



Cwikel’s bound reloaded

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Abstract There are several proofs by now for the famous Cwikel–Lieb–Rozenblum (CLR) bound, which is a semiclassical bound on the number of bound states for a Schrödinger operator, proven in the 1970s. Of the rather distinct proofs by Cwikel, Lieb, and Rozenblum, the one by Lieb gives the best constant, the one by Rozenblum does not seem to yield any reasonable estimate for the constants, and Cwikel’s proof is said to give a constant which is at least about 2 orders of magnitude off the truth. This situation did not change much during the last 40+ years. It turns out that this common belief, i.e., Cwikel’s

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approach yields bad constants, is not set in stone: We give a substantial refinement of Cwikel's original approach which highlights a natural but overlooked connection of the CLR bound with bounds for maximal Fourier multipliers from harmonic analysis. Moreover, it gives an astonishingly good bound for the constant in the CLR inequality. Our proof is also quite flexible and leads to rather precise bounds for a large class of Schrödinger-type operators with generalized kinetic energies.

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1 Introduction

We want to find natural bounds, with the right semi-classical behavior, for the number of negative eigenvalues of Schrödinger operators $P^2 + V$, with momentum operator $P = -i\nabla$, or more general operators like polyharmonic Schrödinger operators $|P|^{2\alpha} + V$, including the ultra-relativistic operator $|P| + V$. We also consider operator-valued potentials V .

For the one-particle Schrödinger operator $P^2 + V$ with a real-valued potential V , this type of bound goes back to Cwikel, Lieb, and Rozenblum [9,31,32,43,44], with very different proofs. They prove

$$N(P^2 + V) \leq L_{0,d} \int_{\mathbb{R}^d} V_-(x)^{d/2} dx \quad (1.1)$$

for the number of negative eigenvalues of a Schrödinger operator, where $L_{0,d}$ is a constant depending only on the dimension. This bound is a semi-classical bound since a simple scaling argument shows that the classical phase-space

volume of the region of negative energy is given by

$$N^{\text{cl}}(\eta^2 + V) = \iint_{\eta^2 + V(x) < 0} 1 \frac{d\eta dx}{(2\pi)^d} = \frac{|B_1^d|}{(2\pi)^d} \int_{\mathbb{R}^d} V_-(x)^{d/2} dx. \quad (1.2)$$

where $|B_1^d|$ is the volume of the unit ball in \mathbb{R}^d .

Rozenblum's paper [43] was an announcement of his result and, typically for the journal, did not contain any proofs. The version with full proofs was published in [44]. Similarly, Lieb's paper [31] is an announcement of his result and the details of his proof were published later in [32, 50]. The approach of Rozenblum was strongly motivated by the St. Petersburg school of mathematical physics around Birman and Solomyak, whose work had been virtually unnoticed in the west until the mid 1970s, see the "Added notes" on page 378 in [49]. The proofs of Cwikel and Lieb were strongly motivated by Simon [49]. Cwikel's approach was developed into a more general scheme by Birman and Solomyak, see e.g. [2, 5]. They were able to obtain more general versions of Cwikel's result in which the L^p and weak- L^p spaces appearing in [9] could be replaced by more general spaces. For the most recent developments in this direction, see [28], which builds upon earlier work by Weidl [53, 54].

The intuition behind semi-classical bounds is that the uncertainty principle forces a quantum particle to occupy roughly a classical phase-space volume $(2\pi)^d$. Thus the phase-space volume $N^{\text{cl}}(\eta^2 + V)$ where the classical Hamiltonian energy $H(\eta, x) = \eta^2 + V(x)$ is negative, should control $N(P^2 + V)$. The CLR bound (1.1) shows that this is the case up to a factor¹ $C_{0,d} = L_{0,d}(2\pi)^d/|B_1^d|$. Simon's profound insights connecting bounds on $N(P^2 + V)$ with known and conjectured interpolation properties of weak operator ideals,² and, in particular, his Conjecture 1 on page 372 in [49], were a major motivation for Cwikel's work. The discussion in [49] suggested that perhaps some new and more powerful interpolation theorem might yield the weak trace ideal bounds of Conjecture 1 of [49], which would suffice to prove the CLR inequality. As he informed us [10], Cwikel initially tried to see if one of the bilinear interpolation theorems in fundamental papers of Calderón [6, p. 118] and Lions–Peetre [35, p. 14] about interpolation spaces, or some variant of them, might prove Simon's Conjecture 1. Indeed Proposition 4.2 of [49] can also be obtained from [6, p. 118].

Unfortunately, as shown on page 97 in [9], a proof of Simon's Conjecture 1 cannot be obtained by any kind of bilinear interpolation. However, as Cwikel

¹ We write $L_{0,d}$ etc., since there are a class of inequalities due to Lieb and Thirring for the γ th moment of the negative eigenvalues with associated constants $L_{\gamma,d}$, see [33, 34] and the reviews [22, 29].

² Parts of this connection were already known to the St. Petersburg school of mathematical physics around Birman and Solomyak, see the above mentioned "Added notes" in [49].

strongly emphasized to us [10], some elements of his proof evolved and benefitted greatly from ideas around Lions and Peetre's Théorème 4.1 of [35, p. 14].

One of our main new contributions is that the CLR bound is intimately related to the fact that certain maximal Fourier multipliers are bounded on $L^2(\mathbb{R}^d)$. This leads to a new class of variational problems, see Theorem 1.3, which allows us to improve Lieb's constants in dimensions $d \geq 5$. The original bounds on $C_{0,d}$ in [9] and [31] were explicitly dimension dependent with a considerable growth in the dimension d . The bound due to Lieb grows like $C_{0,d} = \sqrt{\pi d}(1 + O(d^{-1}))$. See [50] or [42, Chapter 3.4] for an excellent discussion of Lieb's method and Remark 1.2 below for some explicit numbers. However, it is expected that semi-classical arguments work better in high dimensions. In particular, the constant $C_{0,d}$ should not grow in d . The first dimension independent bound $C_{0,d} \leq 81$ was derived by extending Cwikel's method to operator-valued potentials in 2002 in [21]. This work extended an *induction in the dimension* argument³ by Laptev and Weidl [27], who were the first to derive Lieb–Thirring bounds with the sharp classical Lieb–Thirring constant in all dimensions in some cases. Although the upper bound from [21] is dimension independent, it is certainly too large for small dimensions.

For the last 40-plus years it has been believed that any approach based on Cwikel's method cannot yield any bounds on $C_{0,d}$ which are comparable to the ones obtained by Lieb in low dimensions. This is wrong, as we will show by drastically simplifying and, at the same time, generalizing the important ideas of Cwikel. A typical result which can be easily achieved with our method is

Theorem 1.1 *The number $N(P^2 + V)$ of negative energy bound states of $P^2 + V$ obeys the semiclassical bound*

$$N(P^2 + V) \leq C_{0,d} \frac{|B^d|}{(2\pi)^d} \int_{\mathbb{R}^d} V_-(x)^{d/2} dx \quad (1.3)$$

for all $d \geq 3$, where B^d is the unit ball in \mathbb{R}^d , $|B^d|$ its volume, and the constant $C_{0,d}$ given in Table 1 below.

Moreover, the same bounds with the same constants also hold in the operator-valued case, see Theorem 1.8.

Remark 1.2 (i) Table 1 below compares the upper bounds on $C_{0,d}$, obtained with our method, with the best known ones so far for scalar and operator-valued potentials. All bounds on $C_{0,d}$ in the third column of the table were obtained already in the original work of Lieb more than 40 years ago.⁴ Our

³ See also [25] for some indication of the *induction in dimension* trick.

⁴ The numbers are taken from Roepstorff's book [42, Table 3.1].

Table 1 Comparison between the upper bounds on $C_{0,d}$ obtained by our method with the best known ones so far

d	Our results scalar and operator valued	Best known so far	
		Scalar	Operator-valued
3	7.55151	6.86924	
4	6.32791	6.03398	
5	5.95405	5.96677	
6	5.77058	6.07489	10.332
7	5.67647	6.24464	
8	5.63198	6.43921	
9	5.62080	6.64378	

bounds on $C_{0,d}$ also hold in the operator-valued case, see Sect. 6 below. The value in the last column is due to Frank, Lieb and Seiringer [19] and holds for all $d \geq 3$. Our result also gives the bound $C_{0,d} \leq 5.62080$ for $d \geq 9$, see the discussion in Appendix A. For dimensions $3 \leq d \leq 9$ our upper bounds are compared with the values of the lower bound (1.10) achievable by our method in Table 2 below.

- (ii) There have been several previous attempts to improve Lieb’s result, for example, due to Conlon [8], Li and Yau [30], Frank [16], and Weidl [53, 54]. All these very different proofs shed a new light on the Cwikel–Lieb–Rozenblum bound, but failed to give better bounds on the involved constants than already achieved by Lieb.

From the point of view of physics, the other important case is the ultra-relativistic Schrödinger operator $|P| + V$. For more general so-called polyharmonic Schrödinger operators our method yields the following bound for scalar potentials, which involves an interesting variational problem.

Theorem 1.3 *Let $P = -i\nabla$ be the momentum operator, $V = V_+ - V_-$ be a real-valued potential with positive part $V_+ \in L^1_{loc}$ and negative part $V_- \in L^{d/\alpha}(\mathbb{R}^d)$ with $0 < \alpha < d/2$, and $P^{2\alpha} + V$ the Schrödinger-type operator defined via quadratic form methods on $L^2(\mathbb{R}^d)$.*

Furthermore, consider the minimization problem

$$\begin{aligned}
 M_\gamma = \inf \left\{ \left(\|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \right)^{\gamma-2} \right. \\
 \left. \times \int_0^\infty (1 - t^{-1}m(t))^2 t^{1-\gamma} dt \right\}, \tag{1.4}
 \end{aligned}$$

where $\gamma > 2$, the infimum is taken over all $m_1, m_2 \in L^2(\mathbb{R}_+, \frac{ds}{s})$, and $m = m_1 * m_2$ denotes the convolution of m_1, m_2 on \mathbb{R}_+ with measure $\frac{ds}{s}$ and let

$$C_\gamma = \frac{\gamma^{\gamma+1}}{4(\gamma-2)^{\gamma-2}} M_\gamma. \quad (1.5)$$

Then the number $N(P^{2\alpha} + V)$ of negative energy bound states of $P^{2\alpha} + V$ is bounded by

$$N(P^{2\alpha} + V) \leq C_{d/\alpha} \frac{|B_1^d|}{(2\pi)^d} \int_{\mathbb{R}^d} V_-(x)^{\frac{d}{2\alpha}} dx, \quad (1.6)$$

with constant $C_{d/\alpha}$ given by (1.5) for $\gamma = \frac{d}{\alpha}$.

For $\alpha = 1/2$ and in three dimensions we get the upper bound

$$N(|P| + V) \leq 5.77058 \int_{\mathbb{R}^3} V_-(x)^3 dx \quad (1.7)$$

which improves the result of Daubechies [12], who gets $N(|P| + V) \leq 6.08 \int_{\mathbb{R}^3} V_-(x)^3 dx$.

A similar result, with the same constants, also holds for operator-valued potentials, see Theorem 1.7.

Remark 1.4 The minimisation problem for M_γ in (1.4) is crucial for getting good bounds on the constant in the Cwikel–Lieb–Rozenblum bound. It allows us to obtain the first improvement, in more than 40 years, on the constants derived originally by Lieb [31] in dimensions $d \geq 5$.

A simple, but not optimal, choice for m_1, m_2 is $m_1(s) = s \mathbf{1}_{\{0 < s \leq 1\}}$ and $m_2(s) = 2s^{-1} \mathbf{1}_{\{s > 1\}}$, in which case $\|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} = 1$ and $m(t) = m_1 * m_2(t) = \min(t, t^{-1})$, so

$$\int_0^\infty (1 - t^{-1}m(t))^2 t^{1-\gamma} dt = \int_1^\infty (1 - t^{-2})^2 t^{1-\gamma} dt = \frac{8}{(\gamma-2)\gamma(\gamma+2)}.$$

This gives

$$C_{0,d} = \frac{2d^d}{(d-2)^{d-1}(d+2)}$$

as a possible constant in the CLR inequality and yields $C_{0,3} \leq 10.8$, already an order of a magnitude smaller than Cwikel's bound. To get the uniform bound claimed in Theorem 1.1 we have to choose better candidates for m_1 and m_2 . We can achieve this in small dimensions, see Appendix D. Moreover, combining

this with ‘stripping-off-dimensions’ ideas, see Appendix A, with the help of similar bounds for operator-valued potentials presented in Sect. 6, one can get this bound also uniformly in the dimension for the important special case of non-relativistic Schrödinger operators, where $\alpha = 1$.

Choosing $m_1(s) = s\mathbf{1}_{\{0 < s < 1\}}$, we can actually solve the minimization problem for m_2 , see Propositions C.1 and C.4 in Appendix C. This leads to the upper bound in

Proposition 1.5 *For all $\gamma > 2$*

$$\frac{2}{\gamma(\gamma - 1)(\gamma - 2)} \leq M_\gamma \leq \frac{4}{(\gamma - 2)\gamma^2} \frac{1}{\Gamma(\frac{2}{\gamma})^\gamma} \left(\frac{\gamma - 2}{2} \frac{\pi}{\sin(\frac{2\pi}{\gamma})} \right)^{\frac{\gamma}{2}}. \tag{1.8}$$

For the proof of the lower bound see Sect. 5.

Remark 1.6 (i) So far the best known bound for polyharmonic Schrödinger operators is due to Frank [16], who proved

$$N(P^{2\alpha} + V) \leq \left(\frac{d(d + 2\alpha)}{(d - 2\alpha)^2} \right)^{(d-2\alpha)/(2\alpha)} \frac{d}{d - 2\alpha} \frac{|B_1^d|}{(2\pi)^d} \int_{\mathbb{R}^d} V_-(x)^{\frac{d}{2\alpha}} dx, \tag{1.9}$$

based on ideas of Rumin [46,47]. Even the simple upper bound on M_γ from Remark 1.4 yields better results than (1.9). Computing the ratio of the constants in Frank’s bound and the one from (1.6), using the upper bound in (1.8), one sees that our bound from Theorem 1.3 is better in the whole allowed range of $0 < \alpha < d/2$.

(ii) For the constant C_γ in (1.5), the lower bound from (1.8) yields

$$C_\gamma \geq \frac{\gamma^\gamma}{2(\gamma - 1)(\gamma - 2)^{\gamma-1}} =: C_\gamma^{\text{lower}},$$

where C_γ^{lower} is a probably non-sharp lower bound for the best possible constant achievable by our method.⁵ Thus the upper bound on M_γ from Remark 1.4 gives

$$\frac{C_\gamma}{C_\gamma^{\text{lower}}} \leq 4 \frac{\gamma - 1}{\gamma + 2} < 4,$$

⁵ Which is of course not necessarily the best possible constant.

where $\gamma = d/\alpha > 2$. This shows that our easy upper bound is less than a factor of 4 off the lower bound.⁶

(iii) The above lower bound also gives the lower bound

$$C_{0,d}^{\text{lower}} = C_d^{\text{lower}} = \frac{d^d}{2(d-1)(d-2)^{d-1}} \tag{1.10}$$

achievable by our method for the constant in Theorem 1.1. In dimensions $3 \leq d \leq 9$ our results are summarized in Table 2.

In addition,

$$C_{0,d}^{\text{lower}} = \frac{d^2}{2(d-1)(d-2)} \left(1 + \frac{2}{d-2}\right)^{d-2} \rightarrow \frac{e^2}{2} \geq 3.69452.$$

This comparison shows that there is not too much room to improve on the upper bounds we obtained, even if one finds the sharp value in the minimization problem for M_γ in (1.4).

(iv) It is known that if $\alpha \geq d/2$, the operator $P^{2\alpha} - U$ always has bound states for nontrivial $U \geq 0$, so a quantitative bound of the form $N(P^{2\alpha} - U) \lesssim \int_{\mathbb{R}^d} U(x)^{d/\alpha}$ cannot hold if $\alpha \geq d$. For $\alpha = 1$ see [48] or [24, Problem 2 in §45]. For more general cases, see [26, 37, 39], and [20] for a simple proof of how the existence/ non-existence of a CLR type bound for operators of the form $T(P) + V$ for a large class of functions $T : \mathbb{R}^d \rightarrow [0, \infty)$ is related to the behavior of the symbol T close to its zero set.

As mentioned before, our method can be generalized to operator-valued potentials. To formulate this, we need some additional notation. An operator-valued potential V is a map $V : \mathbb{R}^d \rightarrow \mathcal{B}(\mathcal{G})$ with $V(x) : \mathcal{G} \rightarrow \mathcal{G}$ a bounded self-adjoint operator on an auxiliary Hilbert space⁷ \mathcal{G} for almost all $x \in \mathbb{R}^d$. We denote by $\mathcal{B}(\mathcal{G})$ the set of bounded operators on \mathcal{G} and by $\mathcal{S}_p(\mathcal{G})$ the

⁶ Using the upper bound on M_γ given in Proposition 1.5, one can actually derive the better estimate

$$\frac{C_\gamma}{C_\gamma^{\text{lower}}} \leq \frac{2(\gamma-1)}{\gamma} \frac{1}{\Gamma(\frac{2}{\gamma})^\gamma} \left(\frac{\gamma-2}{2} \frac{\pi}{\sin(\frac{2\pi}{\gamma})}\right)^{\frac{\gamma}{2}} = \frac{2(\gamma-1)}{\gamma} \left(\frac{\Gamma(2-\frac{2}{\gamma})}{\Gamma(1+\frac{2}{\gamma})}\right)^{\frac{\gamma}{2}}.$$

The right-hand side can be shown to be increasing in γ with limit $\lim_{\gamma \rightarrow \infty} \frac{2(\gamma-1)}{\gamma} \left(\frac{\Gamma(2-\frac{2}{\gamma})}{\Gamma(1+\frac{2}{\gamma})}\right)^{\frac{\gamma}{2}} = 2e^{2\gamma^*-1} \leq 2.34$, where γ^* is the Euler-Mascheroni constant. We will however not elaborate this further.

⁷ We follow the convention that all Hilbert spaces are considered to be separable, unless stated otherwise ;-). Physically, this auxiliary Hilbert space corresponds to other degrees of freedom, for example spin.

Table 2 Comparison of our results from Appendix D and the lower bound on the constant achievable by our method (derived from Proposition 1.5)

d	Our results	Lower bound
3	7.55151	6.75000
4	6.32791	5.33333
5	5.95405	4.82253
6	5.77058	4.55625
7	5.67647	4.39229
8	5.63198	4.28088
9	5.62080	4.20028

von Neumann–Schatten ideal of compact operators on \mathcal{G} with p -summable singular values, see for example [51] for a background on von Neumann–Schatten ideals.

Theorem 1.7 (Operator-valued version of Theorem 1.3) *Let \mathcal{G} be a Hilbert space and $V : \mathbb{R}^d \rightarrow \mathcal{B}(\mathcal{G})$ an operator valued potential with positive part $V_+ \in L^1_{loc}(\mathbb{R}^d, \mathcal{B}(\mathcal{G}))$ and negative part $V_- \in L^{d/(2\alpha)}(\mathbb{R}^d, \mathcal{S}_{d/(2\alpha)}(\mathcal{G}))$. Then the number of negative energy bound states of $P^{2\alpha} \otimes \mathbf{1}_{\mathcal{G}} + V$ is bounded by*

$$N(P^{2\alpha} \otimes \mathbf{1}_{\mathcal{G}} + V) \leq C_{d/\alpha} \frac{|B_1^d|}{(2\pi)^d} \int_{\mathbb{R}^d} \text{tr}_{\mathcal{G}}[V_-(x)^{\frac{d}{2\alpha}}] dx \tag{1.11}$$

with the same constant $C_{d/\alpha}$ as in Theorem 1.3.

For the physically most interesting case $\alpha = 1$ this enables us to get considerable improvements on the constants in the Cwikel–Lieb–Rozenblum bound.

Theorem 1.8 (Operator-valued version of Theorem 1.1) *Let \mathcal{G} be a Hilbert space and $V : \mathbb{R}^d \rightarrow \mathcal{B}(\mathcal{G})$ an operator valued potential with positive part $V_+ \in L^1_{loc}(\mathbb{R}^d, \mathcal{B}(\mathcal{G}))$ and negative part $V_- \in L^{d/2}(\mathbb{R}^d, \mathcal{S}_{d/2}(\mathcal{G}))$. Then the number of negative energy bound states of $P^2 \otimes \mathbf{1}_{\mathcal{G}} + V$ is bounded by*

$$N(P^2 \otimes \mathbf{1}_{\mathcal{G}} + V) \leq C_{0,d}^{\text{op}} \frac{|B_1^d|}{(2\pi)^d} \int_{\mathbb{R}^d} \text{tr}_{\mathcal{G}}[V_-(x)^{\frac{d}{2}}] dx \tag{1.12}$$

with

$$C_{0,d}^{\text{op}} = \min_{3 \leq n \leq d} C_{0,n}^{\text{op}} \leq \min_{3 \leq n \leq d} C_n, \tag{1.13}$$

where C_n is given by (1.5) for $\gamma = n$.

Remark 1.9 Table 1 lists upper bounds on $C_{0,d}^{\text{op}}$ for dimensions $3 \leq d \leq 9$, see also Appendix D. The constant for $d = 9$ is also an upper bound on $C_{0,d}^{\text{op}}$ in any dimension $d \geq 10$ by (1.13).

The structure of the paper is as follows. In Sect. 2 we present the main ideas of our method in the case of a standard non-relativistic Schrödinger operator. The extension to more general kinetic energies is done in Sect. 3.

In Sect. 4 we explain the surprising connection of semiclassical bounds and maximal Fourier multiplier estimates, which is probably the most important new part of our method.

Although we cannot explicitly find minimizers of the variational problem from Theorem 1.3, there is a natural lower bound, which is discussed in Sect. 5. The numerical study to find reasonable upper bounds for this variational problem is presented in Appendix D.

The extension to the operator-valued setting is done in Sects. 6 and 7. In particular, in Sect. 7 we prove a fully operator-valued version of Cwikel’s original weak trace ideal bound.

2 The splitting trick

Let $U := V_- \geq 0$. As quadratic forms $P^2 + V \geq P^2 - U$. This and the Birman–Schwinger principle shows

$$N(P^2 + V) \leq N(P^2 - U) = n(U^{1/2}|P|^{-2}U^{1/2}; 1),$$

where $n(A; \kappa)$ is the number of singular values $(s_j(A))_{j \in \mathbb{N}}$ greater than $\kappa > 0$ of a compact operator A .

We denote by \mathcal{F} the Fourier transform and by \mathcal{F}^{-1} its inverse, by M_h the operator of multiplication with a function h , and $A = A_{f,g} = M_f \mathcal{F}^{-1} M_g$ for f, g non-negative (measurable) functions on \mathbb{R}^d . When $f(x) = U(x)^{1/2}$ and $g(\eta) = |\eta|^{-1}$, then $AA^* = U^{1/2}|P|^{-2}U^{1/2}$, which has the same non-zero eigenvalues as A^*A . Thus

$$N(P^2 - U) = n(A_{f,g}; 1).$$

In particular, the Chebyshev–Markov inequality gives

$$N(P^2 - U) = n(A_{f,g}; 1) \leq \sum_j \frac{(s_j(A_{f,g}) - \mu)_+^2}{(1 - \mu)^2}$$

for any $0 < \mu < 1$. The first main idea, going already back to Cwikel [9], is to split $A_{f,g} = B_{f,g} + H_{f,g}$, where $B_{f,g}$ is bounded and $H_{f,g}$ is a Hilbert–Schmidt operator, and note that Ky Fan’s inequality for the singular values

[51, Theorem 1.7] yields

$$s_j(A_{f,g}) = s_j(B_{f,g} + H_{f,g}) \leq \|B_{f,g}\| + s_j(H_{f,g})$$

for all $j \in \mathbb{N}$. So if $\|B_{f,g}\| \leq \mu < 1$ we get

$$N(P^2 - U) \leq (1 - \mu)^{-2} \sum_{j \in \mathbb{N}} s_j(H_{f,g})^2 = (1 - \mu)^{-2} \|H_{f,g}\|_{HS}^2, \tag{2.1}$$

where $\|H\|_{HS}$ denotes the Hilbert–Schmidt norm of the operator H .

In order to make the above argument work, one has to be able to split $A_{f,g} = B_{f,g} + H_{f,g}$ in such a way that the Hilbert–Schmidt norm of $H_{f,g}$ is easy to calculate and one has a good bound on the operator norm of $B_{f,g}$. Writing out the inverse Fourier transform, one sees that $A_{f,g}$ has a kernel

$$A_{f,g}(x, \eta) = (2\pi)^{-d/2} e^{ix \cdot \eta} f(x)g(\eta), \tag{2.2}$$

that is,

$$A_{f,g}\varphi(x) = f(x)\mathcal{F}^{-1}(g\varphi)(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix \cdot \eta} f(x)g(\eta)\varphi(\eta) \, d\eta, \tag{2.3}$$

at least for nice enough φ . In order to write $A_{f,g}$ as a sum of a bounded and a Hilbert–Schmidt operator, set $t = f(x)g(\eta)$, split $t = m(t) + t - m(t)$ for some bounded, measurable function $m : [0, \infty) \rightarrow \mathbb{R}$, and define $B_{f,g,m}$ and $H_{f,g,m}$ via their kernels

$$B_{f,g,m}(x, \eta) = (2\pi)^{-d/2} e^{ix \cdot \eta} m(f(x)g(\eta)), \tag{2.4}$$

$$H_{f,g,m}(x, \eta) = (2\pi)^{-d/2} e^{ix \cdot \eta} (f(x)g(\eta) - m(f(x)g(\eta))). \tag{2.5}$$

It is then clear that $A_{f,g} = B_{f,g,m} + H_{f,g,m}$. Our starting point is that the Hilbert–Schmidt norm of $H_{f,g,m}$ is straightforward to calculate; the main difficulty is to get an explicit bound on the operator norm of $B_{f,g,m}$ on L^2 under suitable assumptions on m . For the special choice $g(\eta) = |\eta|^{-1}$ one has $\|H_{f,g}\|_{HS}^2 = c \int_{\mathbb{R}^d} f(x)^d \, dx$, see (2.9), so the right hand side of (2.1) has exactly the right (semi-classical) scaling in f . But, in order to use this in (2.1), it also enforces that the upper bound μ on the operator norm of $B_{f,g}$ has to be *independent* of f . This has an important consequence:

Since for a given $\varphi \in L^2$ one can freely choose $f \geq 0$ as to make $|B_{f,g,m}\varphi|$ as big as possible, this leads naturally to the associated maximal operator $\mathcal{B}_{g,m}(\varphi) := \sup_{f \geq 0} |B_{f,g,m}\varphi|$. Although this is not explicitly

written in the paper by Cwikel, getting a useful bound on such a type of maximal operator is exactly what he achieved in [9], using a dyadic decomposition in the ranges of f and g and collecting suitable terms. We will do this in a much simpler and more efficient way. This enables us to get a constant which is more than 10 times smaller than the original constant by Cwikel.

It turns out that one can always calculate the Hilbert–Schmidt norm of $H_{f,g,m}$. The maximal operator $\mathcal{B}_{g,m}$ corresponding to $B_{f,g,m}$ can be bounded in operator norm under an additional structural assumption on m , which we present first.

Theorem 2.1 *Let g be a measurable non-negative function on \mathbb{R}^d for $d \geq 1$ and assume that m is given by a convolution,*

$$m(t) = m_1 * m_2(t) = \int_0^\infty m_1(t/s)m_2(s) \frac{ds}{s}$$

with $m_1, m_2 \in L^2(\mathbb{R}_+, \frac{ds}{s})$. Then the maximal operator given by $\mathcal{B}_{g,m}(\varphi) := \sup_{f \geq 0} |B_{f,g,m}\varphi|$ extends to a bounded operator on $L^2(\mathbb{R}^d)$ with

$$\|\mathcal{B}_{g,m}\| \leq \left(\int_0^\infty |m_1(s)|^2 \frac{ds}{s} \right)^{1/2} \left(\int_0^\infty |m_2(s)|^2 \frac{ds}{s} \right)^{1/2} \tag{2.6}$$

for its operator norm.

We emphasize that this maximal operator bound provides an upper bound for the operator norm of $B_{f,g,m}$ independently of the choice of f , as it has to be. It also turns out to be independent of g . The maximal operator bound is a natural consequence of the convolution structure of m , see Sect. 4, where we show that it is equivalent to maximal Fourier multiplier bounds. Concerning the Hilbert–Schmidt norm of $H_{f,g,m}$ we have

Theorem 2.2 *Let f, g be non-negative measurable functions on \mathbb{R}^d , $d \geq 1$, and m be a measurable function on \mathbb{R}_+ . The Hilbert–Schmidt norm of $H_{f,g,m}$ is given by*

$$\|H_{f,g,m}\|_{HS}^2 = \int_{\mathbb{R}^d} G_{g,m}(f(x)) \, dx, \tag{2.7}$$

where the function $G_{g,m}$ is given by

$$G_{g,m}(u) = \int_{\mathbb{R}^d} |ug(\eta) - m(ug(\eta))|^2 \frac{d\eta}{(2\pi)^d}. \tag{2.8}$$

Remark 2.3 In its applications to nonrelativistic Schrödinger operators P^2+V , the function g is given by $g(\eta) = |\eta|^{-1}$. We would like to emphasize that g is never in $L^2(\mathbb{R}^d)$, due to its slow decay at infinity, which is an ultraviolet problem. Choosing m with $m(t) \sim t$ for small $t > 0$ makes the integrand in (2.7) vanish for large frequencies. This can be thought of as an *ultraviolet regularization*: the right hand side of (2.7) is finite if and only if g is locally square integrable (near its singularity), which is an infrared problem. Clearly, $g(\eta) = |\eta|^{-1}$ is locally square integrable only in dimension $d \geq 3$. This explains the well-known fact that the CLR bound for non-relativistic Schrödinger operators holds only in dimensions $d \geq 3$.

For a generalized Schrödinger operator $T(P) + V$, where the kinetic energy (frequency–energy relation of the free particle) is given by a measurable function $T \geq 0$, we have $g = T^{-1/2}$. In this case a CLR-type bound holds if T^{-1} is locally integrable near the zero set of T . This is sharp, since we know from [20] that weakly coupled negative energy bound states of $T(P) + V$ exist for arbitrary weak attractive potentials V when T^{-1} is not locally integrable near the zero set of T .

Proof of Theorem 2.2 Since the operator $H_{f,g,m}$ has a kernel given by the right-hand side of (2.5), we compute its Hilbert–Schmidt norm as

$$\begin{aligned} \|H_{f,g,m}\|_{HS}^2 &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} |H_{f,g,m}(x, \eta)|^2 dx d\eta \\ &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} |f(x)g(\eta) - m(f(x)g(\eta))|^2 \frac{dx d\eta}{(2\pi)^d} \\ &= \int_{\mathbb{R}^d} G_{g,m}(f(x)) dx, \end{aligned}$$

using the Fubini–Tonelli Theorem and the definition of $G_{g,m}$. □

In the rest of this section we will discuss how Theorems 2.1, 2.2, and the bound (2.1) lead to the Cwikel–Lieb–Rozenblum bound for a non-relativistic single-particle Schrödinger operator. In this case $g(\eta) = |\eta|^{-1}$, and a simple scaling in the η integral gives

$$\begin{aligned} \|H_{f,g,m}\|_{HS}^2 &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{f(x)}{|\eta|} - m\left(\frac{f(x)}{|\eta|}\right) \right)^2 \frac{dx d\eta}{(2\pi)^d} \\ &= \int_{\mathbb{R}^d} f(x)^d dx \int_{\mathbb{R}^d} (|\eta|^{-1} - m(|\eta|^{-1}))^2 \frac{d\eta}{(2\pi)^d} \end{aligned} \tag{2.9}$$

Going to spherical coordinates shows

$$\int_{\mathbb{R}^d} (|\eta|^{-1} - m(|\eta|^{-1}))^2 \frac{d\eta}{(2\pi)^d} = \frac{|S^{d-1}|}{(2\pi)^d} \int_0^\infty (r^{-1} - m(r^{-1}))^2 r^{d-1} dr$$

$$= \frac{d|B_1^d|}{(2\pi)^d} \int_0^\infty (1 - t^{-1}m(t))^2 t^{1-d} dt,$$

where $|S^{d-1}|$ is the surface area of the unit sphere in \mathbb{R}^d and $|B_1^d| = |S^{d-1}|/d$ is the volume of the unit ball in \mathbb{R}^d .

Now we repeat the derivation of (2.1), except that we also scale f by $\kappa > 0$, using $\kappa A_{f,g} = A_{\kappa f,g} = B_{\kappa f,g,m} + H_{\kappa f,g,m}$. The argument leading to (2.1) then gives

$$N(P^2 - U) = n(A_{\kappa f,g}; \kappa) \leq (\kappa - \mu)^{-2} \sum_j \|H_{\kappa f,g,m}\|_{HS}^2 \tag{2.10}$$

$$= \frac{\kappa^d}{(\kappa - \mu)^2} \frac{d|B_1^d|}{(2\pi)^d} \int_0^\infty (1 - t^{-1}m(t))^2 t^{1-d} dt \int_{\mathbb{R}^d} U(x)^{d/2} dx, \tag{2.11}$$

as long as $\kappa > \mu \geq \|B_{\kappa f,g,m}\|$. Clearly, the last factor in (2.11) has the correct dependence on the potential U . Thanks to Theorem 2.1, we can use $\mu = \|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})}$ as an upper bound for $\|B_{f,g,m}\|$, which is independent of f , so the same bound holds for $\|B_{\kappa f,g,m}\|$ for any $\kappa > 0$. Using this, we can now freely optimize (2.11) in $\kappa > \mu$ to get

$$N(P^2 - U) \leq C \frac{|B_1^d|}{(2\pi)^d} \int_{\mathbb{R}^d} U(x)^{d/2} dx \tag{2.12}$$

with the constant

$$C = C_{d,m} = \frac{d^{d+1}}{4(d-2)^{d-2}} \mu^{d-2} \int_0^\infty (1 - t^{-1}m(t))^2 t^{1-d} dt. \tag{2.13}$$

This gives most of the main ideas of our proof of Theorem 1.1. The last new idea, which is crucially important for the proof of Theorem 2.1, is the connection between the bound on the norm of the operator $B_{f,g,m}$, more precisely, the bound (2.6) on the operator norm of the associated maximal operator $\mathcal{B}_{g,m}(\varphi) := \sup_{f \geq 0} |B_{f,g,m}\varphi|$, and bounds for maximal Fourier multipliers on L^2 . This is explained in Sect. 4.

Before we do this let us point out that our approach leads to new results also for more general kinetic energies.

3 General kinetic energies

First we consider the case where P^2 is replaced by $P^{2\alpha}$ and give the

Proof of Theorem 1.3 Replacing $g(\eta) = |\eta|^{-1}$ by $g(\eta) = |\eta|^{-\alpha}$ one simply reruns the argument from the previous section. Calculating, again by scaling,

$$\begin{aligned} \|H_{f,g,m}\|_{HS}^2 &= \iint_{\mathbb{R} \times \mathbb{R}^d} \left(\frac{f(x)}{|\eta|^\alpha} - m \left(\frac{f(x)}{|\eta|^\alpha} \right) \right)^2 \frac{dx d\eta}{(2\pi)^d} \\ &= \int_{\mathbb{R}^d} f(x)^{d/\alpha} dx \int_{\mathbb{R}^d} (|\eta|^{-\alpha} - m(|\eta|^{-\alpha}))^2 \frac{d\eta}{(2\pi)^d} \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^d} (|\eta|^{-\alpha} - m(|\eta|^{-\alpha}))^2 \frac{d\eta}{(2\pi)^d} &= \frac{|S^{d-1}|}{(2\pi)^d} \int_0^\infty (r^{-\alpha} - m(r^{-\alpha}))^2 r^{d-1} dr \\ &= \frac{d|B_1^d|}{\alpha(2\pi)^d} \int_0^\infty (1 - t^{-1}m(t))^2 t^{1-\frac{d}{\alpha}} dt, \end{aligned}$$

one sees that the argument leading to (2.11) remains virtually unchanged, only d gets replaced by d/α . Thus

$$N(P^{2\alpha} + V) \leq C \frac{d|B_1^d|}{\alpha(2\pi)^d} \int_{\mathbb{R}^d} V_-(x)^{\frac{d}{2\alpha}} dx$$

with constant

$$\begin{aligned} C &= \frac{\left(\frac{d}{\alpha}\right)^{\frac{d}{\alpha}+1}}{4\left(\frac{d}{\alpha} - 2\right)^{\frac{d}{\alpha}-2}} \left(\|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \right)^{\frac{d}{\alpha}-2} \\ &\quad \times \int_0^\infty (1 - t^{-1}m(t))^2 t^{1-\frac{d}{\alpha}} dt \end{aligned}$$

For m_1 and m_2 we make the simple choice from Remark 1.4. Then $m(t) = m_1 * m_2(t) = \min(t, t^{-1})$ and $\mu = \|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} = 1$. Hence,

$$\int_0^\infty (1 - t^{-1}m(t))^2 t^{1-\frac{d}{\alpha}} dt = \int_1^\infty (1 - t^{-2})^2 t^{1-\frac{d}{\alpha}} dt = \frac{8}{\left(\frac{d}{\alpha} - 2\right) \frac{d}{\alpha} \left(\frac{d}{\alpha} + 2\right)}$$

and collecting terms finishes the proof of Theorem 1.3. □

Remark 3.1 For the number of negative energy bound states of $P^{2\alpha} + U$ the so-far best bounds are due to Frank [16, 17]. Using ideas from Rumin [46, 47], he got the bound

$$N(P^{2\alpha} + V) \leq \left(\frac{\frac{d}{\alpha} \left(\frac{d}{\alpha+2}\right)}{\left(\frac{d}{\alpha} - 2\right)^2} \right)^{\frac{d}{2\alpha}-1} \frac{\frac{d}{\alpha}}{\frac{d}{\alpha} - 2} \frac{|B_1^d|}{(2\pi)^d} \int_{\mathbb{R}^d} V_-(x)^{\frac{d}{2\alpha}} dx.$$

Even with the non-optimal choice of m_1 and m_2 above, one sees that the bound from Theorem 1.3 is better as long as $2 < (1 + 2\alpha/d)^{d/(2\alpha)}$. Since $0 < \delta \mapsto (1 + 1/\delta)^\delta$ is strictly increasing, this is the case as soon as $d > 2\alpha$, that is, the whole range of allowed values of α .

For more general kinetic energies of the form $T(P)$ with T a non-negative measurable function which is locally bounded we have

Theorem 3.2 *The number of negative energy bound states of a Schrödinger-type operator $T(P) + V$, defined suitably with the help of quadratic form methods on L^2 , obeys the bound*

$$N(T(P) + V) \leq \lambda^{-2} \int_{\mathbb{R}^d} G_T((\lambda + 1)^2 V_-(x)) \, dx \tag{3.1}$$

for any $\lambda > 0$, with $V_- = \max(-V, 0)$, the negative part of V and

$$\begin{aligned} G_T(u) &= \int \left[\left(\frac{u}{T(\eta)} \right)^{1/2} - \left(\frac{u}{T(\eta)} \right)^{-1/2} \right]_+^2 \frac{d\eta}{(2\pi)^d} \\ &= \int_{T < u} \left[\frac{u}{T(\eta)} + \frac{T(\eta)}{u} - 2 \right] \frac{d\eta}{(2\pi)^d} \end{aligned} \tag{3.2}$$

where $\alpha_+ = \max(\alpha, 0)$ is the positive part.

Proof In this case we use $g(\eta) = T(\eta)^{-1/2}$, $f(x) = V_-(x)$, and again make the choice $m_1(s) = s \mathbf{1}_{\{0 < s \leq 1\}}$ and $m_2(s) = 2s^{-1} \mathbf{1}_{\{s \geq 1\}}$. So $\mu = \|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} = 1$. With $\lambda = \kappa - \mu = \kappa - 1$, the same argument leading to (2.10) now gives

$$N(T(P) + V) \leq N(T(P) - V_-) \leq \lambda^{-2} \|H_{(\lambda+1)f,g,m}\|_{HS}^2.$$

for any $\lambda > 0$. Using Theorem 2.2 to calculate the Hilbert–Schmidt norm shows

$$\|H_{(\lambda+1)f,g,m}\|_{HS}^2 = \int_{\mathbb{R}^d} G_T((\lambda + 1)^2 V_-(x)) \, dx,$$

since $m(t) = m_1 * m_2(t) = \min(t, t^{-1})$. □

Remark 3.3 (i) The bound given in Theorem 3.2 improves the bound from [20], which was based on Cwikel’s original method. Clearly, G_T given by (3.2) is increasing in $u > 0$. Moreover, since T is assumed to be locally bounded it is easy to see that $G_T(u)$ is finite if and only if $\eta \mapsto T(\eta)^{-1}$ is integrable over the set $\{T < u\}$. The result proven in [20] shows that under

some rather mild general conditions on the kinetic energy symbol T the operator $T(P) + V$ has weakly coupled bound states for any non-trivial potential $V \leq 0$, no matter how small $|V|$ is, if T^{-1} is not integrable over the set $\{T < u\}$ for all small $u > 0$, which is equivalent to $G_T(u) = \infty$ for all small $u > 0$ and, by monotonicity, equivalent to $G_T(u) = \infty$ for all $u > 0$. This shows that the bound given by Theorem 1.1 is quite natural.

- (ii) Let $g(u) = (u^{1/2} - u^{-1/2})_+^2$. Then $g'(t) = 0$ for $0 < t < 1$ and $g'(t) = 1 - t^{-2}$ for $t > 1$. The layer cake principle yields

$$\begin{aligned} \int G_T(V_-(x)) \, dx &= \int_0^\infty g'(t) \iint \mathbf{1}_{\{T(\eta) < V_-(x)/t\}} \frac{dx \, d\eta}{(2\pi)^d} \, dt \\ &= \int_0^\infty g'(t) N^{cl}(T + t^{-1}V) \, dt \end{aligned}$$

with the *classical phase-space volume*

$$N^{cl}(T + V) := \iint \mathbf{1}_{\{T(\eta) + V(x) < 0\}} \frac{dx \, d\eta}{(2\pi)^d}. \tag{3.3}$$

Hence, in terms of the classical phase-space volume Theorem 3.2 gives an upper bound of the form

$$N(T(P) + V) \leq \lambda^{-2} \int_1^\infty N^{cl}(T + t^{-1}(\lambda + 1)^2V) (1 - t^{-2}) \, dt \tag{3.4}$$

for any $\lambda > 0$. One can interpret (3.4) as a *quantum correction* to the classical phase-space guess (3.3). The integral on the right hand side is finite if and only if the classical phase-space volume is small enough for small potentials. A bound of the form (3.4), with $(1 - t^{-2})$ replaced by 1, was also derived in [20]. In most cases where one can explicitly calculate or find explicit upper bounds for G_T , one shows, in fact, that

$$\int_1^\infty N^{cl}(T + t^{-1}V)(1 - t^{-2}) \, dt \lesssim N^{cl}(T + V), \tag{3.5}$$

see the discussion in Section 6 of [20]. In these cases, Theorem 3.2 gives an upper bound for the number of negative bound states of $T(P) + V$, under very weak conditions on the dispersion relation T , solely in terms of the classical phase-space volume,

$$N(T(P) + V) \leq C\lambda^{-2}N^{cl}(T + (1 + \lambda)^2V), \tag{3.6}$$

for some constant C and all $\lambda > 0$. However, the bound (3.5), hence also the bound (3.6), does not hold in critical cases, where it is known that logarithmic corrections to the classical phase space guess appear [3, 4, 52].

4 The connection with maximal Fourier multipliers

In this section we give the proof of Theorem 2.1. The important observation is the connection to maximal Fourier multipliers, as we discuss now. Recall that given functions $f, g : \mathbb{R}^d \rightarrow [0, \infty)$ and a bounded, measurable function $m : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, the operator $B_{f,g,m}$ is given by

$$B_{f,g,m}\varphi(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\eta} m(f(x)g(\eta))\varphi(\eta) \, d\eta, \tag{4.1}$$

at least for nice enough φ , e.g., Schwartz functions. We would like to conclude that $B_{f,g,m}$ is a bounded operator on $L^2(\mathbb{R}^d)$, which might suggest to look for results which show that a pseudo-differential operator with symbol $a(x, \eta) = m(f(x)g(\eta))$ is bounded. A classical example of such a result is the Calderón–Vaillancourt theorem, see for instance [36, Proposition 9.4]. However, typical in the study of pseudo-differential operators, this needs high enough differentiability of the symbol a , which we do not have. More importantly, we need an estimate independent of f , which one cannot get without looking more closely into the structure of the problem. To see how the product structure $f(x)g(\eta)$ helps in the operator bound, we rewrite $B_{f,g,m}$ as

$$\begin{aligned} B_{f,g,m}\varphi(x) &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\eta} m(tg(\eta))\varphi(\eta) \, d\eta \Big|_{t=f(x)} \\ &= \mathcal{F}^{-1} [m(tg(\cdot))\varphi(\cdot)](x) \Big|_{t=f(x)}. \end{aligned} \tag{4.2}$$

This suggest to look at the Fourier multiplier $B_{t,g,m}$ defined by

$$B_{t,g,m}\varphi := \mathcal{F}^{-1} [m(tg(\cdot))\varphi(\cdot)] \tag{4.3}$$

and the associated maximal operator

$$B_{g,m}^*(\varphi)(x) := \sup_{t>0} |B_{t,g,m}\varphi(x)|. \tag{4.4}$$

It is clear that one has $|B_{f,g,m}(\varphi)| \leq B_{g,m}^*(\varphi)$, hence also $\mathcal{B}(\varphi) = \sup_{f \geq 0} |B_{f,g,m}(\varphi)| \leq |B_{g,m}^*(\varphi)|$, for any Schwarz function φ . On the other hand, choosing $f(x)$ in such a way as to make $|B_{f,g,m}\varphi(x)|$ arbitrarily close

to $B_{g,m}^* \varphi(x)$, shows the ‘reverse bound’ $\mathcal{B}_{g,m}(\varphi) = \sup_{f \geq 0} |B_{f,g,m} \varphi| \geq B_{g,m}^*(\varphi)$ for a given fixed Schwartz function φ . Thus $\mathcal{B}_{g,m}(\varphi) = B_{g,m}^*(\varphi)$.

In particular, $\|\mathcal{B}_{g,m}\| = \|B_{g,m}^*\|$ for the corresponding operator norms on L^2 . So a bound for the maximal operator $\mathcal{B}_{g,m}(\varphi) = \sup_{f \geq 0} |B_{f,g,m}(\varphi)|$ – which yields a bound for the operator norm of $B_{f,g,m}$ which is *uniform in the choice of the function f* – is equivalent to having a bound for the maximal Fourier multiplier $B_{g,m}^*$. This is our starting point for the proof of Theorem 2.1.

Remark 4.1 One should be a little bit careful in the definition (4.4) of the maximal operator $B_{g,m}^*$. If φ is a Schwartz function and $m : [0, \infty) \rightarrow \mathbb{R}$ is bounded and measurable, then both $B_{f,g,m} \varphi(x)$ and $B_{t,g,m} \varphi(x)$ are well-defined for all $x \in \mathbb{R}^d$, $t \geq 0$, and $f, g \geq 0$ measurable. To ensure measurability of $x \mapsto B_{g,m}^* \varphi(x)$ one has to impose stronger conditions on m , for example $m : [0, \infty) \rightarrow \mathbb{R}$ bounded and continuous is enough. In this case, $t \mapsto B_{t,g,m} \varphi(x)$ is continuous for each $x \in \mathbb{R}^d$ and the supremum in t can be taken over any dense subset. For example, $B_{g,m}^* \varphi(x) = \sup_{t \in \mathbb{Q}_+} |B_{t,g,m} \varphi(x)|$, with \mathbb{Q}_+ the positive rationals. Note that for the choice of m in Theorem 2.1 the function m is continuous. Indeed, if m is given by a convolution of $m_1, m_2 \in L^2(\mathbb{R}_+, \frac{ds}{s})$, then it is easy to see that it has a canonical continuous representative with $\lim_{t \rightarrow 0} m(t) = 0 = \lim_{t \rightarrow \infty} m(t)$.

Theorem 4.2 *Let g be a measurable non-negative function on \mathbb{R}^d and assume that m is given by a convolution,*

$$m(t) = m_1 * m_2(t) = \int_0^\infty m_1(t/s)m_2(s) \frac{ds}{s}$$

with $m_1, m_2 \in L^2(\mathbb{R}_+, \frac{ds}{s})$. Then the maximal Fourier multiplier $B_{g,m}^$, defined in (4.4), extends to a bounded operator on $L^2(\mathbb{R}^d)$ with*

$$\|B_{g,m}^*\| \leq \|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})}$$

for its operator norm.

Remark 4.3 There are several different but related proofs of boundedness of maximal Fourier multipliers available in the literature, see, e.g., [7, 11, 45]. These works concentrate on getting L^p bounds and do not care much about the involved constants. For us the L^2 boundedness is important, with good bounds on the operator norm.

Proof When m is given by a convolution and φ is a Schwartz function, we have

$$B_{t,g,m}\varphi(x) = \int_0^\infty \mathcal{F}^{-1} [m_1(tg/s)\varphi] (x) m_2(s) \frac{ds}{s}.$$

Interchanging the integrals, applying the triangle, and then the Cauchy–Schwarz inequality for the ds/s integration yields

$$\begin{aligned} |B_{t,g,m}\varphi(x)| &\leq \int_0^\infty |\mathcal{F}^{-1} [m_1(tg/s)\varphi] (x)| |m_2(s)| \frac{ds}{s} \\ &\leq \left(\int_0^\infty |\mathcal{F}^{-1} [m_1(tg/s)\varphi] (x)|^2 \frac{ds}{s} \right)^{1/2} \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})}. \end{aligned} \tag{4.5}$$

Since the measure ds/s is invariant under scaling, we can scale s by a fixed factor t to see that

$$\int_0^\infty |\mathcal{F}^{-1} [m_1(tg/s)\varphi] (x)|^2 \frac{ds}{s} = \int_0^\infty |\mathcal{F}^{-1} [m_1(g/s)\varphi] (x)|^2 \frac{ds}{s},$$

that is, the right hand side of (4.5) is independent of $t > 0$. So

$$\begin{aligned} B_{g,m}^*\varphi(x) &= \sup_{t>0} |B_{t,g,m}\varphi(x)| \\ &\leq \left(\int_0^\infty |\mathcal{F}^{-1} [m_1(g/s)\varphi] (x)|^2 \frac{ds}{s} \right)^{1/2} \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})}. \end{aligned}$$

In particular,

$$\|B_{g,m}^*\varphi\|_2^2 \leq \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})}^2 \int_{\mathbb{R}^d} \int_0^\infty |\mathcal{F}^{-1} [m_1(g/s)\varphi] (x)|^2 \frac{ds}{s} dx.$$

Using Fubini–Tonelli to interchange the integrals and Plancherel’s theorem for the L^2 norm of the Fourier transform, one sees that

$$\begin{aligned} &\int_{\mathbb{R}^d} \int_0^\infty |\mathcal{F}^{-1} [m_1(g/s)\varphi] (x)|^2 \frac{ds}{s} dx \\ &= \int_0^\infty \int_{\mathbb{R}^d} |m_1(g(\eta)/s)|^2 |\varphi(\eta)|^2 d\eta \frac{ds}{s}. \end{aligned}$$

Assume for the moment that $0 < g < \infty$ everywhere. Then interchanging the integration and using the same scaling argument as before to scale out $g(\eta)$ yields

$$\int_0^\infty \int_{\mathbb{R}^d} |m_1(g(\eta)/s)|^2 |\varphi(\eta)|^2 d\eta \frac{ds}{s} = \int_{\mathbb{R}^d} \int_0^\infty |m_1(s^{-1})|^2 |\varphi(\eta)|^2 \frac{ds}{s} d\eta = \|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})}^2 \|\varphi\|_2^2.$$

Hence

$$\|B_{g,m}^* \varphi\|_2 \leq \|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \|\varphi\|_2,$$

so $B_{g,m}^*$ is continuous at zero in $L^2(\mathbb{R}^d)$. Since this maximal operator is the supremum of linear operators, it is sublinear and continuity at zero implies that it is locally uniformly continuous. Thus $B_{g,m}^*$ can be extended to a bounded operator on $L^2(\mathbb{R}^d)$.

If g attains the values 0 or ∞ , we set $\tilde{\varphi} = \mathbf{1}_{\{0 < g < \infty\}} \varphi$. Since $m(0) = m(\infty) = 0$, we have $B_{t,g,m} \varphi = B_{t,g,m} \tilde{\varphi}$, hence also $B_{g,m}^* \varphi = B_{g,m}^* \tilde{\varphi}$ and with $\|\tilde{\varphi}\|_{L^2} \leq \|\varphi\|_{L^2}$ the above argument proves the claim in the case of general g . □

The next result, which also yields the proof of Theorem 2.1, is a direct consequence of Theorem 4.2.

Corollary 4.4 *Let f, g be measurable non-negative functions on \mathbb{R}^d and assume that $m : \mathbb{R}_+ \rightarrow \mathbb{R}$ is given by a convolution $m = m_1 * m_2$, with $m_1, m_2 \in L^2(\mathbb{R}_+, \frac{ds}{s})$. Then the operator $B_{f,g,m}$, defined by (2.4), i.e., given by the kernel*

$$B_{f,g,m}(x, \eta) = (2\pi)^{-d/2} e^{ix \cdot \eta} m(f(x)g(\eta)),$$

is bounded on $L^2(\mathbb{R}^d)$ with

$$\sup_{g \geq 0} \left\| \sup_{f \geq 0} |B_{f,g,m} \varphi| \right\|_2 \leq \|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \|\varphi\|_2.$$

Proof By definition of the maximal Fourier multiplier we have $|B_{f,g,m} \varphi(x)| \leq B_{g,m}^* \varphi(x)$ and thus also $\sup_{f \geq 0} |B_{f,g,m} \varphi(x)| \leq B_{g,m}^* \varphi(x)$ for almost every $x \in \mathbb{R}^d$.

Since the L^2 -bound from Theorem 4.2 is independent of $g \geq 0$, we can also take the supremum in $g \geq 0$, after taking the L^2 -norm. □

5 A lower bound for the variational problem M_γ

Recall that the variational problem, which comes up in a natural way in our bound on the number of bound states is

$$M_\gamma = \inf \left\{ (\|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})})^{\gamma-2} \int_0^\infty (1 - t^{-1} m_1 * m_2(t))^2 t^{1-\gamma} dt \right\}, \tag{5.1}$$

where the convolution $m_1 * m_2$ is on \mathbb{R}_+ with its scaling invariant measure $\frac{ds}{s}$, and the infimum is taken over all functions $m_1, m_2 : \mathbb{R}_+ \rightarrow \mathbb{R}$.

Theorem 5.1 *For all $\gamma > 2$ we have the lower bound*

$$M_\gamma \geq \frac{2}{(\gamma - 2)(\gamma - 1)\gamma}.$$

Proof Notice that $\|m\|_\infty \leq \|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})}$ for $m = m_1 * m_2$. Thus

$$\begin{aligned} M_\gamma &\geq \inf_m \left\{ \|m\|_\infty^{\gamma-2} \int_0^\infty (t - m(t))^2 t^{-\gamma-1} dt \right\} \\ &= \inf_{\ell > 0} \left\{ \ell^{\gamma-2} \inf_{\|m\|_\infty = \ell} \int_0^\infty (t - m(t))^2 t^{-\gamma-1} dt \right\}. \end{aligned}$$

In order to minimize the integral $\int_0^\infty (t - m(t))^2 t^{-\gamma-1} dt$ under the pointwise constraint $\ell = \|m\|_{L^\infty} \geq |m|$ for $\ell > 0$, one has to choose m in such a way that $(t - m(t))^2$ is as small as possible for each $t > 0$. Thus, for fixed $\ell > 0$, the minimizer is given by $m_\ell(t) = \min(t, \ell)$. Since

$$\begin{aligned} \int_0^\infty (t - m_\ell(t))^2 t^{-\gamma-1} dt &= \int_\ell^\infty (t - \ell)^2 t^{-\gamma-1} dt \\ &= \ell^{2-\gamma} \frac{2}{(\gamma - 2)(\gamma - 1)\gamma}, \end{aligned}$$

this yields the lower bound for M_γ . □

6 Extension to operator-valued potentials

In this section we extend our method to operator-valued potentials and give the proof of Theorem 1.7, i.e. we prove that the number of negative bound states of $P^{2\alpha} \otimes \mathbf{1}_{\mathcal{G}} + V$ is bounded by

$$N(P^{2\alpha} \otimes \mathbf{1}_{\mathcal{G}} + V) \leq C_{d/\alpha} \frac{|B_1^d|}{(2\pi)^d} \int_{\mathbb{R}^d} \text{tr}_{\mathcal{G}} [V_-(x)^{\frac{d}{2\alpha}}] dx,$$

where $V : \mathbb{R}^d \rightarrow \mathcal{B}(\mathcal{G})$ is an operator valued potential with positive part $V_+ \in L^1_{\text{loc}}(\mathbb{R}^d, \mathcal{B}(\mathcal{G}))$ and negative part $V_- \in L^{d/(2\alpha)}(\mathbb{R}^d, \mathcal{S}_{d/(2\alpha)}(\mathcal{G}))$.

Let $U(x) = V(x)_-$ be the negative part of $V(x)$ defined by spectral calculus. The Birman–Schwinger operator corresponding to $|P|^{2\alpha} \otimes \mathbf{1}_{\mathcal{G}} - U$ is given by

$$K = \sqrt{U}(|P|^{-2\alpha} \otimes \mathbf{1}_{\mathcal{G}})\sqrt{U}$$

and we again have

$$N(|P|^{2\alpha} \otimes \mathbf{1}_{\mathcal{G}} + V) \leq N(|P|^{2\alpha} \otimes \mathbf{1}_{\mathcal{G}} - U) = n(K; 1).$$

Now we factor K as $K = \tilde{A}_{f,g}^* \tilde{A}_{f,g}$ where $\tilde{A}_{f,g}$ has kernel

$$\tilde{A}_{f,g}\varphi(\eta) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\eta \cdot x} g(\eta) f(x)\varphi(x) \, dx ,$$

$g(\eta) = |\eta|^{-\alpha}$ is real-valued (even positive), and $f(x) = \sqrt{U(x)}$ takes values in the self-adjoint positive operators on \mathcal{G} . We split this as

$$\tilde{A}_{f,g} = \tilde{B}_{f,g,m} + \tilde{H}_{f,g,m}$$

with a function $m : [0, \infty) \rightarrow \mathbb{R}$, so that

$$\begin{aligned} \tilde{B}_{f,g,m}\varphi(\eta) &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\eta \cdot x} m(g(\eta) f(x))\varphi(x) \, dx \\ &= \mathcal{F} [m(tf)\varphi] (\eta) \Big|_{t=g(\eta)} \end{aligned} \tag{6.1}$$

and

$$\tilde{H}_{f,g,m}\varphi(\eta) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\eta \cdot x} [g(\eta) f(x) - m(g(\eta) f(x))]\varphi(x) \, dx , \tag{6.2}$$

where φ is a function from a nice dense subset of $L^2(\mathbb{R}^d, \mathcal{G})$, so that the integrals converge and $m(tf(x))$ is an operator on \mathcal{G} defined via functional calculus.

Remark 6.1 With a slight abuse of notation, we write \mathcal{F} in the definition of $\tilde{B}_{f,g,m}$, which strictly speaking denotes the Fourier transform on $L^2(\mathbb{R}^d)$, instead of $\mathcal{F} \otimes \mathbf{1}_{\mathcal{G}}$, the Fourier transform on $L^2(\mathbb{R}^d, \mathcal{G}) = L^2(\mathbb{R}^d) \otimes \mathcal{G}$. In addition, in the definition of $\tilde{B}_{f,g,m}$ and $\tilde{H}_{f,g,m}$ above we swapped the role of f and g compared to the discussion in Sect. 4. This is convenient, since by assumption $g(\eta)$ is a multiplication operator on \mathcal{G} , and this makes a maximal

Fourier multiplier estimate, now with g instead of f , easier. The general case can be reduced to this setting, see Sect. 7 below.

The following theorem is the replacement of Theorems 2.1 and 2.2 in the operator-valued setting.

Theorem 6.2 $\tilde{H}_{f,g,m}$ is a Hilbert–Schmidt operator on $\mathcal{H} = L^2(\mathbb{R}^d, \mathcal{G})$ with Hilbert–Schmidt norm given by

$$\|\tilde{H}_{f,g,m}\|_{\mathcal{S}_2(\mathcal{H})}^2 = \int_{\mathbb{R}^d} \text{tr}_{\mathcal{G}} [G_{g,m}(f(x))] \, dx, \tag{6.3}$$

where $G_{g,m}$ is again given by

$$G_{g,m}(u) = \int_{\mathbb{R}^d} |ug(\eta) - m(ug(\eta))|^2 \frac{d\eta}{(2\pi)^d}. \tag{6.4}$$

If, moreover, $m = m_1 * m_2$ then for all measurable non-negative functions g and non-negative operator-valued functions f the operator $\tilde{B}_{f,g,m}$ is bounded on \mathcal{H} with

$$\|\tilde{B}_{f,g,m}\varphi\|_{\mathcal{H}} \leq \|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \|\varphi\|_{\mathcal{H}} \tag{6.5}$$

for all $\varphi \in \mathcal{H}$.

Proof To prove (6.3), we note that the Hilbert–Schmidt operators on $\mathcal{H} = L^2(\mathbb{R}^d, \mathcal{G})$ are isomorphic to operators with kernels in $L^2(\mathbb{R}^d \times \mathbb{R}^d, \mathcal{S}_2(\mathcal{G}))$ and

$$\|\tilde{H}\|_{\mathcal{S}_2(\mathcal{H})}^2 = \text{tr}_{\mathcal{H}} [\tilde{H}^* \tilde{H}] = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \|\tilde{H}(\eta, x)\|_{\mathcal{S}_2(\mathcal{G})}^2 \, dx \, d\eta,$$

see Lemma B.3.

Using the explicit form of the ‘kernel’ of $\tilde{H}_{f,g,m}$ given in (6.2) this shows

$$\begin{aligned} \|\tilde{H}\|_{\mathcal{S}_2(\mathcal{H})}^2 &= (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \text{tr}_{\mathcal{G}} [|g(\eta)f(x) - m(g(\eta)f(x))|^2] \, d\eta \, dx \\ &= \int_{\mathbb{R}^d} \text{tr}_{\mathcal{G}} [G_{g,m}(f(x))] \, dx \end{aligned}$$

by the definition of $G_{g,m}$ and the spectral theorem.

Concerning the boundedness of $\tilde{B}_{f,g,m}$ we recall (6.1) and, if $m = m_1 * m_2$,

$$\tilde{B}_{f,t,m}\varphi(\eta) = \mathcal{F} [m(tf)\varphi] (\eta) = \int_0^\infty \mathcal{F} [m_1(f/s)\varphi] (\eta) m_2(ts) \frac{ds}{s}.$$

Thus,

$$\begin{aligned} \|\tilde{B}_{f,t,m}\varphi(\eta)\|_{\mathcal{G}} &\leq \int_0^\infty \|\mathcal{F}[m_1(f/s)\varphi](\eta)\|_{\mathcal{G}} |m_2(ts)| \frac{ds}{s} \\ &\leq \left(\int_0^\infty \|\mathcal{F}[m_1(f/s)\varphi](\eta)\|_{\mathcal{G}}^2 \frac{ds}{s}\right)^{1/2} \\ &\quad \times \left(\int_0^\infty |m_2(ts)|^2 \frac{ds}{s}\right)^{1/2} \\ &= \left(\int_0^\infty \|\mathcal{F}[m_1(f/s)\varphi](\eta)\|_{\mathcal{G}}^2 \frac{ds}{s}\right)^{1/2} \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \end{aligned}$$

due to the scaling invariance of ds/s . We therefore have a maximal operator bound

$$\begin{aligned} \tilde{B}_{f,m}^*\varphi(\eta) &:= \sup_{t>0} \|\tilde{B}_{f,t,m}\varphi(\eta)\|_{\mathcal{G}} \\ &\leq \left(\int_0^\infty \|\mathcal{F}[m_1(f/s)\varphi](\eta)\|_{\mathcal{G}}^2 \frac{ds}{s}\right)^{1/2} \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})}. \end{aligned}$$

In particular,

$$\|\tilde{B}_{f,m}^*\varphi\|_{L^2(\mathbb{R}^d)}^2 \leq \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})}^2 \int_{\mathbb{R}^d} \int_0^\infty \|\mathcal{F}[m_1(f/s)\varphi](\eta)\|_{\mathcal{G}}^2 \frac{ds}{s} d\eta,$$

and

$$\begin{aligned} &\int_{\mathbb{R}^d} \int_0^\infty \|\mathcal{F}[m_1(f/s)\varphi](\eta)\|_{\mathcal{G}}^2 \frac{ds}{s} d\eta \\ &= \int_0^\infty \int_{\mathbb{R}^d} \langle \mathcal{F}[m_1(f/s)\varphi](\eta), \mathcal{F}[m_1(f/s)\varphi](\eta) \rangle_{\mathcal{G}} d\eta \frac{ds}{s} \\ &= \int_0^\infty \int_{\mathbb{R}^d} \langle m_1(f(x)/s)\varphi(x), m_1(f(x)/s)\varphi(x) \rangle_{\mathcal{G}} dx \frac{ds}{s} \\ &= \int_{\mathbb{R}^d} \left\langle \varphi(x), \int_0^\infty m_1(f(x)/s)^2 \frac{ds}{s} \varphi(x) \right\rangle_{\mathcal{G}} dx \\ &= \|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})}^2 \int_{\mathbb{R}^d} \|\varphi(x)\|_{\mathcal{G}}^2 dx = \|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})}^2 \|\varphi\|_{\mathcal{H}}^2, \end{aligned}$$

where we again used that, by scaling $\int_0^\infty m_1(r/s)^2 \frac{ds}{s} = \|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})}^2$ for all $r > 0$, so by functional calculus

$$\int_0^\infty m_1(f(x)/s)^2 \frac{ds}{s} = \|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})}^2 \mathbf{1}_{\mathcal{G}}.$$

Altogether, we get the operator-valued version of our previous maximal Fourier multiplier bound in the form

$$\|\tilde{B}_{f,m}^* \varphi\|_{L^2(\mathbb{R}^d)}^2 \leq \|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \|\varphi\|_{\mathcal{H}},$$

and it is easy to see that

$$\|\tilde{B}_{f,g,m} \varphi\|_{\mathcal{H}} \leq \|\tilde{B}_{f,m}^* \varphi\|_{L^2(\mathbb{R}^d)},$$

which completes the proof of Theorem 6.2. □

The proof of Theorem 1.7 is straightforward: one simply does the same steps as in the scalar case with (2.10) replaced by

$$N(P^{2\alpha} \otimes \mathbf{1}_{\mathcal{G}} - U) = n(\tilde{A}_{\kappa f,g}; \kappa) \leq (\kappa - \mu)^{-2} \sum_j \|\tilde{H}_{\kappa f,g,m}\|_{S_2(\mathcal{H})}^2,$$

where now $\mu \geq \|\tilde{B}_{\kappa f,g,m} \varphi\|_{\mathcal{H}}$. As before, Theorem 6.2 gives a bound for $\|\tilde{B}_{\kappa f,g,m} \varphi\|_{\mathcal{H}}$ independent of κ , in particular, we can take any $\mu \geq \|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})}$. It also allows us to calculate the Hilbert-Schmidt norm. For $g(\eta) = |\eta|^{-\alpha}$ we get

$$G_{g,m}(u) = u^{d/\alpha} \int_{\mathbb{R}^d} (|\eta|^{-\alpha} - m(|\eta|^{-\alpha}))^2 \frac{d\eta}{(2\pi)^d},$$

so

$$\|\tilde{H}_{\kappa f,g,m}\|_{S_2(\mathcal{H})}^2 = \kappa^{d/\alpha} \int_{\mathbb{R}^d} (|\eta|^{-\alpha} - m(|\eta|^{-\alpha}))^2 \frac{d\eta}{(2\pi)^d} \int_{\mathbb{R}^d} \text{tr}_{\mathcal{G}} [f(x)^{d/\alpha}] dx.$$

Using this in the above bound for $N(P^{2\alpha} \otimes \mathbf{1}_{\mathcal{G}} - U)$ and minimizing over κ , as in the scalar case, finishes the proof of Theorem 1.7.

7 Trace ideal bounds

In this section we show how the ideas developed so far can be used to prove a fully operator-valued version of Cwikel’s theorem. Such an inequality was first proved in [16].

In this setting let (X, dx) and (Y, dy) be sigma-finite measure spaces and \mathcal{H}, \mathcal{G} (separable) Hilbert spaces. We denote by $L^p(X, \mathcal{S}_p(\mathcal{H}))$ the set of measurable functions $f : X \rightarrow \mathcal{S}_p(\mathcal{H})$, where $\mathcal{S}_p(\mathcal{H})$ is the space of p -summable compact operators, i.e. the von Neumann–Schatten class, on \mathcal{H} , such that

$$\|f\|_{L^p(X, \mathcal{S}_p(\mathcal{H}))}^p := \int_X \|f(x)\|_{\mathcal{S}_p(\mathcal{H})}^p dx < \infty .$$

Similarly, we denote by $L^p_w(Y, \mathcal{B}(\mathcal{G}))$ the set of all measurable functions $g : Y \rightarrow \mathcal{B}(\mathcal{G})$, with values in the bounded operators on \mathcal{G} , such that

$$\|g\|_{L^p_w(Y, \mathcal{B}(\mathcal{G}))}^p := \sup_{t>0} t^p \left| \{y \in Y : \|g(y)\|_{\mathcal{B}(\mathcal{G})} > t\} \right| < \infty .$$

A map $A : L^2(X, \mathcal{H}) \rightarrow L^2(Y, \mathcal{G})$ is in the weak trace-ideal $\mathcal{S}_{p,w} = \mathcal{S}_{p,w}(L^2(X, \mathcal{H}), L^2(Y, \mathcal{G}))$ if

$$\|f\Phi^*g\|_{p,w} := \sup_{n \in \mathbb{N}} \left(n^{\frac{1}{p}} s_n(A) \right) < \infty, \tag{7.1}$$

where $s_n(A)$ are the singular values of A , i.e. the eigenvalues of $A^*A : L^2(X, \mathcal{H}) \rightarrow L^2(X, \mathcal{H})$.

Theorem 7.1 (Fully operator valued version of Cwikel’s theorem) *Let $\Phi : L^2(X, \mathcal{H}) \rightarrow L^2(Y, \mathcal{G})$ be a unitary operator, which is also bounded from $L^1(X, \mathcal{H})$ into $L^\infty(Y, \mathcal{G})$.*

*If $p > 2$ and $f \in L^p(X, \mathcal{S}_p(\mathcal{H}))$ and $g \in L^p_w(Y, \mathcal{B}(\mathcal{G}))$, then $f\Phi^*g$ is in the weak trace ideal $\mathcal{S}_{p,w}(L^2(X, \mathcal{H}), L^2(Y, \mathcal{G}))$ and*

$$\|f\Phi^*g\|_{p,w}^p \leq \frac{p}{4} \frac{p^p}{(p-2)^{p-2}} Q_p \|\Phi\|_{L^1 \rightarrow L^\infty}^2 \|f\|_{L^p(X, \mathcal{S}_p(\mathcal{H}))}^p \|g\|_{L^p_w(Y, \mathcal{B}(\mathcal{G}))}^p, \tag{7.2}$$

where Q_p is given in (C.2).

Remark 7.2 Theorem 7.1 improves the result of Frank in [16],

$$\|f\Phi^*g\|_{p,w}^p \leq \frac{p}{2} \left(\frac{p}{p-2} \right)^{p-1} \|\Phi\|_{L^1 \rightarrow L^\infty}^2 \|f\|_{L^p(X, \mathcal{S}_p(\mathcal{H}))}^p \|g\|_{L^p_w(Y, \mathcal{B}(\mathcal{G}))}^p .$$

The value of Q_p comes from choosing $m_1(s) = s\mathbf{1}_{\{0 < s \leq 1\}}$ and then finding an optimal m_2 , see Appendix C. Making the simple choice of Remark 1.4 for m_2 leads to an upper bound for the weak–trace ideal norm with Q_p replaced by $8(p(p-2)(p+2))^{-1}$ in (7.2). It is easy to see that this simple choice of

m_1 and m_2 yields a bound which is already a factor of $(p + 2)/4$ smaller than the one in [16]. In addition, the bound in [16] in the scalar case, when Φ is the usual Fourier transform, is worse than the one in Theorem 7.1, with the above easy choice for m_1 and m_2 , by a factor of $\frac{1}{2}(1 + 2/p)^{p/2} > 1$ in the allowed range $p > 2$.

Proof First we note that one can reduce the result to the case when g is pointwise a positive multiple of the identity operator on \mathcal{G} . As operators on \mathcal{G} one has $g(y)g(y)^* \leq \|g(y)\|_{\mathcal{B}(\mathcal{G})}^2 \mathbf{1}_{\mathcal{G}}$. Thus with $A_1 = f\Phi^*g$ we have

$$A_1 A_1^* = f\Phi^*g g^* \Phi f^* \leq f\Phi^*(\|g\|_{\mathcal{B}(\mathcal{G})} \mathbf{1}_{\mathcal{G}})^2 \Phi f^* = A_2 A_2^*$$

with $A_2 = f\Phi^*\|g\|_{\mathcal{B}(\mathcal{G})} \mathbf{1}_{\mathcal{G}} = f\Phi^*\|g\|_{\mathcal{B}(\mathcal{G})}$ where, for simplicity, we wrote $\|g\|_{\mathcal{B}(\mathcal{G})}$ for $\|g\|_{\mathcal{B}(\mathcal{G})} \mathbf{1}_{\mathcal{G}}$. Since the singular values of A_1 are the square roots of the eigenvalues of $A_1^* A_1$, which has the same non-zero-eigenvalues as $A_1 A_1^*$ we see that the nonzero singular values of A_1 obey the bound $s_n(A_1) \leq s_n(A_2)$.

Similarly, $|f(x)| := \sqrt{f(x)^* f(x)}$ is a non negative operator on \mathcal{H} and

$$A_2^* A_2 = \|g\|_{\mathcal{B}(\mathcal{G})} \Phi^* f^* f \Phi \|g\|_{\mathcal{B}(\mathcal{G})} = \|g\|_{\mathcal{B}(\mathcal{G})} \Phi^* |f|^2 \Phi \|g\|_{\mathcal{B}(\mathcal{G})} = A_3^* A_3$$

with $A_3 = |f| \Phi^* \|g\|_{\mathcal{B}(\mathcal{G})}$. So the singular values of A_2 are the same as the singular values of A_3 and without loss of generality, we can assume that g is a non-negative function and f takes values in the non-negative operators on \mathcal{H} . By scaling, we can also assume that $\|f\|_{L^p(X, S_p(\mathcal{H}))} = \|g\|_{L^p_w(Y)}^p = 1$.

Since $\Phi : L^1(X, \mathcal{H}) \rightarrow L^\infty(Y, \mathcal{G})$ is bounded, Lemma B.4 shows that it has a kernel $\Phi(\cdot, \cdot)$ such that for all $f \in L^2(X, \mathcal{H})$ and almost all $y \in Y$

$$\Phi f(y) = \int_X \Phi(y, x) f(x) dx .$$

Moreover, $\sup_{(y,x) \in Y \times X} \|\Phi(y, x)\|_{\mathcal{B}(\mathcal{H}, \mathcal{G})} = \|\Phi\|_{L^1 \rightarrow L^\infty}$ Having reduced the estimate to scalar non-negative functions g and non-negative operator-valued functions f we can rewrite $\tilde{A}_{f,g} = g\Phi f$ as

$$\tilde{A}_{f,g}\varphi(y) = \int_X g(y)\Phi(y, x)f(x)\varphi(x) dx = \int_X \Phi(y, x)g(y)f(x)\varphi(x) dx \tag{7.3}$$

using that $g(y)$ is now a non-negative scalar. Thus, we can take again an arbitrary function $m : \mathbb{R}_+ \rightarrow \mathbb{R}$ with $m(0) = 0$ and split

$$\tilde{B}_{f,g,m}\varphi(y) := \int_X \Phi(y, x)m(g(y)f(x))\varphi(x) dx, \tag{7.4}$$

$$\tilde{H}_{f,g,m}\varphi(y) := \int_X \Phi(y, x) [g(y)f(x) - m(g(y)f(x))] \varphi(x) dx. \tag{7.5}$$

The above expressions are well-defined by the spectral theorem, since g is a non-negative function and f takes values in the non-negative operators on \mathcal{H} , so $m(g(y)f(x))$ is a bounded operator on \mathcal{H} for almost all y and x , when m is bounded. Thus the integrals in (7.4) and (7.4) converge for all φ from a dense subset of $L^2(X, \mathcal{H})$, for example the piecewise constant functions.

Scaling in f by $\kappa > 0$, we get from Ky Fan’s inequality

$$\begin{aligned} s_n(g\Phi f) &= \kappa^{-1} s_n(\tilde{A}_{\kappa f,g}) \leq \kappa^{-1} [\|\tilde{B}_{\kappa f,g,m}\| + s_n(\tilde{H}_{\kappa f,g,m})] \\ &\leq \kappa^{-1} [\mu + n^{-1/2} \|\tilde{H}_{\kappa f,g,m}\|_{HS}] \end{aligned} \tag{7.6}$$

where we take $\mu = \|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})}$, the upper bound on the norm of $\tilde{B}_{\kappa f,g,m}$ from Lemma 7.3 below and we used $s_n(H) \leq n^{-1} \sum_{j=1}^n s_j(H)^2 \leq n^{-1} \|H\|_{HS}^2$, for any Hilbert–Schmidt operator, due to the monotonicity of its singular values. Thus using the bound (7.7) one gets

$$s_n(g\Phi f) \leq \kappa^{-1} [\mu + n^{-1/2} p^{1/2} \|\Phi\|_{L^1 \rightarrow L^\infty} D^{1/2} \kappa^{p/2}]$$

with $D = \int_0^\infty (1 - t^{-1}m(t))^2 t^{1-p} dt$, and minimizing this over $\kappa > 0$ we have

$$s_n(g\Phi f) \leq p^{1/p} \|\Phi\|_{L^1 \rightarrow L^\infty}^{2/p} \frac{p}{p-2} \left(\frac{p-2}{2}\right)^{2/p} (\mu^{p-2} D)^{1/p} n^{-1/p}$$

for the singular values for all $n \in \mathbb{N}$.

Now we make the choice $m_1(s) = s \mathbf{1}_{\{0 < s \leq 1\}}$ and minimize over all admissible m_2 . Proposition C.4 shows that this leads to $\mu^{p-2} D = Q_p$, with Q_p defined in (C.2). In view of Remark 7.4 (ii), the minimizer for Q_p is admissible in Lemma 7.3. □

Lemma 7.3 *Let $p > 2$, \mathcal{H} and \mathcal{G} auxiliary Hilbert spaces, (X, dx) and (Y, dy) σ -finite measure spaces, $0 \leq g \in L^p_w(Y)$, $0 \leq f \in L^p(X, \mathcal{S}_p(\mathcal{H}))$, $\Phi : L^2(X, \mathcal{H}) \rightarrow L^2(Y, \mathcal{G})$ unitary and also bounded from $L^1(X, \mathcal{H}) \rightarrow L^\infty(Y, \mathcal{G})$. Then for all continuous and piecewise differentiable bounded functions $m : \mathbb{R}_+ \rightarrow \mathbb{R}$ with $m(0) = 0$ and $\partial_t(t - m(t))^2 \geq 0$ for all $t > 0$, the operator $\tilde{H}_{f,g,m}$ defined in (7.5) is a Hilbert–Schmidt operator and*

$$\begin{aligned} \|\tilde{H}_{f,g,m}\|_{\mathcal{S}_2(L^2(X,\mathcal{H}) \rightarrow L^2(Y,\mathcal{G}))}^2 &= \text{tr}_{L^2(X,\mathcal{H})} [\tilde{H}_{f,g,m}^* \tilde{H}_{f,g,m}] \\ &\leq p \|\Phi\|_{L^1 \rightarrow L^\infty}^2 \int_0^\infty (1 - t^{-1}m(t))^2 t^{1-p} dt \|g\|_{L^p_w(Y)}^p \|f\|_{L^p(X, \mathcal{S}_p(\mathcal{H}))}^p. \end{aligned} \tag{7.7}$$

Moreover, if $m = m_1 * m_2$, then the operator $\tilde{B}_{f,g,m}$ defined in (7.4) is bounded from $L^2(X, \mathcal{H})$ to $L^2(Y, \mathcal{G})$ and

$$\|\tilde{B}_{f,g,m}\|_{L^2 \rightarrow L^2} \leq \|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})}. \tag{7.8}$$

Remark 7.4 (i) As the proof of Lemma 7.3 shows one even has a bound on $\tilde{B}_{f,g,m}$ of the form

$$\sup_{f \geq 0} \|\sup_{g \geq 0} \|\tilde{B}_{f,g,m} \varphi\|_{\mathcal{G}}\|_{L^2(Y)} \leq \|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \|\varphi\|_{L^2(X, \mathcal{H})}$$

where the first supremum is taken over all functions $g : Y \rightarrow [0, \infty)$ and the second supremum is taken over all non-negative operator-valued functions $f : X \rightarrow \mathcal{B}(\mathcal{H})$.

- (ii) The condition $\partial_t(t - m(t))^2 \geq 0$ might look weird at first, but there is a large class of functions m for which it holds: A simple choice is $m_1(s) = s \mathbf{1}_{\{0 < s \leq 1\}}$ and $m_2(s) = 2s^{-1} \mathbf{1}_{\{s \geq 1\}}$. In this case $m(t) = m_1 * m_2(t) = \min(t, t^{-1})$, so this simple choice of m_1 and m_2 is admissible in Lemma 7.3. More generally, setting $m_2(t) = -h'(t^{-1})$ for some absolutely continuous function h with $h(0) = 1$ and $\lim_{t \rightarrow \infty} h(t) = 0$, the proof of Proposition C.4 shows that $t - m(t) = th(t^{-1})$ for all $t > 0$,

$$\int_0^\infty (t - m(t))^2 t^{1-p} dt = \int_0^\infty h(t)^2 t^{p-2} \frac{dt}{t}, \tag{7.9}$$

and

$$\|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} = \left(\frac{1}{2} \int_0^\infty h'(s)^2 \frac{ds}{s} \right)^{\frac{p-2}{2}}. \tag{7.10}$$

Such a choice for m_1 and m_2 then leads to the variational problem (C.1), which we solve in Proposition C.1. Moreover, $\partial_t(t - m(t))^2 = \partial_t(th(t^{-1}))^2 = 2th(t^{-1})(h(t^{-1}) - t^{-1}h'(t^{-1})) \geq 0$ for any decreasing function $h \geq 0$. Fortunately, the minimizers for the variational problem (C.1) have this property and thus can be used in Lemma 7.3 which leads to the constant in Theorem 7.1.

Proof We freely use results for the operator-valued setting given in Appendix B. For notational simplicity we set

$$C = \|\Phi\|_{L^1(X, \mathcal{H}) \rightarrow L^\infty(Y, \mathcal{G})} = \operatorname{ess\,sup}_{(y,x) \in Y \times X} \|\Phi(x, y)\|_{\mathcal{B}(\mathcal{H}, \mathcal{G})}.$$

and note

$$\begin{aligned} & \|\tilde{H}_{f,g,m}\|_{S_2(L^2(X,\mathcal{H})\rightarrow L^2(Y,\mathcal{G}))}^2 \\ &= \iint_{Y\times X} \text{tr}_{\mathcal{H}} [\tilde{H}_{f,g,m}(y,x)^* \tilde{H}_{f,g,m}(y,x)] \, dy \, dx. \end{aligned}$$

Because g is real-valued, even positive, and f takes values in the non-negative, hence self-adjoint, operators

$$\begin{aligned} & \tilde{H}_{f,g,m}(y,x)^* \tilde{H}_{f,g,m}(y,x) \\ &= [g(y)f(x) - m(g(y)f(x))] \Phi(y,x)^* \\ &\quad \times \Phi(y,x) [g(y)f(x) - m(g(y)f(x))] \\ &\leq C^2 [g(y)f(x) - m(g(y)f(x))]^2, \end{aligned}$$

so, setting $G(u) := \int_Y [ug(y) - m(ug(y))]^2 \, dy$, we have

$$\|\tilde{H}_{f,g,m}\|_{S_2(L^2(X,\mathcal{H})\rightarrow L^2(Y,\mathcal{G}))}^2 \leq C^2 \int_X \text{tr}_{\mathcal{H}} G(f(x)) \, dx.$$

With $k(t) = (t - m(t))^2$, the layer-cake principle shows

$$G(u) = \int_0^\infty k'(t) |\{y \in Y : g(y) > t/u\}| \, dt.$$

By definition $|\{y \in Y : g(y) > t\}| \leq t^{-p} \|g\|_{L_w^p(Y)}^p$ for all $t > 0$. By assumption, $k' \geq 0$, thus

$$G(u) \leq u^p \|g\|_{L_w^p(Y)}^p \int_0^\infty k'(t) t^{-p} \, dt.$$

An integration by parts argument would show that $\int_0^\infty k'(t) t^{-p} \, dt = p \int_0^\infty k(t) t^{1-p} \, dt$, but due to the singularity of the integrand this requires that k vanishes at zero fast enough and that k does not grow too fast at infinity. Instead, we prefer to use non-negativity of k' . Note that

$$p \int_0^\infty k(t) t^{1-p} \, dt = \int_0^\infty \int_0^\infty k'(s) \mathbf{1}_{\{s < t\}} p t^{-p} \, ds \, dt.$$

Since the integrand in the double integral is non-negative, we can use the Fubini–Tonelli Theorem to freely interchange the order of integration. Hence

$$p \int_0^\infty k(t)t^{1-p} dt = \int_0^\infty k'(s) \int_s^\infty pt^{-p} dt ds = \int_0^\infty k'(s)s^{-p} ds. \tag{7.11}$$

Thus the formal integration by parts argument is justified. Moreover, this argument shows that if one side is infinite, so is the other. With (7.11) we get

$$\text{tr}_{\mathcal{H}} G(f(x)) \leq p \int_0^\infty k(t)t^{-1-p} dt \|g\|_{L^p_{\mathbb{W}}(Y)}^p \text{tr}_{\mathcal{H}}(f(x)^p).$$

Integrating this over X finishes the proof of (7.7).

To prove (7.8) we introduce

$$\tilde{B}_{f,t,m}\varphi(y) := \int_X \Phi(y, x)m(tf(x))\varphi(x) dx = \Phi[m(tf)\varphi](y) \tag{7.12}$$

for $t \geq 0$ (note that $\tilde{B}_{f,0,m}\varphi = 0$ since $m(0) = 0$). If $m = m_1 * m_2$, then a by now familiar calculation yields

$$\tilde{B}_{f,t,m}\varphi(y) = \int_0^\infty \Phi[m_1(sf)\varphi](y) m_2(t/s) \frac{ds}{s}$$

and therefore the Cauchy–Schwarz inequality gives

$$\begin{aligned} \|\tilde{B}_{f,t,m}\varphi(y)\|_{\mathcal{G}} &\leq \int_0^\infty \|\Phi[m_1(sf)\varphi](y)\|_{\mathcal{G}} |m_2(t/s)| \frac{ds}{s} \\ &\leq \left(\int_0^\infty \|\Phi[m_1(sf)\varphi](y)\|_{\mathcal{G}}^2 \frac{ds}{s} \right)^{1/2} \\ &\quad \times \left(\int_0^\infty |m_2(t/s)|^2 \frac{ds}{s} \right)^{1/2}. \end{aligned}$$

By scaling, the right hand side above does not depend on $t > 0$ anymore. Hence we get the bound

$$\begin{aligned} \tilde{B}_{f,m}^*\varphi(y) &= \sup_{t \geq 0} \|\tilde{B}_{f,t,m}\varphi(y)\|_{\mathcal{G}} \\ &\leq \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \left(\int_0^\infty \|\Phi[m_1(sf)\varphi](y)\|_{\mathcal{G}}^2 \frac{ds}{s} \right)^{1/2} \end{aligned}$$

for the associated maximal operator $\tilde{B}_{f,m}^* \varphi(y) := \sup_{t \geq 0} \|\tilde{B}_{f,t,m} \varphi(y)\|_{\mathcal{G}}$. In particular,

$$\|\tilde{B}_{f,m}^* \varphi\|_{L^2(Y,dy)}^2 \leq \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})}^2 \int_Y \int_0^\infty \|\Phi[m_1(sf)\varphi](y)\|_{\mathcal{G}}^2 \frac{ds}{s} dy. \tag{7.13}$$

Interchanging the integrals, the last factor on the right hand side of (7.13) is given by

$$\begin{aligned} & \int_0^\infty \int_Y \|\Phi[m_1(sf)\varphi](y)\|_{\mathcal{G}}^2 dy \frac{ds}{s} = \int_0^\infty \|\Phi[m_1(sf)\varphi]\|_{L^2(Y,\mathcal{G})}^2 \frac{ds}{s} \\ &= \int_0^\infty \|m_1(sf)\varphi\|_{L^2(X,\mathcal{H})}^2 \frac{ds}{s} \\ &= \int_X \int_0^\infty \langle m_1(sf(x))\varphi(x), m_1(sf(x))\varphi(x) \rangle_{\mathcal{H}} \frac{ds}{s} dx \\ &= \int_X \langle \varphi(x), \int_0^\infty m_1(sf(x))^2 \frac{ds}{s} \varphi(x) \rangle_{\mathcal{H}} dx. \end{aligned}$$

As functions of the real variable $r \geq 0$ the scaling invariance of the measure ds/s on \mathbb{R}_+ and $m_1(0) = 0$ give $\int_0^\infty m_1(sr)^2 \frac{ds}{s} = \|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})}^2 \mathbf{1}_{\{r>0\}}$, so the spectral theorem implies

$$\begin{aligned} \langle \varphi(x), \int_0^\infty m_1(sf(x))^2 \frac{ds}{s} \varphi(x) \rangle_{\mathcal{H}} &= \|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})}^2 \langle \varphi(x), \mathbf{1}_{\{f(x)>0\}} \varphi(x) \rangle_{\mathcal{H}} \\ &\leq \|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})}^2 \|\varphi(x)\|_{\mathcal{H}}^2. \end{aligned}$$

Using this in (7.13) shows

$$\|\tilde{B}_{f,m}^* \varphi\|_{L^2(Y,dy)} \leq \|m_1\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \|m_2\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \|\varphi\|_{L^2(X,\mathcal{H})}, \tag{7.14}$$

which proves (7.8), since $\|\tilde{B}_{f,g,m} \varphi(y)\|_{\mathcal{G}} \leq \tilde{B}_{f,m}^* \varphi(y)$ for all $y \in Y$. □

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Appendix A. Induction in dimension

In this section we prove Theorem 1.8, that is, we prove that the number of negative bound states of $P^2 \otimes \mathbf{1}_{\mathcal{G}} + V$ is bounded by

$$N(P^2 \otimes \mathbf{1}_{\mathcal{G}} + V) \leq C_{0,d}^{\text{op}} \frac{|B_1^d|}{(2\pi)^d} \int_{\mathbb{R}^d} \text{tr}_{\mathcal{G}}[V_-(x)^{\frac{d}{2}}] dx$$

and, moreover,

$$C_{0,d}^{\text{op}} = \min_{3 \leq n \leq d} C_{0,n}^{\text{op}} \leq \min_{3 \leq n \leq d} C_n,$$

where C_n is given by (1.5) for $\gamma = n$. Here, $V : \mathbb{R}^d \rightarrow \mathcal{B}(\mathcal{G})$ is an operator valued potential with positive part $V_+ \in L_{\text{loc}}^1(\mathbb{R}^d, \mathcal{B}(\mathcal{G}))$ and negative part $V_- \in L^{d/2}(\mathbb{R}^d, \mathcal{S}_{d/2}(\mathcal{G}))$.

In order to do this, we need the following operator-valued extension of the well-known Lieb–Thirring bounds for suitable moments θ :

$$\text{tr}_{L^2(\mathbb{R}^d, \mathcal{G})} [P^2 \otimes \mathbf{1}_{\mathcal{G}} + V]_-^\theta \leq L_{\theta,d}^{\text{op}} \int_{\mathbb{R}^d} \text{tr}_{\mathcal{G}} [V_-(x)^{\theta + \frac{d}{2}}] dx, \quad (\text{A.1})$$

where $L_{\theta,d}^{\text{op}} = C_{\theta,d}^{\text{op}} L_{\theta,d}^{\text{cl}}$ with the classical Lieb–Thirring constant

$$L_{\theta,d}^{\text{cl}} = \int_{\mathbb{R}^d} (1 - \eta^2)_+^\theta \frac{d\eta}{(2\pi)^d}. \quad (\text{A.2})$$

It is important that the constant $L_{\theta,d}^{\text{op}}$, respectively, $C_{\theta,d}^{\text{op}}$ does not depend on the auxiliary Hilbert space \mathcal{G} .

The bound (A.1) was first proven in the seminal work of Laptev and Weidl [27] for all dimensions $d \in \mathbb{N}$ and moments $\theta \geq \frac{3}{2}$, moreover, they showed $C_{\theta,d}^{\text{op}} = 1$ in this case. This was later simplified in [1]. For moments $\theta \geq \frac{1}{2}$ and again all dimensions $d \in \mathbb{N}$ the bound (A.1) was shown to hold in [23],

moreover, $C_{\theta,d}^{\text{op}} \leq 2$ for $\frac{1}{2} \leq \theta < \frac{3}{2}$, see also [15] and, recently, [18] for improvements when $\theta = 1$. The limiting case $\theta = 0$, that is, the operator-valued version of the CLR bound was then proven in [21], with improvements on the constant later in [19].

The possibility that a bound of the form A.1 allows to strip off one dimension in the Lieb–Thirring bounds was crucially used in Laptev–Weidl [27], see also [25]. The possibility of stripping off more than one dimension was realized in [21].

In the short proof below, which we give for the convenience of the reader, we follow the discussion in [21].

Lemma A.1 *For $n \leq d$ we have*

$$C_{\theta,d}^{\text{op}} \leq C_{\theta,n}^{\text{op}} C_{\theta+\frac{n}{2},d-n}^{\text{op}}.$$

In particular, for $d \geq 3$,

$$C_{0,d}^{\text{op}} \leq C_{0,n}^{\text{op}} \text{ for all } 3 \leq n \leq d.$$

Proof For $n \leq d$ we factor $\mathbb{R}^d = \mathbb{R}^n \times \mathbb{R}^{d-n}$, that is, $x = (x_<, x_>) \in \mathbb{R}^n \times \mathbb{R}^{d-n}$, and split the the kinetic energy as $P^2 = P_<^2 + P_>^2$, more precisely,

$$P^2 = P_<^2 \otimes \mathbf{1}_{L^2(\mathbb{R}^{d-n})} + \mathbf{1}_{L^2(\mathbb{R}^n)} \otimes P_>^2.$$

Moreover, observe that

$$\begin{aligned} L^2(\mathbb{R}^d, \mathcal{G}) &= L^2(\mathbb{R}^d) \otimes \mathcal{G} = L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^{d-n}) \otimes \mathcal{G} \\ &= L^2(\mathbb{R}^n, L^2(\mathbb{R}^{d-n} \otimes \mathcal{G})). \end{aligned}$$

As quadratic forms on $L^2(\mathbb{R}^d, \mathcal{G})$, we then have

$$\begin{aligned} &P^2 \otimes \mathbf{1}_{\mathcal{G}} + V(x) \\ &= P_<^2 \otimes \mathbf{1}_{L^2(\mathbb{R}^{d-n})} \otimes \mathbf{1}_{\mathcal{G}} + \mathbf{1}_{L^2(\mathbb{R}^n)} \otimes P_>^2 \otimes \mathbf{1}_{\mathcal{G}} + V(x_<, x_>) \quad (\text{A.3}) \\ &\geq P_<^2 \otimes \mathbf{1}_{L^2(\mathbb{R}^{d-n}, \mathcal{G})} - W(x_<) \end{aligned}$$

with the operator-valued potential $W(x_<) = (P_>^2 \otimes \mathbf{1}_{\mathcal{G}} + V(x_<, \cdot))_- : L^2(\mathbb{R}^{d-n}, \mathcal{G}) \rightarrow L^2(\mathbb{R}^{d-n}, \mathcal{G})$. Note that $W(x_<)$ is the negative part of a Schrödinger operator in $d - n$ dimensions where one freezes the $x_<$ coordinate in the potential. Inequality (A.1) can therefore be applied and yields

$$\begin{aligned} \text{tr}_{L^2(\mathbb{R}^{d-n}, \mathcal{G})} W(x_<)^{\theta+\frac{n}{2}} &= \text{tr}_{L^2(\mathbb{R}^{d-n}, \mathcal{G})} (P_>^2 \otimes \mathbf{1}_{\mathcal{G}} + V(x_<, \cdot))_-^{\theta+\frac{n}{2}} \\ &\leq L_{\theta+\frac{n}{2},d-n}^{\text{op}} \int_{\mathbb{R}^{d-n}} \text{tr}_{\mathcal{G}} V_-(x_<, x_>)^{\theta+\frac{d}{2}} dx_<. \end{aligned}$$

Since by assumption $\int_{\mathbb{R}^d} \text{tr}_{\mathcal{G}} V_-(x)^{\theta+\frac{d}{2}} dx < \infty$, the Fubini–Tonelli theorem shows that $W(x_{<})$ is compact (even in the von Neumann–Schatten ideal $\mathcal{S}_{\theta+\frac{n}{2}}(L^2(\mathbb{R}^{d-n}, \mathcal{G}))$) for almost all $x_{<} \in \mathbb{R}^n$. Taking traces in inequality (A.3) gives the estimate

$$\begin{aligned} & \text{tr}_{L^2(\mathbb{R}^d, \mathcal{G})} (P^2 \otimes \mathbf{1}_{\mathcal{G}} + V)_-^\theta \\ & \leq \text{tr}_{L^2(\mathbb{R}^n, L^2(\mathbb{R}^{d-n}, \mathcal{G}))} (P_{<}^2 \otimes \mathbf{1}_{L^2(\mathbb{R}^{d-n}, \mathcal{G})} - W)_-^\theta \\ & \leq L_{\theta, n}^{\text{op}} \int_{\mathbb{R}^n} \text{tr}_{L^2(\mathbb{R}^{d-n}, \mathcal{G})} W(x_{<})^{\theta+\frac{n}{2}} dx_{<} \\ & \leq L_{\theta, n}^{\text{op}} L_{\theta+\frac{n}{2}, d-n}^{\text{op}} \int_{\mathbb{R}^d} \text{tr}_{\mathcal{G}} V_-(x)^{\theta+\frac{d}{2}} dx, \end{aligned}$$

where we also used the operator-valued Lieb–Thirring inequality (A.1) and combined the integrals using the Fubini–Tonelli theorem. It follows that

$$L_{\theta, d}^{\text{op}} \leq L_{\theta, n}^{\text{op}} L_{\theta+\frac{n}{2}, d-n}^{\text{op}}. \tag{A.4}$$

A short calculation, see below, shows

$$L_{\theta, d}^{\text{cl}} = L_{\theta, n}^{\text{cl}} L_{\theta+\frac{n}{2}, d-n}^{\text{cl}}, \tag{A.5}$$

so (A.4) and the definition of $C_{\theta, d}^{\text{op}}$ imply the sub-multiplicativity

$$C_{\theta, d}^{\text{op}} \leq C_{\theta, n}^{\text{op}} C_{\theta+\frac{n}{2}, d-n}^{\text{op}}$$

which proves is the first claim of Lemma A.1. In particular, for $\theta = 0$ and $3 \leq n \leq d - 1$, we get

$$C_{0, d}^{\text{op}} \leq C_{0, n}^{\text{op}} C_{\frac{n}{2}, d-n}^{\text{op}} = C_{0, n}^{\text{op}}$$

since Laptev–Weidl [27] showed $C_{\theta, m}^{\text{op}} = 1$ if $m \in \mathbb{N}$ and $\theta \geq \frac{3}{2}$. This proves the second claim in Lemma A.1.

It remains to show (A.5), which follows from the definition of the classical Lieb–Thirring constant and the Fubini–Tonelli Theorem:

$$\begin{aligned} L_{\theta, d}^{\text{cl}} &= \int_{\mathbb{R}^d} (1 - \eta^2)_+^\theta \frac{d\eta}{(2\pi)^d} = \iint_{\mathbb{R}^n \times \mathbb{R}^{d-n}} (1 - \eta_{<}^2 - \eta_{>}^2)_+^\theta \frac{d\eta_{<} d\eta_{>}}{(2\pi)^n (2\pi)^{d-n}} \\ &= \int_{\mathbb{R}^{d-n}} \int_{\mathbb{R}^n} (1 - \eta_{>})_+^{\theta+\frac{n}{2}} (1 - \xi^2)_+^\theta \frac{d\xi}{(2\pi)^n} \frac{d\eta_{>}}{(2\pi)^{d-n}} = L_{\theta, n}^{\text{cl}} L_{\theta+\frac{n}{2}, d-n}^{\text{cl}}. \end{aligned}$$

The third equality follows from scaling, setting $\eta_{<} = (1 - \eta_{>}^2)_+^{1/2} \xi$ with $\xi \in \mathbb{R}^n$. □

Proof of Theorem 1.8 Lemma A.1 shows that

$$C_{0,d}^{\text{op}} \leq \min_{3 \leq n \leq d} C_{0,n}^{\text{op}}$$

and the reverse inequality clearly holds. Moreover, the case $\alpha = 1$ in Theorem 1.7 shows the bound

$$C_{0,n}^{\text{op}} \leq C_n$$

with the constant $C_{\gamma=n}$ from (1.5). □

Appendix B. Auxiliary bounds for the operator-valued case

In this appendix we gather three results, which we needed for extending our method from the scalar case to the operator-valued case. These results are probably well-known to specialists; we give short proofs for the convenience of the reader.

First we consider operators of the form A^*A and AA^* for some bounded operator $A : \mathcal{H} \rightarrow \mathcal{G}$, where \mathcal{H}, \mathcal{G} are two auxiliary (separable) Hilbert spaces. Let $N(A) = \{f \in \mathcal{H} : Af = 0\} \subset \mathcal{H}$ be the null space of A , $N(A^*) = \{g \in \mathcal{G} : A^*g = 0\} \subset \mathcal{G}$ the null space of the adjoint $A^* : \mathcal{G} \rightarrow \mathcal{H}$, and $N(A)^\perp := \{h \in \mathcal{H} : \langle h, f \rangle_{\mathcal{G}} = 0 \text{ for all } f \in N(A)\} \subset \mathcal{H}$, respectively $N(A^*)^\perp := \{g \in \mathcal{G} : \langle g, f \rangle_{\mathcal{H}} = 0 \text{ for all } f \in N(A^*)\} \subset \mathcal{G}$, the orthogonal complement of $N(A)$ in \mathcal{H} , respectively $N(A^*)$ in \mathcal{G} .

Lemma B.1 *Let \mathcal{H}, \mathcal{G} be Hilbert spaces and $A : \mathcal{H} \rightarrow \mathcal{G}$ be a bounded operator. Then $A^*A|_{N(A)^\perp}$ is unitarily equivalent to $AA^*|_{N(A^*)^\perp}$. In particular, if $A : \mathcal{H} \rightarrow \mathcal{G}$ is compact, then its non-zero singular values, including multiplicities, are the same as the non-zero singular values of $A^* : \mathcal{G} \rightarrow \mathcal{H}$.*

Remark B.2 In Theorem 3 in [13] a stronger result, which allows for unbounded operators is proven, we need it only for bounded operators $A : \mathcal{H} \rightarrow \mathcal{G}$.

Proof The polar decomposition, e.g., Theorem VI.10 in [41], of a bounded operator easily extends to a two Hilbert space situation: For a bounded operator $A : \mathcal{H} \rightarrow \mathcal{G}$ there exists a partial isometry $U : \mathcal{H} \rightarrow \mathcal{G}$ with $N(U) = N(A)$ and range $\text{Ran}(U) = \overline{\text{Ran}(A)}$, and a symmetric operator $|A|$ with $|A|^2 = A^*A$ such that $A = U|A|$.

Moreover, $U : \overline{\text{Ran}(A^*)} = N(A)^\perp \rightarrow \overline{\text{Ran}(A)} = N(A^*)^\perp$ is an isometry, and

$$AA^* = U|A|^2U^* = UA^*AU^*,$$

so $AA^*|_{N(A^*)^\perp}$ is unitarily equivalent to $A^*A|_{N(A)^\perp}$.

Since the singular values of A are the square roots of the eigenvalues of A^*A and the singular values of A^* the square roots of the eigenvalues of AA^* , the last claim in Lemma B.1 is evident from the unitary equivalence above. \square

Given a Hilbert space \mathcal{H} and a σ -finite measure space (X, dx) we denote by $L^p(X, \mathcal{H})$ the space of measurable functions $f : X \rightarrow \mathcal{H}$ for which

$$\|f\|_p := \|f\|_{L^p(X, \mathcal{H})} := \left(\int_X \|f(x)\|_{\mathcal{H}}^p dx \right)^{1/p} < \infty, \tag{B.1}$$

when $1 \leq p < \infty$, respectively,

$$\|f\|_\infty := \|f\|_{L^\infty(X, \mathcal{H})} := \text{ess sup}_{x \in X} \|f(x)\|_{\mathcal{H}} < \infty, \tag{B.2}$$

when $p = \infty$. Since \mathcal{H} is assumed to be separable, Pettis’ measurability theorem [40], see also [14], shows that the weak and strong notions of measurability for functions $X \ni x \mapsto f(x)$ coincide. If $\mathcal{H} = \mathbb{C}$, we simply write $L^p(X, \mathbb{C}) = L^p(X)$. Moreover, we denote by $\mathcal{S}_2(L^2(X, \mathcal{H}), L^2(Y, \mathcal{G}))$, the space of Hilbert–Schmidt operators $H : L^2(X, \mathcal{H}) \rightarrow L^2(Y, \mathcal{G})$ with scalar-product

$$\langle H_1, H_2 \rangle_{\mathcal{S}_2} := \text{tr}_{L^2(X, \mathcal{H})} [H_1^* H_2] \tag{B.3}$$

and associated norm $\|H\|_{\mathcal{S}_2} := \langle H, H \rangle_{\mathcal{S}_2}^{1/2}$ and by $L^2(Y \times X, \mathcal{S}_2(\mathcal{H}, \mathcal{G}))$, the L^2 -space of operator-valued kernels $K : Y \times X \rightarrow \mathcal{S}_2(\mathcal{H}, \mathcal{G})$ with scalar product

$$\begin{aligned} \langle K_1, K_2 \rangle_{L^2(Y \times X, \mathcal{S}_2(\mathcal{H}, \mathcal{G}))} &:= \iint_{Y \times X} \|K(y, x)\|_{\mathcal{S}_2(\mathcal{H}, \mathcal{G})}^2 dy dx \\ &= \iint_{Y \times X} \|K(y, x)\|_{\mathcal{S}_2(\mathcal{H}, \mathcal{G})}^2 dy dx. \end{aligned}$$

The next result extends the well-known one-to-one correspondence of Hilbert–Schmidt operators from $L^2(X)$ to $L^2(Y)$ with kernels in $L^2(Y \times X)$ to the operator-valued setting.

Lemma B.3 *Let (X, dx) and (Y, dy) be σ -finite measure spaces and \mathcal{H}, \mathcal{G} two auxiliary Hilbert spaces. Then $\mathcal{S}_2(L^2(X, \mathcal{H}), L^2(Y, \mathcal{G}))$ is isomorphic to $L^2(Y \times X, \mathcal{S}_2(\mathcal{H}, \mathcal{G}))$, that is, for any $H \in \mathcal{S}_2(L^2(X, \mathcal{H}), L^2(Y, \mathcal{G}))$ there exists a unique $K_H \in L^2(Y \times X, \mathcal{S}_2(\mathcal{H}, \mathcal{G}))$ such that for any $f \in L^2(X, \mathcal{H})$ and almost all $y \in Y$*

$$Hf(y) = \int_X K_H(y, x) f(x) dx$$

and vice versa. Moreover, the Hilbert–Schmidt norm of the operator $H \in \mathcal{S}_2(L^2(X, \mathcal{H}), L^2(Y, \mathcal{G}))$ can be calculated as

$$\|H\|_{\mathcal{S}_2}^2 = \iint_{Y \times X} \text{tr}_{\mathcal{H}} [K_H(y, x)^* K_H(y, x)] dx dy.$$

Proof The proof is a modification of the proof in the scalar-valued case. We sketch it for the convenience of the reader. Any kernel $K \in L^2(Y \times X, \mathcal{S}_2(\mathcal{H}, \mathcal{G}))$ yields a bounded operator $H_K : L^2(X, \mathcal{H}) \rightarrow L^2(Y, \mathcal{G})$ by defining

$$H_K f(x) := \int_X K(y, x) f(x) dx.$$

Indeed, since

$$\begin{aligned} \|H_K f(y)\|_{\mathcal{G}} &\leq \int_X \|K(y, x) f(x)\|_{\mathcal{G}} dx \leq \int_X \|K(y, x)\|_{\mathcal{B}(\mathcal{H}, \mathcal{G})} \|f(x)\|_{\mathcal{H}} dx \\ &\leq \left(\int_X \|K(y, x)\|_{\mathcal{B}(\mathcal{H}, \mathcal{G})}^2 dx \right)^{1/2} \|f\|_{L^2(X, \mathcal{H})}, \end{aligned}$$

by Cauchy–Schwarz, we get

$$\begin{aligned} \|H_K f\|_{L^2(Y, \mathcal{G})}^2 &= \int_Y \|Hf(y)\|_{\mathcal{G}}^2 dy \\ &\leq \iint_{Y \times X} \|K(y, x)\|_{\mathcal{B}(\mathcal{H}, \mathcal{G})}^2 dx dy \|f\|_{L^2(X, \mathcal{H})}^2 \tag{B.4} \\ &\leq \iint_{Y \times X} \|K(y, x)\|_{\mathcal{S}_2(\mathcal{H}, \mathcal{G})}^2 dx dy \|f\|_{L^2(X, \mathcal{H})}^2 \\ &= \|K\|_{L^2}^2 \|f\|_{L^2(X, \mathcal{H})}^2 \end{aligned}$$

since the Hilbert–Schmidt norm bounds the operator norm. So the map $K \mapsto H_K$ from kernels to operators $L^2(X, \mathcal{H}) \rightarrow L^2(Y, \mathcal{G})$ is bounded with norm $\leq \|K\|_{L^2}$, and it is clearly injective.

Given two orthonormal bases $(\alpha_m)_{m \in \mathbb{N}}$ of \mathcal{H} and $(\beta_m)_{m \in \mathbb{N}}$ of \mathcal{G} , the space $S_2(\mathcal{H}, \mathcal{G})$ has a basis for almost all given by the rank-one operators $|\beta_m\rangle\langle\alpha_n| : \mathcal{H} \rightarrow \mathcal{G}, f \mapsto \beta_m\langle\alpha_n, f\rangle_{\mathcal{H}}$. Furthermore, let $(\varphi_j)_{j \in \mathbb{N}}$ and $(\psi_l)_{l \in \mathbb{N}}$ be bases for $L^2(Y)$ and $L^2(X)$. Then $(\Psi_{l,n})_{l,n \in \mathbb{N}}$, given by the \mathcal{H} -valued functions $X \ni x \mapsto \Psi_{l,n}(x) = \psi_l(x)|\alpha_n\rangle$, is a basis for $L^2(X, \mathcal{H}) = L^2(X) \otimes \mathcal{H}$ and $(\Phi_{l,m})_{k,m \in \mathbb{N}}$, given by the \mathcal{G} -valued functions $Y \ni y \mapsto \Phi_{l,n}(y) = \varphi_k(y)|\beta_m\rangle$, is a basis for $L^2(Y, \mathcal{G})$. Thus any kernel $K \in L^2(Y \times X, S_2(\mathcal{H}, \mathcal{G})) = L^2(Y) \otimes L^2(X) \otimes S_2(\mathcal{H}, \mathcal{G})$ can be written in the form

$$K(y, x) = \sum_{k,l,m,n \in \mathbb{N}} a_{k,l,m,n} \varphi_k(y) \overline{\psi_l(x)} |\beta_m\rangle\langle\alpha_n|$$

and a short calculation shows

$$\|K\|_{L^2}^2 = \iint_{Y \times X} \text{tr} [K(y, x)^* K(y, x)] \, dx dy = \sum_{k,l,m,n \in \mathbb{N}} |a_{k,l,m,n}|^2. \tag{B.5}$$

Let $R \in \mathbb{N}$ and

$$K_R(y, x) = \sum_{k,l,m,n=1}^R a_{k,l,m,n} \varphi_k(y) \overline{\psi_l(x)} |\beta_m\rangle\langle\alpha_n|, \tag{B.6}$$

which is the kernel of the finite rank operator

$$\begin{aligned} H_{K_L} &= \sum_{k,l,m,n=1}^R a_{k,l,m,n} |\Phi_{k,m}\rangle\langle\Psi_{l,n}| \\ &= \sum_{k,l,m,n=1}^R a_{k,l,m,n} \Phi_{k,m} \langle\Psi_{l,n}, \cdot\rangle_{L^2(X, \mathcal{H})}. \end{aligned} \tag{B.7}$$

Since $\|K - K_R\|_{L^2} \rightarrow 0$ the bound (B.4) shows $\|H_K - H_{K_R}\| \rightarrow 0$ as $R \rightarrow \infty$, so any H_K is the limit in the operator norm of finite-rank operators, hence a compact operator. Using the basis $(\Psi_{l,n})_{l,n \in \mathbb{N}}$ to calculate the trace, a straightforward calculation shows

$$\text{tr}_{L^2(X, \mathcal{H})} [H_K^* H_K] = \sum_{l,n} \|H_K \Psi_{l,n}\|_{\mathcal{G}}^2 = \sum_{k,l,m,n \in \mathbb{N}} |a_{k,l,m,n}|^2 = \|K\|_{L^2}^2$$

so $H_K \in \mathcal{S}_2(L^2(X, \mathcal{H}), L^2(Y, \mathcal{G}))$ and $\|H_K\|_{\mathcal{S}_2} = \|K\|_{L^2}$.

So far we have shown that the map $K \mapsto H_K$ is an isometry from $L^2(Y \times X, \mathcal{S}_2(\mathcal{H}, \mathcal{G}))$ into $\mathcal{S}(L^2(X, \mathcal{H}), L^2(Y, \mathcal{G}))$ so its range is closed. The finite rank operators $F : L^2(X, \mathcal{H}) \rightarrow L^2(Y, \mathcal{G})$ are of the form

$$F = \sum_{r,s \in \mathbb{N}} c_{r,s} |\tilde{\Phi}_r\rangle \langle \tilde{\Psi}_s| = \sum_{r,s \in \mathbb{N}} c_{r,s} \tilde{\Phi}_r \langle \tilde{\Psi}_s, \cdot \rangle_{L^2(X, \mathcal{H})}$$

with $c_{r,s} \neq 0$ for finitely many $r, s \in \mathbb{N}$ and $\tilde{\Phi}_r \in L^2(Y, \mathcal{G}), \tilde{\Psi}_s \in L^2(X, \mathcal{H})$. Expanding $\tilde{\Psi}_s$ in the basis $(\Phi_{l,n})_{l,n \in \mathbb{N}}$ and similarly for $\tilde{\Phi}_r$, one sees that finite rank operators of the above form can be arbitrarily well approximated, in operator norm, by finite rank operators of the form (B.7). Since the finite rank operators are dense in the Hilbert–Schmidt operators, the operators of the form (B.7) are also dense and hence the range of $K \mapsto H_K$ is all of $\mathcal{S}(L^2(X, \mathcal{H}), L^2(Y, \mathcal{G}))$. \square

The last result concerns an operator-valued version of Dunford’s theorem. For this we need some more notation. For background on integration in Banach spaces, we refer to [14].

We denote by $\mathcal{B}(\mathcal{H}, \mathcal{G})$ the Banach space of bounded operators from \mathcal{H} to \mathcal{G} equipped with the operator norm.

We write $L_s^\infty(Y \times X, \mathcal{B}(\mathcal{H}, \mathcal{G}))$ for the space of functions $K : Y \times X \rightarrow \mathcal{B}(\mathcal{H}, \mathcal{G})$ such that

$$\text{ess sup}_{(y,x) \in Y \times X} \|K(y, x)\|_{\mathcal{B}(\mathcal{H}, \mathcal{G})} < \infty,$$

and for all $h \in \mathcal{H}$ the map

$$Y \times X \ni (y, x) \mapsto K(y, x)h \in \mathcal{G}$$

is strongly measurable (with respect to the topology on \mathcal{G}). Since \mathcal{G} is a separable Hilbert space, Pettis’ measurability theorem implies that this the case if and only if it is weakly measurable, i.e., for any $\psi \in \mathcal{G}$,

$$Y \times X \ni (y, x) \mapsto \langle \psi, K(y, x)h \rangle_{\mathcal{G}}$$

is measurable. In this case, for $f \in L^1(X, \mathcal{H})$, integrals of the form

$$\Phi_K f(y) := \int_X K(y, x) f(x) \, dx \tag{B.8}$$

are well-defined elements in \mathcal{G} for almost all $y \in Y$, with

$$\begin{aligned} \|\Phi_K f(y)\|_{\mathcal{G}} &= \left\| \int_X K(y, x) f(x) \, dx \right\|_{\mathcal{G}} \leq \int_X \|K(y, x) f(x)\|_{\mathcal{G}} \, dx \\ &\leq \operatorname{ess\,sup}_{(y,x) \in Y \times X} \|K(y, x)\|_{\mathcal{B}(\mathcal{H}, \mathcal{G})} \|f\|_{L^1(X, \mathcal{H})}. \end{aligned}$$

Thus, for $K \in L^\infty_s(Y \times X, \mathcal{B}(\mathcal{H}, \mathcal{G}))$, the map $\Phi_K : L^1(X, \mathcal{H}) \rightarrow L^\infty(Y, \mathcal{G})$ is bounded with

$$\|\Phi_K\|_{L^1 \rightarrow L^\infty} \leq \operatorname{ess\,sup}_{(y,x) \in Y \times X} \|K(y, x)\|_{\mathcal{B}(\mathcal{H}, \mathcal{G})}.$$

The next Lemma shows that the map $K \mapsto \Phi_K$ is even an isometry.

Lemma B.4 *For any bounded operator $\Phi : L^1(X, \mathcal{H}) \rightarrow L^\infty(Y, \mathcal{G})$ there exists a kernel $K_\Phi \in L^\infty_s(Y \times X, \mathcal{B}(\mathcal{H}, \mathcal{G}))$ such that*

$$\Phi f(y) = \int_X K_\Phi(y, x) f(x) \, dx$$

for any $f \in L^1(X, \mathcal{H})$ and almost all $y \in Y$. Moreover,

$$\|\Phi\| = \operatorname{ess\,sup}_{(y,x) \in Y \times X} \|K_\Phi(y, x)\|_{\mathcal{B}(\mathcal{H}, \mathcal{G})}.$$

Proof If $K \in L^\infty_s(Y \times X, \mathcal{B}(\mathcal{H}, \mathcal{G}))$, the discussion above shows that the map Φ_K defined in (B.8) is bounded from $L^1(X, \mathcal{H})$ to $L^\infty(Y, \mathcal{G})$ and

$$\|\Phi_K\|_{L^1 \rightarrow L^\infty} \leq \operatorname{ess\,sup}_{(y,x) \in Y \times X} \|K(y, x)\|_{\mathcal{B}(\mathcal{H}, \mathcal{G})} =: \|K\|_{L^\infty}. \tag{B.9}$$

Conversely, assume that Φ is a bounded map from $L^1(X, \mathcal{H})$ into $L^\infty(Y, \mathcal{G})$ and choose orthonormal bases $(\alpha_n)_{n \in \mathbb{N}}$ in \mathcal{H} and $(\beta_m)_{m \in \mathbb{N}}$ in \mathcal{G} . Then any function $f \in L^1(X, \mathcal{H})$ can be identified with a sequence of functions $f = (f_1, f_2, \dots)$, where $f_l \in L^1(X)$ and $\|f\|_{L^1(X, \mathcal{H})} = \|(\sum_{l \in \mathbb{N}} |f_l|^2)^{1/2}\|_{L^1(X)}$, and similarly for $L^1(Y, \mathcal{G})$. So without loss of generality, we can assume that $\mathcal{H} = \mathcal{G} = l^2(\mathbb{N})$, i.e., the bounded operators from $\mathcal{H} \rightarrow \mathcal{G}$ correspond to infinite matrices which map $l^2(\mathbb{N})$ boundedly into itself. Finally, let $(e_j)_{j \in \mathbb{N}}$ be the canonical basis of $l^2(\mathbb{N})$.

For $N \in \mathbb{N}$ and $g_l \in L^1(Y)$, $f_l \in L^1(X)$, $l = 1, \dots, N$, the finite linear combinations⁸ of the form

$$\sum_{l=1}^N g_l \otimes f_l \in L^1(Y) \otimes L^1(X) = L^1(Y \times X)$$

are dense in $L^1(Y \times X)$. Now assume that $\Phi : L^1(X, l^2(\mathbb{N})) \rightarrow L^\infty(Y, l^2(\mathbb{N}))$ is bounded. For $m, n \in \mathbb{N}$ let

$$S_{m,n} \left(\sum_{l=1}^N g_l \otimes f_l \right) := \sum_{l=1}^N \langle g_l \otimes e_m, \Phi f_l \otimes e_n \rangle$$

which defines a linear functional on the finite linear combinations and is bounded by $\|S_{m,n}\| \leq \|\Phi\|$. Thus it has a continuous extension to all of $L^1(Y \times X)$ and since the dual $L^1(Y \times X)^* = L^\infty(Y \times X)$, there exist measurable functions $K_\Phi^{m,n} \in L^\infty(Y \times X)$, $m, n \in \mathbb{N}$, such that

$$\langle g \otimes e_m, \Phi f \otimes e_n \rangle = \iint_{Y \times X} K_\Phi^{m,n}(y, x) \overline{g(y)} f(x) \, dx \, dy.$$

Taking unions of countably many zero sets, we can assume that the kernels $K_\Phi^{m,n}(\cdot, \cdot)$ are well-defined for any $m, n \in \mathbb{N}$, up to a common zero set in $Y \times X$.

Let $l_{\text{fin}}^2(\mathbb{N})$ be the set of sequences $\alpha = (\alpha_1, \alpha_2, \dots)$ with only finitely many α_j non-zero, which is dense in $l^2(\mathbb{N})$. For $\alpha \in l_{\text{fin}}^2(\mathbb{N})$ and $(y, x) \in Y \times X$ we define the sequence $K_\Phi(y, x)\alpha \in \mathbb{C}^{\mathbb{N}}$ as

$$(K_\Phi(y, x)\alpha)_m := \sum_{n \in \mathbb{N}} K_\Phi^{m,n}(y, x)\alpha_n, \quad \text{for } m \in \mathbb{N}.$$

From the construction it is clear that for $\alpha, \beta \in l_{\text{fin}}^2(\mathbb{N})$, the map $(y, x) \mapsto \langle \beta, K_\Phi(y, x)\alpha \rangle_{l^2}$ is measurable. The next step is to show that for almost all

⁸ For the equality $L^1(Y) \otimes L^1(X) = L^1(Y \times X)$ one should be a wee bit more precise about the involved topologies in the tensor products: For a Banach space E , the algebraic tensor product $L^1(Y) \otimes_{\text{alg}} E$ is the vector space of finite linear combinations $\sum_{l=1}^N g_l \otimes f_l$, where $g_l \in L^1(Y)$ and $f_l \in E$. One equips this vector space with the norm $\|z\|_\pi := \inf\{\sum_l \|g_l\|_{L^1(Y)} \|f_l\|_E : z = \sum_l g_l \otimes f_l\}$. Then for the closure $L^1(Y) \widehat{\otimes} E := \overline{L^1(Y) \otimes_{\text{alg}} E}^{\|\cdot\|_\pi}$, called the projective tensor product, one has $L^1(Y) \widehat{\otimes} E = L^1(Y, E)$, see [55, Proposition III.B.28] or [14, Example VIII.10]. In particular, one has $L^1(Y) \widehat{\otimes} L^1(X) = L^1(Y, L^1(X)) = L^1(Y \times X)$. We will not dwell on this fine point any further :-).

$(y, x) \in Y \times X$ one has $K_\Phi(y, x) \in \mathcal{B}(l^2(\mathbb{N}), l^2(\mathbb{N}))$. Since $l^2_{\text{fin}}(\mathbb{N})$ is dense in $l^2(\mathbb{N})$ one has

$$\begin{aligned} \|K_\Phi(y, x)\|_{\mathcal{B}} &= \|K_\Phi(y, x)\|_{\mathcal{B}(l^2(\mathbb{N}), l^2(\mathbb{N}))} \\ &= \sup \left\{ \operatorname{Re} \langle \beta, K_\Phi(y, x)\alpha \rangle \mid \alpha, \beta \in l^2_{\text{fin}}(\mathbb{N}), \|\alpha\|_{l^2} = \|\beta\|_{l^2} = 1 \right\} \\ &= \sup \left\{ \sum_{m,n} \operatorname{Re} (\bar{\beta}_m, K_\Phi^{m,n}(y, x)\alpha_n) \mid \alpha, \beta \in l^2_{\text{fin}}(\mathbb{N}), \|\alpha\|_{l^2} = \|\beta\|_{l^2} = 1 \right\}. \end{aligned}$$

Moreover, let $L^1_{\text{fin}}(X, l^2(\mathbb{N}))$ be the set of functions $f = (f_1, f_2, \dots) \in L^1(X, l^2(\mathbb{N}))$ with only finitely many nonzero f_j , which is dense in $L^1(X, l^2(\mathbb{N}))$, and similarly for $L^1_{\text{fin}}(Y, l^2(\mathbb{N}))$. For any $g \in L^1_{\text{fin}}(Y, l^2(\mathbb{N}))$, $f \in L^1_{\text{fin}}(X, l^2(\mathbb{N}))$, we clearly have from the above

$$\begin{aligned} \langle g, \Phi f \rangle &= \iint_{Y \times X} \sum_{m,n} \overline{g_m(y)} K_\Phi^{m,n}(y, x) f_n(x) \, dx \, dy \\ &= \iint_{Y \times X} \langle g(y), K_\Phi(y, x) f(x) \rangle_{l^2(\mathbb{N})} \, dx \, dy. \end{aligned} \tag{B.10}$$

and with $A = \{(g, f) \in L^1_{\text{fin}}(Y, l^2(\mathbb{N})) \times L^1_{\text{fin}}(X, l^2(\mathbb{N})) \mid \|g\|_{L^1(Y, l^2(\mathbb{N}))} = \|f\|_{L^1(X, l^2(\mathbb{N}))} = 1\}$, which is dense in $L^1(Y, l^2(\mathbb{N})) \times L^1(X, l^2(\mathbb{N}))$, one sees

$$\begin{aligned} \operatorname{ess\,sup}_{(y,x) \in Y \times X} \|K_\Phi(y, x)\|_{\mathcal{B}} &= \sup_{(g,f) \in A} \iint_{Y \times X} \operatorname{Re} \langle g(y), K_\Phi(y, x) f(x) \rangle_{l^2(\mathbb{N})} \, dy \, dx \\ &= \sup_{(g,f) \in A} \operatorname{Re} \langle g, \Phi f \rangle \leq \|\Phi\|_{L^1 \rightarrow L^\infty} \|g\|_{L^1(Y, l^2(\mathbb{N}))} \|f\|_{L^1(X, l^2(\mathbb{N}))} \end{aligned}$$

Thus the kernel $K_\Phi(y, x)$ maps $l^2(\mathbb{N})$ boundedly into itself uniformly in $(y, x) \in Y \times X$. Taking limits, measurability of $(y, x) \mapsto \langle \beta, K_\Phi(y, x)\alpha \rangle_{l^2}$ extends from $\alpha, \beta \in l^2_{\text{fin}}(\mathbb{N})$ to all of $l^2(\mathbb{N})$. Thus K_Φ is weakly, hence strongly measurable. From (B.10) one also gets $\Phi = \Phi_{K_\Phi}$. In addition, the last bound together with (B.9) shows

$$\operatorname{ess\,sup}_{(y,x) \in Y \times X} \|K_\Phi(y, x)\|_{\mathcal{B}} = \|\Phi\|_{L^1 \rightarrow L^\infty},$$

so the map $L^\infty(Y \times X, \mathcal{B}(l^2(\mathbb{N}), l^2(\mathbb{N}))) \ni K \mapsto \Phi_K \in \mathcal{B}(L^1(X, l^2(\mathbb{N})), L^\infty(Y, l^2(\mathbb{N})))$ is an isometry. □

Appendix C. Solution of an auxiliary minimization problem

In this section we introduce an auxiliary minimization problem Q_γ which on one hand can be solved explicitly, and on the other hand provides an upper bound on the minimization problem M_γ defined in (1.4).

Proposition C.1 *For any $\gamma > 2$ the minimization problem*

$$Q_\gamma = \inf \left\{ \left(\frac{1}{2} \int_0^\infty h'(s)^2 \frac{ds}{s} \right)^{\frac{\gamma-2}{2}} \int_0^\infty s^{\gamma-2} h(s)^2 \frac{ds}{s} : \right. \\ \left. h(0) = 1 \text{ and } \lim_{s \rightarrow \infty} h(s) = 0 \right\} \tag{C.1}$$

has the solution

$$Q_\gamma = \frac{4}{(\gamma - 2)\gamma^2} \frac{1}{\Gamma\left(\frac{2}{\gamma}\right)^\gamma} \left(\frac{\gamma - 2}{2} \frac{\pi}{\sin\left(\frac{2\pi}{\gamma}\right)} \right)^{\frac{\gamma}{2}}. \tag{C.2}$$

Moreover, h is a minimizer if and only if $h(s) = h_*(\lambda s)$ for arbitrary $\lambda > 0$, where

$$h_*(s) = \frac{2^{1-\frac{2}{\gamma}}}{\Gamma\left(\frac{2}{\gamma}\right)} s K_{\frac{2}{\gamma}}\left(s^{\frac{\gamma}{2}}\right),$$

and K_α denotes the modified Bessel function of the second kind with parameter $\alpha \in (0, 1)$.

Remark C.2 As the form of the minimization problem suggests, any minimizer should be decreasing and, using known properties of Bessel functions, one can see that the above minimiser h_* is strictly monotone decreasing.

The point is that the above minimization problem is quadratic, hence it can be solved by completing the square. First we make the change of coordinates $s = t^{\frac{2}{\gamma}}$, which gives

$$Q_\gamma = \left(\frac{\gamma}{2}\right)^{\frac{\gamma-4}{2}} \left(\frac{1}{2}\right)^{\frac{\gamma-2}{2}} \inf \left\{ \left(\int_0^\infty g'(t)^2 t^{1-\frac{4}{\gamma}} dt \right)^{\frac{\gamma-2}{2}} \right. \\ \left. \times \int_0^\infty g(t)^2 t^{1-\frac{4}{\gamma}} dt : g(0) = 1, \lim_{t \rightarrow \infty} g(t) = 0 \right\}.$$

This is immediate upon setting $g(t) = h(t^{\frac{2}{\gamma}})$ in the above integrals. Defining the variational problem

$$q_{u,\gamma} := \inf \left\{ \int_0^\infty g'(t)^2 t^{1-\frac{4}{\gamma}} dt : \int_0^\infty g(t)^2 t^{1-\frac{4}{\gamma}} dt = u, g(0) = 1, \right. \\ \left. \text{and } \lim_{t \rightarrow \infty} g(t) = 0 \right\}, \quad (\text{C.3})$$

we obtain

$$Q_\gamma = \left(\frac{\gamma}{2}\right)^{\frac{\gamma-4}{2}} \left(\frac{1}{2}\right)^{\frac{\gamma-2}{2}} \inf_{u>0} \left(u q_{u,\gamma}\right)^{\frac{\gamma-2}{2}}. \quad (\text{C.4})$$

Hence, Proposition C.1 is a direct consequence of

Lemma C.3 *For any $\gamma > 2$ and $u > 0$, the variational problem (C.3) has the solution*

$$q_{u,\gamma} = u^{-\frac{2}{\gamma-2}} \frac{\gamma-2}{2} \left(\frac{2^{2-\frac{4}{\gamma}}}{\Gamma(\frac{2}{\gamma})^2} \frac{\pi}{\gamma \sin(\frac{2\pi}{\gamma})} \right)^{\frac{\gamma}{\gamma-2}}. \quad (\text{C.5})$$

The unique minimizer is given by

$$g_\lambda(t) := \frac{2}{\Gamma(\frac{2}{\gamma})} \left(\frac{t\sqrt{\lambda}}{2} \right)^{\frac{2}{\gamma}} K_{\frac{2}{\gamma}}(t\sqrt{\lambda}), \\ \text{with } \lambda = u^{-\frac{\gamma}{\gamma-2}} \left(\frac{2^{2-\frac{4}{\gamma}}}{\Gamma(\frac{2}{\gamma})^2} \frac{\pi}{\gamma \sin(\frac{2\pi}{\gamma})} \right)^{\frac{\gamma}{\gamma-2}}. \quad (\text{C.6})$$

Proof Given a real Hilbert space \mathcal{H} with scalar product $\langle \cdot, \cdot \rangle$ and linear operators A, B on \mathcal{H} consider the functionals

$$F(\varphi) := \langle A\varphi, A\varphi \rangle, \quad G(\varphi) := \langle B\varphi, B\varphi \rangle \quad (\text{C.7})$$

and the associated constrained minimization problem

$$Q_u := \inf \{ F(\varphi) : G(\varphi) = u \} \quad (\text{C.8})$$

for $u > 0$. Note that directional derivatives of F and G are given by

$$D_h F(\varphi) = \langle Ah, A\varphi \rangle, \quad D_h G(\varphi) = \langle Bh, B\varphi \rangle$$

when h, φ are in the domains of A and B , but we are, intentionally, a bit vague at this point concerning domain questions.

Assume that ψ is a weak solution of the Euler–Lagrange equation

$$\langle Ah, A\psi \rangle = -\lambda \langle Bh, B\psi \rangle \tag{C.9}$$

for some $\lambda \geq 0$ and all h , more precisely, all h in the intersection of the domains of A and B and also assume that ψ fulfils the constraint: $G(\psi) = u$. Given an arbitrary $\varphi \in \mathcal{H}$ with $G(\varphi) = u$, we write it as $\varphi = \psi + h$. Then

$$\begin{aligned} u &= G(\varphi) = \langle B(\psi + h), B(\psi + h) \rangle \\ &= \langle B\psi, B\psi \rangle + 2\langle Bh, B\psi \rangle + \langle Bh, Bh \rangle \end{aligned}$$

so, since $\langle B\psi, B\psi \rangle = u$, we have $2\langle Bh, B\psi \rangle = -\langle Bh, Bh \rangle$ and from (C.9) we get

$$2\langle Ah, A\psi \rangle = \lambda \langle Bh, Bh \rangle. \tag{C.10}$$

Thus

$$\begin{aligned} F(\varphi) &= F(\psi + h) = \langle A(\psi + h), A(\psi + h) \rangle \\ &= \langle A\psi, A\psi \rangle + 2\langle Ah, A\psi \rangle + \langle Ah, Ah \rangle \\ &= F(\psi) + \lambda \langle Bh, Bh \rangle + \langle Ah, Ah \rangle \geq F(\psi) \end{aligned} \tag{C.11}$$

so ψ is a minimizer. Moreover, if equality holds, i.e., if $F(\varphi) = F(\psi)$, then $\langle Ah, Ah \rangle = 0$, i.e., h is in the kernel of A , and if in addition $\lambda > 0$, the h is also in the kernel of B .

We apply the above with the choice

$$\mathcal{H} = L^2(\mathbb{R}_+, t^{1-\frac{4}{\gamma}} dt)$$

of real-valued functions on R_+ , which are square integrable w.r.t. the weighted Lebesgue measure $t^{1-\frac{4}{\gamma}} dt$. The operator B is the identity on $L^2(\mathbb{R}_+, t^{1-\frac{4}{\gamma}} dt)$ and A is the (weak) derivative,

$$A\varphi = \varphi'$$

with domain $\mathcal{D}(A) = \{\varphi \in L^2(\mathbb{R}_+, t^{1-\frac{4}{\gamma}} dt) : \varphi' \in L^2(\mathbb{R}_+, t^{1-\frac{4}{\gamma}} dt)\}$.

In this setting, we have $q_{u,\gamma} = Q_u$. Integration by parts shows that the Euler–Lagrange equation is given by

$$t^2 g''(t) + \left(1 - \frac{4}{\gamma}\right) t g'(t) - \lambda t^2 g(t) = 0, \tag{C.12}$$

which can be transformed into a modified Bessel differential equation upon setting $g(t) = (t\sqrt{\lambda})^{\frac{2}{\gamma}} \tilde{g}(t\sqrt{\lambda})$. Then \tilde{g} satisfies the modified Bessel equation

$$t^2 \tilde{g}''(t) + t \tilde{g}'(t) - \left(t^2 + \frac{4}{\gamma^2} \right) \tilde{g}(t) = 0,$$

with solution space spanned by the modified Bessel functions $I_{\frac{2}{\gamma}}, K_{\frac{2}{\gamma}}$. Using the well-known asymptotics of modified Bessel functions,⁹ it is easy to see that the function

$$g_\lambda(t) := \frac{2}{\Gamma(\frac{2}{\gamma})} \left(\frac{t\sqrt{\lambda}}{2} \right)^{\frac{2}{\gamma}} K_{\frac{2}{\gamma}}(t\sqrt{\lambda})$$

is the unique solution of (C.12) satisfying $g_\lambda(0) = 1$ and $\lim_{t \rightarrow \infty} g_\lambda(t) = 0$. We now use that for $\alpha \in (0, 1)$,

$$\int t K_\alpha(t)^2 dt = \frac{t^2}{2} (K_\alpha(t)^2 - K_{1-\alpha}(t)K_{1+\alpha}(t)), \tag{C.13}$$

which together with the asymptotics of Bessel functions (see Footnote 9) implies that

$$\int_0^\infty t K_\alpha(t)^2 dt = \frac{1}{2} \Gamma(1 - \alpha) \Gamma(1 + \alpha). \tag{C.14}$$

Identity (C.13) follows from integration by parts and $\frac{d}{dt} [\frac{t^2}{2} K_{1-\alpha}(t) K_{1+\alpha}(t)] = t^2 K_\alpha(t) K'_\alpha(t)$, which is a consequence of the relations $K_{1-\alpha}(t) + K_{\alpha+1}(t) = -2K'_\alpha(t)$ and $K'_\alpha(t) = \frac{\alpha}{t} K_\alpha(t) - K_{\alpha+1}(t) = -\frac{\alpha}{t} K_\alpha(t) - K_{1-\alpha}(t)$ [38, Eqs. 10.29.1, 10.29.2] for modified Bessel functions. Hence, using that $\Gamma(1 + \alpha) = \alpha \Gamma(\alpha)$ and $\Gamma(\alpha) \Gamma(1 - \alpha) = \frac{\pi}{\sin(\pi\alpha)}$ for $\alpha \in (0, 1)$, we obtain

$$\begin{aligned} \int_0^\infty g_\lambda(t)^2 t^{1-\frac{4}{\gamma}} dt &= \frac{2^{2-\frac{4}{\gamma}}}{\Gamma(\frac{2}{\gamma})^2} \lambda^{\frac{2}{\gamma}-1} \int_0^\infty s K_{\frac{2}{\gamma}}(s)^2 ds \\ &= \frac{2^{2-\frac{4}{\gamma}}}{\Gamma(\frac{2}{\gamma})^2} \lambda^{\frac{2}{\gamma}-1} \frac{\pi}{\gamma \sin(\frac{2\pi}{\gamma})}. \end{aligned} \tag{C.15}$$

⁹ For $\alpha \in (0, 1)$, one has $K_\alpha(t) \sim \sqrt{\frac{\pi}{2t}} e^{-t}$ as $t \rightarrow \infty$ [38, Eq. 10.25.3] and $K_\alpha(t) \sim \frac{1}{2} \Gamma(\alpha) (\frac{t}{2})^{-\alpha}$ as $t \rightarrow 0$ [38, Eq. 10.30.2], while $I_\alpha(t)$ grows exponentially fast as $t \rightarrow \infty$ [38, Eq. 10.30.4], so it can never satisfy the boundary condition at infinity.

Note that (C.15) determines the relation between the Lagrange multiplier $\lambda > 0$ and the constraint $u > 0$ in (C.6). Similarly, again using the above relation for the derivative of the modified Bessel function K_α , we have

$$g'_\lambda(t) = -\frac{2\sqrt{\lambda}}{\Gamma(\frac{2}{\gamma})} \left(\frac{t\sqrt{\lambda}}{2}\right)^{\frac{2}{\gamma}} K_{1-\frac{2}{\gamma}}(t\sqrt{\lambda}).$$

Therefore, (C.14) and the above functional equations for the Γ -function give

$$\begin{aligned} &\int_0^\infty g'_\lambda(t)^2 t^{1-\frac{4}{\gamma}} dt \\ &= \frac{2^{2-\frac{4}{\gamma}}}{\Gamma(\frac{2}{\gamma})^2} \lambda^{\frac{2}{\gamma}} \int_0^\infty s K_{1-\frac{2}{\gamma}}(s)^2 ds = \frac{2^{2-\frac{4}{\gamma}}}{\Gamma(\frac{2}{\gamma})^2} \lambda^{\frac{2}{\gamma}} \frac{\gamma-2}{2} \frac{\pi}{\gamma \sin(\frac{2\pi}{\gamma})} \\ &= \frac{\gamma-2}{2} \lambda u \stackrel{(C.6)}{=} \frac{\gamma-2}{2} u^{-\frac{2}{\gamma-2}} \left(\frac{2^{2-\frac{4}{\gamma}}}{\Gamma(\frac{2}{\gamma})^2} \frac{\pi}{\gamma \sin(\frac{2\pi}{\gamma})}\right)^{\frac{\gamma}{\gamma-2}}. \end{aligned}$$

This proves (C.5). To show uniqueness of the minimizer g_λ , note that since $\lambda > 0$, we have $0 = Bh = h$ if $F(\varphi) = F(g_\lambda)$ and $G(\varphi) = u = G(g_\lambda)$ by (C.11). □

Proposition C.4 For any $\gamma > 2$,

$$M_\gamma \leq Q_\gamma.$$

Proof The choice $m_1(s) = s \mathbf{1}_{\{0 < s \leq 1\}}$ in the minimization problem for M_γ gives

$$m(t) = m_1 * m_2(t) = \int_0^\infty m_1(ts) m_2(s^{-1}) \frac{ds}{s} = t \int_0^{\frac{1}{t}} m_2(s^{-1}) ds,$$

so

$$\begin{aligned} M_\gamma \leq &\inf_{m_2 \in L^2(\mathbb{R}_+; \frac{ds}{s})} \left\{ \left(\frac{1}{2} \int_0^\infty m_2(s)^2 \frac{ds}{s}\right)^{\frac{\gamma-2}{2}} \right. \\ &\left. \times \int_0^\infty t^{\gamma-2} \left(1 - \int_0^t m_2(s^{-1}) ds\right)^2 \frac{dt}{t} \right\}, \end{aligned}$$

where we used that $\int_0^\infty m_1(s)^2 \frac{ds}{s} = \frac{1}{2}$. Setting

$$h(t) = 1 - \int_0^t m_2(s^{-1})ds, \tag{C.16}$$

it follows that

$$\int_0^\infty t^{\gamma-2} \left(1 - \int_0^t m_2(s^{-1}) ds\right)^2 \frac{dt}{t} = \int_0^\infty t^{\gamma-2} h(t)^2 \frac{dt}{t}. \tag{C.17}$$

Moreover, h is absolutely continuous with $h'(t) = -m_2(t^{-1})$, and

$$\int_0^\infty h'(t)^2 \frac{dt}{t} = \int_0^\infty m_2(s)^2 \frac{ds}{s} < \infty,$$

so h cannot oscillate too fast at infinity. Finiteness of $\int_0^\infty t^{\gamma-2} h(t)^2 \frac{dt}{t}$ then implies that $h(t) \rightarrow 0$ as $t \rightarrow \infty$. Indeed, let $t_2 \geq t_1 \geq 1$. Then, using Cauchy–Schwarz,

$$\begin{aligned} |h(t_2)^2 - h(t_1)^2| &\leq 2 \int_{t_1}^{t_2} |h(s)h'(s)| ds \\ &\leq 2 \left(\int_{t_1}^{t_2} s^{\gamma-2} h(s)^2 \frac{ds}{s}\right)^{1/2} \left(\int_{t_1}^{t_2} s^{2-\gamma} h'(s)^2 \frac{ds}{s}\right)^{1/2} \\ &\leq 2 \left(\int_{t_1}^\infty s^{\gamma-2} h(s)^2 \frac{ds}{s}\right)^{1/2} \left(\int_1^\infty h'(s)^2 \frac{ds}{s}\right)^{1/2}, \end{aligned}$$

where we also used $\gamma > 2$. Since $0 < t \mapsto t^{\gamma-2} h(t)^2$ is integrable on $(0, \infty)$ w.r.t. $\frac{dt}{t}$, the above bound shows that $h(t)^2$ is Cauchy in the limit $t \rightarrow \infty$. Hence $\lim_{t \rightarrow \infty} h(t)^2$ exists. Moreover, using again that $t \mapsto t^{\gamma-2} h(t)^2 \in L^1((0, \infty), \frac{dt}{t})$ and $\gamma > 2$, this forces $\lim_{t \rightarrow \infty} h(t)^2 = 0$, i.e., $\lim_{t \rightarrow \infty} h(t) = 0$.

In addition, using again the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \left|\int_0^t m_2(s^{-1}) ds\right| &\leq \left(\int_0^t m_2(s^{-1})^2 \frac{ds}{s}\right)^{\frac{1}{2}} \left(\int_0^t s^2 \frac{ds}{s}\right)^{\frac{1}{2}} \\ &\leq \left(\int_0^\infty m_2(s)^2 \frac{ds}{s}\right)^{\frac{1}{2}} \frac{t}{\sqrt{2}} \xrightarrow{t \rightarrow 0} 0, \end{aligned}$$

so $\lim_{t \rightarrow 0} h(t) = 1$. Hence,

$$M_\gamma \leq \left(\frac{1}{2} \int_0^\infty h'(s)^2 \frac{ds}{s} \right)^{\frac{\gamma-2}{2}} \int_0^\infty s^{\gamma-2} h(s)^2 \frac{ds}{s},$$

for any absolutely continuous function $h : \mathbb{R}_+ \rightarrow \mathbb{R}$ with $h' \in L^2(\mathbb{R}_+; \frac{ds}{s})$ satisfying the boundary conditions $h(0) = 1$ and $\lim_{s \rightarrow \infty} h(s) = 0$. The bound $M_\gamma \leq Q_\gamma$ follows by taking the infimum over these functions. \square

Appendix D. Numerical results

In this section we derive upper bounds on the the constants in Theorem 1.3 and 1.7, in particular, the constant $C_{0,d}$ in the bound for the number of bound states of a non-relativistic one-particle Schrödinger operator from Corollary 1.1, given in Table 1.

Recall that the best constant in our approach is related to the minimization problem for

$$M_\gamma = \inf_{m_1, m_2 \in L^2(\mathbb{R}_+, \frac{ds}{s})} \left\{ (\|m_1\|_{L^2} \|m_2\|_{L^2})^{\gamma-2} \int_0^\infty \left(1 - \frac{(m_1 * m_2)(s)}{s} \right)^2 s^{2-\gamma} \frac{ds}{s} \right\}.$$

The choice of m_1, m_2 is quite arbitrary. It is important, however, to have $m_1 * m_2(s) \sim s$ for small s , in order to make the integral $\int_0^\infty \left(1 - \frac{(m_1 * m_2)(s)}{s} \right)^2 s^{2-\gamma} \frac{ds}{s}$ finite.

We reformulate the above problem by making the ansatz

$$m_1(s) = s \int_s^\infty \xi(r) \frac{dr}{r}, \quad m_2(s) = s \psi(s),$$

where $\xi, \psi : \mathbb{R}_+ \rightarrow \mathbb{R}$ are such that $\int_0^\infty \xi(r) \frac{dr}{r} = \int_0^\infty \psi(r) \frac{dr}{r} = 1$.

Then the convolution of m_1 and m_2 is given by

$$m_1 * m_2(t) = \int_0^\infty m_1(t/s) m_2(s) \frac{ds}{s} = t \int_0^\infty \int_0^\infty \xi(r) \psi(s) 1_{\{r>t/s\}} \frac{dr}{r} \frac{ds}{s}$$

and a short calculation, taking into account the above normalization of ξ and ψ , shows

$$\begin{aligned} & \int_0^\infty \left(1 - \frac{(m_1 * m_2)(t)}{t}\right)^2 t^{2-\gamma} \frac{dt}{t} \\ &= \int_0^\infty \left(\iint_0^\infty 1_{\{r \leq t/s\}} \xi(r) \psi(s) \frac{dr}{r} \frac{ds}{s}\right)^2 t^{2-\gamma} \frac{dt}{t} \\ &= \frac{1}{\gamma - 2} \underbrace{\iiint_0^\infty \xi(r_1) \xi(r_2) \psi(s_1) \psi(s_2) \max\{r_1 s_1, r_2 s_2\}^{2-\gamma} \frac{dr_1}{r_1} \frac{dr_2}{r_2} \frac{ds_1}{s_1} \frac{ds_2}{s_2}}_{=: I_\gamma[\xi, \psi]}. \end{aligned} \tag{D.1}$$

The L^2 -norms of m_1, m_2 can be expressed in terms of ξ and ψ by

$$\begin{aligned} \int_0^\infty m_1(s)^2 \frac{ds}{s} &= \int_0^\infty \left(s \int_0^\infty \xi(r) \frac{dr}{r}\right)^2 \frac{ds}{s} \\ &= \frac{1}{2} \iint_0^\infty \xi(r_1) \xi(r_2) \min\{r_1, r_2\}^2 \frac{dr_1}{r_1} \frac{dr_2}{r_2} \end{aligned}$$

and

$$\int_0^\infty m_2(s)^2 \frac{ds}{s} = \int_0^\infty s^2 \psi(s)^2 \frac{ds}{s}.$$

Thus, an upper bound on M_γ can be obtained by minimizing the functional

$$\begin{aligned} & \left(\int_0^\infty s^2 \psi(s)^2 \frac{ds}{s}\right)^{\frac{\gamma-2}{2}} \\ & \times \left(\frac{1}{2} \iint_0^\infty \xi(r_1) \xi(r_2) \min\{r_1, r_2\}^2 \frac{dr_1}{r_1} \frac{dr_2}{r_2}\right)^{\frac{\gamma-2}{2}} I_\gamma[\xi, \psi] \end{aligned} \tag{D.2}$$

over all functions $\psi, \xi \in L^1(\mathbb{R}_+, \frac{ds}{s})$ satisfying the constraint

$$\int_0^\infty \xi(r) \frac{dr}{r} = \int_0^\infty \psi(r) \frac{dr}{r} = 1. \tag{D.3}$$

Finding the minimizer, even finding that a minimizer exists for the new minimization problem given by (D.2) and (D.3), is a very challenging problem, as challenging as for the original minimization problem. However, to get a reasonable upper bound on the minimal value, it suffices to take suitable trial functions. To get the constants given in Table 1, in our calculations, which

Table 3 Numerical values of the constants $C_{0,d}$ and the values of the corresponding parameters of the trial functions

d	$C_{0,d}$	Value of parameters in (D.4)			
		p	α	q	β
3	7.55151	2	2.93254	3	2.49795
4	6.32791	2	3.69214	3	2.78716
5	5.95405	3	5.46494	2	2.39433
6	5.77058	3	6.41334	2	2.51583
7	5.67647	3	7.35963	2	2.61721
8	5.63198	3	8.30512	2	2.70368
9	5.62080	3	9.25042	2	2.77865

where done with Mathematica, we used the following family of trial functions

$$\xi(s) = \frac{\alpha^p}{\Gamma(p)} s^{-\alpha} (\log s)^{p-1} 1_{\{s>1\}}, \quad \psi(s) = \frac{\beta^q}{\Gamma(q)} s^{-\beta} (\log s)^{q-1} 1_{\{s>1\}}, \tag{D.4}$$

with parameters $\alpha, p, \beta, q > 0$, i.e., Gamma distributions on \mathbb{R}_+ .

The normalization condition is easily verified. For integer $p, q \geq 1$, the calculation of $I[\xi, \psi]$ can be reduced to calculating the integral

$$J(\alpha_1, \alpha_2, \beta_1, \beta_2) = \iiint\limits_1^\infty r_1^{-\alpha_1} r_2^{-\alpha_2} s_1^{-\beta_1} s_2^{-\beta_2} \max\{r_1 s_1, r_2 s_2\}^{2-\gamma} \frac{dr_1}{r_1} \frac{dr_2}{r_2} \frac{ds_1}{s_1} \frac{ds_2}{s_2},$$

as from J we can get $I[\xi, \psi]$ by taking derivatives,

$$I[\xi, \psi] = \frac{1}{\gamma - 2} \frac{\alpha^{2p} \beta^{2q}}{\Gamma(p)^2 \Gamma(q)^2} (\partial_{\alpha_1} \partial_{\alpha_2})^{p-1} (\partial_{\beta_1} \partial_{\beta_2})^{q-1} \times J(\alpha_1, \alpha_2, \beta_1, \beta_2) \Big|_{\substack{\alpha_1=\alpha_2=\alpha \\ \beta_1=\beta_2=\beta}}$$

Similarly, the “ L^2 -norm integrals” are given by

$$\int_0^\infty s^2 \psi(s)^2 \frac{ds}{s} = \frac{\beta^{2q}}{2^{2q-1} (\beta - 1)^{2q-1}} \frac{\Gamma(2q - 1)}{\Gamma(q)^2}$$

for $q \in \mathbb{N}$ and $\beta > 1$, as well as

$$\frac{1}{2} \iint_0^\infty \xi(r_1) \xi(r_2) \min\{r_1, r_2\}^2 \frac{dr_1}{r_1} \frac{dr_2}{r_2}$$

$$= \frac{1}{2} \frac{\alpha^{2p}}{\Gamma(p)^2} (\partial_{\alpha_1} \partial_{\alpha_2})^{p-1} K(\alpha_1, \alpha_2) \Big|_{\alpha_1=\alpha_2=\alpha},$$

where

$$K(\alpha_1, \alpha_2) = \iint_1^\infty r_1^{-\alpha_1} r_2^{-\alpha_2} \min\{r_1, r_2\}^2 \frac{dr_1}{r_1} \frac{dr_2}{r_2} = \frac{\alpha_1 + \alpha_2}{\alpha_1 \alpha_2 (\alpha_1 + \alpha_2 - 2)}$$

for $\alpha > 1$, $p \in \mathbb{N}$.

In our numerical calculations with *Mathematica*, we made the choice $p = 2$, $q = 3$, for dimensions $d = 3, 4$, and optimized in the parameters $\alpha, \beta > 1$, while for dimensions $d \geq 5$ the values were obtained with $p = 3$, $q = 2$, and minimization in $\alpha, \beta > 1$. More specifically, we got the values in Table 1 by the choice of parameters listed in Table 3.

References

1. Benguria, R., Loss, M.: A simple proof of a theorem of Laptev and Weidl. *Math. Res. Lett.* **7**, 195–203 (2000). <https://doi.org/10.4310/MRL.2000.v7.n2.a5>
2. Birman, M.S., Karadzhov, G.E., Solomyak, M.Z.: Boundedness conditions and spectrum estimates for the operators $b(X)a(D)$ and their analogs. In: *Estimates and Asymptotics for Discrete Spectra of Integral and Differential Equations (Leningrad, 1989–1990)*. *Advances in Soviet Mathematics*, vol. 7, pp. 85–106. American Mathematical Society, Providence (1991)
3. Birman, M.S., Laptev, A.: The negative discrete spectrum of a two-dimensional Schrödinger operator. *Commun. Pure Appl. Math.* **49**, 967–997 (1996)
4. Birman, MSh., Laptev, A., Solomyak, M.: The negative discrete spectrum of the operator $(-\Delta)^l - \alpha V$ in $L_2(\mathbb{R}^d)$ for d even and $2l \geq d$. *Ark. Mat.* **35**, 87–126 (1997). <https://doi.org/10.1007/BF02559594>
5. Birman, MSh., Solomyak, M.: Estimates for the singular numbers of integral operators. *Uspekhi Mat. Nauk* **32**, 17–84 (1977). (Russian)
6. Calderón, A.P.: Intermediate spaces and interpolation, the complex method. *Stud. Math.* **24**, 113–190 (1964). <https://doi.org/10.4064/sm-24-2-113-190>
7. Carbery, A.: Radial Fourier multipliers and associated maximal functions. In: *Recent Progress in Fourier Analysis (El Escorial, 1983)*. *North-Holland Mathematics Studies*, vol. 111, pp. 49–56. North-Holland, Amsterdam (1985). [https://doi.org/10.1016/S0304-0208\(08\)70279-2](https://doi.org/10.1016/S0304-0208(08)70279-2)
8. Conlon, J.G.: A new proof of the Cwikel–Lieb–Rosenbljum bound. *Rocky Mt. J. Math.* **15**, 117–122 (1985). <https://doi.org/10.1216/RMJ-1985-15-1-117>
9. Cwikel, M.: Weak type estimates for singular values and the number of bound states of Schrödinger operators. *Ann. Math. (2)* **106**, 93–100 (1977). <https://doi.org/10.2307/1971160>
10. Cwikel, M.: Private communication (2019)
11. Dappa, H., Trebels, W.: On maximal functions generated by Fourier multipliers. *Ark. Mat.* **23**, 241–259 (1985). <https://doi.org/10.1007/BF02384428>
12. Daubechies, I.: An uncertainty principle for fermions with generalized kinetic energy. *Commun. Math. Phys.* **90**, 511–520 (1983). <https://doi.org/10.1007/BF01216182>
13. Deift, P.A.: Applications of a commutation formula. *Duke Math. J.* **45**, 267–310 (1978). <https://doi.org/10.1215/S0012-7094-78-04516-7>

14. Diestel, J., Uhl, J.J.: Vector Measures. Mathematical Surveys No. 15. American Mathematical Society, Providence (1977)
15. Dolbeault, J., Laptev, A., Loss, M.: Lieb–Thirring inequalities with improved constants. *J. Eur. Math. Soc.* **10**, 1121–1126 (2008). <https://doi.org/10.4171/JEMS/142>
16. Frank, R.L.: Cwikel's theorem and the CLR inequality. *J. Spectr. Theory* **4**, 1–21 (2014). <https://doi.org/10.4171/JST/59>
17. Frank, R.L.: Eigenvalue bounds for the fractional Laplacian: a review. In: Palatucci, G., Kuusi, T. (eds.) *Recent Developments in Nonlocal Theory*, pp. 210–235. De Gruyter Open, Warsaw (2017). <https://doi.org/10.1515/9783110571561>
18. Frank, R.L., Hundertmark, D., Jex, M., Nam, P.T.: The Lieb–Thirring inequality revisited. *J. Eur. Math. Soc.* **23**(8), 2583–2600 (2021). <https://doi.org/10.4171/jems/1062>
19. Frank, R.L., Lieb, E.H., Seiringer, R.: Number of bound states of Schrödinger operators with matrix-valued potentials. *Lett. Math. Phys.* **82**, 107–116 (2007). <https://doi.org/10.1007/s11005-007-0211-x>
20. Hoang, V., Hundertmark, D., Richter, J., Vugalter, S.: Quantitative bounds versus existence of weakly coupled bound states for Schrödinger type operators. [arXiv:1610.09891](https://arxiv.org/abs/1610.09891)
21. Hundertmark, D.: On the number of bound states for Schrödinger operators with operator-valued potentials. *Ark. Mat.* **40**, 73–87 (2002). <https://doi.org/10.1007/BF02384503>
22. Hundertmark, D.: Some bound state problems in quantum mechanics. In: *Spectral Theory and Mathematical Physics: A Festschrift in Honor of Barry Simon's 60th Birthday*. Proceedings of Symposia in Pure Mathematics, vol. 76, Part 1, pp. 463–496. American Mathematical Society, Providence (2007). <https://doi.org/10.1090/pspum/076.1>
23. Hundertmark, D., Laptev, A., Weidl, T.: New bounds on the Lieb–Thirring constants. *Invent. Math.* **140**, 693–704 (2000). <https://doi.org/10.1007/s002220000077>
24. Landau, L.D., Lifshitz, E.M.: Quantum mechanics: non-relativistic theory. In: *Course of Theoretical Physics*, vol. 3. Translated from the Russian by J.B. Sykes and J.S. Bell. 3rd edn. Pergamon Press, London (1977). <https://doi.org/10.1016/C2013-0-02793-4>
25. Laptev, A.: Dirichlet and Neumann eigenvalue problems on domains in Euclidean spaces. *J. Funct. Anal.* **151**, 531–545 (1997). <https://doi.org/10.1006/jfan.1997.3155>
26. Laptev, A., Safronov, O., Weidl, T.: Bound state asymptotics for elliptic operators with strongly degenerated symbols. In: *Nonlinear Problems in Mathematical Physics and Related Topics I*, pp. 233–245. Kluwer, New York (2002). https://doi.org/10.1007/978-1-4615-0777-2_14
27. Laptev, A., Weidl, T.: Sharp Lieb–Thirring inequalities in high dimensions. *Acta Math.* **184**, 87–111 (2000). <https://doi.org/10.1007/BF02392782>
28. Levitina, G., Sukochev, F., Zanin, D.: Cwikel estimates revisited. *Proc. Lond. Math. Soc.* Third Ser. **120**, 265–304 (2020). <https://doi.org/10.1112/plms.12301>
29. Laptev, A., Weidl, T.: Recent results on Lieb–Thirring inequalities. *Journées “Équations aux dérivées partielles”* (2000) Exp. No. 20, 14 p. Université de Nantes, Nantes (2000). http://www.numdam.org/item?id=JEDP_2000_A20_0
30. Li, P., Yau, S.T.: On the Schrödinger equation and the eigenvalue problem. *Comm. Math. Phys.* **88**, 309–318 (1983). <https://doi.org/10.1007/BF01213210>
31. Lieb, E.H.: Bounds on the eigenvalues of the Laplace and Schroedinger operators. *Bull. Am. Math. Soc.* **82**, 751–753 (1976). <https://doi.org/10.1090/S0002-9904-1976-14149-3>
32. Lieb, E.H.: The number of bound states of one-body Schroedinger operators and the Weyl problem. In: *Geometry of the Laplace operator, Honolulu/Hawaii: Proceedings of Symposia in Pure Mathematics*, vol. 36 (1980), pp. 241–252 (1979)
33. Lieb, E.H., Thirring, W.E.: Bound for the kinetic energy of fermions which proves the stability of matter. *Phys. Rev. Lett.* **35**, 687–689 (1975). <https://doi.org/10.1103/PhysRevLett.35.687>. Erratum *Phys. Rev. Lett.* **35**, 1116. <https://doi.org/10.1103/PhysRevLett.35.1116>
34. Lieb, E.H., Thirring, W.E.: Inequalities for the moments of the eigenvalues of the Schrodinger Hamiltonian and their Relation to Sobolev inequalities. In: Lieb, E.H., et al.

- (eds.) *Studies in Mathematical Physics, Essays in Honor of Valentine Bargmann*, pp. 269–303. Princeton University Press, Princeton (1976)
35. Lions, J.L., Peetre, J.: Sur une classe d'espaces d'interpolation. *Publ. Math. l'IHÉS* **19**, 5–68 (1964)
 36. Muscalu, C., Schlag, W.: *Classical and Multilinear Harmonic Analysis, Volume 1*. Cambridge Studies in Advanced Mathematics, vol. 137. Cambridge University Press, Cambridge (2013). <https://doi.org/10.1017/CBO9781139047081>
 37. Netrusov, Y., Weidl, T.: On Lieb–Thirring inequalities for higher order operators with critical and subcritical powers. *Commun. Math. Phys.* **182**, 355–370 (1996). <https://doi.org/10.1007/BF02517894>
 38. NIST Digital Library of Mathematical Functions. <http://dlmf.nist.gov/>. Release 1.1.5 of 2022-03-15. F.W.J. Olver, A.B. Olde Daalhuis, D.W. Lozier, B.I. Schneider, R.F. Boisvert, C.W. Clark, B.R. Miller, B.V. Saunders, H.S. Cohl, M.A. McClain (eds.)
 39. Pankrashkin, K.: Variational principle for Hamiltonians with degenerate bottom. In: Beltita, I., Nenciu, G., Purice, R. (eds.) *Mathematical Results in Quantum Mechanics*, pp. 231–240. World Scientific Publishing, Hackensack (2008). https://doi.org/10.1142/9789812832382_0016
 40. Pettis, B.J.: On integration in vector spaces. *Trans. Am. Math. Soc.* **44**, 277–304 (1938). <https://doi.org/10.2307/1989973>
 41. Reed, M., Simon, B.: *Methods of Modern Mathematical Physics. I. Functional Analysis*, 2nd edn. Academic Press, New York (1980)
 42. Roepstorff, G.: *Path Integral Approach to Quantum Physics. An introduction*. Texts and Monographs in Physics. Springer, Berlin (1994). <https://doi.org/10.1007/978-3-642-57886-1>
 43. Rozenblum, G.V.: Distribution of the discrete spectrum of singular differential operators. *Dokl. Akad. Nauk SSSR* **202**, 1012–1015 (1972) (**Russian**). English translation: *Soviet Math. Dokl.* **13**(1972), 245–249
 44. Rozenblum, G.V.: Distribution of the discrete spectrum of singular differential operators. *Izv. Vysš. Učebn. Zaved. Mat.* **1**(164), 75–86 (1976) (**Russian**). English translation: *Sov. Math. (Iz. VUZ)* **20**(1), 63–71 (1976)
 45. Rubio de Francia, J.L.: Maximal functions and Fourier transforms. *Duke Math. J.* **53**, 395–404 (1986). <https://doi.org/10.1215/S0012-7094-86-05324-X>
 46. Rumin, M.: Spectral density and Sobolev inequalities for pure and mixed states. *Geom. Funct. Anal.* **20**, 817–844 (2010). <https://doi.org/10.1007/s00039-010-0075-6>
 47. Rumin, M.: Balanced distribution-energy inequalities and related entropy bounds. *Duke Math. J.* **160**, 567–597 (2011). <https://doi.org/10.1215/00127094-1444305>
 48. Simon, B.: The bound state of weakly coupled Schrödinger operators in one and two dimensions. *Ann. Phys.* **97**, 279–288 (1976). [https://doi.org/10.1016/0003-4916\(76\)90038-5](https://doi.org/10.1016/0003-4916(76)90038-5)
 49. Simon, B.: Analysis with weak trace ideals and the number of bound states of Schrödinger operators. *Trans. Am. Math. Soc.* **224**(2), 367–380 (1976). <https://doi.org/10.2307/1997482>
 50. Simon, B.: *Functional Integration and Quantum Physics*, 2nd edn. AMS Chelsea Publishing, Providence (2005)
 51. Simon, B.: *Trace Ideals and Their Applications*. Mathematical Surveys and Monographs, vol. 120, 2nd edn. American Mathematical Society, Providence (2005). <https://doi.org/10.1090/surv/120>
 52. Solomyak, M.: Piecewise-polynomial approximation of functions from $H^\ell((0, 1)^d)$, $2\ell = d$, and applications to the spectral theory of the Schrödinger operator. *Isr. J. Math.* **86**, 253–275 (1994). <https://doi.org/10.1007/BF02773681>
 53. Weidl, T.: Another look at Cwikel's inequality. In: *Differential Operators and Spectral Theory*. American Mathematical Society Translations: Series 2, vol. 189, pp. 247–254. American Mathematical Society, Providence (1999). <https://doi.org/10.1090/trans2/189>

54. Weidl, T.: Nonstandard Cwikel type estimates. In: Interpolation Theory and Applications. Contemporary Mathematics, vol. 445, pp. 337–357. American Mathematical Society, Providence (2007). <https://doi.org/10.1090/conm/445/08611>
55. Wojtaszczyk, P.: Banach Spaces for Analysts. Cambridge Studies in Advanced Mathematics, vol. 25. Cambridge University Press, Cambridge (1991). <https://doi.org/10.1017/CBO9780511608735>

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