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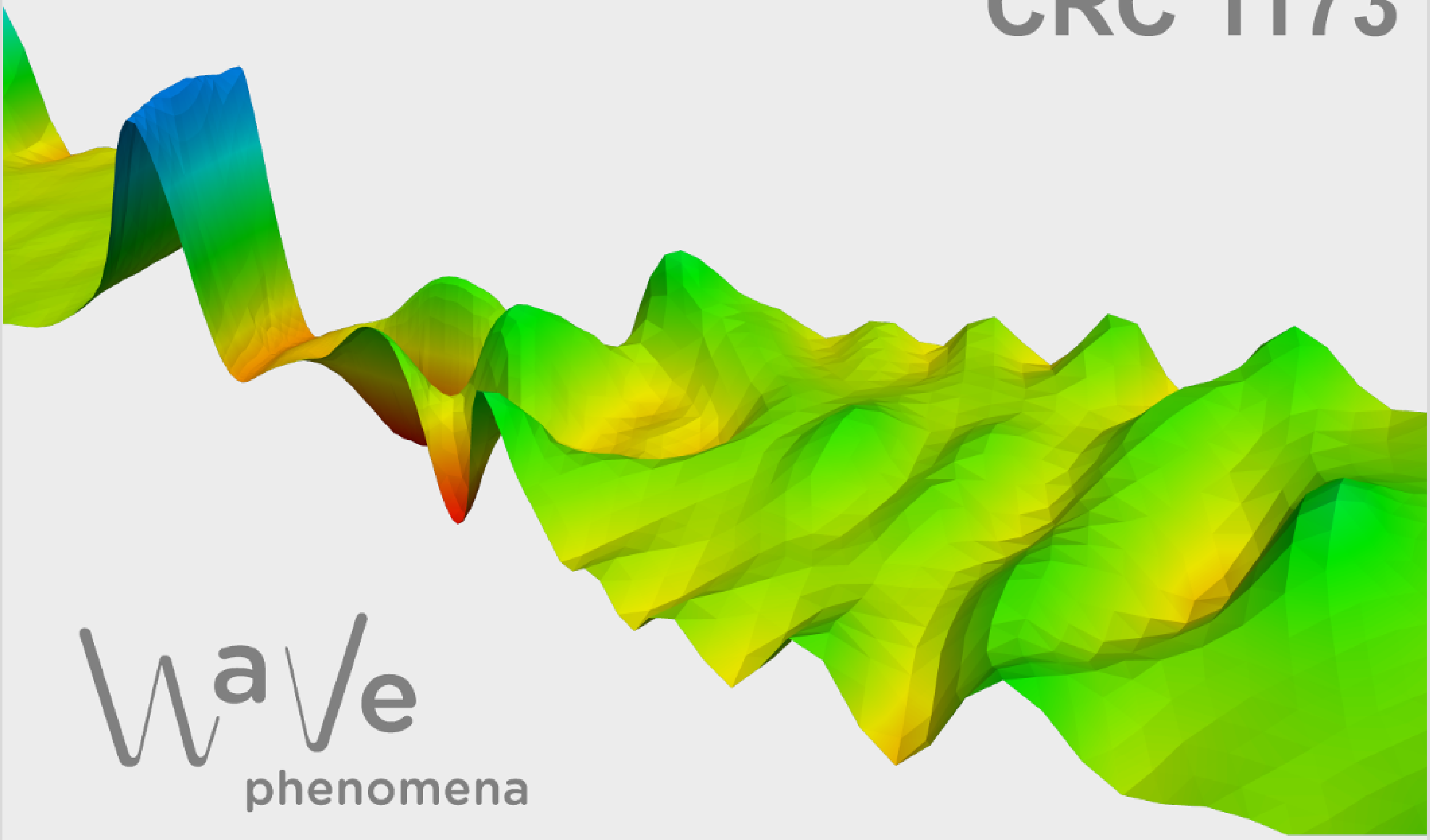

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SEMICLASSICAL ANALYSIS OF QUANTUM ASYMPTOTIC FIELDS IN THE YUKAWA THEORY

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ABSTRACT. In this article, we study the asymptotic fields of the Yukawa particle-field model of quantum physics, in the semiclassical regime $\hbar \to 0$, with an interaction subject to an ultraviolet cutoff. We show that the transition amplitudes between final (respectively initial) states converge towards explicit quantities involving the outgoing (respectively incoming) wave operators of the nonlinear Schrödinger–Klein–Gordon (S-KG) equation. Thus, we rigorously link the scattering theory of the Yukawa model to that of the Schrödinger–Klein–Gordon equation. Moreover, we prove that the asymptotic vacuum states of the Yukawa model have a phase space concentration property around classical radiationless solutions. Under further assumptions, we show that the S-KG energy admits a unique minimizer modulo symmetries and identify exactly the semiclassical measure of Yukawa ground states. Some additional consequences of asymptotic completeness are also discussed, and some further open questions are raised.

1. INTRODUCTION

The fundamental interaction between matter and the electromagnetic field forms the underlying theoretical basis of various branches of physics and engineering (quantum optics, atomic spectroscopy, magnetized plasma, spacecraft semi-conductors, ...). Such interaction is modeled by a variety of reduced mathematical models. For instance, in classical mechanics the light-matter interaction is simply described via Newton's law and the Lorentz force, while in quantum mechanics it is described by quantum electrodynamics (QED). An alternative approach for studying light-matter interactions, depending on the scale of the system, is provided by semiclassical electrodynamics, where quantum matter interacts with classical electromagnetic fields, according for example to the Maxwell–Schrödinger or Maxwell–Bloch equations. Similarly in high energy physics, the quantum and semiclassical descriptions provide two different models of the strong nuclear force via the Yukawa theory, depending on whether the meson field is treated as a quantum or classical object. Although there is consensus about the fact that the quantum and the semiclassical descriptions of matter-field interactions are intimately related by Bohr's correspondence principle, there are few rigorous derivations of such relationship [see, *e.g.*, AF14, AF17, LP18, LP20, CCFO21, CFO19, CFO20, CFO21, and references therein contained]. From both a conceptual and a practical standpoint, it is interesting to question the consistency of these different models of particle-field interactions and to introduce an asymptotic analysis for *transition* and *scattering amplitudes* in terms of

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classical quantities. Recall that the scattering amplitudes correspond to the main measurable quantities in experimental physics, and their study is quite challenging particularly for realistic models.

According to the standard formalism of quantum field theory, the time-asymptotics of quantum dynamics is encoded in the asymptotic fields and in the S-matrix [SS62]. In particular, the transition amplitudes for various scattering processes are described by the following expectations values

$$\langle f_\hbar^\alpha, f_\hbar^\beta \rangle \quad \text{and} \quad \langle i_\hbar^\alpha, i_\hbar^\beta \rangle \,, \tag{1}$$

where $|f_\hbar^\alpha\rangle$ are final (out) states and $|i_\hbar^\beta\rangle$ are initial (in) states defined through the asymptotic fields as,

$$|f_\hbar^\alpha\rangle = \prod_j a_\hbar^{+,*}(\alpha_j)|\Omega_\hbar\rangle, \qquad \text{and} \qquad |i_\hbar^\beta\rangle = \prod_k a_\hbar^{-,*}(\beta_k)|\Omega_\hbar\rangle \,, \tag{2}$$

where $|\Omega_\hbar\rangle$ is a prepared vector state, α_j, β_k are smooth functions and $a_\hbar^{\pm,*}, a_\hbar^{\pm}$ are the asymptotic creation-annihilation operators respectively. On the other hand, the scattering amplitudes are defined generally as the quantities

$$\langle f_\hbar^\alpha, i_\hbar^\beta \rangle \,. \tag{3}$$

Specifically, if one takes $|\Omega_\hbar\rangle$ to be an asymptotic vacuum (ground) state of the interacting system, the transition amplitudes (1) can be computed exactly thanks to the canonical commutation relations

$$a_\hbar^{\pm}(\alpha_j)a_\hbar^{\pm,*}(\beta_k) - a_\hbar^{\pm,*}(\beta_k)a_\hbar^{\pm}(\alpha_j) = \hbar\langle \alpha_j, \beta_k\rangle \mathbb{1} \,,$$

and Wick's theorem. However, for interacting quantum field theories it is not possible in general to give exact closed formulas for these transition and scattering amplitudes. Therefore, one resorts to perturbation theory and semiclassical analysis in order to provide expansions of these amplitudes respectively in terms of the coupling constant or the semiclassical parameter. One of the remarkable results of QFT is the *Lehmann-Symanzik-Zimmermann* (LSZ) reduction formula which relates the scattering amplitudes (3) to time-ordered correlation functions, thus formally allowing to derive a perturbative expansion by means of Feynman diagrams. An important question is then to prove rigorously an expansion formula for the transition and scattering amplitudes (1)-(3) with respect to the semiclassical parameter $\hbar$. It is worth noting that in principle the semiclassical limit $\hbar \to 0$ in (3) yields the tree diagrams of the S-matrix elements while the loop diagrams provide higher order corrections in terms of $\hbar$.

In the present article, we focus on the rigorous computation of the first order of the transition amplitudes (1) given by the limit $\hbar \to 0$. In this topic various quantum field theories can be considered. For relevance and convenience, we focus on the nonlocal Yukawa particle-field model. In particular, the spectral and scattering theory for such system is thoroughly investigated [see, *e.g.*, DG99, Amm00, FGS01].

The Yukawa theory describes the nuclear force as the interaction of a Dirac (fermion) field ψ with a Klein-Gordon (boson) field ϕ to be given by the expression

$$g \int_{\mathbb{R}^d} \overline{\psi}(x)\phi(x)\psi(x)dx \,, \tag{4}$$

where g is the corresponding coupling constant. Considering non-relativistic nucleons and fixing their number in the above theory, one obtains the so-called Nelson model, that was studied in the land-marking article [Nel64] by Edward Nelson. The Hamiltonian that we shall consider here is a reduction of the original Yukawa theory, in the sense that we impose a nonlocal interaction through an ultraviolet cutoff and a boson-boson coupling instead of fermion-boson one. Under such simplification, the first two authors proved that the semiclassical limit $\hbar \to 0$ yields the Schrödinger-Klein-Gordon equation, see [Fal13, AF14] and also [AF17] where the ultraviolet cutoff is removed. The above mentioned works concern either stationary solutions or finite time dynamics, and do not address the problem of large times and scattering theory. In fact, as far as we know, there are no rigorous results on the $\hbar$-asymptotics of the scattering and transition amplitudes for the Nelson and Yukawa models.

Let us briefly overview our main contribution. The Hamiltonian describing the reduced Yukawa theory considered here is given by

$$H_\hbar = H_\hbar^0 + \int_{\mathbb{R}^d} \psi_\hbar^*(x)(a_\hbar^*(\lambda_x) + a_\hbar(\lambda_x))\, \psi_\hbar(x)\, \mathrm{d}x\,,$$

with the non-interacting Hamiltonian defined as

$$H_\hbar^0 = \int_{\mathbb{R}^d} \psi_\hbar^*(x)(-\Delta_x + V(x))\, \psi_\hbar(x)\, \mathrm{d}x + \int_{\mathbb{R}^d} a_\hbar^*(k)\omega(k)\, a_\hbar(k)\, \mathrm{d}k\,.$$

Here $\psi_\hbar^\sharp$ and $a_\hbar^\sharp$ denote respectively the particle and meson creation-annihilation operators while

$$\omega(k) = \sqrt{k^2 + m^2}\,,\ m > 0\,,$$

is the meson dispersion relation and λ_x is a form factor (see Section 2 for more details). Then the asymptotic fields are defined as follows,

$$a_\hbar^{\pm,\sharp}(\eta) = \lim_{t\to\pm\infty} e^{-i\frac{t}{\hbar}H_\hbar}\, e^{i\frac{t}{\hbar}H_\hbar^0}\, a_\hbar^\sharp(\eta)\, e^{-i\frac{t}{\hbar}H_\hbar^0}\, e^{i\frac{t}{\hbar}H_\hbar}\,, \tag{5}$$

and they are known to exist for a dense subset of functions $\eta \in L^2(\mathbb{R}^d)$, see [HK68, HK69a, HS95, DG99]. Our first main result gives the semiclassical limit of the transition amplitudes under some natural assumptions,

$$\lim_{\hbar\to 0} \langle f_\hbar^\alpha, f_\hbar^\beta\rangle = \int_{L^2(\mathbb{R}^d)\oplus L^2(\mathbb{R}^d)} \prod_j \langle \alpha_j, \Lambda^+(u,z)\rangle_{L^2(\mathbb{R}^d)} \prod_k \langle \Lambda^+(u,z), \beta_k\rangle_{L^2(\mathbb{R}^d)}\, d\mu_0(u,z)\,, \tag{6}$$

and

$$\lim_{\hbar\to 0} \langle i_\hbar^\alpha, i_\hbar^\beta\rangle\,, = \int_{L^2(\mathbb{R}^d)\oplus L^2(\mathbb{R}^d)} \prod_j \langle \alpha_j, \Lambda^-(u,z)\rangle_{L^2(\mathbb{R}^d)} \prod_k \langle \Lambda^-(u,z), \beta_k\rangle_{L^2(\mathbb{R}^d)}\, d\mu_0(u,z)\,, \tag{7}$$

where $\Lambda^\pm$ are the in-out-going wave operators of the nonlinear Schrödinger-Klein-Gordon equation (45), constructed in Section 4 (Definition 4.1) and μ_0 is a semiclassical or Wigner measure of the family of states $\{\Omega_\hbar\}_{\hbar\in(0,1)}$, which is a Borel probability measure over the phase-space $L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$. Such result is stated throughout Theorem 5.2, Proposition 5.4, and Corollaries 5.5 and 5.7. We further show that if $\{\Omega_\hbar\}_{\hbar\in(0,1)}$ are the ground states of the interacting system, then the semiclassical measure μ_0 concentrates on the lowest classical energy level of the Schrödinger-Klein-Gordon system. More generally, if we replace $\{\Omega_\hbar\}_{\hbar\in(0,1)}$ by excited or bound sates of the quantum Yukawa system, we then show that μ_0 concentrates on a set of classical asymptotic radiationless states, from which no radiation is coming out or in (see Theorem 6.1). The later notion is similar to that of trapped trajectories in finite dimensional semiclassical analysis, and here it is defined by means of the wave operators $\Lambda^{\pm}$, through the condition

$$\Lambda^{\pm}(u_0, z_0) = 0 .$$

Our final result shows that the energy functional of the Schrödinger-Klein-Gordon system (41) admits a unique minimizer (u_δ, z_δ), up to invariance, under the constraint $\|u\|_{L^2(\mathbb{R}^d)} = \delta$ for δ sufficiently small. This leads to the identification of the semiclassical measure μ_0 in Corollary 6.10 when $\{\Omega_\hbar\}_{\hbar\in(0,1)}$ are the ground states of $H_\hbar$ restricted to the space of finite number of nucleons $n_\hbar$ such that $\hbar n_\hbar \to \delta$.

The key point in the proofs of the results above is to combine dispersive and uniform energy estimates with semiclassical analysis and scattering theory.

Outlook of the article: The reduced particle-field Yukawa model is introduced in Section 2. A technical part of the article lays in Section 3, where uniform energy and dispersive estimates are given for both the classical S-KG equation and the quantum Yukawa model. Then scattering theory of the Schrödinger-Klein-Gordon system is discussed in Section 4, where we construct the classical out-in-coming wave operators. Our main results on transition amplitudes are proved in Section 5. Precise semiclassical concentration properties of asymptotic vacuum and bound states are given in Section 6. In the same section, uniqueness of minimizers for the S-KG energy functional and consequences of the asymptotic completeness for the quantum Yukawa model are addressed. A list of perspectives and open problems is provided in Section 7.

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2. Yukawa model, semiclassical analysis and asymptotic meson fields

2.1. **Yukawa's reduced Hamiltonian.** The reduced Yukawa model is composed by two subsystems mutually interacting: the first one being an arbitrary number of boson particles, considered as a non-relativistic field obeying Bose-Einstein statistics, while the second one is a scalar meson field. The Hilbert space of the fully interacting dynamical system is

$$\mathscr{H} := \Gamma_s(\mathfrak{H}) \otimes \Gamma_s(\mathfrak{H}) \cong \Gamma_s(\mathscr{Z}) ,$$

where $\Gamma_s(\cdot)$ denotes the symmetric Fock space, $\mathfrak{H} = L^2(\mathbb{R}^d)$, and $\mathscr{Z} := \mathfrak{H} \oplus \mathfrak{H}$. The latter shall be interpreted as the classical phase-space for the particle-field system. For convenience, we denote the sector with n non-relativistic particles by

$$\mathscr{H}^{(n)} := L^2_s(\mathbb{R}^{dn}) \otimes \Gamma_s(\mathfrak{H}) ,$$

and the sector with n particles and m mesons by

$$\mathscr{H}^{(n,m)} = L_s^2(\mathbb{R}^{dn}) \otimes L_s^2(\mathbb{R}^{dm}) \,.$$

It follows that

$$\mathscr{H} = \bigoplus_{n\in\mathbb{N}} \mathscr{H}^{(n)} = \bigoplus_{n,m\in\mathbb{N}} \mathscr{H}^{(n,m)}.$$

Consider now $(\psi_\hbar^*, \psi_\hbar)$ and $(a_\hbar^*, a_\hbar)$ to be the creation and annihilation operators for the particle and meson fields, respectively. These are $\hbar$-dependent couples of quantum observables, both satisfying the canonical commutation relations:

$$[\psi_\hbar(x), \psi_\hbar^*(y)] = \hbar\,\delta(x-y), \qquad [\,a_\hbar(k)\,, a_\hbar^*(k')] = \hbar\,\delta(k-k'), \qquad [a_\hbar^\natural(k), \psi_\hbar^\natural(x)] = 0 \,.$$

The creation and annihilation operators above are in fact considered “operator-valued distributions”, in the sense that for any $\eta \in L^2(\mathbb{R}^d)$,

$$\begin{aligned} \psi_\hbar(\eta) &= \int_{\mathbb{R}^d} \bar\eta(x)\psi_\hbar(x)\mathrm{d}x \,, & \psi_\hbar^*(\eta) &= \int_{\mathbb{R}^d} \eta(x)\psi_\hbar^*(x)\mathrm{d}x \,; \\ a_\hbar(\eta) &= \int_{\mathbb{R}^d} \bar\eta(k)a_\hbar(k)\mathrm{d}k \,, & a_\hbar^*(\eta) &= \int_{\mathbb{R}^d} \eta(k)a_\hbar^*(k)\mathrm{d}k \,; \end{aligned}$$

are closed operators on $\mathscr{H}$, respectively one adjoint to the other. Recall as well that $\psi_\hbar^\natural$ and $a_\hbar^\natural$ can be written in terms of the standard $\hbar$-independent creation and annihilation operators [see, *e.g.*, DG13, for an introductory reference]:

$$\psi_\hbar^\natural(\cdot) = \sqrt{\hbar}\,\psi^\natural(\cdot)\,; a_\hbar^\natural(\cdot) = \sqrt{\hbar}\,a^\natural(\cdot)\,.$$

Given any self-adjoint operator A on $\mathfrak{H}$, we denote its second quantizations by

$$\mathrm{d}\Gamma_\hbar^{(1)}(A) := \int_{\mathbb{R}^d} \mathrm{d}x\,\psi_\hbar^*(x)A\,\psi_\hbar(x) \quad , \quad \mathrm{d}\Gamma_\hbar^{(2)}(A) := \int_{\mathbb{R}^d} \mathrm{d}k\,a_\hbar^*(k)A\,a_\hbar(k)\,.$$

In particular, the rescaled number operators are respectively

$$N_1 := \mathrm{d}\Gamma_\hbar^{(1)}(1) \otimes \mathbb{1}\,, \quad N_2 := \mathbb{1} \otimes \mathrm{d}\Gamma_\hbar^{(2)}(1)\,,$$

and the total rescaled number operator is

$$N = N_1 + N_2\,.$$

Remark that N_1 and N_2 count $\hbar$-times the number of particles in each respective sector.

With the above notations, the particle-field Yukawa Hamiltonian is given by

$$H_\hbar := H_\hbar^0 + H_\hbar^I\,, \tag{8}$$

such that

$$\begin{aligned} H_\hbar^0 &:= \mathrm{d}\Gamma_\hbar^{(1)}(-\Delta + V) \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma_\hbar^{(2)}(\omega)\,, \\ H_\hbar^I &:= \mathrm{d}\Gamma_\hbar^{(1)}(\phi_\hbar(\lambda_x))\,, \end{aligned}$$

and

$$\phi_\hbar(\lambda_x) := a_\hbar(\lambda_x) + a_\hbar^*(\lambda_x)\,. \tag{9}$$

Remark 2.1. The Yukawa Hamiltonian $H_\hbar$ strongly commutes with the nucleon number operator N_1 [see, *e.g.*, AN15, Appendix A], therefore it can be fibered to each $\mathscr{H}^{(n)}$:

$$H_\hbar = \bigoplus_{n\in\mathbb{N}} H_\hbar^{(n)} = \bigoplus_{n\in\mathbb{N}} \Big(H_\hbar^{0,(n)} + H_\hbar^{I,(n)} \Big)\,.$$

The above potential V, dispersion relation ω, and form factor $\lambda_x(\cdot)$ need to satisfy the following regularity assumptions.

Assumption 1.

- $V : \mathbb{R}^d \to \mathbb{R}^+$ is locally integrable. In addition, there exist $C_0 > 0$ and $\nu > 0$ such that:
$$V(x) \geq C_0 \langle x \rangle^{1+\nu} . \tag{A1}$$
Here $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$ stands for the so-called Japanese bracket.

- ω is the dispersion of a scalar relativistic field of mass m:
$$\omega(k) = \sqrt{k^2 + m^2}, \quad m > 0 . \tag{A2}$$

- There exists a cut-off function $\chi \in C_0^\infty(\mathbb{R}^d)$ such that
$$\lambda_x(k) = e^{ik\cdot x}\omega^{-1/2}(k)\chi(k) . \tag{A3}$$

From now on, we will always assume implicitly (A1)-(A3) to be satisfied.

2.2. **Semiclassical limit and Wigner measures.** The classical limit procedure, known also as Bohr's correspondence principle, can be interpreted as the approximation of the behavior of a quantum system by its classical counterpart in a certain effective regime. There are several ways to give a mathematical foundation to this principle. Among them, one can mention the methods based on coherent states, path integration, Schwinger-Dyson expansion and Wigner measures. The latter is the one that we shall follow here. In fact, the Wigner measure approach initiated in [AN08] for quantum field and many-body theories proved to be quite effective [see, *e.g.*, ABN19, AF14, AF17, AN15, and references therein contained]. More precisely, consider a family of normalized vectors $\{\Psi_\hbar\}_{\hbar\in(0,1)} \subseteq \mathscr{H}$. The main point consists in studying the cluster points of the expectations
$$\langle \Psi_\hbar, W_\hbar(\eta)\Psi_\hbar\rangle_{\mathscr{H}} \ ,$$
where $\eta = \eta_1 \oplus \eta_2 \in \mathfrak{X}$ and $W_\hbar(\eta)$ is the *Weyl operator*. Recall that the Weyl operator is a unitary operator on $\mathscr{H}$, defined by
$$W_\hbar(\eta) = W_\hbar^{(1)}(\eta_1) \otimes W_\hbar^{(2)}(\eta_2) \equiv e^{\frac{i}{\sqrt{2}}\left(\psi_\hbar^*(\eta_1)+\psi_\hbar(\eta_1)\right)+\frac{i}{\sqrt{2}}\left(a_\hbar^*(\eta_2)+a_\hbar(\eta_2)\right)} . \tag{10}$$
As proved in [AN08, Theorem 6.2], if there exists $C > 0$ such that
$$\forall \hbar \in (0,1), \quad \langle \Psi_\hbar, N\Psi_\hbar\rangle_{\mathscr{H}} \leq C , \tag{11}$$
then there is at least one sequence $(\hbar_n)_{n\in\mathbb{N}}$, $\hbar_n \to 0$, and a Borel probability measure μ on $\mathfrak{X}$ such that for all $\eta \in \mathfrak{X}$,
$$\lim_{n\to\infty} \langle \Psi_{\hbar_n}, W_{\hbar_n}(\eta)\Psi_{\hbar_n}\rangle_{\mathscr{H}} = \int_{\mathfrak{X}} e^{\sqrt{2}i\mathrm{Re}\langle \eta_1\oplus\eta_2, u\oplus z\rangle_{\mathfrak{X}}}\, d\mu(u,z) . \tag{12}$$
The expectation on the left-hand side is the Fourier-Wigner transform of a (bosonic) quantum state while the average on the right-hand side is the (inverse) Fourier transform of a measure μ. Thus the convergence given in (12) is interpreted as a semiclassical limit emphasizing the

relationship between quantum field generating functionals and classical characteristic functions. The formal infinitesimal version of the limit (12) yields the the convergence of quantum correlation functions towards the classical ones. Symbolically, we denote this convergence by $\Psi_{\hbar_n} \rightharpoonup \mu$. The probability measures μ such that $\Psi_{\hbar_k} \rightharpoonup \mu$ for some sequence $\hbar_k \to 0$, are called Wigner or semiclassical measures of the family $\{\Psi_\hbar\}_{\hbar\in(0,1)}$. The set of all Wigner measures associated with the family $\{\Psi_\hbar\}_{\hbar\in(0,1)}$ is denoted by

$$\mathscr{M}\big(\Psi_\hbar; \hbar \in (0,1)\big) .$$

From a dynamical standpoint, it is quite relevant to understand whether a classical dynamics in the phase space $\mathscr{X}$ is a good approximation in the semiclassical regime to the microscopic dynamics of quantum fields. This can be rigorously done giving a characterization of the Wigner measures associated to the time-evolved vectors $\Psi_\hbar(t) = e^{-i\frac{t}{\hbar}H_\hbar}\Psi_\hbar$. Indeed, it is proved in [AF14, Theorem 1.1] for the Yukawa model that if $\mathscr{M}\big(\Psi_{\hbar_n}; n \in \mathbb{N}\big) = \{\mu\}$ at time zero, then for all later or previous times

$$\mathscr{M}\big(\Psi_{\hbar_n}(t); n \in \mathbb{N}\big) = \{(\Phi_t)_*\,\mu\} ,$$

where Φ_t is the nonlinear Hamiltonian flow of the Schrödinger-Klein-Gordon system given in (45) and $(\Phi_t)_*\,\mu$ is the push-forward measure defined for any Borel set B in $\mathscr{X}$ by

$$(\Phi_t)_*\,\mu(B) = \mu\big((\Phi_t)^{-1}(B)\big) .$$

We refer the reader to Section 4 for more details on the S-KG equation, and to our previous papers [AN08, AF14] on Wigner measures and semiclassical limits.

2.3. **Asymptotic meson fields.** Knowing the behavior of the quantum system in the semiclassical limit at fixed times $t \in \mathbb{R}$, it is natural to ask whether the approximation by the classical dynamics holds true for asymptotically long times. In order to deal with such a question, it is necessary to introduce the concept of *asymptotic fields* [see HK68, HK69a, HK69b, DG99].

Definition 2.2. *The asymptotic Weyl operators for the meson field, denoted by $W_\hbar^\pm(\eta)$, are defined for any $\eta \in \mathfrak{H}$ by*

$$\begin{aligned} W_\hbar^\pm(\eta) &:= \operatorname*{s-lim}_{t\to\pm\infty} e^{i\frac{t}{\hbar}H_\hbar} e^{-i\frac{t}{\hbar}H_\hbar^0}\, W_\hbar^{(2)}(\eta) e^{i\frac{t}{\hbar}H_\hbar^0}\, e^{-i\frac{t}{\hbar}H_\hbar} \\ &= \operatorname*{s-lim}_{t\to\pm\infty} e^{i\frac{t}{\hbar}H_\hbar} W_\hbar^{(2)}(\eta_t) e^{-i\frac{t}{\hbar}H_\hbar} , \end{aligned} \tag{13}$$

and exist as $\mathscr{H}$-strong limits. In the above equation, $\eta_t = e^{-it\omega}\eta$.

The existence of the asymptotic Weyl operators for the relativistic meson field in the Yukawa model was proved firstly in [HK68], see also [DG99] for additional details. The asymptotic Weyl operators are the fundamental objects in the study of the scattering properties of the Yukawa model, playing a role analogous to wave operators in the scattering theory of Schrödinger operators. They satisfy indeed the Weyl relations

$$W_\hbar^\pm(\eta) W_\hbar^\pm(\eta') = e^{\frac{i}{2}\mathrm{Im}\langle\eta,\eta'\rangle_{\mathfrak{H}}}\, W_\hbar^\pm(\eta+\eta') .$$

The collection $\{W_\hbar^\pm(\eta), \eta \in \mathfrak{H}\}$ generate thus, in the algebraic sense, a representation of the time-zero canonical commutation relations for the relativistic free field. It is also possible to define the associated asymptotic fields $\phi_\hbar^\pm(\eta)$, and creation-annihilation operators $a_\hbar^{\pm,\natural}(\eta)$.

In order to do that, observe that according to Remark 2.1, the asymptotic Weyl operators $W_\hbar^\pm(\eta)$ strongly commute with N_1, hence they can be fibered on each $\mathscr{H}^{(n)}$,

$$W_\hbar^\pm(\eta) = \bigoplus_{n\in\mathbb{N}} W_\hbar^\pm(\eta)^{(n)} \, . \tag{14}$$

Moreover, notice that the maps

$$\mathfrak{H} \ni \eta \mapsto W_\hbar^\pm(\eta)^{(n)} = W_\hbar^\pm(\eta)_{|\mathscr{H}^{(n)}} \in \mathscr{B}(\mathscr{H}^{(n)})$$

are continuous with respect to the strong operator topology, and the maps

$$\mathfrak{H} \ni \eta \mapsto W_\hbar^\pm(\eta)^{(n)}(H_\hbar^{(n)} + i)^{-\varepsilon} \in \mathscr{B}(\mathscr{H}^{(n)})$$

are continuous, for any $\varepsilon > 0$, with respect to the uniform operator topology [DG99]. Hence, by Stone's theorem, it is possible to define, for any $\eta \in \mathfrak{H}$, $\phi_\hbar^\pm(\eta)^{(n)}$ as the generators of the asymptotic Weyl operators

$$W_\hbar^\pm(\eta)^{(n)} = e^{\frac{i}{\sqrt{2}}\phi_\hbar^\pm(\eta)^{(n)}} \, .$$

Additionally, one concludes for instance using [AN15, Proposition A1] that the operators

$$\phi_\hbar^\pm(\eta) = \bigoplus_{n\in\mathbb{N}} \phi_\hbar^\pm(\eta)^{(n)} \, . \tag{15}$$

are self-adjoint and satisfy for all $\eta \in \mathfrak{H}$,

$$W_\hbar^\pm(\eta) = e^{\frac{i}{\sqrt{2}}\phi_\hbar^\pm(\eta)} \, .$$

Finally, note that the operators $\{\phi_\hbar^\pm(\eta)^{(n)}, \eta \in \mathfrak{H}\}$ have a common dense domain, since the following inequality holds for any $\eta \in \mathfrak{H}$ [DG99, Theorem 5.2]:

$$\left\| \phi_\hbar^\pm(\eta)^{(n)}(H_\hbar^{(n)} + i)^{-\frac{1}{2}} \right\|_{\mathscr{B}(\mathscr{H}^{(n)})} \leq C\|\eta\|_{\mathfrak{H}} \, . \tag{16}$$

Therefore the form domain $\mathbb{Q}(H_\hbar^{(n)}) \subset \mathscr{D}(\phi_\hbar^\pm(\eta)^{(n)})$ for all $\eta \in \mathfrak{H}$. The creation and annihilation operators

$$a_\hbar^\pm(\eta)^{(n)} = \frac{1}{2}\Big(\phi_\hbar^\pm(\eta)^{(n)} + i\phi_\hbar^\pm(i\eta)^{(n)}\Big) \, , \tag{17}$$

$$a_\hbar^{\pm,*}(\eta)^{(n)} = \frac{1}{2}\Big(\phi_\hbar^\pm(\eta)^{(n)} - i\phi_\hbar^\pm(i\eta)^{(n)}\Big) \, , \tag{18}$$

are then densely defined and closed on $\mathscr{D}(\phi_\hbar^\pm(\eta)^{(n)}) \cap \mathscr{D}(\phi_\hbar^\pm(i\eta)^{(n)}) \supset \mathbb{Q}(H_\hbar^{(n)})$ for all $\eta \in \mathfrak{H}$ [DG99, Theorem 5.3]. Henceforth, one defines

$$a_\hbar^{\pm,\natural}(\eta) = \bigoplus_{n\in\mathbb{N}} a_\hbar^{\pm,\natural}(\eta)^{(n)} \, , \tag{19}$$

as closed operators on the fibred Hilbert space $\mathscr{H}$.

We conclude this section by introducing the notion of *asymptotic completeness* for the Yukawa model. The n-nucleon asymptotic vacuum spaces are defined respectively as

$$\mathscr{K}_\hbar^{\pm,(n)} = \Big\{\psi^{(n)} \in \mathscr{H}^{(n)} \mid \forall\eta \in \mathfrak{H} \, , \, a_\hbar^\pm(\eta)^{(n)}\psi^{(n)} = 0\Big\} \, . \tag{20}$$

Physically, the spaces $\mathscr{K}_\hbar^{\pm,(n)}$ can be thought as the configurations in which only the n asymptotic ("dressed") nucleons are present, for there is no asymptotically free meson in this case. Asymptotic completeness is formulated as a property of asymptotic vacua. Indeed, the

Yukawa theory is said to be asymptotically complete iff (each) $\mathscr{K}_\hbar^{\pm,(n)}$ equals the spectral subspace of bound states of $H_\hbar^{(n)}$, *i.e.*

$$\mathscr{K}_\hbar^{\pm,(n)} = \mathbb{1}_{\mathrm{pp}}(H_\hbar^{(n)})\mathscr{H}^{(n)} , \tag{21}$$

where $\mathbb{1}_{\mathrm{pp}}(H_\hbar^{(n)})$ denotes the spectral projection over the pure point spectrum $\sigma_{\mathrm{pp}}(H_\hbar^{(n)})$. This guarantees in particular that for each $n \in \mathbb{N}$, $\mathscr{K}_\hbar^{+,(n)} = \mathscr{K}_\hbar^{-,(n)}$ (future and past asymptotic vacua coincide), and that the asymptotic vacua are bound states of the Yukawa Hamiltonian.

3. Technical estimates

In this section we collect some important technical estimates that are used to prove our main results.

3.1. **Energy estimates.**

We recall here some uniform estimates for the Hamiltonian $H_\hbar^{(n)}$. For each fixed $\hbar$, such estimates are well-known; however, for our analysis of the limit $\hbar \to 0$, uniformity with respect to $\hbar$ has to be kept into account.

Lemma 3.1. *There exists $C > 0$ such that for all $\hbar \in (0, 1)$,*

$$\|(\mathrm{d}\Gamma_\hbar^{(2)}(\omega) + 1)^{-1/2} a_\hbar^\natural(\lambda_x)\|_{\mathscr{B}(\mathscr{H}^{(1)})} \leq C.$$

Proof. Let us start with the annihilation operator, and let $\Psi \in \mathscr{H}^{(1,n)}$. Then,

$$\begin{aligned}
\|a_\hbar(\lambda_x)\Psi\|^2_{\mathscr{H}^{(1,n-1)}} &= \int_{\mathbb{R}^d\times\mathbb{R}^{d(n-1)}} \mathrm{d}x\,\mathrm{d}k_2\ldots\mathrm{d}k_n\,\hbar n \left|\int_{\mathbb{R}^d} \mathrm{d}k_1\,\overline{\lambda_x(k_1)}\,\Psi(x, k_1, \ldots, k_n)\right|^2 \\
&= \int_{\mathbb{R}^d\times\mathbb{R}^{d(n-1)}} \mathrm{d}x\,\mathrm{d}k_2\ldots\mathrm{d}k_n\,\hbar n \left|\int_{\mathbb{R}^d} \mathrm{d}k_1\,\omega(k_1)^{-\frac{1}{2}}\overline{\lambda_x(k_1)}\,\omega(k_1)^{\frac{1}{2}}\Psi(x, k_1, \ldots, k_n)\right|^2 \\
&\leq \int_{\mathbb{R}^d\times\mathbb{R}^{d(n-1)}} \mathrm{d}x\,\mathrm{d}k_1\ldots\mathrm{d}k_n\,\big\|\omega^{-\frac{1}{2}}\lambda_x\big\|^2_{\mathfrak{H}}\hbar n\,\omega(k_1)\,\big|\Psi(x, k_1, \ldots, k_n)\big|^2 \\
&\leq \|\omega^{-\frac{1}{2}}\lambda_{(\cdot)}\|^2_{L^\infty(\mathbb{R}^d;\mathfrak{H})}\|\mathrm{d}\Gamma_\hbar^{(2)}(\omega)^{\frac{1}{2}}\Psi\|^2_{\mathscr{H}^{(1,n)}} \leq \frac{1}{m}\|\lambda_{(\cdot)}\|^2_{L^\infty(\mathbb{R}^d;\mathfrak{H})}\|\mathrm{d}\Gamma_\hbar^{(2)}(\omega)^{\frac{1}{2}}\Psi\|^2_{\mathscr{H}^{(1,n)}} .
\end{aligned}$$

The proof for the creation operator makes use of the canonical commutation relations, of the result for the annihilation operator, and of the fact that $\hbar \in (0, 1)$:

$$\begin{aligned}
\|a_\hbar^*(\lambda_x)\Psi\|^2_{\mathscr{H}^{(1)}} &= \|a_\hbar(\lambda_x)\Psi\|^2_{\mathscr{H}^{(1)}} + \langle\Psi, \hbar\|\lambda_x\|^2_{\mathfrak{H}}\Psi\rangle_{\mathscr{H}^{(1)}} \\
&\leq (1 + \tfrac{1}{m})\|\lambda_{(\cdot)}\|^2_{L^\infty(\mathbb{R}^d;\mathfrak{H})}\big\langle\Psi, \big(\mathrm{d}\Gamma_\hbar^{(2)}(\omega) + 1\big)\Psi\big\rangle_{\mathscr{H}^{(1)}} .
\end{aligned}$$

⊣

Lemma 3.2. *Let $0 < \alpha < \beta$ given. Then there exist $b, C > 0$ such that for any $n \in \mathbb{N}$ and $\hbar \in (0, 1)$ satisfying $n\hbar \in (\alpha, \beta)$, one has:*

$$\|H_\hbar^{(n)}\,(H_\hbar^{0,(n)} + 1)^{-1}\|_{\mathscr{B}(\mathscr{H})} \leq C , \tag{22}$$

$$\|H_\hbar^{0,(n)}\,(H_\hbar^{(n)} + b)^{-1}\|_{\mathscr{B}(\mathscr{H})} \leq C . \tag{23}$$

Proof. Thanks to the estimate of Lemma 3.1, there exists $C > 0$ such that

$$\Big\|\hbar\sum_{j=1}^n \phi_\hbar(\lambda_{x_j})\,(H_\hbar^{0,(n)} + 1)^{-1/2}\Big\|_{\mathscr{B}(\mathscr{H})} \leq C, \tag{24}$$

uniformly in n and $\hbar$ such that $n\hbar \in (\alpha, \beta)$. This proves (22), since

$$H_\hbar^{(n)} = H_\hbar^{0,(n)} + H_\hbar^{I,(n)} = H_\hbar^{0,(n)} + \hbar \sum_{j=1}^{n} \phi_\hbar(\lambda_{x_j}) \,.$$

In particular, it follows that for any $\gamma \in (0,1)$ there exists $b > 0$ such that, for any $\varphi \in \mathscr{D}(H_\hbar^{0,(n)})$,

$$\|H_\hbar^{I,(n)}\varphi\| \le \gamma \, \|H_\hbar^{0,(n)}\varphi\| + b\,\|\varphi\| \,, \tag{25}$$

uniformly in n and $\hbar$ as before. As a consequence of the Kato-Rellich Theorem, $H_\hbar^{(n)}$ is self-adjoint on $\mathscr{D}(H_\hbar^{0,(n)})$, uniformly bounded from below and (23) holds true for some $b > 0$ and $C > 0$. ⊣

Concerning the total Hamiltonian $H_\hbar$, the following uniform inequalities are very useful.

Lemma 3.3. *There exist $c_1, c_2, \underline{\alpha}, \underline{\beta} > 0$ such that for all $\hbar \in (0,1)$,*

$$\pm H_\hbar^I \le c_1 \,(H_\hbar^0 + N_1^2 + 1) \,, \tag{26}$$

$$H_\hbar^0 \le c_2 \,(H_\hbar + \underline{\alpha} N_1^2 + \underline{\beta}) \,. \tag{27}$$

Proof. Let $\varphi^{(n)} \in \mathscr{H}^{(n)}$. By symmetry of the function $\varphi^{(n)}$, we have

$$\begin{aligned}
\big|\langle \varphi^{(n)}, H_\hbar^I \varphi^{(n)}\rangle\big| &= \hbar n \big|\langle \varphi^{(n)}, \, \phi_\hbar(\lambda_{x_1})\varphi^{(n)}\rangle\big| \\
&\le \|(\mathrm{d}\Gamma_\hbar^{(2)}(\omega)+1)^{1/2}\varphi^{(n)}\|_{\mathscr{H}^{(n)}} \, \|(\mathrm{d}\Gamma_\hbar^{(2)}(\omega)+1)^{-1/2}\, \hbar n \,\phi_\hbar(\lambda_{x_1})\varphi^{(n)}\|_{\mathscr{H}^{(n)}} \\
&\le C\,\|(\mathrm{d}\Gamma_\hbar^{(2)}(\omega)+1)^{1/2}\varphi^{(n)}\|_{\mathscr{H}^{(n)}} \|N_1\varphi^{(n)}\|_{\mathscr{H}^{(n)}} \,,
\end{aligned} \tag{28}$$

where Lemma 3.1 is used and $C > 0$ is a constant independent of $n \in \mathbb{N}$.

Now, to prove (26) we use (28) to show that

$$\begin{aligned}
|\langle \varphi^{(n)}, H_\hbar^I \varphi^{(n)}\rangle| &\le C\,\|(\mathrm{d}\Gamma_\hbar^{(2)}(\omega)+1)^{1/2}\varphi^{(n)}\|_{\mathscr{H}^{(n)}} \|N_1\varphi^{(n)}\|_{\mathscr{H}^{(n)}} \\
&\le C\,\langle \varphi^{(n)}, \big(\mathrm{d}\Gamma_\hbar^{(2)}(\omega) + N_1^2 + 1\big)\varphi^{(n)}\rangle \\
&\le C\,\langle \varphi^{(n)}, \big(H_\hbar^0 + N_1^2 + 1\big)\varphi^{(n)}\rangle \,.
\end{aligned}$$

Now, since $H_\hbar^I$ and $H_\hbar^0 + N_1^2 + 1$ commute with N_1, we can sum over n and conclude the proof.

To prove (27), we use again (28), and the inequality $2ab \le \frac{1}{\eta^2}a^2 + \eta^2 b^2$ that yield

$$\begin{aligned}
|\langle \varphi^{(n)}, H_\hbar^I \varphi^{(n)}\rangle| &\le C\,\|(\mathrm{d}\Gamma_\hbar^{(2)}(\omega)+1)^{1/2}\varphi^{(n)}\|_{\mathscr{H}^{(n)}} \|N_1\varphi^{(n)}\|_{\mathscr{H}^{(n)}} \\
&\le \frac{C}{\eta^2}\langle \varphi^{(n)}, \big(\mathrm{d}\Gamma_\hbar^{(2)}(\omega) + 1\big)\varphi^{(n)}\rangle + C\eta^2 \langle \varphi^{(n)}, N_1^2\varphi^{(n)}\rangle \,.
\end{aligned}$$

Hence,

$$H_\hbar^I \ge -\frac{C}{\eta^2}(\mathrm{d}\Gamma_\hbar^{(2)}(\omega)+1) - C\eta^2 N_1^2 \,.$$

Finally, adding $H_\hbar^0$ on both sides one obtains

$$H_\hbar + C\eta^2 N_1^2 \ge \mathrm{d}\Gamma_\hbar^{(1)}(-\Delta + V) + \left(1 - \frac{C}{\eta^2}\right) \mathrm{d}\Gamma_\hbar^{(2)}(\omega) - \frac{C}{\eta^2}.$$

Now choosing $\eta > 0$ such that $C\eta^{-2} < 1$, it follows that

$$(1 - C\eta^{-2})\, H_\hbar^0 \le H_\hbar + C\eta^2 N_1^2 + C\eta^{-2} \,.$$

⊣

The regularity property below is a straightforward consequence of the above Lemma 3.3. Beforehand, denote

$$\Psi_\hbar(t) = e^{-i\frac{t}{\hbar}H_\hbar}\Psi_\hbar \,, \tag{29}$$

$$S = H_\hbar^0 + N_1^2 + 1 \,. \tag{30}$$

Corollary 3.4. *Assume that* $\{\Psi_\hbar\}_{\hbar\in(0,1)} \subseteq \mathscr{H}$ *is a family of normalized vectors satisfying for some constant* $C > 0$ *and all* $\hbar \in (0,1)$*,*

$$\langle \Psi_\hbar, S\,\Psi_\hbar\rangle \le C.$$

Then there exists a constant $c_3 > 0$ *such that, uniformly in* $\hbar \in (0,1)$ *and in* $t \in \mathbb{R}$*,*

$$\langle \Psi_\hbar(t), S\,\Psi_\hbar(t)\rangle \le c_3. \tag{31}$$

We conclude this section by proving *uniform energy-number* estimates, inspired by the ones in [DG99, Section 3.5] and which will be useful in studying correlation functions in § 5.2.

Proposition 3.5. *Let* $0 < \alpha < \beta$*. For* $k, r \in \mathbb{N}$ *there exist* $c, b > 0$ *such that for any* $n \in \mathbb{N}$ *and any* $\hbar \in (0,1)$ *satisfying* $n\hbar \in (\alpha,\beta)$*, the following estimates hold true :*

i) $\|(N_2+1)^k(H_\hbar^{(n)}+b)^{-k}\| \le c.$

ii) $\|(N_2+1)^{k+r}(H_\hbar^{(n)}+b)^{-k}(N_2+1)^{-r}\| \le c.$

iii) $\|H_\hbar^{0,(n)}(N_2+1)^k(H_\hbar^{(n)}+b)^{-(k+1)}\| \le c.$

Proof. Throughout the proof, we restrict to n and $\hbar$ satisfying the assumption in the proposition. Observe that by (27) of Lemma 3.3 there exits a constant $b > 0$ such that $H_\hbar^{(n)} + b \ge 1$. For simplicity, let us denote

$$A = N_2 + 1, \qquad\qquad B = (H_\hbar^{(n)} + b)^{-1} \,,$$

and $\mathrm{ad}_A^j(B)$ the adjoint action defined recursively as

$$\mathrm{ad}_A^0(B) = B \,,$$

$$\mathrm{ad}_A^j(B) = [A, \mathrm{ad}_A^{j-1}(B)] \,.$$

Recall the Leibniz's formula

$$A^k B = \sum_{j=0}^{k}\binom{k}{j}\,\mathrm{ad}_A^j(B)\,A^{k-j} \,. \tag{32}$$

First, observe that for all $j \in \mathbb{N}$

$$\mathrm{ad}_A^j(B) = (i\hbar)^j\sum_{p=1}^{j} c_p\, B\phi_\hbar(r_{p,1})\ldots B\phi_\hbar(r_{p,p})B, \tag{33}$$

with $r_{p,q} \in \{\hbar\sum_{s=1}^n \lambda_{x_s}, i\hbar\sum_{s=1}^n \lambda_{x_s}\}$ for $q = 1,\ldots,p$, and c_p are real coefficients independent of $\hbar$ and n. Such identity is proved by induction on j and using the commutation relations

$$\mathrm{ad}_A^1(B) = [A,B] = -i\hbar\, B\,\phi_\hbar\Big(i\hbar\sum_{s=1}^{n}\lambda_{x_s}\Big)\,B \,,$$

and

$$[A, \phi_\hbar(r_{p,q})] = -i\hbar\,\phi_\hbar(ir_{p,q}) \,.$$

In particular, the Leibniz formula (32) make sense as an operator equality on $\mathscr{D}(A^k)$.

We prove now i) by induction. For $k = 1$, the inequality is a consequence of Lemma 3.2, and the fact that $(H_\hbar^{0,(n)} + 1)^{-1} N_2$ is bounded uniformly with respect to n and $\hbar$. The induction step goes as follows.

$$
\begin{aligned}
A^{k+1}B^{k+1} &= A\left(A^k B\right) B^k \\
&= A\left(\sum_{j=0}^{k} \binom{k}{j} \mathrm{ad}_A^j(B)\, A^{k-j}\right) B^k .
\end{aligned}
$$

Using Lemma 3.2 and the inequality (24), one shows that there exists $c > 0$ such that for all $p \in \mathbb{N}, q = 1, \dots, p$,

$$
\|\phi_\hbar(r_{p,q})\; B\|_{\mathscr{B}(\mathscr{H}^{(n)})} \leq c\,, \tag{34}
$$

uniformly with respect to n and $\hbar$. Therefore,

$$
A^{k+1}B^{k+1} = ABA^kB^k + AB\sum_{j=1}^{k}(i\hbar)^j\binom{k}{j}\left(\sum_{p=1}^{j} c_p\, \phi_\hbar(r_{p,1})\dots B\phi_\hbar(r_{p,p})B\right) A^{k-j}B^k . \tag{35}
$$

Since all the operators on the right hand side are bounded uniformly with respect to n and $\hbar$, the proof of i) is completed.

Let us now prove ii). We use the following algebraic operator identity:

$$
A^{k+r}B^kA^{-r} = A^{k+r}BA^{-k-r+1}\dots A^{r+1}BA^{-r} . \tag{36}
$$

The Leibniz's formula and the identity (33) yield

$$
A^\gamma BA^{-\gamma+1} = \sum_{j=0}^{\gamma}(i\hbar)^j\binom{\gamma}{j}\left(\sum_{p=1}^{j} c_p\, B\phi_\hbar(r_{p,1})\dots B\phi_\hbar(r_{p,p})B\right)A^{1-j}\ , \tag{37}
$$

for any $\gamma \in \mathbb{N}$. Since all the terms in the right hand side of (37), and consequently (36), are uniformly bounded with respect to n and $\hbar$, the proof is concluded.

It remains to prove iii). It follows from i), using Leibniz's formula. In fact, one has

$$
\begin{aligned}
H_\hbar^{0,(n)}A^kB^{k+1} &= H_\hbar^{0,(n)}\left(A^kB\right)B^k \\
&= H_\hbar^{0,(n)}BA^kB^k + H_\hbar^{0,(n)}B\sum_{j=1}^{k}(i\hbar)^j\binom{k}{j}\left(\sum_{p=1}^{j} c_p\phi_\hbar(r_{p,1})\cdots B\phi_\hbar(r_{p,p})B\right)A^{k-j}B^k.
\end{aligned}
$$

Hence combining i) with (34) and with the fact that $H_\hbar^{0,(n)}B$ is uniformly bounded according to Lemma 3.2, it follows that all the terms on the right hand side are uniformly bounded with respect to n and $\hbar$. ⊣

3.2. **Dispersive estimates.** Let us now discuss the dispersive properties of the Yukawa models, both quantum and classical. Recall that we are supposing that Assumption 1 is satisfied. We start with a first elementary dispersive estimate.

Lemma 3.6. *For every $\xi \in \mathscr{C}_0^\infty(\mathbb{R} \smallsetminus \{0\})$, there exists $C > 0$ such that, for all $t \in \mathbb{R}$,*

$$\left\| \langle x\rangle^{-1-\nu} \, \langle \xi_t, \lambda_x\rangle \right\|_{L^\infty} \leq \frac{C}{\langle t\rangle^{1+\nu}} \,, \tag{38}$$

$$\|(-\Delta + V + 1)^{-1/2} \, \mathrm{Im} \, \langle \xi_t, \lambda_x\rangle \, (-\Delta + V + 1)^{-1/2}\|_{\mathscr{B}(L^2(\mathbb{R}^d))} \leq \frac{C}{\langle t\rangle^{1+\nu}} \,, \tag{39}$$

where $\xi_t = e^{-it\omega}\xi$.

Proof. Observe that

$$\begin{aligned} &\|(-\Delta + V + 1)^{-1/2} \, \mathrm{Im} \, \langle \xi_t, \lambda_x\rangle \, (-\Delta + V + 1)^{-1/2}\|_{\mathscr{B}(L^2(\mathbb{R}^d))} \\ &\qquad \leq \|\langle \cdot \rangle^{-1-\nu} \left\langle \xi_t, \lambda_{(\cdot)} \right\rangle \|_{L^\infty} \, \|(-\Delta + V + 1)^{-1/2} \langle x\rangle^{\frac{1+\nu}{2}} \|^2_{\mathscr{B}(L^2(\mathbb{R}^d))} \,. \end{aligned} \tag{40}$$

The last term on the right-hand side is bounded by a constant, thanks to the assumption (A1) on V. Moreover, $\xi \in \mathscr{C}_0^\infty(\mathbb{R}^d \smallsetminus \{0\})$ and thus the stationary point $\{k : \nabla\omega(k) = 0\} = \{0\}$ does not belong to the support of ξ. Hence, the non-stationary phase method yields [see, *e.g.*, RS79, Theorem XI.14]:

$$\langle x\rangle^{-1-\nu} \big|\langle \xi_t, \lambda_x\rangle\big| = \langle x\rangle^{-1-\nu} \left| \int_{\mathbb{R}^d} \mathrm{d}k \, e^{it\omega(k)} \, \overline{\xi}(k) \, \lambda_x(k) \right| \leq \frac{C}{\langle t\rangle^{1+\nu}} \,.$$

⊣

We prove below the main $\hbar$-uniform decay estimate for the Yukawa model. Recall that the operator S is defined according to (30).

Proposition 3.7. *For any $\xi \in \mathscr{C}_0^\infty(\mathbb{R}^d \smallsetminus \{0\})$ there exists $c > 0$ such that, for all $t \in \mathbb{R}$,*

$$\left\| S^{-1/2} \, \mathrm{d}\Gamma_\hbar^{(1)}\big(\mathrm{Im} \, \langle \xi_t, \lambda_{(\cdot)}\rangle\big) \, S^{-1/2} \right\|_{\mathscr{B}(\mathscr{H})} \leq \frac{c}{\langle t\rangle^{1+\nu}} \,,$$

uniformly with respect to $\hbar \in (0,1)$.

Proof. Consider $\Phi^{(n)}, \Psi^{(n)} \in \mathscr{H}^{(n)}$, then using symmetry one obtains

$$\begin{aligned} &\left|\left\langle \Phi^{(n)}, S^{-1/2} \, \mathrm{d}\Gamma_\hbar^{(1)}(\mathrm{Im}\langle \xi_t, \lambda_{(\cdot)}\rangle) \, S^{-1/2}\Psi^{(n)}\right\rangle\right| \\ &\quad = \left|\hbar n\left\langle S^{-1/2}\Phi^{(n)}, \mathrm{Im}\langle \xi_t, \lambda_{x_1}\rangle S^{-1/2}\Psi^{(n)}\right\rangle\right| \\ &\quad \leq \left\| \frac{\mathrm{Im} \, \langle \xi_t, \lambda_{x_1}\rangle}{\langle x_1\rangle^{1+\nu}} \right\|_{\mathscr{B}(L^2(\mathbb{R}^d))} \left\| \sqrt{(n\hbar)\langle x_1\rangle^{1+\nu}} S^{-1/2}\Phi^{(n)} \right\|_{\mathscr{H}^{(n)}} \left\| \sqrt{(n\hbar)\langle x_1\rangle^{1+\nu}} S^{-1/2}\Psi^{(n)} \right\|_{\mathscr{H}^{(n)}} \\ &\quad \leq \frac{C}{\langle t\rangle^{1+\nu}} \| \sqrt{(n\hbar)\langle x_1\rangle^{1+\nu}} S^{-1/2}\Phi^{(n)}\|_{\mathscr{H}^{(n)}} \| \sqrt{(n\hbar)\langle x_1\rangle^{1+\nu}} S^{-1/2}\Psi^{(n)}\|_{\mathscr{H}^{(n)}} \,, \end{aligned}$$

where in the last inequality Lemma 3.6 is used. Using symmetry again and assumption (A1),

$$\begin{aligned} \| \sqrt{(n\hbar)\langle x_1\rangle^{1+\nu}} S^{-1/2}\Psi^{(n)}\|^2_{\mathscr{H}^{(n)}} &= \left\langle \Psi^{(n)}, S^{-1/2} \, \mathrm{d}\Gamma_\hbar^{(1)}(\langle x\rangle^{1+\nu}) \, S^{-1/2}\Psi^{(n)}\right\rangle \\ &\leq C' \| \, \Psi^{(n)}\|^2_{\mathscr{H}^{(n)}} \,. \end{aligned}$$

An analogous inequality holds for $\Phi^{(n)}$ as well, and therefore

$$\left|\left\langle \Phi^{(n)}, S^{-1/2} \, \mathrm{d}\Gamma_\hbar^{(1)}(\mathrm{Im}\langle \xi_t, \lambda_{(\cdot)}\rangle) \, S^{-1/2}\Psi^{(n)}\right\rangle\right| \leq \frac{c}{\langle t\rangle^{1+\nu}} \| \, \Phi^{(n)}\|_{\mathscr{H}^{(n)}} \| \, \Psi^{(n)}\|_{\mathscr{H}^{(n)}} \,.$$

⊣

4. Scattering theory of the Schrödinger-Klein-Gordon equation

In the sequel, we discuss the classical scattering theory of the Schrödinger-Klein-Gordon system. Since we assume that the Schrödinger particle is confined, only Klein-Gordon waves can be asymptotically free. Therefore, in our context the diffusion scheme differs from the traditional translation invariant case studied for instance in [OT94, Shi03].

The S-KG equation is an infinite dimensional classical Hamiltonian system described by its Hamiltonian function

$$\mathscr{E}(u,z)=\langle u,(-\Delta+V)u\rangle_{\mathfrak{H}}+\langle z,\omega z\rangle_{\mathfrak{H}}+\int_{\mathbb{R}^{2d}}\big(\lambda_x(k)\bar{z}(k)+\bar{\lambda}_x(k)z(k)\big)|u(x)|^2\mathrm{d}x\mathrm{d}k\;, \tag{41}$$

defined on the energy space

$$\mathfrak{X}=\mathscr{D}(\sqrt{-\Delta+V})\oplus\mathscr{D}(\sqrt{\omega})\;\subset\mathscr{X}=\mathfrak{H}\oplus\mathfrak{H}\,, \tag{42}$$

with $\mathfrak{H}=L^2(\mathbb{R}^d)$. The energy of the non-interacting system is given by

$$\mathscr{E}_0(u,z)=\langle u,(-\Delta+V)u\rangle_{\mathfrak{H}}+\langle z,\omega z\rangle_{\mathfrak{H}}\;.$$

Then, the following rough inequalities compare $\mathscr{E}$ and $\mathscr{E}_0$. Indeed, there exist $c,\alpha>0$ such that for all $(u,z)\in\mathfrak{X}$,

$$\big|\mathscr{E}(u,z)-\mathscr{E}_0(u,z)\big|\leq c\,(\mathscr{E}_0(u,z)+\|u\|_{L^2}^4)\,, \tag{43}$$

$$0\leq\mathscr{E}_0(u,z)\leq c\,(\mathscr{E}(u,z)+\alpha\|u\|_{L^2}^4)\,. \tag{44}$$

With these notations, the Schrödinger-Klein-Gordon equation is a system of PDE given by:

$$\begin{cases} i\partial_t u=(-\Delta+V)\,u+\varphi_\chi\,u\\ i\partial_t z=\omega z+\omega^{-1/2}\chi\,\widehat{|u|^2}\end{cases} \tag{45}$$

where

$$\varphi_\chi(x)=\int_{\mathbb{R}^d}\frac{1}{\sqrt{\omega(k)}}\big(e^{ik\cdot x}\chi(k)\bar{z}(k)+e^{-ik\cdot x}\bar{\chi}(k)z(k)\big)\mathrm{d}k$$

is the smeared Klein-Gordon field and $\widehat{|u|^2}$ is the Fourier transform of $|u|^2\in L^1(\mathbb{R}^d)$. In particular, the S-KG equation is globally well-posed over $\mathfrak{X}$ with mass $\|u\|_{L^2(\mathbb{R}^d)}$ and energy $\mathscr{E}(u,z)$ as conserved quantities [see, *e.g.*, Bac84, FT75]. In the following, we denote by $\Phi_t:\mathfrak{X}\to\mathfrak{X}$ the solution flow associated to the S-KG equation.

In order to discuss the long-time asymptotics for the Klein-Gordon field, it is convenient to rewrite the solution $z(t)$ using the Duhamel's integral formula:

$$z(t)=e^{-it\omega}z_0-i\int_0^t e^{-i(t-\tau)\omega}\omega^{-1/2}\chi\,\widehat{|u(\tau)|^2}\,\mathrm{d}\tau\,. \tag{46}$$

The corresponding wave operator is then defined as follows.

Definition 4.1. *The classical forward and backward wave operators of the S-KG equation are defined as the maps:*

$$\begin{aligned}\Lambda^\pm:\;\mathfrak{X}\;&\longrightarrow\;\mathfrak{H}\\ (u_0,z_0)\;&\longmapsto\;z^\pm:=\operatorname*{w-lim}_{t\to\pm\infty}\,e^{it\omega}z(t)\,,\end{aligned}$$

where $(u(\cdot,t),z(\,\cdot\,,t))=\Phi_t(u_0,z_0)$ *is the unique solution of the S-KG equation* (45) *satisfying the initial condition* (u_0,z_0) *at time* $t=0$ *and the limit in the right hand side is with respect to the weak* $L^2(\mathbb{R}^d)$*-topology.*

The above classical wave operators exist according to the proposition below.

Proposition 4.2. *The wave operators* $\Lambda^\pm$ *of the S-KG equation are well defined.*

Proof. Consider a smooth function $\varphi\in\mathscr{C}_0^\infty(\mathbb{R}^d\smallsetminus\{0\})$, then according to (46),

$$\begin{aligned}\langle\varphi,e^{it\omega}z(t)\rangle_{\mathfrak{H}}&=\langle\varphi,z_0\rangle_{\mathfrak{H}}-i\big\langle\varphi,\int_0^t e^{i\tau\omega}\omega^{-1/2}\chi\,\widehat{|u(\tau)|^2}\mathrm{d}\tau\big\rangle_{\mathfrak{H}}\\&=\langle\varphi,z_0\rangle_{\mathfrak{H}}-i\int_0^t\big\langle\varphi,e^{i\tau\omega}\omega^{-1/2}\chi\,\widehat{|u(\tau)|^2}\big\rangle_{\mathfrak{H}}\mathrm{d}\tau\,.\end{aligned}\tag{47}$$

Using the dispersive estimate in Lemma 3.6, one obtains

$$\begin{aligned}\big|\big\langle\varphi,e^{i\tau\omega}\omega^{-1/2}\chi\,\widehat{|u(\tau)|^2}\big\rangle_{\mathfrak{H}}\big|&\leq\big\|\langle\,\cdot\,\rangle^{\frac{1+\nu}{2}}u(\tau)\big\|_{L^2}^2\,\big\|\langle\,\cdot\,\rangle^{-1-\nu}\langle e^{-i\tau\omega}\varphi,\lambda_{(\cdot)}\rangle_{\mathfrak{H}}\big\|_{L^\infty}\\&\leq\frac{C}{\langle\tau\rangle^{1+\nu}}\,\big\|\langle\,\cdot\,\rangle^{\frac{1+\nu}{2}}u(\tau)\big\|_{\mathfrak{H}}^2\,.\end{aligned}$$

Now, the bound (44) yields

$$\begin{aligned}\big\|\langle\,\cdot\,\rangle^{\frac{1+\nu}{2}}u(\tau)\big\|_{\mathfrak{H}}^2&\leq\big\langle u(\tau),(-\Delta+V)u(\tau)\big\rangle_{\mathfrak{H}}\\&\leq\mathscr{E}_0(u(\tau),z(\tau))\\&\leq c\Big(\mathscr{E}(u(\tau),z(\tau))+\alpha\|u(\tau)\|_{\mathfrak{H}}^4\Big)\,.\end{aligned}$$

Thus, the energy and mass conservation guarantees that the integral in (47) converges as $t\to\pm\infty$. Such convergence is lifted to all $\varphi\in L^2(\mathbb{R}^d)$ by a density argument, using also the fact that the norm $\|z(t)\|_{L^2(\mathbb{R}^d)}$ is uniformly bounded in time thanks to the energy estimate (44). $\dashv$

Let us make a couple of remarks.

(i) The classical wave operators $\Lambda^\pm$ have the following representation, intended as a distribution equality in $\mathscr{D}'(\mathbb{R}^d\smallsetminus\{0\})$:

$$\Lambda^\pm(u_0,z_0)=z_0-i\int_0^{\pm\infty}e^{i\tau\omega}\omega^{-1/2}\chi\,\widehat{|u(\tau)|^2}\mathrm{d}\tau\,.\tag{48}$$

(ii) For all $\zeta\in\mathfrak{H}$,

$$\lim_{t\to\pm\infty}\langle e^{-it\omega}\zeta,z(t)-e^{-it\omega}z^\pm\rangle_{\mathfrak{H}}=0\,.$$

In other words, the nonlinear evolution of the field z_0 can be approximated for long times, in a weak sense, by the free evolution of $z^\pm=\Lambda^\pm(u_0,z_0)$.

Note also that despite the name, $\Lambda^\pm$ are not linear maps.

Corollary 4.3.

$$\operatorname{Ran}\Lambda^\pm\subseteq\mathscr{D}(\sqrt{\omega})\,.$$

Proof. The solution $z(t)\in\mathscr{D}(\sqrt{\omega})$ for all $t\in\mathbb{R}$. In addition, by (44) and the conservation of energy, $\|\sqrt{\omega}z(t)\|_{L^2}$ is uniformly bounded in time. Therefore, the convergence in Proposition 4.2 holds when testing with $\varphi\in\omega^{-1/2}L^2(\mathbb{R}^d)$ as well. $\dashv$

As for the quantum theory and asymptotic vacuum states (20), at the classical level there is a notion of asymptotic radiationless states. These are the phase space points in the kernel of the classical wave operator, *i.e.*

$$\mathscr{K}_0^{\pm} = \{(u,z) \in \mathfrak{X} \mid \Lambda^{\pm}(u,z) = 0\} . \tag{49}$$

In the next sections, we will see how to relate the notions of classical wave operators and the space of asymptotic radiationless states to the corresponding ones for the quantum Yukawa model.

5. Semiclassical limit of meson fields

In this section we study the behavior of the asymptotic fields of the Yukawa theory, as $\hbar \to 0$. As already recalled in § 2.2, at any finite time $t \in \mathbb{R}$ the semiclassical behavior of the family $\{\Psi_\hbar(t) = e^{-i\frac{t}{\hbar}H_\hbar}\Psi_\hbar\}_{\hbar\in(0,1)}$ can be characterized by their semiclassical measures at a fixed given time (usually $t = 0$), pushed forward by the nonlinear flow associated to the classical S-KG equations (45) [see AF14, for a detailed analysis]. Combining the techniques used for finite times with the dispersive properties of both quantum and classical Yukawa models, one extends the analysis to infinite times $t \to \pm\infty$.

5.1. **Meson fields.** For convenience, recall the operator S already introduced in § 3.1:

$$S := H_\hbar^0 + N_1^2 + 1 . \tag{50}$$

This operator emerges naturally, for this particular model, as the right tool that encodes the regularity properties needed to characterize explicitly the semiclassical limit of asymptotic fields. To this extent, we will make extensive use of the following assumption.

Assumption 2. Given a family $\{\Psi_\hbar\}_{\hbar\in(0,1)} \subseteq \mathscr{H}$ of normalized vectors, there exists $C > 0$ such that, uniformly with respect to $\hbar \in (0,1)$,

$$\langle \Psi_\hbar, S\,\Psi_\hbar\rangle \leq C. \tag{A4}$$

The integral formula below is a consequence of such assumption.

Proposition 5.1. *Let $\{\Psi_\hbar\}_{\hbar\in(0,1)}$ be a family of vectors satisfying* (A4)*. Denoting $\Psi_\hbar(t) = e^{-i\frac{t}{\hbar}H_\hbar}\Psi_\hbar$, we have that for all $\xi \in \mathscr{C}_0^\infty(\mathbb{R}^d \smallsetminus \{0\})$,*

$$\big\langle \Psi_\hbar, W_\hbar^{\pm}(\xi)\Psi_\hbar\big\rangle = \langle \Psi_\hbar, W_\hbar(\xi)\Psi_\hbar\rangle + \sqrt{2}i\int_0^{\pm\infty} \Big\langle \Psi_\hbar(\tau), \mathrm{d}\Gamma_\hbar^{(1)}\big(\mathrm{Im}\langle \xi_\tau, \lambda_{(\cdot)}\rangle_{\mathfrak{H}}\big)\Psi_\hbar(\tau)\Big\rangle\, \mathrm{d}\tau\ . \tag{51}$$

Proof. The operator $e^{i\frac{t}{\hbar}H_\hbar}W_\hbar(\xi_t)\, e^{-i\frac{t}{\hbar}H_\hbar}$ is weakly differentiable on $\mathscr{D}(\sqrt{S}) \subset \mathbb{Q}(H_\hbar)$. This is a consequence of Lemma 3.3 and Corollary 3.4, together with the following properties of Weyl operators:

$$W_\hbar(\xi_t) = e^{-i\frac{t}{\hbar}H_\hbar^0}\, W_\hbar(\xi)\, e^{i\frac{t}{\hbar}H_\hbar^0}\ , \quad W_\hbar(\xi_t)\mathscr{D}(\sqrt{S}) \subset \mathscr{D}(\sqrt{S})\ , \tag{52}$$

for any $\xi \in \mathscr{C}_0^\infty(\mathbb{R}^d \smallsetminus \{0\})$. The time derivative yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\, e^{i\frac{t}{\hbar}H_\hbar}W_\hbar(\xi_t)\, e^{-i\frac{t}{\hbar}H_\hbar} = \sqrt{2}i\, e^{i\frac{t}{\hbar}H_\hbar}\, \mathrm{d}\Gamma_\hbar^{(1)}\big(\mathrm{Im}\,\langle \xi_t, \lambda_{(\cdot)}\rangle_{\mathfrak{H}}\big) W_\hbar(\xi_t)\, e^{-i\frac{t}{\hbar}H_\hbar}\ . \tag{53}$$

By the fundamental theorem of calculus, we obtain

$$\begin{aligned}\big\langle \Psi_\hbar, e^{i\frac{t}{\hbar}H_\hbar} W_\hbar(\xi_t)\, e^{-i\frac{t}{\hbar}H_\hbar}\Psi_\hbar\big\rangle &= \big\langle \Psi_\hbar, W_\hbar(\xi)\Psi_\hbar\big\rangle \\ &\quad + \sqrt{2}i\int_0^t \big\langle \Psi_\hbar(\tau), \mathrm{d}\Gamma_\hbar^{(1)}\big(\mathrm{Im}\langle \xi_\tau, \lambda_{(\cdot)}\rangle_{\mathfrak{H}}\big) W_\hbar(\xi_\tau)\,\Psi_\hbar(\tau)\big\rangle\,\mathrm{d}\tau\,.\end{aligned} \tag{54}$$

Using Corollary 3.4 and the canonical commutation relations [see, *e.g.*, DG99, Lemma 2.5], it then follows that there exists $c>0$ such that for all $\tau\in\mathbb{R}$,

$$\langle W_\hbar(\xi_\tau)\Psi_\hbar(\tau), SW_\hbar(\xi_\tau)\,\Psi_\hbar(\tau)\rangle \le c\,.$$

Hence we can use Proposition 3.7, to find a (possibly different) $c>0$ such that

$$\big|\langle \Psi_\hbar(\tau), \mathrm{d}\Gamma_\hbar^{(1)}\big(\mathrm{Im}\langle \xi_\tau, \lambda_{(\cdot)}\rangle_{\mathfrak{H}}\big) W_\hbar(\xi_\tau)\,\Psi_\hbar(\tau)\rangle\big| \le c\,\langle \tau\rangle^{-1-\nu}\,. \tag{55}$$

This ensures integrability on the whole positive (negative) real line, and therefore we can take the limit $t\to\pm\infty$ in both sides of (54). ⊣

Thanks to the above proposition and using the techniques of infinite dimensional semiclassical analysis developed in [AN08], we prove below a semiclassical characterization for asymptotic Weyl operators. We recommend the reading of paragraph § 2.2 before going through this part.

Theorem 5.2. *Let* $\{\Psi_\hbar\}_{\hbar\in(0,1)}\subseteq\mathscr{H}$ *be a family of normalized vectors satisfying* (A4) *and assume that* $\mathscr{M}\big(\Psi_{\hbar_n}; n\in\mathbb{N}\big)=\{\mu\}$ *for a sequence* $\hbar_n\to 0$*. Then the Wigner measure* μ *is concentrated on* $\mathfrak{X}=\mathscr{D}(\sqrt{-\Delta+V})\oplus\mathscr{D}(\sqrt{\omega})$*, and for all* $\xi\in\mathfrak{H}$*,*

$$\lim_{n\to\infty}\big\langle \Psi_{\hbar_n}, W^{\pm}_{\hbar_n}(\xi)\Psi_{\hbar_n}\big\rangle = \int_{\mathfrak{X}} e^{i\sqrt{2}\mathrm{Re}\langle \xi,\Lambda^{\pm}(u,z)\rangle_{\mathfrak{H}}}\,\mathrm{d}\mu(u,z)\,, \tag{56}$$

where $\Lambda^{\pm}$ *are the classical wave operators of Definition 4.1.*

Additionally, the following integral representation holds true for all $\xi\in\mathscr{C}_0^\infty(\mathbb{R}^d\setminus\{0\})$*:*

$$\begin{aligned}\lim_{n\to\infty}\big\langle \Psi_{\hbar_n}, W^{\pm}_{\hbar_n}(\xi)\Psi_{\hbar}\big\rangle &= \int_{\mathfrak{X}} e^{i\sqrt{2}\mathrm{Re}\langle \xi,z\rangle_{\mathfrak{H}}}\,\mathrm{d}\mu(u,z) \\ &\quad + \sqrt{2}i\int_0^{\pm\infty}\int_{\mathfrak{X}} e^{i\sqrt{2}\mathrm{Re}\langle \xi_\tau,z\rangle_{\mathfrak{H}}}\,\mathrm{Im}\big\langle \xi_\tau, \omega^{-1/2}\chi\,\widehat{|u|^2}\big\rangle_{\mathfrak{H}}\mathrm{d}\mu_\tau(u,z)\mathrm{d}\tau\,,\end{aligned} \tag{57}$$

where $\mu_\tau=(\Phi_\tau)_*\,\mu$ *is the measure pushed forward by the nonlinear flow* $\Phi_\tau:\mathfrak{X}\to\mathfrak{X}$ *solving the S-KG equation* (45)*.*

Proof. Firstly, let us fix $\xi\in\mathscr{C}_0^\infty(\mathbb{R}^d\setminus\{0\})$. By [AF14, Theorem 1.1], it follows that $\Psi_{\hbar_n}\rightharpoondown\mu$ if and only if for all $t\in\mathbb{R}$, $\Psi_{\hbar_n}(t)\rightharpoondown\mu_t=(\Phi_t)_*\,\mu$. In addition, thanks to (A4), it also follows that μ is concentrated on $\mathfrak{X}$. Now, it is sufficient to take the limit $n\to\infty$ on both sides of (51) (with $\hbar_n$ in place of $\hbar$) to obtain (57). The strategy for the proof is analogous to the one used to prove the aforementioned [AF14, Theorem 1.1], let us however briefly comment on how to deal with the term containing the integral over all positive times τ. The crucial point is that the bound for the integrand given in (55) is uniform with respect to $\hbar$. Therefore, Lebesgue's dominated convergence theorem shall be applied, exchanging the integral with the limit $n\to\infty$, and then the convergence follows from the same arguments as in the proofs of Lemma 3.14 and Proposition 4.10 in [AF14].

It remains to prove (56) for all $\xi \in \mathfrak{H}$. However, for the moment keep $\xi \in \mathscr{C}_0^\infty(\mathbb{R}^d \smallsetminus \{0\})$ as above. Remark that, by Definition 4.1,

$$\begin{aligned}\int_{\mathfrak{X}} e^{i\sqrt{2}\mathrm{Re}\langle \xi, \Lambda^{\pm}(u,z)\rangle_{\mathfrak{H}}} \mathrm{d}\mu(u,z) &= \lim_{t\to\pm\infty} \int_{\mathfrak{X}} e^{i\sqrt{2}\mathrm{Re}\langle \xi_t, z(t)\rangle_{\mathfrak{H}}} \mathrm{d}\mu(u,z) \\ &= \int_{\mathfrak{X}} e^{i\sqrt{2}\mathrm{Re}\langle \xi, z\rangle} \mathrm{d}\mu(u,z) + \lim_{t\to\pm\infty} \int_0^t \int_{\mathfrak{X}} \frac{\mathrm{d}}{\mathrm{d}\tau}\Big(e^{i\sqrt{2}\mathrm{Re}\langle \xi_\tau, z(\tau)\rangle_{\mathfrak{H}}}\Big) \mathrm{d}\mu(u,z)\mathrm{d}\tau \,.\end{aligned} \tag{58}$$

Now, S-KG equation (45) yields precisely

$$\frac{\mathrm{d}}{\mathrm{d}\tau} e^{i\sqrt{2}\mathrm{Re}\langle \xi_\tau, z(\tau)\rangle_{\mathfrak{H}}} = \sqrt{2} i e^{i\sqrt{2}\mathrm{Re}\langle \xi_\tau, z(\tau)\rangle_{\mathfrak{H}}} \mathrm{Im}\big\langle \xi_\tau, \omega^{-1/2}\chi\, \widehat{|u(\tau)|^2}\big\rangle_{\mathfrak{H}} \,.$$

Hence (56) is proved for all $\xi \in \mathscr{C}_0^\infty(\mathbb{R}^d \smallsetminus \{0\})$. To extend the proof to all $\xi \in \mathfrak{H}$, a density argument is used, observing that by Lemma 3.3 and [AN08, Lemma 3.1] it follows that, uniformly with respect to $\hbar$, there exists $c > 0$ such that

$$\Big\| \big(W_\hbar^{\pm}(\xi) - W_\hbar^{\pm}(\eta)\big) S^{-1/2} \Big\|_{\mathscr{B}(\mathscr{H})} \le c\, \|\xi - \eta\|_{\mathfrak{H}} \,. \tag{59}$$

$\dashv$

Remark 5.3. The asymptotic Weyl operators have the following intertwining property [see DG99, Theorem 5.1]:

$$e^{i\frac{t}{\hbar}H_\hbar}\, W_\hbar^{\pm}(\xi)\, e^{-i\frac{t}{\hbar}H_\hbar} = W_\hbar^{\pm}(\xi_{-t}) \,.$$

Theorem 5.2 yields that the intertwining property is also inherited by the classical wave operator:

$$\Lambda^{\pm}(\Phi_t(u,z)) = e^{it\omega}\Lambda^{\pm}(u,z) \,.$$

Of course, such relation can also be proved using directly Definition 4.1 and Proposition 4.2.

The previous results connect, via the semiclassical limit, the quantum and classical scattering theories of the Yukawa model and the S-KG system, respectively. For the moment, we have shown that the time asymptotic limit of the Fourier-Wigner transforms of quantum states converge to the characteristic functions of probability measures, that are none other than the initial Wigner measures at time $t = 0$ pushed forward by the classical wave operators $\Lambda^{\pm}$. Now, we prove that quantum asymptotic meson fields converge to classical asymptotic Klein-Gordon fields.

Recall that starting from the asymptotic Weyl operators $W_\hbar^{\pm}$ it is possible to define the asymptotic fields $\phi_\hbar^{\pm}$ as their self-adjoint generators:

$$W_\hbar^{\pm}(\eta) = e^{\frac{i}{\sqrt{2}}\phi_\hbar^{\pm}(\eta)} \,.$$

These operators have a common dense core, given by the square root of S: for all $\xi \in \mathfrak{H}$, $\mathscr{D}(\sqrt{S}) \subset \mathscr{D}(\phi_\hbar^{\pm}(\xi))$.

Proposition 5.4. *Let $\{\Psi_\hbar\}_{\hbar\in(0,1)} \subseteq \mathscr{H}$ be a family of normalized vectors satisfying* (A4) *and assume that $\Psi_{\hbar_n} \rightharpoonup \mu$ for a sequence $\hbar_n \to 0$. Then for all $\xi \in \mathfrak{H}$,*

$$\lim_{n\to\infty} \langle \Psi_{\hbar_n}, \phi_{\hbar_n}^{\pm}(\xi)\Psi_{\hbar_n}\rangle = 2\int_{\mathfrak{X}} \mathrm{Re}\langle \xi, \Lambda^{\pm}(u,z)\rangle_{\mathfrak{H}}\, \mathrm{d}\mu(u,z) \,. \tag{60}$$

Proof. The function $t \mapsto e^{i\frac{t}{\hbar}H_\hbar}\,\phi_\hbar(\xi_t)\,e^{-i\frac{t}{\hbar}H_\hbar}$ is differentiable as a quadratic form on $\mathscr{D}(\sqrt{S})$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{i\frac{t}{\hbar}H_\hbar}\,\phi_\hbar(\xi_t)\,e^{-i\frac{t}{\hbar}H_\hbar} = 2\,e^{i\frac{t}{\hbar}H_\hbar}\,\mathrm{d}\Gamma_\hbar^{(1)}(\mathrm{Im}\langle \xi_t,\lambda_x\rangle_{\mathfrak{H}})\,e^{-i\frac{t}{\hbar}H_\hbar}\,.$$

Therefore, we get

$$\big\langle \Psi_\hbar, e^{i\frac{t}{\hbar}H_\hbar}\,\phi_\hbar(\xi_t)\,e^{-i\frac{t}{\hbar}H_\hbar}\Psi_\hbar\big\rangle = \big\langle \Psi_\hbar,\phi_\hbar(\xi)\Psi_\hbar\big\rangle + 2\int_0^t \big\langle \Psi_\hbar(\tau),\mathrm{d}\Gamma_\hbar^{(1)}(\mathrm{Im}\langle \xi_\tau,\lambda_x\rangle_{\mathfrak{H}})\Psi_\hbar(\tau)\big\rangle \mathrm{d}\tau\,.$$

Using Proposition 3.7, the integrand in the previous expression is in $L^1_\tau(\mathbb{R}^\pm)$ for all $\xi \in \mathscr{C}_0^\infty(\mathbb{R}^d \smallsetminus \{0\})$. Hence, taking the limit $t\to\pm\infty$ yields:

$$\begin{aligned}\big\langle \Psi_{\hbar_n},\phi^\pm_{\hbar_n}(\xi)\Psi_{\hbar_n}\big\rangle =& \big\langle \Psi_{\hbar_n},\phi_{\hbar_n}(\xi)\Psi_{\hbar_n}\big\rangle \\ &+ 2\int_0^{\pm\infty}\big\langle \Psi_{\hbar_n}(\tau),\mathrm{d}\Gamma_{\hbar_n}^{(1)}(\mathrm{Im}\langle \xi_\tau,\lambda_x\rangle_{\mathfrak{H}})\Psi_{\hbar_n}(\tau)\big\rangle\,\mathrm{d}\tau\,.\end{aligned} \tag{61}$$

Using again [AF14, Theorem 1.1], we observe that $\mathscr{M}(\Psi_{\hbar_n}(\tau); n\in\mathbb{N}) = \{\mu_\tau = (\Phi_\tau)_*\,\mu\}$. Hence, by Corollary 3.4, and the analogous of [AF14, Lemma 3.15 and Proposition 4.10], the limit $\hbar_n\to 0$ of (61) yields:

$$\lim_{n\to\infty}\big\langle \Psi_{\hbar_n},\phi^\pm_{\hbar_n}(\xi)\Psi_{\hbar_n}\big\rangle = 2\int_{\mathfrak{X}}\mathrm{Re}\langle \xi,z\rangle_{\mathfrak{H}}\mathrm{d}\mu(u,z) + 2\int_0^{\pm\infty}\int_{\mathfrak{X}}\mathrm{Im}\big\langle \xi_\tau,\omega^{-1/2}\chi\,\widehat{|u|^2}\big\rangle_{\mathfrak{H}}\mathrm{d}\mu_\tau(u,z)\mathrm{d}\tau\,.$$

Recalling the representation formula (48) for the classical wave operators, it follows that

$$\mathrm{Re}\langle \xi,z\rangle_{\mathfrak{H}} + \int_0^{\pm\infty}\mathrm{Im}\big\langle \xi_\tau,\omega^{-1/2}\chi\,\widehat{|u(\tau)|^2}\big\rangle_{\mathfrak{H}}\mathrm{d}\tau = \mathrm{Re}\langle \xi,\Lambda^\pm(u,z)\rangle_{\mathfrak{H}}\,. \tag{62}$$

The result extends to any $\xi\in\mathfrak{H}$ by means of the uniform bound

$$\Big\|\big(\phi^\pm_\hbar(\xi)-\phi^\pm_\hbar(\eta)\big)S^{-1/2}\Big\| \leq c\,\|\xi-\eta\|_{\mathfrak{H}}\,. \tag{63}$$

Let us remark that the energy estimates (43)-(44) imply the bound

$$\|\Lambda^\pm(u,z)\|_{\mathfrak{H}} \leq \frac{1}{m}\mathscr{E}_0(u,z) \leq c\,(\mathscr{E}(u,z) + \alpha\|u\|^4_{\mathfrak{H}})\,,$$

and that by assumption (A4) we have that

$$\int_{\mathfrak{X}}\mathscr{E}_0(u,z)\,\mathrm{d}\mu(u,z) < \infty\,.$$

$\dashv$

Corollary 5.5. *Under the same assumptions of Proposition 5.4, we have that for any $\xi\in\mathfrak{H}$,*

$$\begin{aligned}\lim_{n\to\infty}\langle \Psi_{\hbar_n}, a^\pm_{\hbar_n}(\xi)\Psi_{\hbar_n}\rangle &= \int_{\mathfrak{X}}\langle \xi,\Lambda^\pm(u,z)\rangle_{\mathfrak{H}}\;\mathrm{d}\mu(u,z)\,,\\ \lim_{n\to\infty}\langle \Psi_{\hbar_n}, a^{\pm,*}_{\hbar_n}(\xi)\Psi_{\hbar_n}\rangle &= \int_{\mathfrak{X}}\langle \Lambda^\pm(u,z),\xi\rangle_{\mathfrak{H}}\;\mathrm{d}\mu(u,z)\,.\end{aligned} \tag{64}$$

5.2. **Correlation functions and transition amplitudes.** In the quantum Yukawa model, the number of nucleons is invariant (*i.e.*, the corresponding operator N_1 strongly commutes with the Hamiltonian $H_\hbar$). As a consequence, the asymptotic operators $W^\pm_\hbar, \phi^\pm_\hbar, a^{\pm,\natural}_\hbar$ all commute in a strong sense with N_1, and they can be decomposed to a direct sum of operators on each fiber $\mathscr{H}^{(n)}$, see (14), (15) and (19). Hence, it is natural to consider the following type of quantum states.

Assumption 3. Let $\{\Psi_{\hbar_k}^{(n_k)}\}_{k\in\mathbb{N}}$ be a family of normalized vectors on $\mathscr{H}$ such that there exists $\delta > 0$ such that

$$N_1\Psi_{\hbar_k}^{(n_k)} = \hbar_k\, n_k\, \Psi_{\hbar_k}^{(n_k)}\,, \qquad \lim_{k\to\infty} \hbar_k = 0\,, \qquad \lim_{k\to\infty} \hbar_k n_k = \delta^2\,, \qquad \text{and} \qquad \Psi_{\hbar_k}^{(n_k)} \rightharpoondown \mu. \tag{A5}$$

Recall that the last statement $\Psi_{\hbar_k}^{(n_k)} \rightharpoondown \mu$ means that the sequence $\{\Psi_{\hbar_k}^{(n_k)}\}_{k\in\mathbb{N}}$ admits a unique Wigner measure according to (12). On such families of vectors it is possible to study the semiclassical behavior of the asymptotic p-point correlation functions and to deduce relevant informations on asymptotic vacuum vectors, bound states (see § 6.1) and ground states (see § 6.2). The asymptotic p-point correlation functions are defined as follows. Let $\eta \in \mathcal{S}(\mathbb{R}^d, \mathbb{R}) \subset L^2(\mathbb{R}^d) = \mathfrak{H}$, then by (63), or analogously (16), the asymptotic fields $\phi_\hbar^\pm(\cdot)$ can be seen as operator valued distributions in momentum space. That is usually written as

$$\phi_\hbar^\pm(\eta) = \int_{\mathbb{R}^d} \phi_\hbar^\pm(k)\eta(k)\mathrm{d}k\ ,$$

with $\phi_\hbar^\pm(k)$ the aforementioned operator-valued distributions. Taking the Fourier transform on all L^2-wavefunctions, a unitary transformation $\mathcal{F}$ is induced on the meson field's Fock space. Using such a unitary transformation, it is possible to define the fields in position space as operator valued distributions:

$$\varphi_\hbar^\pm(h) = \int_{\mathbb{R}^d} \varphi_\hbar^\pm(x)h(x)\mathrm{d}x\ ,$$

where

$$\varphi_\hbar^\pm(\check{\eta}) = \mathcal{F}^{-1}\phi_\hbar^\pm(\eta)\mathcal{F}\ . \tag{65}$$

We have here denoted by $\check{\eta}$ the inverse Fourier transform of η.

Given a vector $\Psi_\hbar^{(n)} \in \mathscr{H}^{(n)}$ (in the field's momentum Fock space), the p-point asymptotic correlation functions for the meson field are distributions in $\mathcal{S}'(\mathbb{R}^{dp})$ usually defined as

$$\langle\varphi_\hbar^\pm(x_1)\cdots\varphi_\hbar^\pm(x_p)\rangle_{\Psi_\hbar^{(n)}} = \langle\mathcal{F}^{-1}\Psi_\hbar^{(n)}, \varphi_\hbar^\pm(x_1)\cdots\varphi_\hbar^\pm(x_p)\mathcal{F}^{-1}\Psi_\hbar^{(n)}\rangle\ . \tag{66}$$

We remark that in (66) the signs are either all $+$ or all $-$. Using again (16), it follows that the distribution $\langle\varphi_\hbar^\pm(x_1)\cdots\varphi_\hbar^\pm(x_p)\rangle_{\Psi_\hbar^{(n)}}$ is well defined for all $\Psi_\hbar^{(n)} \in \mathbb{Q}((H_\hbar^{(n)})^p)$, and it is a square integrable function:

$$\langle\varphi_\hbar^\pm(x_1)\cdots\varphi_\hbar^\pm(x_p)\rangle_{\Psi_\hbar^{(n)}} \in L^2_{x_1,\dots,x_p}(\mathbb{R}^{dp})\ . \tag{67}$$

We can characterize explicitly, using the tools introduced above, the leading order (*i.e.*, the $\hbar^0$ contribution) of the asymptotic correlation functions for the meson field.

Proposition 5.6. *Let* $\{\Psi_{\hbar_k}^{(n_k)}\}_{k\in\mathbb{N}}$ *be family of normalized vectors satisfying* (A5). *Assume that there exist* $p \geq 1$ *and* $c > 0$ *such that:*

$$\forall k \in \mathbb{N}, \qquad \langle\Psi_{\hbar_k}^{(n_k)},\, (H_{\hbar_k}^{(n_k)} + b)^p\, \Psi_{\hbar_k}^{(n_k)}\rangle \leq c\ . \tag{68}$$

Then the semiclassical measure μ *in* (A5) *is concentrated on the set* $\{(u,z) \in \mathcal{X}\ ,\ \|u\|_{\mathfrak{H}} = \delta\} \subset \mathcal{X}$. *Moreover, for all integers* $\underline{p} \in \mathbb{N}$, $0 < \underline{p} \leq 2p-1$ *and all* $\xi_1, \dots, \xi_{\underline{p}} \in \mathfrak{H}$,

$$\lim_{k\to\infty} \langle\Psi_{\hbar_k}^{(n_k)}, \prod_{j=1}^{\underline{p}} \phi_{\hbar_k}^\pm(\xi_j)\Psi_{\hbar_k}^{(n_k)}\rangle = \int_{\mathcal{X}} \prod_{j=1}^{\underline{p}} \Big(\langle\xi_j, \Lambda^\pm(u,z)\rangle_{\mathfrak{H}} + \langle\Lambda^\pm(u,z), \xi_j\rangle_{\mathfrak{H}}\Big)\mathrm{d}\mu(u,z)\ , \tag{69}$$

and

$$\langle \varphi^{\pm}_{\hbar_k}(x_1)\cdots\varphi^{\pm}_{\hbar_k}(x_{\underline{p}})\rangle_{\Psi^{(n_k)}_{\hbar_k}} = \int_{\mathfrak{X}} \prod_{j=1}^{\underline{p}} \Big(\check{\Lambda}^{\pm}(u,z) + \bar{\check{\Lambda}}^{\pm}(u,z)\Big)(x_j)\,\mathrm{d}\mu(u,z) + o_{\hbar_k}(1)\;, \tag{70}$$

where $o_{\hbar_k}(1)$ *is converging to zero in the weak* $L^2_{x_1,\dots,x_{\underline{p}}}(\mathbb{R}^{d\underline{p}})$ *topology.*

Proof. The concentration property of the measure μ follows from well-known semiclassical considerations [see AN08, AF14, AFP16]. Using (65), it is straightforward to see that, in order to prove (70), it suffices to prove the convergence (69).

Let us omit, for convenience, the explicit k-dependence of n_k and $\hbar_k$, and denote

$$\Psi(t) = e^{-i\frac{t}{\hbar}H^{(n)}_{\hbar}}\Psi^{(n)}_{\hbar}\;.$$

Let us also remark that

$$\partial_t \prod_{j=1}^{\underline{p}} \phi_\hbar(\xi_{j,t}) = -\frac{i}{\hbar}\big[H^{0,(n)}_{\hbar}, \prod_{j=1}^{\underline{p}} \phi_\hbar(\xi_{j,t})\big]\;. \tag{71}$$

Hence

$$\big\langle \Psi(t), \prod_{j=1}^{\underline{p}} \phi_\hbar(\xi_{j,t})\Psi(t)\big\rangle = \big\langle \Psi, \prod_{j=1}^{\underline{p}} \phi_\hbar(\xi_j)\Psi\big\rangle + \frac{i}{\hbar}\int_0^t \big\langle \Psi(\tau), \big[H^{I,(n)}_{\hbar}, \prod_{j=1}^{\underline{p}} \phi_\hbar(\xi_{j,\tau})\big]\Psi(\tau)\big\rangle \mathrm{d}\tau. \tag{72}$$

The commutator yields

$$\big[H^{I,(n)}_{\hbar}, \prod_{j=1}^{\underline{p}} \phi_\hbar(\xi_{j,\tau})\big] = \hbar\sum_{\ell=1}^{n}\big[\phi_\hbar(\lambda_{x_\ell}), \prod_{j=1}^{\underline{p}} \phi_\hbar(\xi_{j,\tau})\big] = i\hbar^2 \sum_{\ell=1}^{n}\sum_{j=1}^{\underline{p}} \mathrm{Im}(\langle \lambda_{x_\ell}, \xi_{j,\tau}\rangle) \prod_{r\neq j} \phi_\hbar(\xi_{r,\tau})\;.$$

In particular, Lemma 3.6 and Proposition 3.5 and (68) yield

$$\mathcal{A}(\tau) = \Big|\frac{i}{\hbar}\big\langle \Psi(\tau), \big[H^{I,(n)}_{\hbar}, \prod_{j=1}^{\underline{p}} \phi_\hbar(\xi_{j,\tau})\big]\Psi(\tau)\big\rangle\Big| \leq c\langle\tau\rangle^{-1-\nu}\;. \tag{73}$$

Indeed, one has

$$\begin{aligned}
\mathcal{A}(\tau) \;&\leq\; \hbar\sum_{\ell=1}^{n}\sum_{j=1}^{\underline{p}}\Big|\big\langle \Psi(\tau), \mathrm{Im}(\langle \lambda_{x_\ell}, \xi_{j,\tau}\rangle)\prod_{r\neq j} \phi_\hbar(\xi_{r,\tau})\Psi(\tau)\big\rangle\Big| \\
&\leq\; c\,\hbar\sum_{\ell=1}^{n}\sum_{j=1}^{\underline{p}} \|(H^{(n)}_{\hbar}+b)^{-\frac{p}{2}}\mathrm{Im}(\langle \lambda_{x_\ell}, \xi_{j,\tau}\rangle)\prod_{r\neq j} \phi_\hbar(\xi_{r,\tau})(H^{(n)}_{\hbar}+b)^{-\frac{p}{2}}\|\,.
\end{aligned}$$

Moreover, using an interpolation argument, for instance Hadamard's three lines theorem, one deduces from Proposition 3.5 (iii) the inequality:

$$\big\|(H^{(n)}_{\hbar}+b)^{-\frac{p}{2}}(H^{0,(n)}_{\hbar}+1)^{\frac{1}{2}}(N_2+1)^{\frac{p-1}{2}}\big\| \leq c\,.$$

Hence by standard number estimates one gets

$$\begin{aligned}
&\|(H^{(n)}_{\hbar}+b)^{-\frac{p}{2}}\mathrm{Im}(\langle \lambda_{x_\ell}, \xi_{j,\tau}\rangle)\prod_{r\neq j} \phi_\hbar(\xi_{r,\tau})(H^{(n)}_{\hbar}+b)^{-\frac{p}{2}}\| \\
&\leq C\|(H^{0,(n)}_{\hbar}+1)^{-1/2}(N_2+1)^{-\frac{p-1}{2}}\mathrm{Im}(\langle \lambda_{x_\ell}, \xi_{j,\tau}\rangle)\prod_{r\neq j} \phi_\hbar(\xi_{r,\tau})(H^{0,(n)}_{\hbar}+1)^{-1/2}(N_2+1)^{-\frac{p-1}{2}}\|\;,
\end{aligned}$$

and finally by Lemma 3.6, one concludes (for possibly different constants $C>0$):

$$\begin{aligned}
&\|(H_\hbar^{(n)}+b)^{-\frac{p}{2}}\mathrm{Im}(\langle\lambda_{x_\ell},\xi_{j,\tau}\rangle)\prod_{r\neq j}\phi_\hbar(\xi_{r,\tau})(H_\hbar^{(n)}+b)^{-\frac{p}{2}}\|\\
&\leq C\left\|\langle x_\ell\rangle^{-1-\nu}\langle\lambda_{x_\ell},\xi_{j,\tau}\rangle\right\|_{L^\infty}\|(N_2+1)^{-\frac{p-1}{2}}\prod_{r\neq j}\phi_\hbar(\xi_{r,\tau})(N_2+1)^{-\frac{p-1}{2}}\|\\
&\leq\frac{C}{\langle\tau\rangle^{1+\nu}}\,.
\end{aligned}$$

Thus, it is possible to exchange the limits $t\to\pm\infty$ and $k\to\infty$ in (72) for any $\xi_1,\dots,\xi_{\underline{p}}\in\mathscr{C}_0^\infty(\mathbb{R}^d\smallsetminus\{0\})$. This leads to

$$\lim_{k\to 0}\Big\langle\Psi(t),\prod_{j=1}^{\underline{p}}\phi_{\hbar_k}^{\pm}(\xi_j)\Psi(t)\Big\rangle=2^{\underline{p}}\int_{\mathfrak{X}}\prod_{j=1}^{\underline{p}}\mathrm{Re}\langle\xi_j,z\rangle_{\mathfrak{H}}\mathrm{d}\mu(u,z)+2^{\underline{p}}\sum_{j=1}^{\underline{p}}\int_0^{\pm\infty}\int_{\mathfrak{X}}\mathrm{Im}\Big\langle\xi_{j,\tau},\omega^{-1/2}\chi\,\widehat{|u|^2}\Big\rangle_{\mathfrak{H}}\prod_{r\neq j}^{\underline{p}}\mathrm{Re}\langle\xi_{r,\tau},z\rangle_{\mathfrak{H}}\mathrm{d}\mu_\tau(u,z)\mathrm{d}\tau\,.$$

Recalling the property (48) of $\Lambda^\pm$, we have that

$$\prod_{j=1}^{\underline{p}}\mathrm{Re}\langle\xi_j,\Lambda^\pm(u,z)\rangle_{\mathfrak{H}}=\prod_{j=1}^{\underline{p}}\mathrm{Re}\langle\xi_j,z\rangle_{\mathfrak{H}}+\sum_{j=1}^{\underline{p}}\int_0^{\pm\infty}\mathrm{Im}\Big\langle\xi_{j,\tau},\omega^{-1/2}\chi\,\widehat{|u(\tau)|^2}\Big\rangle_{\mathfrak{H}}\prod_{k\neq j}^{\underline{p}}\mathrm{Re}\langle\xi_{j,\tau},z(\tau)\rangle_{\mathfrak{H}}\mathrm{d}\tau\,,$$

where $(u(\tau),z(\tau))=\Phi_\tau(u,z)$ is the solution at time τ of the S-KG equations (45) with initial datum (u,z). A density argument concludes the proof. ⊣

As a consequence of the above result, one obtains the $\hbar$-limit of the transition amplitudes.

Corollary 5.7. *Under the same assumptions as in Proposition 5.6, for all integers $\underline{p}\in\mathbb{N}$, $0<\underline{p}\leq 2p-1$ and for all $\xi_1,\dots,\xi_{\underline{p}}\in\mathfrak{H}$:*

$$\lim_{k\to\infty}\langle\Psi_{\hbar_k}^{(n_k)},a_{\hbar_k}^{\pm,\natural}(\xi_1)\cdots a_{\hbar_k}^{\pm,\natural}(\xi_{\underline{p}})\Psi_{\hbar_k}^{(n_k)}\rangle=\int_{\mathfrak{X}}\prod_{j=1}^{\underline{p}}\langle\xi_j,\Lambda^\pm(u,z)\rangle_{\mathfrak{H}}^\natural\,\mathrm{d}\mu(u,z)\,,$$

where the signs are either all $+$ or all $-$, $a^{\pm,\natural}$ is either $a^\pm$ or $a^{\pm,*}$ and respectively $\langle\xi_j,\Lambda^\pm(\cdot)\rangle_{\mathfrak{H}}^\natural$ is $\langle\xi_j,\Lambda^\pm(\cdot)\rangle_{\mathfrak{H}}$ or $\overline{\langle\xi_j,\Lambda^\pm(\cdot)\rangle}_{\mathfrak{H}}$.

6. Semiclassical properties of asymptotic vacuum states

6.1. **Bound states.** In this section we study the semiclassical concentration properties of *bound states* (*i.e.*, states belonging to $\mathrm{Ran}\,\mathbb{1}_{\mathrm{pp}}(H_\hbar^{(n)})$). Recall that, as proved in [DG99] and already mentioned in (21), these bound states correspond exactly to the asymptotic vacuum states in $\mathscr{K}_\hbar^{\pm,(n)}$ defined in (20). Hence, we are able to prove the following semiclassical characterization.

Theorem 6.1. *Let $\{\Psi_{\hbar_k}^{(n_k)}\}_{k\in\mathbb{N}}$ be a sequence of normalized bound states satisfying* (A5). *Assume further that there exists $c>0$ such that:*

$$\forall k\in\mathbb{N},\qquad\langle\Psi_{\hbar_k}^{(n_k)},\,(H_{\hbar_k}^{(n_k)}+b)^{3/2}\;\Psi_{\hbar_k}^{(n_k)}\rangle\leq c\,.\tag{74}$$

Then its Wigner measure μ concentrates on the set

$$\mathscr{K}_0^+\cap\mathscr{K}_0^-\cap\{(u,z)\in\mathfrak{X}\,,\,\|u\|_{\mathfrak{H}}=\delta\}\,,$$

where we recall that the space of classical asymptotic radiationless states is defined by

$$\mathscr{K}_0^{\pm} = \{(u,z) \in \mathfrak{X} \mid \Lambda^{\pm}(u,z) = 0\} . \tag{75}$$

Proof. On one hand, by Corollary 5.7 (applied with $p = 3/2$) for all $\eta \in \mathfrak{H}$ if $\Psi_{\hbar_k}^{(n_k)} \rightharpoonup \mu$,

$$\lim_{k\to\infty} \langle \Psi_{\hbar_k}^{(n_k)}, a_{\hbar_k}^{\pm,*}(\eta) a_{\hbar_k}^{\pm}(\eta) \Psi_{\hbar_k}^{(n_k)} \rangle = \int_{\mathfrak{X}} \big| \langle \eta, \Lambda^{\pm}(u,z) \rangle_{\mathfrak{H}} \big|^2 \mathrm{d}\mu(u,z) .$$

On the other hand, since $\Psi_{\hbar_k}^{(n_k)}$ is an asymptotic vacuum,

$$\langle \Psi_{\hbar_k}^{(n_k)}, a_{\hbar_k}^{\pm,*}(\eta) a_{\hbar_k}^{\pm}(\eta) \Psi_{\hbar_k}^{(n_k)} \rangle = \| a_{\hbar_k}^{\pm}(\eta) \Psi_{\hbar_k}^{(n_k)} \|^2 = 0 .$$

Hence, for μ-a.a. $(u,z) \in \mathfrak{X}$, and for all $\eta \in \mathfrak{H}$,

$$\langle \eta, \Lambda^{\pm}(u,z) \rangle_{\mathfrak{H}} = 0 .$$

This proves the result, since as discussed in Proposition 5.6 the measure μ is concentrated on $\{(u,z) \in \mathfrak{X} \ , \ \|u\|_{\mathfrak{H}} = \delta\}$. ⊣

Corollary 6.2. *Let* $\{\Psi_{\hbar_k}^{(n_k)}\}_{k\in\mathbb{N}}$ *be a sequence of normalized bound states satisfying* (A5) *and the bound* (68) *for some* $p \geq 1$. *Then all asymptotic correlation functions are purely quantum:*

$$\langle \varphi_{\hbar}^{\pm}(x_1) \cdots \varphi_{\hbar}^{\pm}(x_p) \rangle_{\Psi_{\hbar_k}^{(n_k)}} = o_{\hbar}(1) .$$

6.2. **Ground states.** We study in this subsection the semiclassical behavior of *ground states*. It is well-known that the massive quantum Yukawa model under consideration, with trapped particles, has ground states by a HVZ type theorem [see, *e.g.*, DG99, Theorem 4.1]. Let us denote

$$E_\delta = \inf_{\substack{(u,z)\in\mathfrak{X} \\ \|u\|_{\mathfrak{H}}=\delta}} \mathscr{E}(u,z) . \tag{76}$$

Theorem 6.3. *Let* $\{\Phi_{\hbar_k}^{(n_k)}\}_{k\in\mathbb{N}}$ *be a sequence of ground states of* $H_{\hbar_k}^{(n_k)}$ *such that there exists* $\delta > 0$,

$$N_1 \Phi_{\hbar_k}^{(n_k)} = \hbar_k \, n_k \, \Phi_{\hbar_k}^{(n_k)} , \qquad \lim_{k\to\infty} \hbar_k = 0 , \qquad \lim_{k\to\infty} \hbar_k n_k = \delta^2 .$$

Then any Wigner measure $\mu \in \mathscr{M}\big(\Phi_{\hbar_k}^{(n_k)}; k \in \mathbb{N}\big)$ *concentrates on the set*

$$\{(u,z) \in \mathfrak{X} \ , \ \mathscr{E}(u,z) = E_\delta \ , \ \text{and} \ \|u\|_{\mathfrak{H}} = \delta\} .$$

In particular, it follows that the set $\{(u,z) \in \mathfrak{X} \ , \ \mathscr{E}(u,z) = E_\delta \ , \ \text{and} \ \|u\|_{\mathfrak{H}} = \delta\}$ *is not empty, and thus the variational problem* (76) *has minimizers for all* $\delta > 0$.

Proof. First of all, let us remark that $\mathscr{M}\big(\Phi_{\hbar_k}^{(n_k)}; k \in \mathbb{N}\big) \neq \varnothing$ since ground states satisfy (A4). In addition, thanks to the identity

$$\langle \Phi_{\hbar_k}^{(n_k)}, N_1 \Phi_{\hbar_k}^{(n_k)} \rangle = \hbar_k n_k \to \delta^2 ,$$

one deduces from [AF14, Lemma 5.6] that any Wigner measure $\mu \in \mathscr{M}\big(\Phi_{\hbar_k}^{(n_k)}; k \in \mathbb{N}\big)$ concentrates on the set

$$\mathfrak{X}_\delta := \{(u,z) : \|u\|_{\mathfrak{H}} = \delta\} .$$

Let now $E_{\hbar_k} = \langle \Phi_{\hbar_k}^{(n_k)}, H_{\hbar_k}^{(n_k)} \Phi_{\hbar_k}^{(n_k)} \rangle$. In [AF14, Theorem 1.2] it was proved that

$$\lim_{k\to\infty} E_{\hbar_k} = E_\delta .$$

In addition, an *a posteriori* argument as the one used in obtaining a lower bound ([AF14, Lemma 5.7]) shows that if one takes a subsequence of ground states $\{\Phi_{\hbar_k}^{(n_k)}\}_{k\in\mathbb{N}}$, which we still denote the same, such that $\Phi_{\hbar_k}^{(n_k)} \rightharpoonup \mu$, then

$$\int_{\mathfrak{X}_\delta} \mathscr{E}(u,z)\mathrm{d}\mu(u,z) \leq E_\delta \ .$$

This concludes the proof, since μ is a probability measure, and thus its action is that of a convex combination of energies $\mathscr{E}(u,z)$. ⊣

The existence of minimizers for (76) can also be proved by direct investigation of the functional $\mathscr{E}$, as showed by the next proposition. Beforehand, let us prove a preparatory result.

Lemma 6.4. *For any $(u,z)\in\mathfrak{X}$ with $\|u\|_{\mathfrak{H}}=\delta$, the following inequalities are satisfied:*

i) $\mathscr{E}_0(u,z) - 2\delta^2\|\omega^{-1/2}\chi\|_{\mathfrak{H}}\|z\|_{\mathfrak{H}} \leq \mathscr{E}(u,z) \leq \mathscr{E}_0(u,z) + 2\delta^2\|\omega^{-1/2}\chi\|_{\mathfrak{H}}\|z\|_{\mathfrak{H}}$,

ii)
$$\mathscr{E}(u,z) \geq \inf_{\substack{u\in\mathscr{D}(\sqrt{-\Delta+V}),\\ \|u\|_{\mathfrak{H}}=\delta}} \langle u,(-\Delta+V)u\rangle_{\mathfrak{H}} - \delta^4\|\omega^{-1}\chi\|_{\mathfrak{H}} \ .$$

Proof. i): Such inequalities are a consequence of Cauchy-Schwarz inequality:

$$\begin{aligned}\Big|2\mathrm{Re}\big\langle u(x),\langle\lambda_x(k),z(k)\rangle_{\mathfrak{H}_k}u(x)\big\rangle_{\mathfrak{H}_x}\Big| &\leq 2\int_{\mathbb{R}^{2d}}|u(x)|^2|\omega^{-1/2}(k)\chi(k)z(k)|\mathrm{d}x\mathrm{d}k\\ &\leq 2\|u\|_{\mathfrak{H}}^2\,\|\omega^{-1/2}\chi\|_{\mathfrak{H}}\,\|z\|_{\mathfrak{H}} = 2\delta^2\|\omega^{-1/2}\chi\|_{\mathfrak{H}}\,\|z\|_{\mathfrak{H}}.\end{aligned}$$

ii): It is convenient to complete the square in the expression of the energy functional

$$\begin{aligned}\langle z,\omega z\rangle_{\mathfrak{H}} + 2\mathrm{Re}\big\langle u,\langle\lambda_x,z\rangle_{\mathfrak{H}}u\big\rangle_{\mathfrak{H}} =&\int_{\mathbb{R}^{2d}}|u(x)|^2\Big(\big(\overline{z(k)}+\delta^2e^{-ik\cdot x}\tfrac{\bar\chi(k)}{\omega^{3/2}(k)}\big)\tfrac{\omega(k)}{\delta^2}\big(z(k)\\ &+\delta^2e^{ik\cdot x}\tfrac{\chi(k)}{\omega^{3/2}(k)}\big)\Big)\mathrm{d}x\mathrm{d}k - \delta^4\|\omega^{-1}\chi\|_{\mathfrak{H}}^2 \geq -\delta^4\|\omega^{-1}\chi\|_{\mathfrak{H}}^2 \ .\end{aligned}$$

Therefore

$$\mathscr{E}(u,z) \geq \langle u,(-\Delta+V)u\rangle_{\mathfrak{H}} - \delta^4\|\omega^{-1}\chi\|_{\mathfrak{H}}^2 \ . \tag{77}$$

⊣

Proposition 6.5. *For any $\delta>0$ there exists $(u_0,z_0)\in\{(u,z)\in\mathfrak{X}\ ,\ \|u\|_{\mathfrak{H}}=\delta\}$ such that*

$$\mathscr{E}(u_0,z_0) = \inf_{\substack{(u,z)\in\mathfrak{X}\\ \|u\|_{\mathfrak{H}}=\delta}} \mathscr{E}(u,z) \ . \tag{78}$$

Proof. The functional $\mathscr{E}$ is bounded from below, thanks to Lemma 6.4. Now consider a minimizing sequence (u_n,z_n):

$$\mathscr{E}(u_n,z_n) = E_\delta + \frac{1}{n} \ . \tag{79}$$

Since $\|u_n\|_{\mathfrak{H}}^2 = \delta$, there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ and $u_0\in\mathfrak{H}$ such that $u_{n_k}\rightharpoonup u_0$ converges weakly in $\mathfrak{H}$. By (77) it also exists $C>0$ such that

$$\langle u_{n_k},(-\Delta+V)u_{n_k}\rangle_{\mathfrak{H}} \leq C \ . \tag{80}$$

By lower semi-continuity of the induced norm, this yields

$$\langle u_0, (-\Delta + V)u_0\rangle_{\mathfrak{H}} \leq \liminf_{k\to+\infty} \langle u_{n_k}, (-\Delta + V)u_{n_k}\rangle_{\mathfrak{H}} \leq C \,. \tag{81}$$

Therefore, $u_0 \in \mathscr{D}(\sqrt{-\Delta + V})$. In addition, $\|(-\Delta + V)^{1/2}u_{n_k}\|^2_{\mathfrak{H}} \leq C$, thus there exists a subsubsequence $\{u_{n_{k_j}}\}_{j\in\mathbb{N}}$ and $v \in \mathfrak{H}$ such that

$$(-\Delta + V)^{1/2}u_{n_{k_j}} \rightharpoonup v \,. \tag{82}$$

Let now $\varphi \in \mathfrak{H}$:

$$0 = \lim_{j\to+\infty}\langle \varphi, u_{n_{k_j}} - u_0\rangle_{\mathfrak{H}} = \lim_{j\to+\infty}\langle (-\Delta + V)^{-1/2}\varphi, (-\Delta + V)^{1/2}(u_{n_{k_j}} - u_0)\rangle_{\mathfrak{H}} \,, \tag{83}$$

that implies $v = (-\Delta + V)^{1/2}u_0$. Since the potential V is confining and $(-\Delta + V)^{-1/2}$ is compact, one has

$$\mathfrak{H}\text{-}\lim_{j\to\infty} u_{n_{k_j}} = u_0 \,. \tag{84}$$

In addition, $\|u_0\|_{\mathfrak{H}} = \delta$.

Consider now the sequence $\{z_{n_{k_j}}\}_{j\in\mathbb{N}}$. By Lemma 6.4 together with the fact that for all $k \in \mathbb{R}^d$, $\omega(k) \geq m > 0$ we have that for all $n \in \mathbb{N}$:

$$C > \mathscr{E}(u_n, z_n) \geq \mathscr{E}_0(u_n, z_n) - 2\delta^2\|\omega^{-1/2}\chi\|_{\mathfrak{H}}\|z_n\|_{\mathfrak{H}} \geq m\|z_n\|^2_{\mathfrak{H}} - 2\delta^2\|\omega^{-1/2}\chi\|_{\mathfrak{H}}\|z_n\|_{\mathfrak{H}} \,.$$

Hence $\|z_{n_{k_j}}\|_{\mathfrak{H}}$ is bounded, so there exist a subsubsequence which we still denote by $\{z_{n_{k_j}}\}_{j\in\mathbb{N}}$ and a function $z_0 \in \mathfrak{H}$ such that $z_{n_{k_j}} \rightharpoonup z_0$. As a byproduct, lower semi-continuity for $\langle z, \omega z\rangle_{\mathfrak{H}}$ yields

$$\langle z_0, \omega z_0\rangle_{\mathfrak{H}} \leq \liminf_{j\to+\infty}\langle z_{n_{k_j}}, \omega z_{n_{k_j}}\rangle_{\mathfrak{H}} \,, \tag{85}$$

i.e., $z_0 \in \mathscr{D}(\sqrt{\omega})$. In addition, thanks to Fubini's theorem and weak convergence,

$$\lim_{j\to+\infty}\big\langle u_0, \langle z_{n_{k_j}}, \lambda_x\rangle_{\mathfrak{H}} u_0\big\rangle_{\mathfrak{H}} = \big\langle u_0, \langle z_0, \lambda_x\rangle_{\mathfrak{H}} u_0\big\rangle_{\mathfrak{H}} \,. \tag{86}$$

Furthermore,

$$\begin{aligned}\Big|\big\langle u_{n_{k_j}}, \mathrm{Re}\langle z_{n_{k_j}}, \lambda_x\rangle_{\mathfrak{H}} u_{n_{k_j}}\big\rangle_{\mathfrak{H}} - \big\langle u_0, \mathrm{Re}\langle z_0, \lambda_x\rangle_{\mathfrak{H}} u_0\big\rangle_{\mathfrak{H}}\Big| &\leq C\big\|\lambda_{(\cdot)}\big\|_{L^\infty(\mathbb{R}^d;\mathfrak{H})}\big\|u_{n_{k_j}} - u_0\big\|^2_{\mathfrak{H}} \\ &\quad + \big\|\langle z_{n_{k_j}} - z_0, \lambda_{(\cdot)}\rangle_{\mathfrak{H}}\big\|_{L^\infty_x}\big\|u_0\big\|^2_{\mathfrak{H}} \,,\end{aligned}$$

where the right hand side converges to zero thanks to the strong convergence of $u_{n_{k_j}}$ and (86). Thus,

$$\lim_{j\to+\infty}\big\langle u_{n_{k_j}}, \mathrm{Re}\langle z_{n_{k_j}}, \lambda_x\rangle_{\mathfrak{H}} u_{n_{k_j}}\big\rangle_{\mathfrak{H}} = \big\langle u_0, \mathrm{Re}\langle z_0, \lambda_x\rangle_{\mathfrak{H}} u_0\big\rangle_{\mathfrak{H}} \,. \tag{87}$$

Hence, we conclude the proof:

$$E_\delta \leq \mathscr{E}(u_0, z_0) \leq \liminf_{j\to+\infty}\mathscr{E}(u_{n_{k_j}}, z_{n_{k_j}}) = E_\delta + \liminf_{j\to+\infty}\frac{1}{n_{k_j}} = E_\delta \,. \tag{88}$$

$\dashv$

The next natural question is whether the minimizer (u_0, z_0) is unique, apart from the trivial $U(1)$ symmetry on the nucleon part. That amounts to ask whether the ground state is invariant with respect to the eventual symmetries of the external potential V. In order to investigate such question it is convenient to recast our minimization problem in the variables (u, z) in an equivalent way involving only the variable u. In order to do that, let us state (without proof) a preparatory lemma.

Lemma 6.6. *The energy functional $\mathscr{E} : \mathscr{X} \longrightarrow \mathbb{R}$ is continuous and Gâteaux differentiable.*

The derivative of $\mathscr{E}$ yields:

$$\begin{aligned}\frac{\partial}{\partial\alpha}\Big|_{\alpha=0}\mathscr{E}(u+\alpha v, z+\alpha y) = 2\mathrm{Re}\langle u, (-\Delta+V)v\rangle_{\mathfrak{H}} + 2\mathrm{Re}\langle z, \omega y\rangle_{\mathfrak{H}} + 2\mathrm{Re}\big\langle u, \langle y, \lambda_x\rangle_{\mathfrak{H}} u\big\rangle_{\mathfrak{H}} \\ +2\mathrm{Re}\big\langle v, 2\mathrm{Re}\langle \lambda_x, z\rangle_{\mathfrak{H}} u\big\rangle_{\mathfrak{H}} \ .\end{aligned} \tag{89}$$

The above is true for any variation $(v, y) \in \mathscr{X}$, and in particular for $v = 0$:

$$\frac{\partial}{\partial\alpha}\Big|_{\alpha=0}\mathscr{E}(u, z+\alpha y) = 2\mathrm{Re}\langle z, \omega y\rangle_{\mathfrak{H}} + 2\mathrm{Re}\big\langle u, \langle y, \lambda_x\rangle_{\mathfrak{H}} u\big\rangle_{\mathfrak{H}} \ . \tag{90}$$

Proposition 6.7. *If $(u_0, z_0) \in \mathscr{X}$ is a minimizer of the energy functional $\mathscr{E}(u, z)$ under the constraint $\|u_0\|_{\mathfrak{H}} = \delta$, then*

$$z_0(\,\cdot\,) = -\frac{\chi}{\omega^{3/2}}(-\,\cdot\,)\,\overline{\widehat{|u_0|^2}}(\,\cdot\,)\ . \tag{91}$$

Furthermore,

$$\mathscr{E}(u_0, z_0) = \inf_{\substack{(u,z)\in\mathscr{X}\\ \|u\|_{\mathfrak{H}}=\delta}} \mathscr{E}(u, z) = \inf_{\substack{u\in\mathscr{D}(\sqrt{-\Delta+V})\\ \|u\|_{\mathfrak{H}}=\delta}} \mathcal{E}(u) = \mathcal{E}(u_0)\ , \tag{92}$$

where $\mathcal{E}$ is the Hartree functional

$$\mathcal{E}(u) = \langle u, (-\Delta+V)u\rangle_{\mathfrak{H}} - \int_{\mathbb{R}^{2d}} |u(x)|^2\, W(x-y)\, |u(y)|^2 \mathrm{d}x\mathrm{d}y \tag{93}$$

with pair potential $W \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$ satisfying

$$W = \left(\widehat{\frac{\bar\chi(\,\cdot\,)\chi(-\,\cdot\,)}{\omega^2}}\right)\ .$$

Proof. According to (90), $(u_0, z_0) \in \mathscr{X}$ are critical points of the functional $\mathscr{E}(u, z)$ and have to satisfy, for all $y \in \mathscr{D}(\sqrt{\omega})$,

$$\mathrm{Re}\big\langle y, \omega z_0 + \langle u_0, \lambda_x u_0\rangle_{\mathfrak{H}_x}\big\rangle_{\mathfrak{H}} = 0\ . \tag{94}$$

This implies

$$z_0(k) = -\omega^{-1}(k)\langle u_0, \lambda_x(k) u_0\rangle_{\mathfrak{H}_x} = -\frac{\chi(-k)}{\omega^{3/2}(k)}\widehat{|u_0|^2}(-k) \in \mathscr{D}(\sqrt{\omega})\ ,$$

and proves (91). Computing $\mathscr{E}(u, z)$ with the constraint

$$z(\,\cdot\,) = -\frac{\chi}{\omega^{3/2}}(-\,\cdot\,)\,\widehat{|u|^2}(-\,\cdot\,)\ , \tag{95}$$

yields

$$\mathscr{E}(u, z) = \mathcal{E}(u)\ .$$

Hence, it follows that

$$\mathscr{E}(u_0, z_0) = \inf_{\substack{(u,z)\in\mathscr{X}\\ \|u\|_{\mathfrak{H}}=\delta}} \mathscr{E}(u, z) \le \inf_{\substack{u\in\mathscr{D}(\sqrt{-\Delta+V})\\ (95)\,,\,\|u\|_{\mathfrak{H}}=\delta}} \mathscr{E}(u, z) = \inf_{\substack{u\in\mathscr{D}(\sqrt{-\Delta+V})\\ \|u\|_{\mathfrak{H}}=\delta}} \mathcal{E}(u) \le \mathcal{E}(u_0)\ .$$

⊣

Lemma 6.8. *If $u_0 \in \mathscr{D}(\sqrt{-\Delta+V})$ is a minimizer of the Hartree energy (93),*

$$E_\delta = \inf_{\substack{u\in\mathscr{D}(\sqrt{-\Delta+V})\\ \|u\|_{\mathfrak{H}}=\delta}} \mathcal{E}(u)\ , \tag{96}$$

then $u_0 \in \mathscr{D}(-\Delta + V)$ *and there exists* $\lambda = \delta^{-2} E_\delta$ *such that*

$$(-\Delta + V)u_0 - W * |u_0|^2 u_0 = \lambda u_0 \,. \tag{97}$$

Proof. Follows by the generalized Lagrange multiplier theorem on Banach spaces, see [see *e.g.* LS74, Theorem 2 §8.11]. ⊣

The uniqueness of ground states for the Hartree functional $\mathcal{E}$ has been thoroughly investigated, and it is related to symmetry breaking phenomena [see AFG+02, CRDY07, CPRY12, Sei11, and references therein contained].

Lemma 6.9. *There exists* $\delta^* > 0$ *such that for any* $0 < \delta < \delta^*$*, the Hartree functional* $\mathcal{E}(\cdot)$ *given in* (93) *with the constraint* $\|\cdot\|_{\mathfrak{H}} = \delta$ *admits a unique minimizer up to* $U(1)$ *symmetry.*

Proof. The proof follows for instance by [AFG+02, Theorem 2] although the assumptions there are slightly different. So for completeness, we sketch the main argument. By rescaling, the original variational problem is put in a perturbative form,

$$\inf_{\substack{u\in\mathscr{D}(\sqrt{-\Delta+V})\\ \|u\|_{\mathfrak{H}}=\delta}} \mathcal{E}(u) = \delta^2 \inf_{\substack{u\in\mathscr{D}(\sqrt{-\Delta+V})\\ \|u\|_{\mathfrak{H}}=1}} \mathcal{E}_\delta(u)\,,$$

with

$$\mathcal{E}_\delta(u) \quad = \quad \langle u, (-\Delta + V)u\rangle_{\mathfrak{H}} - \delta^2 \int_{\mathbb{R}^{2d}} |u(x)|^2\, W(x-y)\, |u(y)|^2 \mathrm{d}x\mathrm{d}y\,.$$

Define, for $u \in \mathfrak{H}$, the self-adjoint operator

$$H_\delta^{(u)} = -\Delta + V - \delta^2 \left(W * |u|^2\right).$$

Thanks to the hypothesis (A1), the ground state energy of the Schrödinger operator $H_0^{(u)} = -\Delta + V$ is non-degenerate (see for instance [FS75, Theorem 1]). Moreover, for all $u \in \mathfrak{H}, \|u\|_{\mathfrak{H}} \leq 1$, $\delta \mapsto H_\delta^{(u)}$ is an analytic family of type A with a potential satisfying

$$\|W * |u|^2\|_{L^\infty} \leq \|W\|_{L^\infty}\,.$$

Hence, according to regular perturbation theory there exists $\delta^* > 0$ sufficiently small and independent from u such that the ground state energy of $H_\delta^{(u)}$ stays non-degenerate for all $0 < \delta < \delta^*$. So, let us denote by $P_\delta^{(u)}$ the one dimensional orthogonal projector on the ground state of $H_\delta^{(u)}$. Then using Lemma 6.8, one remarks that any minimizer of $\mathcal{E}_\delta(\cdot)$ is up to a phase factor a fixed point of the following map

$$\begin{aligned} T : \mathcal{S} = \{u \in \mathfrak{H}, \|u\|_{\mathfrak{H}} = 1\} \quad &\longrightarrow \quad \mathcal{S} \\ u \quad &\longrightarrow \quad \frac{P_\delta^{(u)}\psi_0}{\|P_\delta^{(u)}\psi_0\|_{\mathfrak{H}}}\,, \end{aligned}$$

where ψ_0 is a ground state of $H_0^{(u)}$. Again thanks to perturbation theory, one proves that T is a well defined (strict) contraction admitting therefore a unique fixed point. Combining this to the existence of minimizers given in Proposition 6.7, proves the claimed result.

⊣

Using Lemma 6.9, we can further characterize the Wigner measures of quantum ground states when the classical minimizer is unique.

Corollary 6.10 (of Theorem 6.3). *Consider* $\{\Phi_{\hbar_k}^{(n_k)}\}_{k\in\mathbb{N}}$ *to be a sequence of ground states of* $H_{\hbar_k}^{(n_k)}$ *such that there exists* $0<\delta<\delta^*$ *(with* δ^* *given in Lemma 6.9) such that:*

$$N_1\Phi_{\hbar_k}^{(n_k)} = \hbar_k\, n_k\, \Phi_{\hbar_k}^{(n_k)}\,, \qquad \lim_{k\to\infty}\hbar_k = 0\,, \qquad \textit{and} \qquad \lim_{k\to\infty}\hbar_k n_k = \delta^2\,. \tag{98}$$

In addition, let u_0 *be the unique minimizer modulo* $U(1)$*-invariance of* $\mathcal{E}(u)$ *with* $\|u_0\|_{\mathfrak{H}}=\delta$*. Then the corresponding Wigner measure is unique, and explicit:*

$$\mathscr{M}\big(\Phi_{\hbar_k}^{(n_k)}; k\in\mathbb{N}\big) = \{\mu\} = \Big\{\frac{1}{2\pi}\int_0^{2\pi} \delta_{e^{i\theta}u_0}\otimes\delta_{-\frac{\chi(-\,\cdot\,)}{\omega^{3/2}}\widehat{|u_0|^2}(\,\cdot\,)}\,\mathrm{d}\theta\Big\}\,.$$

Proof. According to Theorem 6.3 and Lemma 6.9, any Wigner measure μ of a sequence of ground states $\{\Phi_{\hbar_k}^{(n_k)}\}_{k\in\mathbb{N}}$ satisfying (98), concentrates on the set of minimizers

$$\mathfrak{X}_\delta^0 := \Big\{\big(e^{i\theta}u_0, -\frac{\chi(-\,\cdot\,)}{\omega^{3/2}}\widehat{|u_0|^2}(\,\cdot\,)\big), \theta\in[0,2\pi[\Big\}\,.$$

On the other hand, the probability measure μ is $U(1)$ invariant, *i.e.*, for all $\theta\in[0,2\pi[$

$$(e^{i\theta})_*\,\mu = \mu\,. \tag{99}$$

Such property is indeed a consequence of (12) and the following relation which holds true at least for a subsequence:

$$\begin{aligned}\widehat{(e^{i\theta})_*\,\mu}(\eta_1\oplus\eta_2) &= \lim_{k\to\infty}\langle e^{i\theta N_1}\Phi_{\hbar_k}^{(n_k)}, W(\eta_1\oplus\eta_2)e^{i\theta N_1}\Phi_{\hbar_k}^{(n_k)}\rangle\\ &= \lim_{k\to\infty}\langle \Phi_{\hbar_k}^{(n_k)}, W(\eta_1\oplus\eta_2)\Phi_{\hbar_k}^{(n_k)}\rangle\\ &= \widehat{\mu}(\eta_1\oplus\eta_2)\,,\end{aligned}$$

where here $\widehat{(e^{i\theta})_*\,\mu}$ and $\widehat{\mu}$ denote respectively the generating functions of $(e^{i\theta})_*\,\mu$ and μ. So, this implies that μ is translation-invariant with respect to the variable θ and hence μ is uniformly distributed over the circle $U(1)$. Therefore, one concludes

$$\mu = \frac{1}{2\pi}\int_0^{2\pi} \delta_{e^{i\theta}u_0}\otimes\delta_{-\frac{\chi(-\,\cdot\,)}{\omega^{3/2}}\widehat{|u_0|^2}(\,\cdot\,)}\,\mathrm{d}\theta\,.$$

⊣

7. Some open problems

We end up this paper by discussing some open problems, in the hope they will stimulate a renewed interest on such fundamental and mathematically challenging topic.

i) *Asymptotic completeness for the S-KG system:* The scattering theory for the S-KG equation (45) discussed in Section 4 is far from complete. In particular, it will be interesting to characterize the out/in classical asymptotic radiationless solutions in $\mathscr{K}_0^{\pm}$ and to prove the "weak" asymptotic completeness

$$\mathscr{K}_0^+ = \mathscr{K}_0^-\,,$$

or the "strong" asymptotic completeness

$$\mathscr{K}_0^+ = \{\text{nonlinear bound states or solitons}\} = \mathscr{K}_0^-\,.$$

Actually, in light of J. Dereziński and C. Gérard result [DG99] for quantum asymptotic completeness, such statements are expected to hold true.

ii) *Radiative decay:* Radiative decay is the fundamental physical mechanism that explains relaxation of excited atoms to their radiationless ground states. According to the work of M. Hubnert and H. Spohn [HS95], it can be stated roughly as the convergence

$$\lim_{t\to\infty}\langle e^{itH_\hbar^{(n)}}\Psi_\hbar^{(n)},\, A\, e^{itH_\hbar^{(n)}}\Psi_\hbar^{(n)}\rangle = \langle \Phi_\hbar^{(n)},\, A\, \Phi_\hbar^{(n)}\rangle\,, \tag{100}$$

where A and $\Psi_\hbar^{(n)}$ are respectively some given observables and states, and $\Phi_\hbar^{(n)}$ is the ground state. For the above identity to be true, one assumes further that the pure point spectrum space $\mathscr{H}_{pp}^{(n)} = \mathbb{1}_{\rm pp}(H_\hbar^{(n)})\mathscr{H}^{(n)}$ is one dimensional. Thus, a formal semiclassical limit in (100) suggests the identity

$$\lim_{t\to\infty}\int_{\mathscr{X}} a(z)\, d\mu_t(u,z) = \int_{\mathscr{X}} a(z)\, d\mu_0(u,z)\,, \tag{101}$$

where μ_0 is a measure concentrated on the classical ground state of the S-KG equation, given for instance by Corollary 6.10. Proving rigorously (101) will highlight a relaxation phenomena at the level of the nonlinear classical system.

iii) *Ground states convergence:* Convergence of the ground state energy of the Yukawa model towards the infimum of the classical energy of the S-KG system is discussed in § 6.2. It is interesting to extend such result with the concentration property in Theorem 6.3 to various models of particle-field interactions, which could carry the following major difficulties:

   a) *Infrared problems:* Ground states of models with massless fields are well studied [see, *e.g.*, Spo98, GGM04, HHS05]. The massless assumption amounts to consider $m = 0$ in (A2) and it induces some lack of compactness with respect to the field variables. In particular, the proof of convergence in [AF14] fails, and a different argument needs to be worked out.

   b) *Ultraviolet problems:* The removal of the ultraviolet cutoff (*i.e.*, taking $\chi \equiv 1$ in assumption (A3)) requires a renormalization procedure, due to the nucleons' self-energy divergence. Here as well the semiclassical convergence of the ground state energy needs to be worked out with different arguments [see, *e.g.*, Miy19].

   c) *Translation invariance problem:* In the absence of a confining potential V, the Yukawa model becomes translation-invariant and the quantum Hamiltonian as well as the classical energy can be fibered with respect to its total momentum. Therefore, it is natural to ask whether the ground state energy and ground states at fixed momentum converge, in the semiclassical limit, towards their classical counterparts.

iv) *Transition and scattering amplitudes:* Is it possible to derive semiclassical limits for transition and scattering amplitudes of various models (carrying in particular the infrared, ultraviolet or translation-invariance difficulties discussed in the previous point)? It is worth noting that the quantum scattering theory is studied for instance in

[Ara83, FGS01, GGM04, FS14] for massless fields, in [GMR11, DM15] for the translation invariant Nelson model, and in [Amm00] for the renormalized Nelson model without ultraviolet cutoff. Another model where such questions could be addressed, perhaps in a simpler way, is the so-called (massive or massless) spin-boson model [see, *e.g.*, BDFH20].

v) *Time decay and resolvent estimates:* By drawing further the parallel between semiclassical analysis of infinite and finite dimensional systems, we can speculate about the characterization of uniform time decay and resolvent estimates for the quantum Yukawa theory. In particular, it is quite interesting to prove the following uniform resolvent estimate for all $\hbar \in (0,1]$:

$$\|\langle A\rangle^{-s}(H_\hbar^{(n)} - \lambda \pm i0)^{-1}\langle A\rangle^{-s}\|_{\mathscr{B}(\mathscr{H}^{(n)})} \leq C_1(\hbar)\,, \tag{102}$$

where A is a given selfadjoint operator and the $\hbar$-dependence $C_1(\hbar)$ is to be quantitatively determined. Moreover, it is also interesting to derive the following uniform decay estimate for all $t \in \mathbb{R}$ and $\hbar \in (0,1]$:

$$\|\langle A\rangle^{-s}\,\chi(H_\hbar^{(n)})\,e^{it\hbar^{-1}H_\hbar^{(n)}}\langle A\rangle^{-s}\|_{\mathscr{B}(\mathscr{H}^{(n)})} \leq C_2(\hbar)\,\langle t\rangle^{-\varepsilon}\,, \tag{103}$$

where $\chi \in \mathscr{C}_0^\infty(\mathbb{R})$. The exponents $s, \varepsilon > 0$ as well as the $\hbar$-dependence $C_2(\hbar)$ need to be determined as well. Such estimates (102)-(103) are well known for the Schrödinger operator [see, *e.g.*, Wan88, Nak92, and references therein], and they are related to Mourre's theory, where A represents a conjugate or a position operator. Note that (102)-(103) are expected to hold when the cutoff function χ, respectively λ, are localized on radiating ("non trapping") energy levels of the S-KG equation. Indeed, the latter inequalities (102)-(103) can in principle be derived from a semiclassical positive commutator (Mourre) estimate

$$\chi(H_\hbar^{(n)})\,[H, iA]\,\chi(H_\hbar^{(n)}) \geq C(\hbar)\,\chi^2(H_\hbar^{(n)})\,, \quad \forall \hbar \in (0,1]\,,$$

and a semiclassical limiting absorption principle. Note that a Mourre estimate is proved in [DG99] for the ultraviolet-cutoff Nelson model with $\hbar = 1$, and a fixed number of nucleons n.

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