilon)^2\, {}_2F_1(1, -2\epsilon, -4\epsilon, 1-t)}{(1-t)^4\Gamma(-4\epsilon)} \Bigg) \\ &- \frac{16\epsilon(2\epsilon-1)(1+t)t^\epsilon\Gamma(1-2\epsilon)^2\, {}_2F_1(1, 1-2\epsilon, 2-4\epsilon, 1-t)}{(1-t)^2\Gamma(2-4\epsilon)} \\ &+ \frac{32\epsilon(2\epsilon-1)t^\epsilon\Gamma(1-2\epsilon)^2}{(1-t)^2\Gamma(2-4\epsilon)} \Bigg]. \end{aligned} \tag{4.3.2}$$

The $z_1$ integration is straightforward while the remaining $t$ integration seems to exhibit a power-like divergences in $t = 1$. However, upon expanding the expression in square brackets in Eq. (4.3.2) around $t = 1$, we obtain

$$\begin{aligned} \lim_{t\to 1}[\dots] = &-(1-t)^{-1-2\epsilon}\frac{2^{6\epsilon+1}(2(\epsilon-1)\epsilon+1)\Gamma(2-2\epsilon)\Gamma(1-\epsilon)\Gamma\left(\epsilon-\frac{1}{2}\right)}{\Gamma\left(\frac{5}{2}-2\epsilon\right)} \\ &+ \mathcal{O}\left((1-t)^0\right) \end{aligned} \tag{4.3.3}$$

Hence, we observe that the $t = 1$ singularity only leads to a logarithmic divergence. We subtract the divergence at $t = 1$, expand the integrand in a Laurent series in $\epsilon$ using HypExp [70] and compute the integral order by order in $\epsilon$ with the help of `HyperInt` [68]. The final result reads

$$S^{(2)}_{q\bar{q},A} = [N]^2\,\tau^{-1-4\epsilon}\Bigg[-\frac{8}{3\epsilon^2}-\frac{40}{9\epsilon}-\frac{8\pi^2}{9}-\frac{152}{27} + \epsilon\left(-\frac{64\zeta(3)}{3}-\frac{952}{81}-\frac{40\pi^2}{27}\right)+\epsilon^2\left(-\frac{320\zeta(3)}{9}-\frac{3848}{243}-\frac{368\pi^2}{81}-\frac{32\pi^4}{45}\right) + \epsilon^3\left(-\frac{2944\zeta(3)}{27}+\frac{416\pi^2\zeta(3)}{9}-672\zeta(5)+\frac{17576}{729}-\frac{3328\pi^2}{243}-\frac{32\pi^4}{27}\right) + \mathcal{O}\left(\epsilon^4\right)\Bigg]. \tag{4.3.4}$$

Next we discuss the contribution $S^{(2)}_{q\bar{q},B}$ due to an emission of a quark anti-quark pair in configuration $B$. Written in terms of master integrals, this contribution reads

$$\begin{aligned} S^{(2)}_{q\bar{q},B} = 2\int_0^1 \mathrm{d}z_1\int_0^{z_1}\mathrm{d}z_2\Bigg[&\frac{32\epsilon(4\epsilon-1)z_1z_2}{\tau^2(z_1z_2-1)^2}\,I_{B1} \\ &-\frac{8\tau z_2\left(\epsilon^2(z_2+1)\left(z_1^2z_2^2-1\right)+\epsilon(z_1-1)z_2(z_1z_2+1)-z_1(z_2-1)z_2\right)}{(z_2-1)^3(z_1z_2-1)^3}\,I_{B6} \\ &+\Bigg(\frac{16z_1z_2(z_1(-z_2)+z_1+z_2-1)}{(z_1-1)^2(z_2-1)^2(z_1z_2-1)^2} \\ &+\epsilon\frac{8\left(z_1^3(-(z_2-3))z_2^2+z_1^2z_2\left(3z_2^2-11z_2+6\right)+z_1\left(6z_2^2-11z_2+3\right)+3z_2-1\right)}{(z_1-1)^2(z_2-1)^2(z_1z_2-1)^2} \\ &+\epsilon^2\frac{32\left(z_1^3z_2^2+z_1^2z_2\left(z_2^2-4z_2+2\right)+z_1\left(2z_2^2-4z_2+1\right)+z_2\right)}{(z_1-1)^2(z_2-1)^2(z_1z_2-1)^2}\Bigg)\,I_{B3} \\ &+\frac{8\tau z_1\left(\epsilon z_1^2z_2^2(\epsilon z_1+\epsilon+1)-(\epsilon+1)(z_1-1)z_1z_2-\epsilon(\epsilon z_1+\epsilon+z_1)\right)}{(z_1-1)^3(z_1z_2-1)^3}\,I_{B4} \\ &+\frac{8\epsilon z_1z_2(2\epsilon(z_1+1)(z_1z_2-1)-(z_1-1)(z_1z_2+1))}{\tau(z_1-1)(z_1z_2-1)^3}\,I_{B2} \\ &-\frac{8\epsilon z_1z_2(2\epsilon(z_2+1)(z_1z_2-1)-(z_2-1)(z_1z_2+1))}{\tau(z_2-1)(z_1z_2-1)^3}\,I_{B6}\Bigg]. \end{aligned} \tag{4.3.5}$$

Although the expression in Eq. (4.3.5) looks more complicated than the one in Eq. (4.3.1), it is actually much simpler. Indeed, the integrand in Eq. (4.3.5) is finite in the whole integration region and we simply expand it in a Laurent series in $\epsilon$ and integrate using `HyperInt`. We obtain

$$S^{(2)}_{q\bar{q},B} = \left[\frac{\Omega^{(d-2)}}{4(2\pi)^{d-1}}\right]^2\tau^{-1-4\epsilon}\Bigg[-\frac{8}{3}+\frac{16\pi^2}{9}+\epsilon\left(\frac{224\zeta(3)}{3}+\frac{88}{9}-\frac{128\pi^2}{27}\right) + \epsilon^2\left(-\frac{1792\zeta(3)}{9}+\frac{904}{27}+\frac{688\pi^2}{81}+\frac{136\pi^4}{45}\right) \tag{4.3.6}$$

$$+\epsilon^3\left(\frac{9920\zeta(3)}{27}-\frac{352\pi^2\zeta(3)}{3}+\frac{6560\zeta(5)}{3}+\frac{5752}{81}-\frac{2672\pi^2}{243}-\frac{1088\pi^4}{135}\right)$$
$$+\mathcal{O}\left(\epsilon^4\right)\Bigg].$$

While it was not necessary for the above computations, we can actually predict and understand the singularities of the $z_i$ integration from physical arguments. To understand this point we consider the Sudakov decomposition of one of the emitted partons

$$k_i^\mu = \alpha_i \frac{n^\mu}{2} + \beta_i \frac{\bar{n}^\mu}{2} + k_{i\perp}^\mu. \tag{4.3.7}$$

After using the on-shell constraint $k_i^2 = 0$ as well as the relation encoded by the delta function $\beta_i = \alpha_i z_i$, we obtain

$$k_i^\mu = \alpha_i \frac{n^\mu}{2} + \beta_i \frac{\bar{n}^\mu}{2} + \sqrt{\alpha_i \beta_i} e_{i\perp}^\mu = \alpha_i \left(\frac{n^\mu}{2} + z_i \frac{\bar{n}^\mu}{2} + \sqrt{z_i} e_{i\perp}^\mu\right). \tag{4.3.8}$$

From the representation in Eq. (4.3.8) it becomes clear that $\alpha_i \to 0$ encodes the soft-limit $k_i^\mu \to 0$ while the limit $z_i \to 0$ describes kinematic configuration where $k_i$ becomes collinear to the beam axis $n$, i.e. $k_i^\mu \propto n^\mu$.[2] Let us now consider the scalar product between the momenta of two emitted partons $k_i$ and $k_j$ in the representation Eq. (4.3.8)

$$k_i \cdot k_j = \frac{\alpha_i \alpha_j}{2}\left(z_j + z_i - 2\sqrt{z_i z_j}\, \vec{e}_i \cdot \vec{e}_j\right). \tag{4.3.9}$$

We are interested in finding the limit where the partons $i$ and $j$ become collinear i.e. $k_i \cdot k_j = 0$. To this end we require the expression in parentheses to vanish

$$\left(z_j + z_i - 2\sqrt{z_i z_j}\, \vec{e}_i \cdot \vec{e}_j\right) \stackrel{!}{=} 0. \tag{4.3.10}$$

After changing variables $\vec{e}_i \cdot \vec{e}_j = 1 - 2y$ with $y \in [0,1]$ we obtain

$$\left(\sqrt{z_j} - \sqrt{z_i}\right)^2 + 4\sqrt{z_i\, z_j}\, y \stackrel{!}{=} 0, \tag{4.3.11}$$

and see that this condition is only satisfied when $z_i = z_j$ and $y = 0$.

Thus the collinear limit $\boldsymbol{k}_i \parallel \boldsymbol{k}_j$, after $\phi_{ij}$ integration appears as the singularity at $z_i = z_j$. This is not surprising, as $z_i$ controls the emission angle in the $n\bar{n}$ plane. Hence, if this quantity has the same value for two emitters after $\phi_{ij}$ integration, both emissions are in the same direction and thus collinear to each other.

From this discussion we can infer that the $z_1 = z_2$ singularity in Eq. (4.3.1) is the collinear $\boldsymbol{k}_1 \parallel \boldsymbol{k}_2$ singularity of the original amplitude. After the change of variables $z_2 = t\, z_1$ in Eq. (4.3.2), the singularity at $t = 1$ describes the limit where the quark and anti-quark become collinear to each other, while the singularity at $z_1 = 0$ describes the kinematic configuration

2 In configuration $B$ the limit $z_2 \to 0$ would correspond to $k_2^\mu \propto \bar{n}^\mu$ instead.

in which the gluon, that emits the pair, becomes collinear to the light-like directions $n^\mu$. Conversely, in Eq. (4.3.5) the $z_i$ integration is free of any singularities as in configuration $B$ the quark and anti-quark should be emitted into different hemispheres and collinear singularities are thus absent.

We now discuss contributions due to the emission of two gluons $S^{(2)}_{gg}$. Similar to the quark case, we split it into $A$ and $B$ contributions. We find

$$\begin{aligned} S^{(2)}_{gg} &= \frac{2}{2!}\int \mathrm{d}z_1 \int \mathrm{d}z_2 \int \mathrm{dPS}^{(2)}_S\; M_{Az}\,\xi^{(2)}_{gg} + \frac{2}{2!}\int \mathrm{d}z_1 \int \mathrm{d}z_2 \int \mathrm{dPS}^{(2)}_S\; M_{Bz}\,\xi^{(2)}_{gg} \\ &= C_a^2\left( S^{(2)}_{Ca,gg,A} + S^{(2)}_{Ca,gg,B}\right) + C_a C_A \left( S^{(2)}_{CA,gg,A} + S^{(2)}_{CA,gg,B}\right), \end{aligned} \tag{4.3.12}$$

where we separated contributions proportional to $C_a^2$ and $C_aC_A$. They read

$$S^{(2)}_{Ca,gg,A,B} = \int \mathrm{d}z_1 \int \mathrm{d}z_2 \int \mathrm{dPS}^{(2)}_S\; M_{Az,Bz}\; 4\,\xi_{12}(k_1)\,\xi_{12}(k_2), \tag{4.3.13}$$

$$S^{(2)}_{CA,gg,A,B} = \int \mathrm{d}z_1 \int \mathrm{d}z_2 \int \mathrm{dPS}^{(2)}_S\; M_{Az,Bz}\,(2\xi_{12} - \xi_{11} - \xi_{22})\,. \tag{4.3.14}$$

The calculation of $S^{(2)}_{CA,gg}$ is fully analogous to the calculation of $S^{(2)}_{q\bar{q}}$. Although, in comparison with the $q\bar{q}$ case, the double gluon emission includes an additional singular configuration, the pole structure of the $z_i$-integration is the same as before. Indeed, the single soft divergence, which is absent in $q\bar{q}$ emission, is accounted for by an additional factor $1/\epsilon$ originating from IBP reduction. We find

$$\begin{aligned} S^{(2)}_{CA,gg,A} = [N]^2\,\tau^{-1-4\epsilon}\Bigg[ &\frac{8}{\epsilon^3} + \frac{44}{3\epsilon^2} + \left(\frac{268}{9} - \frac{4\pi^2}{3}\right)\frac{1}{\epsilon} - 72\zeta(3) + \frac{44\pi^2}{9} \\ &+ \frac{1544}{27} + \epsilon\left(\frac{352\zeta(3)}{3} + \frac{9568}{81} + \frac{268\pi^2}{27} - \frac{10\pi^4}{3}\right) \\ &+ \epsilon^2\Bigg(\frac{2144\zeta(3)}{9} + 16\pi^2\zeta(3) - 1208\zeta(5) + \frac{55424}{243} \\ &+ \frac{1760\pi^2}{81} + \frac{176\pi^4}{45}\Bigg) \\ &+ \epsilon^3\Bigg(\frac{14080\zeta(3)}{27} - \frac{2288\pi^2\zeta(3)}{9} + 424\zeta(3)^2 + 3696\zeta(5) + \frac{297472}{729} \\ &+ \frac{11296\pi^2}{243} + \frac{1072\pi^4}{135} - \frac{3596\pi^6}{945}\Bigg) + \mathcal{O}\left(\epsilon^4\right)\Bigg], \end{aligned} \tag{4.3.15}$$

$$S^{(2)}_{CA,gg,B} = [N]^2\,\tau^{-1-4\epsilon}\Bigg[-\frac{8\pi^2}{3\epsilon}-\frac{88\pi^2}{9}+\frac{8}{3} + \epsilon\left(-\frac{1232\zeta(3)}{3}-\frac{112}{9}+\frac{536\pi^2}{27}+\frac{32\pi^4}{45}\right) + \epsilon^2\left(\frac{7504\zeta(3)}{9}+480\zeta(5)-\frac{640}{27}-\frac{3232\pi^2}{81}-\frac{748\pi^4}{45}\right) + \epsilon^3\Bigg(-\frac{45536\zeta(3)}{27}+\frac{1936\pi^2\zeta(3)}{3}-912\zeta(3)^2-\frac{36080\zeta(5)}{3}-\frac{3040}{81} + \frac{17696\pi^2}{243}+\frac{4556\pi^4}{135}+\frac{2984\pi^6}{945}\Bigg)+\mathcal{O}\left(\epsilon^4\right)\Bigg]. \tag{4.3.16}$$

The contribution which is proportional to the color factor $C_a^2$, $S^{(2)}_{Ca,gg}$ is completely fixed by the NLO results $S^{(1)}_A$ and $S^{(1)}_B$ in Eq. (4.1.19).
Indeed, we find

$$\begin{aligned} S^{(2)}_{Ca,gg,A} &= 2\times\int\frac{\mathrm{d}^d k_2}{(2\pi)^{d-1}}\delta^+\left(k_2^2\right)\,\theta\left(k_2\cdot\bar{n}-k_2\cdot n\right)\,\xi_{12}(k_2) \\ &\quad\times\int\frac{\mathrm{d}^d k_1}{(2\pi)^{d-1}}\delta^+\left(k_1^2\right)\,\delta\left(\tau-k_1\cdot n-k_2\cdot n\right)\theta\left(k_1\cdot\bar{n}-k_1\cdot n\right)\,2\,\xi_{12}(k_1) \\ &= 2\times\int\frac{\mathrm{d}^d k_2}{(2\pi)^{d-1}}\delta^+\left(k_2^2\right)\,\theta\left(k_2\cdot\bar{n}-k_2\cdot n\right)\,\frac{n\cdot\bar{n}}{(k_2\cdot n)\,(k_2\cdot\bar{n})} \\ &\quad\times\frac{S^{(1)}_A(\tau-k_2\cdot n)}{C_a}\theta\left(\tau-k_2\cdot n\right) \\ &= -\,[N]\,\frac{4\tau^{-1-4\epsilon}\Gamma(1-2\epsilon)^2}{\epsilon^2\Gamma(1-4\epsilon)}\frac{S^{(1)}_A(1)}{C_a} \\ &= -\,[N]^2\,\frac{16\tau^{-1-4\epsilon}}{\epsilon^3}\frac{\Gamma(1-2\epsilon)^2}{\Gamma(1-4\epsilon)}. \end{aligned} \tag{4.3.17}$$

Similarly, we find the following relation for the configuration $B$

$$\begin{aligned} S^{(2)}_{Ca,gg,B} &= -\,[N]\,\frac{4\tau^{-1-4\epsilon}\Gamma(1-2\epsilon)^2}{\epsilon^2\Gamma(1-4\epsilon)}\frac{S^{(1)}_B(1)}{C_a} \\ &= -\,[N]^2\,\frac{16\tau^{-1-4\epsilon}}{\epsilon^3}\frac{\Gamma(1-2\epsilon)^2}{\Gamma(1-4\epsilon)}. \end{aligned} \tag{4.3.18}$$

## 4.4 RESULTS AND SUMMARY

We now present our final result for the bare soft function $S^{(2)}$ through $\mathcal{O}(\epsilon^2)$ at NNLO QCD. To this end, we write

$$S^{(2)} = \tau^{-1-4\epsilon} \left[ \frac{\Omega^{(d-2)}}{4(2\pi)^{d-1}} \right]^2 4 \left( C_a^2 \, S_A^{(2)} + C_a T_F n_f \, S_B^{(2)} + C_A C_a \, S_C^{(2)} \right). \tag{4.4.1}$$

The individual contributions shown in Eq. (4.4.1) read

$$\begin{aligned} S_A^{(2)} &= -\frac{8}{\epsilon^3} + \frac{16\pi^2}{3\epsilon} + 128\zeta(3) + \epsilon\frac{16\pi^4}{5} + \epsilon^2 \left(1536\zeta(5) - \frac{256\pi^2\zeta(3)}{3}\right) \\ &+ \epsilon^3 \left(\frac{2528\pi^6}{945} - 1024\zeta(3)^2\right), \end{aligned} \tag{4.4.2}$$

$$\begin{aligned} S_B^{(2)} &= -\frac{4}{3\epsilon^2} - \frac{20}{9\epsilon} + \frac{4\pi^2}{9} - \frac{112}{27} + \epsilon \left(\frac{80\zeta(3)}{3} - \frac{80}{81} - \frac{28\pi^2}{9}\right) \\ &+ \epsilon^2 \left(-\frac{352\zeta(3)}{3} + \frac{2144}{243} + \frac{160\pi^2}{81} + \frac{52\pi^4}{45}\right) \\ &+ \epsilon^3 \left(\frac{3488\zeta(3)}{27} - \frac{320\pi^2\zeta(3)}{9} + \frac{2272\zeta(5)}{3} + \frac{34672}{729} - \frac{1000\pi^2}{81} - \frac{208\pi^4}{45}\right), \end{aligned} \tag{4.4.3}$$

$$\begin{aligned} S_C^{(2)} &= \frac{11}{3\epsilon^2} + \frac{1}{\epsilon}\left(\frac{67}{9} - \frac{\pi^2}{3}\right) - 14\zeta(3) + \frac{404}{27} - \frac{11\pi^2}{9} \\ &+ \epsilon \left(-\frac{220\zeta(3)}{3} + \frac{2140}{81} + \frac{67\pi^2}{9} - \frac{49\pi^4}{90}\right) \\ &+ \epsilon^2 \left(268\zeta(3) + \frac{8\pi^2\zeta(3)}{3} - 170\zeta(5) + \frac{12416}{243} - \frac{368\pi^2}{81} - \frac{143\pi^4}{45}\right) \\ &+ \epsilon^3 \left(-\frac{7864\zeta(3)}{27} + \frac{880\pi^2\zeta(3)}{9} - 126\zeta(3)^2 - \frac{6248\zeta(5)}{3} \right. \\ &\left. + \frac{67528}{729} + \frac{2416\pi^2}{81} + \frac{469\pi^4}{45} - \frac{10\pi^6}{63}\right). \end{aligned} \tag{4.4.4}$$

We compare the result Eqs. (4.4.1) - (4.4.4) with the $\mathcal{O}(\epsilon)$ results in Refs. [48, 49] and find full agreement.

Thus we achieved the first goal of this chapter, namely the calculation of the NNLO zero jettiness soft function to higher orders in $\epsilon$. Our second goal was to develop an approach that can be extended to N3LO. To this end, we proposed to write Heaviside functions as follows

$$\theta(b-a) = \int_0^1 \mathrm{d}z_i \, \delta\left(z_i \, b - a\right) \; b. \tag{4.4.5}$$

We then noticed that, since Eq. (4.4.5) allows us to trade Heaviside functions for delta functions, and since delta functions can be dealt with using reverse unitarity, this representation enables a straightforward reduction to master integrals and generalization to higher orders.

We found that the NNLO zero-jettiness soft function can be expressed through just 10 master integrals. These master integrals are remarkably easy to compute, as the integration over the relative azimuthal angle between two partons completely decouples from the remaining integrations. As a final step we had to integrate the sum of reduction coefficients and master integrals over the auxiliary parameters $z_i$ introduced through Eq. (4.4.5). All in all we saw that the use of Eq. (4.4.5) led to a great simplification of the calculation compared to the earlier calculations of Refs. [48, 49].

Unfortunately, it turns out that the application of this approach to the computation of the N3LO correction is less straightforward. Indeed, the N3LO soft function requires the computation of master integrals that involve a propagator $(k_1 + k_2 + k_3)^{-2}$. In this case the angular integration does not factorize and the introduction of three auxiliary parameters complicates the calculation of these master integrals. For this reason we will recalculate the NNLO soft function with a different approach, that does not require us to introduce auxiliary parameters, in the next chapter.

# 5

# NNLO SOFT FUNCTION AND IBPS FOR THETA FUNCTIONS

*This chapter is based on Reference [77].*

In this chapter we re-calculate parts of the NNLO soft function using a different approach that does not require introduction of auxiliary parameters. The idea is to derive integration-by-parts identities for integrals with Heaviside functions and solve the resulting system of linear equations to obtain a reduction to master integrals.
This chapter is organized as follows. In Section 5.1 we explain the general setup of the calculation, illustrating how to construct IBP relations for integrals with Heaviside functions. We proceed by applying this method to the NNLO soft function in Section 5.2. We then discuss how to calculate master integrals obtained using this modified IBP approach in Section 5.3. Finally, we conclude in Section 5.4, summarizing the approach and comparing it to the earlier calculation in Chapter 4.

## 5.1 GENERAL SETUP AND IBPS FOR THETA FUNCTIONS

In the previous chapter we turned phase-space integrals into loop integrals to allow for a reduction with standard multi-loop tools. To accomplish that, we had to write Heaviside functions as integrals over delta-functions because standard reduction programs do not allow for the reduction of integrals containing Heaviside functions. These reduction programs use the IBP relations as a starting point to create linear relations between integrals. The relation reads

$$\int \mathrm{d}^d k_1 \int \mathrm{d}^d k_2 \frac{\partial}{\partial k_m^\mu} \left[ v^\mu \frac{1}{\prod_i D_i^{\alpha_i}} \right] = 0\,, \qquad v \in \{k_j, n, \bar{n}\} \tag{5.1.1}$$

where $m \in \{1, 2\}$ and $D_i$ are any linear or quadratic propagators. Since Eq. (5.1.1) is just Gauss' theorem in $d$-dimensions supplemented with the statement, that boundary terms of any dimensionally-regularized integral vanish, it should also hold true if one adds Heaviside functions to the integrand. Thus, we can write

$$\int \mathrm{d}^d k_1 \int \mathrm{d}^d k_2 \frac{\partial}{\partial k_m^\mu} \left[ v^\mu \frac{\prod_i \theta_i}{\prod_i D_i^{\alpha_i}} \right] = 0\,. \qquad v \in \{k_j, n, \bar{n}\} \tag{5.1.2}$$

The modified IBP identity Eq. (5.1.2) turns out to be quite useful as it can be related to the standard, well understood, IBP relation in Eq. (5.1.1). To illustrate this point we consider Eq. (5.1.2) in case there is only one Heaviside function. We compute the derivative in Eq. (5.1.2) and split the resulting expression into two pieces: the so-called homogeneous term, which is obtained when the derivative does not act on the Heaviside function, and the inhomogeneous term, which appears when the derivative acts on the Heaviside function. It is clear that, in case of the homogeneous term, Eq. (5.1.2) is identical to Eq. (5.1.1). In case of the inhomogeneous term, the derivative of the Heaviside function yields a delta function $\partial/\partial x\ \theta(x) = \delta(x)$. This delta function is then treated as a cut propagator through the use of reverse unitarity. Therefore, the class of integrals obtained from the inhomogeneous term can again be studied with the help of reverse unitarity and the standard IBP relation Eq. (5.1.1). While this approach is straightforward, there is one caveat that makes its practical realization complicated. Indeed, when the delta function is written as a cut propagator, partial fractioning may be necessary if this new propagator and propagators $D_i$ that are present in the integrand are linearly dependent.

To illustrate this process consider the following integral

$$I^{\theta}_{\{1,1,1\}} = \int \frac{\mathrm{d}^d k}{(2\pi)^{d-1}} \frac{\theta\left(k\cdot\bar{n} - k\cdot n\right)}{[k\cdot\bar{n}]\,[\tau - k\cdot n]_c\,[k^2]_c}. \tag{5.1.3}$$

In Eq. (5.1.3) we used the following notation for cut propagators

$$\delta(x) = \frac{i}{2\pi}\left(\frac{1}{x+i0} - \frac{1}{x-i0}\right) = \frac{1}{[x]_c}, \tag{5.1.4}$$

with $x = k^2, \tau - k\cdot n$. We will use the following shorthand notations for the integrals

$$I^{\theta}_{\{n_1,n_2,n_3\}} = \int \frac{\mathrm{d}^d k}{(2\pi)^{d-1}} \frac{\theta\left(k\cdot\bar{n} - k\cdot n\right)}{[k^2]_c^{n_1}\,[\tau - k\cdot n]_c^{n_2}\,[k\cdot\bar{n}]^{n_3}}, \tag{5.1.5}$$

$$I^{\delta}_{\{n_1,n_2,n_3\}} = \int \frac{\mathrm{d}^d k}{(2\pi)^{d-1}} \frac{\delta\left(k\cdot\bar{n} - k\cdot n\right)}{[k^2]_c^{n_1}\,[\tau - k\cdot n]_c^{n_2}\,[k\cdot\bar{n}]^{n_3}}. \tag{5.1.6}$$

To express the integral in Eq. (5.1.3) through simpler integrals we have to choose a set of integrals and apply the identity Eq. (5.1.2) to them. These so-called seed integrals are given by $I^{\theta}_{\{1,1,1\}}$ and $I^{\theta}_{\{1,1,0\}}$. To simplify the computation, we re-write the derivatives in Eq. (5.1.2) through suitable scalar products. We find

$$\begin{aligned}\frac{\partial}{\partial k^\mu}\left(v^\mu f\left(k^2, k\cdot n, k\cdot\bar{n}\right)\right) &= \left(\frac{\partial v^\mu}{\partial k^\mu} + v^\mu \frac{\partial k^2}{\partial k^\mu}\frac{\partial}{\partial k^2} + v^\mu \frac{\partial k\cdot n}{\partial k^\mu}\frac{\partial}{\partial k\cdot n}\right.\\ &\qquad \left. + v^\mu \frac{\partial k\cdot\bar{n}}{\partial k^\mu}\frac{\partial}{\partial k\cdot\bar{n}}\right) f \\ &= \left(d\delta_{v,k} + 2k\cdot v\frac{\partial}{\partial k^2} + n\cdot v\frac{\partial}{\partial k\cdot n} + \bar{n}\cdot v\frac{\partial}{\partial k\cdot\bar{n}}\right) f.\end{aligned} \tag{5.1.7}$$

We now apply Eq. (5.1.7) to our seed integrals and use the fact that $\partial/\partial x\ \theta(x) = \delta(x)$. We obtain the following linear relations

$$v = n: \quad 0 = (n\cdot\bar{n})\left[I^{\delta}_{\{1,1,1\}} - I^{\theta}_{\{1,1,2\}}\right] - 2\tau I^{\theta}_{\{2,1,1\}}, \tag{5.1.8}$$

$$v = \bar{n}: \quad 0 = (n\cdot\bar{n})\left[-I^{\delta}_{\{1,1,1\}} + I^{\theta}_{\{1,2,1\}}\right] - 2I^{\theta}_{\{2,1,0\}}, \tag{5.1.9}$$

$$v=k: \quad 0=\left[-\tau I^{\delta}_{\{1,1,1\}}-I^{\theta}_{\{1,1,1\}}+\tau I^{\theta}_{\{1,2,1\}}\right]+\left[I^{\delta}_{\{1,1,0\}}-I^{\theta}_{\{1,1,1\}}\right]-2I^{\theta}_{\{1,1,1\}} \\ +dI^{\theta}_{\{1,1,1\}}, \tag{5.1.10}$$

and

$$v=n: \quad 0=(n\cdot\bar{n})\,I^{\delta}_{\{1,1,0\}}-2\tau I^{\theta}_{\{2,1,0\}}, \tag{5.1.11}$$

$$v=\bar{n}: \quad 0=(n\cdot\bar{n})\left[-I^{\delta}_{\{1,1,0\}}+I^{\theta}_{\{1,2,0\}}\right]-2I^{\theta}_{\{2,1,-1\}}, \tag{5.1.12}$$

$$v=k: \quad 0=dI^{\theta}_{\{1,1,0\}}+\left[-\tau I^{\delta}_{\{1,1,0\}}-I^{\theta}_{\{1,1,0\}}+\tau I^{\theta}_{\{1,2,0\}}\right]+I^{\delta}_{\{1,1,0\}}-2I^{\theta}_{\{1,1,0\}}, \tag{5.1.13}$$

for $I^{\theta}_{\{1,1,1\}}$ and $I^{\theta}_{\{1,1,0\}}$ respectively.

These six equations are sufficient to express the integral $I^{\theta}_{\{1,1,1\}}$ through simpler integrals. Indeed, we solve the system of linear equations for $I^{\theta}_{\{1,1,1\}}$ and find the following result

$$I^{\theta}_{\{1,1,1\}}=\frac{2}{(4-d)}I^{\delta}_{\{1,1,0\}}=\frac{1}{\epsilon}I^{\delta}_{\{1,1,0\}}\,. \tag{5.1.14}$$

Eq. (5.1.14) expresses a "complicated" integral $I^{\theta}_{\{1,1,1\}}$ through a simpler integral $I^{\delta}_{\{1,1,0\}}$. We verify Eq. (5.1.14) through a direct calculation of both integrals. We find

$$\begin{aligned} I^{\theta}_{\{1,1,1\}} &= \int \mathrm{dPS}^{(1)}_{S}\,\delta\left(\tau-k_1\cdot n\right)\frac{\theta\left(k_1\cdot\bar{n}-k_1\cdot n\right)}{k_1\cdot\bar{n}} \\ &= [N]\int \mathrm{d}\alpha\mathrm{d}\beta\frac{(\alpha\beta)^{-\epsilon}}{\alpha}\delta\left(\tau-\beta\right)\theta\left(\alpha-\beta\right) \\ &= [N]\,\tau^{-\epsilon}\int_{\tau}^{\infty}\mathrm{d}\alpha\alpha^{-1-\epsilon}=[N]\,\frac{\tau^{-2\epsilon}}{\epsilon}, \end{aligned} \tag{5.1.15}$$

as well as

$$\begin{aligned} I^{\delta}_{\{1,1,0\}} &= \int \mathrm{dPS}^{(1)}_{S}\,\delta\left(\tau-k_1\cdot n\right)\delta\left(k_1\cdot\bar{n}-k_1\cdot n\right) \\ &= [N]\int \mathrm{d}\alpha\mathrm{d}\beta\,(\alpha\beta)^{-\epsilon}\,\delta\left(\tau-\beta\right)\delta\left(\alpha-\beta\right)=[N]\,\tau^{-2\epsilon}, \end{aligned} \tag{5.1.16}$$

where the constant $[N]$ was defined in Eq. (4.2.5) . Thus, we confirm the correctness of Eq. (5.1.14).

While Eq. (5.1.14) demonstrates that IBPs and reductions can be constructed for integrals with Heaviside functions, the example we just considered is clearly way too simple. To better illustrate some technical aspects that arise in more complex situations, let us consider the NNLO case. At NNLO we obtain two copies of Eq. (5.1.7), one for $k_1$ and another one for $k_2$

$$\frac{\partial}{\partial k_1^{\mu}}v^{\mu}=d\delta_{v,k_1}+2k_1\cdot v\frac{\partial}{\partial k_1^2}+n\cdot v\frac{\partial}{\partial k_1\cdot n}+\bar{n}\cdot v\frac{\partial}{\partial k_1\cdot\bar{n}}+k_2\cdot v\frac{\partial}{\partial k_1\cdot k_2}, \tag{5.1.17}$$

$$\frac{\partial}{\partial k_2^{\mu}}v^{\mu}=d\delta_{v,k_2}+2k_2\cdot v\frac{\partial}{\partial k_2^2}+n\cdot v\frac{\partial}{\partial k_2\cdot n}+\bar{n}\cdot v\frac{\partial}{\partial k_2\cdot\bar{n}}+k_1\cdot v\frac{\partial}{\partial k_1\cdot k_2}. \tag{5.1.18}$$

While it was not necessary for the NLO example, an important technical aspect of any loop calculation is the definition of integral families.

Integral families define bases of linear independent propagators which allow one to unambiguously map any scalar product onto a particular basis of propagators. Since there are two internal momenta $\{k_1, k_2\}$ and two external momenta $\{n, \bar{n}\}$ we have seven scalar products involving the internal momenta. Thus, to define a family we need to identify seven independent propagators. For example a family[1] without Heaviside functions reads

$$\mathcal{T}^{\text{ex}}_{a_1\ldots a_7} = \int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2}{[k_1^2]_c^{a_1} [k_2^2]_c^{a_2} [1 - k_{12} \cdot n]_c^{a_3} (k_1 \cdot k_2)^{a_4} (k_2 \cdot n)^{a_5} (k_1 \cdot \bar{n})^{a_6} (k_{12} \cdot \bar{n})^{a_7}}, \tag{5.1.19}$$

where we introduced $k_{12} = k_1 + k_2$. The family representation of an integral Eq. (5.1.19) now becomes useful when deriving IBPs, as it allows to write the derivatives in Eqs. (5.1.17) and (5.1.18) in a convenient way. Indeed, we apply Eq. (5.1.17) with $v = \bar{n}$ to Eq. (5.1.19) and focus on the last term. We find

$$\begin{aligned}
&\int \frac{\partial}{\partial k_1^\mu} \bar{n}^\mu \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2}{[k_1^2]_c^{a_1} [k_2^2]_c^{a_2} [1 - k_{12} \cdot n]_c^{a_3} (k_1 \cdot k_2)^{a_4} (k_2 \cdot n)^{a_5} (k_1 \cdot \bar{n})^{a_6} (k_{12} \cdot \bar{n})^{a_7}} = \ldots \\
&+ \int k_2 \cdot \bar{n} \frac{\partial}{\partial k_1 \cdot k_2} \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2}{[k_1^2]_c^{a_1} [k_2^2]_c^{a_2} [1 - k_{12} \cdot n]_c^{a_3} (k_1 \cdot k_2)^{a_4} (k_2 \cdot n)^{a_5} (k_1 \cdot \bar{n})^{a_6} (k_{12} \cdot \bar{n})^{a_7}} \\
&= \cdots + \int k_2 \cdot \bar{n} \frac{(-a_4) \mathrm{d}^d k_1 \mathrm{d}^d k_2}{[k_1^2]_c^{a_1} [k_2^2]_c^{a_2} [1 - k_{12} \cdot n]_c^{a_3} (k_1 \cdot k_2)^{a_4+1} (k_2 \cdot n)^{a_5} (k_1 \cdot \bar{n})^{a_6} (k_{12} \cdot \bar{n})^{a_7}} \\
&= \cdots + a_4 \left( \mathcal{T}^{\text{ex}}_{a_1,a_2,a_3,a_4+1,a_5,a_6-1,a_7} - \mathcal{T}^{\text{ex}}_{a_1,a_2,a_3,a_4+1,a_5,a_6 a_7-1} \right),
\end{aligned} \tag{5.1.20}$$

where we used the fact that the scalar product $k_2 \cdot \bar{n}$ can be expressed through the propagator basis

$$k_2 \cdot \bar{n} = -\frac{1}{(k_1 \cdot \bar{n})^{-1}} + \frac{1}{(k_1 \cdot \bar{n} + k_2 \cdot \bar{n})^{-1}}. \tag{5.1.21}$$

Thus, we observe that taking a derivative is equivalent to shifting indices when working with integral families. We can make this property even more apparent by writing

$$\begin{aligned}
0 &= \int \frac{\partial}{\partial k_1^\mu} \bar{n}^\mu \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2}{[k_1^2]_c^{a_1} [k_2^2]_c^{a_2} [1 - k_{12} \cdot n]_c^{a_3} (k_1 \cdot k_2)^{a_4} (k_2 \cdot n)^{a_5} (k_1 \cdot \bar{n})^{a_6} (k_{12} \cdot \bar{n})^{a_7}} \\
&= \left[ a_4 \hat{4}^+ (\hat{6}^- - \hat{7}^-) + 2 a_3 \hat{3}^+ - a_1 \hat{1}^+ \hat{6}^- \right] \mathcal{T}^{\text{ex}}_{a_1,a_2,a_3,a_4,a_5,a_6,a_7},
\end{aligned} \tag{5.1.22}$$

where we defined operators $\hat{i}^{+(-)}$ that increase (decrease) the index $a_i$ of the integral $\mathcal{T}^{\text{ex}}_{a_1,a_2,a_3,a_4,a_5,a_6,a_7}$ by one.

We note that while integral families serve as a useful tool to manage IBP relations, they are strictly speaking not necessary for a reduction. Indeed, the shorthand notation in Eq. (5.1.6) did not constitute a well-defined family definition, as it contained linearly-dependent propagators.

1 We note that we can set $\tau = 1$, in all integrals, because of the uniform scaling of the soft function. We can recover the full $\tau$ dependence at the end of the calculation.

We now add two Heaviside functions to the family definition Eq. (5.1.19) as required for the NNLO calculation of the soft function

$$\mathcal{T}^{\text{ex},\theta\theta}_{a_1\ldots a_7} = \int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2\, \theta(k_1\cdot\bar{n} - k_1\cdot n)\theta(k_2\cdot\bar{n} - k_2\cdot n)}{[k_1^2]_c^{a_1}[k_2^2]_c^{a_2}[1-k_{12}\cdot n]_c^{a_3}(k_1\cdot k_2)^{a_4}\,(k_2\cdot n)^{a_5}\,(k_1\cdot\bar{n})^{a_6}\,(k_{12}\cdot\bar{n})^{a_7}}. \tag{5.1.23}$$

Clearly, the two additional Heaviside functions in Eq. (5.1.23) do not count towards the total number of propagators and Eq. (5.1.23) thus remains a valid family definition with seven propagators. We re-derive the IBP relation Eq. (5.1.22) using the family definition in Eq. (5.1.23) and obtain

$$\begin{aligned}
0 &= \int \frac{\partial}{\partial k_1^\mu}\bar{n}^\mu \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2\, \theta(k_1\cdot\bar{n} - k_1\cdot n)\theta(k_2\cdot\bar{n} - k_2\cdot n)}{[k_1^2]_c^{a_1}[k_2^2]_c^{a_2}[1-k_{12}\cdot n]_c^{a_3}(k_1\cdot k_2)^{a_4}\,(k_2\cdot n)^{a_5}\,(k_1\cdot\bar{n})^{a_6}\,(k_{12}\cdot\bar{n})^{a_7}} \\
&= \left[a_4\hat{4}^+(\hat{6}^- - \hat{7}^-) + 2a_3\hat{3}^+ - a_1\hat{1}^+\hat{6}^-\right]\mathcal{T}^{\text{ex},\theta\theta}_{a_1,a_2,a_3,a_4,a_5,a_6,a_7} \\
&- (n\cdot\bar{n})\int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2\, \delta(k_1\cdot\bar{n} - k_1\cdot n)\theta(k_2\cdot\bar{n} - k_2\cdot n)}{[k_1^2]_c^{a_1}[k_2^2]_c^{a_2}[1-k_{12}\cdot n]_c^{a_3}(k_1\cdot k_2)^{a_4}\,(k_2\cdot n)^{a_5}\,(k_1\cdot\bar{n})^{a_6}\,(k_{12}\cdot\bar{n})^{a_7}}.
\end{aligned} \tag{5.1.24}$$

As already discussed, the IBP equation can be split into two parts. The first part, which we call the homogeneous part of the IBP equation, is identical to Eq. (5.1.22) and all terms present there can be expressed through integrals defined in Eq. (5.1.23). The second part, which we refer to as the inhomogeneous part of the IBP equation, appears when derivatives act on the Heaviside functions; the resulting integrals can not be expressed through integrals in Eq. (5.1.23). To describe them we can not simply introduce a new family

$$\mathcal{T}^{\text{ex},\delta\theta}_{a_1\ldots a_7} = \int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2\, \delta(k_1\cdot\bar{n} - k_1\cdot n)\theta(k_2\cdot\bar{n} - k_2\cdot n)}{[k_1^2]_c^{a_1}[k_2^2]_c^{a_2}[1-k_{12}\cdot n]_c^{a_3}(k_1\cdot k_2)^{a_4}\,(k_2\cdot n)^{a_5}\,(k_1\cdot\bar{n})^{a_6}\,(k_{12}\cdot\bar{n})^{a_7}}, \tag{5.1.25}$$

because the propagators $[k_1\cdot\bar{n} - k_1\cdot n]_c$, $[1-k_1\cdot n - k_2\cdot n]_c$, $k_1\cdot\bar{n}$ and $k_2\cdot n$ are now linearly dependent on each other. While we are unable to resolve this linear dependence for the arbitrary exponents in Eq. (5.1.25), we can perform a partial fractioning for any fixed values of $a_1$ through $a_7$. For example, if we consider Eq. (5.1.24) for the seed integral $\mathcal{T}^{\text{ex},\theta\theta}_{1,1,1,1,1,1,1}$, we obtain

$$\begin{aligned}
0 &= \int \frac{\partial}{\partial k_1^\mu}\bar{n}^\mu \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2\, \theta(k_1\cdot\bar{n} - k_1\cdot n)\theta(k_2\cdot\bar{n} - k_2\cdot n)}{[k_1^2]_c[k_2^2]_c[1-k_{12}\cdot n]_c(k_1\cdot k_2)\,(k_2\cdot n)\,(k_1\cdot\bar{n})\,(k_{12}\cdot\bar{n})} \\
&= \left[\hat{4}^+(\hat{6}^- - \hat{7}^-) + 2\,\hat{3}^+ - \hat{1}^+\hat{6}^-\right]\mathcal{T}^{\text{ex},\theta\theta}_{1,1,1,1,1,1,1} \\
&- (n\cdot\bar{n})\int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2\, \delta(k_1\cdot\bar{n} - k_1\cdot n)\theta(k_2\cdot\bar{n} - k_2\cdot n)}{[k_1^2]_c[k_2^2]_c[1-k_{12}\cdot n]_c(k_1\cdot k_2)\,(k_2\cdot n)\,(k_1\cdot\bar{n})\,(k_{12}\cdot\bar{n})}.
\end{aligned} \tag{5.1.26}$$

The linear dependence in the last term of Eq. (5.1.26) is now resolved by replacing $\delta(k_1\cdot\bar{n} - k_1\cdot n) \to 1/[k_1\cdot\bar{n} - k_1\cdot n]_c$ and multiplying with the partial fractioning identity[2]

$$1 = \frac{(k_1\cdot\bar{n}) - [k_1\cdot\bar{n} - k_1\cdot n]_c}{(k_1\cdot n)} \times \left\{[1-k_{12}\cdot n]_c + (k_1\cdot n) + (k_2\cdot n)\right\}. \tag{5.1.27}$$

2 We use the algorithm outlined by A. Pak in Ref. [78] to implement partial fraction identities in our code.

We obtain

$$\begin{aligned} &\int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2\, \theta(k_2\cdot\bar{n} - k_2\cdot n)\, [k_1\cdot\bar{n} - k_1\cdot n)]_c^{-1}}{[k_1^2]_c[k_2^2]_c[1-k_{12}\cdot n]_c\,(k_1\cdot k_2)\,(k_1\cdot\bar{n})\,(k_2\cdot n)\,(k_{12}\cdot\bar{n})} \\ &= \int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2\, \theta(k_2\cdot\bar{n} - k_2\cdot n)\, [k_1\cdot\bar{n} - k_1\cdot n)]_c^{-1}}{[k_1^2]_c[k_2^2]_c[1-k_{12}\cdot n]_c\,(k_1\cdot k_2)\,(k_{12}\cdot\bar{n})} \left[\frac{1}{(k_1\cdot n)} + \frac{1}{(k_2\cdot n)}\right], \end{aligned} \tag{5.1.28}$$

where we discarded terms without the full set of cut propagators. After defining two new families

$$\mathcal{T}^{\mathrm{ex1},\delta\theta}_{a_1\ldots a_7} = \int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2\, \theta(k_2\cdot\bar{n} - k_2\cdot n)\, [k_1\cdot\bar{n} - k_1\cdot n)]_c^{-a_4}}{[k_1^2]_c^{a_1}[k_2^2]_c^{a_2}[1-k_{12}\cdot n]_c^{a_3}(k_1\cdot k_2)^{a_5}(k_{12}\cdot\bar{n})^{a_6}(k_1\cdot n)^{a_7}}, \tag{5.1.29}$$

$$\mathcal{T}^{\mathrm{ex2},\delta\theta}_{a_1\ldots a_7} = \int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2\, \theta(k_2\cdot\bar{n} - k_2\cdot n)\, [k_1\cdot\bar{n} - k_1\cdot n)]_c^{-a_4}}{[k_1^2]_c^{a_1}[k_2^2]_c^{a_2}[1-k_{12}\cdot n]_c^{a_3}(k_1\cdot k_2)^{a_5}(k_{12}\cdot\bar{n})^{a_6}(k_2\cdot n)^{a_7}}, \tag{5.1.30}$$

we finally write Eq. (5.1.26) as

$$0 = \left[\hat{4}^+(\hat{6}^- - \hat{7}^-) + 2\,\hat{3}^+ - \hat{1}^+\hat{6}^-\right] \mathcal{T}^{\mathrm{ex},\theta\theta}_{1,1,1,1,1,1,1} - 2\left(\mathcal{T}^{\mathrm{ex1},\delta\theta}_{1,1,1,1,1,1,1} + \mathcal{T}^{\mathrm{ex2},\delta\theta}_{1,1,1,1,1,1,1}\right). \tag{5.1.31}$$

We can now compare the normal IBP relation Eq. (5.1.22) and the modified IBP Eq. (5.1.31). As already mentioned, the first part of the modified IBP is equivalent to the normal IBP. Since these normal IBP relations form a closed system of linear equations which can be studied on their own, we call them the homogeneous part of the modified IBP equation. The inhomogeneous part is given by integrals of different families where one of the Heaviside functions is replaced by a delta function. Thus, the IBP identity Eq. (5.1.31) relates integrals with two Heaviside functions to integrals with one Heaviside function and one delta function. If we repeat this derivation for an integral with one Heaviside function and one delta function as a seed integral, we obtain integrals with two delta functions as inhomogeneous terms. Extrapolating, we conclude that all IBPs are of the form

$$0 = \sum_i b_i^{\mathrm{hom}}(\epsilon)\mathcal{T}^{k,\theta\theta}_{\{a_i\}} + \sum_i\sum_k b_{i,k}^{\mathrm{inhom}}(\epsilon)\mathcal{T}^{k,\delta\theta}_{\{a_i\}}, \tag{5.1.32}$$

$$0 = \sum_i c_i^{\mathrm{hom}}(\epsilon)\mathcal{T}^{k,\theta\theta}_{\{a_i\}} + \sum_i\sum_k c_{i,k}^{\mathrm{inhom}}(\epsilon)\mathcal{T}^{k,\theta\delta}_{\{a_i\}}, \tag{5.1.33}$$

$$0 = \sum_i d_i^{\mathrm{hom}}(\epsilon)\mathcal{T}^{k,\delta\theta}_{\{a_i\}} + \sum_i\sum_k d_{i,k}^{\mathrm{inhom}}(\epsilon)\mathcal{T}^{k,\delta\delta}_{\{a_i\}}, \tag{5.1.34}$$

$$0 = \sum_i e_i^{\mathrm{hom}}(\epsilon)\mathcal{T}^{k,\theta\delta}_{\{a_i\}} + \sum_i\sum_k e_{i,k}^{\mathrm{inhom}}(\epsilon)\mathcal{T}^{k,\delta\delta}_{\{a_i\}}, \tag{5.1.35}$$

$$0 = \sum_i f_i^{\mathrm{hom}}(\epsilon)\mathcal{T}^{k,\delta\delta}_{\{a_i\}}, \tag{5.1.36}$$

where the notations $\mathcal{T}^{k,\theta\theta}$, $\mathcal{T}^{k,\delta\theta}$, $\mathcal{T}^{k,\theta\delta}$ and $\mathcal{T}^{k,\delta\delta}$ are self-explanatory. Eqs. (5.1.32) - (5.1.36) define a hierarchical structure of families. In this hierarchy, integrals with fewer Heaviside functions and more $\delta$ functions are easier to compute. For this reason we will always try to reduce integrals with more Heaviside functions to integrals with more $\delta$ functions.

We would like to make a final remark concerning the derivation of modified IBP equations. We have seen that inhomogeneous parts of the IBP equations can not be derived for arbitrary powers of propagators due to the necessity of partial fractioning. This makes the generation of IBPs considerably slower than what is the case normally. We note that this forces us to choose a seed lists in a conservative manner when generating IBPs for complicated problems, such as the calculation of the N3LO soft function.

This concludes our discussion of modified IBPs. We apply them to the NNLO soft function in the next section.

## 5.2 REDUCTION OF THE SOFT FUNCTION

In the last section we derived modified IBP equations for NNLO integrals with Heaviside functions.
To compare the modified IBP approach with the auxiliary parameter method in Chapter 4, we calculate the maximally non-Abelian contribution to the NNLO soft function $S^{(2)}_{Ca,gg,A}$ and $S^{(2)}_{Ca,gg,B}$. They are defined as

$$S^{(2)}_{CA,gg,A,B} = \int \mathrm{dPS}^{(2)}_S \; M_{A,B} \left(2\xi_{12} - \xi_{11} - \xi_{22}\right), \tag{5.2.1}$$

where

$$M_A = \delta\left(\tau - n\cdot k_1 - n\cdot k_2\right)\theta\left(\bar{n}\cdot k_1 - n\cdot k_1\right)\theta\left(\bar{n}\cdot k_2 - n\cdot k_2\right), \tag{5.2.2}$$

$$M_B = \delta\left(\tau - n\cdot k_1 - \bar{n}\cdot k_2\right)\theta\left(\bar{n}\cdot k_1 - n\cdot k_1\right)\theta\left(n\cdot k_2 - \bar{n}\cdot k_2\right), \tag{5.2.3}$$

and the integrand $\xi_{ij}$ was defined in Eq. (4.1.25). We implement derivation of modified IBPs in a `Mathematica` code. The output is then fed to `Kira` where we use the“user-defined system” feature to solve the system of linear equations. Proceeding in this way, we obtain the maximally non-Abelian contribution in terms of master integrals. The relation reads

$$\begin{aligned} S_{\mathrm{NA},gg} &= S^{(2)}_{CA,gg,A} + S^{(2)}_{CA,gg,B} \\ &= \tau^{-1-4\epsilon}\Bigg\{\Bigg[\frac{\left(192\epsilon^5+48\epsilon^4-736\epsilon^3+1336\epsilon^2-376\epsilon+33\right)}{3\epsilon^3(2\epsilon-3)(2\epsilon-1)}\mathcal{I}^{nn}_1 \\ &\quad - \frac{8(4\epsilon-1)\left(12\epsilon^4-25\epsilon^2+41\epsilon-3\right)}{3\epsilon^2(2\epsilon-3)(2\epsilon-1)}\mathcal{I}^{nn}_2 + \frac{3}{\epsilon}\mathcal{I}^{nn}_3 + \frac{2}{\epsilon}\mathcal{I}^{nn}_4\Bigg] \\ &\quad + \Bigg[\frac{128\epsilon^7+864\epsilon^6-848\epsilon^5-1680\epsilon^4+152\epsilon^3+770\epsilon^2-163\epsilon+3}{\epsilon^3(\epsilon+1)(2\epsilon-1)(2\epsilon+1)(2\epsilon+3)}\mathcal{I}^{n\bar{n}}_1 \\ &\quad - \frac{8\left(64\epsilon^7+120\epsilon^6-164\epsilon^5-246\epsilon^4+69\epsilon^3+126\epsilon^2-46\epsilon+3\right)}{\epsilon^2(\epsilon+1)(2\epsilon-1)(2\epsilon+1)(2\epsilon+3)}\mathcal{I}^{n\bar{n}}_2 \\ &\quad + \frac{\left(16\epsilon^5+16\epsilon^3+36\epsilon^2+11\epsilon-9\right)}{\epsilon(\epsilon+1)(2\epsilon+1)(2\epsilon+3)}\mathcal{I}^{n\bar{n}}_3 + \frac{2}{\epsilon}\mathcal{I}^{n\bar{n}}_4 \end{aligned} \tag{5.2.4}$$

$$-\frac{8(4\epsilon-1)\left(2\epsilon^3+3\epsilon^2+3\epsilon-3\right)}{\epsilon(2\epsilon+1)(2\epsilon+3)}\mathcal{I}_5^{n\bar{n}}+2\,\mathcal{I}_6^{n\bar{n}}+4\mathcal{I}_7^{n\bar{n}}\Bigg]\Bigg\}.$$

The master integrals that appear in Eq. (5.2.4) are defined as follows

$$\begin{aligned}
\mathcal{I}_1^{nn} &= \int \mathrm{d}\Phi_{\delta\delta}^{nn}\,, & \mathcal{I}_2^{nn} &= \int \frac{\mathrm{d}\Phi_{\delta\theta}^{nn}}{(k_{12}\cdot\bar{n})}\,,\\
\mathcal{I}_3^{nn} &= \int \frac{\mathrm{d}\Phi_{\delta\theta}^{nn}}{(k_1\cdot k_2)(k_2\cdot\bar{n})}\,, & \mathcal{I}_4^{nn} &= \int \frac{\mathrm{d}\Phi_{\delta\theta}^{nn}}{(k_1\cdot k_2)(k_2\cdot n)(k_{12}\cdot\bar{n})}\,,
\end{aligned} \tag{5.2.5}$$

and

$$\begin{aligned}
\mathcal{I}_1^{n\bar{n}} &= \int \mathrm{d}\Phi_{\delta\delta}^{n\bar{n}}\,, & \mathcal{I}_2^{n\bar{n}} &= \int \frac{\mathrm{d}\Phi_{\delta\theta}^{n\bar{n}}}{(k_{12}\cdot n)}\,,\\
\mathcal{I}_3^{n\bar{n}} &= \int \frac{\mathrm{d}\Phi_{\delta\theta}^{n\bar{n}}}{(k_1\cdot k_2)(k_2\cdot n)}\,, & \mathcal{I}_4^{n\bar{n}} &= \int \frac{\mathrm{d}\Phi_{\delta\theta}^{n\bar{n}}}{(k_1\cdot k_2)(k_2\cdot\bar{n})(k_{12}\cdot n)}\,,\\
\mathcal{I}_5^{n\bar{n}} &= \int \frac{\mathrm{d}\Phi_{\theta\theta}^{n\bar{n}}}{(k_{12}\cdot n)(k_{12}\cdot\bar{n})}\,, & \mathcal{I}_6^{n\bar{n}} &= \int \frac{\mathrm{d}\Phi_{\theta\theta}^{n\bar{n}}}{(k_1\cdot k_2)(k_1\cdot\bar{n})(k_2\cdot n)}\,,\\
\mathcal{I}_7^{n\bar{n}} &= \int \frac{\mathrm{d}\Phi_{\theta\theta}^{n\bar{n}}}{(k_1\cdot k_2)(k_2\cdot n)(k_{12}\cdot\bar{n})}\,.
\end{aligned} \tag{5.2.6}$$

In Eqs. (5.2.5) and (5.2.6) we introduced the shorthand notations

$$\mathrm{d}\Phi_{fg}^{nn} = [\mathrm{d}k_1][\mathrm{d}k_2]\delta\left(1-k_1\cdot n-k_2\cdot n\right)f\left(k_1\cdot\bar{n}-k_1\cdot n\right)g\left(k_2\cdot\bar{n}-k_2\cdot n\right), \tag{5.2.7}$$

$$\mathrm{d}\Phi_{fg}^{n\bar{n}} = [\mathrm{d}k_1][\mathrm{d}k_2]\delta\left(1-k_1\cdot n-k_2\cdot\bar{n}\right)f\left(k_1\cdot\bar{n}-k_1\cdot n\right)g\left(k_2\cdot n-k_2\cdot\bar{n}\right), \tag{5.2.8}$$

$$[\mathrm{d}k_i] = \frac{\mathrm{d}^d k_i}{(2\pi)^{d-1}}\,\delta^+(k_i^2). \tag{5.2.9}$$

We note that not all of these master integrals are independent of each other. In fact it holds true that $\mathcal{I}_i^{n\bar{n}} = \mathcal{I}_i^{nn}$ for $i=1,2,3,4$. This is due to the fact that these integrals are symmetric under the exchange of $\alpha_i \leftrightarrow \beta_i$ (or $n \leftrightarrow \bar{n}$). For example, consider the integral $\mathcal{I}_2^{n\bar{n}}$. We find

$$\begin{aligned}
\mathcal{I}_2^{n\bar{n}} &= \int \frac{[\mathrm{d}k_1][\mathrm{d}k_2]\delta(1-k_1\cdot n-k_2\cdot\bar{n})\delta(k_1\cdot\bar{n}-k_1\cdot n)\theta(k_2\cdot n-k_2\cdot\bar{n})}{(k_{12}\cdot n)}\\
&\overset{n\leftrightarrow\bar{n}}{=} \int \frac{[\mathrm{d}k_1][\mathrm{d}k_2]\delta(1-k_1\cdot\bar{n}-k_2\cdot n)\delta(k_1\cdot n-k_1\cdot\bar{n})\theta(k_2\cdot\bar{n}-k_2\cdot n)}{(k_{12}\cdot\bar{n})}\\
&= \int \frac{[\mathrm{d}k_1][\mathrm{d}k_2]\delta(1-k_1\cdot n-k_2\cdot n)\delta(k_1\cdot\bar{n}-k_1\cdot n)\theta(k_2\cdot\bar{n}-k_2\cdot n)}{(k_{12}\cdot\bar{n})} = \mathcal{I}_2^{nn}\,.
\end{aligned} \tag{5.2.10}$$

We further note, that there are no master integrals with two Heaviside functions in configuration $A$. To understand this we note that, since the homogeneous parts of the IBPs is unaffected by the Heaviside functions, we should find all master integrals with two Heaviside functions by removing the Heaviside functions and solving the homogeneous IBP equations. However, all integrals belonging to configuration $A$ would vanish in this case, because they are scaleless, leaving us without master integrals.

To illustrate this, we note that eikonal propagator needs to be homogeneous in the external momenta $n$ and $\bar{n}$. Thus at NNLO we only find the following propagators

$$D \in \{k_1 \cdot n, k_2 \cdot n, k_{12} \cdot n, k_1 \cdot \bar{n}, k_2 \cdot \bar{n}, k_{12} \cdot \bar{n}, k_1 \cdot k_2\}, \tag{5.2.11}$$

and any integral we need to consider in configuration $A$ is of the form

$$\begin{aligned} &\int \frac{[\mathrm{d}k_1][\mathrm{d}k_2]\delta(1 - k_1 \cdot n - k_2 \cdot n)}{\prod_i D_i} \\ &= \int \left( \prod_{i=1}^{2} \mathrm{d}\alpha_i \mathrm{d}\beta_i \mathrm{d}\Omega_i^{d-2} (\alpha_i \beta_i)^{-\epsilon} \right) \frac{\delta(1 - \beta_1 - \beta_2)}{\prod_i D_i}. \end{aligned} \tag{5.2.12}$$

We note that such integrals would vanish if they are homogeneous functions of $\alpha_i$. Then

$$\int \mathrm{d}\alpha_i \mathrm{d}\beta_i f(\alpha_i, \beta_i) \overset{\alpha_i \to \lambda\alpha_i}{=} \lambda^n \int \mathrm{d}\alpha_i \mathrm{d}\beta_i f(\alpha_i, \beta_i), \tag{5.2.13}$$

$$\Rightarrow \int \mathrm{d}\alpha_i \mathrm{d}\beta_i f(\alpha_i, \beta_i) = 0. \tag{5.2.14}$$

To check scaling properties of propagators $\{D_i\}$ with $\alpha_i$, we write

$$D \in \{\beta_1, \beta_2, \beta_{12}, \alpha_1, \alpha_2, \alpha_{12}, \beta_1\alpha_2 + \beta_2\alpha_1 + \sqrt{\beta_1\beta_2\alpha_1\alpha_2}\vec{e}_1 \cdot \vec{e}_2\}. \tag{5.2.15}$$

From Eq. (5.2.15) it becomes clear that all propagators $D_i$, are homogeneous functions under a simultaneous scaling of $\alpha_1$ and $\alpha_2$

$$D_i(\lambda\alpha_1, \lambda\alpha_2) = \lambda^{n_i} D_i(\alpha_1, \alpha_2), \tag{5.2.16}$$

where $n_i$ is an integer. Thus any integral in Eq. (5.2.12) is equal to zero.

Similarly, we can predict which integrals do not vanish in configuration $B$ and thus which master integrals to expect. The integrals we need to consider in this case are of the form

$$\begin{aligned} &\int \frac{[\mathrm{d}k_1][\mathrm{d}k_2]\delta(1 - k_1 \cdot n - k_2 \cdot \bar{n})}{\prod_i D_i} \\ &= \int \left( \prod_{i=1}^{2} \mathrm{d}\alpha_i \mathrm{d}\beta_i \mathrm{d}\Omega_i^{d-2} (\alpha_i \beta_i)^{-\epsilon} \right) \frac{\delta(1 - \beta_1 - \alpha_2)}{\prod_i D_i}, \end{aligned} \tag{5.2.17}$$

and we immediately see that the only possible homogeneous scaling is given by $\alpha_1 \to \lambda\alpha_1$, $\beta_2 \to \lambda\beta_2$. However, from Eq. (5.2.15) it becomes clear that the three propagators $k_{12} \cdot n$, $k_{12} \cdot \bar{n}$ and $k_1 \cdot k_2$ are not homogeneous functions under this scaling and thus at least one of them has to be part of each $n\bar{n}$ master integral with two Heaviside functions. A glance at Eq. (5.2.6) shows that this is indeed the case.

## 5.3 CALCULATION OF MASTER INTEGRALS

Having discussed the reduction to master integrals in the previous sections, we will now explain how to calculate the master integrals shown in Eqs. (5.2.5) and (5.2.6).
We begin by discussing the easiest integral, the empty phase space with two delta functions first. It reads

$$\mathcal{I}_1^{nn} = \int [\mathrm{d}k_1][\mathrm{d}k_2]\delta\left(1 - k_1\cdot n - k_2\cdot n\right)\delta\left(k_1\cdot\bar{n} - k_1\cdot n\right)\delta\left(k_2\cdot\bar{n} - k_2\cdot n\right). \tag{5.3.1}$$

We employ the Sudakov decomposition

$$k_i^\mu = \alpha_i \frac{n^\mu}{2} + \beta_i \frac{\bar{n}^\mu}{2} + k_{i\perp}^\mu, \tag{5.3.2}$$

introduce spherical coordinates for $k_{i\perp}$ and integrate over its absolute value. The phase-space element reads

$$[\mathrm{d}k_i] = \frac{\mathrm{d}\Omega_i^{(d-2)}}{4(2\pi)^{d-1}}\,\mathrm{d}\alpha_i\,\mathrm{d}\beta_i\,(\alpha_i\beta_i)^{-\epsilon}, \qquad \alpha_i\,,\beta_i \in [0,\infty)\,. \tag{5.3.3}$$

Combining Eq. (5.3.3) with Eq. (5.3.1) we immediately obtain

$$\begin{aligned}
\mathcal{I}_1^{nn} &= [N]^2 \int \prod_{i=1}^{2} \mathrm{d}\alpha_i\,\mathrm{d}\beta_i(\alpha_i\beta_i)^{-\epsilon}\,\delta(1-\beta_{12})\,\delta(\alpha_1-\beta_1)\,\delta(\alpha_2-\beta_2)\\
&= [N]^2 \int_0^1 \mathrm{d}\beta_1\,\mathrm{d}\beta_2(\beta_1\beta_2)^{-2\epsilon}\,\delta(1-\beta_{12})\\
&= [N]^2\,\frac{\Gamma^2(1-2\epsilon)}{\Gamma(2-4\epsilon)},
\end{aligned} \tag{5.3.4}$$

where the constant $[N]$ was defined in Eq. (4.2.5). The next integral is $\mathcal{I}_2^{nn}$; it reads

$$\mathcal{I}_2^{nn} = \int [\mathrm{d}k_1][\mathrm{d}k_2]\frac{\delta\left(1 - k_1\cdot n - k_2\cdot n\right)}{k_{12}\cdot\bar{n}}\delta\left(k_1\cdot\bar{n} - k_1\cdot n\right)\theta\left(k_2\cdot\bar{n} - k_2\cdot n\right). \tag{5.3.5}$$

We repeat the steps discussed in the calculation of $\mathcal{I}_1^{nn}$ and obtain

$$\begin{aligned}
\mathcal{I}_2^{nn} &= [N]^2 \int_0^\infty \mathrm{d}\alpha_2\,\mathrm{d}\beta_1\,\mathrm{d}\beta_2\,\frac{(\alpha_2\beta_1^2\beta_2)^{-\epsilon}\,\delta(1-\beta_{12})\,\theta(\alpha_2-\beta_2)}{\beta_1+\alpha_2}\\
&\overset{\alpha_2\to\beta_2/\xi_2}{=} [N]^2 \int_0^1 \mathrm{d}\xi_2 \int_0^1 \mathrm{d}\beta_1\,\frac{\beta_1^{-2\epsilon}(1-\beta_1)^{1-2\epsilon}\xi_2^{\epsilon-1}}{1-\beta_1(1-\xi_2)}\\
&= [N]^2\,\frac{\Gamma(1-2\epsilon)\,\Gamma(2-2\epsilon)}{\epsilon\,\Gamma(3-4\epsilon)}\,{}_3F_2\left(1,1,1-2\epsilon,3-4\epsilon,1+\epsilon,1\right).
\end{aligned} \tag{5.3.6}$$

Integral $\mathcal{I}_3^{nn}$ is the first one to include the scalar product $k_1\cdot k_2$. It reads

$$\mathcal{I}_3^{nn} = \int [\mathrm{d}k_1][\mathrm{d}k_2]\frac{\delta\left(1 - k_1\cdot n - k_2\cdot n\right)}{k_1\cdot k_2\; k_2\cdot\bar{n}}\delta\left(k_1\cdot\bar{n} - k_1\cdot n\right)\theta\left(k_2\cdot\bar{n} - k_2\cdot n\right). \tag{5.3.7}$$

Using Eq. (5.3.3) in Eq. (5.3.7) we obtain

$$\mathcal{I}_3^{nn} \stackrel{\alpha_2 \to \beta_2/\xi_2}{=} \left[\frac{1}{4(2\pi)^{d-1}}\right]^2 2\int \mathrm{d}\Omega_{12}^{(d-2)} \int \frac{\mathrm{d}\xi_2\, \mathrm{d}\beta_1\, \xi_2^{\epsilon}\, [\beta_1(1-\beta_1)]^{-1-2\epsilon}}{\left[\xi_2 + 1 - 2\sqrt{\xi_2}\cos\varphi_{12}\right]}, \tag{5.3.8}$$

where $\mathrm{d}\Omega_{12}^{(d-2)} = \mathrm{d}\Omega_1^{(d-2)}\,\mathrm{d}\Omega_2^{(d-2)}$ and $\varphi_{12}$ is the relative angle between transverse components of $k_1$ and $k_2$. We write the angular integration as

$$\mathrm{d}\Omega_{12}^{(d-2)} = 2\,\mathrm{d}\Omega^{(d-2)}\,\mathrm{d}\Omega^{(d-3)}\,\mathrm{d}y\,[4y(1-y)]^{-1/2-\epsilon}\,, \tag{5.3.9}$$

where we introduced

$$y = \frac{1-\cos\varphi_{12}}{2}\,. \tag{5.3.10}$$

We use Eq. (5.3.9) in Eq. (5.3.8), integrate over $y$ and find

$$2\int_0^1 \mathrm{d}y \frac{[4y(1-y)]^{-1/2-\epsilon}}{\left[\xi_2 + 1 - 2\sqrt{\xi_2}(1-2y)\right]} = \frac{\Omega^{(d-2)}}{\Omega^{d-3}} \frac{{}_2F_1\left(1, 1/2-\epsilon, 1-2\epsilon, \frac{-4\sqrt{\xi_2}}{(1-\sqrt{\xi_2})^2}\right)}{(1-\sqrt{\xi_2})^2}\,. \tag{5.3.11}$$

We rewrite the hypergeometric function using the identity

$${}_2F_1\left(a, b, 2b, z\right) = \left(1-\frac{z}{2}\right)^{-a} {}_2F_1\left(\frac{a}{2}, \frac{a+1}{2}, b+\frac{1}{2}, \frac{z^2}{4(1-z/2)^2}\right), \tag{5.3.12}$$

integrate over $\xi_2$ and obtain

$$\mathcal{I}_3^{nn} = -[N]^2 \frac{2\,\Gamma^2(1-2\epsilon)}{\epsilon(1+\epsilon)\,\Gamma(1-4\epsilon)}\, {}_3F_2\left(1, 1+\epsilon, 1+\epsilon, 1-\epsilon, 2+\epsilon, 1\right)\,. \tag{5.3.13}$$

The calculation of $I_4$ proceeds in full analogy. We find

$$\begin{aligned}\mathcal{I}_4^{nn} &= \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{\delta\left(1 - k_1\cdot n - k_2\cdot n\right)}{k_1\cdot k_2\; k_2\cdot n\; k_{12}\cdot \bar{n}}\,\delta\left(k_1\cdot\bar{n} - k_1\cdot n\right)\theta\left(k_2\cdot\bar{n} - k_2\cdot n\right) \\ &= -[N]^2\, 2\frac{\Gamma^2(1-2\epsilon)}{\epsilon\,\Gamma(1-4\epsilon)} \int_0^1 \mathrm{d}\xi_2\, \xi_2^{-1-\epsilon}(1-\xi_2)^{-1-2\epsilon}\mathcal{X}_4(\xi_2)\,,\end{aligned} \tag{5.3.14}$$

where

$$\mathcal{X}_4(\xi_2) = {}_2F_1\left(-1-4\epsilon, -2\epsilon, -4\epsilon, 1-\xi_2\right) {}_2F_1\left(-\epsilon, -2\epsilon, 1-\epsilon, \xi_2\right)\,. \tag{5.3.15}$$

While the integrand in Eq. (5.3.14) diverges at the integration boundaries $\xi_2 = 0, 1$, the function $\mathcal{X}_4(\xi_2)$ is regular at these points

$$\mathcal{X}_4(0) = \frac{\Gamma(1-4\epsilon)\,\Gamma(1+2\epsilon)}{2\,\Gamma(1-2\epsilon)}\,, \quad \mathcal{X}_4(1) = \frac{\Gamma(1-\epsilon)\,\Gamma(1+2\epsilon)}{\Gamma(1+\epsilon)}\,. \tag{5.3.16}$$

Therefore, we compute the integral by subtracting the divergent contributions at the two endpoints and adding them back. Specifically, we write

$$
\begin{aligned}
\mathcal{I}_4^{nn} = &- [N]^2\, 2\frac{\Gamma^2(1-2\epsilon)}{\epsilon\,\Gamma(1-4\epsilon)}\Big\{\mathcal{X}_4(0)\int_0^1 \mathrm{d}\xi_2\,\xi_2^{-1-\epsilon} + \mathcal{X}_4(1)\int_0^1 \mathrm{d}\xi_2\,(1-\xi_2)^{-1-2\epsilon} \\
&+ \int_0^1 \mathrm{d}\xi_2 \Big[\xi_2^{-1-\epsilon}(1-\xi_2)^{-1-2\epsilon}\mathcal{X}_4(\xi_2) - \xi_2^{-1-\epsilon}\mathcal{X}_4(0) \\
&- (1-\xi_2)^{-1-2\epsilon}\mathcal{X}_4(1)\Big]\Big\}.
\end{aligned} \tag{5.3.17}
$$

The first two integrals on the r.h.s of Eq. (5.3.17) can be trivially computed, while the last integral is now regular in the integration domain $\xi_2 \in [0,1]$ and can be computed as an expansion in $\epsilon$. We perform the expansion with the package `HypExp` [79] and use the program `HyperInt` [68] to integrate the result over $\xi_2$. We find

$$
\begin{aligned}
\mathcal{I}_4^{nn} = [N]^2 &\left\{\frac{2}{\epsilon^2} + \frac{\pi^2}{3} - \frac{17\pi^4\epsilon^2}{90} + \epsilon^3\left[-6\pi^2\zeta_3 - 26\zeta_5\right] - \epsilon^4\left[\frac{193\pi^6}{810} + 64\zeta_3^2\right]\right\} \\
&+ \mathcal{O}(\epsilon^5),
\end{aligned} \tag{5.3.18}
$$

where we have discarded contributions of transcendental weight seven and higher. The remaining integrals are all part of configuration $B$. The computation is analogous to that of configuration $A$ and explicit calculations can be found in Appendix C.2.

This concludes the computation of master integrals required for the maximally non-Abelian contribution to the zero-jettiness soft function at NNLO. We present the result in the next section.

## 5.4 RESULT AND SUMMARY

We insert the results for master integrals computed in Section 5.3 into Eq. (5.2.4) and obtain

$$
\begin{aligned}
S^{gg}_{\mathrm{NA}} = [N]^2\,\tau^{-1-4\epsilon} 4 &\Bigg\{\frac{2}{\epsilon^3} + \frac{11}{3\epsilon^2} + \frac{1}{\epsilon}\left[\frac{67}{9} - \pi^2\right] + \left[\frac{404}{27} - \frac{11\pi^2}{9} - 18\zeta_3\right] \\
&+ \epsilon\left[\frac{2140}{81} + \frac{67\pi^2}{9} - \frac{59\pi^4}{90} - \frac{220\zeta_3}{3}\right] + \epsilon^2\Bigg[\frac{12416}{243} - \frac{368\pi^2}{81} \\
&- \frac{143\pi^4}{45} + 268\zeta_3 + 4\pi^2\zeta_3 - 182\zeta_5\Bigg] + \epsilon^3\Bigg[\frac{67528}{729} + \frac{2416\pi^2}{81} \\
&+ \frac{469\pi^4}{45} - \frac{17\pi^6}{105} - \frac{7864\zeta_3}{27} + \frac{880\pi^2\zeta_3}{9} - 122\zeta_3^2 - \frac{6248\zeta_5}{3}\Bigg] + \mathcal{O}(\epsilon^4)\Bigg\}.
\end{aligned} \tag{5.4.1}
$$

Eq. (5.4.1) agrees with our earlier results for the same quantity computed with a different method, see Eqs. (4.3.15) and (4.3.16).

Having computed the same quantity with two different approaches, we can compare the auxiliary parameter method of Chapter 4 and the modified IBPs discussed in this chapter.

While the number of master integrals was roughly equal for both approaches, the computation of these integrals was somewhat easier if the auxillary parameters were used. Indeed, the factorization of angular integration in this case made the computation of master integrals next to trivial. However, the integrals encountered in this chapter were still remarkably simple. In addition, no complicated integration over auxillary parameters was required. Thus it appears that, the modified IBP approach is a significant simplification as compared to the original computations of the NNLO soft function in Refs. [48, 49] and is of comparable simplicity to the auxillary parameter method of Chapter 4.

As already mentioned in Section 4.4, the auxillary parameter method does become quite complicated at N3LO in the presence of the propagator $(k_1 + k_2 + k_3)^{-2}$ for which angular integrations do not factorize. Even worse, introduction of auxillary parameters complicates the evaluation of master integrals; and, additionally, one needs to integrate over three auxillary parameters afterwards. Thus. it appears to us that, the modified IBP approach is much better suited to calculate the zero-jettiness soft function at N3LO. We discuss the calculation of the same-hemisphere three-gluon-emission contribution to the zero-jettiness soft function at N3LO QCD with the modified IBP approach in the next chapter.

# 6

# N3LO SOFT FUNCTION

*This chapter is based on References [77, 80].*

In this chapter we apply the modified IBP approach introduced in Chapter 5 to calculate the same hemisphere triple gluon contribution to the N3LO zero-jettiness soft function. The chapter is structured as follows. We begin by discussing technical aspects of modified IBPs at N3LO in Section 6.1. We proceed to explain the general setup of the calculation in Section 6.2. The required eikonal function is extracted from the literature [74] where it is organized into four different parts according to their singularity structure. For this reason, our computation is also split into these four different pieces: $\omega^{(3),a}$, $\omega^{(3),b}$, $\omega^{(3),c}$ and $\omega^{(3),d}$. We discuss parts $\omega^{(3),a}$, $\omega^{(3),b}$ and $\omega^{(3),c}$ one after another in Sections 6.3 - 6.5, explaining the reduction process and how to compute master integrals for each of them. We proceed with the computation of the most complex contribution $\omega^{(3),d}$ in Section 6.6, explaining how to set up differential equations and calculate boundary conditions for the most complex master integrals. Finally, we conclude in Section 6.7.

We note that we also applied the auxiliary parameter approach of Chapter 4 to the computations of $\omega^{(3),a}$, $\omega^{(3),b}$ and $\omega^{(3),c}$. While we did not finalize the calculation, the approach worked and we derived first results. However, it became clear that this method is not suited for the computation of $\omega^{(3),d}$, as master integrals containing the propagator $(k_1+k_2+k_3)^{-2}$ need to be calculated, which could not be done in this approach.

## 6.1 MODIFIED IBPS AT N3LO

In this section we extend the previous discussion of the modified IBP approach in Chapter 5 to the N3LO case. Just like at NNLO, our starting point are the IBP relations which state that any total derivative of a dimensionally regularized integral vanishes. The relation reads

$$\int \mathrm{d}^d k_1 \int \mathrm{d}^d k_2 \int \mathrm{d}^d k_3 \, \frac{\partial}{\partial k_m^\mu} \left[ v^\mu \frac{\prod_i \theta_i}{\prod_i D_i^{\alpha_i}} \right] = 0\,, \qquad v \in \{k_1, k_2, k_3, n, \bar{n}\} \tag{6.1.1}$$

where $m \in \{1,2,3\}$. After rewriting the derivatives in Eq. (6.1.1) in terms of scalar products, we obtain three sets of equations one for each $k_m$

$$\begin{aligned} \frac{\partial}{\partial k_1^\mu} v^\mu = d\delta_{v,k_1} + 2k_1 \cdot v \frac{\partial}{\partial k_1^2} + n \cdot v \frac{\partial}{\partial k_1 \cdot n} + \bar{n} \cdot v \frac{\partial}{\partial k_1 \cdot \bar{n}} \\ + k_2 \cdot v \frac{\partial}{\partial k_1 \cdot k_2} + k_3 \cdot v \frac{\partial}{\partial k_1 \cdot k_3}, \end{aligned} \tag{6.1.2}$$

$$\frac{\partial}{\partial k_2^\mu} v^\mu = d\delta_{v,k_2} + 2k_2 \cdot v \frac{\partial}{\partial k_2^2} + n \cdot v \frac{\partial}{\partial k_2 \cdot n} + \bar{n} \cdot v \frac{\partial}{\partial k_2 \cdot \bar{n}} + k_1 \cdot v \frac{\partial}{\partial k_1 \cdot k_2} + k_3 \cdot v \frac{\partial}{\partial k_3 \cdot k_2}. \tag{6.1.3}$$

$$\frac{\partial}{\partial k_3^\mu} v^\mu = d\delta_{v,k_3} + 2k_3 \cdot v \frac{\partial}{\partial k_3^2} + n \cdot v \frac{\partial}{\partial k_3 \cdot n} + \bar{n} \cdot v \frac{\partial}{\partial k_3 \cdot \bar{n}} + k_1 \cdot v \frac{\partial}{\partial k_1 \cdot k_3} + k_2 \cdot v \frac{\partial}{\partial k_3 \cdot k_2}. \tag{6.1.4}$$

Accounting for all the choices of $v$ in Eq. (6.1.1), we obtain a total of 15 relations per seed integral. Deriving IBP relations for seed integrals, starting with integrals with three Heaviside functions, we again obtain a hierarchical structure of IBP relations

$$0 = \sum_i b_i^{\text{hom}}(\epsilon) \mathcal{T}^{k,\theta\theta\theta}_{\{a_i\}} + \sum_i \sum_k b_{i,k}^{\text{inhom}}(\epsilon) \mathcal{T}^{k,\delta\theta\theta}_{\{a_i\}}, \tag{6.1.5}$$

$$0 = \sum_i c_i^{\text{hom}}(\epsilon) \mathcal{T}^{k,\delta\theta\theta}_{\{a_i\}} + \sum_i \sum_k c_{i,k}^{\text{inhom}}(\epsilon) \mathcal{T}^{k,\delta\delta\theta}_{\{a_i\}}, \tag{6.1.6}$$

$$\vdots$$

$$0 = \sum_i f_i^{\text{hom}}(\epsilon) \mathcal{T}^{k,\delta\delta\delta}_{\{a_i\}}. \tag{6.1.7}$$

As we already noticed, integrals with fewer Heaviside functions and more delta functions are easier to compute. For this reason we will always try to express integrals with Heaviside functions through integrals with delta functions to the extent possible.

This hierarchical structure of IBP relations is depicted in Fig. 6.1. As we move down this graph using IBP relations the number of families increases. We start with just 15 $\theta\theta\theta$ integral families, however by the time we reach the $\delta\delta\delta$ level this number has grown to 105. Since generation of IBP relations is slow because of the need to perform partial fractioning, having to deal with 105 integral families becomes a significant burden. To minimize the reduction time, we implement inter-family relations already at the seed and IBP generation level. The way in which integral families are connected to each other through inter-family relations is shown in Fig. 6.1. Like the other pieces of the IBP code, the inter-family relations are implemented in `Mathematica`.

To further decrease the time needed to produce IBP relations we choose seedlists much more conservatively than what is the case normally. To this end, we exploit the fact that the homogeneous terms in Eqs. (6.1.5) - (6.1.7) can be studied on their own by removing the Heaviside functions. We can thus use `KIRA` to find the master integrals for these homogeneous equations. If we find more master integrals than `KIRA` we know that we have chosen too few seeds and we add more seeds to our list until the number of master integrals agrees with `KIRA`.

We end of this section with a few remarks. First, one might be tempted by the hierarchical structure Eqs. (6.1.5) - (6.1.7) to run a reduction stepwise, i.e. first only reduce all $\theta\theta\theta$ integrals to $\theta\theta\theta$ master integrals and inhomogeneous terms with one delta function. Then inhomogeneous terms with one delta function can again be reduced to master integrals and

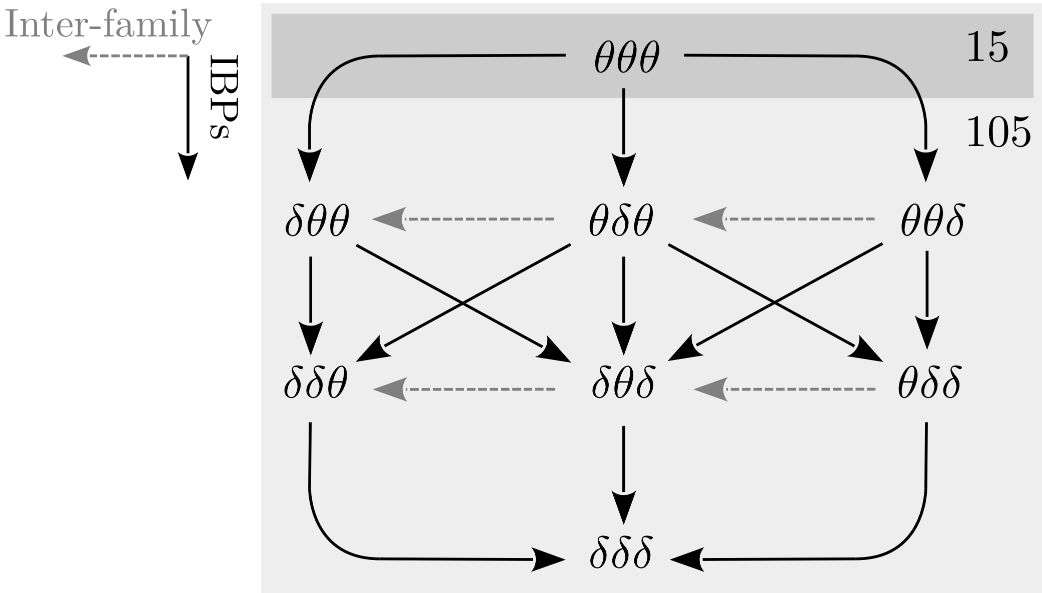


**Figure 6.1:** Hierarchical structure of integral families for the N3LO soft function. IBPs and inter-family relations connect the different families as indicated by arrows. A total of 105 integral families are required in the course of the reduction, while only 15 integral families are required to describe the amplitude with three Heaviside functions.

inhomogeneous terms with two delta functions. The process is then repeated until one reaches integrals with three delta functions. While one indeed can perform a reduction like this, it is actually disadvantageous to do that.

To understand this point consider the reduction of $\delta\delta\delta$ master integrals required for the calculation of the same-hemisphere three-gluon-emission contribution to the zero-jettiness soft function at N3LO. There are five relevant integral families which can be found in Appendix D.2. We determine the master integrals of the top sector for these five families using `KIRA` and obtain six master integrals. We determine the same set of master integrals with our modified IBP setup, by only using seed integrals with three delta functions and find the same six integrals. However, if we repeat this process and also use integrals with one Heaviside function as seed integrals, the number of master integrals reduces to five. The reason for this decrease is the following relation between supposed master integrals

$$\int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\delta}}{k_{123}^2\, k_1\cdot n\, (k_1\cdot n + k_2\cdot n)} = \frac{1}{2}\int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\delta}}{k_{123}^2\, k_1\cdot n\, k_2\cdot n}, \tag{6.1.8}$$

where the phase space $\Phi^{nnn}_{\delta\delta\delta}$ is defined in Appendix D.1. We can understand this relation through a simple calculation. Indeed, upon multiplying the l.h.s. of Eq. (6.1.8) with the partial fraction relation

$$1 = \frac{(k_1\cdot n + k_2\cdot n) - k_1\cdot n}{k_2\cdot n}, \tag{6.1.9}$$

and exploiting $k_1 \leftrightarrow k_2$ symmetry in the second term , we reproduce Eq. (6.1.8). Thus we observe that we find additional relations by working with all integral families at once and we may miss relations between integrals by performing a reduction piece-wise.

Finally, we will need to introduce an analytic regulator in Section 6.4 since not all occuring integrals are regularized dimensionally. We introduce this regulator by multiplying every integral by a factor $(k_1 \cdot n)^\nu (k_2 \cdot n)^\nu (k_3 \cdot n)^\nu$. Since an additional partial fractioning step may be necessary when a derivative acts on this extra term, it can simply be treated as an additional inhomogeneous term in our setup. In this case the IBP relations in Eqs. (6.1.5) - (6.1.7) read

$$0 = \sum_i b_i^{\text{hom}}(\epsilon) \mathcal{T}^{k,\theta\theta\theta}_{\{a_i\}} + \sum_i \sum_k b_{i,k}^{\text{inhom}}(\epsilon) \mathcal{T}^{k,\delta\theta\theta}_{\{a_i\}} + \nu \sum_i \sum_k b_{i,k}^{\text{inhom},\nu}(\epsilon) \mathcal{T}^{k,\theta\theta\theta}_{\{a_i\}} \,, \tag{6.1.10}$$

$$0 = \sum_i c_i^{\text{hom}}(\epsilon) \mathcal{T}^{k,\delta\theta\theta}_{\{a_i\}} + \sum_i \sum_k c_{i,k}^{\text{inhom}}(\epsilon) \mathcal{T}^{k,\delta\delta\theta}_{\{a_i\}} + \nu \sum_i \sum_k c_{i,k}^{\text{inhom},\nu}(\epsilon) \mathcal{T}^{k,\delta\theta\theta}_{\{a_i\}} \,, \tag{6.1.11}$$

$$\vdots$$

$$0 = \sum_i f_i^{\text{hom}}(\epsilon) \mathcal{T}^{k,\delta\delta\delta}_{\{a_i\}} + \nu \sum_i f_i^{\text{inhom},\nu}(\epsilon) \mathcal{T}^{k,\delta\delta\delta}_{\{a_i\}} \,. \tag{6.1.12}$$

We notate that the additional terms in Eqs. (6.1.10) - (6.1.12) increase the number of integral families from 105 to 109 and in general make the computation more complex because of the new parameter $\nu$.

## 6.2 GENERAL SETUP

In Chapter 4 we discussed how the soft function can be obtained by integrating eikonal QCD functions over a suitable phase space. We found that the bare soft function $S$ is related to the eikonal function $\xi_{\{m\}}$ that describes the soft emission of $m$ partons (cf. Eqs. (4.1.1) - (4.1.3)).

We now specialize to the N3LO case, setting $m = 3$ in Eq. (4.1.3). To this end we again introduce Heaviside functions to enable unambiguous calculation of the minimum functions. We obtain

$$M_3 = M_3^{nnn} + M_3^{nn\bar{n}} + M_3^{n\bar{n}n} + M_3^{\bar{n}nn} + [n \leftrightarrow \bar{n}] \;, \tag{6.2.1}$$

where we defined

$$M_3^{ijk} = \delta(\tau - i \cdot k_1 - j \cdot k_2 - k \cdot k_3)\, \theta\big(\bar{i} \cdot k_1 - i \cdot k_1\big)\, \theta\big(\bar{j} \cdot k_2 - j \cdot k_2\big)\, \theta\big(\bar{k} \cdot k_3 - k \cdot k_3\big) \,, \tag{6.2.2}$$

and we set $\bar{\bar{n}} = n$. Exploiting the invariance of the eikonal function under exchange of $n$ and $\bar{n}$, we again only need to consider two configurations, which we refer to as $A$ and $B$. Hence we write

$$M_3 = 2\, M_A + 6\, M_B, \tag{6.2.3}$$

$$M_A = \delta(\tau - n \cdot k_1 - n \cdot k_2 - n \cdot k_3)\, \theta(\bar{n} \cdot k_1 - n \cdot k_1)\, \theta(\bar{n} \cdot k_2 - n \cdot k_2) \times \theta(\bar{n} \cdot k_3 - n \cdot k_3) \,, \tag{6.2.4}$$

$$M_B = \delta(\tau - n\cdot k_1 - n\cdot k_2 - \bar{n}\cdot k_3)\,\theta(\bar{n}\cdot k_1 - n\cdot k_1)\,\theta(\bar{n}\cdot k_2 - n\cdot k_2) \times \theta(n\cdot k_3 - \bar{n}\cdot k_3)\,. \tag{6.2.5}$$

Again, configuration $A$ describes the case where all partons are emitted into the same hemisphere. Configuration $B$ describes the case where two partons are emitted into one hemisphere and the third parton into the opposite hemisphere. The two different configurations are illustrated in Fig. 6.2.

Similar to the NNLO case the different contributions to the N3LO soft function can be split according to partonic final states. We find

$$S^{(3)} = S^{(3)}_g + S^{(3)}_{gg} + S^{(3)}_{q\bar{q}} + S^{(3)}_{ggg} + S^{(3)}_{gq\bar{q}}, \tag{6.2.6}$$

where $S^{(3)}_g$ is the RVV contribution, $S^{(3)}_{q\bar{q}}$ and $S^{(3)}_{gg}$ are RRV contributions. While all of these pieces need to be calculated eventually, we focus on the most difficult piece, $S^{(3)}_{ggg}$, in this thesis. The different contributions to $S^{(3)}_{ggg}$ can again be split into configuration $A$ and $B$

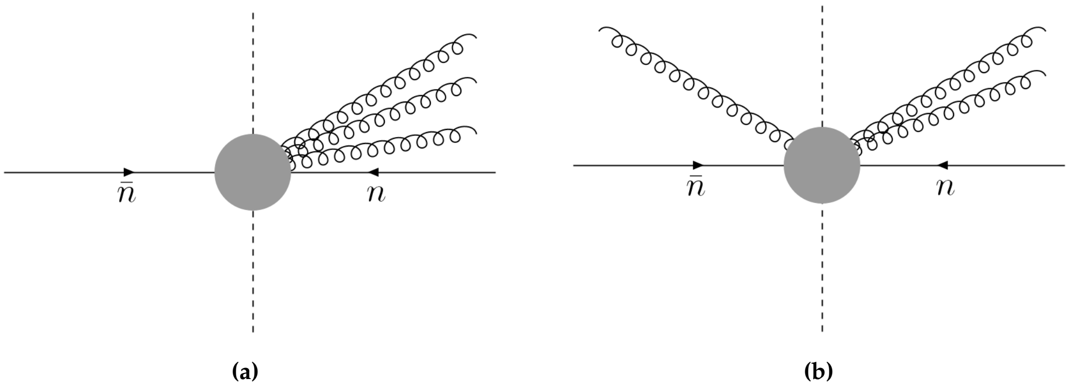


**Figure 6.2:** The two different configurations $A$ and $B$ required for the calculation of the triple gluon emission soft function. The configuration $A$ where all gluons are emitted into the same hemisphere is shown in Fig. 6.2a, while the configuration $B$ where one gluon is emitted into the other hemisphere is shown in Fig. 6.2b.

$$\begin{aligned} S^{(3)}_{ggg} &= \frac{1}{3!}\int \mathrm{dPS}^{(3)}_S\; M_3\, \xi_{ggg} \\ &= 2\frac{1}{3!}\int \mathrm{dPS}^{(3)}_S\; M_A\, \xi_{ggg} + 6\frac{1}{3!}\int \mathrm{dPS}^{(3)}_S\; M_B\, \xi_{ggg} \\ &= 2\, S^{nnn}_{ggg} + 6\, S^{nn\bar{n}}_{ggg}, \end{aligned} \tag{6.2.7}$$

where the eikonal function $\xi_{ggg}$ can be extracted from Eq. (7.10) of Ref. [74]. We focus on the most singular part of Eq. (6.2.7), the same hemisphere contribution $S^{nnn}_{ggg}$. To facilitate the computation, we further split the eikonal function according to the number of correlated gluon emissions

$$
\begin{aligned}
(3!)\; S_{ggg}^{nnn} &= C_a^3 \int \mathrm{dPS}_S^{(3)}\; M_A\; \omega_{n\bar{n}}^{(1)}(k_1)\; \omega_{n\bar{n}}^{(1)}(k_2)\; \omega_{n\bar{n}}^{(1)}(k_3) \\
&\quad + C_a^2 C_A \int \mathrm{dPS}_S^{(3)}\; M_A\; \left[\omega_{n\bar{n}}^{(1)}(k_1)\omega_{n\bar{n}}^{(2)}(k_2,k_3) + (1 \leftrightarrow 2) + (1 \leftrightarrow 3)\right] \\
&\quad + C_a C_A^2 \int \mathrm{dPS}_S^{(3)}\; M_A\; \omega_{n\bar{n}}^{(3)}(k_1,k_2,k_3). \\
&= C_a^3\, S_{ggg}^{nnn,1} + C_a^2 C_A\, S_{ggg}^{nnn,2} + C_a C_A^2\, S_{ggg}^{nnn,3},
\end{aligned}
\tag{6.2.8}
$$

where $C_a = C_F(C_A)$ if the incoming particles are quarks (gluons), respectively. The functions $S_{ggg}^{nnn,1}$ and $S_{ggg}^{nnn,2}$ are easy to compute as they just present convolutions of the NLO and NNLO results. For this reason, we move their computation to Appendix D.5 and focus on the maximal non-abelian contribution $S_{ggg}^{nnn,3}$. Following Ref. [74] we split the integrand one last time. The function $\omega_{n\bar{n}}^{(3)}(k_1,k_2,k_3)$ in Eq. (6.2.8) reads

$$
\omega_{n\bar{n}}^{(3)}(k_1,k_2,k_3) = \sum_{t\in\{a,b,c,d\}} \omega_{n\bar{n}}^{(3),t}(k_1,k_2,k_3)\,,
\tag{6.2.9}
$$

$$
\begin{aligned}
\omega_{n\bar{n}}^{(3),t} = &\left[\overline{\mathcal{S}}_{n\bar{n}}^{(t)}(k_1,k_2,k_3) + \overline{\mathcal{S}}_{\bar{n}n}^{(t)}(k_1,k_2,k_3) - \overline{\mathcal{S}}_{nn}^{(t)}(k_1,k_2,k_3) - \overline{\mathcal{S}}_{\bar{n}\bar{n}}^{(t)}(k_1,k_2,k_3)\right] \\
&+ \text{permutations}\{k_1,k_2,k_3\}\,,
\end{aligned}
\tag{6.2.10}
$$

where "permutations$\{k_1,k_2,k_3\}$" describes all possible permutations of the gluon momenta $k_i$. The four terms $\overline{\mathcal{S}}_{ik}^{(t)}$, $t = a,b,c,d$ in Eq. (6.2.10) are contributions to the soft eikonal function that are ordered according to the structure of their collinear singularities [74]. This has direct consequences for the complexity of the respective calculations. Indeed, $\omega_{n\bar{n}}^{(3),a}$ does not involve any propagator of the form $1/k_i \cdot k_j$, while $\omega_{n\bar{n}}^{(3),b}$ already involves one and $\omega_{n\bar{n}}^{(3),c}$ two of these propagators. Finally, $\omega_{n\bar{n}}^{(3),d}$ is the most difficult contribution to compute as it involves the propagators $1/(k_1+k_2+k_3)^2$. Example diagrams for all four contributions are shown in Fig. 6.3. We calculate these contributions starting with the easiest and proceeding to hardest.

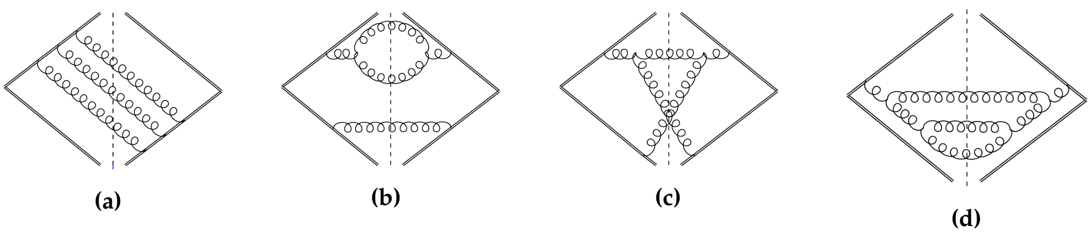

**Figure 6.3:** Example diagrams for contributions $\omega^{(3),a}$ through $\omega^{(3),d}$ to the N3LO soft function. Straight lines denote eikonal lines, the dashed line represents a "cut" so that all particles crossing it are on the mass-shell.

## 6.3 THE $\omega^{(3),a}$ CONTRIBUTION

In this section we consider the simplest contribution to the N3LO zero-jettiness soft function, the integral over $\omega_{n\bar{n}}^{(3),a}$. According to Eq. (6.2.10), this function is constructed from the function $\overline{\mathcal{S}}_{ik}^{(a)}$, which reads [74][1]

$$\begin{aligned}
\overline{\mathcal{S}}_{ik}^{(a)}(q_1,q_2,q_3) &= \frac{31}{144}\frac{(n\bar{n})^3}{nk_1\,\bar{n}k_1\,nk_2\,\bar{n}k_2\,nk_3\,\bar{n}k_3} \\
&+ \frac{(n\bar{n})^3}{32\,\bar{n}k_{12}\,\bar{n}k_3\,nk_1\,nk_3}\left(\frac{1}{\bar{n}k_1}-\frac{1}{\bar{n}k_2}\right)\left(\frac{6}{nk_{12}}+\frac{n(k_3-k_1)}{nk_{13}\,nk_2}\right) \\
&+ \frac{(n\bar{n})^3}{288\,\bar{n}k_{123}\,nk_{123}}\left(\frac{1}{\bar{n}k_1}-\frac{1}{\bar{n}k_2}\right)\times\left\{\frac{2}{nk_1}\left(\frac{1}{nk_3}-\frac{3}{nk_{12}}\right)\left(\frac{1}{\bar{n}k_3}-\frac{3}{\bar{n}k_{12}}\right)\right. \\
&\left.+\left(\frac{1}{nk_2}-\frac{3}{nk_{13}}\right)\left(\frac{1}{nk_1}-\frac{1}{nk_3}\right)\left(\frac{1}{\bar{n}k_3}-\frac{3}{\bar{n}k_{12}}\right)\right],
\end{aligned} \tag{6.3.1}$$

where we used $k_{12} = k_1 + k_2$ and $k_{123} = k_1 + k_2 + k_3$. It is easy to check that, $\overline{\mathcal{S}}_{ik}^{(a)}$ is not singular when any of the two gluons become collinear to each other. As already mentioned, this feature reduces the number of scalar products that appear in the denominators of that function, making integration over three-gluon phase space simpler.

Applying integration-by-parts identities, we find that the integral of $\omega_{n\bar{n}}^{(3),a}$ over the phase space of configuration $A$ can be expressed through six master integrals. The result of the reduction reads

$$\begin{aligned}
&\int \mathrm{d}\Phi_{\theta\theta\theta}^{nnn}\,\omega_{n\bar{n}}^{(3),a}(k_1,k_2,k_3) \\
&= \frac{1}{(1-4\epsilon)(1-5\epsilon)}\left\{\left[\frac{182}{5\epsilon^5}-\frac{3392}{5\epsilon^4}+\frac{23268}{5\epsilon^3}-\frac{69432}{5\epsilon^2}+\frac{75728}{5\epsilon}\right]I_1\right. \\
&+\left[\frac{8}{5\epsilon^3}-\frac{72}{5\epsilon^2}+\frac{32}{\epsilon}\right]I_2+\left[-\frac{112}{5\epsilon^4}+\frac{2016}{5\epsilon^3}-\frac{13328}{5\epsilon^2}+\frac{38304}{5\epsilon}-8064\right]I_3 \\
&+\left[-\frac{4}{\epsilon^4}+\frac{356}{5\epsilon^3}-\frac{2248}{5\epsilon^2}+\frac{5968}{5\epsilon}-\frac{5664}{5}\right]I_4+\left[-\frac{8}{5\epsilon^3}+\frac{104}{5\epsilon^2}-\frac{448}{5\epsilon}+128\right]I_5 \\
&\left.+\left[-\frac{36}{5\epsilon^4}+\frac{648}{5\epsilon^3}-\frac{4428}{5\epsilon^2}+2592\epsilon+\frac{14328}{5\epsilon}-\frac{22032}{5}\right]I_6\right\},
\end{aligned} \tag{6.3.2}$$

where the phase space $\Phi_{\theta\theta\theta}^{nnn}$ is defined in Appendix D.1. Calculation of these master integrals is straightforward and we only discuss the computation of $I_1$ and $I_5$ in detail. The definition of the remaining master integrals can be found in Appendix D.3. Note, that none of the master integrals contains the full set of three Heaviside functions. We explained why this happens in detail when discussing the NNLO soft function in Chapter 5. We now calculate the integral $I_1$ explicitly. It reads

$$\begin{aligned}
I_1 = \int \mathrm{d}\Phi_{\delta\delta\delta}^{nnn} = \int [\mathrm{d}k_1][\mathrm{d}k_2][\mathrm{d}k_3]\,\delta\left(1-k_1\cdot n-k_2\cdot n\right) \\
\times\,\delta\left(k_1\cdot\bar{n}-k_1\cdot n\right)\delta\left(k_2\cdot\bar{n}-k_2\cdot n\right)\delta\left(k_3\cdot\bar{n}-k_3\cdot n\right).
\end{aligned} \tag{6.3.3}$$

1 We note that we took the emitters to be massless.

We use the parametrization

$$[\mathrm{d}k_i] = \frac{\mathrm{d}\Omega_i^{(d-2)}}{4(2\pi)^{d-1}}\,\mathrm{d}\alpha_i\,\mathrm{d}\beta_i\,(\alpha_i\beta_i)^{-\epsilon}, \qquad \alpha_i\,,\beta_i \in [0,\infty)\,, \tag{6.3.4}$$

and directly integrate over all three angels $\Omega_i^{(d-2)}$. We then remove all $\delta(\alpha_i - \beta_i)$-functions by integrating over $\alpha_i$, $i = 1,2,3$, and find

$$I_1 = [N]^3 \int \prod_{i=1}^{3} \mathrm{d}\beta_i \beta_i^{-2\epsilon}\;\; \delta(1-\beta_{123}) = [N]^3\,\frac{\Gamma^3(1-2\epsilon)}{\Gamma(3-6\epsilon)}. \tag{6.3.5}$$

For a less trivial example consider the integral $I_5$, it reads

$$I_5 = \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\theta}}{(k_{13}\cdot n)(k_{123}\cdot \bar{n})}. \tag{6.3.6}$$

We again use the parameterization in Eq. (6.3.4) and integrate over $\alpha_1$ and $\alpha_2$ removing two delta functions. Finally, we integrate over $\beta_1$ by removing $\delta(1-\beta_{123})$ and change variables as $\beta_2 = (1-\beta_3)\xi$ and $\alpha_3 = \beta_3/\rho$. We obtain

$$\begin{aligned} I_5 &= [N]^3 \int_0^1 \mathrm{d}\beta_3 \beta_3^{1-2\epsilon}(1-\beta_3)^{1-4\epsilon} \int_0^1 \frac{\mathrm{d}\rho\;\rho^{-1+\epsilon}}{\beta_3 + (1-\beta_3)\rho} \int_0^1 \frac{\mathrm{d}\xi\;\xi^{-2\epsilon}(1-\xi)^{-2\epsilon}}{1-(1-\beta_3)\xi}, \\ &= [N]^3\,\frac{\Gamma^2(1-2\epsilon)}{\epsilon\Gamma(2-4\epsilon)} \int_0^1 \mathrm{d}u\;(1-u)^{1-2\epsilon} u^{1-4\epsilon}\,{}_2F_1\left(1, 1-2\epsilon, 2-4\epsilon, u\right) \\ &\quad \times {}_2F_1\left(1,1,1+\epsilon,u\right), \end{aligned} \tag{6.3.7}$$

where we used the transformation $u = 1-\beta_3$ in the last line. Although both hypergeometric functions in Eq. (6.3.7) are divergent at $u = 1$, the whole expression is integrable due to the explicit factor $(1-u)^{1-2\epsilon}$ in the integrand. Therefore we directly expand the integrand in Eq. (6.3.7) and integrate using `HyperInt` [68]. We obtain

$$\begin{aligned} I_5 = [N]^3 \Bigg\{ &\frac{1}{\epsilon} + 8 + \epsilon\left(2\zeta_3 - \frac{7\pi^2}{6} + 46\right) + \epsilon^2\left(-26\zeta_3 + \frac{83\pi^4}{360} - \frac{23\pi^2}{3} + 216\right) \\ &+ \epsilon^3\left(-\frac{1}{6}89\pi^2\zeta_3 - 180\zeta_3 + \frac{585\zeta_5}{2} + \frac{17\pi^4}{180} - \frac{86\pi^2}{3} + 776\right) \\ &+ \epsilon^4\left(-271\zeta_3^2 - \frac{89\pi^2\zeta_3}{3} - 536\zeta_3 + 729\zeta_5 + \frac{7739\pi^6}{7560} + \frac{4\pi^2}{3} - \frac{149\pi^4}{90} + 1024\right) \\ &+ \mathcal{O}(\epsilon^5)\Bigg\}, \end{aligned} \tag{6.3.8}$$

where we have discarded contributions of transcendental weight seven and higher.
The remaining integrals in Eq. (6.3.3) can be obtained along similar lines. Results for all master

integrals can be found in the ancillary file of Ref. [77]. Inserting these results into Eq. (6.3.3) we find the final result for $\omega_{n\bar{n}}^{(3),a}$ contribution. It reads

$$\int \mathrm{d}\Phi_{\theta\theta\theta}^{nnn}\, \omega_{n\bar{n}}^{(3),a}(k_1,k_2,k_3) = [N]^3 \left\{ \frac{8}{\epsilon^5} - \frac{20\pi^2}{\epsilon^3} - \frac{584\zeta_3}{\epsilon^2} - \frac{86\pi^4}{15\epsilon} + (1472\pi^2\zeta_3 - 12480\zeta_5) + \epsilon\left(22288\zeta_3^2 - \frac{3796\pi^6}{945}\right)\right\}. \tag{6.3.9}$$

We compute the next-to-easiest contribution, $\omega_{n\bar{n}}^{(3),b}$ in the next section.

## 6.4 THE $\omega^{(3),b}$ CONTRIBUTION

In this section we consider the next-to easiest contribution to the zero-jettiness soft function, the integral of the function $\omega_{n\bar{n}}^{(3),b}$. This function is constructed from the function $\overline{\mathcal{S}}_{ik}^{(b)}$, which reads [74]

$$\begin{aligned}
\overline{\mathcal{S}}_{ik}^{(b)} &= \frac{1}{16\, k_1k_2\, nk_{12}} \left\{ \frac{7\, n\bar{n}}{nk_1\, nk_3\, \bar{n}k_2\, \bar{n}k_3} (-n\bar{n}\, nk_{12}) \right. \\
&+ \frac{(n\bar{n})^2}{nk_3} \left[ \frac{12\, n(k_1-k_2)}{\bar{n}k_{12}\, nk_1\, \bar{n}k_3} + \frac{1}{\bar{n}k_{13}} \left( \frac{1}{\bar{n}k_3} - \frac{1}{\bar{n}k_1} \right) \left( 3 + \frac{nk_1}{nk_2} - \frac{2\,\bar{n}k_1}{\bar{n}k_2} \right) \right] \Bigg] \\
&+ \frac{1}{48\, k_1k_2\, \bar{n}k_{123}} \left\{ \frac{3}{\bar{n}k_1\, nk_2} \left( \frac{1}{\bar{n}k_{12}} - \frac{1}{\bar{n}k_3} \right) \left( \frac{n\bar{n}}{nk_3} \right) (n\bar{n}\, \bar{n}k_{12}) \right. \\
&+ 3 \left( \frac{1}{\bar{n}k_3} - \frac{1}{\bar{n}k_{12}} \right) (n\bar{n}\, \bar{n}k_{12}) \left[ \frac{n\bar{n}}{nk_{23}\, \bar{n}k_1} \left( \frac{1}{nk_2} - \frac{1}{nk_3} \right) \right] \Bigg] \\
&+ \frac{n\bar{n}}{48\, k_1k_2\, \bar{n}k_{123}\, nk_{123}} \left( \frac{1}{nk_{12}} - \frac{1}{nk_3} \right) \left( 2\, n\bar{n}\, \bar{n}(k_2-k_1) \right) \\
&\times \left[ \left( \frac{1}{\bar{n}k_3} - \frac{3}{\bar{n}k_{12}} \right) \left( \frac{1}{\bar{n}k_1} - \frac{1}{\bar{n}k_2} \right) + \left( \frac{1}{\bar{n}k_2} - \frac{3}{\bar{n}k_{13}} \right) \left( \frac{1}{\bar{n}k_1} - \frac{1}{\bar{n}k_3} \right) \right],
\end{aligned} \tag{6.4.1}$$

where we used $k_{12} = k_1 + k_2$ and $k_{123} = k_1 + k_2 + k_3$. Unlike $\overline{\mathcal{S}}_{ik}^{(a)}$, the function $\overline{\mathcal{S}}_{ik}^{(b)}$ contains the scalar product of two gluon momenta $k_1 \cdot k_2$ which results in a collinear singularity in the limit $\boldsymbol{k}_1 \parallel \boldsymbol{k}_2$. As already discussed, this scalar product makes it more difficult to calculate the integral of $\omega_{n\bar{n}}^{(3),b}$ compared to the $\omega_{n\bar{n}}^{(3),a}$ case.

The remainder of this chapter is organized as follows. In Section 6.4.1 we discuss the reduction of the integral $\omega_{n\bar{n}}^{(3),b}$ to master integrals. It turns out that we have to introduce an analytic regulator $\nu$ since not all integrals that appear in the IBP relations are regularized dimensionally. We proceed to calculate an example master integral that is divergent in Section 6.4.2. In doing so, we show that the $1/\nu$ pole arises because of an unregularized collinear divergence. Finally, we conclude and give a result for $\omega_{n\bar{n}}^{(3),b}$ in Section 6.4.3.

### 6.4.1 *Reduction*

To compute $\omega_{n\bar{n}}^{(3),b}$, we proceed by applying modified IBPs as discussed in Section 6.1 and express $\omega_{n\bar{n}}^{(3),b}$ through master integrals. The result reads

$$
\begin{aligned}
&\int \mathrm{d}\Phi_{\theta\theta\theta}^{nnn}\, \omega_{n\bar{n}}^{(3),b}(k_1,k_2,k_3) = \\
&\frac{\left(10249456\epsilon^5 - 6479980\epsilon^4 + 713856\epsilon^3 + 268429\epsilon^2 - 67966\epsilon + 4287\right)}{200\epsilon^5(2\epsilon-1)(4\epsilon-1)(5\epsilon-1)} I_1 \\
&+ \frac{7}{5\epsilon^3} I_2 - \frac{38\left(504\epsilon^3 - 270\epsilon^2 + 37\epsilon - 1\right)}{5\epsilon^4(2\epsilon-1)} I_3 \\
&- \frac{\left(4260192\epsilon^5 - 3531008\epsilon^4 + 674380\epsilon^3 + 124140\epsilon^2 - 49897\epsilon + 3923\right)}{50\epsilon^4\left(40\epsilon^3 - 38\epsilon^2 + 11\epsilon - 1\right)} I_4 \\
&+ \frac{\left(704\epsilon^2 - 280\epsilon + 26\right)}{5\epsilon^3 - 10\epsilon^4} I_5 + \frac{9\left(9288\epsilon^4 - 7308\epsilon^3 + 858\epsilon^2 + 347\epsilon - 55\right)}{5\epsilon^4(4\epsilon-1)} I_6 - \frac{237}{4\epsilon^2} I_7 \\
&- \frac{6(4\epsilon+1)}{\epsilon^3} I_8 + \frac{10}{3\epsilon^2} I_9 - \frac{10}{\epsilon^2} I_{10} + \frac{6}{\epsilon^2} I_{11} + \frac{6}{5\epsilon^2} I_{12} - \frac{6}{5\epsilon^2} I_{13} + \frac{18}{\epsilon^2} I_{14} - \frac{9}{5\epsilon^2} I_{15} \\
&+ \frac{(14 - 84\epsilon)}{5\epsilon^2} I_{16} + \frac{12}{5\epsilon} I_{17} + \frac{93}{5\epsilon} I_{18} - \frac{12}{\epsilon} I_{19} - \frac{39}{\epsilon} I_{20} + \frac{22(4\epsilon-1)}{\epsilon^2} I_{21} \\
&- \frac{7}{\epsilon} I_{22}\,,
\end{aligned}
\tag{6.4.2}
$$

where the integrals $I_{1,\ldots,22}$ can be found in Appendix D.3.

Although Eq. (6.4.2) looks totally normal, it is actually wrong. We have discovered this issue while numerically checking Eq. (6.4.2). The reason for this equation being wrong is subtle; it is related to the fact that *not all integrals that appear in the integration-by-parts identities in the course of the reduction are regulated dimensionally*. However, we emphasize that *before* and *after the reduction* all integrals are dimensionally regularized, making this problem hard to detect.

In Section 6.4.2 we will show that integrals in IBP relations become unregularized due to a collinear divergence and that this divergence can be regularized by changing the phase-space measure $\Phi_{fgh}^{nnn} \to \Phi_{fgh}^{nnn}(k_1 n)^\nu (k_2 n)^\nu (k_3 n)^\nu$.

We note that care needs to be taken when choosing a basis while working with a non-vanishing analytic regulator. Indeed, the automatically chosen basis (preferring dots over scalar products) leads to reduction coefficients of the form $1/\nu^n$ where $n \in \mathbb{N}$ and thus complicating the eventual $\nu \to 0$ limit that needs to be taken. A more convenient basis can be found by first performing a wrong reduction with $\nu = 0$, identifying the wrong master integrals, and setting these integrals as a preferred basis for a correct $\nu \neq 0$ reduction. The choice of the reduction basis is best illustrated in the computation of $\omega_{n\bar{n}}^{(3),c}$, and we present a direct comparison of the bases in Section 6.5.

Proceeding as explained above we obtain the following relation

$$
\int \mathrm{d}\Phi_{\theta\theta\theta}^{nnn}\, \omega_{n\bar{n}}^{(3),b}(k_1,k_2,k_3) =
$$

$$
\begin{aligned}
&\frac{\left(10249456\epsilon^5-6479980\epsilon^4+713856\epsilon^3+268429\epsilon^2-67966\epsilon+4287\right)}{200\epsilon^5(2\epsilon-1)(4\epsilon-1)(5\epsilon-1)}I_1\\
&+\frac{7}{5\epsilon^3}I_2-\frac{38\left(504\epsilon^3-270\epsilon^2+37\epsilon-1\right)}{5\epsilon^4(2\epsilon-1)}I_3\\
&-\frac{\left(4260192\epsilon^5-3531008\epsilon^4+674380\epsilon^3+124140\epsilon^2-49897\epsilon+3923\right)}{50\epsilon^4\left(40\epsilon^3-38\epsilon^2+11\epsilon-1\right)}I_4\\
&+\frac{\left(704\epsilon^2-280\epsilon+26\right)}{5\epsilon^3-10\epsilon^4}I_5+\frac{9\left(9288\epsilon^4-7308\epsilon^3+858\epsilon^2+347\epsilon-55\right)}{5\epsilon^4(4\epsilon-1)}I_6-\frac{237}{4\epsilon^2}I_7\\
&-\frac{6(4\epsilon+1)}{\epsilon^3}I_8+\frac{10}{3\epsilon^2}I_9-\frac{10}{\epsilon^2}I_{10}+\frac{6}{\epsilon^2}I_{11}+\frac{6}{5\epsilon^2}I_{12}-\frac{6}{5\epsilon^2}I_{13}+\frac{18}{\epsilon^2}I_{14}-\frac{9}{5\epsilon^2}I_{15}\\
&+\frac{(14-84\epsilon)}{5\epsilon^2}I_{16}+\frac{12}{5\epsilon}I_{17}+\frac{93}{5\epsilon}I_{18}-\frac{12}{\epsilon}I_{19}-\frac{39}{\epsilon}I_{20}+\frac{22(4\epsilon-1)}{\epsilon^2}I_{21}\\
&-\frac{7}{\epsilon}I_{22}-\frac{4}{3\epsilon^2}\left[\lim_{\nu\to0}\nu J_\nu(\epsilon,\nu)\right].
\end{aligned}
\tag{6.4.3}
$$

We note that Eq. (6.4.3) is identical to the original $\nu = 0$ reduction of Eq. (6.4.2) except for a new integral $J_\nu(\epsilon,\nu)$. This new master integral exhibits a $1/\nu$ pole and after extracting this pole the limit in Eq. (6.4.3) can be taken in a straightforward way.
The remaining integrals appearing in this equation can be calculated following the discussion in Section 6.3. Definitions for all integrals are again given in Appendix D.3.
In the next subsection we extract the $1/\nu$ pole from $J_\nu(\epsilon,\nu)$ and discuss the origin of the corresponding divergence.

### 6.4.2 *Calculation of master integrals*

To understand how dimensional regularization may fail to regularize an integral and how to fix this problem, consider the master integral

$$
J_\nu(\epsilon,\nu) = \int \frac{d\Phi^{nnn}_{\theta\delta\theta}\,(k_1n)^\nu(k_2n)^\nu(k_3n)^\nu}{(k_1k_3)(k_1n)(k_{12}\bar{n})(k_3\bar{n})}. \tag{6.4.4}
$$

We will now explicitly show that the factor $\nu$ in Eq. (6.4.4) serves as a regulator of a divergence which is not regularized dimensionally. To this end, we replace both Heaviside functions in Eq. (6.4.4) by integrals of delta functions

$$
\theta(\alpha_i-\beta_i) = \int_0^1 dz_i\,\delta\left(z_i\,\alpha_i-\beta_i\right)\,\alpha_i, \tag{6.4.5}
$$

and proceed by splitting the $z_i$ integration regions into two pieces $z_1 > z_3$ and $z_3 > z_1$. We call the two contributions $J_{1,\nu}(\epsilon,\nu)$ and $J_{2,\nu}(\epsilon,\nu)$ respectively.

For $z_1 > z_3$, we find

$$
\int \frac{d\Omega_1^{(d-2)}d\Omega_3^{(d-2)}}{2k_1\cdot k_3}\bigg|_{\alpha_i\to\frac{\beta_i}{z_i}} = \frac{\left[\Omega^{(d-2)}\right]^2 z_3}{\beta_1\beta_3}\,{}_2F_1\left(1,1+\epsilon,1-\epsilon,\frac{z_3}{z_1}\right). \tag{6.4.6}
$$

We use Eq. (6.4.6) together with

$$[\mathrm{d}k_i] = \frac{\mathrm{d}\Omega_i^{(d-2)}}{4(2\pi)^{d-1}}\,\mathrm{d}\alpha_i\,\mathrm{d}\beta_i\,(\alpha_i\beta_i)^{-\epsilon}, \qquad \alpha_i\,,\beta_i \in [0,\infty)\,, \tag{6.4.7}$$

and obtain

$$\begin{aligned} J_{1,\nu}(\epsilon,\nu) = {}& [N]^3 \int \mathrm{d}z_1 \int \mathrm{d}z_3 \prod_i \mathrm{d}\alpha_i \mathrm{d}\beta_i \frac{2z_3\alpha_1^{1-\epsilon}\alpha_3^{-\epsilon}\beta_1^{-\epsilon-2+\nu}\beta_2^{-2\epsilon+\nu}\beta_3^{-1-\epsilon+\nu}}{\alpha_1+\beta_2} \\ & \times \delta(\alpha_1 z_1 - \beta_1)\delta(\alpha_3 z_3 - \beta_3)\delta(1-\beta_{123})\, {}_2F_1\left(1,1+\epsilon,1-\epsilon,\frac{z_3}{z_1}\right). \end{aligned} \tag{6.4.8}$$

We remove delta functions by integrating $\alpha_1$, $\alpha_3$ and $\beta_1$ and change variables to $x$ and $y$ such that $b_3 = x\,y$ and $b_2 = x\,(1-y)$. We obtain

$$\begin{aligned} J_{1,\nu}(\epsilon,\nu) = {}& 2\,[N]^3 \int \mathrm{d}x\mathrm{d}y\, \frac{(1-x)^{-1-2\epsilon+\nu}\, x^{-4\epsilon+2\nu}\,(1-y)^{-2\epsilon+\nu}\, y^{-1-2\epsilon+\nu}}{x(1-y)+\frac{1-x}{z_1}} \\ & \times \int \mathrm{d}z_1\,\mathrm{d}z_3\, z_1^{\epsilon-2}\, z_3^{\epsilon}\, {}_2F_1\left(1,1+\epsilon,1-\epsilon,\frac{z_3}{z_1}\right). \end{aligned} \tag{6.4.9}$$

We next change variables $z_3 = tz_1$ and integrate over $x$. The expression reads

$$\begin{aligned} J_{1,\nu}(\epsilon,\nu) = {}& 2\,[N]^3 \int \mathrm{d}y\mathrm{d}z_1\,\mathrm{d}t\, t^{\epsilon}\, z_1^{2\epsilon}\, {}_2F_1(1,1+\epsilon,1-\epsilon,t)(1-y)^{\nu-2\epsilon} y^{-1-2\epsilon+\nu} \\ & \times \frac{\Gamma(\nu-2\epsilon)\Gamma(1-4\epsilon+2\nu)}{\Gamma(1-6\epsilon+3\nu)} \\ & \times\, {}_2F_1(1,1-4\epsilon+2\nu,1-6\epsilon+3\nu,1-(1-y)z_1). \end{aligned} \tag{6.4.10}$$

To proceed further, we write the last hypergeometric function in Eq. (6.4.10) as a Mellin-Barnes integral

$$\begin{aligned} & {}_2F_1(1,1-4\epsilon+2\nu;1-6\epsilon+3\nu;1-(1-y)z_1) \\ & = \frac{\Gamma(1-6\epsilon+3\nu)}{\Gamma(\nu-2\epsilon)\Gamma(1-4\epsilon+2\nu)\Gamma(3\nu-6\epsilon)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \mathrm{d}z_M ((1-y)z_1)^{-z_M}\Gamma(1-z_M) \\ & \times \Gamma(z_M)\Gamma(1-6\epsilon+3\nu)\Gamma(1-4\epsilon+2\nu-z_M)\Gamma(-1-2\epsilon+\nu+z_M). \end{aligned} \tag{6.4.11}$$

Using Eq. (6.4.11) in Eq. (6.4.10) and integrating over $y$ and $z_M$ we obtain

$$\begin{aligned} J_{1,\nu}(\epsilon,\nu) = {}& 2\,[N]^3 \int \mathrm{d}z_1 \mathrm{d}t\, t^{\epsilon}\, {}_2F_1(1,1+\epsilon;1-\epsilon;t) \frac{\Gamma(\nu-2\epsilon)}{\Gamma(3\nu-6\epsilon)} \\ & \times z_1^{2\epsilon} \frac{\Gamma(\nu-2\epsilon)\Gamma(1-2\epsilon+\nu)\Gamma(2\nu-4\epsilon)}{\Gamma(1-4\epsilon+2\nu)} \\ & \times {}_2F_1(1,1-2\epsilon+\nu,1-4\epsilon+2\nu,1-z_1), \end{aligned} \tag{6.4.12}$$

Finally, we re-write the hypergeometric function to isolate the $1/\nu$ pole by using the following identity

$$\begin{aligned}&{}_2F_1(1,1-2\epsilon+\nu;1-4\epsilon+2\nu;1-z_1)\\&=\frac{(1-z_1)^{4\epsilon-2\nu}z_1^{-1-2\epsilon+\nu}\Gamma(1+2\epsilon-\nu)\Gamma(1-4\epsilon+2\nu)}{\Gamma(1-2\epsilon+\nu)}\\&\quad+\frac{\Gamma(-1-2\epsilon+\nu)\Gamma(1-4\epsilon+2\nu)}{\Gamma(\nu-2\epsilon)\Gamma(2\nu-4\epsilon)}\,{}_2F_1(1,1-2\epsilon+\nu,2+2\epsilon-\nu,z_1).\end{aligned}\tag{6.4.13}$$

We find

$$\begin{aligned}J_{1,\nu}(\epsilon,\nu)&=2\,[N]^3\int dz_1dt\;z_1^{-1+\nu}\;t^{\epsilon}\frac{\Gamma(1+2\epsilon-\nu)\Gamma(\nu-2\epsilon)^2\Gamma(2\nu-4\epsilon)}{\Gamma(3\nu-6\epsilon)}\\&\quad\times\,{}_2F_1(1,1+\epsilon,1-\epsilon,t)(1-z_1)^{4\epsilon-2\nu}\\&\quad+2\,[N]^3\int dz_1\;dt\;t^{\epsilon}\;z_1^{2\epsilon}\frac{\Gamma(\nu-2\epsilon)\Gamma(-1-2\epsilon+\nu)\Gamma(1-2\epsilon+\nu)}{\Gamma(3\nu-6\epsilon)}\\&\quad\times\,{}_2F_1(1,1+\epsilon,1-\epsilon,t)\,{}_2F_1(1,1-2\epsilon+\nu,2+2\epsilon-\nu,z_1).\end{aligned}\tag{6.4.14}$$

The second integral in Eq. (6.4.14) is well-defined in dimensional regularization. In the first integral the $t$ integration is also well-defined while the $z_1$ integration clearly suffers from an unregulated divergence if the analytic regulator is not present. From the discussion in Chapter 4 we know that the $z_1\to 0$ limit describes the kinematic configuration in which both $k_1$ and $k_3$ become collinear to the beam axis $n$. Thus, the additional singularity we observe is a collinear divergence, for which the dimensional regulator is not effective in this case.
We proceed by explicitly extracting the singular part of Eq. (6.4.14). To this end, we integrate over $z_1$ and $t$ and expand in $\nu$. We obtain

$$\nu\frac{J_{1,\nu}(\epsilon,\nu)}{[N]^3}=-\frac{3}{8\epsilon^3}+15\zeta(3)+\epsilon\frac{121\pi^4}{120}+\frac{\epsilon^2\left(5\pi^2\zeta(3)+957\zeta(5)\right)}{2}+\mathcal{O}\left(\nu^1,\epsilon^3\right).\tag{6.4.15}$$

The second contribution $J_{2,\nu}(\epsilon,\nu)$ for which $z_3>z_1$, is computed similarly. The calculation can be found in Appendix D.6. The result reads

$$\begin{aligned}J_{2,\nu}(\epsilon,\nu)&=2\,[N]^3\int dz_3dt\;z_3^{-1+\nu}\;t^{-\epsilon+\nu-1}\;(1-tz_3)^{4\epsilon-2\nu}\\&\quad\times\frac{\Gamma(2\epsilon-\nu+1)\Gamma(\nu-2\epsilon)^2\Gamma(2\nu-4\epsilon)}{\Gamma(3\nu-6\epsilon)}\,{}_2F_1(1,\epsilon+1;1-\epsilon;t)\\&\quad+2\,[N]^3\int dz_3dt\;t^{\epsilon}\;z_3^{2\epsilon}\frac{\Gamma(\nu-2\epsilon)\Gamma(-2\epsilon+\nu-1)\Gamma(-2\epsilon+\nu+1)}{\Gamma(3\nu-6\epsilon)}\\&\quad\times\;{}_2F_1(1,\epsilon+1,1-\epsilon,t)\;\;{}_2F_1(1,-2\epsilon+\nu+1,2\epsilon-\nu+2,tz_3).\end{aligned}\tag{6.4.16}$$

Thus we find the same singularity structure as in Eq. (6.4.14) with $z_1$ exchanged for $z_3$.

We proceed by extracting the singular part of Eq. (6.4.16), by integrating over $z_3$ and $t$ and expanding in $\nu$. We obtain

$$\nu\frac{J_{2,\nu}(\epsilon,\nu)}{[N]^3} = -\frac{9}{8\epsilon^3}+\frac{\pi^2}{\epsilon}+60\zeta(3)+\epsilon\frac{341\pi^4}{120}-\epsilon^2\frac{3\left(35\pi^2\zeta(3)-1031\zeta(5)\right)}{2} + \mathcal{O}(\nu,\epsilon^3). \tag{6.4.17}$$

Combining the results of both sectors we find

$$\nu\frac{J_{\nu}(\epsilon,\nu)}{[N]^3} = -\frac{3}{2\epsilon^3}-25\epsilon^2\left(2\pi^2\zeta(3)-81\zeta(5)\right)+\frac{77\pi^4\epsilon}{20}+\frac{\pi^2}{\epsilon}+75\zeta(3) + \mathcal{O}(\nu,\epsilon^3). \tag{6.4.18}$$

### 6.4.3 *Results*

Finally, combining the reduction in Eq. (6.4.3) with the results for master integrals found in the ancillary file of Ref. [77], we obtain

$$\begin{aligned}\int \mathrm{d}\Phi^{nnn}_{\theta\theta\theta}\,\omega^{(3),b}_{n\bar{n}}(k_1,k_2,k_3) &= [N]^3\left\{\frac{8}{\epsilon^5}+\frac{32}{\epsilon^4}+\frac{1}{\epsilon^3}\left(64-\frac{41\pi^2}{3}\right)\right.\\ &+\frac{1}{\epsilon^2}\left(128-64\pi^2-774\zeta_3\right)+\frac{1}{\epsilon}\left(256-128\pi^2-\frac{581\pi^4}{10}-2144\zeta_3\right)\\ &+\left(512-256\pi^2-\frac{1688\pi^4}{15}-4288\zeta_3+\frac{1306\pi^2\zeta_3}{3}-28770\zeta_5\right)\\ &+\epsilon\left(1024-512\pi^2-\frac{3376\pi^4}{15}-\frac{616\pi^6}{5}-8576\zeta_3+2304\pi^2\zeta_3\right.\\ &\left.\left.+19480\zeta_3^2-68736\zeta_5\right)\right\}.\end{aligned} \tag{6.4.19}$$

We discuss the calculation of the next-to-hardest contribution to the soft function, $\omega^{(3),c}_{n\bar{n}}$, in the next subsection.

## 6.5 THE $\omega^{(3),c}$ CONTRIBUTION

The third contribution to the zero-jettiness soft function can be written as an integral over the function $\omega_{n\bar{n}}^{(3),c}$. We construct it from the function $\overline{\mathcal{S}}_{ik}^{(c)}$, which reads [74]

$$\begin{aligned}
\overline{\mathcal{S}}_{ik}^{(c)} &= \frac{1}{8\,(k_1k_2)^2\, nk_{12}\, \bar{n}k_3}\Bigg\{\Big((4-d)\, nk_1 + d\, nk_2\Big)\left[\frac{n\bar{n}\,\bar{n}k_1}{2\,\bar{n}k_{123}\, nk_{123}}\left(\frac{\bar{n}k_3}{\bar{n}k_{12}}-1\right)\left(\frac{nk_{12}}{nk_3}-1\right)\right.\\
&\left.+\frac{n\bar{n}}{nk_3}\left(\frac{nk_1}{nk_{12}}-\frac{3}{2}\frac{\bar{n}k_1}{\bar{n}k_{12}}\right)\right]\\
&+\frac{n\bar{n}}{nk_{123}}\left(\frac{1}{nk_3}-\frac{1}{nk_{12}}\right)\Big((4-d)\,(nk_1)^2 + d\, nk_1\, nk_2\Big)\Bigg]\\
&+\frac{1}{32\, k_1k_2\, k_1k_3\, nk_{12}}\left\{\frac{n\bar{n}}{\bar{n}k_2}\left[\frac{4\,\bar{n}(k_2-k_1)}{\bar{n}k_3}+\frac{2\, nk_{12}}{nk_3}+2\,\frac{nk_2\, nk_3 + nk_1\, nk_{123}}{nk_{13}\, nk_3}\right.\right.\\
&\left.+\frac{1}{\bar{n}k_{13}\, nk_3}\Big(nk_1\,\bar{n}(5k_1 - 8k_2 + 2k_3) - 3\, nk_2\,\bar{n}k_3 + 4\, n\bar{n}\, k_2k_3\Big)\right]\\
&+\frac{2}{nk_{123}\,\bar{n}k_3}\left(\frac{1}{nk_2}-\frac{1}{nk_{13}}\right)\Big(n\bar{n}\, nk_{12}\, nk_{13}\Big)\\
&+\frac{2}{\bar{n}k_{123}\,\bar{n}k_3}\left(\frac{1}{\bar{n}k_{13}}-\frac{1}{\bar{n}k_2}\right)\Big[2\, n\bar{n}\,\bar{n}k_{13}\,\bar{n}(k_2-k_1)\Big]+\frac{1}{\bar{n}k_{123}\, nk_{123}}\left(1-\frac{nk_{12}}{nk_3}\right)\\
&\times\left(\frac{1}{\bar{n}k_{13}}-\frac{1}{\bar{n}k_2}\right)\Big[4\,(n\bar{n})^2\, k_2k_3 + n\bar{n}\Big(nk_1\,\bar{n}(5k_1 - 8k_2 + 2k_3) - 3\, nk_2\,\bar{n}k_3\Big)\Big]\Big]. 
\end{aligned} \tag{6.5.1}$$

Unlike $\overline{\mathcal{S}}_{ik}^{(a)}$ and $\overline{\mathcal{S}}_{ik}^{(b)}$, the function $\overline{\mathcal{S}}_{ik}^{(c)}$ contains propagators $1/k_1 \cdot k_2$ and $1/k_2 \cdot k_3$ at the same time. While these propagators make the evaluation of master integrals more complicated, no additional issues arise when modified IBP relations are constructed.

The remainder of this section is organized as follows. In Section 6.5.1 we explain the reduction to master integrals comparing the two choices of bases introduced in the last section. In Section 6.5.2 we discuss the calculation of an example master integral. Finally, we present the result for the integral of $\omega_{n\bar{n}}^{(3),c}$ in Section 6.5.3.

### 6.5.1 *Reduction*

We proceed by applying modified IBPs as discussed in Section 6.1 to obtain a reduction for $\omega_{n\bar{n}}^{(3),c}$. In doing so we compare the two different choices of bases introduced in Section 6.4.

We start with the reduction to master integrals where we choose the $\nu = 0$ integrals as a preferred basis. We obtain

$$\begin{aligned}
&\int \mathrm{d}\Phi_{\theta\theta\theta}^{nnn}\, \omega_{n\bar{n}}^{(3),c}(k_1,k_2,k_3) = \\
&\left(-\frac{5152}{675\epsilon^5}+\frac{60883}{1350\epsilon^4}+\frac{2218663}{5400\epsilon^3}-\frac{33423797}{10800\epsilon^2}-\frac{498502532330\epsilon}{466560}+\frac{44313583}{12960\epsilon}+\frac{40023347}{15552}\right)I_1\\
&+\left(-\frac{262}{135\epsilon^3}+\frac{8}{135\epsilon^2}-\frac{1280\epsilon}{9}-\frac{16}{9\epsilon}+\frac{160}{9}\right)I_2+\left(\frac{2242}{135\epsilon^4}-\frac{23954}{135\epsilon^3}+\frac{74967872\epsilon^2}{1215}\right.\\
&\left.+\frac{14596}{45\epsilon^2}+\frac{2845312\epsilon}{405}+\frac{69832}{45\epsilon}-\frac{733168}{135}\right)I_3+\left(-\frac{47683}{675\epsilon^4}+\frac{27427}{25\epsilon^3}-\frac{7899529}{1350\epsilon^2}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{11094485353\epsilon^2}{12960}-\frac{119650151\epsilon}{2160}+\frac{7137263}{900\epsilon}+\frac{5243129}{360}\Big)I_4+\Big(\frac{334}{135\epsilon^3}-\frac{600703\epsilon^2}{81}\\
&-\frac{794}{45\epsilon^2}+\frac{38854\epsilon}{27}+\frac{5036}{135\epsilon}-\frac{2024}{9}\Big)I_5+\Big(-\frac{1202}{15\epsilon^4}+\frac{4022}{3\epsilon^3}+\frac{6077660177\epsilon^2}{6480}-\frac{121631}{15\epsilon^2}\\
&-\frac{76894303\epsilon}{1080}+\frac{170291}{10\epsilon}+\frac{534641}{180}\Big)I_6+\Big(-3211\epsilon^4+1409\epsilon^3-151\epsilon^2-\frac{9}{2\epsilon^2}+119\epsilon\\
&+\frac{3}{2\epsilon}+44\Big)I_7+\Big(\frac{16640\epsilon^4}{3}-\frac{3968\epsilon^3}{3}+\frac{1088\epsilon^2}{3}-\frac{14}{3\epsilon^2}-\frac{224\epsilon}{3}-\frac{8}{3\epsilon}+\frac{80}{3}\Big)I_9+\frac{65}{9\epsilon^2}I_{10}\\
&+\Big(-\frac{430971\epsilon^4}{32}+\frac{238589\epsilon^3}{48}-\frac{21817\epsilon^2}{24}-\frac{47}{9\epsilon^2}+\frac{1527\epsilon}{4}+\frac{337}{9\epsilon}+\frac{47}{6}\Big)I_{11}+\Big(\frac{1613075\epsilon^4}{96}\\
&-\frac{118055\epsilon^3}{48}+\frac{8195\epsilon^2}{24}-\frac{13}{15\epsilon^2}-\frac{515\epsilon}{12}+\frac{25}{6}\Big)\Big[I_{12}-I_{13}\Big]+\Big(-\frac{127360\epsilon^4}{3}+\frac{18560\epsilon^3}{3}\\
&-\frac{2560\epsilon^2}{3}-\frac{578}{45\epsilon^2}+\frac{320\epsilon}{3}+\frac{16}{45\epsilon}-\frac{32}{3}\Big)I_{14}+\Big(-\frac{377080\epsilon^4}{9}+\frac{54680\epsilon^3}{9}-\frac{7480\epsilon^2}{9}\\
&+\frac{19}{15\epsilon^2}+\frac{920\epsilon}{9}+\frac{8}{45\epsilon}-\frac{88}{9}\Big)I_{15}+\Big(-\frac{110384\epsilon^3}{3}+\frac{21808\epsilon^2}{3}-\frac{92}{15\epsilon^2}-\frac{4496\epsilon}{3}\\
&+\frac{212}{15\epsilon}+\frac{832}{3}\Big)I_{16}+\Big(-\frac{1075877\epsilon^5}{16}+\frac{114497\epsilon^4}{8}-\frac{10397\epsilon^3}{4}+\frac{1217\epsilon^2}{2}-17\epsilon\\
&+\frac{68}{5\epsilon}+14\Big)I_{17}+\Big(\frac{1111653\epsilon^5}{16}-\frac{127713\epsilon^4}{8}+\frac{9933\epsilon^3}{4}-\frac{1593\epsilon^2}{2}-87\epsilon+\frac{12}{5\epsilon}-36\Big)I_{18}
\end{aligned}
\tag{6.5.2}
$$

$$
\begin{aligned}
&+\Big(\frac{2546205\epsilon^5}{16}-\frac{275865\epsilon^4}{8}+\frac{22005\epsilon^3}{4}-\frac{3465\epsilon^2}{2}-15\epsilon+\frac{18}{\epsilon}-150\Big)I_{19}\\
&+\Big(-172\epsilon^3+44\epsilon^2+32\epsilon+\frac{2}{\epsilon}-14\Big)I_{20}+\Big(64\epsilon^3+32\epsilon^2-\frac{4}{3\epsilon^2}+16\epsilon+\frac{4}{3\epsilon}+8\Big)I_{21}\\
&+\Big(1968\epsilon^3-320\epsilon^2+\frac{172\epsilon}{3}+\frac{40}{9\epsilon}-\frac{68}{9}\Big)I_{22}+\Big(-312\epsilon^3+\frac{151}{60\epsilon^3}-\frac{4532\epsilon^2}{3}+\frac{73}{6\epsilon^2}\\
&-270\epsilon-\frac{69}{10\epsilon}-\frac{353}{3}\Big)I_{23}+\Big(3840\epsilon^4-672\epsilon^3+96\epsilon^2-24\epsilon-\frac{2}{\epsilon}\Big)I_{24}+\Big(69719\epsilon^4\\
&-9413\epsilon^3+2419\epsilon^2+\frac{1}{3\epsilon^2}-107\epsilon-\frac{125}{18\epsilon}+\frac{920}{9}\Big)I_{25}+\Big(\frac{83680\epsilon^4}{3}-\frac{13280\epsilon^3}{3}+\frac{2080\epsilon^2}{3}\\
&+\frac{4}{45\epsilon^2}-\frac{320\epsilon}{3}-\frac{104}{45\epsilon}+16\Big)I_{26}+\Big(-96\epsilon^4-48\epsilon^3-24\epsilon^2-12\epsilon-\frac{3}{\epsilon}-6\Big)I_{27}\\
&+\Big(\frac{4480\epsilon^5}{3}+\frac{2240\epsilon^4}{3}+\frac{1120\epsilon^3}{3}+\frac{560\epsilon^2}{3}+\frac{4}{3\epsilon^2}+\frac{280\epsilon}{3}+\frac{91}{6\epsilon}+\frac{140}{3}\Big)I_{28}\\
&+\Big(117120\epsilon^5-19200\epsilon^4+3360\epsilon^3-480\epsilon^2-\frac{2}{\epsilon^2}+120\epsilon+\frac{2}{3\epsilon}\Big)I_{29}+\Big(34560\epsilon^5\\
&-5760\epsilon^4+960\epsilon^3-160\epsilon^2+\frac{80\epsilon}{3}+\frac{20}{9\epsilon}-\frac{40}{9}\Big)I_{30}+\Big(-96576\epsilon^5+15216\epsilon^4\\
&-2976\epsilon^3+276\epsilon^2+\frac{2}{3\epsilon^2}-156\epsilon-\frac{9}{\epsilon}-29\Big)I_{31}+\Big(-\frac{40832\epsilon^5}{3}+\frac{7232\epsilon^4}{3}-\frac{992\epsilon^3}{3}\\
&+\frac{272\epsilon^2}{3}+\frac{7}{18\epsilon^2}+\frac{8\epsilon}{3}+\frac{29}{18\epsilon}+\frac{76}{9}\Big)I_{32}+\Big(-5728\epsilon^4+1048\epsilon^3-128\epsilon^2-\frac{37}{18\epsilon^2}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{134\epsilon}{3} - \frac{9}{2\epsilon} + \frac{38}{9}\Big) I_{33} + \Big(\frac{93152\epsilon^5}{3} - \frac{15632\epsilon^4}{3} + \frac{2552\epsilon^3}{3} - \frac{452\epsilon^2}{3} - \frac{7}{18\epsilon^2} + \frac{62\epsilon}{3} - \frac{1}{6\epsilon} \\
&- \frac{17}{3}\Big) I_{34} + \Big(34695\epsilon^5 - 5805\epsilon^4 + 975\epsilon^3 - 165\epsilon^2 + \frac{85\epsilon}{3} + \frac{5}{3\epsilon} - 5\Big) \Big[I_{35} - I_{36} + I_{37}\Big] \\
&+ \frac{4}{3\epsilon}\Big[I_{38} - I_{39}\Big] + \Big(-\frac{16640\epsilon^4}{3} + \frac{3968\epsilon^3}{3} - \frac{1088\epsilon^2}{3} + \frac{8}{3\epsilon^2} + \frac{224\epsilon}{3} + \frac{8}{3\epsilon} \\
&- \frac{80}{3}\Big) \Big[\lim_{\nu\to 0} \nu J_\nu\Big] + \mathcal{O}(\epsilon^2)\,.
\end{aligned}
$$

The master integrals $I_1$ through $I_{39}$ can be evaluated at $\nu = 0$. The only $\nu$-dependent term is again given by $J_\nu$ which we already calculated for the $\omega_{n\bar{n}}^{(3),b}$ contribution.

In comparison, the reduction to the automatically chosen basis (preferring dots over scalar products) will lead to an expression where $\nu$-finite integrals are multiplied by $1/\nu$ coefficients. For example

$$
\begin{aligned}
&\int \mathrm{d}\Phi_{\theta\theta\theta}^{nnn}\, \omega_{n\bar{n}}^{(3),c}(k_1,k_2,k_3) = \\
&\frac{1}{\nu}\Bigg\{ \Big(-\frac{11}{36\epsilon^4} + \frac{1373}{180\epsilon^3} - \frac{59}{\epsilon^2} + \frac{108085\epsilon}{18} + \frac{16073}{90\epsilon} - \frac{14369}{18}\Big) I_1 \\
&+ \Big(\frac{13}{3\epsilon^3} - \frac{750476\epsilon^2}{15} - \frac{1274}{15\epsilon^2} + \frac{123352\epsilon}{15} + \frac{6764}{15\epsilon} - \frac{6356}{5}\Big) I_4 \\
&+ \Big(\frac{5}{\epsilon^3} - 68292\epsilon^2 - \frac{103}{\epsilon^2} + 11252\epsilon + \frac{626}{\epsilon} - 2104\Big) I_6 \\
&+ \Big(\frac{13933\epsilon^5}{6} - \frac{2671\epsilon^4}{6} + \frac{277\epsilon^3}{6} - \frac{397\epsilon^2}{18} - \frac{61\epsilon}{18} + \frac{5}{36\epsilon} - \frac{4}{3}\Big) I_{44} \Bigg\} + \mathcal{O}(\nu^0).
\end{aligned}
\tag{6.5.3}
$$

Calculation of the $\nu \to 0$ limit of Eq. (6.5.3) is clearly complicated. Hence, it follows from this comparison that the $\nu = 0$ integral basis is the preferred choice.

### 6.5.2 *Calculation of master integrals*

We now discuss the calculation of master integrals appearing in Eq. (6.5.2). Evaluation of these integrals closely follows earlier discussions. However, the two propagators $1/k_1 \cdot k_2$ and $1/k_1 \cdot k_3$ make the calculation of master integrals significantly more complicated.

To illustrate this point, we compute the master integral $I_{30}$ explicitly. The integral reads

$$
I_{30} = \int \frac{\mathrm{d}\Phi_{\theta\delta\theta}^{nnn}}{(k_1k_2)(k_1k_3)(k_{12}n)(k_{13}\bar{n})}. \tag{6.5.4}
$$

We integrate over the relative azimuthal angle between $\vec{k}_{1,\perp}$ and $\vec{k}_{3,\perp}$, using the formula

$$\int \frac{d\Omega^{(d-2)}}{\Omega^{(d-2)}} \frac{1}{k_i \cdot k_j} = 2\Bigg\{ \frac{\theta(\alpha_i\beta_j - \alpha_j\beta_i)}{\alpha_i\beta_j} \, {}_2F_1\left(1, 1+\epsilon, 1-\epsilon, \frac{\alpha_j\beta_i}{\alpha_i\beta_j}\right) + \frac{\theta(\alpha_j\beta_i - \alpha_i\beta_j)}{\alpha_j\beta_i} \, {}_2F_1\left(1, 1+\epsilon, 1-\epsilon, \frac{\alpha_i\beta_j}{\alpha_j\beta_i}\right) \Bigg\}, \tag{6.5.5}$$

where $i = 1$ and $j = 3$. We proceed by changing variables $\alpha_{1,3} \to \beta_{1,3}/\xi_{1,3}$ and obtain

$$\begin{aligned} I_{30} = [N]^3\, 4 \int \prod_{i=1}^{3} d\beta_i\, \beta_i^{-2\epsilon-1}\, \delta(1-\beta_{123})\, & \frac{d\xi_1 d\xi_3\, \xi_1^{\epsilon}\xi_3^{\epsilon-1}\, \beta_3}{(\beta_1+\beta_2)(\beta_1\xi_3+\beta_3\xi_1)} \\ \times {}_2F_1(1, 1+\epsilon, 1-\epsilon, \xi_1) & \Bigg[ \xi_1\theta(\xi_3-\xi_1)\, {}_2F_1\left(1, 1+\epsilon, 1-\epsilon, \frac{\xi_1}{\xi_3}\right) \\ + \xi_3\theta(\xi_1-\xi_3)\, {}_2F_1\left(1, 1+\epsilon, 1-\epsilon, \frac{\xi_3}{\xi_1}\right) \Bigg]. \end{aligned} \tag{6.5.6}$$

We next integrate over $\beta_2$ by removing the delta function $\delta(1-\beta_{123})$ and further change variables $(\beta_1, \beta_3) \to (x, y)$ using $\beta_1 = x(1-y)$ and $\beta_3 = x\, y$. The integration over $x$ yields a hypergeometric function. Finally, we obtain

$$\begin{aligned} I_{30} = [N]^3 \frac{4\Gamma(-4\epsilon)\Gamma(-2\epsilon)}{\Gamma(-6\epsilon)} \int dy\, d\xi_1 d\xi_3 & \frac{y^{-2\epsilon}(1-y)^{-2\epsilon-1}\xi_1^{\epsilon}\xi_3^{\epsilon}}{(1-y)\xi_3 + y\xi_1} \\ \times {}_2F_1(1, -4\epsilon, -6\epsilon, y)\, & {}_2F_1(1, 1+\epsilon, 1-\epsilon, \xi_1) \\ \times \Bigg[ \frac{\xi_1}{\xi_3}\theta(\xi_3-\xi_1)\, & {}_2F_1\left(1, 1+\epsilon, 1-\epsilon, \frac{\xi_1}{\xi_3}\right) \\ + \theta(\xi_1-\xi_3)\, & {}_2F_1\left(1, 1+\epsilon, 1-\epsilon, \frac{\xi_3}{\xi_1}\right) \Bigg]. \end{aligned} \tag{6.5.7}$$

Multiplying out the squared bracket we write the integral as the sum of two terms

$$I_{30} = [N]^3 \frac{4\Gamma(-4\epsilon)\Gamma(-2\epsilon)}{\Gamma(-6\epsilon)} \left( \mathcal{I}_{30}^{(a)} + \mathcal{I}_{30}^{(b)} \right). \tag{6.5.8}$$

We change variables as follows: $\xi_1 = r\xi$, $\xi_3 = \xi$ in $\mathcal{I}_{30}^{(a)}$, and $\xi_3 = r\xi$, $\xi_1 = \xi$ in $\mathcal{I}_{30}^{(b)}$. Finally, we transform the hypergeometric functions appearing in the squared brackets using the following identity

$${}_2F_1(a, b, c, z) = (1-z)^{c-a-b}\, {}_2F_1(c-a, c-b, c, z). \tag{6.5.9}$$

We obtain

$$\mathcal{I}_{30}^{(a)} = \int dy\, d\xi\, dr\, \frac{y^{-2\epsilon}(1-y)^{-4\epsilon-2}\, \xi^{2\epsilon}\, r^{1+\epsilon}\, [(1-r)(1-r\xi)]^{-1-2\epsilon}}{1-y(1-r)}$$

$$\times\, {}_2F_1\left(-1-6\epsilon,-2\epsilon,-6\epsilon,y\right)\ {}_2F_1\left(-\epsilon,-2\epsilon,1-\epsilon,r\xi\right) \\ \times\, {}_2F_1\left(-\epsilon,-2\epsilon,1-\epsilon,r\right), \tag{6.5.10}$$

and

$$\mathcal{I}_{30}^{(b)} = \int \mathrm{d}y\mathrm{d}\xi\mathrm{d}r\, \frac{y^{-2\epsilon}(1-y)^{-4\epsilon-2}\,\xi^{2\epsilon}\,r^{\epsilon}(1-r)^{-2\epsilon-1}}{r+y(1-r)} \\ \times\, {}_2F_1\left(-1-6\epsilon,-2\epsilon,-6\epsilon,y\right)\ {}_2F_1\left(1,1+\epsilon,1-\epsilon,\xi\right) \\ \times\ {}_2F_1\left(-\epsilon,-2\epsilon,1-\epsilon,r\right). \tag{6.5.11}$$

The two integrals in Eqs. (6.5.10) and (6.5.11) are difficult to compute due to a power-like divergence at $y=1$. To isolate and extract this divergence, we transform the $y$-dependent hypergeometric function in the following way

$${}_2F_1\left(-1-6\epsilon,-2\epsilon,-6\epsilon,y\right) = \frac{\Gamma(-6\epsilon)\Gamma(1+2\epsilon)}{\Gamma(-4\epsilon)}y^{1+6\epsilon} \\ +(1-y)^{1+2\epsilon}\frac{\Gamma(-1-2\epsilon)\Gamma(-6\epsilon)}{\Gamma(-1-6\epsilon)\Gamma(-2\epsilon)}\,{}_2F_1\left(1,-4\epsilon,2+2\epsilon,1-y\right). \tag{6.5.12}$$

The first term in Eq. (6.5.12) can now be straightforwardly integrated and the second term only leads to a logarithmic divergence in $y=1$. Thus, to proceed further we consider the integral $\mathcal{I}_{30}^{(a)}$ and write it as

$$\mathcal{I}_{30}^{(a)} = \mathcal{I}_{30}^{(a,1)} + \mathcal{I}_{30}^{(a,2)}, \tag{6.5.13}$$

where $\mathcal{I}_{30}^{(a,1)}$ and $\mathcal{I}_{30}^{(a,2)}$ correspond to the two terms of Eq. (6.5.12). To compute $\mathcal{I}_{30}^{(a,1)}$ we integrate over $y$ and find

$$\int \mathrm{d}y \frac{y^{1+4\epsilon}(1-y)^{-4\epsilon-2}}{1-y(1-r)} = \Gamma(2+4\epsilon)\Gamma(-1-4\epsilon)\ r^{-2-4\epsilon}. \tag{6.5.14}$$

The full expression for $\mathcal{I}_{30}^{(a,1)}$ now reads

$$\mathcal{I}_{30}^{(a,1)} = -\frac{\Gamma(2+4\epsilon)\Gamma(-6\epsilon)\Gamma(1+2\epsilon)}{(1+4\epsilon)}\int \mathrm{d}\xi\ \mathrm{d}r\ \xi^{2\epsilon}r^{-1-3\epsilon}(1-r)^{-2\epsilon-1} \\ \times(1-\xi r)^{-2\epsilon-1}\,{}_2F_1\left(-\epsilon,-2\epsilon,1-\epsilon,r\xi\right)\ {}_2F_1\left(-\epsilon,-2\epsilon,1-\epsilon,r\right). \tag{6.5.15}$$

This integral has a singularity at $r=0$ and an overlapping singularity at $r=1,\xi=1$. We separate these two singularities by multiplying the integrand with $1=(1-r)+r$. The first term removes the $r=1$ singularity so that the remaining $r=0$ singularity can easily be extracted. The second term still suffers from the overlapping $r=1,\xi=1$ singularities. To deal with this case we subtract the product of hypergeometric functions at $r = 1$ and add it back. When this difference of hypergeometric functions

$$\begin{aligned}&{}_2F_1\left(-\epsilon,-2\epsilon,1-\epsilon,r\xi\right)\,{}_2F_1\left(-\epsilon,-2\epsilon,1-\epsilon,r\right)\\&-{}_2F_1\left(-\epsilon,-2\epsilon,1-\epsilon,\xi\right)\,{}_2F_1\left(-\epsilon,-2\epsilon,1-\epsilon,1\right),\end{aligned} \tag{6.5.16}$$

appears in the integrand, the $r = 1$ singularity is removed and the integrand can be expanded in a series in $\epsilon$ and integrated. On the other hand, when only the subtraction term appears, the hypergeometric function does not depend on $r$ anymore, so that the remaining integration over $r$ can be carried out in a straightforward manner. The resulting one-dimensional integration over $\xi$ is divergent in $\xi = 1$. However, this divergence is logarithmic and can be easily isolated and subtracted. Combining all the different contributions, we find

$$\begin{aligned}\mathcal{I}_{30}^{(a,1)} =& -\frac{1}{18\epsilon^2}+\frac{1}{48\epsilon^3}+\frac{1}{\epsilon}\left(\frac{5}{18}+\frac{\pi^2}{6}\right)+\frac{43\zeta_3}{12}+\frac{5\pi^2}{108}-\frac{25}{18}\\&+\epsilon\left(\frac{61\zeta_3}{6}+\frac{437\pi^4}{540}+\frac{125}{18}-\frac{13\pi^2}{108}\right)+\epsilon^2\left(\frac{89\pi^2\zeta_3}{9}-\frac{85\zeta_3}{2}\right.\\&\left.+\frac{845\zeta_5}{4}+\frac{209\pi^4}{162}+\frac{17\pi^2}{108}-\frac{625}{18}\right)+\epsilon^3\left(-32\zeta_3^2+\frac{304\pi^2\zeta_3}{9}+\frac{1075\zeta_3}{6}\right.\\&\left.+\frac{3059\zeta_5}{6}+\frac{57637\pi^6}{17010}+\frac{107\pi^2}{108}+\frac{3125}{18}-\frac{2194\pi^4}{405}\right).\end{aligned} \tag{6.5.17}$$

The computation of $\mathcal{I}_{30}^{(a,2)}$ proceeds in the following way. There are two singularities one at $y = 1$ and another one at $r = 1, \xi = 1$. To disentangle the overlapping singularity at $r = 1, \xi = 1$ we make the following replacement

$$\begin{aligned}&\frac{r^{1+\epsilon}}{1-y(1-r)}\,{}_2F_1\left(-\epsilon,-2\epsilon,1-\epsilon,\xi r\right){}_2F_1\left(-\epsilon,-2\epsilon,1-\epsilon,r\right)\to\\&{}_2F_1\left(-\epsilon,-2\epsilon,1-\epsilon,\xi\right){}_2F_1\left(-\epsilon,-2\epsilon,1-\epsilon,1\right),\end{aligned} \tag{6.5.18}$$

in Eq. (6.5.10) and add the difference of the two terms back. In the difference the $r = 1$ singularity is regulated such that we only need to extract the logarithmic singularity at $y = 1$. To compute the contribution of the subtraction term, we simply integrate over $y$ and $r$ to obtain yet another hypergeometric function of $\xi$. The resulting $\xi$ integration only has a logarithmic divergence at $\xi = 1$ which can easily be isolated and subtracted. We find

$$\begin{aligned}\mathcal{I}_{30}^{(a,2)} =& -\frac{1}{16\epsilon^3}-\frac{1}{4\epsilon^2}+\frac{1}{\epsilon}\left(1-\frac{\pi^2}{4}\right)-\frac{15\zeta_3}{4}-\frac{2\pi^2}{3}-4+\epsilon\left(-23\zeta_3+\frac{8\pi^2}{3}\right.\\&\left.-\frac{23\pi^4}{36}+16\right)+\epsilon^2\left(\frac{23\pi^2\zeta_3}{6}+104\zeta_3-\frac{595\zeta_5}{4}-\frac{32\pi^2}{3}-\frac{47\pi^4}{30}-64\right)\\&+\epsilon^3\left(249\zeta_3^2+\frac{2\pi^2\zeta_3}{3}-476\zeta_3-577\zeta_5+\frac{67\pi^4}{10}+\frac{128\pi^2}{3}-\frac{4756\pi^6}{2835}+256\right).\end{aligned} \tag{6.5.19}$$

Computation of $\mathcal{I}_{30}^{(b)}$ proceeds along the same lines, however this time the $\xi$ integration can be directly performed. We find

$$\begin{aligned}\mathcal{I}_{30}^{(b)} &= \frac{1}{1+2\epsilon}\, {}_3F_2\left(1, 1+\epsilon, 1+2\epsilon, 1-\epsilon, 2+2\epsilon, 1\right) \\ &\times \int \mathrm{d}y\, \mathrm{d}r\, \frac{y^{-2\epsilon}(1-y)^{-4\epsilon-2}\, r^{\epsilon}(1-r)^{-2\epsilon-1}}{r+y(1-r)} \\ &\times {}_2F_1\left(-1-6\epsilon, -2\epsilon, -6\epsilon, y\right)\ {}_2F_1\left(-\epsilon, -2\epsilon, 1-\epsilon, r\right).\end{aligned} \tag{6.5.20}$$

We then re-write ${}_2F_1\left(-1-6\epsilon, -2\epsilon, -6\epsilon, y\right)$ using Eq. (6.5.12) and integrate over the two resulting terms separately. The first integration is straightforward as it leads to yet another ${}_3F_2$-function. Integration over the second term is also straightforward since no additional resolution of overlapping singularities is required.

We finally put everything together and obtain the final result for $I_{30}$

$$\begin{aligned}I_{30} = [N]^3 \Bigg\{ &\frac{3}{8\epsilon^4} + \frac{13}{6\epsilon^3} + \frac{1}{\epsilon^2}\left(-\frac{109}{12} - \frac{\pi^2}{12}\right) + \frac{1}{\epsilon}\left(-31\zeta_3 + \frac{5\pi^2}{9} + \frac{461}{12}\right) - 76\zeta_3 \\ &- \frac{1969}{12} - \frac{23\pi^2}{18} - \frac{1211\pi^4}{360} + \epsilon\left(-\frac{311}{6}\pi^2\zeta_3 + 292\zeta_3 - \frac{3111\zeta_5}{2} + \frac{7\pi^2}{18} + \frac{8501}{12} \right. \\ &\left. - \frac{355\pi^4}{36}\right) + \epsilon^2\left(18\zeta_3^2 - \frac{727\pi^2\zeta_3}{3} - 1108\zeta_3 - 4035\zeta_5 + \frac{1147\pi^4}{30} \right. \\ &\left. + \frac{397\pi^2}{18} - \frac{37129}{12} - \frac{10729\pi^6}{810}\right)\Bigg\}.\end{aligned} \tag{6.5.21}$$

Definitions of the remaining master integrals that appear Eq. (6.5.2) can be found in Appendix D.3 and their analytic results are again given in the ancillary file of Ref. [77].

### 6.5.3 *Results*

Substituting results for the master integrals into Eq. (6.5.2) we arrive at the following result for the integral of $\omega_{n\bar{n}}^{(3),c}$ over the $nnn$ phase space

$$\begin{aligned}&\int \mathrm{d}\Phi_{\theta\theta\theta}^{nnn}\, \omega_{n\bar{n}}^{(3),c}(k_1, k_2, k_3) \\ &= [N]^3 \Bigg\{ -\frac{4}{\epsilon^5} + \frac{70}{3\epsilon^4} + \frac{1}{\epsilon^3}\left(\frac{920}{9} + \frac{19\pi^2}{3}\right) + \frac{1}{\epsilon^2}\left(\frac{8527}{27} + \frac{122\pi^2}{3} + 162\zeta_3\right) \\ &+ \frac{1}{\epsilon}\left(\frac{67193}{81} + \frac{1280\pi^2}{9} + \frac{197\pi^4}{90} + \frac{2732\zeta_3}{3}\right) + \left(\frac{558745}{243} + \frac{10990\pi^2}{27} + \frac{439\pi^4}{9} \right. \\ &\left. + \frac{32032\zeta_3}{9} - \frac{1604\pi^2\zeta_3}{3} + 4204\zeta_5\right) + \epsilon\left(\frac{4074557}{729} + \frac{89138\pi^2}{81} + \frac{28024\pi^4}{135} \right. \\ &\left. - \frac{23029\pi^6}{2835} + \frac{288992\zeta_3}{27} - \frac{9224\pi^2\zeta_3}{3} - 7604\zeta_3^2 + 50296\zeta_5\right)\Bigg\}.\end{aligned} \tag{6.5.22}$$

We explain the calculation of the last contribution $\omega_{n\bar{n}}^{(3),d}$ to the zero-jettiness soft function in the next section.

## 6.6 THE $\omega^{(3),d}$ CONTRIBUTION

The last contribution to the zero-jettiness soft function can be written as an integral over the function $\omega_{n\bar{n}}^{(3),d}$. We construct it from the function $\overline{\mathcal{S}}_{ik}^{(d)}$, which reads [74]

$$
\begin{aligned}
\overline{\mathcal{S}}_{ik}^{(d)} &= \frac{-(n\bar{n})^2}{4\,k_{123}^2\,\bar{n}k_{123}\,\bar{n}k_1\,nk_2\,nk_3} + \frac{1}{4\,k_{123}^2\,\bar{n}k_{123}}\Bigg\{\frac{1}{k_1k_2}\Bigg[\frac{1}{\bar{n}k_1\,nk_2}\Bigg(n\bar{n}\,(k_2k_3-k_1k_3)\left(\frac{n\bar{n}}{nk_3}\right) \\
&+2\,\frac{n\bar{n}}{nk_3}\,(nk_{12}\,\bar{n}k_{123}-nk_3\,\bar{n}k_2-nk_2\,\bar{n}k_3)\Bigg) \\
&+\frac{1}{2\,nk_{13}\,\bar{n}k_2}\Bigg\{\left(\frac{1}{nk_1}-\frac{1}{nk_3}\right) \\
&\times\Big[n\bar{n}\Big(\bar{n}k_1\,n(4\,k_3-3\,k_{12})-3\,\bar{n}k_2\,nk_{12}+\bar{n}k_3\,n(k_1-3k_2)+2\,n\bar{n}\,(k_2k_3-k_1k_3)\Big)\Big] \\
&+4\,n\bar{n}\,\bar{n}(k_3-k_1)\Bigg\} \\
&+\frac{1}{nk_{12}\,\bar{n}k_3}\Bigg\{\left(\frac{1}{nk_1}-\frac{1}{nk_2}\right)\Big(-2\,n\bar{n}\,nk_1\,\bar{n}k_{123}\Big) \\
&+n\bar{n}\,\bar{n}(k_1-3\,k_2-k_3)\Bigg\}+\frac{1}{\bar{n}k_{13}\,nk_2}\Bigg\{n\bar{n}\,\bar{n}k_{13} \\
&+\left(\frac{1}{\bar{n}k_3}-\frac{1}{\bar{n}k_1}\right)\left[\frac{n\bar{n}}{2}\,\bar{n}k_{123}\,\bar{n}k_{12}\right]\Bigg\}+\frac{1}{\bar{n}k_{12}\,nk_3}\Bigg\{n\bar{n}\,\bar{n}(3\,k_1+k_2-3\,k_3) \\
&+\left(\frac{1}{\bar{n}k_2}-\frac{1}{\bar{n}k_1}\right)\Big[2\,n\bar{n}\,\bar{n}k_1\,\bar{n}k_{13}\Big]\Bigg\} \\
&+\frac{2}{nk_{123}}\Bigg\{\left(\frac{\bar{n}k_{13}}{\bar{n}k_2}-1\right)\left(\frac{n\bar{n}}{2}\right)+\left(\frac{1}{\bar{n}k_{12}}-\frac{1}{\bar{n}k_3}\right)\Big(-n\bar{n}\,\bar{n}(k_2+2\,k_3)\Big) \\
&+\frac{1}{3}\left(\frac{1}{\bar{n}k_3}-\frac{3}{\bar{n}k_{12}}\right)\left(\frac{1}{\bar{n}k_1}-\frac{1}{\bar{n}k_2}\right)\Big(2\,n\bar{n}\,\bar{n}k_2\,\bar{n}(k_{12}-k_3)\Big) \\
&+\frac{1}{12}\left(\frac{1}{\bar{n}k_2}-\frac{3}{\bar{n}k_{13}}\right)\left(\frac{1}{\bar{n}k_1}-\frac{1}{\bar{n}k_3}\right)\Big[4\,n\bar{n}\,\bar{n}(k_1-k_2)\,\bar{n}(k_3-k_{12})\Big]\Bigg\}\Bigg] \\
&+\frac{1}{(k_1k_2)^2}\Bigg[\frac{2}{nk_{123}}\left(\frac{1}{\bar{n}k_3}-\frac{1}{\bar{n}k_{12}}\right) \\
&\times\Big[n\bar{n}\,k_1k_3\Big((d-4)\,\bar{n}k_1-d\,\bar{n}k_2\Big)+2\,(d-2)\,nk_2\,(\bar{n}k_1)^2+\bar{n}k_1\,\bar{n}k_2\Big((4-d)\,nk_1+\frac{d}{2}\,nk_3\Big)\Big] \\
&+\frac{1}{nk_{12}\,\bar{n}k_3}\Big\{(d-2)\,n(k_2-k_1)\,\bar{n}k_1\,\bar{n}k_{13}\Big\} \\
&+\frac{1}{\bar{n}k_{12}\,nk_3}\Big\{2\,n\bar{n}\,k_1k_3\Big((4-d)\,\bar{n}k_1+d\,\bar{n}k_2\Big)+(d-2)\,nk_1\,\bar{n}(k_1-k_2)\,\bar{n}(k_{13}-3\,k_2)\Big\}\Bigg] \\
&+\frac{1}{2\,k_1k_2\,k_1k_3}\Bigg[\frac{1}{nk_{12}\,\bar{n}k_3}\Big\{4\,n\bar{n}\,k_2k_3\,\bar{n}(2\,k_1+k_3) \\
&+(\bar{n}k_1)^2\Big((7-2\,d)\,nk_1+(2\,d+1)\,nk_2-4\,nk_3\Big)+2\,\bar{n}k_1\,\bar{n}k_3\Big((5-d)\,nk_1+(d-5)\,nk_2-2\,nk_3\Big) \\
&+\bar{n}k_1\,\bar{n}k_2\Big((2\,d-3)\,nk_1+(9-2\,d)\,nk_2-2\,nk_3\Big)+3\,(\bar{n}k_3)^2\,n(k_1-k_2) \\
&+\bar{n}k_2\,\bar{n}k_3\,n(9\,k_1-3\,k_2+2\,k_3)+2\,(\bar{n}k_2)^2\,n(k_3-k_1)\Big\} \\
&+\frac{1}{\bar{n}k_{12}\,nk_3}\Big\{2\,n\bar{n}\,k_2k_3\,\bar{n}(k_{12}+2\,k_3)+(\bar{n}k_1)^2\Big((2\,d-7)\,nk_1+2\,nk_2+nk_3\Big) \\
&+\bar{n}k_1\,\bar{n}k_3\Big((2\,d-7)\,nk_1+4\,nk_2-3\,nk_3\Big)+2\,\bar{n}k_1\,\bar{n}k_2\,(nk_2-2\,d\,nk_1)+2\,(\bar{n}k_3)^2\,n(2\,k_1-k_2) \\
&+\bar{n}k_2\,\bar{n}k_3\Big(2\,nk_2+3\,nk_3-(2\,d+1)\,nk_1\Big)+(\bar{n}k_2)^2\Big((2\,d-9)\,nk_1-nk_3\Big)\Big\}
\end{aligned}
\tag{6.6.1}
$$

$$+ \frac{4}{nk_{123}} \left( \frac{1}{\bar{n}k_{12}} - \frac{1}{\bar{n}k_3} \right) \Big\{ k_2k_3 \Big[ \frac{n\bar{n}}{2} \bar{n}(5\,k_1 - 3\,k_2 + 4\,k_3) \Big] + (\bar{n}k_2)^2\, nk_3$$
$$+ (\bar{n}k_1)^2 \Big( nk_1 + (3-d)\, nk_2 \Big) + (d-2)\, \bar{n}k_1\, \bar{n}k_2\, nk_2 + \bar{n}k_1\, \bar{n}k_3\, n(k_1 - 3\,k_2) + \bar{n}k_2\, \bar{n}k_3\, n(3\,k_1 - k_2) \Big\} \Big] \Big\}$$
$$+ \frac{1}{2\,(k_{123}^2)^2\, nk_{123}\, \bar{n}k_{123}} \Big\{ (3\,d - 10)\, n\bar{n} + \frac{2\, n\bar{n}\, k_1k_3}{(k_1k_2)^2} \Big( (d-4)\, k_1k_3 - d\, k_2k_3 \Big)$$
$$+ \frac{1}{k_1k_2} \Big[ \bar{n}k_1 \Big( (8 - 3\,d)\, nk_1 + (16 - 7\,d)\, nk_2 \Big) - \frac{d}{2}\, \bar{n}k_3\, nk_3 - 2\, n\bar{n}\, (2\,k_1k_3 + 3\,k_2k_3) \Big]$$
$$+ \frac{k_2k_3}{k_1k_2\, k_1k_3} \Big[ n\bar{n}\, k_2k_3 + 4\,(d-4)\, \bar{n}k_1\, nk_1 - 16\, \bar{n}k_1\, nk_2 + 4\,(2-d)\, \bar{n}k_2\, nk_2 \Big] \Big\} .$$

A distinct feature of this contribution is the propagator $1/k_{123}^2$, which was absent in all other contributions. Master integrals containing this propagator are quite complex and we decided not to calculate them through a direct integration. Instead, we can use the fact that we are able to write down IBPs for integrals with Heaviside functions to construct differential equations for these master integrals. Solving differential equations, allows us then to compute the master integrals.

We begin by expressing $\omega_{n\bar{n}}^{(3),d}$ through master integrals, which can be found in Appendix D.3. We split them into two categories. The first category includes all master integrals without $1/k_{123}^2$. Such integrals can be computed following the discussion in Section 6.5.

For integrals with a $1/(k_{123}^2)$ propagator, we replace $1/(k_{123}^2)$ by $1/(k_{123}^2 + m^2)$ and derive a differential equation with respect to the mass parameter $m^2$. The resulting system of differential equations is solved numerically following the discussion in Ref. [81].

We fix the boundary condition at $m \to \infty$, since this limit leads to a simplification in the propagator $1/(k_{123}^2 + m^2)$. This makes the boundary conditions calculable, as explained in the text. However, the introduction of the mass parameter also complicates the derivation of IBPs and introduces more integrals to close the system of differential equations. In the remainder of this section we explain how to calculate $m \to \infty$ boundary conditions in Section 6.6.1. Afterwards we discuss how to solve the system of differential equations using a simple example in Section 6.6.2.

### 6.6.1 *Boundary conditions*

In the previous sections we have seen that, even though IBP reductions have to be carried out with full $\nu$ dependence, the $\nu \to 0$ limit can often be taken in a straightforward manner after the reduction. This also holds true for integrals with the additional mass parameter. For this reason we will discuss the boundary conditions for $\nu = 0$. We will explain how to take the $\nu \to 0$ limit for the integrals, where this limit is non-trivial, in Section 6.6.2.1.

As we will see, the complexity of boundary conditions strongly dependents on the number of Heaviside functions for any given integral. Thus, the discussion of boundary integrals naturally splits into three pieces. Integrals with no Heaviside functions, integrals with one Heaviside function, and finally, integrals with two Heaviside functions[2].

2 We remind the reader that there are no master integrals with three Heaviside functions

We begin by discussing the easiest case, boundary condition for integrals with zero Heaviside functions. Thus, we consider the following integral

$$I_{\delta\delta\delta}(m) = \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\delta}}{(k_{123}^2 + m^2)^i \dots}. \tag{6.6.2}$$

In Eq. (6.6.2) the ellipses stand for scalar products which are independent of the mass parameter and $i$ is an integer. In the $nnn$ configuration all $\beta_i$ variables are constrained $0 < \beta_i < 1$, because of the delta function $\delta(1 - \beta_{123})$. Furthermore there are three delta functions constraining the values of the $\alpha_i$ to $\alpha_i = \beta_i$. Finally, since the $k_i$ are emitted on-shell, it holds that $k_{i,\perp}^2 = \alpha_i \beta_i$. Hence, the integration in Eq. (6.6.2) is performed over a *finite* region of the three-particle phase space and the $m \to \infty$ limit is simply given by the Taylor expansion of the propagator $1/(k_{123}^2 + m^2)$ in $k_{123}^2/m^2$. Thus, the asymptotic mass dependence of Eq. (6.6.2) is given by

$$\lim_{m\to\infty} I_{\delta\delta\delta}(m) \sim m^{-2i}. \tag{6.6.3}$$

It is clear that a Taylor expansion of $I_{\delta\delta\delta}(m)$ integrals in $k_{123}^2/m^2$ leads to integrals where the massive propagator $1/(k_{123}^2 + m^2)$ is absent. Using IBP relations, these integrals can be further expressed through *massless* master integrals that were discussed in the previous sections. Thus no additional calculations for this class of boundary integrals are needed.

Consider now the class of boundary integrals that contain two delta functions and one Heaviside function. We choose four momenta $k_{1,2,3}$ in such a way that the Heaviside function is dependent on $k_1$ and write

$$I_{\theta\delta\delta}(m) = \int \frac{\mathrm{d}\Phi^{nnn}_{\theta\delta\delta}}{(k_{123}^2 + m^2)^i \dots}. \tag{6.6.4}$$

At variance with $I_{\delta\delta\delta}(m)$ the phase space in Eq. (6.6.4) is *not restricted*. Indeed, the $\alpha_1$ integration is only constrained by the Heaviside functions $\theta(\alpha_1 - \beta_1)$ so that $\alpha_1 \in [0, \infty)$. The integral Eq. (6.6.4) thus has two contributions in the $m \to \infty$ limit. The first one is given for $\alpha_1 \sim \beta_1 \sim 1 \ll m^2$. The calculation of this contribution is completely analogous to that of $I_{\delta\delta\delta}(m)$ and is given by a Taylor-expansion in powers of $k_{123}^2/m^2$.
On the other hand, since $\alpha_1$ is unbounded, it can be of order $m^2$, $\alpha_1 \sim m^2$. Since the phase space scales as $\alpha_1^{-\epsilon}$ and since in the $\alpha_1 \to \infty$ limit it holds that $k_{123}^2 + m^2 \sim \alpha_1(\beta_2 + \beta_3) + m^2$, we find that all integrals $I_{\theta\delta\delta}$ have the following asymptotic dependence on the mass parameter $m$ in the $m \to \infty$ limit

$$\lim_{m\to\infty} I_{\theta\delta\delta}(m) \sim m^{-2i_1 - 2\epsilon} A_2 + m^{-2i} A_1. \tag{6.6.5}$$

In Eq. (6.6.5) $i_1$ and $i$ are integers particular to the integral under consideration, $A_1$ is the Taylor-expansion contribution, and $A_2$ is the new contribution from the region where $\alpha_1 \sim m^2$. To compute $A_2$ we need to discard the $\theta(\alpha_1 - \beta_1)$ constraint, as it is only relevant for small values of $\alpha_1$, and simplify the integrand under the assumption that $\alpha_1 \sim m^2 \gg \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$. We note that in this case all integrals evaluate to hypergeometric functions.

Finally, we need to calculate boundary conditions for integrals with two Heaviside functions

$$I_{\theta\theta\delta} = \int \frac{\mathrm{d}\Phi^{nnn}_{\theta\theta\delta}}{(k_{123}^2 + m^2)^i \ldots}. \tag{6.6.6}$$

The asymptotic $m \to \infty$ limit of such an integral reads

$$\lim_{m\to\infty} I_{\theta\theta\delta}(m) \sim m^{-2i_2-4\epsilon} A_3 + m^{-2i_1-2\epsilon} A_2 + m^{-2i} A_1. \tag{6.6.7}$$

In Eq. (6.6.7) $A_1$ originates from the Taylor-expansion of the integral in powers of $k_{123}^2/m^2$, $A_2$ from the region where $\alpha_1 \sim m^2$, $\alpha_2 \sim 1$ or $\alpha_1 \sim 1$, $\alpha_2 \sim m^2$ and $A_3$ from the region where $\alpha_1 \sim \alpha_2 \sim m^2$. As before, $A_2$ is computed by removing the corresponding constraints given by Heaviside functions, where appropriate, and simplifying the integrand in the required limit, i.e. $\alpha_1 \sim m^2 \gg \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ and $\alpha_2 \sim m^2 \gg \alpha_1, \alpha_3, \beta_1, \beta_2, \beta_3$. Similarly, $A_3$ can be computed by removing both Heaviside functions and simplifying the integrand in the limit $\alpha_1 \sim \alpha_2 \sim m^2 \gg \alpha_3, \beta_1, \beta_2, \beta_3$

In the following, we illustrate the computation of all non-trivial branches by considering an explicit example. We discuss the computation of the $m^{-2\epsilon}$-branch in Section 6.6.1.1 and proceed by illustrating the computation of the $m^{-4\epsilon}$-branch in Section 6.6.1.2. Finally, we continue with the discussion of the differential equation in Section 6.6.2.

#### 6.6.1.1 $m^{-2\epsilon}$-branch

To illustrate how contributions of different regions to the boundary conditions can be computed, we consider the following integral

$$B_{1,\delta\theta\theta} = \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2 + m^2)(k_1k_3)(k_{12}n)(k_3\bar{n})}. \tag{6.6.8}$$

We would like to extract a contribution to this integral which, in the $m \to \infty$ limit, scales as $m^{-2\epsilon}$. We begin by inserting the Sudakov decomposition for all propagators except $1/(k_{123}^2 + m^2)$. We obtain

$$\begin{aligned} B_{1,\delta\theta\theta} = & \frac{2}{[4(2\pi)^{d-1}]^3} \prod_{i=1}^{3} \int \mathrm{d}\alpha_i \mathrm{d}\beta_i (\alpha_i\beta_i)^{-\epsilon} \mathrm{d}\Omega_i^{(d-2)} \frac{\delta(\alpha_1 - \beta_1)\, \theta(\alpha_2 - \beta_2)}{(k_{123}^2 + m^2)} \\ & \times \frac{\theta(\alpha_3 - \beta_3)}{(\alpha_1\beta_3 + \alpha_3\beta_1 + 2\sqrt{\alpha_1\alpha_3\beta_1\beta_3}\cos\phi_{13})\ (\beta_1 + \beta_2)\ \alpha_3}. \end{aligned} \tag{6.6.9}$$

We would like to calculate the leading $m^{-2\epsilon}$-branch contribution to this integral. To this end we consider two regions $\alpha_2 \sim m^2 \gg 1$ and $\alpha_3 \sim m^2 \gg 1$. In the first case it holds that $k_{123}^2 + m^2 \sim \alpha_2(\beta_1+\beta_3) + m^2$ and together with Eq. (6.6.9) we find that

$$\lim_{m\to\infty} B_{1,\delta\theta}\Big|_{\alpha_2 \sim m^2} \sim \int \frac{\mathrm{d}\alpha_2\, \alpha_2^{-\epsilon}}{\alpha_2(\beta_1+\beta_3)+m^2} \sim m^{-2\epsilon}. \tag{6.6.10}$$

However, in case $\alpha_3 \sim m^2 \gg 1$ it holds that $k_{123}^2 + m^2 \sim \alpha_3(\beta_1+\beta_2) + m^2$ as well as $k_3 k_1 \sim \alpha_3\beta_1 \sim m^2$. This implies

$$\lim_{m\to\infty} B_{1,\delta\theta}\Big|_{\alpha_3 \sim m^2} \sim \int \frac{\mathrm{d}\alpha_3\, \alpha_3^{-\epsilon}}{(\alpha_3(\beta_1+\beta_2)+m^2)\alpha_3\alpha_3} \sim m^{-2\epsilon-4}. \tag{6.6.11}$$

We conclude that in the limit $m \to \infty$ the leading $m^{-2\epsilon}$ behavior of the integral $B_{1,\delta\theta}$ comes from the region where $\alpha_2 \sim m^2 \gg 1$. The second region $\alpha_3 \sim m^2 \gg 1$ only contributes to the $m^{-2\epsilon}$-branch with power suppressed terms.

We now extract the leading $\mathcal{O}(m^{-2\epsilon})$ dependence of $B_{1,\delta\theta}$ by simplifying Eq. (6.6.9) in the $\alpha_2 \sim m^2 \gg 1$, $\alpha_3 \sim 1$ limit. First note that $B_{1,\delta\theta}$ only depends on $\alpha_2$ through

$$B_{1,\delta\theta} \sim \int_0^\infty \frac{\mathrm{d}\alpha_2\, \alpha_2^{-\epsilon}\theta(\alpha_2-\beta_2)}{k_{123}^2+m^2}. \tag{6.6.12}$$

Because we want to extract the contributions where $\alpha_2 \gg 1$, we split the integration region in Eq. (6.6.12) introducing an auxiliary parameter $\Lambda$. We find

$$\int_0^\infty \frac{\mathrm{d}\alpha_2\, \alpha_2^{-\epsilon}\theta(\alpha_2-\beta_2)}{k_{123}^2+m^2} = \int_0^\Lambda \frac{\mathrm{d}\alpha_2\, \alpha_2^{-\epsilon}\theta(\alpha_2-\beta_2)}{k_{123}^2+m^2} + \int_\Lambda^\infty \frac{\mathrm{d}\alpha_2\, \alpha_2^{-\epsilon}\theta(\alpha_2-\beta_2)}{k_{123}^2+m^2}. \tag{6.6.13}$$

The parameter $\Lambda$ is chosen to be $1 \ll \Lambda \ll m^2$ so that the region where $\alpha_2 \sim 1$ is confined to the first term and the region where $\alpha_2 \sim m^2$ to the second term in Eq. (6.6.13). To extract the $m^{-2\epsilon}$ contribution from the second term in Eq. (6.6.13) we replace $1/(k_{123}^2+m^2)$ by $1/(\alpha_2\beta_{13}+m^2)$ and neglect the Heaviside function, since $\Lambda \gg 1$ and $\beta_2 \in [0,1]$. We obtain

$$\begin{aligned}
\int_\Lambda^\infty \frac{\mathrm{d}\alpha_2\, \alpha_2^{-\epsilon}\theta(\alpha_2-\beta_2)}{k_{123}^2+m^2} &\approx \int_\Lambda^\infty \frac{\mathrm{d}\alpha_2\, \alpha_2^{-\epsilon}}{\alpha_2\beta_{13}+m^2} \\
&\approx \int_0^\infty \frac{\mathrm{d}\alpha_2\, \alpha_2^{-\epsilon}}{\alpha_2\beta_{13}+m^2} - \int_0^\Lambda \frac{\mathrm{d}\alpha_2\, \alpha_2^{-\epsilon}}{m^2} \\
&= m^{-2\epsilon} \int_0^\infty \frac{\mathrm{d}\alpha_2\, \alpha_2^{-\epsilon}}{\alpha_2\beta_{13}+1} - \frac{1}{m^2}\int_0^\Lambda \mathrm{d}\alpha_2\, \alpha_2^{-\epsilon}.
\end{aligned} \tag{6.6.14}$$

where we used that $\alpha_2\beta_{13} \ll m^2$ in the second term. The first term in Eq. (6.6.14) contains the complete contribution to the sought after $m^{-2\epsilon}$ branch.

The second term in Eq. (6.6.14) depends on the parameter $\Lambda$. To show that the dependence on this parameter disappears in the full integral in Eq. (6.6.13), we need to calculate the $\alpha_2 \sim 1$ contribution, i.e. the first term in Eq. (6.6.13).
Considering this term, we safely neglect $k_{123}^2$ compared to $m^2$ and obtain

$$\int_0^\Lambda \frac{d\alpha_2\, \alpha_2^{-\epsilon}\theta(\alpha_2-\beta_2)}{k_{123}^2+m^2} \approx \int_0^\Lambda \frac{d\alpha_2\, \alpha_2^{-\epsilon}\theta(\alpha_2-\beta_2)}{m^2}$$
$$\approx \frac{1}{m^2}\int_0^\infty d\alpha_2\, \alpha_2^{-\epsilon}\theta(\alpha_2-\beta_2) - \frac{1}{m^2}\int_\Lambda^\infty d\alpha_2\, \alpha_2^{-\epsilon}, \tag{6.6.15}$$

where we again neglected the Heaviside function in the second term since there $\alpha > \Lambda \gg 1 > \beta_2$. The first term in Eq. (6.6.15) is just the Taylor $m^0$-branch of $B_{1,\delta\theta\theta}$, that we already discussed. The second term in Eq. (6.6.15) now explicitly cancels again the second term in Eq. (6.6.14). Indeed, summing both terms we find

$$-\frac{1}{m^2}\int_\Lambda^\infty d\alpha_2\, \alpha_2^{-\epsilon} - \frac{1}{m^2}\int_0^\Lambda d\alpha_2\, \alpha_2^{-\epsilon} = -\frac{1}{m^2}\int_0^\infty d\alpha_2\, \alpha_2^{-\epsilon} = 0. \tag{6.6.16}$$

Finally, combining all the contributions discussed above we obtain

$$\int_0^\infty \frac{d\alpha_2\, \alpha_2^{-\epsilon}\theta(\alpha_2-\beta_2)}{k_{123}^2+m^2}\bigg|_{m^2\to\infty} = m^{-2\epsilon}\int_0^\infty \frac{d\alpha_2\, \alpha_2^{-\epsilon}}{\alpha_2\beta_{13}+1} + \frac{1}{m^2}\int_0^\infty d\alpha_2\, \alpha_2^{-\epsilon}\theta(\alpha_2-\beta_2). \tag{6.6.17}$$

Thus, as stated earlier, we obtain the $m^{-2\epsilon}$-branch by simply replacing the propagator $1/(k_{123}^2 + m^2)$ with $1/(\alpha_2\beta_{13}+m^2)$ and neglecting the Heaviside function in the full integral in Eq. (6.6.9).

We continue the computation of the $m^{-2\epsilon}$ branch by substituting Eq. (6.6.17) into Eq. (6.6.9) and neglecting the Taylor contribution. After integrating over the relative azimuthal angles of the massless partons we obtain

$$B_{1,\delta\theta\theta}\bigg|_{\alpha_2\sim m^2} = 2[N]^3 m^{-2\epsilon}\int \frac{d\beta_1 d\beta_2 d\beta_3\, \beta_1^{-2\epsilon}\beta_2^{-\epsilon}\beta_3^{-\epsilon}\delta(1-\beta_{123})\, d\alpha_3\alpha_3^{-\epsilon}}{(\beta_1+\beta_2)\beta_1\alpha_3^2}$$
$$\times\, \theta(\alpha_3-\beta_3)\ {}_2F_1\left(1, 1+\epsilon, 1-\epsilon, \frac{\beta_3}{\alpha_3}\right)\int_0^\infty \frac{d\alpha_2\, \alpha_2^{-\epsilon}}{\alpha_2\beta_{13}+1}. \tag{6.6.18}$$

The $\alpha_2$ integration is straightforward, we find

$$\int_0^\infty \frac{d\alpha_2\, \alpha_2^{-\epsilon}}{\alpha_2\beta_{13}+1} = \beta_{13}^{-1+\epsilon}\,\Gamma(\epsilon)\Gamma(1-\epsilon)\,. \tag{6.6.19}$$

After changing variables $\alpha_3 \to \xi$ where $\alpha_3 = \beta_3/\xi$ the $\xi$ integration factorizes and we obtain

$$\begin{aligned} B_{1,\delta\theta\theta}\Big|_{\alpha_2\sim m^2} &= 2[N]^3\, m^{-2\epsilon}\, \frac{\Gamma(1-\epsilon)\Gamma^2(1+\epsilon)}{\epsilon\Gamma(2+\epsilon)}\ {}_3F_2\left(1,1+\epsilon,1+\epsilon,1-\epsilon,2+\epsilon,1\right) \\ &\times \int \mathrm{d}\beta_1\mathrm{d}\beta_2\mathrm{d}\beta_3\ \delta(1-\beta_{123})\ \beta_1^{-2\epsilon-1}\beta_2^{-\epsilon}\beta_3^{-\epsilon}\beta_{12}^{-1}\beta_{13}^{\epsilon-1}\beta_3^{-\epsilon-1}. \end{aligned} \tag{6.6.20}$$

To compute the remaining integral, we remove the delta function by integrating over $\beta_2$ and change variables $\beta_1 = xy$ and $\beta_3 = x(1-y)$. The integration of $x$ and $y$ is straightforward and yields just another hypergeometric function. Finally, we obtain the result for the required branch

$$\begin{aligned} B_{1,\delta\theta\theta}\Big|_{\alpha_2\sim m^2} &= 2[N]^3\, m^{-2\epsilon}\, \frac{\Gamma^2(1-\epsilon)\Gamma^2(1+\epsilon)\Gamma(-1-3\epsilon)\Gamma^2(-2\epsilon)}{\epsilon\Gamma(2+\epsilon)\Gamma^2(-4\epsilon)} \\ &\times {}_3F_2\left(1,1+\epsilon,1+\epsilon,1-\epsilon,2+\epsilon,1\right)\ {}_3F_2\left(1,-1-3\epsilon,-2\epsilon,-4\epsilon,-4\epsilon,1\right). \end{aligned} \tag{6.6.21}$$

The calculation of the $m^{-2\epsilon}$-branch described above is representative for the computation of all other boundary conditions contributing to this branch. Indeed, all such boundary conditions can be calculated in terms of hypergeometric functions, in a straightforward fashion.

#### 6.6.1.2 $m^{-4\epsilon}$-branch

At variance with the simplicity of the previous computation, the calculation of $m^{-4\epsilon}$-branches turns out to be quite difficult. For such branches we need to consider the asymptotic limits $\alpha_1 \sim \alpha_2 \sim m^2 \gg \alpha_3, \beta_1, \beta_2, \beta_3$. In this limit the propagator $1/(k_{123}^2 + m^2)$ simplifies to $1/(2k_1 \cdot k_2 + k_3 \cdot n(k_{12} \cdot \bar{n}))$, where the remaining dependence on the scalar product $k_1 \cdot k_2$ makes a calculation difficult.

To understand how such integrals can be computed, we consider the generic integral

$$I_{\theta\theta\delta} = \int \frac{\mathrm{d}\Phi^{nnn}_{\theta\theta\delta}}{(k_{123}^2 + m^2)^i \ldots}. \tag{6.6.22}$$

Similar to the discussion of the $m^{-2\epsilon}$ branch, we obtain the $m^{-4\epsilon}$-branch by neglecting the Heaviside functions in Eq. (6.6.22) and simplifying the propagators in the limit $\alpha_1 \sim \alpha_2 \sim m^2 \gg \alpha_3, \beta_1, \beta_2, \beta_3$. We obtain

$$I_{\theta\theta\delta}\Big|_{\alpha_1\sim\alpha_2\sim m^2} = \int [\mathrm{d}k_1][\mathrm{d}k_2][\mathrm{d}k_3] \frac{\delta(1-k_{123}\cdot n)\ \delta(k_3\cdot\bar{n} - k_3\cdot n)}{2k_1\cdot k_2 + k_3\cdot n(k_{12}\cdot\bar{n})} \ldots, \tag{6.6.23}$$

where ellipses stand for other propagators that need to be simplified in the given limit. To facilitate the computation, we re-write Eq. (6.6.23) by introducing the vector $q = k_1 + k_2$. We find

$$\begin{aligned} I_{\theta\theta\delta}\Big|_{\alpha_1\sim\alpha_2\sim m^2} &= \int \mathrm{d}^d q\, [\mathrm{d}k_3] \frac{\delta(1-q\cdot n-k_3\cdot n)\delta(k_3\cdot\bar{n}-k_3\cdot n)}{(q^2+(1-q\cdot n)(q\cdot\bar{n})+m^2)^i} \\ &\times \int [\mathrm{d}k_1][\mathrm{d}k_2]\delta(q-k_1-k_2)\ldots. \end{aligned} \tag{6.6.24}$$

Therefore, to determine the $m^{-4\epsilon}$-branch of any $I_{\theta\theta\delta}$ integral, we need to compute integrals of the form

$$I_{\theta\theta\delta}\Big|_{\alpha_1\sim\alpha_2\sim m^2} = \int \frac{\mathrm{d}^d q\, \theta(1-q\cdot n)}{(q^2+(1-q\cdot n)(q\cdot\bar{n})+m^2)^i}\, F(q^2, q\cdot n, q\cdot\bar{n}), \tag{6.6.25}$$

$$\begin{aligned} F(q^2, q\cdot n, q\cdot\bar{n}) &= \int [\mathrm{d}k_3]\delta(1-q\cdot n-k_3\cdot n)\delta(k_3\cdot\bar{n}-k_3\cdot n) \\ &\times \int [\mathrm{d}k_1][\mathrm{d}k_2]\delta(q-k_1-k_2)\ldots. \end{aligned} \tag{6.6.26}$$

To simplify the computation of many required functions $F(q^2, q\cdot n, q\cdot\bar{n})$, we can derive IBP relations and use them to express any function $F$ as a sum of master integrals. Upon doings so, we find that there are five master integrals in total. They read

$$\begin{aligned} &\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5\} \\ &= \int [\mathrm{d}k_3]\delta(1-q\cdot n-k_3\cdot n)\delta(k_3\cdot\bar{n}-k_3\cdot n) \int [\mathrm{d}k_1][\mathrm{d}k_2]\delta(q-k_1-k_2) \\ &\times \left\{1, \frac{1}{k_2\cdot\bar{n}}, \frac{1}{(k_2\cdot n)(k_2\cdot\bar{n})}, \frac{1}{(k_2\cdot n)(k_1\cdot\bar{n})}, \frac{1}{1-k_1\cdot n}\right\}. \end{aligned} \tag{6.6.27}$$

Note that these integrals are fairly easy to compute, as they do not involve the scalar products of gluon four momenta. Indeed, any such scalar product is either simple, as is the case of $k_1\cdot k_2 = q^2/2$ or it gets simplified in the limit under consideration, as e.g. in $k_1\cdot k_3 \to k_3\cdot n\, k_1\cdot\bar{n}$, $k_2\cdot k_3 \to k_3\cdot n\, k_2\cdot\bar{n}$.

To compute the master integrals $\mathcal{B}_{1,..,5}$ we switch to the rest frame of $q$, where $k_1$ and $k_2$ are back-to-back. In this frame the angular integration yields

$$\begin{aligned} &\int \frac{\mathrm{d}\Omega_k^{(d-1)}}{(1-\vec{n}_k\cdot\vec{n})(1-\vec{n}_k\cdot\underline{\vec{n}})} \\ &= 2^{-2\epsilon}\Omega^{(d-2)}\frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}\, {}_2F_1\left(1,1,1-\epsilon, \frac{1+\vec{n}\cdot\underline{\vec{n}}}{2}\right), \end{aligned} \tag{6.6.28}$$

where we used $n = (1, \vec{n})$ , $\bar{n} = (1, \underline{\vec{n}})$ and $k = k_0(1, \vec{n}_k)$. We find

$$
\begin{aligned}
\mathcal{B}_1 &= \tilde{N}_\epsilon (q^2)^{-\epsilon}(1-q\cdot n)^{-2\epsilon}, \\
\mathcal{B}_2 &= \tilde{N}_\epsilon \frac{(1-2\epsilon)}{(-\epsilon)} \frac{(q^2)^{-\epsilon}(1-q\cdot n)^{-2\epsilon}}{q\cdot\bar{n}}, \\
\mathcal{B}_3 &= \tilde{N}_\epsilon \frac{(1-2\epsilon)}{(-\epsilon)} \frac{2(q^2)^{-\epsilon}(1-q\cdot n)^{-2\epsilon}}{(q\cdot n)(q\cdot\bar{n})^2}\, {}_2F_1\left(1,1,1-\epsilon,1-\frac{q^2}{(q\cdot n)(q\cdot\bar{n})}\right), \\
\mathcal{B}_4 &= \tilde{N}_\epsilon \frac{(1-2\epsilon)}{(-\epsilon)} \frac{2(q^2)^{-\epsilon}(1-q\cdot n)^{-2\epsilon}}{(q\cdot n)(q\cdot\bar{n})^2}\, {}_2F_1\left(1,1,1-\epsilon,\frac{q^2}{(q\cdot n)(q\cdot\bar{n})}\right), \\
\mathcal{B}_5 &= \tilde{N}_\epsilon q^{-2\epsilon}(1-q\cdot n)^{-2\epsilon}\, {}_2F_1\left(1,1-\epsilon,2-2\epsilon,q\cdot n\right),
\end{aligned}
\tag{6.6.29}
$$

where

$$
\tilde{N}_\epsilon = \frac{1}{(2\pi)^{d-1}} \left[\frac{\Omega^{(d-2)}}{4(2\pi)^{d-1}}\right]^2 \frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}. \tag{6.6.30}
$$

Having computed the master integrals $\mathcal{B}_{1,..,5}$ we next need to integrate the function $F(q^2, q\cdot n, q\cdot\bar{n})$ over the remaining phase space (cf. Eq. (6.6.25) ). For example, for one of the boundary conditions that we will refer to as $B_2$ this function reads

$$
\begin{aligned}
F_2(q^2, q\cdot n, q\cdot\bar{n}) = \Bigg[&\frac{4(1-2\epsilon)^2(2-q\cdot n)(1-q\cdot n)}{(1+\epsilon)m^6(q\cdot n)^2}\,\mathcal{B}_1 + \frac{4\epsilon^2}{(1+\epsilon)m^4}\mathcal{B}_2 \\
&- \frac{2(1+2\epsilon)(1-q\cdot n)q\cdot\bar{n}}{m^2q^2}\,\mathcal{B}_4\Bigg],
\end{aligned}
\tag{6.6.31}
$$

where

$$
B_2 = \int \frac{\mathrm{d}^d q\, \theta(1-q\cdot n)}{q^2 + q\cdot\bar{n}(1-q\cdot n) + m^2} F_2(q^2, q\cdot n, q\cdot\bar{n}). \tag{6.6.32}
$$

To compute the integral $B_2$ we introduce the Sudakov decomposition for the vector $q$ and write $q = \frac{1}{2}\alpha_q n + \frac{1}{2}\beta_q \bar{n} + q_\perp$. In this parameterization the phase-space measure reads

$$
\mathrm{d}^d q\, \theta(1-q\cdot n) = \frac{1}{4}\mathrm{d}\alpha_q \mathrm{d}\beta_q \mathrm{d}q_\perp^2 (q_\perp^2)^{-\epsilon} \mathrm{d}\Omega^{(d-2)}\, \theta(1-\beta_q). \tag{6.6.33}
$$

In addition we require the four-vector $q$ to be time-like

$$
q^2 = 2k_1\cdot k_2 = 2|\vec{k}_1||\vec{k}_2|[1-\cos(\phi_{12})] > 0. \tag{6.6.34}
$$

This implies that

$$
q^2 = \alpha_q\beta_q - q_\perp^2 > 0. \tag{6.6.35}
$$

Thus the integration boundaries can be chosen to be $0 < \beta_q < 1$, $0 < \alpha_q < \infty$ and $0 < q_\perp^2 < \alpha_q\beta_q$. To simplify the mass-dependent denominator in Eq. (6.6.32) we perform the variable transformation $q_\perp^2 \to t$, where $t = \alpha_q - q_\perp^2$ and obtain.

$$q^2 + q\cdot\bar{n}(1 - q\cdot n) + m^2 = \alpha_q - q_\perp^2 + m^2 = t + m^2. \tag{6.6.36}$$

Transforming the integration measure and boundaries of Eq. (6.6.33) we find

$$\mathrm{d}^d dq\theta(1 - q\cdot n) = \frac{1}{4}\mathrm{d}\alpha_q \mathrm{d}\beta_q \mathrm{d}t(\alpha_q - t)^{-\epsilon}\mathrm{d}\Omega^{(d-2)}\theta(1-\beta_q), \tag{6.6.37}$$

where the integration boundaries are $0 < \beta_q < 1$, $0 < \alpha_q < \infty$ and $\alpha_q(1-\beta_q) < t < \alpha_q$.

However, we would like to choose a parameterization of this integration volume, such that the $t$ integration is not bounded by the other integration variables. In this case we find $0 < t < \infty$ , $t < \alpha_q < \infty$ and $1 - t/\alpha_q < \beta_q < 1$. We perform a last variable transform $\beta_q \to \beta_q'$ with $\beta_q' = 1 - \beta_q$ such that $0 < \beta_q' < t/\alpha_q$.

We proceed by substituting Eqs. (6.6.31), (6.6.36) and (6.6.37) into Eq. (6.6.32) and integrating over $\mathrm{d}\Omega^{(d-2)}$. We obtain

$$\begin{aligned} B_2 = \frac{\Omega^{(d-2)}}{4}\int \frac{\mathrm{d}\alpha_q \mathrm{d}\beta_q' \mathrm{d}t(\alpha_q - t)^{-\epsilon}}{t+m^2}\Bigg[\frac{4(1-2\epsilon)^2(1+\beta_q')\beta_q'}{(1+\epsilon)m^6(1-\beta_q')^2}\,\mathcal{B}_1 \\ + \frac{4\epsilon^2}{(1+\epsilon)m^4}\mathcal{B}_2 - \frac{2(1+2\epsilon)\beta_q'\alpha_q}{m^2(t-\alpha_q\beta_q')}\,\mathcal{B}_4\Bigg]. \end{aligned} \tag{6.6.38}$$

To integrate further we change variables $\alpha_q \to \xi$ with $\alpha_q = t/\xi$ and $0 < \xi < 1$, and thus $0 < \beta_q' < \xi$. Then, we change variables one more time $\beta_q' \to r$ with $\beta_q' = r\xi$, such that $r \in [0,1]$.

Upon changing variables and using explicit expressions for integrals $\mathcal{B}_{1,2,4}$, we note that integration over $t$ factorizes and can be performed easily. We obtain

$$\begin{aligned} B_2 &= -\frac{m^{-4-4\epsilon}[N]^3\Gamma^2(1-\epsilon)\Gamma(1+2\epsilon)}{\epsilon^2}\int_0^1 \mathrm{d}\xi\ \mathrm{d}r\ W_2(\xi,r), \\ W_2(\xi,r) &= \Bigg\{2\epsilon\frac{\left(r(1+r\xi)+\epsilon\left(1-2r(1+\xi)-2r^2\xi(1-\xi/2)\right)\right)}{(1+\epsilon)(1-r)^\epsilon r^{2\epsilon}(1-\xi)^\epsilon\xi^\epsilon(1-r\xi)^2} \\ &+ 2(2\epsilon+1)\frac{r^{1-2\epsilon}\xi^{1-\epsilon}(1-r\xi)^\epsilon}{(1-r)^{1+\epsilon}(1-\xi)^{1+2\epsilon}}\ {}_2F_1\left(-\epsilon,-\epsilon,1-\epsilon,\frac{(1-r)\xi}{1-r\xi}\right)\Bigg\}. \end{aligned} \tag{6.6.39}$$

After integrating over $\xi$ and $r$, we find

$$\begin{aligned} B_2 =& m^{-4\epsilon-4}[N]^3\Bigg(-\frac{3}{2\epsilon^4} - \frac{6}{\epsilon^3} - \frac{12}{\epsilon^2} + \frac{15\zeta_3 - 36}{\epsilon} + 60\zeta_3 + \frac{\pi^4}{4} - 84 + \epsilon\,(120\zeta_3 \\ &+81\zeta_5 + \pi^4 - 204\Big) + \epsilon^2\left(-75\zeta_3^2 + 360\zeta_3 + 324\zeta_5 + \frac{11\pi^6}{63} + 2\pi^4 - 468\right)\Bigg). \end{aligned} \tag{6.6.40}$$

Calculations of all other $m^{-4\epsilon}$-branches required for boundary conditions can be performed similarly. This concludes the discussion of boundary conditions. We explain how to solve the differential equation in the next section.

### 6.6.2 *Numerical solution of the differential equation*

The system of differential equations that we need to solve involves 265 integrals. For illustration, we discuss a subset spanned by only eleven integrals. For a large subset of the 265 integrals and for the subset under consideration, the $\nu \to 0$ limit can be smoothly taken and we set $\nu = 0$ in the following discussion. We will discuss the $\nu \to 0$ limit for integrals that require it, afterwards.

We consider the subset of the following eleven integrals

$$
\begin{aligned}
\mathcal{J}_1 &= \int \mathrm{d}\Phi^{nnn}_{\delta\delta\delta}, & \mathcal{J}_2 &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\delta}}{k^2_{123}+m^2}, \\
\mathcal{J}_3 &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\delta}}{(k^2_{123}+m^2)k_1\cdot n}, & \mathcal{J}_4 &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\theta}}{k^2_{123}+m^2}, \\
\mathcal{J}_5 &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\theta}}{(k^2_{123}+m)^2}, & \mathcal{J}_6 &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta'\theta}}{k^2_{123}+m^2}, \\
\mathcal{J}_7 &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\theta}}{(k^2_{123}+m^2)(k_1 n)}, & \mathcal{J}_8 &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\theta}}{(k^2_{123}+m^2)(k_3 \bar{n})}, \\
\mathcal{J}_9 &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\theta\theta}}{k^2_{123}+m^2}, & \mathcal{J}_{10} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\theta\theta}}{(k^2_{123}+m^2)(k_2 \bar{n})}, \\
\mathcal{J}_{11} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta'\theta\theta}}{(k^2_{123}+m^2)(k_2 \bar{n})}.
\end{aligned}
\tag{6.6.41}
$$

We note that in Eq. (6.6.41) we defined the new shorthand notation $\delta'$ for derivatives of delta functions. They read $\mathrm{d}\delta(k_2\bar{n}-k_2 n)/\mathrm{d}(k_2 n)$ and $\mathrm{d}\delta(1-k_{123}n)/\,\mathrm{d}(k_{123}n)$ in the equations for $\mathcal{J}_6$ and $\mathcal{J}_{11}$ respectively. Calculating these derivatives is straightforward after writing the delta functions as a sum of two propagators with the help of reverse unitarity.

We now derive a differential equation for the vector $\boldsymbol{\mathcal{J}}$ consisting of the eleven master integrals in Eq. (6.6.41). To this end, we differentiate $\boldsymbol{\mathcal{J}}$ w.r.t. $m^2$ and reduce the resulting integrals back to master integrals in Eq. (6.6.41). The differential equation can be written as a matrix equation

$$
\frac{\partial}{\partial m^2}\boldsymbol{\mathcal{J}} = \left[\frac{\boldsymbol{M}_1}{m^2+1} + \frac{\boldsymbol{M}_2}{m^2+\frac{1}{4}} + \frac{\boldsymbol{M}_3}{m^2} + \boldsymbol{M}_4 + m^2\boldsymbol{M}_5\right]\boldsymbol{\mathcal{J}}. \tag{6.6.42}
$$

The matrices $\boldsymbol{M}_{1,..,5}$ are independent of $m^2$ but depend on $\epsilon$. They can be found in Appendix D.7. We note that the matrices $M_{1,\ldots,5}$, boundary conditions and solutions to differential equation Eq. (6.6.42) are also given in digital form in the ancillary file of Ref.[77].

We have already explained how to calculate the boundary conditions for the integrals $\mathcal{J}_{1,\ldots,11}$ in the previous sections. A list of all boundary conditions required for the N3LO soft function can be found in Appendix D.4. We solve the differential equation by expanding all master integrals at $m^2 \to \infty$ in a power logarithmic series in $y = 1/m^2$

$$\mathcal{J} = \sum C_\infty(n_1, n_2, n_3) y^{n_1+n_2\epsilon} \ln^{n_3} y. \tag{6.6.43}$$

The coefficients $C_\infty(n_1, n_2, n_3)$ are series expansions in $\epsilon$ and are fixed with the help of the boundary conditions and the differential equation Eq. (6.6.42). We use these series solutions to evaluate integrals in the upper complex half-plane within the radius of convergence. The radius of convergence is determined from the singularities of the differential equation. According to Eq. (6.6.42) they are located at $m^2 = 0, -1/4, -1$. We keep constructing and matching series solutions within the radius of convergence until we reach the physical point $m^2 = 0$.

Suppose we want to find the solution of $\mathcal{J}$ at the point $m^2 = m_0^2 = 1/y_0$ which is still within the radius of convergence of Eq. (6.6.43). We construct another series solution at this point

$$\mathcal{J} = \sum C_{y_0}(n_1)(y - y_0)^{n_1}, \tag{6.6.44}$$

and fix the coefficient $C_{y_0}(0)$ by matching Eq. (6.6.44) with Eq. (6.6.41) at $y = y_0$. The remaining coefficients $C_{y_0}(n_1)$ with $n_1 \neq 0$ are then determined through the expansion of the differential equation around $y = y_0$. The series solution in Eq. (6.6.44) now has its own radius of convergence which allows us to move past the original radius of the series in Eq. (6.6.41). We repeat this process until we reach the vicinity of the physical point $m^2 = 0$.

For the differential equation Eq. (6.6.42) the poles and matching points are shown in Fig. 6.4. Note, that for the example differential equation Eq. (6.6.42), there are no poles on the real axis, and we can simply move along it while never leaving the radius of convergence of the first matching point at $m^2 = 2$. For the complete set of 265 integrals we will also encounter the situation where there are poles on the real axis. In this case one can move around poles by going into the complex plane where necessary. However, it is simpler to exploit the fact that the power logarithmic in Eq. (6.6.43) are the same in both limits $m \to \infty$ and $m \to i\infty$ and move along the positive imaginary axis instead.

Once we arrive in the vicinity of the physical point $m^2 = 0$, we construct another power-logarithmic series

$$\mathcal{J} = \sum C_0(n_1, n_2, n_3)(m^2)^{n_1+n_2\epsilon} \ln^{n_3} m^2. \tag{6.6.45}$$

We again fix the coefficients $C_0(n_1, n_2, n_3)$ by matching Eq. (6.6.45) to Eq. (6.6.44). The physical solution is now obtained by taking the $m^2 \to 0$ limit while keeping $\epsilon$ fixed in Eq. (6.6.45). This is only possible if the coefficients $C_0(n_1, 0, n_3)$ with $n_1 < 0$ and $n_1 = 0, n_3 > 0$ vanish such that there are no $1/m^2$ and $\ln m^2$ terms that are not multiplied by powers of $m^{2\epsilon}$. This condition is fulfilled for our eleven example master integrals and indeed all other integrals required for the computation of $\omega_{n\bar{n}}^{(3),d}$. In this case, the physical limit is simply retained by extracting the coefficient $C_0(0, 0, 0)$.

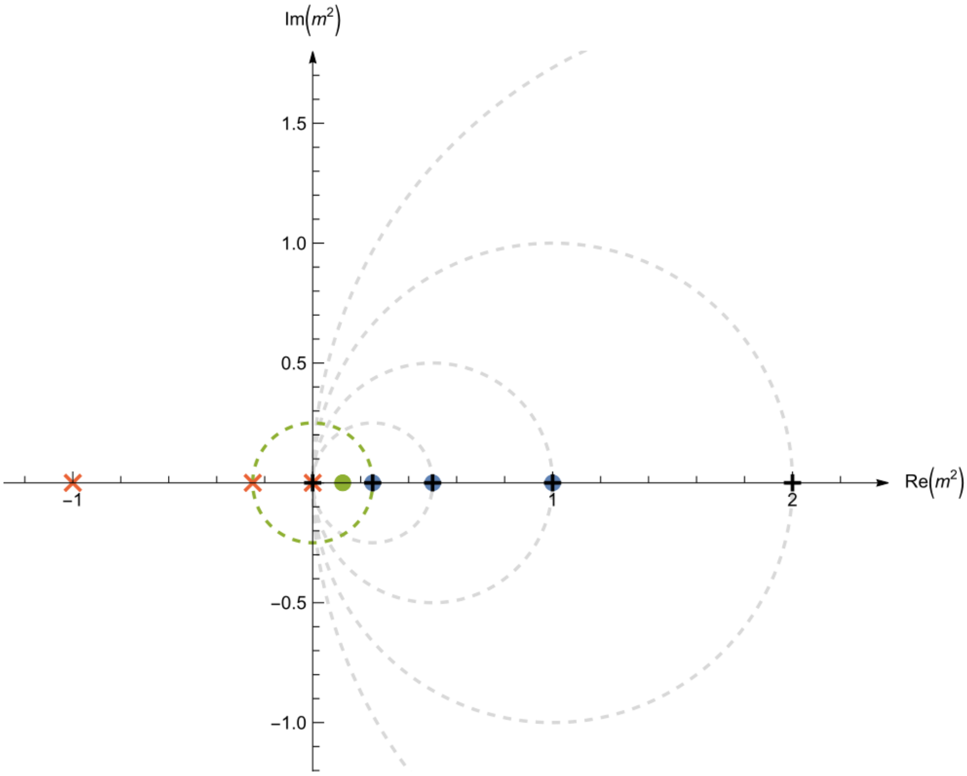


**Figure 6.4:** Poles and matching points for the example differential equation. Red crosses indicate poles, blue dots indicate matching points, black pluses indicate expansion points and the grey circles around them are the corresponding convergence radii. For the physical region around $m^2 = 0$ the color of the matching point and convergence radius is changed to green.

While the procedure outlined above only gives a numerical result, we stress that these solutions can be obtained to *arbitrary* precision. To illustrate this point we present the solution to two of the integrals in Eq. (6.6.41) at the physical point $m^2 \to 0$ computed up to at least 15 significant digits through weight six. They read

$$\begin{aligned}\mathcal{J}_9|_{m^2=0} = \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\theta\theta}}{k^2_{123}} &= -\frac{0.5}{\epsilon} - 5 - 28.17026373260709\,\epsilon - 119.43143332972728\,\epsilon^2 \\ &- 430.4404286909044\,\epsilon^3 - 1410.1679482808422\,\epsilon^4 \\ &- 4372.111524529197\,\epsilon^5 - 13148.701437210732\,\epsilon^6 + \mathcal{O}(\,\epsilon^7), \qquad (6.6.46)\end{aligned}$$

$$\begin{aligned}\mathcal{J}_{10}|_{m^2=0} = \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\theta\theta}}{(k^2_{123})(k_2\bar{n})} &= \frac{0.5}{\varepsilon^2} + \frac{5.5}{\varepsilon} + 34.34926305180546 + 170.0583525098628\,\epsilon \\ &+ 758.7443815516605\,\epsilon^2 + 3238.222100561864\,\epsilon^3 \\ &+ 13535.346184323936\,\epsilon^4 + \mathcal{O}(\,\epsilon^5). \qquad (6.6.47)\end{aligned}$$

We note, that we reconstructed analytic results for all master integrals required for $\omega^{(3),d}$ using the PSLQ[82] and LLL [83] algorithms together with a basis of transcendental constants. We have computed the master integrals to more than two thousand digits to check the validity of the analytic results obtained in this way.

#### 6.6.2.1 *The $\nu \to 0$ limit in the differential equations*

Finally, we comment on the $\nu \to 0$ limit of the DEQ. We first note that the differential equation contains additional integrals $\mathcal{J}_{\nu,\text{div}}$ which are divergent in the $\nu \to 0$ limit and contain a $k_{123}^2$ propagator. However, it is sufficient to simply change the basis of the DEQ $\{\mathcal{J}_{\nu,\text{div}}, \dots\} \to \{\nu\mathcal{J}_{\nu,\text{div}}, \dots\}$ to obtain a system that is free of $1/\nu$ poles where the $\nu \to 0$ limit can be taken smoothly. Although not straightforward, we can determine which integrals are singular in the $\nu \to 0$ limit by studying the differential equations and boundary conditions together with numerical evaluations of the integrals at finite values of $m^2$. For the numerical evaluations we employed Mellin-Barnes representations of relevant integrals as well as the programs `MB` [84] and `MBresolve` [85]. We also used the program `pySecDec` [86, 87] for an alternative numeric check. We find that there is only one additional master integral which is singular in the $\nu \to 0$ limit. It reads

$$J_2 = \int \frac{\mathrm{d}\Phi_{\theta\delta\theta}(k_1 \cdot n)^\nu \ (k_2 \cdot n)^\nu (k_3 \cdot n)^\nu}{(k_{123}^2 + m^2)(k_1 \cdot k_3)(k_1 \cdot n)(k_{12} \cdot \bar{n})}. \tag{6.6.48}$$

After determining all divergent integrals and multiplying them with a factor of $\nu$, it is beneficial to take the $\nu \to 0$ limit because the number of integrals that span the system of differential equations shrinks from 265 to 173 in this limit. However this step is not straightforward. Indeed, after solving the system of differential equations in $m^2$ the correct order of limits would be $m^2 \to 0$, $\nu \to 0$ and finally $\epsilon \to 0$. Exchanging the $m^2$ and the $\nu$ limit we may face two different problems.

First, since the parameter $m^2$ may also serve as a regulator of infra-red divergences the $\nu$ and $m^2$ limits might not commute, i.e. taking the $\nu \to 0$ limit first, the $m^2 \to 0$ limit might be divergent. Second, additional terms may mix into the Taylor $m^0$-branch at $m^2 \to 0$ which we identified as the sought after solution of the DEQ (cf. Eq. (6.6.45)).

We already addressed the first problem in the previous section by checking that the $m^2 \to 0$ limit in Eq. (6.6.45) can be taken for all integrals while keeping $\epsilon$ fixed. The second problem can be understood by studying Eq. (6.6.45) in its generalized form for $\nu \neq 0$. It reads

$$\mathcal{J} \sim \sum_{n_1,n_2,n_3,n_4} c_{n_1 n_2 n_3 n_4} (m^2)^{n_1 + n_2\epsilon + n_3\nu} \ln^{n_4}(m^2). \tag{6.6.49}$$

Taking the correct sequence of limits $m \to 0$, $\nu \to 0$, $\epsilon \to 0$ we would identify the solution of the DEQ with the coefficient in front of $m^0$, $c_{0000}$. However, if we take the $\nu \to 0$ limit first the $m^{n_3\nu}$-branches will start mixing with the $m^0$-branch, that is $m^{n_3\nu} \to m^0$ in the $\nu \to 0$ limit. If we now again identify the physical solution with the coefficient of $m^0$ we would obtain $c_{0000} + \sum_{n_3=1} c_{00n_30}$ and thus a wrong solution if $c_{00n_30} \neq 0$.

Studying the differential equation for small values of $m^2$ but with full $\nu$ and $\epsilon$ dependence we find that solutions of the form $m^{n_3\nu}$ without additional powers of $\epsilon$ or $m^2$ are not possible.

We can therefore, safely take the $\nu \to 0$ limit without any mixing of additional contributions with the $m^0$-branch.

While this mixing with the $m^0$-branch does not appear in the $m^2 \to 0$ limit, it actually appears at $m^2 \to \infty$ and needs to be accounted for when calculating boundary conditions. Indeed, the integral in $J_2$ in Eq. (6.6.48) is not only peculiar because it is the only master integral with a $k_{123}^2$ propagator and a $1/\nu$ pole, but also because it provides a contribution in the $m^2 \to \infty$ limit proportional to $m^{-2\nu}/\nu$. After multiplying this integral with $\nu$ and taking the $\nu \to 0$ limit this additional contribution will mix with the $m^0$-branch.

This additonal $m^{-2\nu}$-branch does not originate from any of the asymptotic limits discussed in Section 6.6.1, but from a *fourth* region where $\alpha_3 \sim m^2$, $\alpha_1 \sim 1$ and $\beta_1 \sim m^{-2}$. However, we stress that this integral is the *only one* which requires the computation of this additional contribution. For all other integrals the *three regions* discussed in Section 6.6.1 *are sufficient*.

We proceed by extracting the $m^{-2\nu}$ branch from $J_2$. To this end, we first simplify the integrand in Eq. (6.6.48) in the limit $\alpha_3 \sim m^2 \gg \alpha_1 \sim 1 \gg \beta_1 \sim m^{-2}$. We obtain

$$J_2\Bigg|_{\alpha_3\sim m^2,\beta_1\sim m^{-2}} = \widetilde{J}_2 = \int \frac{\mathrm{d}\Phi_{\theta\delta\theta}(k_1\cdot n)^\nu\ (k_2\cdot n)^\nu (k_3\cdot n)^\nu}{((k_3\cdot\bar{n})\ (k_2\cdot n)+m^2)(k_1\cdot k_3)(k_1\cdot n)(k_{12}\cdot\bar{n})}\,. \tag{6.6.50}$$

Expanding in this asymptotic limit we further need to remove both Heaviside functions, neglect the $\beta_1$ contribution to the "zero-jettiness" delta function and extend the $\beta_1$ integration to infinity. We rescale $\alpha_3 \sim m^2, \beta_1 \sim m^{-2}$, and find

$$\begin{aligned}\widetilde{J}_2 = 2[N]^3 m^{-2\nu} &\int_0^\infty \mathrm{d}\beta_1 \int \mathrm{d}\beta_2 \mathrm{d}\beta_3\ \mathrm{d}\alpha_1 \mathrm{d}\alpha_3\ \beta_1^{-\epsilon+\nu}\beta_2^{-2\epsilon+\nu}\beta_3^{-\epsilon+\nu}\alpha_1^{-\epsilon}\alpha_3^{-\epsilon}\\ &\times \frac{\delta(1-\beta_{23})}{(\alpha_3\beta_2+1)\beta_1(\alpha_1+\beta_2)}\\ &\times\left[\frac{\theta(\beta_1/\alpha_1-\beta_3/\alpha_3)}{\beta_1\alpha_3}{}_2F_1\left(1,1+\epsilon;1-\epsilon;\frac{\alpha_1\beta_3}{\alpha_3\beta_1}\right)\right.\\ &\left.+\frac{\theta(\beta_3/\alpha_3-\beta_1/\alpha_1)}{\beta_3\alpha_1}{}_2F_1\left(1,1+\epsilon;1-\epsilon;\frac{\alpha_3\beta_1}{\alpha_1\beta_3}\right)\right].\end{aligned} \tag{6.6.51}$$

Changing integration variables to $\alpha_1 = \beta_1/\xi_1$ and $\alpha_3 = \beta_3/\xi_3$ we obtain

$$\begin{aligned}\widetilde{J}_2^\nu = 2[N]^3 m^{-2\nu} &\int \mathrm{d}\beta_2 \mathrm{d}\beta_3 \mathrm{d}\xi_1 \mathrm{d}\xi_3 \beta_2^{-2\epsilon+\nu}\beta_3^{-2\epsilon+\nu}\xi_1^{\epsilon-1}\xi_3^{\epsilon-1}\frac{\delta(1-\beta_{23})}{\beta_3\beta_2+\xi_3}\\ &\times\left(\xi_3\theta(\xi_1-\xi_3)\,{}_2F_1\left(1,1+\epsilon,1-\epsilon,\frac{\xi_3}{\xi_1}\right)\right.\\ &\left.+\xi_1\theta(\xi_3-\xi_1)\,{}_2F_1\left(1,1+\epsilon,1-\epsilon,\frac{\xi_1}{\xi_3}\right)\right)\int_0^\infty \mathrm{d}\beta_1\frac{\beta_1^{-2\epsilon+\nu-1}}{\beta_1+\beta_2\xi_1}\,.\end{aligned} \tag{6.6.52}$$

Integrating over $\beta_1$ we arrive at

$$\int_0^\infty d\beta_1 \frac{\beta_1^{-2\epsilon+\nu-1}}{\beta_1+\beta_2\xi_1} = (\beta_2\xi_1)^{-2\epsilon+\nu-1}\Gamma(-2\epsilon+\nu)\Gamma(2\epsilon-\nu+1). \quad (6.6.53)$$

We use this result in Eq. (6.6.52), change variables $\xi_1 = r\xi_3$, and obtain

$$\widetilde{J}_2 = 2[N]^3 m^{-2\nu}\Gamma(-2\epsilon+\nu)\Gamma(2\epsilon-\nu+1)\int d\beta_2 d\beta_3 \beta_2^{-4\epsilon+2\nu-1}\beta_3^{-2\epsilon+\nu}\delta(1-\beta_{23}) \quad (6.6.54)$$
$$\times \int_0^\infty \frac{d\xi_3 \xi_3^{\nu-1}}{\beta_3\beta_2+\xi_3} \int_0^1 dr \left[r^{\epsilon-\nu}+r^{-\epsilon+\nu-1}\right] {}_2F_1\left(1,1+\epsilon,1-\epsilon,r\right) .$$

It follows that $\widetilde{J}_2$ has a $1/\nu$ pole, which originates from the point $\xi_3 = 0$ in Eq. (6.6.54). As discussed earlier, we only need to extract this $1/\nu$ pole. We find

$$\int_0^\infty d\xi_3 \frac{\xi_3^{\nu-1}}{\beta_3\beta_2+1\xi_3} = \frac{1}{\nu}(\beta_3\beta_2)^{-1} + \mathcal{O}(\nu^0) . \quad (6.6.55)$$

Upon further integration, we obtain the $1/\nu$ pole of $\widetilde{J}_2$. It reads

$$\widetilde{J}_2 = \frac{C_2}{\nu}m^{-2\nu} + \mathcal{O}(\nu^0), \quad (6.6.56)$$

where

$$C_2 = [N]^3\frac{2\Gamma^2(-2\epsilon)\Gamma(-4\epsilon-1)\Gamma(2\epsilon+1)}{\Gamma(-6\epsilon-1)}\left(\frac{{}_3F_2\left(1,1+\epsilon,1+\epsilon,1-\epsilon,2+\epsilon,1\right)}{1+\epsilon}\right.$$
$$\left. - \frac{{}_3F_2\left(1,-\epsilon,1+\epsilon,1-\epsilon,1-\epsilon,1\right)}{\epsilon}\right). \quad (6.6.57)$$

Adding the additional contribution $C_2/\nu$ in Eq. (6.6.57) to the $m^0$-branch of $J_2$ is the last particularity that arises when taking the $\nu \to 0$ limit.

Combining the above discussion with the discussion of numerical evaluations of the system of DEQs, we compute all 48 master integrals, that have a $1/k_{123}^2$ propagator, numerically. We find analytic solutions for all of them using the PSLQ and LLL algorithms, and choosing appropriate basis of transcendental numbers [88]. We present the result for the $\omega^{(3),d}$ contribution in the next section.

### 6.6.3 *Results*

Checking all 48 master integrals containing the $1/k_{123}^2$ propagator with the help of IBP relations derived in the $m = 0$ case, we find that all master integrals fulfill the relations to their full precision. In addition, all master integrals that do not contain the $1/k_{123}^2$ propagator have been numerically checked by constructing Mellin-Barnes representations and the use of public programs `MB` [84] and `MBresolve` [85]. We also used the program `pySecDec` [86, 87] for an alternative numeric check. Using these programs, we found good agreement between the two up to the default precision of the numerical programs that, in practice, can vary between 3 and 10 digits. For master integrals that contain the $1/k_{123}^2$ propagator we were not able to derive any sensible numeric results using the programs mentioned above.

Combining the reduction with the master integrals, listed in Appendix D.3, we obtain

$$
\begin{aligned}
&\int \mathrm{d}\Phi_{\theta\theta\theta}^{nnn}\, \omega_{n\bar{n}}^{(3),d}(k_1,k_2,k_3) = [N]^3 \Bigg\{ \frac{12}{\epsilon^5} + \frac{142}{3\epsilon^4} + \frac{1}{\epsilon^3}\left(\frac{46\pi^2}{3} + \frac{628}{3}\right) \\
&+ \frac{1}{\epsilon^2}\left(196\zeta_3 + \frac{650\pi^2}{9} + \frac{18161}{27}\right) + \frac{1}{\epsilon}\left(\frac{397\pi^4}{45} + 1380\zeta_3 + \frac{6808\pi^2}{27} + \frac{165323}{81}\right) \\
&+ \Bigg(8982\zeta_5 - \frac{2146\zeta_3\pi^2}{3} + \frac{191\pi^4}{9} + 4224\,\mathrm{Li}_4\left(\frac{1}{2}\right) + 3696\zeta_3\ln(2) - 176\pi^2\ln^2(2) \\
&+176\ln^4(2) + \frac{46184\zeta_3}{9} + \frac{66614\pi^2}{81} + 96\ln(2) + \frac{413971}{81}\Bigg) \\
&+ \epsilon\Bigg(2304\,\zeta_{-5,-1} - 4464\zeta_5\ln(2) - 8380\zeta_3^2 + \frac{46934\pi^6}{2835} - 6336G_R(0,0,r_2,1,-1) \\
&\quad - 6336G_R(0,0,1,r_2,-1) - 3168G_R(0,0,1,r_2,r_4) - 6336G_R(0,0,r_2,-1)\ln(2) \\
&\quad + \frac{324215\zeta_5}{3} - 45056\,\mathrm{Li}_5\left(\frac{1}{2}\right) - 45056\,\mathrm{Li}_4\left(\frac{1}{2}\right)\ln(2) + 176\,\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\pi \\
&\quad - 1056\zeta_3\,\mathrm{Li}_2\left(\frac{1}{4}\right) - \frac{9634\zeta_3\pi^2}{3} - 21824\zeta_3\ln^2(2) + 2112\zeta_3\ln(2)\ln(3) \\
&\quad - 1584\,\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\ln(3) - \frac{4400\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\pi^3}{27} + \frac{88\pi^4\ln(2)}{45} - \frac{616\pi^4\ln(3)}{27} \\
&\quad + \frac{11264\pi^2\ln^3(2)}{9} - \frac{22528\ln^5(2)}{15} + 8576\,\mathrm{Li}_4\left(\frac{1}{2}\right) + 7504\zeta_3\ln(2) + \frac{4646\pi^4}{27} \\
&\quad - \frac{1072\pi^2\ln^2(2)}{3} + \frac{1072\ln^4(2)}{3} + \frac{496592\zeta_3}{27} - 32\pi^2\ln(2) + \frac{587380\pi^2}{243} \\
&\quad - 384\ln^2(2) + 832\ln(2) + \frac{7857076}{729} + \sqrt{3}\Bigg(192\,\mathrm{Im}\left\{\mathrm{Li}_3\left(\frac{\exp(i\pi/3)}{2}\right)\right\} \\
&\quad +160\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln(2) - 16\pi\ln^2(2) - \frac{560\pi^3}{81}\Bigg)\Bigg)\Bigg\} + \mathcal{O}(\epsilon^2)\,,
\end{aligned}
\tag{6.6.58}
$$

where $\zeta_{-5,-1} \approx -0.029902$ is a multiple zeta value, and $\mathrm{Cl}_n(x)$ are Clausen functions. $G_R(a_1,\ldots,a_w)$ is the real part of the multiple polylogarithm $G(a_1,\ldots,a_w;z)$ evaluated at $z=1$ [88]

$$
G_R(a_1,\ldots,a_w) = \mathrm{Re}\{G(a_1,\ldots,a_w;1)\}\,. \tag{6.6.59}
$$

Finally, $r_2 = \exp(-i\pi/3)$ and $r_4 = \exp(-i2\pi/3)$. We note that we have computed the master integrals to more than two thousand digits to check the validity of the analytic result.
We present the result for the same-hemisphere triple gluon zero-jettiness soft function in the next section.

## 6.7 RESULTS AND CONCLUSION

We are in a position to present the result for the same-hemisphere *three-gluon*-emission contribution to the N3LO soft function. To this end, we write

$$\begin{aligned} S_{ggg}^{nnn} &= \int \mathrm{d}\Phi_{\theta\theta\theta}^{nnn} \, |J(k_1,k_2,k_3)|^2 \\ &= \tau^{-1-6\epsilon} \frac{1}{3!} \left[ C_a^3 S_{ggg}^{nnn,1} + C_a^2 C_A S_{ggg}^{nnn,2} + C_a C_A^2 S_{ggg}^{nnn,3} \right], \end{aligned} \tag{6.7.1}$$

where we re-introduced the dependence on $\tau$ and $C_a = C_F(C_A)$ if the emitter is a quark(gluon)-pair respectively. The Abelian contributions $S_{ggg}^{nnn,1}$ and $S_{ggg}^{nnn,2}$ are obtained by convoluting NLO and NNLO results. We compute them explicitly in Appendix D.5. $S_{ggg}^{nnn,3}$ is obtained by summing the contributions $\omega^{(3),i}$, $i \in \{a,b,c,d\}$ calculated in the previous sections.
Combining all these results, we find

$$\frac{S_{ggg}^{nnn,1}}{[N]^3} = \frac{48\,\Gamma^3(1-2\epsilon)}{\epsilon^5 \Gamma(1-6\epsilon)}, \tag{6.7.2}$$

$$\begin{aligned} \frac{S_{ggg}^{nnn,2}}{[N]^3} &= -\frac{9\,\Gamma(1-4\epsilon)\,\Gamma(1-2\epsilon)}{\epsilon^2 \Gamma(1-6\epsilon)} \times \left[ \frac{8}{\epsilon^3} + \frac{44}{3\epsilon^2} + \frac{1}{\epsilon}\left(\frac{268}{9} - 8\zeta_2\right) \right. \\ &+ \left(\frac{1544}{27} + \frac{88}{3}\zeta_2 - 72\zeta_3\right) + \epsilon\left(\frac{9568}{81} + \frac{536\zeta_2}{9} + \frac{352}{3}\zeta_3 - 300\zeta_4\right) \\ &+ \epsilon^2\left(\frac{55424}{243} + \frac{3520\zeta_2}{27} + \frac{2144\zeta_3}{9} + 352\zeta_4 + 96\zeta_2\zeta_3 - 1208\zeta_5\right) \\ &+ \epsilon^3\left(\frac{297472}{729} + \frac{22592\zeta_2}{81} + \frac{14080\zeta_3}{27} + \frac{2144}{3}\zeta_4 - \frac{4576}{3}\zeta_2\zeta_3 + 3696\zeta_5 \right. \\ &\left.\left. + 424\zeta_3^2 - 3596\zeta_6\right) + \mathcal{O}\left(\epsilon^4\right) \right], \end{aligned} \tag{6.7.3}$$

$$\begin{aligned} \frac{S_{ggg}^{nnn,3}}{[N]^3} &= \frac{24}{\epsilon^5} + \frac{308}{3\epsilon^4} + \frac{1}{\epsilon^3}\left(-12\pi^2 + \frac{3380}{9}\right) + \frac{1}{\epsilon^2}\left(-1000\zeta_3 + \frac{440\pi^2}{9} + \frac{10048}{9}\right) \\ &+ \frac{1}{\epsilon}\left(-\frac{2377\pi^4}{45} + \frac{440\zeta_3}{3} + \frac{7192\pi^2}{27} + \frac{253252}{81}\right) \\ &+ \left(-28064\zeta_5 + \frac{1972\zeta_3\pi^2}{3} - \frac{638\pi^4}{15} + 4224\,\mathrm{Li}_4\left(\frac{1}{2}\right) + 3696\zeta_3\ln(2) \right. \\ &\left. - 176\pi^2\ln^2(2) + 176\ln^4(2) + \frac{13208\zeta_3}{3} + \frac{78848\pi^2}{81} + 96\ln(2) + \frac{1925074}{243}\right) \\ &+ \epsilon\left(2304\ \zeta_{-5,-1} - 4464\zeta_5\ln(2) + 25784\zeta_3^2 - \frac{67351\pi^6}{567} \right. \\ &- 6336G_R(0,0,r_2,1,-1) - 6336G_R(0,0,1,r_2,-1) - 3168G_R(0,0,1,r_2,r_4) \\ &- 6336G_R(0,0,r_2,-1)\ln(2) + \frac{268895\zeta_5}{3} - 45056\,\mathrm{Li}_5\left(\frac{1}{2}\right) \\ &- 45056\,\mathrm{Li}_4\left(\frac{1}{2}\right)\ln(2) + 176\,\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\pi - 1056\zeta_3\,\mathrm{Li}_2\left(\frac{1}{4}\right) - 3982\zeta_3\pi^2 \\ &- 21824\zeta_3\ln^2(2) + 2112\zeta_3\ln(2)\ln(3) - 1584\,\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\ln(3) \end{aligned} \tag{6.7.4}$$

$$
\begin{aligned}
&-\frac{4400\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\pi^3}{27}+\frac{88\pi^4\ln(2)}{45}-\frac{616\pi^4\ln(3)}{27}+\frac{11264\pi^2\ln^3(2)}{9}\\
&-\frac{22528\ln^5(2)}{15}+8576\,\mathrm{Li}_4\left(\frac{1}{2}\right)+7504\zeta_3\ln(2)+\frac{4174\pi^4}{27}-\frac{1072\pi^2\ln^2(2)}{3}\\
&+\frac{1072\ln^4(2)}{3}+\frac{554032\zeta_3}{27}-32\pi^2\ln(2)+\frac{730378\pi^2}{243}-384\ln^2(2)\\
&+832\ln(2)+\frac{1408681}{81}+\sqrt{3}\left(192\;\mathrm{Im}\left\{\mathrm{Li}_3\left(\frac{\exp(i\pi/3)}{2}\right)\right\}\right.\\
&\left.+160\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln(2)-16\pi\ln^2(2)-\frac{560\pi^3}{81}\right)\Bigg)+\mathcal{O}\left(\epsilon^2\right).
\end{aligned}
$$

In this chapter we discussed the computation of the same-hemisphere three-gluon-emission contribution to the zero-jettiness soft function at N3LO in perturbative QCD. To this end, we used the approach of modified IBPs developed in Chapter 5. This approach allowed us to apply integration-by-parts identities and the method of differential equations to phase-space integrals containing Heaviside functions. The appearance of integrals that are not regularized dimensionally required the introduction of an analytic regulator. While this feature complicated the use of differential equations and the computation of master integrals, we were able to bypass these problems efficiently.

The missing configuration in which one of the gluons is emitted into a different hemisphere can be computed in similar fashion. However, it remains unclear if this computation will be simpler or more complex than the one for the same-hemisphere emission contribution. On the one hand, the computation of master integrals should be easier, due to the absence of some collinear divergences. On the other hand, we will have to compute master integrals with three Heaviside functions, which were absent for the same-hemisphere emission contribution. From the first preliminary results for the different-hemisphere emission contribution, it seems that this configuration is indeed easier to compute and may be calculated with the methods outlined in this chapter.

Finally, once a full result for three-gluon-emission contribution to the zero-jettiness soft function is known, the contributions that arise from the emission of a soft $q\bar{q}$-pair and a soft gluon can be computed straightforwardly. Similarly, we expect that contributions due to virtual corrections to two real emissions can also be dealt with using the method developed in this chapter.

# CONCLUSION

# 7 CONCLUSION

In this thesis we calculated various contributions to the N3LO soft and beam functions as required for the zero-jettiness slicing scheme at N3LO. Specifically, in Chapter 2 we extended the already known results for the NNLO beam functions [60, 61] by calculating all NNLO beam functions through second order in the dimensional regularization parameter $\epsilon$. These results are required for the renormalization of the N3LO beam function. We further computed contributions to the beam function due to real-virtual emission at N3LO in Chapter 3. These contributions are needed for the ongoing calculating of the N3LO beam functions in Refs. [44–46] as an independent cross-check of the N3LO beam functions presented in Ref. [47]. The calculations in Chapters 2 and 3 could be carried out with standard multi-loop techniques such as IBP relations and reverse unitary.

In contrast to this the computation of the soft function is complicated by Heaviside functions that appear in the definition of the zero-jettiness observable, which prevent the standard use of IBP relations to obtain a reduction to master integrals. To remedy this problem we developed two different methods in Chapters 4 and 5.

In Chapter 4 we wrote Heaviside functions as integrals of auxiliary parameters over delta functions. After writing delta functions as cut propagators using reverse unitarity, integration-by-parts identities could be applied. At variance with this, in Chapter 5 we derived modified integration-by-parts identities that can be directly applied to integrals containing Heaviside functions. To this end, we used the fact that the starting point for IBP relations - the statement that in dimensional regularization an integral of the total derivative vanishes, is independent of whether Heaviside functions appear in the integrand or not. We further used the fact, that when this derivative acts on a Heaviside function the result is a delta function which can again be treated in the IBP framework using reverse unitarity.

Both of these approaches proved successful and we calculated the NNLO zero-jettiness soft function through second order in $\epsilon$, as required for the renormalization of the N3LO soft function. Both methods led to a huge simplification compared to the original computations presented in Refs [48, 49], where the soft function was computed up to finite contributions in $\epsilon$. However, the modified IBP approach of Chapter 5 appeared more suitable for an N3LO calculation. Therefore, we applied it to the computation of the same-hemisphere three-gluon-emission contribution to the N3LO zero-jettiness soft function in Chapter 6. While this calculation was complicated by the need to introduce an additional analytic regulator, it ultimately proved successful and we obtained the first contributions to the N3LO zero-jettiness soft function.

To complete the computation of the N3LO zero-jettiness soft function additional contributions are required, such as the contribution due to the emission of three-gluons in different hemispheres, the emission of quark-anti quark pairs together with a gluon as well as all real-real-virtual emissions. [1] While we only calculated a partial contribution to the N3LO soft function, we are confident that it is the most general one and all other pieces can be calculated with the methods developed in this thesis.

When these missing contributions are computed, all ingredients for the zero-jettiness slicing scheme at N3LO would become available. This scheme would allow for a fully differential description for the production of a colorless final state in proton-proton collisions $pp \to V$ where $V \in \{H, W, Z, \gamma^*, WW, ZZ, \gamma\gamma, \dots\}$ at N3LO.

The extension of this scheme to a one-jettiness slicing scheme at N3LO would only require the computation of the one-jettiness soft function, as the required beam and jet functions are already known [90, 91]. While the modified IBP approach can also be applied to determine the one-jettiness soft function, such a computation would be complicated by the more complex definition of the one-jettiness observable. A one-jettiness slicing scheme would allow for the fully differential description of the production of colorful final states $pp \to V + j$ where $V \in \{H, W, Z, \dots\}$ at N3LO.

Finally, the two different methods of dealing with phase-space integrals containing Heaviside functions developed in thesis may be used for other problems, for example for computing differential fiducial cross-sections. Thus, the two methods developed in this thesis neatly complement the standard toolbox of reverse unitarity and integration-by-parts identities for phase-space integrations.

1 In the final stages of the preparation of this thesis Ref. [89] appeared which presents the results for the RRV and RVV contributions to the N3LO soft function.

# Part III

# APPENDIX

# A USEFUL FORMULAS

In this Appendix we collect useful formulas that are needed in Chapters 1 - 3.

## A.1 INTEGRATION-BY-PARTS IDENTITIES AND REVERSE UNITARITY

When computing multi-loop amplitudes one is confronted with the need to evaluate a large number of integrals. These loop integrals are of the form

$$\int \mathrm{d}^d l \frac{1}{\prod_i D_i^{n_i}}, \tag{A.1.1}$$

where $D_i$ are propagators which are quadratic or linear in $l$. It was pointed out in Ref. [51] that there exists linear relations between loop integrals that can be established using IBP relations. These relations follow from Gauss' theorem in $d$-dimensions supplemented with the statement, that boundary terms of dimensionally-regularized integrals vanish. Thus, the relations read

$$\int \mathrm{d}^d l \frac{\partial}{\partial l_\mu} \left[ v_\mu \frac{1}{\prod_i D_i^{n_i}} \right] = 0, \tag{A.1.2}$$

where $v^\mu$ is an arbitrary vector. After computing the derivative in Eq. (A.1.2), one obtains a sum of integrals which is equal to zero. From there, one derives a system of linear equations which can be solved to express more complicated integrals through easier ones.

To demonstrate the usefulness of IBPs we consider the following relation

$$\int \mathrm{d}^d l \frac{\partial}{\partial l_\mu} \left[ l_\mu \frac{1}{l^2 - m^2} \right] = 0. \tag{A.1.3}$$

We compute the derivative explicitly and find a relation between the two integrals

$$\int \mathrm{d}^d l \frac{1}{[l^2 - m^2]^2} = \frac{\frac{d}{2} - 1}{m^2} \int \mathrm{d}^d l \frac{1}{l^2 - m^2}, \tag{A.1.4}$$

without the need to compute either of them.

While quite useful, IBP relations were originally derived for loop-integrals and are not applicable to the real-emission phase spaces we are interested in. Indeed, real-emission phase spaces contain delta functions which can not be included in the structure in Eq. (A.1.2). As pointed out in Ref. [50] one can circumvent this problem by using reverse unitarity.

Reverse unitarity is based on the observation that the following formula holds for any real-valued $X$

$$\delta(X) = \frac{\mathrm{i}}{2\pi}\left(\frac{1}{X+\mathrm{i}\epsilon} - \frac{1}{X-\mathrm{i}\epsilon}\right) \equiv \frac{1}{[X]_c}, \tag{A.1.5}$$

where $1/[X]_c$ is referred to as a "cut" propagator. With the help of Eq. (A.1.5) any delta function appearing in the phase space can now be written as the sum of two "propagators" with different $i\epsilon$ prescription and thus in the form of Eq. (A.1.2). Since the derivatives in Eq. (A.1.2) are independent of the different $i\epsilon$ prescription the cut propagator in Eq. (A.1.5) behaves like a normal propagator with one exception. When this cut "propagator" disappears, the integral vanishes. Indeed,

$$\int \mathrm{d}^d l \frac{1}{[l^2]_c} \frac{l \cdot l}{\prod_i D_i^{n_i}} = \int \mathrm{d}^d l \; \delta^+(l^2) \, \frac{l \cdot l}{\prod_i D_i^{n_i}} = 0. \tag{A.1.6}$$

## A.2 FEYNMAN PARAMETERS

Feynman found a useful identity to combine factors of a denominator. This identity is often used to simplify loop-integrals. We will use the special case of the identity

$$\frac{1}{AB} = \int_0^1 \frac{\mathrm{d}x}{[x\;A + (1-x)\;B]^2}. \tag{A.2.1}$$

When working with linear propagators a different representation of the Feynman parametrization becomes useful. We make the substitution $y = \frac{x}{1-x}$ in Eq. (A.2.1) and obtain

$$\frac{1}{AB} = \int_0^\infty \mathrm{d}y \frac{1}{[A + yB]^2}. \tag{A.2.2}$$

When using this formula we will let A be a normal propagator and B a linear one.

## A.3 PLUS DISTRIBUTION

The plus distribution is defined as

$$\int_0^1 \mathrm{d}z \frac{f(z)}{(1-z)_+} = \int_0^1 \mathrm{d}z \left(\frac{f(z)}{1-z} - \frac{f(1)}{1-z}\right). \tag{A.3.1}$$

While the integral $\int_0^1 \frac{\mathrm{d}z}{1-z}$ is divergent it can be calculated in dimensional regularization

$$\int_0^1 \mathrm{d}z \frac{1}{(1-z)^{1-\epsilon}} = \int_0^1 \mathrm{d}z \frac{1}{z^{1-\epsilon}} = \left.\frac{z^\epsilon}{\epsilon}\right|_0^1 = \frac{1}{\epsilon} \quad \text{with} \quad \epsilon > 0. \tag{A.3.2}$$

We combine the previous results to split any test function $f(z)$ into a finite part, expressed through plus distributions, and an explicit pole in $\epsilon$. To this end we subtract the divergent term at $z = 1$ and add it back

$$\int_0^1 \mathrm{d}z \frac{f(z)}{(1-z)^{1-\epsilon}} = \int_0^1 \mathrm{d}z \frac{f(z) - f(1)}{(1-z)^{1-\epsilon}} + f(1) \int_0^1 \mathrm{d}z \frac{1}{(1-z)^{1-\epsilon}} \tag{A.3.3}$$

$$= \int_0^1 \mathrm{d}z \frac{f(z)}{[(1-z)^{1-\epsilon}]_+} + \frac{f(1)}{\epsilon} \tag{A.3.4}$$

$$= \int_0^1 \mathrm{d}z \;\; f(z) \left[ \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \frac{\ln^n (1-z)}{1-z} \right]_+ + \frac{f(1)}{\epsilon}. \tag{A.3.5}$$

Omitting the integral sign Eq. (A.3.5) reads

$$\frac{f(z)}{(1-z)^{1-\epsilon}} = f(z) \left[ \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} L_n(1-z) \right] + \frac{f(z)\delta(1-z)}{\epsilon}, \tag{A.3.6}$$

where we introduced

$$L_n(1-z) = \left[ \frac{\ln^n (1-z)}{1-z} \right]_+ . \tag{A.3.7}$$

We note that the result is completely independent of the test function and we simply write

$$\frac{1}{(1-z)^{1-\epsilon}} = \left[ \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} L_n(1-z) \right] + \frac{\delta(1-z)}{\epsilon}. \tag{A.3.8}$$

## A.4 SUBTRACTIONS AT DIVERGENT ENDPOINTS

Imagine again we want to compute the integral

$$\int_0^1 \mathrm{d}z \frac{f(z)}{(1-z)^{1-\epsilon}}, \tag{A.4.1}$$

where the function $f(z)$ is too complicated to be integrated directly but finite in $z = 1$. In dimensional regularization it is sufficient to determine this integral as an expansion in $\epsilon$. To exploit this fact, we again subtract and add back the divergence at $z = 1$

$$\int_0^1 \mathrm{d}z \frac{f(z)}{(1-z)^{1-\epsilon}} = \int_0^1 \mathrm{d}z \frac{f(z) - f(1)}{(1-z)^{1-\epsilon}} + f(1) \int_0^1 \mathrm{d}z \frac{1}{(1-z)^{1-\epsilon}}. \tag{A.4.2}$$

The second integral in Eq. (A.4.2) can trivially be integrated while the first integral is now finite in the $z \to 1$ limit and can thus be expanded in $\epsilon$. Typical functions that appear in $f(z)$ such as hypergeometric functions can be expanded using `HypExp`. Typical functions that appear after expansion are iterated logarithms, which can be integrated using `HyperInt`.

## A.5 CONVOLUTIONS OF PLUS DISTRIBUTIONS

To renormalize the beam functions in Section 2.3 we have to calculate convolutions of the form

$$\int_0^t \mathrm{d}t' L_n(t-t')\ L_{n'}(t), \tag{A.5.1}$$

where

$$L_n(t) = \left[\frac{\ln^n(t)}{t}\right]_+ . \tag{A.5.2}$$

All required convolutions can be obtained by expanding the generating equation in $\epsilon$ and matching coefficients

$$\int_0^t \mathrm{d}t'(t-t')^{-1-\beta\epsilon}\ t'^{\ -1-\alpha\epsilon} = \frac{\Gamma(-\alpha\epsilon)\Gamma(-\beta\epsilon)}{\Gamma(-(\alpha+\beta)\epsilon)} t^{-1-(\alpha+\beta)\epsilon}. \tag{A.5.3}$$

Specifically, we need the following convolutions

$$\int_0^t \mathrm{d}t' L_0(t-t')\ L_0(t) = 2L_1(t) - \frac{\pi^2}{6}\delta(t), \tag{A.5.4}$$

$$\int_0^t \mathrm{d}t' L_0(t-t')\ L_1(t) = \frac{3}{2}L_2(t) - \frac{\pi^2}{6}L_0(t) + \zeta(3)\delta(t), \tag{A.5.5}$$

$$\int_0^t \mathrm{d}t' L_0(t-t')\ L_2(t) = \frac{4}{3}L_3(t) - \frac{\pi^2}{3}L_1(t) + 2\zeta(3)L_0(t) - \frac{\pi^4}{45}\delta(t), \tag{A.5.6}$$

$$\begin{aligned}\int_0^t \mathrm{d}t' L_0\left(t-t'\right)\ L_3\left(t'\right) = &-((4\pi^4 L_0\left(t\right) + 15(2\pi^2 L_2\left(t\right) - 5L_4\left(t\right) + 12L_1\left(t\right)\zeta(3) \\ &+ \delta\left(t\right)\zeta(5)))/60,\end{aligned} \tag{A.5.7}$$

$$\begin{aligned}\int_0^t \mathrm{d}t' L_0\left(t-t'\right) L_4\left(t'\right) = &-((8\pi^6\delta\left(t\right) + 21(4\pi^4 L_1\left(t\right) + 10\pi^2 L_3\left(t\right) - 18L_5\left(t\right) + 90L_2\left(t\right)\zeta(3) \\ &+ 15L_0\left(t\right)\zeta(5)))/315,\end{aligned} \tag{A.5.8}$$

$$\int_0^t \mathrm{d}t' L_1\left(t-t'\right)\ L_1\left(t'\right) = (-(\pi^4\delta\left(t\right))/360 - (\pi^2 L_1\left(t\right))/3 + L_3\left(t\right) - L_0\left(t\right)\zeta(3)) \tag{A.5.9}$$

$$\begin{aligned}\int_0^t \mathrm{d}t' L_1\left(t-t'\right)\ L_2\left(t'\right) = &((-(\pi^4 L_0\left(t\right)) - 6(3\pi^2 L_2\left(t\right) - 5L_4\left(t\right) - \pi^2\delta\left(t\right)\zeta(3) \\ &+ 18L_1\left(t\right)\zeta(3) + \delta\left(t\right)\zeta(5)))/36,\end{aligned} \tag{A.5.10}$$

$$\begin{aligned}\int_0^t \mathrm{d}t' L_1\left(t-t'\right) L_3\left(t'\right) = &-((63\pi^4 L_1\left(t\right) + \delta\left(t\right)\left(2\pi^6 - 315\zeta(3)^2\right) + 35(8\pi^2 L_3\left(t\right) - 9L_5\left(t\right) \\ &- 6\pi^2 L_0\left(t\right)\zeta(3) + 72L_2\left(t\right)\zeta(3) + 9L_0\left(t\right)\zeta(5)))/420,\end{aligned} \tag{A.5.11}$$

$$\begin{aligned}\int_0^t \mathrm{d}t' L_2\left(t-t'\right) L_2\left(t'\right) = &-((\delta\left(t\right)\left(23\pi^6 - 3780\zeta(3)^2\right) + 420(\pi^4 L_1\left(t\right) + 6(\pi^2 L_3\left(t\right) - L_5\left(t\right) \\ &- \pi^2 L_0\left(t\right)\zeta(3) + 9L_2\left(t\right)\zeta(3) + L_0\left(t\right)\zeta(5))))/3780.\end{aligned} \tag{A.5.12}$$

# B

# BEAM FUNCTION FEYNMAN DIAGRAMS

In this Appendix we present all Feynman diagrams required for the various NNLO and N3LO beam functions calculated in Chapters 2 and 3. They are shown below

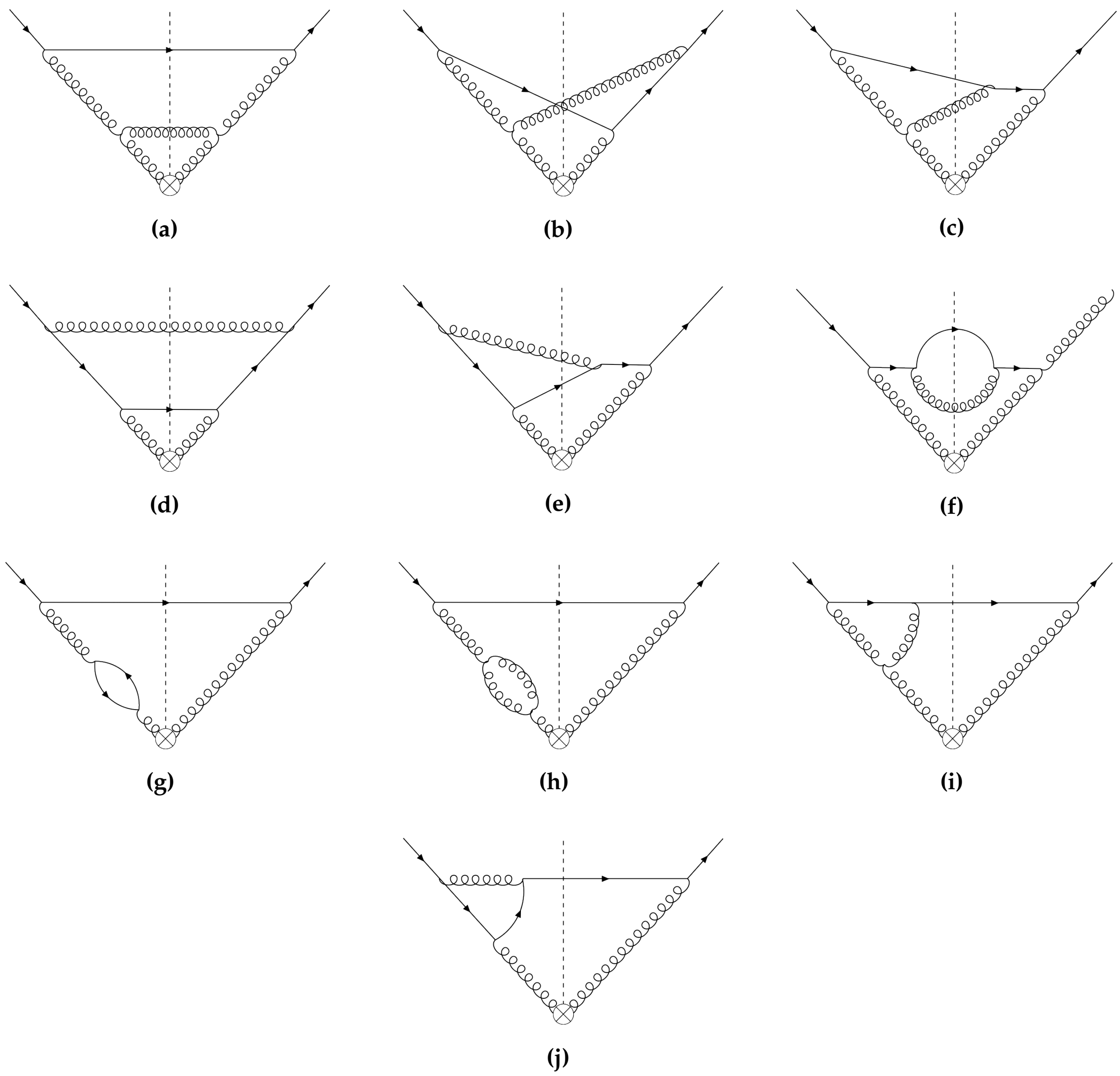


**Figure B.1:** Diagrams contributing to the $B_{gq}^{(b)}$ beam function. Diagrams for which the fermion flow is reversed and left-right mirror diagrams are not shown. The dashed line represents a "cut" so that all particles crossing it are on the mass-shell. The vertex $\otimes$ denotes the insertion of the projection operator defined in Eq. (2.1.5). Diagrams (a)-(f) are double real emission diagrams, (g)-(j) are real-virtual emission diagrams.

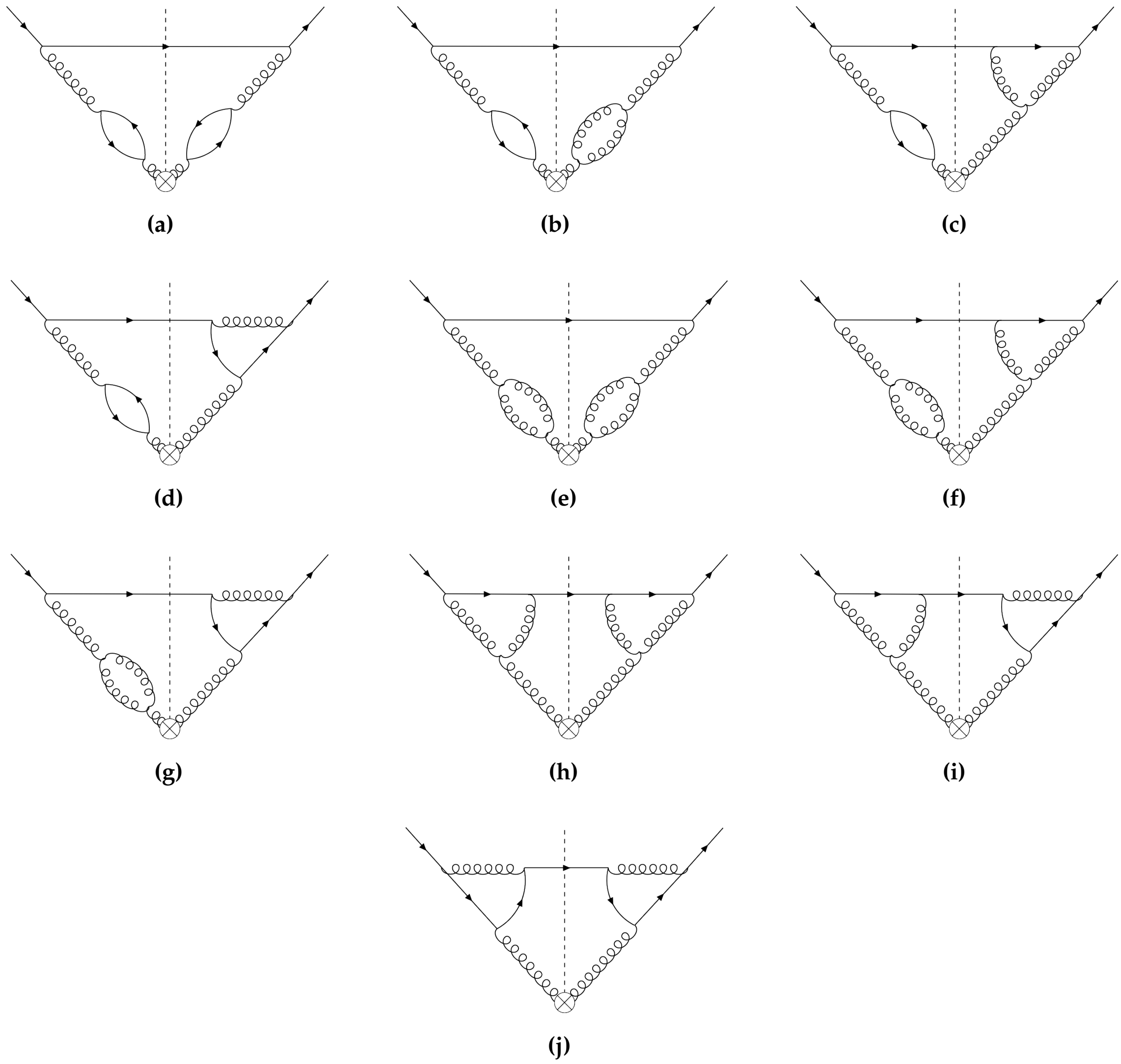

**Figure B.2:** Diagrams for the RV emission contribution to the N3LO $B_{gq}^{(b)}$ beam function. The dashed line represents a "cut" so that all particles crossing it are on the mass-shell. The vertex $\otimes$ denotes the insertion of the projection operator defined in Eq. (2.1.5). Diagrams which can be obtained by reversing any of the fermion flows and/or mirroring the diagram at the cut are not shown.

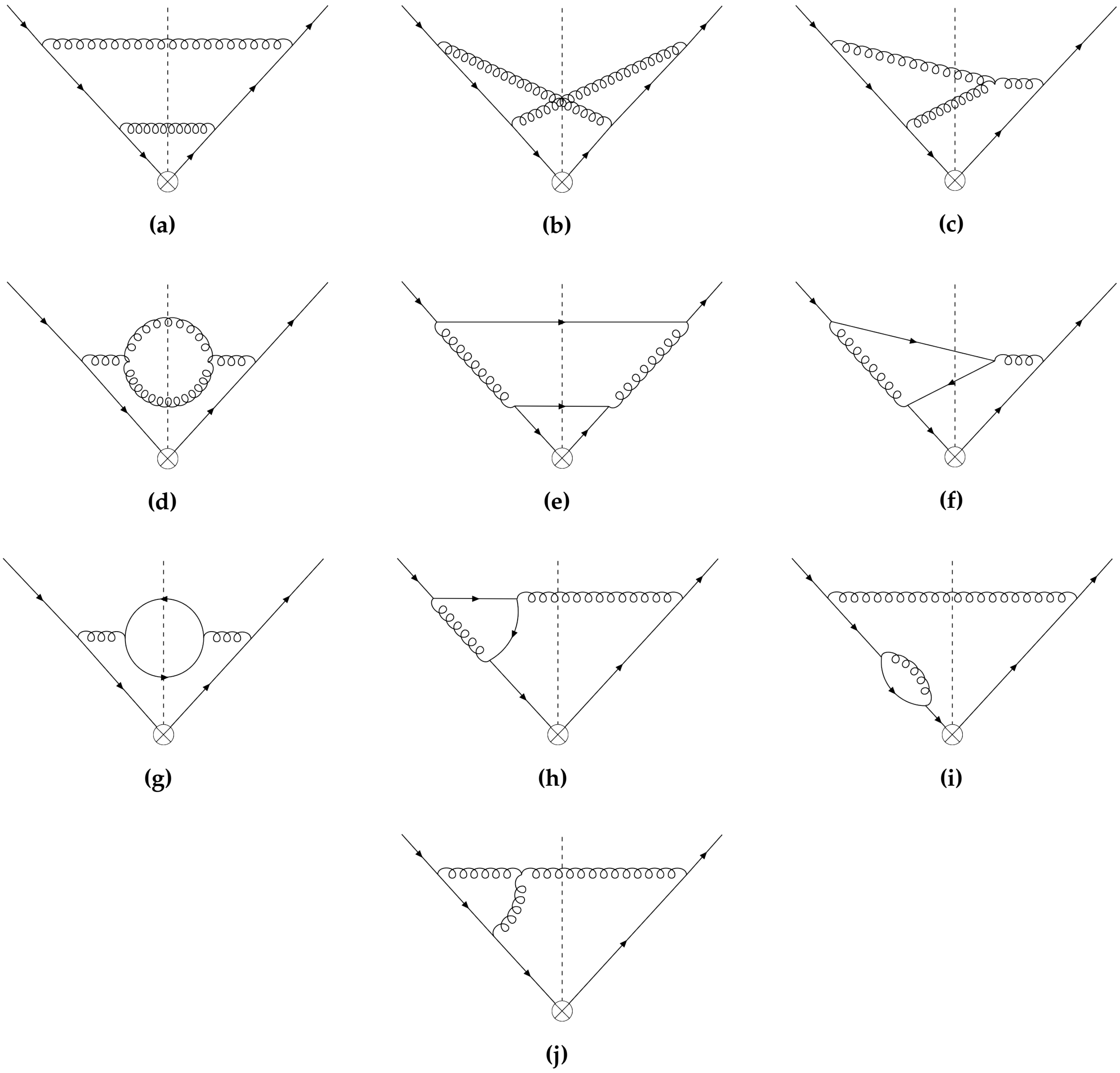


**Figure B.3:** Diagrams contributing to the $B_{qq}^{(b)}$ beam function. Diagrams for which the fermion flow is reversed and left-right mirror diagrams are not shown. The dashed line represents a "cut" so that all particles crossing it are on the mass-shell. The vertex $\otimes$ denotes the insertion of the projection operator defined in Eq. (2.1.5). Diagrams (a)-(d) are double real gluon emission diagrams, (e)-(g) are quark anti-quark emission diagrams, and (h)-(j) are real-virtual emission diagrams.

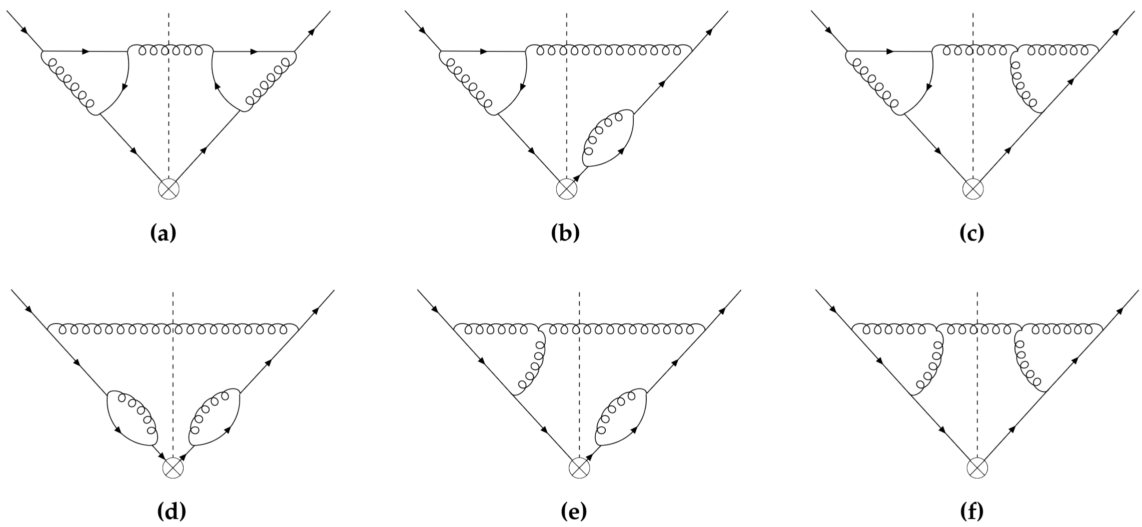


**Figure B.4:** Diagrams for the RV emission contribution to the N3LO $B_{qq}^{(b)}$ beam function. The dashed line represents a "cut" so that all particles crossing it are on the mass-shell. The vertex $\otimes$ denotes the insertion of the projection operator defined in Eq. (2.1.5). Diagrams which can be obtained by reversing any of the fermion flows and/or mirroring the diagram at the cut are not shown.

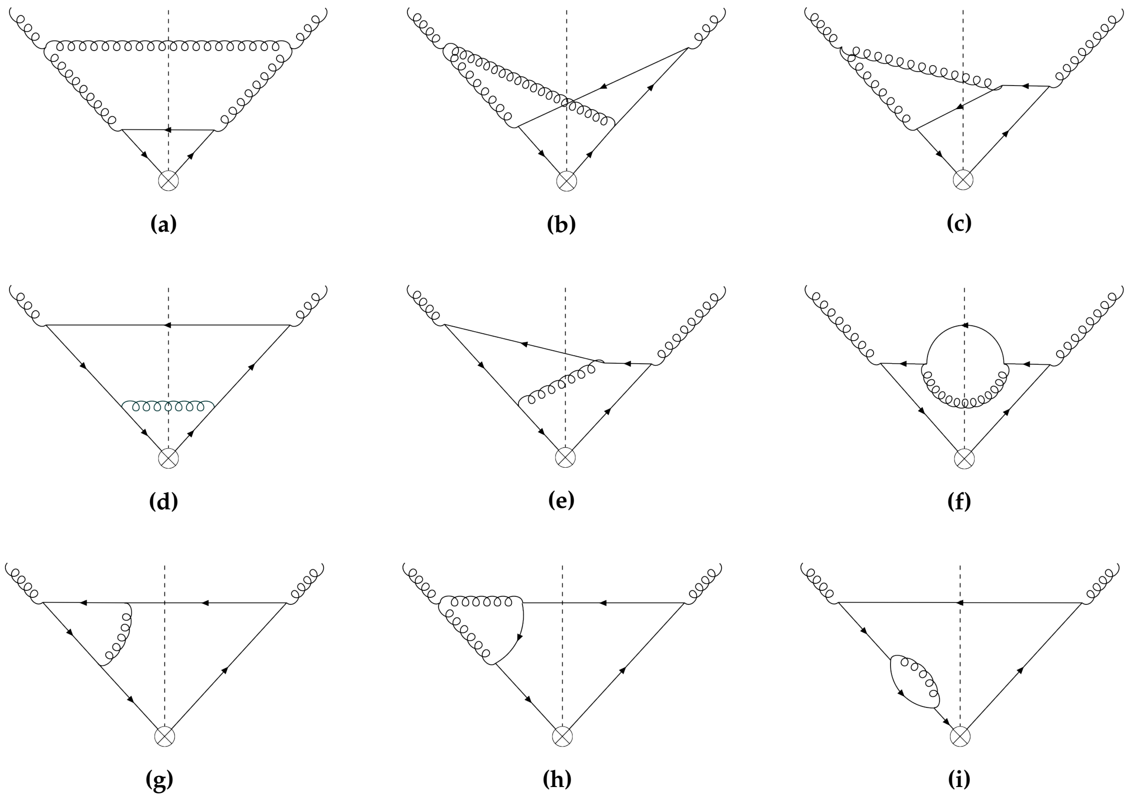


**Figure B.5:** Diagrams contributing to the $B_{qg}^{(b)}$ beam function. Diagrams for which the fermion flow is reversed and left-right mirror diagrams are not shown. The dashed line represents a "cut" so that all particles crossing it are on the mass-shell. The vertex $\otimes$ denotes the insertion of the projection operator defined in Eq. (2.1.5). Diagrams (a)-(f) are double real emission diagrams, (g)-(i) are real-virtual emission diagrams.

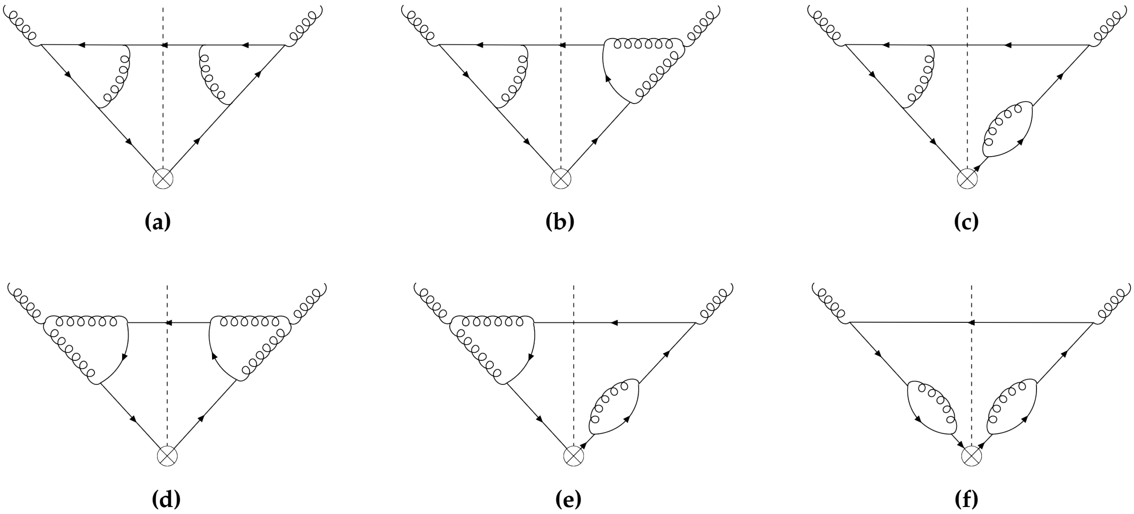


**Figure B.6:** Diagrams for the RV emission contribution to the N3LO $B_{qg}^{(b)}$ beam function. The dashed line represents a "cut" so that all particles crossing it are on the mass-shell. The vertex $\otimes$ denotes the insertion of the projection operator defined in Eq. (2.1.5). Diagrams which can be obtained by reversing any of the fermion flows and/or mirroring the diagram at the cut are not shown.

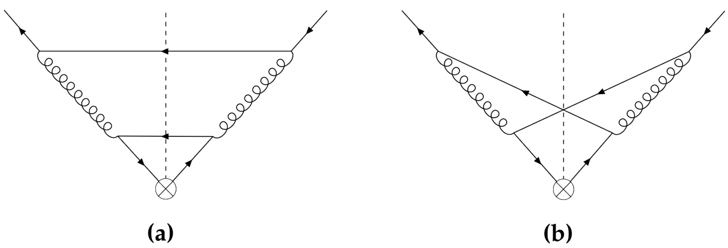


**Figure B.7:** Diagrams contributing to the $B_{q\bar{q}}^{(b)}$ beam function. Diagrams for which the fermion flow is reversed and left-right mirror diagrams are not shown. The dashed line represents a "cut" so that all particles crossing it are on the mass-shell. The vertex $\otimes$ denotes the insertion of the projection operator defined in Eq. (2.1.5).

C

# MASTER INTEGRALS FOR DIFFERENT HEMISPHERE EMISSION CONTRIBUTIONS TO THE NNLO SOFT FUNCTION

In this Appendix we present the explicit calculation of master integrals in configuration $B$ for the soft functions calculated in Chapters 4 and 5.

## C.1 MASTER INTEGRALS FOR CONFIGURATION $B$ IN THE AUXILIARY PARAMETER APPROACH

In this section we discuss the computation of master integrals required for the NNLO soft function in the auxiliary parameter approach, that contribute to configuration $B$ and were not yet discussed in Chapter 4.

### C.1.1 *Master integral* $I_{B1}$

The first inegral reads

$$I_{B1} = \int \mathrm{dPS}_S^{(2)}\ \delta\left(\tau - n\cdot k_1 - \bar{n}\cdot k_2\right)\ \delta(n\cdot k_1 - z_1\bar{n}\cdot k_1)\ \delta(\bar{n}\cdot k_2 - z_2 n\cdot k_2). \tag{C.1.1}$$

We follow the calculation outlined in Section 4.2.1 and obtain

$$\begin{aligned} I_{B1} &= [N]^2 \int \mathrm{d}\beta_1 \mathrm{d}\alpha_2 \frac{(\beta_1\alpha_2)^{-2\epsilon}}{(z_1 z_2)^{1-\epsilon}} \delta(\tau - \beta_1 - \alpha_2) \\ &= [N]^2 \frac{1}{(z_1 z_2)^{1-\epsilon}} \int\limits_0^\tau \mathrm{d}\beta_1 \beta_1^{-2\epsilon} \left(\tau - \beta_1\right)^{-2\epsilon} \\ &= I_{A1}. \end{aligned} \tag{C.1.2}$$

### C.1.2 *Master integral* $I_{B2}$

The next integral to consider is $I_{B2}$. It reads

$$\begin{aligned} I_{B2} = &\int \mathrm{dPS}_S^{(2)}\ \delta\left(\tau - n\cdot k_1 - \bar{n}\cdot k_2\right)\ \delta(n\cdot k_1 - z_1\bar{n}\cdot k_1)\ \delta(\bar{n}\cdot k_2 - z_2 n\cdot k_2) \\ &\times \left(\frac{n\cdot k_1}{2} + \frac{\tau z_1}{2(1-z_1)}\right)^{-1}. \end{aligned} \tag{C.1.3}$$

The additional propagator only depends on $\beta_1$ and the integration is identical to the one of $I_{B1}$ up to the very last step. We find

$$
\begin{aligned}
I_{B2} &= [N]^2 \int \mathrm{d}\beta_1 \mathrm{d}\alpha_2 \frac{(\beta_1\alpha_2)^{-2\epsilon}}{(z_1 z_2)^{1-\epsilon}} \frac{\delta(\tau-\beta_1-\alpha_2)}{\frac{\beta_1}{2}+\frac{\tau z_1}{2(1-z_1)}} \\
&= [N]^2 \frac{2\tau^{-4\epsilon}}{(z_1 z_2)^{1-\epsilon}} \frac{1-z_1}{z_1} \int_0^1 \mathrm{d}\beta_1 \frac{(\beta_1(1-\beta_1))^{-2\epsilon}}{1+\frac{1-z_1}{z_1}\beta_1} \\
&= [N]^2 \frac{2\tau^{-4\epsilon}}{(z_1 z_2)^{1-\epsilon}} \frac{1-z_1}{z_1} \frac{\Gamma^2(1-2\epsilon)}{\Gamma(2-4\epsilon)} \, {}_2F_1\left(1, 1-2\epsilon, 2-4\epsilon, -\frac{1-z_1}{z_1}\right).
\end{aligned}
\tag{C.1.4}
$$

### C.1.3 *Master integral $I_{B3}$*

We now calculate the master integral $I_{B3}$. It reads

$$
\begin{aligned}
I_{B3} = &\int \mathrm{dPS}_S^{(2)} \, \delta\left(\tau - n\cdot k_1 - \bar{n}\cdot k_2\right) \, \delta(n\cdot k_1 - z_1 \bar{n}\cdot k_1) \, \delta(\bar{n}\cdot k_2 - z_2 n\cdot k_2) \\
&\times (k_1\cdot k_2)^{-1}.
\end{aligned}
\tag{C.1.5}
$$

We start by writing the expression for the scalar product $k_1 \cdot k_2$ in the configuration $B$. We find

$$
2k_1\cdot k_2 = \alpha_1\beta_2 + \beta_1\alpha_2 - 2\sqrt{\alpha_1\beta_2\alpha_2\beta_1}\cos\phi_{12} = \alpha_1\beta_2\left(1 + z_1 z_2 - 2\sqrt{z_2 z_1}\cos\phi_{12}\right), \tag{C.1.6}
$$

Following the calculation described in Eq. (4.2.14), we obtain

$$
\begin{aligned}
&\int \frac{\mathrm{d}\Omega_1^{(d-2)}\mathrm{d}\Omega_2^{(d-2)}}{2k_1\cdot k_2} = \frac{1}{\alpha_1\beta_2}\int \frac{\mathrm{d}\Omega_1^{(d-2)}\mathrm{d}\Omega_2^{(d-2)}}{1+z_1z_2-2\sqrt{z_2z_1}\cos\phi_{12}} \\
&= \frac{\Omega^{(d-2)}\Omega^{(d-3)}}{\alpha_1\beta_2}\int_0^\pi \frac{\mathrm{d}\theta\left(\sin^2\theta\right)^{-\epsilon}}{1+z_1z_2-2\sqrt{z_2z_1}\cos\phi_{12}} \\
&\overset{x=\cos\phi_{12}}{=} \frac{\Omega^{(d-2)}\Omega^{(d-3)}}{\alpha_1\beta_2}\int_{-1}^1 \frac{\mathrm{d}x\left(1-x^2\right)^{-\epsilon-\frac{1}{2}}}{1+z_1z_2-2\sqrt{z_2z_1}\,x} \\
&\overset{y=\frac{1+x}{2}}{=} \frac{\Omega^{(d-2)}\Omega^{(d-3)}}{\alpha_1\beta_2}4^{-\epsilon}\int_0^1 \frac{\mathrm{d}y\left[y\,(1-y)\right]^{-\epsilon-\frac{1}{2}}}{(1+\sqrt{z_1}\sqrt{z_2})^2-4y\sqrt{z_1\,z_2}} \\
&= \frac{\left[\Omega^{(d-2)}\right]^2}{\alpha_1\beta_2\left(1+\sqrt{z_2z_1}\right)^2}\,{}_2F_1\left(1,\frac{1}{2}-\epsilon,1-2\epsilon,\frac{4\sqrt{z_2z_1}}{\left(1+\sqrt{z_2z_1}\right)^2}\right) \\
&= \frac{\left[\Omega^{(d-2)}\right]^2 z_2\,z_1}{\beta_1\alpha_2}\,{}_2F_1\left(1,1+\epsilon,1-\epsilon,z_2\,z_1\right).
\end{aligned}
\tag{C.1.7}
$$

We now combine the integral representations of Eqs. (C.1.2) and (C.1.7) and find

$$\begin{aligned} I_{B3} &= [N]^2 \frac{2\tau^{-1-4\epsilon}}{z_1^{1+\epsilon} z_2^{1+\epsilon}} \, z_1 \, z_2 \, {}_2F_1(1, 1+\epsilon, 1-\epsilon, z_1 z_2) \\ &\quad \times \int_0^1 \mathrm{d}\beta_1 \, (\beta_1)^{-1-2\epsilon} \, (1-\beta_1)^{-1-2\epsilon} \\ &= [N]^2 \frac{2\tau^{-1-4\epsilon} z_1 z_2}{(z_1 z_2)^{1-\epsilon}} \frac{\Gamma^2(-2\epsilon)}{\Gamma(-4\epsilon)} \, {}_2F_1 \left(1, 1+\epsilon, 1-\epsilon, z_1 z_2\right). \end{aligned} \tag{C.1.8}$$

### C.1.4 *Master integrals $I_{B4}$, $I_{B5}$ and $I_{B6}$*

As we have seen from the above discussion, the evaluation of master integrals is remarkably easy. For this reason, we discuss the evaluation of the master integrals $I_{B4}$, $I_{B5}$ and $I_{B6}$ all at once. They read

$$\begin{aligned} I_{B4} &= \int \mathrm{dPS}_S^{(2)} \, \delta\left(\tau - n \cdot k_1 - \bar{n} \cdot k_2\right) \, \delta(n \cdot k_1 - z_1 \bar{n} \cdot k_1) \, \delta(\bar{n} \cdot k_2 - z_2 n \cdot k_2) \\ &\quad \times \left( \frac{n \cdot k_1}{2} + \frac{\tau z_1}{2(1-z_1)} \right)^{-1} (k_1 \cdot k_2)^{-1}, \end{aligned} \tag{C.1.9}$$

$$\begin{aligned} I_{B5} &= \int \mathrm{dPS}_S^{(2)} \, \delta\left(\tau - n \cdot k_1 - \bar{n} \cdot k_2\right) \, \delta(n \cdot k_1 - z_1 \bar{n} \cdot k_1) \, \delta(\bar{n} \cdot k_2 - z_2 n \cdot k_2) \\ &\quad \times \left( \frac{n \cdot k_1}{2} - \frac{\tau}{2(1-z_2)} \right)^{-1}, \end{aligned} \tag{C.1.10}$$

$$\begin{aligned} I_{B6} &= \int \mathrm{dPS}_S^{(2)} \, \delta\left(\tau - n \cdot k_1 - \bar{n} \cdot k_2\right) \, \delta(n \cdot k_1 - z_1 \bar{n} \cdot k_1) \, \delta(\bar{n} \cdot k_2 - z_2 n \cdot k_2) \\ &\quad \times \left( \frac{n \cdot k_1}{2} - \frac{\tau}{2(1-z_2)} \right)^{-1} (k_1 \cdot k_2)^{-1}. \end{aligned} \tag{C.1.11}$$

To facilitate their computation we again combine the integral representations of Eqs. (C.1.2) and (C.1.7) and find

$$\begin{aligned} I_{B4} &= [N]^2 \frac{4\tau^{-2-4\epsilon}}{(z_1 z_2)^{1-\epsilon}} (z_1 z_2) \, {}_2F_1 \left(1, 1+\epsilon, 1-\epsilon, z_1 z_2\right) \frac{1-z_1}{z_1} \\ &\quad \times \int_0^1 \mathrm{d}\beta_1 \frac{\beta_1^{-1-2\epsilon} \, (1-\beta_1)^{-1-2\epsilon}}{1 + \frac{1-z_1}{z_1}\beta_1}, \end{aligned} \tag{C.1.12}$$

$$I_{B5} = [N]^2 \frac{(-2)\tau^{-4\epsilon}}{(z_1 z_2)^{1-\epsilon}} (1-z_2) \int_0^1 \mathrm{d}\beta_1 \frac{\beta_1^{-2\epsilon} \, (1-\beta_1)^{-2\epsilon}}{1-(1-z_2)\beta_1}, \tag{C.1.13}$$

$$\begin{aligned} I_{B6} &= [N]^2 \frac{(-4)\tau^{-2-4\epsilon}}{(z_1 z_2)^{1-\epsilon}} z_1 z_2 (1-z_2) \, {}_2F_1 \left(1, 1+\epsilon, 1-\epsilon, z_1 z_2\right) \\ &\quad \times \int_0^1 \mathrm{d}\beta_1 \frac{\beta_1^{-1-2\epsilon} \, (1-\beta_1)^{-1-2\epsilon}}{1-(1-z_2)\beta_1}. \end{aligned} \tag{C.1.14}$$

The remaining $\beta_1$ integration is straightforward to perform. We obtain

$$
\begin{aligned}
I_{B4} &= [N]^2 \frac{4\tau^{-2-4\epsilon}(1-z_1)z_2}{(z_1 z_2)^{1-\epsilon}} \frac{\Gamma^2(-2\epsilon)}{\Gamma(-4\epsilon)} \\
&\quad \times {}_2F_1\left(1, 1+\epsilon, 1-\epsilon, z_1 z_2\right) {}_2F_1\left(1, -2\epsilon, -4\epsilon, -\frac{1-z_1}{z_1}\right), \qquad \text{(C.1.15)} \\
I_{B5} &= [N]^2 \frac{(-2)\tau^{-4\epsilon}}{(z_1 z_2)^{1-\epsilon}}(1-z_2)\frac{\Gamma^2(1-2\epsilon)}{\Gamma(2-4\epsilon)} {}_2F_1(1, 1-2\epsilon, 2-4\epsilon, 1-z_2), \qquad \text{(C.1.16)} \\
I_{B6} &= [N]^2 \frac{(-4)\tau^{-2-4\epsilon} z_1 z_2 (1-z_2)}{(z_1 z_2)^{1-\epsilon}} \frac{\Gamma^2(-2\epsilon)}{\Gamma(-4\epsilon)} \\
&\quad \times {}_2F_1\left(1, 1+\epsilon, 1-\epsilon, z_1 z_2\right) {}_2F_1(1, -2\epsilon, -4\epsilon, 1-z_2). \qquad \text{(C.1.17)}
\end{aligned}
$$

## C.2 MASTER INTEGRALS FOR CONFIGURATION $B$ IN THE MODIFIED IBP APPROACH

In this section we discuss the computation of master integrals that were omitted in the computation of the NNLO soft function in the modified IBP approach in Chapter 5. The first integral reads

$$
\mathcal{I}_5^{n\bar{n}} = \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{\delta\left(1 - k_1 \cdot n - k_2 \cdot n\right)}{k_{12} \cdot \bar{n}\; k_{12} \cdot n} \theta\left(k_1 \cdot \bar{n} - k_1 \cdot n\right) \theta\left(k_2 \cdot \bar{n} - k_2 \cdot n\right). \qquad \text{(C.2.1)}
$$

We use

$$
[\mathrm{d}k_i] = \frac{\mathrm{d}\Omega_i^{(d-2)}}{4(2\pi)^{d-1}} \mathrm{d}\alpha_i \, \mathrm{d}\beta_i \, (\alpha_i \beta_i)^{-\epsilon}, \qquad \alpha_i, \beta_i \in [0, \infty), \qquad \text{(C.2.2)}
$$

change variables $\alpha_1 = \beta_1/\xi_1$, $\beta_2 = \alpha_2/\xi_2$ to get rid of Heaviside functions and find

$$
\begin{aligned}
\mathcal{I}_5^{n\bar{n}} &= [N]^2 \frac{2\,\Gamma^2(2-2\epsilon)}{\Gamma(4-4\epsilon)} \\
&\qquad \times \int_0^1 \mathrm{d}\xi_1 \int_0^1 \mathrm{d}\xi_2 \frac{(1-\xi_1)(\xi_1\xi_2)^{-1+\epsilon}}{1-\xi_1\xi_2} {}_2F_1\left(1, 2-2\epsilon, 4-4\epsilon, 1-\xi_1\right) \qquad \text{(C.2.3)} \\
&= [N]^2 \frac{2\,\Gamma^2(2-2\epsilon)}{\epsilon\,\Gamma(4-4\epsilon)} \int_0^1 \mathrm{d}\xi_1 (1-\xi_1)\xi_1^{-1+\epsilon} \mathcal{X}_5(\xi_1),
\end{aligned}
$$

where

$$
\mathcal{X}_5(\xi_1) = {}_2F_1\left(1, \epsilon, 1+\epsilon, \xi_1\right) {}_2F_1\left(1, 2-2\epsilon, 4-4\epsilon, 1-\xi_1\right). \qquad \text{(C.2.4)}
$$

We subtract the singularity at $\xi_1 = 0$, expand in $\epsilon$, where appropriate, and obtain

$$\begin{aligned}\mathcal{I}_5^{n\bar{n}} = [N]^2 \Bigg\{ & \frac{1}{\epsilon^2} + \frac{2}{\epsilon} + \left[2 + \frac{\pi^2}{6}\right] + \epsilon \left[2\zeta_3 - 8 + \pi^2\right] \\ & + \epsilon^2 \left[16\zeta_3 - 64 + 4\pi^2 + \frac{\pi^4}{9}\right] \\ & + \epsilon^3 \left[64\zeta_3 - \frac{2\pi^2\zeta_3}{3} + 30\zeta_5 - 256 + \frac{32\pi^2}{3} + \frac{26\pi^4}{45}\right] \\ & + \epsilon^4 \left[128\zeta_3 + 4\pi^2\zeta_3 + 8\zeta_3^2 + 60\zeta_5 - 512 + \frac{92\pi^4}{45} + \frac{44\pi^6}{945}\right] + \mathcal{O}\left(\epsilon^5\right) \Bigg\} .\end{aligned} \tag{C.2.5}$$

The next integral reads

$$\mathcal{I}_6^{n\bar{n}} = \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{\delta\left(1 - k_1 \cdot n - k_2 \cdot n\right)}{k_1 \cdot k_2 \; k_2 \cdot \bar{n} \; k_{12} \cdot n} \theta\left(k_1 \cdot \bar{n} - k_1 \cdot n\right) \theta\left(k_2 \cdot \bar{n} - k_2 \cdot n\right). \tag{C.2.6}$$

The computation follows along the same lines. We integrate over relative azimuthal angle between $k_1$ and $k_2$ and obtain

$$\begin{aligned}\mathcal{I}_6^{n\bar{n}} &= -[N]^2 \; \frac{2\,\Gamma(1-2\epsilon)^2}{\epsilon\Gamma(1-4\epsilon)} \\ &\quad \times \int_0^1 \mathrm{d}\xi_1 \int_0^1 \mathrm{d}\xi_2 \; \xi_1^\epsilon \; \xi_2^\epsilon \; {}_2F_1(1, \epsilon+1, 1-\epsilon, \xi_1 \; \xi_2) \\ &= -[N]^2 \; \frac{2\,\Gamma^2(1-2\epsilon)}{\epsilon(1+\epsilon)^2\,\Gamma(1-4\epsilon)} \\ &\quad \times {}_4F_3\left(1, 1+\epsilon, 1+\epsilon, 1+\epsilon, 1-\epsilon, 2+\epsilon, 2+\epsilon, 1\right).\end{aligned} \tag{C.2.7}$$

Finally, the last integral reads

$$\mathcal{I}_7^{n\bar{n}} = \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{\delta\left(1 - k_1 \cdot n - k_2 \cdot n\right)}{k_1 \cdot k_2 \; k_2 \cdot n \; k_{12} \cdot \bar{n}} \theta\left(k_1 \cdot \bar{n} - k_1 \cdot n\right) \theta\left(k_2 \cdot \bar{n} - k_2 \cdot n\right). \tag{C.2.8}$$

After the angular integration, we find

$$\begin{aligned}\mathcal{I}_7^{n\bar{n}} &= -[N]^2 \, \frac{\Gamma^2(1-2\epsilon)}{\epsilon(1+\epsilon)\,\Gamma(1-4\epsilon)} \int_0^1 \mathrm{d}\xi_1 \xi_1^\epsilon \\ &\quad \times {}_2F_1\left(1, -2\epsilon, 1-4\epsilon, 1-\xi_1\right) \, {}_3F_2\left(1, 1+\epsilon, 1+\epsilon, 1-\epsilon, 2+\epsilon, \xi_1\right) .\end{aligned} \tag{C.2.9}$$

As the remaining $\xi_1$ integration is finite, we expand the integrand in $\epsilon$ and integrate using `HyperInt`. The result reads

$$\begin{aligned}\mathcal{I}_7^{n\bar{n}} = [N]^2 \Bigg\{ & -\frac{\pi^2}{6\epsilon} + 2\zeta_3 + \frac{\pi^4\epsilon}{12} + \epsilon^2 \left[\frac{5\pi^2\zeta_3}{3} + 19\zeta_5\right] + \epsilon^3 \left[\frac{937\pi^6}{3780} - 82\zeta_3^2\right] \Bigg\} \\ & + \mathcal{O}(\epsilon^4) .\end{aligned} \tag{C.2.10}$$

D

# INTERMEDIATE RESULTS AND DEFINITIONS FOR THE N3LO SOFT FUNCTION

This Appendix contains various definitions and intermediate results that arise in the computation of the N3LO soft function.

## D.1 PHASE SPACE DEFINITION

In the calculation of the N3LO soft function we often integrate over the real emission phase space of the emitted gluons. We introduce the following shorthand notations

$$\begin{aligned} \mathrm{d}\Phi^{nnn}_{fgh} &= [\mathrm{d}k_1][\mathrm{d}k_2][\mathrm{d}k_3]\delta\left(1 - k_1 \cdot n - k_2 \cdot n\right) \\ &\quad \times f\left(k_1 \cdot \bar{n} - k_1 \cdot n\right) g\left(k_2 \cdot \bar{n} - k_2 \cdot n\right) h\left(k_3 \cdot \bar{n} - k_3 \cdot n\right), \end{aligned} \tag{D.1.1}$$

$$[\mathrm{d}k_i] = \frac{\mathrm{d}^d k_i}{(2\pi)^{d-1}}\,\delta^+(k_i^2). \tag{D.1.2}$$

We note that the $\tau$ dependence has already been scaled out of the integral in Eq. (D.1.1); it is reintroduced when presenting the results in Section 6.7. We further note that we always use the Sudakov decomposition when calculating master integrals, in this case Eq. (D.1.2) reads

$$[\mathrm{d}k_i] = \frac{\mathrm{d}\Omega_i^{(d-2)}}{4(2\pi)^{d-1}}\,\mathrm{d}\alpha_i\,\mathrm{d}\beta_i\,(\alpha_i\beta_i)^{-\epsilon}, \qquad \alpha_i\,,\beta_i \in [0,\infty)\,. \tag{D.1.3}$$

## D.2 INTEGRAL FAMILY DEFINITIONS FOR INTEGRALS WITH THREE DELTA FUNCTIONS

For the reduction of the zero-jettiness soft function at N3LO we require five different integral families for integrals with three delta functions. These integral families read

$$T_1^{\delta\delta\delta}(n_1\ldots n_{12}) = \int \frac{\mathrm{d}\Phi^{n_1\ldots n_7}_{\delta\delta\delta}}{(k_1\cdot k_2)^{n_8}(k_1\cdot k_3)^{n_9}(k_{123}^2)^{n_{10}}(k_1\cdot n)^{n_{11}}(k_2\cdot n)^{n_{12}}}, \tag{D.2.1}$$

$$T_2^{\delta\delta\delta}(n_1\ldots n_{12}) = \int \frac{\mathrm{d}\Phi^{n_1\ldots n_7}_{\delta\delta\delta}}{(k_1\cdot k_2)^{n_8}(k_1\cdot k_3)^{n_9}(k_{123}^2)^{n_{10}}(k_2\cdot n)^{n_{11}}(k_3\cdot n)^{n_{12}}}, \tag{D.2.2}$$

$$T_3^{\delta\delta\delta}(n_1\ldots n_{12}) = \int \frac{\mathrm{d}\Phi^{n_1\ldots n_7}_{\delta\delta\delta}}{(k_1\cdot k_2)^{n_8}(k_1\cdot k_3)^{n_9}(k_{123}^2)^{n_{10}}(k_1\cdot n)^{n_{11}}(k_{12}\cdot n)^{n_{12}}}, \tag{D.2.3}$$

$$T_4^{\delta\delta\delta}(n_1\ldots n_{12}) = \int \frac{\mathrm{d}\Phi^{n_1\ldots n_7}_{\delta\delta\delta}}{(k_1\cdot k_2)^{n_8}(k_1\cdot k_3)^{n_9}(k_{123}^2)^{n_{10}}(k_2\cdot n)^{n_{11}}(k_{23}\cdot n)^{n_{12}}}, \tag{D.2.4}$$

$$T_5^{\delta\delta\delta}(n_1\ldots n_{12}) = \int \frac{\mathrm{d}\Phi^{n_1\ldots n_7}_{\delta\delta\delta}}{(k_1\cdot k_2)^{n_8}(k_1\cdot k_3)^{n_9}(k_{123}^2)^{n_{10}}(k_{12}\cdot n)^{n_{11}}(k_{13}\cdot n)^{n_{12}}}, \tag{D.2.5}$$

(D.2.6)

where we defined

$$\begin{aligned} \mathrm{d}\Phi_{\delta\delta\delta}^{n_1 \ldots n_7} = & \frac{\mathrm{d}^d k_1}{(2\pi)^{d-1}} \, \delta^+(k_1^2)^{n_1} \frac{\mathrm{d}^d k_2}{(2\pi)^{d-1}} \, \delta^+(k_2^2)^{n_2} \frac{\mathrm{d}^d k_3}{(2\pi)^{d-1}} \, \delta^+(k_3^2)^{n_3} \\ & \times \delta\left(1 - k_1 \cdot n - k_2 \cdot n\right)^{n_4} \delta\left(k_1 \cdot \bar{n} - k_1 \cdot n\right)^{n_5} \delta\left(k_2 \cdot \bar{n} - k_2 \cdot n\right)^{n_6} \\ & \times \delta\left(k_3 \cdot \bar{n} - k_3 \cdot n\right)^{n_7} . \end{aligned} \tag{D.2.7}$$

## D.3 MASTER INTEGRALS FOR TRIPLE GLUON SAME HEMISPHERE EMISSION TO THE SOFT FUNCTION

The master integrals that are required for the computation of the triple gluon same hemisphere emission to the soft function in Sections 6.3 - 6.6 are defined as follows:

- integrals for $\int \mathrm{d}\Phi_{\theta\theta}^{nnn} \, \omega_{n\bar{n}}^{(3),a}$ :

$$\begin{aligned} I_1 &= \int \mathrm{d}\Phi_{\delta\delta}^{nnn} , & I_2 &= \int \frac{\mathrm{d}\Phi_{\delta\delta}^{nnn}}{(k_{12}n)(k_{13}n)} , \\ I_3 &= \int \frac{\mathrm{d}\Phi_{\delta\theta}^{nnn}}{(k_{13}\bar{n})} , & I_4 &= \int \frac{\mathrm{d}\Phi_{\delta\theta}^{nnn}}{(k_{123}\bar{n})} , \\ I_5 &= \int \frac{\mathrm{d}\Phi_{\delta\theta}^{nnn}}{(k_{13}n)(k_{123}\bar{n})} , & I_6 &= \int \frac{\mathrm{d}\Phi_{\theta\theta}^{nnn}}{(k_{123}\bar{n})} . \end{aligned} \tag{D.3.1}$$

- additional integrals for $\int \mathrm{d}\Phi_{\theta\theta}^{nnn} \, \omega_{n\bar{n}}^{(3),b}$ :

$$\begin{aligned} I_7 &= \int \frac{\mathrm{d}\Phi_{\delta\theta}^{nnn}}{(k_1k_3)(k_3\bar{n})} , & I_8 &= \int \frac{\mathrm{d}\Phi_{\delta\theta}^{nnn}}{(k_1k_3)(k_3n)(k_{13}\bar{n})} , \\ I_9 &= \int \frac{\mathrm{d}\Phi_{\delta\theta}^{nnn}}{(k_1k_3)(k_3n)(k_{23}\bar{n})} , & I_{10} &= \int \frac{\mathrm{d}\Phi_{\delta\theta}^{nnn}}{(k_1k_3)(k_{12}n)(k_{13}\bar{n})} , \\ I_{11} &= \int \frac{\mathrm{d}\Phi_{\delta\theta}^{nnn}}{(k_1k_3)(k_3n)(k_{123}\bar{n})} , & I_{12} &= \int \frac{\mathrm{d}\Phi_{\theta\delta}^{nnn}}{(k_1k_2)(k_{12}n)(k_2\bar{n})(k_{13}n)} , \\ I_{13} &= \int \frac{\mathrm{d}\Phi_{\theta\delta}^{nnn}}{(k_1k_2)(k_{12}n)(k_2\bar{n})(k_{123}\bar{n})} , & I_{14} &= \int \frac{\mathrm{d}\Phi_{\theta\delta}^{nnn}}{(k_1k_3)(k_{12}n)(k_{13}\bar{n})} , \\ I_{15} &= \int \frac{\mathrm{d}\Phi_{\theta\delta}^{nnn}}{(k_1k_2)(k_{12}n)(k_{13}\bar{n})} , & I_{16} &= \int \frac{\mathrm{d}\Phi_{\delta\theta}^{nnn}}{(k_1k_2)(k_{23}\bar{n})} , \\ I_{17} &= \int \frac{\mathrm{d}\Phi_{\delta\theta}^{nnn}}{(k_1k_2)(k_2\bar{n})(k_{13}\bar{n})} , & I_{18} &= \int \frac{\mathrm{d}\Phi_{\delta\theta}^{nnn}}{(k_1k_2)(k_2\bar{n})(k_{123}\bar{n})} , \\ I_{19} &= \int \frac{\mathrm{d}\Phi_{\delta\theta}^{nnn}}{(k_1k_2)(k_2n)(k_{123}\bar{n})} , & I_{20} &= \int \frac{\mathrm{d}\Phi_{\delta\theta}^{nnn}}{(k_1k_3)(k_{12}n)(k_{123}\bar{n})} , \\ I_{21} &= \int \frac{\mathrm{d}\Phi_{\delta\theta}^{nnn}}{(k_1k_2)(k_{23}n)(k_{123}\bar{n})} , & I_{22} &= \int \frac{\mathrm{d}\Phi_{\theta\delta\theta}^{nnn}}{(k_1k_3)(k_{12}n)(k_{123}\bar{n})} . \end{aligned} \tag{D.3.2}$$

- additional integral for $\int \mathrm{d}\Phi_{\theta\theta}^{nnn} \, \omega_{n\bar{n}}^{(3),b}$ with $1/\nu$ behaviour:

$$J_\nu = \int \frac{\mathrm{d}\Phi_{\theta\delta\theta}^{nnn} \, (k_1n)^\nu (k_2n)^\nu (k_3n)^\nu}{(k_1k_3)(k_1n)(k_{12}\bar{n})(k_3\bar{n})} . \tag{D.3.3}$$

- additional integrals for $\int \mathrm{d}\Phi^{nnn}_{\theta\theta\theta}\ \omega^{(3),c}_{n\bar{n}}$ :

$$\begin{aligned}
I_{23} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\theta\delta\delta}}{(k_1k_2)(k_1k_3)}\,, & I_{24} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\theta\delta\delta}}{(k_1k_2)(k_1k_3)(k_{12}\bar{n})}\,,\\
I_{25} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\theta\delta\delta}}{(k_1k_2)(k_1k_3)(k_{123}\bar{n})}\,, & I_{26} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\theta\delta\delta}}{(k_1k_2)(k_1k_3)(k_{12}n)(k_{13}\bar{n})}\,,\\
I_{27} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\theta\delta\delta}}{(k_1k_2)(k_1k_3)(k_{12}n)(k_{123}\bar{n})}\,, & I_{28} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\theta\delta\theta}}{(k_1k_2)(k_1k_3)(k_3\bar{n})}\,,\\
I_{29} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\theta\delta\theta}}{(k_1k_2)(k_1k_3)(k_{12}\bar{n})(k_3\bar{n})}\,, & I_{30} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\theta\delta\theta}}{(k_1k_2)(k_1k_3)(k_{12}n)(k_{13}\bar{n})}\,,\\
I_{31} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\theta\delta\theta}}{(k_1k_2)(k_1k_3)(k_{123}\bar{n})(k_3\bar{n})}\,, & I_{32} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\theta\delta\theta}}{(k_1k_2)(k_1k_3)(k_3n)(k_{123}\bar{n})}\,,\\
I_{33} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\theta\delta\theta}}{(k_1k_2)(k_1k_3)(k_{12}n)(k_{123}\bar{n})}\,, & I_{34} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\theta\delta\theta}}{(k_1k_2)(k_1k_3)(k_3n)(k_{123}\bar{n})^2}\,,\\
I_{35} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\theta\theta}}{(k_1k_2)(k_1k_3)(k_2n)(k_{123}\bar{n})(k_3\bar{n})}\,, & I_{36} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\theta\theta}}{(k_1k_2)(k_1k_3)(k_{12}n)(k_2\bar{n})(k_{13}\bar{n})}\,,\\
I_{37} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\theta\theta}}{(k_1k_2)(k_1k_3)(k_{12}n)(k_{123}\bar{n})(k_2\bar{n})}\,, & I_{38} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\theta\theta\delta}}{(k_1k_2)(k_1k_3)(k_{12}n)(k_2\bar{n})(k_{13}\bar{n})}\,,\\
I_{39} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\theta\theta\delta}}{(k_1k_2)(k_1k_3)(k_{12}n)(k_{123}\bar{n})(k_2\bar{n})}\,.
\end{aligned} \tag{D.3.4}$$

- additional integral for $\int \mathrm{d}\Phi^{nnn}_{\theta\theta\theta}\ \omega^{(3),d}_{n\bar{n}}$ without $1/k^2_{123}$ propagator:

$$I_{40} = \int \frac{\mathrm{d}\Phi^{nnn}_{\theta\delta\theta}}{(k_1k_3)(k_1n)(k_{123}\bar{n})(k_3\bar{n})}\,, \tag{D.3.5}$$

- additional integrals for $\int \mathrm{d}\Phi^{nnn}_{\theta\theta\theta}\ \omega^{(3),d}_{n\bar{n}}$ with $1/k^2_{123}$ propagator:

$$\begin{aligned}
I_{41} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\delta}}{k^2_{123}(k_{12}n)}\,, & I_{42} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\delta}}{k^2_{123}(k_1n)(k_2n)}\,,\\
I_{43} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\theta}}{k^2_{123}}\,, & I_{44} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\theta}}{k^2_{123}(k_3\bar{n})}\,,\\
I_{45} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\theta}}{k^2_{123}(k_{13}\bar{n})}\,, & I_{46} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\theta}}{k^2_{123}(k_{23}n)}\,,\\
I_{47} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\theta}}{k^2_{123}(k_{13}\bar{n})^2}\,, & I_{48} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\theta}}{k^2_{123}(k_1n)(k_3\bar{n})}\,,\\
I_{49} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\theta}}{k^2_{123}(k_1k_3)(k_3\bar{n})}\,, & I_{50} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\theta}}{k^2_{123}(k_1k_3)(k_3n)}\,,\\
I_{51} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\theta}}{k^2_{123}(k_{13}\bar{n})(k_2n)}\,, & I_{52} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\theta}}{k^2_{123}(k_{13}\bar{n})(k_1n)}\,,\\
I_{53} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\theta}}{k^2_{123}(k_1k_3)(k_{23}\bar{n})}\,, & I_{54} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\theta}}{k^2_{123}(k_{12}n)(k_1k_3)}\,,\\
I_{55} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\theta}}{k^2_{123}(k_1k_3)(k_{23}n)}\,, & I_{56} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\theta}}{k^2_{123}(k_{123}\bar{n})(k_1n)}\,,\\
I_{57} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\theta}}{k^2_{123}(k_{123}\bar{n})(k_3n)}\,, & I_{58} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\theta}}{k^2_{123}(k_{123}\bar{n})(k_{13}n)}\,,\\
I_{59} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\theta}}{k^2_{123}(k_1k_3)(k_{23}\bar{n})^2}\,, & I_{60} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\theta}}{k^2_{123}(k_{123}\bar{n})^2(k_{13}n)}\,,\\
I_{61} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\theta}}{k^2_{123}(k_{12}n)(k_1k_3)(k_3\bar{n})}\,, & I_{62} &= \int \frac{\mathrm{d}\Phi^{nnn}_{\delta\delta\theta}}{k^2_{123}(k_{123}\bar{n})(k_1n)(k_3n)}\,,
\end{aligned}$$

$$I_{63} = \int \frac{d\Phi^{nnn}_{\delta\delta\theta}}{k_{123}^2 (k_{123}\bar{n})(k_1k_3)(k_3n)}\,, \qquad I_{64} = \int \frac{d\Phi^{nnn}_{\delta\delta\theta}}{k_{123}^2 (k_{123}\bar{n})(k_1k_3)(k_3\bar{n})}\,, \tag{D.3.6}$$
$$I_{65} = \int \frac{d\Phi^{nnn}_{\delta\delta\theta}}{k_{123}^2 (k_{123}\bar{n})(k_{13}n)(k_1n)}\,, \qquad I_{66} = \int \frac{d\Phi^{nnn}_{\delta\delta\theta}}{k_{123}^2 (k_{123}\bar{n})(k_1k_3)(k_{23}n)}\,,$$
$$I_{67} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{k_{123}^2 (k_2\bar{n})}\,, \qquad I_{68} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{k_{123}^2 (k_{13}\bar{n})}\,,$$
$$I_{69} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{k_{123}^2 (k_{13}\bar{n})(k_1k_2)}\,, \qquad I_{70} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{k_{123}^2 (k_{12}n)(k_3\bar{n})}\,,$$
$$I_{71} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{k_{123}^2 (k_{123}\bar{n})(k_2n)}\,, \qquad I_{72} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{k_{123}^2 (k_{123}\bar{n})(k_{12}n)}\,,$$
$$I_{73} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{k_{123}^2 (k_1k_2)(k_2\bar{n})(k_3\bar{n})}\,, \qquad I_{74} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{k_{123}^2 (k_1k_2)(k_2n)(k_3\bar{n})}\,,$$
$$I_{75} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{k_{123}^2 (k_{13}\bar{n})(k_2n)(k_2\bar{n})}\,, \qquad I_{76} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{k_{123}^2 (k_{13}\bar{n})(k_1k_2)(k_2n)}\,,$$
$$I_{77} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{k_{123}^2 (k_1k_2)(k_{23}\bar{n})(k_2\bar{n})}\,, \qquad I_{78} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{k_{123}^2 (k_{123}\bar{n})(k_2n)(k_2\bar{n})}\,,$$
$$I_{79} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{k_{123}^2 (k_{123}\bar{n})(k_{12}n)(k_1k_3)}\,, \qquad I_{80} = \int \frac{d\Phi^{nnn}_{\theta\delta\theta}}{k_{123}^2 (k_{12}n)(k_1k_3)(k_3\bar{n})}\,,$$
$$I_{81} = \int \frac{d\Phi^{nnn}_{\theta\delta\theta}}{k_{123}^2 (k_1k_3)(k_1n)(k_{23}\bar{n})}\,, \qquad I_{82} = \int \frac{d\Phi^{nnn}_{\theta\delta\theta}}{k_{123}^2 (k_1k_3)(k_1n)(k_{23}\bar{n})^2}\,,$$
$$I_{83} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{k_{123}^2 (k_{123}\bar{n})(k_1k_3)(k_2n)(k_3\bar{n})}\,, \qquad I_{84} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{k_{123}^2 (k_{123}\bar{n})(k_{13}n)(k_1k_2)(k_2\bar{n})}\,,$$
$$I_{85} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{k_{123}^2 (k_{123}\bar{n})(k_1k_3)(k_{23}n)(k_2n)}\,, \qquad I_{86} = \int \frac{d\Phi^{nnn}_{\theta\delta\theta}}{k_{123}^2 (k_{123}\bar{n})(k_1k_3)(k_1\bar{n})(k_{23}n)}\,.$$

- additional integral for $\int d\Phi^{nnn}_{\theta\theta\theta}\ \omega^{(3),d}_{n\bar{n}}$ with $1/k_{123}^2$ propagator and $1/\nu$ behaviour:

$$J_2 = \int \frac{d\Phi^{nnn}_{\theta\delta\theta}\ (k_1n)^\nu (k_2n)^\nu (k_3n)^\nu}{k_{123}^2 (k_1k_3)(k_1n)(k_{12}\bar{n})(k_3\bar{n})}\,. \tag{D.3.7}$$

Analytic solutions to the integrals $I_1$ through $I_{40}$ are given in electronic form in the ancillary file of Ref. [77].

## D.4 BOUNDARY CONDITIONS FOR TRIPLE GLUON SAME HEMISPHERE EMISSION TO THE SOFT FUNCTION

The various boundary conditions needed for the calculation of $\int d\Phi^{nnn}_{\theta\theta\theta}\ \omega^{(3),d}_{n\bar{n}}$ are defined as follows

- integrals for which the leading $m^{-2\epsilon}$-branch needs to be extracted:

$$B_1 = \int \frac{d\Phi^{nnn}_{\delta\delta\theta}}{k_{123}^2 + m^2}\,, \qquad B_2 = \int \frac{d\Phi^{nnn}_{\delta\delta\theta}}{(k_{123}^2 + m^2)^2}\,,$$
$$B_3 = \int \frac{d\Phi^{nnn}_{\delta\delta\theta}}{(k_{123}^2 + m^2)^3}\,, \qquad B_4 = \int \frac{d\Phi^{nnn}_{\delta\delta\theta}}{(k_{123}^2 + m^2)(k_{23}n)}\,,$$
$$B_5 = \int \frac{d\Phi^{nnn}_{\delta\delta\theta}}{(k_{123}^2 + m^2)(k_{23}n)^2}\,, \qquad B_6 = \int \frac{d\Phi^{nnn}_{\delta\delta\theta}}{(k_{123}^2 + m^2)^2(k_{23}n)}\,, \tag{D.4.1}$$
$$B_7 = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2 + m^2)(k_1k_2)(k_2\bar{n})}\,, \qquad B_8 = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2 + m^2)(k_{13}\bar{n})}\,,$$

$$B_9 = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)(k_{13}\bar{n})}\,, \qquad B_{10} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)(k_{13}\bar{n})^2}\,,$$
$$B_{11} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)^2(k_{13}\bar{n})}\,, \qquad B_{12} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)^2(k_{13}\bar{n})}\,,$$
$$B_{13} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)(k_1k_3)(k_3\bar{n})(k_{12}n)}\,,$$

- integrals for which the leading $m^{-4\epsilon}$-branch needs to be extracted:

$$\mathcal{B}_1 = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)(k_1k_2)(k_2n)(k_3n)(k_{123}\bar{n})}\,, \qquad \mathcal{B}_2 = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)(k_1k_2)(k_2n)(k_3n)(k_{123}\bar{n})^2}\,,$$
$$\mathcal{B}_3 = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)(k_{12}n)(k_{123}\bar{n})}\,, \qquad \mathcal{B}_4 = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)(k_{12}n)(k_{123}\bar{n})^2}\,,$$
$$\mathcal{B}_5 = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)(k_{12}n)^2(k_{123}\bar{n})}\,, \qquad \mathcal{B}_6 = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)^2(k_{12}n)(k_{123}\bar{n})}\,,$$
$$\mathcal{B}_7 = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)(k_1k_2)(k_{23}n)}\,, \qquad \mathcal{B}_8 = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)(k_1k_2)(k_2n)(k_{23}n)(k_{123}\bar{n})}\,,$$
$$\mathcal{B}_9 = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)(k_1k_2)(k_2n)(k_{23}n)(k_{123}\bar{n})^2}\,, \qquad \mathcal{B}_{10} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{k_{123}^2+m^2}\,,$$
$$\mathcal{B}_{11} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)(k_2n)(k_2\bar{n})(k_{13}\bar{n})}\,, \qquad \mathcal{B}_{12} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)(k_2n)(k_2\bar{n})(k_{13}\bar{n})^2}\,,$$
$$\mathcal{B}_{13} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)(k_1k_2)(k_2n)(k_{13}\bar{n})}\,, \qquad \mathcal{B}_{14} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)(k_1k_2)(k_2n)(k_{13}\bar{n})^2}\,, \tag{D.4.2}$$
$$\mathcal{B}_{15} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)(k_1k_2)(k_2n)^2(k_{13}\bar{n})}\,, \qquad \mathcal{B}_{16} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)^2(k_1k_2)(k_2n)(k_{13}\bar{n})}\,,$$
$$\mathcal{B}_{17} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)(k_{12}n)(k_3\bar{n})}\,, \qquad \mathcal{B}_{18} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)(k_{12})^2(k_3\bar{n})}\,,$$
$$\mathcal{B}_{19} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)(k_1k_3)(k_{12}n)}\,, \qquad \mathcal{B}_{20} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)(k_2n)(k_{123}\bar{n})}\,,$$
$$\mathcal{B}_{21} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)(k_2n)(k_{123}\bar{n})^2}\,, \qquad \mathcal{B}_{22} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)(k_2n)(k_2\bar{n})(k_{123}\bar{n})}\,,$$
$$\mathcal{B}_{23} = \int \frac{d\Phi^{nnn}_{\delta\theta\theta}}{(k_{123}^2+m^2)(k_2n)(k_2\bar{n})(k_{123}\bar{n})^2}\,, \qquad \mathcal{B}_{24} = \int \frac{d\Phi^{nnn}_{\theta\delta\theta}}{(k_{123}^2+m^2)(k_1k_3)(k_1n)(k_{23}\bar{n})^2}\,,$$
$$\mathcal{B}_{25} = \int \frac{d\Phi^{nnn}_{\theta\delta\theta}}{(k_{123}^2+m^2)^2(k_1k_3)(k_1n)(k_{23}\bar{n})}\,, \qquad \mathcal{B}_{26} = \int \frac{d\Phi^{nnn}_{\theta\delta\theta}}{(k_{123}^2+m^2)(k_1k_3)(k_1n)(k_{23}\bar{n})}\,,$$
$$\mathcal{B}_{27} = \int \frac{d\Phi^{nnn}_{\theta\delta\theta}}{(k_{123}^2+m^2)(k_1k_3)(k_1n)^2(k_{23}\bar{n})}\,,$$

- additonal integral for which the leading $m^{-2\nu}$-branch is required:

$$\mathcal{B}_\nu = \int \frac{d\Phi^{nnn}_{\theta\delta\theta}\,(k_1n)^\nu(k_2n)^\nu(k_3n)^\nu}{(k_{123}^2+m^2)(k_1k_3)(k_1n)(k_{12}\bar{n})(k_3\bar{n})}\,. \tag{D.4.3}$$

## D.5 COMPUTATION OF ABELIAN CONTRIBUTIONS TO THE SOFT FUNCTION

In this section we compute abelian contributions to the N3LO soft function i.e. the first two terms in Eq. (6.7.1). The first term is the so-called fully-abalian contribution, it is defined as

$$S_{ggg}^{nnn,1} = \int \mathrm{d}\Phi_{\theta\theta\theta}^{nnn}\; \omega_{n\bar{n}}^{(1)}(k_1)\; \omega_{n\bar{n}}^{(1)}(k_2)\; \omega_{n\bar{n}}^{(1)}(k_3), \tag{D.5.1}$$

where

$$\omega_{n\bar{n}}^{(1)}(k) = \frac{4}{(n\cdot k)\,(\bar{n}\cdot k)}\,. \tag{D.5.2}$$

The calculation is straightforward and we immediately obtain

$$\begin{aligned} S_{ggg}^{nnn,1} &= [N]^3 64 \int\limits_0^\infty \left( \prod_{i=1}^{3} \mathrm{d}\alpha_i\, \mathrm{d}\beta_i\; (\alpha_i\beta_i)^{-1-\epsilon}\, \theta(\alpha_i-\beta_i) \right) \delta(1-\beta_{123}) \\ &= [N]^3 \frac{64}{\epsilon^3}\, \frac{\Gamma^3(-2\epsilon)}{\Gamma(-6\epsilon)}\,. \end{aligned} \tag{D.5.3}$$

The second abelian contribution is defined as

$$S_{ggg}^{nnn,2} = \int \mathrm{d}\Phi_{\theta\theta\theta}^{nnn}\; \left[\omega_{n\bar{n}}^{(1)}(k_1)\omega_{n\bar{n}}^{(2)}(k_2,k_3) + (1\leftrightarrow 2) + (1 \leftrightarrow 3)\right] \tag{D.5.4}$$

$$= 3\int \mathrm{d}\Phi_{\theta\theta\theta}^{nnn}\; \left[\omega_{n\bar{n}}^{(1)}(k_1)\,\omega_{n\bar{n}}^{(2)}(k_2,k_3)\right]\,. \tag{D.5.5}$$

We write this integral as

$$\begin{aligned} S_{ggg}^{nnn,2} &= 3\int [\mathrm{d}k_1]\, \theta(k_1\bar{n} - k_1 n)\; \omega_{n\bar{n}}^{(1)}(k_1) \\ &\times \int \left( \prod_{i=2}^{3} [\mathrm{d}k_i]\, \theta(k_i\bar{n} - k_i n) \right) \delta(1-k_{123}n)\, \omega_{n\bar{n}}^{(2)}(k_2,k_3)\,. \end{aligned} \tag{D.5.6}$$

The inner integral in Eq. (D.5.6) over $[\mathrm{d}k_2][\mathrm{d}k_3]$ is simply equal to same-hemisphere double-real gluon emission contribution to the NNLO soft function in Eq. (5.4.1) with the replacement $\tau \to 1-\beta_1$. We find

$$\begin{aligned} S_{1+2}^{nnn} = 3[N]\; &\int_0^\infty \mathrm{d}\alpha_1\, \mathrm{d}\beta_1\, \theta(\alpha_1-\beta_1)\, \frac{4(\alpha_1\beta_1)^{-\epsilon}}{\alpha_1\beta_1} \times [N]^2 C_2^{nn}(1-\beta_1)^{-1-4\epsilon} \\ = [N]^3 \frac{12}{\epsilon}\, &\frac{\Gamma(-4\epsilon)\,\Gamma(-2\epsilon)}{\Gamma(-6\epsilon)}\, C_2^{nn}\,, \end{aligned} \tag{D.5.7}$$

where the definition of $C_2^{nn}$ is evident from comparison with Eq. (5.4.1).

## D.6 COMPUTATION OF THE DIVERGENT MASTER INTEGRAL IN CONFIGURATION $B$

In Section 6.4 we split the calculation of the integral

$$J_\nu(\epsilon,\nu) = \int \frac{d\Phi^{nnn}_{\theta\delta\theta}(k_1 n)^\nu (k_2 n)^\nu (k_3 n)^\nu}{(k_1 k_3)(k_1 n)(k_{12}\bar{n})(k_3 \bar{n})}, \tag{D.6.1}$$

into two pieces, $J_{1,\nu}(\epsilon,\nu)$ and $J_{2,\nu}(\epsilon,\nu)$ for $z_1 > z_3$ and $z_3 > z_1$ respectively. We now discuss the second contribution $J_{2,\nu}(\epsilon,\nu)$. We use

$$[dk_i] = \frac{d\Omega_i^{(d-2)}}{4(2\pi)^{d-1}}\, d\alpha_i\, d\beta_i\, (\alpha_i\beta_i)^{-\epsilon}, \qquad \alpha_i\,,\beta_i \in [0,\infty)\,, \tag{D.6.2}$$

together with the angular integration

$$\int \frac{d\Omega_1^{(d-2)} d\Omega_3^{(d-2)}}{2k_1\cdot k_3}\Bigg|_{\alpha_i\to\frac{\beta_i}{z_i}} = \frac{\left[\Omega^{(d-2)}\right]^2 z_1}{\beta_1\beta_3}\; {}_2F_1\left(1, 1+\epsilon, 1-\epsilon, \frac{z_1}{z_3}\right), \tag{D.6.3}$$

to obtain

$$\begin{aligned} J_{2,\nu}(\epsilon,\nu) = [N]^3 \int dz_3 \int dz_1 \prod_i d\alpha_i d\beta_i \frac{2z_1\alpha_1^{1-\epsilon}\alpha_3^{-\epsilon}\beta_1^{-\epsilon-2+\nu}\beta_2^{-2\epsilon+\nu}\beta_3^{-1-\epsilon+\nu}}{\alpha_1+\beta_2} \\ \times\, \delta(\alpha_1 z_1 - \beta_1)\delta(\alpha_3 z_3 - \beta_3)\delta(1-\beta_{123})\, {}_2F_1\left(1, \epsilon+1, 1-\epsilon, \frac{z_1}{z_3}\right). \end{aligned} \tag{D.6.4}$$

We again remove delta functions by integrating $\alpha_1$, $\alpha_3$ and $\beta_1$ and change variables to $x$ and $y$ such that $b_3 = x\, y$ and $b_2 = x\,(1-y)$. After changing variables $z_3 = tz_1$ and integrating over $x$ the expression reads

$$\begin{aligned} J_{2,\nu}(\epsilon,\nu) = 2\,[N]^3 \int dz_3\, dt\, dy\, t^\epsilon\, z_3^{2\epsilon}\, (1-y)^{\nu-2\epsilon}\, y^{-2\epsilon+\nu-1} \\ \times \frac{\Gamma(\nu-2\epsilon)\Gamma(-4\epsilon+2\nu+1)}{\Gamma(-6\epsilon+3\nu+1)}\, {}_2F_1(1,\epsilon+1,1-\epsilon,t) \\ \times\, {}_2F_1(1,-4\epsilon+2\nu+1,-6\epsilon+3\nu+1,t(y-1)z_3+1). \end{aligned} \tag{D.6.5}$$

We rewrite the last hypergeometric function in Eq. (D.6.5) as a Melin-Barnes integral and integrate over $y$. We obtain

$$\begin{aligned}
J_{2,\nu}(\epsilon,\nu) &= 2\,[N]^3 \int \mathrm{d}z_3 \mathrm{d}t\ t^{\epsilon}\ z_3^{2\epsilon}\,{}_2F_1(1,\epsilon+1,1-\epsilon,t)\frac{\Gamma(\nu-2\epsilon)}{\Gamma(3\nu-6\epsilon)} \\
&\quad \times \int_{-i\infty}^{i\infty} \mathrm{d}z_M (t\ z3)^{-z_M}\Gamma(1-z_M)\Gamma(z_M)\Gamma(-2\epsilon+\nu-z_M+1) \\
&\quad \times \Gamma(-2\epsilon+\nu+z_M-1) \\
J_{2,\nu}(\epsilon,\nu) &= 2\,[N]^3 \int \mathrm{d}z_3 \mathrm{d}t\ t^{\epsilon}\ z_3^{2\epsilon}\,{}_2F_1(1,\epsilon+1,1-\epsilon,t)\frac{\Gamma(\nu-2\epsilon)}{\Gamma(3\nu-6\epsilon)} \\
&\quad \times \frac{\Gamma(\nu-2\epsilon)\Gamma(-2\epsilon+\nu+1)\Gamma(2\nu-4\epsilon)}{\Gamma(-4\epsilon+2\nu+1)} \\
&\quad \times\ {}_2F_1(1,-2\epsilon+\nu+1,-4\epsilon+2\nu+1,1-t\ z_3)
\end{aligned} \tag{D.6.6}$$

Finally we re-write the last hypergeometric function in Eq. (D.6.6) using

$$\begin{aligned}
&{}_2F_1(1,-2\epsilon+\nu+1,-4\epsilon+2\nu+1,1-t\ z_3) \\
&= \frac{t^{-2\epsilon+\nu-1} z_3^{-2\epsilon+\nu-1}\Gamma(2\epsilon-\nu+1)\Gamma(-4\epsilon+2\nu+1)(1-tz_3)^{4\epsilon-2\nu}}{\Gamma(-2\epsilon+\nu+1)} \\
&+ \frac{\Gamma(-2\epsilon+\nu-1)\Gamma(-4\epsilon+2\nu+1)}{\Gamma(\nu-2\epsilon)\Gamma(2\nu-4\epsilon)}\,{}_2F_1(1,-2\epsilon+\nu+1,2\epsilon-\nu+2,tz_3),
\end{aligned} \tag{D.6.7}$$

and find

$$\begin{aligned}
J_{2,\nu}(\epsilon,\nu) &= 2\,[N]^3 \int \mathrm{d}z_3 \mathrm{d}t z_3^{-1+\nu}\ t^{-\epsilon+\nu-1}\ (1-tz_3)^{4\epsilon-2\nu} \\
&\quad \times \frac{\Gamma(2\epsilon-\nu+1)\Gamma(\nu-2\epsilon)^2\Gamma(2\nu-4\epsilon)}{\Gamma(3\nu-6\epsilon)}\,{}_2F_1(1,\epsilon+1,1-\epsilon,t) \\
&\quad + 2\,[N]^3 \int \mathrm{d}z_3 \mathrm{d}t\ t^{\epsilon}\ z_3^{2\epsilon}\frac{\Gamma(\nu-2\epsilon)\Gamma(-2\epsilon+\nu-1)\Gamma(-2\epsilon+\nu+1)}{\Gamma(3\nu-6\epsilon)} \\
&\quad \times\ {}_2F_1(1,\epsilon+1,1-\epsilon,t)\ {}_2F_1(1,-2\epsilon+\nu+1,2\epsilon-\nu+2,tz_3).
\end{aligned} \tag{D.6.8}$$

## D.7 MATRIX FOR THE EXAMPLE DEQ

In Section 6.6 we discussed how to solve differential equations w.r.t. the mass parameter for eleven example master integrals. These differential equations could be written as five eleven-by-eleven matrices shown below.

$$\boldsymbol{M}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{2(4\epsilon-1)}{(2\epsilon-1)(5\epsilon-2)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{D.7.1}$$

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1-3\epsilon & -\epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{(3\epsilon-1)(6\epsilon-1)}{2\epsilon} & \frac{1}{2}(1-6\epsilon) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\epsilon-1 & 0 & -\frac{1}{2}(5\epsilon-1)(6\epsilon-1) & 1-6\epsilon & -3\epsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-2\epsilon & 0 & \frac{1}{2}(5\epsilon-1)(6\epsilon-1) & 6\epsilon-1 & 3\epsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1-3\epsilon}{\epsilon} & \frac{1-3\epsilon}{\epsilon} & 0 & \frac{(5\epsilon-1)(6\epsilon-1)}{2\epsilon} & \frac{6\epsilon-1}{\epsilon} & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{2(3\epsilon-1)}{\epsilon} & -\frac{2(3\epsilon-1)}{\epsilon} & 0 & \frac{30\epsilon^2-11\epsilon+1}{\epsilon} & \frac{2(6\epsilon-1)}{\epsilon} & 6 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{D.7.2}$$

$$M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3\epsilon-1}{\epsilon} & \frac{3\epsilon-1}{\epsilon} & \frac{1}{4} & -\frac{(5\epsilon-1)(6\epsilon-1)}{2\epsilon} & -\frac{3(6\epsilon-1)}{8\epsilon} & -3 & 0 & -3\epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2(3\epsilon-1)(7\epsilon-2)}{\epsilon(5\epsilon-2)} & \frac{11\epsilon-4}{2\epsilon} & 0 & \frac{-238\epsilon^3+157\epsilon^2-32\epsilon+2}{\epsilon(5\epsilon-2)} & \frac{-25\epsilon^2+11\epsilon-1}{\epsilon(5\epsilon-2)} & \frac{34-97\epsilon}{2(5\epsilon-2)} & \frac{7\epsilon}{2} & \frac{1}{2}(2\epsilon-1) & 4(2\epsilon-1)(3\epsilon-1) & -\frac{1}{2}(2\epsilon-1)(6\epsilon-1) & \frac{1}{2}(1-8\epsilon) \end{pmatrix} \tag{D.7.3}$$

$$M_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3\epsilon - 1 & -2\epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{(3\epsilon-1)(6\epsilon-1)}{2\epsilon} & \frac{1}{2}(6\epsilon - 1) & \frac{1}{2}(-6\epsilon - 1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 - 2\epsilon & 0 & \frac{1}{2}(5\epsilon - 1)(6\epsilon - 1) & \frac{1}{2}(-4\epsilon - 1) & 3\epsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(2\epsilon - 1) & 0 & -\frac{1}{2}(5\epsilon - 1)(6\epsilon - 1) & 0 & -4\epsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2\epsilon-1}{2\epsilon} & 0 & 0 & 0 & -2 & -3\epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2-5\epsilon} & 0 & 0 & \frac{(4\epsilon-1)^2}{(3\epsilon-1)(5\epsilon-2)} & 0 & 0 & 0 & 0 & 1 - 2\epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1 - 6\epsilon) & \frac{1}{2} \\ -\frac{4(3\epsilon-1)}{5\epsilon-2} & 0 & 0 & \frac{4(4\epsilon-1)^2}{5\epsilon-2} & 0 & 0 & 0 & 0 & -4(2\epsilon - 1)(3\epsilon - 1) & \frac{1}{2}(2\epsilon - 1)(6\epsilon - 1) & \frac{1}{2}(1 - 2\epsilon) \end{pmatrix} \quad (D.7.4)$$

$$M_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(4\epsilon-1)(6\epsilon-1)}{(2\epsilon-1)(5\epsilon-2)} & -\frac{2(4\epsilon-1)}{(3\epsilon-1)(5\epsilon-2)} & \frac{2\epsilon(4\epsilon-1)}{(2\epsilon-1)(3\epsilon-1)(5\epsilon-2)} & -\frac{2\epsilon^2}{(2\epsilon-1)(3\epsilon-1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2(4\epsilon-1)}{5\epsilon-2} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (D.7.5)$$

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## colophon

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