Traveling waves for a quasilinear wave equation

Gabriele Bruell [a], Piotr Idzik [b], Wolfgang Reichel [c],*

[a] *Centre for Mathematical Sciences, Lund University, 22100 Lund, Sweden*
[b] *PII Pipetronix GmbH, Lorenzstraße 10, 76297 Stutensee, Germany*
[c] *Institute for Analysis, Karlsruher Institute of Technology (KIT), D-76128 Karlsruhe, Germany*

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ABSTRACT

We consider a 2+1 dimensional wave equation appearing in the context of polarized waves for the nonlinear Maxwell equations. The equation is quasilinear in the time derivatives and involves two material functions V and Γ. We prove the existence of traveling waves which are periodic in the direction of propagation and localized in the direction orthogonal to the propagation direction. Depending on the nature of the nonlinearity coefficient Γ we distinguish between two cases: (a) $\Gamma \in L^\infty$ being regular and (b) $\Gamma = \gamma \delta_0$ being a multiple of the delta potential at zero. For both cases we use bifurcation theory to prove the existence of nontrivial small-amplitude solutions. One can regard our results as a persistence result which shows that guided modes known for linear wave-guide geometries survive in the presence of a nonlinear constitutive law. Our main theorems are derived under a set of conditions on the linear wave operator. They are subsidized by explicit examples for the coefficients V in front of the (linear) second time derivative for which our results hold.

1. Introduction

Of concern is the following 2+1 dimensional quasilinear wave equation

$$-\Delta u + \partial_t^2 \left(V(\lambda, y)u + \Gamma(y)u^3 \right) = 0, \tag{1.1}$$

which appears in the context of polarized waves for the nonlinear Maxwell equations. Here, $u = u(t, x, y)$ is the unknown depending on time $t \in \mathbb{R}_+$ and the two spatial variables x, y. We assume u to be periodic in x-direction and localized in y-direction. In what follows, we denote by $\mathbb{T}$ the one dimensional flat 2π-periodic torus, so that $(x, y) \in \mathbb{T} \times \mathbb{R}$. The potentials $V(\lambda, \cdot)$ and Γ depend only on y and incorporate material properties. Here, $\lambda \in \mathbb{R}$ is a parameter. The function Γ might be a bounded function (referred to as *regular* Γ) or a multiple of a delta potential at $y = 0$ (referred to as *distributional* Γ).

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* Corresponding author.
E-mail addresses: gabriele.brull@math.lth.se (G. Bruell), piotr.idzik@bakerhughes.com (P. Idzik), wolfgang.reichel@kit.edu (W. Reichel).

To motivate our interest in (1.1), let us explain how it appears in the context of electromagnetics. Recall that the Maxwell equations in the absence of charges and currents are given by

$$\nabla \cdot \mathbf{D} = 0, \qquad \nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \qquad \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}(\mathbf{E}),$$
$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{H} = \partial_t \mathbf{D}, \qquad \mathbf{B} = \mu_0 \mathbf{H}.$$

The modeling of the underlying material is done by making assumptions on the form of the polarization field $\mathbf{P}$. Here, we assume that $\mathbf{P}$ depends instantaneously on the electric field $\mathbf{E}$ as follows

$$\mathbf{P}(\mathbf{E}) = \varepsilon_0 \chi_1(\mathbf{x}) \mathbf{E} + \varepsilon_0 \chi_3(\mathbf{x}) |\mathbf{E}|^2 \mathbf{E}$$

with $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$, cf. [1, Chapter 2.3]. For simplicity we take χ_1, χ_3 as given scalar functions instead of the more general matrix/tensor structure of these quantities. The values ε_0, μ_0 are constant such that $c^2 = (\varepsilon_0 \mu_0)^{-1}$ and c is the speed of light in vacuum. By direct calculations from Maxwell's equations one obtains the second-order quasilinear wave-type equation for the electric field

$$0 = \nabla \times \nabla \times \mathbf{E} + \partial_t^2 \left(V(\mathbf{x}) \mathbf{E} + \Gamma(\mathbf{x}) |\mathbf{E}|^2 \mathbf{E} \right), \tag{1.2}$$

where $V(\mathbf{x}) = \mu_0 \varepsilon_0 (1 + \chi_1(\mathbf{x}))$ and $\Gamma(\mathbf{x}) = \mu_0 \varepsilon_0 \chi_3(\mathbf{x})$. The magnetic induction $\mathbf{B}$ can be retrieved from $\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$ by time-integration and it will satisfy $\nabla \cdot \mathbf{B} = 0$ provided it does so at time $t = 0$. By assumption the magnetic field is given by $\mathbf{H} = \frac{1}{\mu_0} \mathbf{B}$ and it satisfies $\nabla \times \mathbf{H} = \partial_t \mathbf{D}$. It remains to check that the displacement field $\mathbf{D}$ satisfies the Gauss law $\nabla \cdot \mathbf{D} = 0$ in the absence of external charges. This follows directly from the constitutive equation $\mathbf{D} = \varepsilon_0 (1 + \chi_1(\mathbf{x})) \mathbf{E} + \varepsilon_0 \chi_3(\mathbf{x}) |\mathbf{E}|^2 \mathbf{E}$ and the assumption of the polarized form of the electric field

$$\mathbf{E}(\mathbf{x}, t) = (0, 0, u(t, x, y))^T.$$

If we assume additionally that $V(\mathbf{x}) = V(y)$ and $\Gamma(\mathbf{x}) = \Gamma(y)$ then the quasilinear vectorial wave-type Eq. (1.2) turns into the scalar Eq. (1.1) for $u = u(t, x, y)$.

We study the existence of traveling wave solutions of (1.1) propagating in x-direction for certain classes of potentials V. More precisely, we consider potentials of the form

$$V(\lambda, y) = \lambda V_0(y) + V_1(y), \qquad \lambda \in \mathbb{R},$$

where $V_0 \in L^\infty(\mathbb{R})$ and V_1 is a distribution (e.g a δ-potential). If

$$u(t, x, y) = \Phi(x - t, y) \tag{1.3}$$

is a traveling wave solution of (1.1), propagating in x-direction with wave speed $v = 1$, then $\Phi : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ satisfies

$$-\Phi_{yy} - (1 - \lambda V_0(y) - V_1(y)) \Phi_{xx} + \Gamma(y)(\Phi^3)_{xx} = 0. \tag{1.4}$$

The parameter $\lambda \in \mathbb{R}$ will serve as a bifurcation parameter. One might ask why we introduced λ as a bifurcation parameter in the form $V(\lambda, y) = \lambda V_0(y) + V_1(y)$ and not in the more intuitive form $V(\lambda, y) = \lambda(V_0(y) + V_1(y))$ – the latter giving $\sqrt{\lambda}$ the meaning of the propagation speed of the traveling wave. The reason lies in the distributional character of V_1. The choice of the underlying function spaces for our analysis allows to formulate a (suitably defined, cf. Lemma 4.3) bounded inverse of $\tilde{L}_\lambda$, which is a slightly modified version of the wave operator $L_\lambda = -\partial_y^2 - (1 - \lambda V_0(y) - V_1(y)) \partial_x^2$. Extending the multiplication with λ also to the distributional part V_1 causes difficulties when differentiating wit respect to λ, cf. Remark 4.5.

Throughout the paper, a function u is called a traveling wave solution of (1.1) if u takes the form (1.3) and Φ is a solution of (1.4), which is periodic in its first and localized in its second component.

We investigate the existence of nontrivial solutions Φ_λ of (1.4) corresponding to the parameter λ under certain assumptions on the potential V; thereby providing the existence a nontrivial traveling wave solution u_λ of (1.1). We apply bifurcation theory for the parameter λ to find nontrivial solutions of (1.4). Our aim is to analyze the existence of nontrivial solutions in a way as general as possible, finding key properties of associate linear operators, which guarantee the existence of solutions via bifurcation theory. Eventually, we provide examples of specific potentials V for which these properties can be verified. In particular, the following cases are under consideration:

(P1) V **is a** $\delta-$**potential on a constant background**, that is V is a distribution of the form

$$V(\lambda, y) = \lambda + \alpha\delta_0(y).$$

(P2) V **is a** $\delta-$**potential on a step background**, that is V is a distribution of the form

$$V(\lambda, y) = \lambda\mathbf{1}_{|y|\geq b} + \beta\mathbf{1}_{|y|<b} + \alpha\delta_0(y).$$

Concerning the nonlinear potential Γ we distinguish between **regular** Γ, that is $\Gamma \in L^\infty(\mathbb{R})$, and **distributional** Γ, that is $\Gamma = \gamma\delta_0$. The main results for regular Γ are shown in Section 2.1 and for distributional Γ in Section 2.2.

Let us also comment on related work. Problem (1.1) has been considered in [17] where spatially localized traveling wave solutions of the 1+1-dimensional quasi-linear Maxwell model were investigated. The authors assume that $V(y)$ is a periodic arrangement of delta potentials. Using a multiple scale ansatz in fast and slow time, the field profile is expanded into infinitely many modes which are quasiperiodic in time (time-periodic both in the fast and slow time variables). Using local bifurcation methods the authors solve a related system which is homotopically linked to the Maxwell problem written as an infinite coupled system. It is not clear if the local bifurcation connects the related system and the Maxwell problem but numerical results support the existence of spatially localized traveling waves. In [7,9] another approximation (including error-estimates) of a version of (1.1) with periodic coefficients by finitely many coupled modes near band edges has been performed both analytically and numerically.

In the studies of the nonlinear Maxwell-system (1.2), often monochromatic waves $\mathbf{E}(\mathbf{x}) = U(\mathbf{x})e^{i\omega t} +$ *c.c.* are considered. Since a typical cubic nonlinearity generates higher harmonics, they either need to be neglected (leading to an error), or the constitutive equation for $\mathbf{D}$ is replaced by a time-averaged nonlinearity $\mathbf{D} = \varepsilon_0(1 + \chi_1(\mathbf{x}))\mathbf{E} + \varepsilon_0\chi_3(\mathbf{x})\frac{1}{T}\int_0^T |\mathbf{E}|^2 dt\mathbf{E}$, cf. [2,8,14,16,19–21] and particularly the two survey papers [3,15].

In contrast to the previously cited works, our approach is genuinely polychromatic and does not rely on time-averaged material laws. In our solution ansatz we allow for harmonics of arbitrary order and we treat (1.1) without any approximation. Recently in [12] a similar approach was taken and spatially localized, time-periodic solutions of (1.1) were obtained via variational methods. The result of the present paper and [12] are complementary in the following sense: (a) in [12] only distributional Γ is considered whereas in the present paper we also allow for regular $\Gamma \in L^\infty$; (b) in the case of distributional Γ, [12] only treats $V \in L^\infty$ whereas in the present paper we always have a delta potential contributing to V. Variational methods as in [12] have the advantage of producing solutions which may be far away from the trivial solution whereas local bifurcation methods as in the present paper produce solutions in the vicinity of zero. On the other hand, the local bifurcation method leads to more precise information about the actual shape of the bifurcating branch of solutions.

Outline of the paper. We close the introduction with a brief outline of the paper. In Section 2, we collect our main results. In Theorem 2.3 and Theorem 2.10 we provide a set of conditions on the linear wave operator guaranteeing the existence of nontrivial traveling wave solutions of (1.1) for regular and distributional Γ, respectively. Subsequentially, we present in Corollaries 2.5, 2.6, 2.13 and 2.14 particular examples in the

form of $(P1)$ and $(P2)$ for regular and distributional Γ. In Section 3 we fix some notation. The remaining Section 4–7 are devoted to the proofs of our main results. To be more precise, in Section 4 we prove the existence result in Theorem 2.3 for regular Γ followed by the proofs of Corollaries 2.5 and 2.6 in Section 5, which provide specific examples. Similarly, we prove in Section 6 the existence result in Theorem 2.10 for distributional Γ and finalize our studies in Section 7 with the proofs of Corollaries 2.13 and 2.14 on specific examples in the case of distributional Γ. In the appendix we collect auxiliary results.

2. Main results

We are looking for solutions of (1.4) of the form

$$\Phi(x, y; \lambda) = \sum_{k \in \mathbb{N}} \phi_k(y; \lambda) \sin(kx). \tag{2.1}$$

Our analysis is going to be divided into two parts separating the case when Γ is regular in the sense that $\Gamma \in L^\infty(\mathbb{R})$ and the case when Γ is distributional and takes the extreme form of a δ-potential. This is essentially due to the fact that in the former case we are concerned with a nonlinear equation on the domain $\mathbb{T} \times \mathbb{R}$, while in case of Γ being a δ-potential the problem can be viewed as a linear equation on $\mathbb{T} \times \mathbb{R} \setminus \{0\}$ – which can be solved separately – equipped with a nonlinear boundary condition at $x = 0$ induced by the delta potential.

2.1. Main result for regular Γ

Let us start with the definition of a weak solution of (1.4) in the case when $\Gamma \in L^\infty(\mathbb{R})$.

Definition 2.1 (*Weak Solution in the Case of Regular Γ*).
We say that $\Phi \in H^2(\mathbb{T}; L^2(\mathbb{R})) \cap H^1(\mathbb{T}; H^1(\mathbb{R}))$ is a *weak solution* of (1.4) if and only if

$$\int_\mathbb{T} \int_\mathbb{R} \Phi_y \Psi_y - (1 - \lambda V_0(y)) \Phi_{xx} \Psi \, dy \, dx - \int_\mathbb{T} \langle V_1(\cdot) \Phi_x(x, \cdot), \Psi_x(x, \cdot) \rangle \, dx + \int_\mathbb{T} \int_\mathbb{R} (\Phi^3)_{xx} \Psi \, dy \, dx = 0$$

for any $\Psi \in H^1(\mathbb{T}; H^1(\mathbb{R}))$. Here, $\langle \cdot, \cdot \rangle$ is the dual pairing between $H^{-1}(\mathbb{R})$ and $H^1(\mathbb{R})$.

Remark 2.2.

(a) We consider V_1 as a bounded linear operator from $H^1(\mathbb{R})$ into $H^{-1}(\mathbb{R})$. When $V_1 = \delta_0$ this means that for $f, g \in H^1(\mathbb{R})$ we have $\langle V_1 f, g \rangle = f(0)g(0)$, i.e., since $f \in C(\mathbb{R})$ it multiplies δ_0 and generates $f(0)\delta_0$ as a distribution acting on g.
(b) Clearly $\Phi_{xx} \in L^2(\mathbb{T} \times \mathbb{R})$. We shall see in Section 4 (cf. (4.8), (4.9) in Lemma 4.4) that also $\Phi \in L^\infty(\mathbb{T} \times \mathbb{R})$ and $\Phi_x \in L^4(\mathbb{T} \times \mathbb{R})$ so that $(\Phi^3)_{xx} = 3\Phi^2 \Phi_{xx} + 6\Phi \Phi_x^2 \in L^2(\mathbb{T} \times \mathbb{R})$.

If $\Gamma \in L^\infty(\mathbb{R})$, the ansatz in (2.1) allows us to reduce the problem of finding nontrivial solutions of (1.4) to studying spectral properties of the family of linear wave operators

$$L_\lambda^k := -\frac{d^2}{dy^2} + k^2(1 - \lambda V_0(y) - V_1(y)) \qquad \text{for} \quad k \in \mathbb{N}.$$

We prove the following theorem:

Theorem 2.3 (*Existence of Traveling Waves for Regular Γ*). *Assume that $\Gamma \in L^\infty(\mathbb{R})$, the potential V is given by*

$$V(\lambda, y) = \lambda V_0(y) + V_1(y)$$

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and

(L0) $V_0 \in L^\infty(\mathbb{R})$ and $V_1 : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})$ is bounded;

(L1) for every fixed $k \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ the operator $L_\lambda^k : D(L_\lambda^k) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is self-adjoint;

(L2) there exist a wavenumber $k_* \in \mathbb{N}$, a value $\lambda_* \in \mathbb{R}$, and an open interval $I_{\lambda_*} \subset \mathbb{R}$ containing λ_* such that zero is an isolated simple eigenvalue of $L_{\lambda_*}^{k_*}$ and $0 \in \rho(L_\lambda^k)$ for any $(k, \lambda) \in \mathbb{N} \times I_{\lambda_*}$ with $(k, \lambda) \neq (k_*, \lambda_*)$;

(L3) if $L_\lambda^k \phi = f$ for some $f \in L^2(\mathbb{R})$, then

$$\|\phi\|_{L^2(\mathbb{R})} \lesssim \frac{1}{k^2} \|f\|_{L^2(\mathbb{R})} \qquad and \qquad \|\phi'\|_{L^2(\mathbb{R})} \lesssim \frac{1}{k} \|f\|_{L^2(\mathbb{R})}$$

uniformly for $\lambda \in I_{\lambda_*}$ and $k \in \mathbb{N}$ sufficiently large.

If in addition V_0 satisfies the transversality condition

$$\langle V_0 \phi^*, \phi^* \rangle_{L^2(\mathbb{R})} \neq 0, \tag{2.2}$$

where ϕ^* spans the one-dimensional kernel of $L_{\lambda_*}^{k_*}$, then there exists $\varepsilon_0 > 0$ and a smooth curve through $(0, \lambda_*)$,

$$\{(\Phi(\varepsilon), \lambda(\varepsilon)) \mid |\varepsilon| < \varepsilon_0\} \subset (H^2(\mathbb{T}; L^2(\mathbb{R})) \cap H^1(\mathbb{T}; H^1(\mathbb{R}))) \times I_{\lambda_*},$$

of nontrivial solutions of (1.4) *with*

$$\Phi(0) = 0, \qquad\qquad D_\varepsilon \Phi(0)(x, y) = \phi^*(y) \sin(k_* x),$$

$$\lambda(0) = \lambda_*, \qquad\qquad \dot\lambda(0) = 0, \quad \ddot\lambda(0) = -\frac{3}{2} \frac{\int_\mathbb{R} \Gamma(y)(\phi^*)^4(y)\,dy}{\int_\mathbb{R} V_0(y)(\phi^*)^2(y)\,dy}.$$

Remark 2.4. The transversality condition (2.2) is trivially satisfied if $V_0 \geq 0, \not\equiv 0$ or $V_0 \leq 0, \not\equiv 0$.

In Section 4 we prove Theorem 2.3. There are two main requirements on L_λ^k providing the existence of nontrivial solutions via bifurcation theory: The first is that there exists a value $\lambda_* \in \mathbb{R}$ of the bifurcation parameter such that $L_{\lambda_*}^k$ has a one-dimensional kernel if and only if $k = k_*$ for some wave number $k_* \in \mathbb{N}$ (see (L2)); this is a necessary bifurcation condition. Secondly, we demand that for any $k \neq k_*$ the self-adjoint operator L_λ^k has a spectral gap $(-ck^2, ck^2)$ around zero, which ensures the decay properties of $\phi_k(\cdot; \lambda)$ (see (L3)). Eventually, after we have established Theorem 2.3, we turn to the specific case, when $\Gamma \in L^\infty(\mathbb{R})$ is regular, V are potentials of the form as in (P1) and (P2) and formulate tangible assumptions on the triple (k_*, λ_*, α) (see (2.3) and (2.4)), which guarantee that conditions $(L0) - (L3)$ of Theorem 2.3 are satisfied; thereby proving the existence of nontrivial traveling wave solutions of (1.1). In particular, we prove the following corollaries.

Corollary 2.5 (*Case P1, Regular Γ*). *Let $\Gamma \in L^\infty(\mathbb{R})$ and $V(\lambda, y) = \lambda + \alpha \delta_0(y)$. If $k_* \in \mathbb{N}$ and $\lambda_* < 1$ are given and $\alpha > 0$ is determined from*

$$\alpha = \frac{2\sqrt{1 - \lambda_*}}{k_*}, \tag{2.3}$$

then the assumptions in Theorem 2.3 are satisfied with

$$\phi^*(y) = \sqrt{k_* \sqrt{1 - \lambda_*}} e^{-k_* \sqrt{1 - \lambda_*}|y|}$$

and

$$\Phi(0) = 0, \qquad\qquad D_\varepsilon \Phi(0)(x, y) = \phi^*(y) \sin(k_* x),$$

$$\lambda(0) = \lambda_*, \qquad\qquad \dot\lambda(0) = 0, \quad \ddot\lambda(0) = -\frac{3}{2} k_*^2 (1 - \lambda_*) \int_\mathbb{R} \Gamma(y) e^{-4k_* \sqrt{1 - \lambda_*}|y|}\,dy.$$

Corollary 2.6 (*Case P2, Regular Γ*). *Let $\Gamma \in L^\infty(\mathbb{R})$ and $V(\lambda, y) = \lambda \mathbf{1}_{|y| \geq b} + \beta \mathbf{1}_{|y| < b} + \alpha \delta_0(y)$. If $k_* \in \mathbb{N}$, $b > 0$ and $\beta, \lambda_* < 1$ are given and $\alpha > 0$ is determined from*

$$\alpha = \frac{2\sqrt{1-\beta}}{k^*} \cdot \frac{\sqrt{1-\beta}\sinh(k_*\sqrt{1-\beta}b) + \sqrt{1-\lambda_*}\cosh(k_*\sqrt{1-\beta}b)}{\sqrt{1-\beta}\cosh(k_*\sqrt{1-\beta}b) + \sqrt{1-\lambda_*}\sinh(k_*\sqrt{1-\beta}b)}, \tag{2.4}$$

then the assumptions in Theorem 2.3 are satisfied.

Remark 2.7. Details on the construction of ϕ^* in Corollary 2.6 can be taken from Section 5.2.

2.2. Main result for distributional Γ

Again, we start with the definition of a weak solution of (1.4), but now in the case when Γ is given by a δ-potential. We assume that the function $V_1 = W + \alpha \delta_0$ splits into a regular part W and the distributional part $\alpha \delta_0$ so that $V = \lambda V_0 + W + \alpha \delta_0$, where $V_0, W \in L^\infty(\mathbb{R})$ and $\Gamma = \gamma \delta_0$.

Definition 2.8 (*Weak Solution in the Case of Distributional Γ*). We say that $\Phi \in H^2(\mathbb{T}; L^2(\mathbb{R})) \cap H^1(\mathbb{T}; H^1(\mathbb{R})) \cap C(\mathbb{R}; H^2(\mathbb{T}))$ is a *weak solution* of (1.4) if and only if

$$\int_\mathbb{T} \int_\mathbb{R} \Phi_y \Psi_y - (1 - \lambda V_0(y) - W(y)) \, \Phi_{xx} \Psi \, dy \, dx + \int_\mathbb{T} \left(\alpha \Phi_{xx}(x, 0) + \gamma (\Phi^3)_{xx}(x, 0) \right) \Psi(x, 0) \, dx = 0$$

for any $\Psi \in H^1(\mathbb{T}; H^1(\mathbb{R}))$.

Remark 2.9. Clearly $\Phi_{xx}(\cdot, 0) \in L^2(\mathbb{T})$ and $\Phi(\cdot, 0), \Phi_x(\cdot, 0) \in L^\infty(\mathbb{T})$ so that $(\Phi^3(\cdot, 0))_{xx} = 3\Phi(\cdot, 0)^2 \Phi(\cdot, 0)_{xx} + 6\Phi(\cdot, 0)\Phi(\cdot, 0)_x^2 \in L^2(\mathbb{T})$. Moreover, $\Psi \in H^1(\mathbb{T}; H^1(\mathbb{R})) \subset H^1(\mathbb{T} \times \mathbb{R})$ has an L^2-trace at $y = 0$.

Note that (1.4) can be written as a linear partial differential equation on $\mathbb{T} \times \mathbb{R} \setminus \{0\}$ equipped with a nonlinear boundary condition on $\mathbb{T}$:

$$\begin{cases} -\Phi_{yy} - (1 - \lambda V_0 - W)\Phi_{xx} = 0, & (x, y) \in \mathbb{T} \times \mathbb{R} \setminus \{0\}, \\ \Phi_y(x, 0_+) - \Phi_y(x, 0_-) = \partial_x^2 \left(\alpha \Phi + \gamma \Phi^3 \right)(x, 0), & x \in \mathbb{T}. \end{cases} \tag{2.5}$$

As before let

$$L_k^\lambda := -\frac{d^2}{dy^2} + k^2(1 - \lambda V_0(y) - W(y) - \alpha \delta_0(y))$$

be a family of linear wave operators and set

$$L_{0,k}^\lambda := -\frac{d^2}{dy^2} + k^2(1 - \lambda V_0(y) - W(y))$$

to be the regular part of L_k^λ. We prove the following theorem:

Theorem 2.10 (*Existence of Traveling Wave for Distributional Γ*). *Assume that $\Gamma = \gamma \delta_0$, the potential V is given by*

$$V(\lambda, y) = \lambda V_0(y) + W(y) + \alpha \delta_0(y),$$

and

$(\tilde{L}0)$ *$V_0, W \in L^\infty(\mathbb{R})$ are even;*
$(\tilde{L}1)$ *there exists an interval $I \subset \mathbb{R}$ such that for every fixed $k \in \mathbb{N}$ and $\lambda \in I$ the operator $L_{0,\lambda}^k : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ satisfies $0 \in \rho(L_{0,\lambda}^k)$;*

($\tilde{L}2$) *there exist a wavenumber $k_* \in \mathbb{N}$, a value $\lambda_* \in \mathbb{R}$, and an open interval $I_{\lambda_*} \subset I \subset \mathbb{R}$ containing λ_* such that zero is an isolated simple eigenvalue of $L_{\lambda_*}^{k_*}$ and $0 \in \rho(L_\lambda^k)$ for any $(k, \lambda) \in \mathbb{N} \times I_{\lambda_*}$ with $(k, \lambda) \neq (k_*, \lambda_*)$;*

($\tilde{L}3$) *there exist $C > 0$ such that $\|\phi_k(\cdot; \lambda)\|_{L^2(0,\infty)} \leq C$ uniformly for $\lambda \in I_{\lambda_*}$, $k \in \mathbb{N}$, and where $\phi_k \in H^2(0, \infty)$ satisfies[1]*

$$L_{0,k}^\lambda \phi_k(y; \lambda) = 0 \quad on \quad (0, \infty) \quad with \quad \phi_k(0; \lambda) = 1.$$

If in addition V_0 satisfies the transversality condition

$$\langle V_0 \phi^*, \phi^* \rangle_{L^2(\mathbb{R})} \neq 0, \tag{2.6}$$

where ϕ^* spans the one-dimensional kernel of $L_{\lambda_*}^{k_*}$, then there exists $\varepsilon_0 > 0$ and a smooth curve through $(0, \lambda_*)$

$$\{(\Phi(\varepsilon), \lambda(\varepsilon)) \mid |\varepsilon| < \varepsilon_0\} \subset (H^2(\mathbb{T}; L^2(\mathbb{R})) \cap H^1(\mathbb{T}; H^1(\mathbb{R})) \cap \{\Phi(\cdot, 0) \in H^2(\mathbb{T})\}) \times I_{\lambda_*}$$

of nontrivial solutions of (1.4) *with*

$$\Phi(0) = 0, \qquad\qquad D_\varepsilon \Phi(0)(x, y) = \phi^*(y) \sin(k_* x),$$

$$\lambda(0) = \lambda_*, \qquad\qquad \dot{\lambda}(0) = 0, \quad \ddot{\lambda}(0) = -\frac{3\gamma}{4 \int_{\mathbb{R}} V_0(y)(\phi^*)^2(y)\, dy}.$$

Remark 2.11. The transversality condition (2.6) is trivially satisfied if $V_0 \geq 0, \not\equiv 0$ or $V_0 \leq 0, \not\equiv 0$.

Remark 2.12. We can formulate ($\tilde{L}2$) entirely in terms of the ansatz-functions $\phi_k(\cdot; \lambda)$ defined in ($\tilde{L}3$). To this end notice that $\sigma_{ess}(L_{0,\lambda}^k) = \sigma_{ess}(L_\lambda^k)$, cf. Lemma A.5. Since $0 \notin \sigma(L_{0,\lambda}^k)$ by assumption ($\tilde{L}1$) it is clear that $0 \in \sigma(L_\lambda^k)$ is characterized by zero being an eigenvalue of L_λ^k. This, however, in combination with the evenness of V_0, W, means that $\phi_k(|y|; \lambda)$ is up to scalar multiples the unique candidate for the eigenfunction and has to satisfy

$$2\phi_k'(0_+; \lambda) + k^2\alpha = 0.$$

Here $2\phi_k'(0_+; \lambda)$ is the jump of the first derivative of the even function $\phi_k(|y|; \lambda)$ at $y = 0$. Based on this characterization of zero belonging to the spectrum of L_k^λ we can replace ($\tilde{L}2$) by

($\bar{L}2$) *there exists a wavenumber $k_* \in \mathbb{N}$, $\lambda_* > 0$, and an open interval $I_{\lambda_*} \subset \mathbb{R}_+$ containing λ_* such that*

$$2\phi_k'(0_+; \lambda) + k^2\alpha = 0$$

if and only if $(k, \lambda) = (k_*, \lambda_*)$ for any $k \in \mathbb{N}$ and $\lambda \in I_{\lambda_*}$.

In Section 6 we prove Theorem 2.10. Claiming periodicity in one spatial direction and evenness in the transverse direction we make a Fourier ansatz of the form

$$\Phi(x, y; \lambda) = \sum_{k \in \mathbb{N}} a_k(\lambda)\phi_k(|y|; \lambda)\sin(kx), \tag{2.7}$$

where ϕ_k are the decaying functions from assumption ($\tilde{L}3$). In particular, the ansatz (2.7) solves the linear differential equation in (2.5) and thus reduces the problem of finding nontrivial solutions of (1.4) to the following family of *algebraic* equations

$$2a_k(\lambda)\phi_k'(0_+; \lambda) = -k^2\left(\alpha a_k(\lambda) - \frac{1}{4}\gamma\big(a(\lambda) * a(\lambda) * a(\lambda)\big)_k\right) \qquad for\ all \quad k \in \mathbb{N}. \tag{2.8}$$

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[1] The existence and the properties of the functions ϕ_k are detailed in Remark 6.1.

In the spirit of Section 4, we show that if conditions $(\tilde{L}0) - (\tilde{L}3)$ are satisfied, then the nonlinear Eq. (2.8) has a non-trivial solution $(a_k(\lambda))_{k\in\mathbb{Z}}$ with $a_k(\lambda) = -a_{-k}(\lambda)$ and the decay property

$$(k^2 a_k(\lambda))_k \in l^2(\mathbb{R});$$

thereby providing a solution of (1.4) in the form of (2.7). Here, $l^2(\mathbb{R})$ denotes the space of l^2–summable sequences in $\mathbb{R}^{\mathbb{Z}}$. Condition $(\tilde{L}1)$ guarantees the existence of the family of even ansatz-functions $(\phi_k)_{k\in\mathbb{N}}$ with $\phi_k(0;\lambda) = 1$. Condition $(\tilde{L}2)$ assures that there exists $k_* \in \mathbb{N}$, $\lambda_* \in \mathbb{R}$ and an interval $I_{\lambda_*} \subset \mathbb{R}$ including λ_* such that the linearization of (2.8), given by the multiplication operator

$$A_\lambda^k := 2\phi_k'(0_+;\lambda) + k^2\alpha,$$

has a one-dimensional kernel if $(k,\lambda) = (k_*, \lambda_*)$, that is $A_{\lambda_*}^{k*} = 0$; and $A_\lambda^k \neq 0$ for all $(k,\lambda) \in \mathbb{N} \times I_{\lambda_*}$ with $(k,\lambda) \neq (k_*, \lambda_*)$. This is a necessary bifurcation condition. After we have proved Theorem 2.10, we investigate the specific cases, when V is a potential of the form as in (P1) and (P2). The former being a δ-potential on a constant background, while the latter is a δ-potential on the background of a step function. In both cases the δ-potential part in V is essential, guaranteeing sufficient decay properties of the sequence $(a_k(\lambda))_{k\in\mathbb{N}}$. In particular, we prove the following corollaries:

Corollary 2.13 (*Case P1, Distributional Γ*). *Assume that $\Gamma = \gamma\delta_0$ and $V(\lambda, y) = \lambda + \alpha\delta_0(y)$. If $k_* \in \mathbb{N}$ and $\lambda_* < 1$ are given and $\alpha > 0$ is determined from*

$$\alpha = \frac{2\sqrt{1-\lambda_*}}{k_*}$$

then the assumptions in Theorem 2.10 are satisfied with

$$\phi^*(y) = e^{-k_*\sqrt{1-\lambda_*}|y|}$$

and

$$\Phi(0) = 0, \qquad\qquad D_\varepsilon\Phi(0)(x,y) = e^{-k_*\sqrt{1-\lambda_*}|y|}\sin(k_*x),$$
$$\lambda(0) = \lambda_*, \qquad\qquad \dot{\lambda}(0) = 0, \quad \ddot{\lambda}(0) = -\gamma k_*\sqrt{1-\lambda_*}.$$

Moreover, the solutions $\Phi(\varepsilon)$ take the form

$$\Phi(\varepsilon)(x,y) = \sum_{k\in\mathbb{N}} a_k(\varepsilon)e^{-k\sqrt{1-\lambda}|y|}\sin(kx).$$

Corollary 2.14 (*Case P2, Distributional Γ*). *Let $\Gamma \in L^\infty(\mathbb{R})$ and $V(\lambda, y) = \lambda\mathbf{1}_{|y|\geq b} + \beta\mathbf{1}_{|y|<b} + \alpha\delta_0(y)$. Suppose furthermore that $k_* \in \mathbb{N}$ and $\lambda_* < 1$ are given. If*

- *(Case $\beta > 1$) $b, \alpha > 0$ are determined from*

$$b = \frac{\pi}{\sqrt{\beta - 1}}, \quad \alpha = \frac{2\sqrt{1-\lambda_*}}{k_*}$$

- *(Case $\beta < 1$) $b > 0$ is given and $\alpha > 0$ is determined from*

$$\alpha = \frac{2\sqrt{1-\beta}}{k^*} \cdot \frac{\sqrt{1-\beta}\sinh(k_*\sqrt{1-\beta}b) + \sqrt{1-\lambda_*}\cosh(k_*\sqrt{1-\beta}b)}{\sqrt{1-\beta}\cosh(k_*\sqrt{1-\beta}b) + \sqrt{1-\lambda_*}\sinh(k_*\sqrt{1-\beta}b)}$$

- *(Case $\beta = 1$) $b > 0$ is given and $\alpha > 0$ is determined from*

$$\alpha = \frac{2\sqrt{1-\lambda_*}}{k_*(1 + \sqrt{1-\lambda_*}k_*b)}$$

then in all three cases the assumptions in Theorem 2.10 are satisfied.

Remark 2.15. Details on the construction of ϕ^* in Corollary 2.14, the functions ϕ_k and the form of the solutions $\Phi(\varepsilon)(x,y) = \sum_{k\in\mathbb{N}} a_k(\varepsilon)\phi_k(y;\lambda)\sin(kx)$ can be taken from Section 7.2.

3. Notation

If f and g are elements in an ordered Banach space with $f, g \geq 0$, we write $f \lesssim g$ ($f \gtrsim g$) if there exists a constant $c > 0$ such that $f \leq cg$ ($f \geq cg$). Moreover, the notation $f \asymp g$ is used whenever $f \lesssim g$ and $f \gtrsim g$. We write $c = c(p_1, p_2, \ldots) > 0$ if we want to emphasize that the constant $c > 0$ depends on the parameters $p_1, p_2, \ldots$. In Section 6 we are looking for solutions of an infinite dimensional system of nonlinear algebraic equation. We consider solutions in the sequence spaces related to

$$l^2(\mathbb{R}) := \left\{ a = (a_k)_{k\in\mathbb{Z}} \mid a_k \in \mathbb{R} \text{ for all } k \in \mathbb{Z} \text{ and } \|a\|^2_{l^2(\mathbb{R})} := \sum_{k\in\mathbb{Z}} |a_k|^2 < \infty \right\}.$$

Eventually, for any $r \in \mathbb{R}$ we set

$$h^r(\mathbb{R}) := \left\{ a \in l^2(\mathbb{R}) \mid \left((1 + |k|)^r a_k \right)_{k\in\mathbb{Z}} \in l^2(\mathbb{R}) \right\}.$$

and equip the space $h^r(\mathbb{R})$ with the norm

$$\|a\|^2_{h^r(\mathbb{R})} := \sum_{k\in\mathbb{Z}} (1 + |k|)^{2r} |a_k|^2.$$

We also consider the subspaces

$$l^2_\sharp(\mathbb{R}) := \{ a \in l^2(\mathbb{R}) : a_{-k} = -a_k \text{ for } k \in \mathbb{Z} \},$$
$$h^r_\sharp(\mathbb{R}) := h^r(\mathbb{R}) \cap l^2_\sharp(\mathbb{R}).$$

Throughout the paper we use the notation $\langle \cdot, \cdot \rangle_H$ to denote the dual pairing in the Hilbert space H. If $f, g \in L^2(U)$ are real-valued functions, where $U \subset \mathbb{R}^n$ is a domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, then

$$\langle f, g \rangle_{L^2(U)} := \int_U f(z) g(z) \, dz$$

and if $a, b \in l^2(\mathbb{R})$ then

$$\langle a, b \rangle_{l^2(\mathbb{R})} := \sum_{k\in\mathbb{Z}} a_k b_k.$$

If $L : D(L) \subset H \to H$ is a linear operator with domain $D(L)$, we denote by

$$\rho(L) := \{ \lambda \in \mathbb{C} \mid \lambda - L : D(L) \to H \text{ has a bounded inverse} \}$$

the *resolvent set* of L. The *spectrum* of L is given by $\mathbb{C} \setminus \rho(L)$. If L is self-adjoint, then $\sigma(L) \subset \mathbb{R}$ and the spectrum of L can be decomposed as a disjoint union

$$\sigma(L) = \sigma_{ess}(L) \cup \sigma_d(L),$$

where σ_d is the *discrete spectrum* of L consisting of isolated eigenvalues of $\sigma(L)$ of finite multiplicity and $\sigma_{ess}(L) = \sigma(L) \setminus \sigma_d(L)$ is the *essential spectrum*.

4. Existence of traveling waves for bounded potentials Γ

This section is devoted to the proof of Theorem 2.3. Subsequently, we affirm in Section 5 that the conditions are fulfilled for special cases where V takes the form in (P1) or (P2), thereby proving Corollaries 2.5 and 2.6. In the following we restrict ourself to solutions Φ of (1.4) having the form

$$\Phi(x, y; \lambda) = \sum_{k\in\mathbb{N}} \phi_k(y; \lambda) \sin(kx),$$

where the y-dependent Fourier coefficients $\phi := (\phi_k)_{k\in\mathbb{N}}$ are decaying at infinity (suitable function spaces are formulated later). Then, Φ is a solution of

$$- \Phi_{yy} - (1 - \lambda V_0(y) - V_1(y))\, \Phi_{xx} + \Gamma(y)(\Phi^3)_{xx} = 0 \tag{4.1}$$

if and only if

$$-\phi_k'' + k^2(1 - \lambda V_0(y) - V_1(y))\phi_k + \frac{1}{4}k^2\Gamma(y)\,(\phi * \phi * \phi)_k = 0 \qquad \text{for all} \quad k \in \mathbb{N}.$$

Note that $-\frac{1}{4}\,(\phi * \phi * \phi)$ is the kth Fourier coefficient of Φ^3, cf. Lemma A.1. The Fourier ansatz with respect to x decomposes the operator

$$L_\lambda := -\partial_y^2 - (1 - \lambda V_0(y) - V_1(y))\partial_x^2 \tag{4.2}$$

into the sequence of Schrödinger operators

$$L_\lambda^k := -\frac{d^2}{dy^2} + k^2(1 - \lambda V_0(y) - V_1(y)).$$

Recall that we are working under the assumptions $(L0)$–$(L3)$ from Theorem 2.3.

Remark 4.1.

(i) Notice that a necessary condition for $(L3)$ to hold is that the operator L_λ^k satisfies the spectral gap property

$$(-ck^2, ck^2) \subset \rho(L_\lambda^k) \qquad \text{for some constant} \quad c > 0$$

uniformly in $\lambda \in I_{\lambda_*}$ and $k \in \mathbb{N}$ sufficiently large.

(ii) The domain of L_λ^k is a subset of $H^1(\mathbb{R})$, which is the domain of the quadratic form of L_λ^k. As a vector space, it does not depend on λ. However, the graph norm on $D(L_\lambda^k)$ is λ-dependent and the embedding $D(L_\lambda^k) \subset H^1(\mathbb{R})$ is locally uniformly bounded with respect to λ.

The next lemma extends property $(L3)$ to all values of $k \in \mathbb{N}$ by adding a projection to L_λ^k for $k = k_*$. For this purpose let $\ker L_{\lambda_*}^{k_*} = \mathrm{span}\{\phi^*\}$ with $\|\phi^*\|_{L^2(\mathbb{R})} = 1$. Denote by P^{k_*} the projection mapping

$$P^{k_*}\phi := \langle \phi, \phi^* \rangle_{L^2(\mathbb{R})}\phi^* \qquad \text{for any} \quad \phi \in L^2(\mathbb{R})$$

and define by $\tilde{L}_\lambda^k : D(L_\lambda^k) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ for $k \in \mathbb{N}$ and $\lambda \in I_{\lambda_*}$ the family of operators

$$\tilde{L}_\lambda^k = \begin{cases} L_\lambda^{k_*} + P^{k_*} & \text{if} \quad k = k_*, \\ L_\lambda^k & \text{if} \quad k \neq k_*. \end{cases} \tag{4.3}$$

Lemma 4.2. Let $\tilde{L}_\lambda^k\phi = g$ for some $g \in L^2(\mathbb{R})$. Then, by possibly shrinking the interval I_{λ_*}, we have that

$$\|\phi\|_{L^2(\mathbb{R})} \lesssim \frac{1}{k^2}\|g\|_{L^2(\mathbb{R})} \qquad and \qquad \|\phi'\|_{L^2(\mathbb{R})} \lesssim \frac{1}{k}\|g\|_{L^2(\mathbb{R})} \tag{4.4}$$

for all $k \in \mathbb{N}$ uniformly in $\lambda \in I_{\lambda_*}$.

Proof. By Theorem VIII.25 in [18] it follows that the map $\lambda \mapsto L_\lambda^k$ is norm-resolvent continuous, that is $\lambda \mapsto (L_\lambda^k - \mathrm{i})^{-1} \in \mathcal{L}(L^2(\mathbb{R}))$ is continuous with respect to the operator norm. Let us verify that also $\tilde{L}_\lambda^{k_*}$ is norm-resolvent convergent to $\tilde{L}_{\lambda_*}^{k_*}$ as $\lambda \to \lambda_*$. Note that $\mathrm{Id} + P^{k_*}(L_{\lambda_*}^{k_*} - \mathrm{i})^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a compact perturbation of the identity, injective and hence bijective. Then, for λ close to λ_*, also $\mathrm{Id} + P^{k_*}(L_\lambda^{k_*} - \mathrm{i})^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is bijective. Note that we have the identity

$$\left(L_\lambda^{k_*} - \mathrm{i} + P^{k_*}\right)^{-1} = \left(\left(\mathrm{Id} + P^{k_*}(L_\lambda^{k_*} - \mathrm{i})^{-1}\right)(L_\lambda^{k_*} - \mathrm{i})\right)^{-1}.$$

From this we see that

$$(\tilde{L}_\lambda^{k_*} - \mathrm{i})^{-1} = (L_\lambda^{k_*} - \mathrm{i})^{-1} \left(\mathrm{Id} + P^{k_*}(L_\lambda^{k_*} - \mathrm{i})^{-1} \right)^{-1}.$$

Using the assumption that $L_\lambda^{k_*}$ converges to $L_{\lambda_*}^{k_*}$ in the norm resolvent sense, this implies the claim.

Next we show that $0 \in \rho(\tilde{L}_{\lambda_*}^{k_*})$. Since adding a (compact) projection operator only changes the discrete spectrum, we may assume by (L2) for contradiction that 0 is an eigenvalue of $\tilde{L}_{\lambda_*}^{k_*}$, that is $L_{\lambda_*}^{k_*}\phi + P^{k_*}\phi = 0$. Testing with ϕ^*, which spans the kernel of $L_{\lambda_*}^{k_*}$, we get $\langle \phi, \phi^* \rangle_{L^2(\mathbb{R})} = 0$ and hence $P^{k_*}\phi = 0$. Thus, ϕ also belongs to the kernel of $L_{\lambda_*}^{k_*}$, which contradicts $\langle \phi, \phi^* \rangle_{L^2(\mathbb{R})} = 0$ and the simplicity of the 0-eigenvalue of $L_{\lambda_*}^{k_*}$.

Finally, by $(L3)$ we know that there exists $k_0 \in \mathbb{N}$ (we assume w.l.o.g. $k_0 \geq k_*$) such that (4.4) holds for $k > k_0 \geq k_*$. This implies that

$$\inf_{k > k_0} \inf_{|\lambda - \lambda_*| < \delta} \mathrm{dist}(0, \sigma(\tilde{L}_\lambda^k)) > 0. \tag{4.5}$$

Now we want to extend this inequality to the remaining values of $k \in \{1, \ldots, k_0\}$ by possibly diminishing δ. Thus, let $k \in \mathbb{N}$ with $1 \leq k \leq k_0$ and assume for contradiction the existence of a sequence $\lambda_n \to \lambda_*$ as $n \to \infty$ such that there exists $\mu_n \in \sigma(\tilde{L}_{\lambda_n}^k)$ with $\lim_{n \to \infty} \mu_n = 0$. By norm-resolvent convergence this implies $0 \in \sigma(\tilde{L}_{\lambda_*}^k)$, which is impossible for $k \neq k_*$ by (L2) and also impossible for $k = k_*$ as stated above. This contradiction establishes (4.5) for all $k \in \mathbb{N}$. Finally, (4.5) shows that the map $\lambda \mapsto \|(\tilde{L}_\lambda^k)^{-1}\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})}$ is bounded for $\lambda \in (\lambda_* - \delta, \lambda_* + \delta)$ uniformly for $k \in \mathbb{N}$. The same holds for $\lambda \mapsto \|(\tilde{L}_\lambda^k)^{-1}\|_{L^2(\mathbb{R}) \to D(L_\lambda^k)}$, and due to (ii) in Remark 4.1, also for $\lambda \mapsto \|(\tilde{L}_\lambda^k)^{-1}\|_{L^2(\mathbb{R}) \to H^1(\mathbb{R})}$. This establishes (4.4) for all $k \in \mathbb{N}$. $\square$

Now, we introduce suitable function spaces and use Lemma 4.2 to reformulate the nonlinear problem (4.1) in a setting, which makes the local bifurcation theorem due to Crandall–Rabinowitz [5] applicable. Set

$$X := \left\{ \Phi \in H^2(\mathbb{T}; L^2(\mathbb{R})) \cap H^1(\mathbb{T}; H^1(\mathbb{R})) \mid \Phi(x, y) = \sum_{k \in \mathbb{N}} \phi_k(y) \sin(kx) \right\}$$

and

$$Y := \left\{ \Phi \in L^2(\mathbb{T}; L^2(\mathbb{R})) \mid \Phi(x, y) = \sum_{k \in \mathbb{N}} \phi_k(y) \sin(kx) \right\}.$$

Moreover, we set

$$\tilde{L}_\lambda := L_\lambda + P^*,$$

where P^* denotes the L^2-orthogonal projection onto $\ker L_{\lambda_*} = \mathrm{span}\{\Phi^*\}$ with $\Phi^*(x, y) = \frac{1}{\sqrt{\pi}}\phi^*(y)\sin(k_* x)$. Recall, that the operator L_λ is defined in (4.2). As an immediate consequence of $(L0)$–$(L3)$, we obtain the following lemma.

Lemma 4.3. *Assume that $(L0)$–$(L3)$ holds true and let $\lambda \in I_{\lambda_*}$. There exists a bounded linear map $\tilde{L}_\lambda^{-1} : Y \to X$ with the following property: if $f \in Y$ is given and the function $\Phi \in X$ solves*

$$\Phi = \tilde{L}_\lambda^{-1}(f + P^*\Phi) \tag{4.6}$$

then Φ solves

$$L_\lambda \Phi = f$$

in the weak sense, that is

$$\int_\mathbb{T} \int_\mathbb{R} \Phi_y \Psi_y - (1 - \lambda V_0(y)) \Phi_{xx} \Psi \, dy \, dx - \int_\mathbb{T} \langle V_1(\cdot) \Phi_x(x, \cdot), \Psi_x(x, \cdot) \rangle \, dx = \int_\mathbb{T} \int_\mathbb{R} f \Psi \, dy \, dx$$

for any $\Psi \in X$.

Proof. Let $\lambda \in I_{\lambda_*}$. For $g \in Y$ the definition of $\Phi := \tilde{L}_\lambda^{-1} g$ is given by

$$\Phi(x, y) = \sum_{k \in \mathbb{N}} \phi_k(y) \sin(kx) \quad \text{with} \quad \phi_k = (\tilde{L}_\lambda^k)^{-1} g_k.$$

Then Lemma 4.2 implies that $\Phi \in X$ and that $\tilde{L}_\lambda^{-1} : Y \to X$ is bounded. Now suppose that $\Phi \in X$ solves (4.6). Then $\phi_k = (\tilde{L}_\lambda^k)^{-1}(f_k + P^{k*}\phi_k)$ so that $\phi_k \in D(\tilde{L}_\lambda^k) = D(L_\lambda^k)$ for all $k \in \mathbb{N}$. In particular, we know that

$$L_\lambda^k \phi_k = f_k \qquad \text{for all} \qquad k \in \mathbb{N}$$

and thus for $K_0 \in \mathbb{N}$ and $\psi_1, \ldots, \psi_{K_0} \in H^1(\mathbb{R})$ we have

$$\sum_{k=1}^{K_0} \left(\int_\mathbb{R} \phi_k' \psi_k' + k^2 (1 - \lambda V_0(y)) \, dy - \langle V_1 \phi_k, \psi_k \rangle \right) = \sum_{k=1}^{K_0} \int_\mathbb{R} f_k \psi_k \, dy.$$

Taking the limit $K_0 \to \infty$ in the previous equation will lead to

$$\int_\mathbb{T} \int_\mathbb{R} \Phi_y \Psi_y - (1 - \lambda V_0(y)) \, \Phi_{xx} \Psi \, dy \, dx - \int_\mathbb{T} \langle V_1(\cdot) \Phi_x(x, \cdot), \Psi_x(x, \cdot) \rangle \, dx = \int_\mathbb{T} \int_\mathbb{R} f \Psi \, dy \, dx$$

for any $\Psi \in X$ due to the following estimates:

$$\int_\mathbb{T} \int_\mathbb{R} |\Phi_y \Psi_y| \, dy \, dx \leq \sum_{k \in \mathbb{N}} \|\phi_k'\|_{L^2(\mathbb{R})} \|\psi_k'\|_{L^2(\mathbb{R})} \leq \|\Phi\|_{L^2(\mathbb{T}; H^1(\mathbb{R}))} \|\Psi\|_{L^2(\mathbb{T}; H^1(\mathbb{R}))},$$

$$\int_\mathbb{T} \int_\mathbb{R} |V_0(y) \Phi_{xx} \Psi| \, dy \, dx \leq \sum_{k \in \mathbb{N}} \|V_0\|_{L^\infty(\mathbb{R})} k^2 \|\phi_k\|_{L^2(\mathbb{R})} \|\psi_k\|_{L^2(\mathbb{R})}$$

$$\leq \|V_0\|_{L^\infty(\mathbb{R})} \|\Phi\|_{H^2(\mathbb{T}, ;^2(\mathbb{R}))} \|\Psi\|_{L^2(\mathbb{T}; L^2(\mathbb{R}))},$$

$$\int_\mathbb{T} |\langle V_1(\cdot) \Phi_x(x, \cdot), \Psi_x(x, \cdot) \rangle| \, dx \leq \sum_{k \in \mathbb{N}} \|V_1\|_{H^1 \to H^{-1}} \|k\phi_k\|_{H^1(\mathbb{R})} \|k\psi_k\|_{H^1(\mathbb{R})}$$

$$\leq \|V_1\|_{H^1 \to H^{-1}} \|\Phi\|_{H^1(\mathbb{T}; H^1(\mathbb{R}))} \|\Psi\|_{H^1(\mathbb{T}; H^1(\mathbb{R}))},$$

$$\int_\mathbb{T} \int_\mathbb{R} |f \Psi| \, dy \, dx \leq \sum_{k \in \mathbb{N}} \|f_k\|_{L^2(\mathbb{R})} \|\psi_k\|_{L^2(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{T}; L^2(\mathbb{R}))} \|\Psi\|_{L^2(\mathbb{T}; L^2(\mathbb{R}))}. \quad \square$$

Equipped with the above lemma, we use the invertibility of $\tilde{L}_\lambda$ to reformulate (4.1) as

$$F(\Phi, \lambda) = 0, \tag{4.7}$$

where $F : X \times I_{\lambda_*} \to X$ is given by

$$F(\Phi, \lambda) = \Phi + \tilde{L}_\lambda^{-1} \left(\Gamma(y)(\Phi^3)_{xx} - P^* \Phi \right).$$

We want to apply bifurcation theory to Eq. (4.7). Clearly, $F(0, \lambda) = 0$ for any $\lambda \in I_{\lambda_*}$ and the line $\{(0, \lambda) \mid \lambda \in I_{\lambda_*}\}$ constitutes the line of trivial solutions from which we aim to bifurcate at $\lambda = \lambda_*$. The following lemma collects the necessary properties of the map F.

Lemma 4.4. *The map* $F : X \times I_{\lambda_*} \to X$ *is a* C^∞-*map. Moreover the following holds:*

(i) *The linearization of* F *about* $\Phi = 0$, *given by*

$$D_\Phi F(0, \lambda) = \mathrm{Id} - \tilde{L}_\lambda^{-1} P^* : X \to X$$

is a Fredholm operator of index zero. In particular $D_\Phi F(0, \lambda_*) = \mathrm{Id} - P^*$. *The kernel of* $D_\Phi F(0, \lambda)$ *is trivial for* $\lambda \in I_{\lambda_*} \setminus \{\lambda_*\}$ *and it is given by* $\mathrm{span}\{\Phi^*\}$ *if* $\lambda = \lambda_*$.

(ii) *The mixed second derivative of F about $\Phi = 0$ is given by*

$$D^2_{\Phi,\lambda} F(0,\lambda) = \tilde{L}_\lambda^{-1} V_0 \partial_x^2 \tilde{L}_\lambda^{-1} P^* : X \to X.$$

Proof. Let us first verify the mapping properties of F by checking that $(\Phi^3)_{xx} \in Y$ for $\Phi \in X$. First note that $\Phi \in X$ implies

$$\|\Phi\|_\infty = \sup_{x \in \mathbb{R}, y \in \mathbb{T}} \Big| \sum_{k \in \mathbb{N}} \phi_k(y) \sin(kx) \Big| \leq \sum_{k \in \mathbb{N}} \|\phi_k\|_{L^\infty(\mathbb{R})} \leq \sum_{k \in \mathbb{N}} \|\phi_k\|_{H^1(\mathbb{R})}$$
$$\leq \Big(\sum_{k \in \mathbb{N}} \frac{1}{k^2} \Big)^{\frac{1}{2}} \Big(\sum_{k \in \mathbb{N}} k^2 \|\phi_k\|^2_{H^1(\mathbb{R})} \Big)^{\frac{1}{2}} \lesssim \|\Phi\|_{H^1(\mathbb{T}, H^1(\mathbb{R}))} \tag{4.8}$$

and, using $|\cos(kx)| \leq 1$,

$$\Big(\int_\mathbb{T} \int_\mathbb{R} |\Phi_x|^4 \, dy \, dx \Big)^{1/4} \leq \sum_{k \in \mathbb{N}} \|k\phi_k\|_{L^4(\mathbb{T} \times \mathbb{R})} = \sqrt[4]{2\pi} \sum_{k \in \mathbb{N}} |k| \|\phi_k\|_{L^4(\mathbb{R})} \lesssim \sum_{k \in \mathbb{N}} |k| \|\phi_k'\|^{\frac{1}{4}}_{L^2(\mathbb{R})} \|\phi_k\|^{\frac{3}{4}}_{L^2(\mathbb{R})},$$

by the Gagliardo–Nirenberg inequality, cf. [10]. Using a triple Hölder inequality we obtain that

$$\Big(\int_\mathbb{T} \int_\mathbb{R} |\Phi_x|^4 \, dy \, dx \Big)^{\frac{1}{4}} \leq C \Big(\sum_{k \in \mathbb{N}} k^2 \|\phi_k'\|^2_{L^2(\mathbb{R})} \Big)^{\frac{1}{8}} \Big(\sum_{k \in \mathbb{N}} k^4 \|\phi_k\|^2_{L^2} \Big)^{\frac{3}{8}} \Big(\sum_{k \in \mathbb{N}} k^{-3/2} \Big)^{\frac{1}{2}}$$
$$\lesssim \|\Phi\|^{\frac{1}{4}}_{H^1(\mathbb{T};H^1(\mathbb{R}))} \|\Phi\|^{\frac{3}{4}}_{H^2(\mathbb{T};L^2(\mathbb{R}))}. \tag{4.9}$$

Hence, for $\Phi \in X$ we have $(\Phi^3)_{xx} = 3\Phi^2 \Phi_{xx} + 6\Phi \Phi_x^2 \in L^2(\mathbb{T}; L^2(\mathbb{R})) \subset Y$ and thus the mapping properties of F are proved.

The differentiability properties of F with respect to Φ also follow in a similar way from $\Phi \in L^\infty(\mathbb{T} \times \mathbb{R})$ and $\Phi_x \in L^4(\mathbb{T} \times \mathbb{R})$. This can be seen as follows: the (formal) first/second derivatives of F with respect to Φ are linear/bilinear operators and contain terms of the form abc_{xx} or $ab_x c_x$ where $a, b, c \in X$. Based on the estimates

$$\int_\mathbb{T} \int_\mathbb{R} |ab_x c_x|^2 \, dy \, dx \leq \|a\|^2_\infty \|b_x\|^2_{L^4(\mathbb{T} \times \mathbb{R})} \|c_x\|^2_{L^4(\mathbb{T} \times \mathbb{R})},$$
$$\int_\mathbb{T} \int_\mathbb{R} |abc_{xx}|^2 \, dy \, dx \leq \|a\|^2_\infty \|b\|^2_\infty \|c_{xx}\|^2_{L^2(\mathbb{T} \times \mathbb{R})}$$

we find in view of (4.8) and (4.9) that the first/second derivatives of F with respect to Φ exist, are bounded linear/bilinear operators from X to Y, and depend continuously on Φ and λ. Due to the cubic nature of the nonlinearity, derivatives of F of order higher than two with respect to Φ are independent of Φ.

The differentiability properties of F with respect to λ follow from

$$\frac{d}{d\lambda} \tilde{L}_\lambda^{-1} = -\tilde{L}_\lambda^{-1} \frac{d}{d\lambda} \tilde{L}_\lambda \tilde{L}_\lambda^{-1} = -\tilde{L}_\lambda^{-1} V_0 \partial_x^2 \tilde{L}_\lambda^{-1} \tag{4.10}$$

and due to $\tilde{L}_\lambda^{-1} : Y \to X$ and $V_0 \partial_x^2 : X \to Y$, we see that the resulting operator on the right-hand side of (4.10) is indeed a bounded linear map from $Y \to X$. Moreover, (4.10) explains the formula for $D^2_{\Phi,\lambda} F(0,\lambda)$ in (ii).

Finally, the formula in (i) shows that $D_\Phi F(0,\lambda)$ is a compact perturbation of the identity, and hence Fredholm of index zero. Let us compute the kernel of $D_\Phi F(0,\lambda)$. If $\Psi \in X$ satisfies $D_\Phi F(0,\lambda) \Psi = 0$ then according to Lemma 4.3 we have that Ψ is a weak solution of $L_\lambda \Psi = 0$. Then, for $\lambda \neq \lambda_*$ we have $\psi_k = 0$ for all $k \in \mathbb{N}$ and hence $\Psi = 0$. For $\lambda = \lambda_*$ we have $\psi_k = 0$ for all $k \in \mathbb{N} \setminus \{k_*\}$ and $\psi_{k_*} \in \text{span}\{\phi^*\}$ so that $\Psi \in \text{span}\{\Phi^*\}$ as claimed. Notice finally that $D_\Phi F(0,\lambda_*) = \text{Id} - \tilde{L}_{\lambda_*}^{-1} P^* = \text{Id} - P^*$ since range $P^* = \text{span}\{\Phi^*\}$ is the eigenspace of $\tilde{L}_{\lambda_*}$ corresponding to the eigenvalue 1. This finishes the proof. $\square$

Remark 4.5. Let us briefly describe the difficulty that arises when one considers the bifurcation problem for $V(\lambda, y) = \lambda V(y)$ with $V = V_0 + V_1$, i.e., when multiplication with the bifurcation parameter is extended to the distributional potential V_1. In this case one already obtains a problem in verifying the C^1-property of the map F. Formally one finds

$$D_\lambda F(\varPhi, \lambda) = -\tilde{L}_\lambda^{-1} V \partial_x^2 \tilde{L}_\lambda^{-1} (\varGamma(y)(\varPhi^3)_{xx} - P^* \varPhi).$$

As above, we would expect to have $\tilde{L}_\lambda^{-1} V \partial_x^2 \tilde{L}_\lambda^{-1} : Y \to X$ as a bounded linear map. But this is not the case, as a calculation in the case where $V_0(y) \equiv 1$ and $V_1(y) = \alpha\delta_0(y)$ shows. Namely, let $A = \tilde{L}_\lambda^{-1} : Y \to X$ and $B = V\partial_x^2$. Then $B : X \to H^{-1}(\mathbb{T}; H^{-1}(\mathbb{R}))$ and $C = \tilde{L}_\lambda^{-1} : \mathrm{range}(B) \to H^{3/2}(\mathbb{T}; L^2(\mathbb{R})) \cap H^{1/2}(\mathbb{T}; H^1(\mathbb{R})) \not\subset X$, i.e., we are missing a half-derivative in the regularity gain.

The advantage of formulating the problem (4.1) as $F(\varPhi, \lambda) = 0$ relies on the fact that its linearization about $\varPhi = 0$ is of the form identity plus compact operator, which provides the Fredholm property for free. Applying the Crandall–Rabinowitz theorem (cf. e.g. [5] or [11, Theorem I.5.1]), we prove that assumption $(L0)$–$(L3)$ on the family of Schrödinger operators L_λ^k are sufficient to guarantee the existence of nontrivial small-amplitude solutions of (4.1) provided a certain *transversality condition* is satisfied, which we can formulate in terms of the potential V_0, see (2.2).

Proof of Theorem 2.3. Recall from Lemma 4.4 that $D_\varPhi F(0, \lambda_*) = \mathrm{Id} - P^* : X \to X$. Moreover, $D_\varPhi F(0, \lambda_*)$ is a Fredholm operator of index zero with a one-dimensional kernel spanned by $\varPhi^*$. Correspondingly, we can split the underlying space as follows:

$$X = \mathrm{span}\{\varPhi^*\} \oplus \mathrm{span}\{\varPhi^*\}^{\perp_{L^2}} = \ker(D_\varPhi F(0, \lambda_*)) \oplus \mathrm{range}(D_\varPhi F(0, \lambda_*)).$$

Hence, according to the Crandall–Rabinowitz theorem, the existence of a local bifurcation branch of nontrivial solutions of (1.4) follows provided that the transversality condition

$$D_{\varPhi\lambda}^2 F(0, \lambda_*) \varPhi^* \notin \mathrm{range}\, D_\varPhi F(0, \lambda_*) = \mathrm{span}\{\varPhi^*\}^{\perp_{L^2}}$$

is satisfied. In view of $\tilde{L}_{\lambda_*}^{-1} \varPhi^* = \varPhi^*$ and the symmetry of $\tilde{L}_{\lambda_*}^{-1}$ (which follows from the self-adjointness of $L_{\lambda_*}^k, \tilde{L}_{\lambda_*}^k$) the transversality condition holds since

$$\langle D_{\varPhi\lambda}^2 F(0, \lambda_*) \varPhi^*, \varPhi^* \rangle_{L^2(\mathbb{T}\times\mathbb{R})} = \langle \tilde{L}_{\lambda_*}^{-1} V_0 \varPhi_{xx}^*, \varPhi^* \rangle_{L^2(\mathbb{T}\times\mathbb{R})} = \langle V_0 \varPhi_{xx}^*, \varPhi^* \rangle_{L^2(\mathbb{T}\times\mathbb{R})} = -\pi k^2 \langle V_0 \phi_x^*, \phi_x^* \rangle_{L^2(\mathbb{R})} \neq 0$$

due to assumption (2.2) of the theorem. This finishes the proof of Theorem 2.3.

Now, we are going to state the bifurcation formulas with the help of the Lyapunov–Schmidt reduction (cf. [11, Theorem I.2.3]). The Lyapunov–Schmidt reduction theorem in our context reads as follows:

Theorem 4.6 (*Lyapunov–Schmidt Reduction, [11], Theorem I.2.3*). *Let $F : X \times I \to X$ be a C^∞-map and $X = N \oplus N^{\perp_{L^2}}$ with $N = \mathrm{span}\{\varPhi^*\} = \ker D_\varPhi F(0, \lambda_*)$ and $\lambda_* \in I$. There exists a neighborhood $O \times I' \subset \{(\varPhi, \lambda) \in X \times \mathbb{R}_+\}$ of the bifurcation point $(0, \lambda_*)$ such that the problem*

$$F(\varPhi, \lambda) = 0 \qquad for \quad (\varPhi, \lambda) \in O \times I'$$

is equivalent to the finite-dimensional problem

$$\eta(\varepsilon\varPhi^*, \lambda) := P^* F(\varepsilon\varPhi^* + \psi(\varepsilon\varPhi^*, \lambda), \lambda) = 0 \tag{4.11}$$

for functions $\eta \in C^\infty(O_N \times I'; N)$, $\psi \in C^\infty(O_N \times I'; N^{\perp_{L^2}})$ *where* $O_N \subset N$ *is an open neighborhood of the zero element in* N. *One has that*

$$\eta(0, \lambda_*) = \psi(0, \lambda_*) = D_\Phi \psi(0, \lambda_*) = 0$$

and solving (4.11) *provides a solution*

$$\Phi = \varepsilon \Phi^* + \psi(\varepsilon \Phi^*, \lambda)$$

of the infinite-dimensional problem $F(\Phi, \lambda) = 0$.

We have the following Fréchet derivatives:

$$D_\Phi F(\Phi, \lambda)\Phi^* = \Phi^* + \tilde{L}_\lambda^{-1}\left(\Gamma(y)3(\Phi^2\Phi^*)_{xx} - P^*\Phi^*\right),$$

$$D_{\Phi\Phi}^2 F(\Phi, \lambda)[\Phi^*, \Phi^*] = \tilde{L}_\lambda^{-1}\left(\Gamma(y)6\Phi(\Phi^*)^2\right)_{xx},$$

$$D_{\Phi\Phi\Phi}^3 F(\Phi, \lambda)[\Phi^*, \Phi^*, \Phi^*] = \tilde{L}_\lambda^{-1}\left(\Gamma(y)6(\Phi^*)^3\right)_{xx}.$$

According to [11, Section I.6], we have that

$$\dot\lambda(0) = -\frac{1}{2}\frac{\langle D_{\Phi\Phi}^2 F(0, \lambda_*)[\Phi^*, \Phi^*], \Phi^* \rangle_{L^2(\mathbb{T}\times\mathbb{R})}}{\langle D_{\Phi\lambda}^2 F(0, \lambda_*)\Phi^*, \Phi^* \rangle_{L^2(\mathbb{T}\times\mathbb{R})}}.$$

In view of F being cubic in Φ it is clear that $\dot\lambda(0) = 0$. In this case the second derivative is given by

$$\ddot\lambda(0) = -\frac{1}{3}\frac{\langle D_{\Phi\Phi\Phi}^3 \eta(0, \lambda_*)[\Phi^*, \Phi^*, \Phi^*], \Phi^* \rangle_{L^2(\mathbb{T}\times\mathbb{R})}}{\langle D_{\Phi\lambda}^2 F(0, \lambda_*)\Phi^*, \Phi^* \rangle_{L^2(\mathbb{T}\times\mathbb{R})}}. \qquad (4.12)$$

Proposition 4.7. *Let* $\{(\Phi(\varepsilon), \lambda(\varepsilon)) \mid |\varepsilon| < \varepsilon_0\} \subset X \times I_{\lambda_*}$ *be the local bifurcation curve found in* Theorem 2.3 *corresponding to the bifurcation point* $(0, \lambda_*)$. *Then*

$$\dot\lambda(0) = 0 \qquad and \qquad \ddot\lambda(0) = -\frac{3\pi}{2}\frac{\int_{\mathbb{R}} \Gamma(y)(\phi^*)^4(y)\,dy}{\int_{\mathbb{R}} V_0(y)(\phi^*)^2(y)\,dy}.$$

Proof. As already mentioned, the cubic nonlinearity of F implies already that $\dot\lambda(0) = 0$. We are left to compute the second derivative of λ at the origin. According to the formula in (4.12) we need to compute the third derivative of η with respect to Φ evaluated at $(0, \lambda_*)$. As for instance in [11, Eq. (I.6.5)] we obtain that

$$D_{\Phi\Phi\Phi}^3 \eta(0, \lambda_*)[\Phi^*, \Phi^*, \Phi^*] = P^* D_{\Phi\Phi\Phi}^3 F(0, \lambda_*)[\Phi^*, \Phi^*, \Phi^*] + 3P^* D_{\Phi\Phi}^2 F(0, \lambda_*)[\Phi^*, D_{\Phi\Phi}^2 \psi(0, \lambda_*)[\Phi^*, \Phi^*]].$$

Again, since F is cubic in Φ, we have that $D_{\Phi\Phi}^2 F(0, \lambda_*) = 0$, whence

$$\ddot\lambda(0) = -\frac{1}{3}\frac{\langle P^* D_{\Phi\Phi\Phi}^3 F(0, \lambda_*)[\Phi^*, \Phi^*, \Phi^*], \Phi^* \rangle_{L^2(\mathbb{T}\times\mathbb{R})}}{\langle D_{\Phi\lambda}^2 F(0, \lambda_*)\Phi^*, \Phi^* \rangle_{L^2(\mathbb{T}\times\mathbb{R})}}$$

$$= -\frac{1}{3}\frac{\langle D_{\Phi\Phi\Phi}^3 F(0, \lambda_*)[\Phi^*, \Phi^*, \Phi^*], \Phi^* \rangle_{L^2(\mathbb{T}\times\mathbb{R})}}{\langle D_{\Phi\lambda}^2 F(0, \lambda_*)\Phi^*, \Phi^* \rangle_{L^2(\mathbb{T}\times\mathbb{R})}}.$$

We have that

$$\langle D_{\Phi\Phi\Phi}^3 F(0, \lambda_*)[\Phi^*, \Phi^*, \Phi^*], \Phi^* \rangle_{L^2(\mathbb{T}\times\mathbb{R})} = \langle \tilde{L}_{\lambda_*}^{-1}\left(\Gamma(y)6(\Phi^*)^3\right)_{xx}, \Phi^* \rangle_{L^2(\mathbb{T}\times\mathbb{R})}.$$

Using the symmetry of $\tilde{L}_{\lambda_*}$ together with $\tilde{L}_{\lambda_*}^{-1}\Phi^* = \Phi^*$, we obtain that

$$\langle D_{\Phi\Phi\Phi}^3 F(0, \lambda_*)[\Phi^*, \Phi^*, \Phi^*], \Phi^* \rangle_{L^2(\mathbb{T}\times\mathbb{R})} = 6\langle \Gamma(y)\left((\Phi^*)^3\right)_{xx}, \Phi^* \rangle_{L^2(\mathbb{T}\times\mathbb{R})} = -\frac{9}{2}\pi k_*^2\int_{\mathbb{R}}\Gamma(y)(\phi^*)^4(y)\,dy$$

and we know already that the denominator in $\ddot{\lambda}(0)$ is given by

$$\langle D_{\Phi\lambda}^2 F(0,\lambda_*)\Phi^*, \Phi^*\rangle_{L^2(\mathbb{T}\times\mathbb{R})} = \langle V_0(y)\Phi_{xx}^*, \Phi^*\rangle_{L^2(\mathbb{T}\times\mathbb{R})} = -\pi k_*^2 \int_{\mathbb{R}} V_0(y)(\phi^*)^2(y)\,dy.$$

Summarizing, we conclude that

$$\ddot{\lambda}(0) = -\frac{3}{2}\frac{\int_{\mathbb{R}} \Gamma(y)(\phi^*)^4(y)\,dy}{\int_{\mathbb{R}} V_0(y)(\phi^*)^2(y)\,dy}. \qquad \square$$

5. Examples for regular Γ

In what follows, we consider specific examples of potentials V and prove Corollaries 2.5 and 2.6, which state the existence of traveling waves of (1.4) in the specific case when the potentials are given as in (P1), (P2), respectively. Both Corollaries 2.5 and 2.6 are immediate consequences of Theorem 2.3 and Proposition 4.7, provided conditions $(L0) - (L3)$ are satisfied.

Recall that $V(\lambda, y) = \lambda V_0(y) + V_1(y)$ where in case (P1) we have $V_0(y) = 1$, $V_1(y) = \alpha\delta_0(y)$ and in case (P2) we have $V_0(y) = \mathbf{1}_{|y|\geq b}$, $V_1(y) = \beta\mathbf{1}_{|y|<b} + \alpha\delta_0(y)$. The transversality condition (2.2) is trivially satisfied, since in both cases $V_0 \geq 0$ and $\not\equiv 0$. It is also clear that $(L0)$ holds true. The beginning of this section will be valid both for (P1) and (P2) since at the general level we may consider (P1) as a special case of (P2) with $\beta = \lambda$. In the subsequent subsections the considerations will split according to the two cases.

Let us consider the operator

$$L_\lambda^k := -\frac{d^2}{dy^2} + k^2(1 - \lambda\mathbf{1}_{|y|\geq b} - \beta\mathbf{1}_{|y|<b} - \alpha\delta_0(y))$$

with $\lambda, \beta < 1$. According to [4] the operator $L_\lambda^k : D(L_\lambda^k) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is self-adjoint on the domain

$$D(L_\lambda^k) = \{\phi \in H^1(\mathbb{R}) \mid \phi \in H^2(-\infty, 0) \cap H^2(0, \infty), \phi'(0_+) - \phi'(0_-) = -k^2\alpha\phi(0)\};$$

thereby $(L1)$ is fulfilled. Moreover, $\sigma_{ess}(L_k) = [k^2(1-\lambda), \infty)$ according to Lemma A.5. Next we consider the point spectrum of L_k, i.e., the eigenvalue problem of finding $\phi \in D(L_\lambda^k)$ with $L_\lambda^k\phi = k^2\mu\phi$ where $\tilde{\mu} = k^2\mu$ is the actual eigenvalue. Setting $\tilde{\lambda} = \lambda + \mu$ and $\tilde{\beta} = \beta + \mu$ the eigenvalue problem then reduces to

$$\begin{cases} -\phi'' + k^2(1 - \tilde{\lambda}\mathbf{1}_{|y|\geq b} - \tilde{\beta}\mathbf{1}_{|y|<b})\phi = 0, & y \in (-\infty, 0) \cup (0, \infty), \\ \phi'(0+) - \phi'(0-) + k^2\alpha\phi(0) = 0. \end{cases} \tag{5.1}$$

For reasons that will become obvious in the subsequent discussion we suppose μ to be so small that $\tilde{\lambda}, \tilde{\beta} < 1$. In Lemma A.6 in the Appendix we show that this problem is solvable (with a one-dimensional eigenspace) if and only if

$$\frac{k\alpha}{2\sqrt{1-\tilde{\beta}}} = \frac{\sqrt{1-\tilde{\beta}}\sinh(k\sqrt{1-\tilde{\beta}}b) + \sqrt{1-\tilde{\lambda}}\cosh(k\sqrt{1-\tilde{\beta}}b)}{\sqrt{1-\tilde{\beta}}\cosh(k\sqrt{1-\tilde{\beta}}b) + \sqrt{1-\tilde{\lambda}}\sinh(k\sqrt{1-\tilde{\beta}}b)}. \tag{5.2}$$

Now we will split the discussion into subsections according to the cases (P1) and (P2), verifying $(L2)$ and $(L3)$ separately.

5.1. (P1) V a δ-potential on a constant background

Here we take $V_0 = 1$ and $V_1 = \alpha\delta_0$ and $V(\lambda, y) = \lambda + \alpha\delta_0(y)$ with $\alpha > 0$ and $\lambda < 1$. In the subsequent results of Lemma 5.1, Lemma 5.2 we verify that the family of linear operators L_λ^k satisfies also the assumptions $(L2)$ and $(L3)$ in Theorem 2.3. Since (P1) is a special case of (P2) with $\lambda = \beta$ we see that the eigenvalue condition (5.2) becomes

$$\frac{k\alpha}{2\sqrt{1-\tilde{\lambda}}} = 1. \tag{5.3}$$

This leads to the following lemma.

Lemma 5.1. *Let us fix a wavenumber $k_* \in \mathbb{N}$ and let $\lambda_* < 1$. We determine $\alpha > 0$ such that*

$$\alpha = \frac{2\sqrt{1 - \lambda_*}}{k_*}. \tag{5.4}$$

Then there exists an open interval $I_{\lambda_*} \subset \mathbb{R}$ containing λ_* such that

$$\dim \ker L^{k_*}_{\lambda_*} = 1,$$

and $\ker L^k_\lambda = \{0\}$ for any $(k, \lambda) \in \mathbb{N} \times I_{\lambda_*}$ with $(k, \lambda) \neq (k_*, \lambda_*)$.

Proof. Since we are considering the zero-eigenvalue of L^k_λ we have $\mu = 0$ and $\tilde\lambda = \lambda$. Together with our choice of α the eigenvalue condition (5.3) becomes

$$\frac{k}{k_*} = \frac{\sqrt{1 - \lambda}}{\sqrt{1 - \lambda_*}}.$$

Recall that $k \in \mathbb{N}$ is integer valued. Therefore, choosing a sufficiently small interval $I_{\lambda_*} \subset (-\infty, 1)$ that contains λ_* the eigenvalue condition is satisfied for $\lambda \in I_{\lambda_*}$ and $k \in \mathbb{N}$ if and only if $\lambda = \lambda_*$ and $k = k_*$. Moreover, for $k = k_*$ and $\lambda = \lambda_*$ the eigenspace is one-dimensional. $\square$

It is clear that the kernel of $L^{k_*}_{\lambda_*}$ is spanned by the $L^2(\mathbb{R})$-unitary element

$$\phi^*(y) := \sqrt{k_* \sqrt{1 - \lambda_*}} \, e^{-k_* \sqrt{1 - \lambda_*} |y|}$$

since $L^{k_*}_{\lambda_*} \phi^* = 0$ in $\mathbb{R} \setminus \{0\}$ and it satisfies $\phi^{*\prime}(0+) - \phi^{*\prime}(0-) + \alpha k_*^2 \phi^*(0) = 0$. The above lemma ensures that assumption $(L2)$ is satisfied. The following lemma concerns the spectral properties of L^k_λ and shows that assumption $(L3)$ holds true.

Lemma 5.2. *There exists an open interval $I_{\lambda_*} \subset \mathbb{R}$ containing λ_* such that the following holds for all $k \geq 3k_*$ and all $\lambda \in I_{\lambda_*}$: if $L^k_\lambda \phi = f$ for some $f \in L^2(\mathbb{R})$, then*

$$\|\phi\|_{L^2(\mathbb{R})} \lesssim \frac{1}{k^2} \|f\|_{L^2(\mathbb{R})} \qquad and \qquad \|\phi'\|_{L^2(\mathbb{R})} \lesssim \frac{1}{k} \|f\|_{L^2(\mathbb{R})}.$$

In particular, there exists a constant $c = c(k_*, |I_{\lambda_*}|) > 0$, depending on k_* and the size of the interval I_{λ_*}, such that

$$(-ck^2, ck^2) \subset \rho(L^k_\lambda) \quad for\ every \quad k \geq 3k_*, \lambda \in I_{\lambda_*}.$$

Proof. We show that for any $\lambda < 1$ we have

$$\begin{aligned}
\|L^k_\lambda \phi\|^2_{L^2(\mathbb{R})} \geq &\frac{1}{2} \left(\|\phi''\|^2_{L^2(-\infty, 0)} + \|\phi''\|^2_{L^2(0, \infty)} \right) \\
&+ 2(1 - \lambda) \left(k^2 - 4k_*^2 - \frac{16(\lambda_* - \lambda)}{\alpha^2} \right) \|\phi'\|^2_{L^2(\mathbb{R})} + \left(\frac{k_*^2 \alpha^2}{4} + \lambda_* - \lambda \right)^2 k^4 \|\phi\|^2_{L^2(\mathbb{R})},
\end{aligned}$$

which proves the assertion. We have that

$$\|L^k_\lambda \phi\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^0 (L^k_\lambda \phi)^2 \, dy + \int_0^\infty (L^k_\lambda \phi)^2 \, dy.$$

For the first integral on the right hand side, we compute

$$
\begin{aligned}
\int_{-\infty}^{0} (L_\lambda^k \phi)^2 \, dy &= \int_{-\infty}^{0} (-\phi'' + k^2(1-\lambda)\phi)^2 \, dy \\
&= \int_{-\infty}^{0} |\phi''|^2 - 2k^2(1-\lambda)\phi''\phi + k^4(1-\lambda)^2\phi^2 \, dy \\
&= \|\phi''\|_{L^2(-\infty,0)}^2 + 2k^2(1-\lambda)\|\phi'\|_{L^2(-\infty,0)}^2 + k^4(1-\lambda)^2\|\phi\|_{L^2(-\infty,0)}^2 - 2k^2(1-\lambda)\phi'(0_-)\phi(0),
\end{aligned}
$$

where we used integration by parts. Similarly, we obtain that

$$
\int_0^{\infty} (L_\lambda^k \phi)^2 \, dy = \|\phi''\|_{L^2(0,\infty)}^2 + 2k^2(1-\lambda)\|\phi'\|_{L^2(0,\infty)}^2 + k^4(1-\lambda)^2\|\phi\|_{L^2(0,\infty)}^2 + 2k^2(1-\lambda)\phi'(0_+)\phi(0).
$$

Taking the sum of the two integrals and using for $\phi \in D(L_\lambda^k)$ that

$$
\phi'(0_+) - \phi'(0_-) = -k^2\alpha\phi(0)
$$

we find that

$$
\begin{aligned}
\|L_\lambda^k \phi\|_{L^2(\mathbb{R})}^2 =&\|\phi''\|_{L^2(-\infty,0)}^2 + \|\phi''\|_{L^2(0,\infty)}^2 + 2k^2(1-\lambda)\|\phi'\|_{L^2(\mathbb{R})}^2 + k^4(1-\lambda)^2\|\phi\|_{L^2(\mathbb{R})}^2 \\
&- \frac{2}{\alpha}(1-\lambda)\left(\phi'(0_+) - \phi'(0_-)\right)^2.
\end{aligned}
\tag{5.5}
$$

A simple computation together with Young's inequality implies that

$$
|\phi'(0_+)|^2 \le 2 \int_0^{\infty} |\phi'\phi''| \, dy \le \left(\varepsilon\|\phi'\|_{L^2(0,\infty)}^2 + \frac{1}{\varepsilon}\|\phi''\|_{L^2(0,\infty)}^2 \right)
$$

for any $\varepsilon > 0$. A similar estimate holds for $|\phi'(0_-)|^2$. Therefore

$$
\begin{aligned}
|\phi'(0_+) - \phi'(0_-)|^2 &\le 2\left(|\phi'(0_+)|^2 + |\phi'(0_-)|^2\right) \\
&\le 2\left(\varepsilon\|\phi'\|_{L^2(\mathbb{R})}^2 + \frac{1}{\varepsilon}\|\phi''\|_{L^2((-\infty,0))}^2 + \frac{1}{\varepsilon}\|\phi''\|_{L^2((0,\infty))}^2 \right).
\end{aligned}
$$

Inserting the latter into (5.5) yields

$$
\begin{aligned}
\|L_\lambda^k \phi\|_{L^2(\mathbb{R})}^2 \ge& \left(1 - \frac{4(1-\lambda)}{\alpha\varepsilon}\right)\left(\|\phi''\|_{L^2(-\infty,0)}^2 + \|\phi''\|_{L^2(0,\infty)}^2\right) \\
&+ 2(1-\lambda)(k^2 - \frac{2\varepsilon}{\alpha})\|\phi'\|_{L^2(\mathbb{R})}^2 + k^4(1-\lambda)^2\|\phi\|_{L^2(\mathbb{R})}^2.
\end{aligned}
$$

The choice $\varepsilon = \frac{8}{\alpha}(1-\lambda) = 2\alpha k_*^2 + \frac{8(\lambda_* - \lambda)}{\alpha}$ implies the claim. $\quad\square$

Collecting Lemmas 5.1 and 5.2, we infer that there exists an open interval $I_{\lambda_*} \subset \mathbb{R}_+$ containing λ_* such that conditions $(L0)$–$(L3)$ are satisfied, which concludes the proof of Corollary 2.5. The formulas for $\dot\lambda(0)$ and $\ddot\lambda(0)$ follow directly from Proposition 4.7.

5.2. (P2) V a δ-potential on a step background

Here we take $V_0(y) = \mathbf{1}_{|y|\ge b}$, $V_1(y) = \beta\mathbf{1}_{|y|<b} + \alpha\delta_0(y)$ and $V(\lambda,y) = \lambda\mathbf{1}_{|y|\ge b} + \beta\mathbf{1}_{|y|<b} + \alpha\delta_0(y)$ with $\alpha > 0$, $\beta, \lambda < 1$. The subsequent two results verify that the family of linear operators L_λ^k satisfies also the assumptions $(L2)$ and $(L3)$ in Theorem 2.3. They are the counterparts to Lemmas 5.1 and 5.2.

Lemma 5.3.   *Let us fix a wavenumber $k_* \in \mathbb{N}$ and let $\lambda_* < 1$. We determine $\alpha > 0$ such that*

$$\frac{k_*\alpha}{2\sqrt{1-\beta}} = \frac{\sqrt{1-\beta}\sinh(k_*\sqrt{1-\beta}b) + \sqrt{1-\lambda_*}\cosh(k_*\sqrt{1-\beta}b)}{\sqrt{1-\beta}\cosh(k_*\sqrt{1-\beta}b) + \sqrt{1-\lambda_*}\sinh(k_*\sqrt{1-\beta}b)}. \tag{5.6}$$

Then there exists an open interval $I_{\lambda_*} \subset \mathbb{R}$ containing λ_* such that

$$\dim \ker L^{k_*}_{\lambda_*} = 1,$$

and $\ker L^k_\lambda = \{0\}$ for any $(k,\lambda) \in \mathbb{N} \times I_{\lambda_*}$ with $(k,\lambda) \neq (k_*,\lambda_*)$.

Proof.   As before we are considering the zero-eigenvalue of L^k_λ. Hence we have $\mu = 0$ and $\tilde\lambda = \lambda$, $\tilde\beta = \beta$. Then (5.6) amounts to $L^{k_*}_{\lambda_*}$ having a simple zero eigenvalue, cf. Lemma A.6. It remains to show that for no other value of $\lambda \in I_{\lambda_*}$ and $k \in \mathbb{N}$ there is a zero eigenvalue of L^k_λ. First note that for λ in a bounded interval of $(-\infty, 1)$ there are only finitely many values of $k \in \{1, \ldots, K\}$ which potentially also fulfill (5.6) since the right-hand side is bounded in k and the left-hand side tends to infinity as $k \to \infty$. Now we observe (by a standard calculation) that for fixed $\lambda = \lambda_*$, the right-hand side of (5.6) divided by k is monotone decreasing in k. Hence, for given λ_* no other value of $k \in \{1, \ldots, K\}$ than k_* fulfills (5.6). Finally, since $k \in \{1, \ldots, K\}$ needs to be integer valued, we can find a sufficiently small open interval $I_{\lambda_*} \subset (-\infty, 1)$ containing λ_* such that (5.6) is fulfilled for $(\lambda, k) \in I_{\lambda_*} \times \mathbb{N}$ if and only if $(\lambda, k) = (\lambda_*, k_*)$. $\square$

Lemma 5.4.   *There exists an open interval $I_{\lambda_*} \subset \mathbb{R}$ containing λ_* such that the following holds for all sufficiently large $k \in \mathbb{N}$ and all $\lambda \in I_{\lambda_*}$: if $L^k_\lambda \phi = f$ for some $f \in L^2(\mathbb{R})$, then*

$$\|\phi\|_{L^2(\mathbb{R})} \lesssim \frac{1}{k^2}\|f\|_{L^2(\mathbb{R})} \qquad and \qquad \|\phi'\|_{L^2(\mathbb{R})} \lesssim \frac{1}{k}\|f\|_{L^2(\mathbb{R})}. \tag{5.7}$$

In particular, there exists a constant $c = c(k_*, |I_{\lambda_*}|) > 0$, depending on k_* and the size of the interval I_{λ_*}, such that

$$(-ck^2, ck^2) \subset \rho(L^k_\lambda) \quad \text{for every } k \text{ sufficiently large}, \lambda \in I_{\lambda_*}.$$

Proof.   The proof consists of two parts. First we determine an interval $(-ck^2, ck^2) \subset \rho(L^k_\lambda)$ for all $\lambda \in I_{\lambda_*}$ and all sufficiently large k. This implies the first part of the estimate in (5.7). In the second part of the proof we will show the remaining part of (5.7).

Part 1: Recall that $\sigma_{ess}(L^k_\lambda) = [k^2(1-\lambda), \infty)$, which is consistent with the desired result provided we choose I_{λ_*} in such a way that it has a positive distance from 1. Subject to this observation we take the bounded interval I_{λ_*} from Lemma 5.3 and diminish it in the following if necessary. Notice that (5.2) describes all eigenvalues of L^k_λ of the form $\tilde\mu = k^2\mu$, where μ is so small that $\sup_{\lambda \in I_{\lambda_*}}\{\tilde\lambda = \lambda + \mu\} < 1$ and $\tilde\beta = \beta + \mu < 1$. Now observe that uniformly for $\lambda \in I_{\lambda_*}$ and $\mu \in [-\mu_0, \mu_0]$ for small $\mu_0 > 0$ the left-hand side of (5.2) tends to ∞ as $k \to \infty$ whereas the right-hand side stays bounded in k. Therefore the set $[-\mu_0 k^2, \mu_0 k^2]$ belongs to the resolvent of L^k_λ for all $\lambda \in I_{\lambda_*}$ and all sufficiently large k.

Part 2: We need to distinguish the operator $L^k_\lambda = -\frac{d^2}{dy^2} + k^2(1-\lambda-\alpha\delta_0)$ of case (P1) from its counterpart in (P2). Within this part of the proof let us denote it by $L^k_{\lambda,\beta} = -\frac{d^2}{dy^2} + k^2(1 - \lambda\mathbf{1}_{|y|\geq b} - \beta\mathbf{1}_{|y|<b} - \alpha\delta_0)$. Using Part 1 we find

$$\|(L^k_{\lambda,\beta})^{-1}\|_{L^2 \to L^2} \leq \frac{1}{\mathrm{dist}(0, \sigma(L^k_{\lambda,\beta}))} \lesssim \frac{1}{k^2}.$$

Therefore, with $f \in L^2(\mathbb{R})$ and ϕ as in the hypothesis of the lemma, we get $\|\phi\|_{L^2(\mathbb{R})} \lesssim \frac{1}{k^2}\|f\|_{L^2(\mathbb{R})}$. The estimate for $\|\phi'\|_{L^2(\mathbb{R})}$ is obtained as follows. We have

$$L^k_{\lambda,\beta}\phi = L^k_\lambda\phi + k^2(-\beta + \lambda)\mathbf{1}_{|y|<b}\phi = f$$

from which we deduce by using $\|(L_\lambda^k)^{-1}\|_{L^2 \to H^1} \lesssim \frac{1}{k}$ for $k \gg 1$ from Lemma 5.2

$$\|\phi'\|_{L^2(\mathbb{R})} \leq \|(L_\lambda^k)^{-1}\|_{L^2 \to H^1} \|f - k^2(-\beta + \lambda)\mathbf{1}_{|y|<b}\phi\|_{L^2(\mathbb{R})} \lesssim \frac{1}{k}(\|f\|_{L^2(\mathbb{R})} + k^2\|\phi\|_{L^2(\mathbb{R})}) \lesssim \frac{1}{k}\|f\|_{L^2(\mathbb{R})}$$

where in the last step we have used the result from Part 1. The finishes the proof of the lemma. $\quad\square$

Due to Lemmas 5.3 and 5.4 conditions $(L0)$–$(L3)$ are satisfied. This concludes the proof of Corollary 2.6.

6. Existence of traveling waves when Γ is a delta potential

Subject of this section is the proof of Theorem 2.10 when $\Gamma = \gamma\delta_0$ is given by a multiple of a delta potential and

$$V(\lambda, y) = \lambda V_0(y) + \underbrace{W(y) + \alpha\delta_0(y)}_{=V_1(y)},$$

where $V_0, W \in L^\infty(\mathbb{R})$ are even. The equation for traveling wave solutions (1.4) is then given by

$$- \Phi_{yy} - (1 - \lambda V_0(y) - W(y) - \alpha\delta_0(y))\,\Phi_{xx} + \gamma\delta_0(y)\,(\Phi^3)_{xx} = 0 \tag{6.1}$$

and can be written as a linear partial differential equation on $\mathbb{T} \times \mathbb{R} \setminus \{0\}$ equipped with a nonlinear boundary condition on $\mathbb{T}$:

$$-\Phi_{yy} - (1 - \lambda V_0 - W(y))\,\Phi_{xx} = 0, \qquad\qquad (x,y) \in \mathbb{T} \times \mathbb{R} \setminus \{0\}, \tag{6.2}$$
$$\Phi_y(x, 0_+) - \Phi_y(x, 0_-) = \partial_x^2\,(\alpha\,\Phi + \gamma\,\Phi^3)\,(x,0), \qquad x \in \mathbb{T}. \tag{6.3}$$

In what follows let us assume that Φ is even with respect to y. We seek for solutions Φ of the form

$$\Phi(x,y) = \sum_{k \in \mathbb{N}} a_k \phi_k(y; \lambda) \sin(kx), \tag{6.4}$$

where $\phi_k(\cdot; \lambda) \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\})$ is an evenly extended solution to the linear problem

$$L_{0,k}^\lambda \phi_k(y; \lambda) = 0 \quad \text{on } (0, \infty) \qquad \text{with} \qquad \phi_k(0; \lambda) = 1 \tag{6.5}$$

and

$$L_{0,k}^\lambda := -\frac{d^2}{dy^2} + k^2(1 - \lambda V_0(y) - W(y)).$$

Thus ansatz (6.4) already solves (6.2) and its remains to determine $a = (a_k)_{k \in \mathbb{N}}$ such that (6.3) is also satisfied. It will be convenient to parameterize the sequence (a_k) over $\mathbb{Z}$ instead of $\mathbb{N}$ by setting $a_k = -a_{-k}$. In this way $\Phi(x,y) = \frac{1}{2}\sum_{k \in \mathbb{Z}} a_k \phi_k(y; \lambda) \sin(kx)$. Here we have defined $\phi_{-k}(\cdot; \lambda) := \phi_k(\cdot; \lambda)$ for $k \in \mathbb{N}$. Then, we shall see that for $s \geq \frac{5}{2}$ the existence of a traveling wave solution Φ in the space

$$X_s = H^s(\mathbb{T}; L^2(\mathbb{R})) \cap H^{s-1}(\mathbb{T}; H^1(\mathbb{R})) \cap C(\mathbb{R}; H^{s-\frac{1}{2}}(\mathbb{T})) \cap C^1(\mathbb{R}; H^{s-\frac{3}{2}}(\mathbb{T}))$$

follows from the existence of a sequence $a \in h_\sharp^s(\mathbb{R})$ satisfying the boundary condition

$$2a_k \phi_k'(0_+; \lambda) = -k^2\left(\alpha a_k - \frac{\gamma}{4}(a * a * a)_k\right) \qquad \text{for all} \quad k \in \mathbb{N}. \tag{6.6}$$

Recall that the kth Fourier coefficient of $\Phi^3(x,0)$ is given by $-\frac{1}{4}(a * a * a)_k$ (cf. Lemma A.1). Notice that for $s \geq \frac{5}{2}$ we have the embedding

$$X_s \hookrightarrow X = H^2(\mathbb{T}; L^2(\mathbb{R})) \cap H^1(\mathbb{T}; H^1(\mathbb{R})) \cap C(\mathbb{R}; H^2(\mathbb{T})) \cap C^1(\mathbb{R}; H^1(\mathbb{T})).$$

As in the previous section we aim to apply bifurcation theory with respect to the parameter λ to obtain the existence of nontrivial solutions $a \in h_\sharp^s(\mathbb{R})$ of (6.6) for $s \geq \frac{5}{2}$ by the Crandall–Rabinowitz theorem. Recall that this time we are working under the assumptions $(\tilde{L}0)$–$(\tilde{L}3)$ from Theorem 2.10.

Remark 6.1. Existence and properties of the decaying solutions $\phi_k(\cdot; \lambda)$ of (6.5):

(i) Due to $(\tilde{L}1)$ the problem $L^k_{0,\lambda} \phi_k = \mathbf{1}_{[-2,-1]}$ on $\mathbb{R}$ has a unique $H^2(\mathbb{R})$ solution. Its restriction to $[0, \infty)$ satisfies (6.5). The fact that $\phi_k(y; \lambda) \to 0$ exponentially as $y \to \infty$ can be seen as follows: Since $L^\lambda_{0,k}$ is a self-adjoint operator with $0 \in \rho(L^\lambda_{0,k})$ and the resolvent set is open in $\mathbb{C}$ there exists $c_{k,\lambda} > 0$ such that $(-c_{k,\lambda}, c_{k,\lambda}) \subset \rho(L^\lambda_{0,k})$. Set $\psi_k(y; \lambda) := e^{\delta_k y} \phi_k(y; \lambda)$. Then

$$L^\lambda_{0,k} \psi_k(y; \lambda) + B_k \psi_k(y; \lambda) = e^{\delta_k y} \mathbf{1}_{[-2,-1]}, \tag{6.7}$$

where $B_k \psi := 2\delta_k \frac{d}{dy} \psi + \delta_k^2 \psi$. One can show that B_k is $L^\lambda_{0,k}$–bounded in the sense that there exist $a_k, b_k > 0$ such that

$$\|B_k \psi\|^2_{L^2(\mathbb{R})} \leq a_k \|\psi\|^2_{L^2(\mathbb{R})} + b_k \|L^\lambda_{0,k} \psi\|^2_{L^2(\mathbb{R})} \qquad \text{for all} \quad \psi \in H^2(\mathbb{R}).$$

In fact, if $b_k > 0$ is fixed, then $a_k := \frac{16\delta_k^4}{b_k} + 8\delta_k^2 k^2 \|1 - \lambda V_0 - W\|_\infty + 2\delta_k^4$. For fixed $b_k \in (0,1)$ let us choose $\delta_k > 0$ so small that

$$a_k^2 + b_k^2 c_{k,\lambda}^2 < c_{k,\lambda}^2.$$

Then $(-\tilde{c}_{k,\lambda}, \tilde{c}_{k,\lambda}) + i\mathbb{R} \subset \rho(L^\lambda_{0,k} + B_k)$, where $\tilde{c}_{k,\lambda} = c_k - \sqrt{a_k^2 + b_k^2 c_{k,\lambda}^2}$, cf. [6, Theorem 2.1 (ii)]. In particular, $0 \in \rho(L^\lambda_{0,k} + B)$ so that there exists a unique solution $\psi_k \in H^2(\mathbb{R})$ of (6.7). The boundedness of ψ_k then implies that $|\phi_k(y; \lambda)| \lesssim e^{-\delta_k y}$ decays exponentially on the half-line $[0, \infty)$. This result is also known as "exponential dichotomy". Assumption $(\tilde{L}3)$ may be interpreted as some kind of generalized uniform exponential dichotomy with respect to $k \in \mathbb{N}$ and $\lambda \in I_{\lambda_*}$.

(ii) In the specific examples (P1) and (P2) which we consider at the end of this section, the family of ansatz-functions $(\phi_k(\cdot; \lambda))_{k \in \mathbb{N}}$ satisfies a true uniform exponential dichotomy with respect to $k \in \mathbb{N}$ and $\lambda \in I_{\lambda_*}$; that is, there exists $C, \delta > 0$ independent of $k \in \mathbb{N}$ and $\lambda \in I_{\lambda_*}$ such that $|\phi_k(y; \lambda)| \leq Ce^{-\delta y}$ for all $y \geq 0$. This leads to an exponential decay in y-direction of the traveling solution Φ of (1.4) and in particular it implies $(\tilde{L}3)$.

(iii) Notice also that $\phi_k(0; \lambda) \neq 0$, since otherwise (by an odd reflection around zero) we would obtain an eigenfunction of $L^\lambda_{0,k}$ for the eigenvalue 0. This is excluded by assumption $(\tilde{L}1)$. Likewise we see that $\phi'_k(0; \lambda) \neq 0$ (using an even reflection around zero).

Remark 6.2. If V_0, W are bounded, even functions and there exists $\bar{v} > 0$ such that

$$1 - \lambda V_0(y) - W(y) \geq \bar{v} \qquad \text{for all } \lambda \in I_{\lambda_*}, y \in \mathbb{R},$$

then assumption $(\tilde{L}0), (\tilde{L}1)$, and $(\tilde{L}3)$ are satisfied. Clearly, if $1 - \lambda V_0 - W \geq \bar{v}$, then $L^\lambda_{0,k}$ is a self-adjoint operator with $\sigma(L^\lambda_{0,k}) \subset [k^2 \bar{v}, \infty)$; thus $0 \in \rho(L^\lambda_{0,k})$ and $(\tilde{L}1)$ is satisfied. As explained in Remark 6.1(i), condition $(\tilde{L}1)$ implies the existence of a solution $\phi_k(\cdot, \lambda) \in H^2(0, \infty)$ with

$$-\phi''_k + k^2(1 - \lambda V_0(y) - W(y))\phi_k = 0 \qquad \text{on} \quad (0, \infty) \tag{6.8}$$

and $\phi_k(0; \lambda) = 1$. Multiplying (6.8) with ϕ_k and integrating over the half line $(0, \infty)$, we obtain that

$$-\phi'_k(0_+; \lambda) = \int_0^\infty |\phi'_k|^2 \, dy + k^2 \int_0^\infty (1 - \lambda V_0 - W)\phi_k^2 \, dy. \tag{6.9}$$

On the other hand, multiplying (6.8) with ϕ_k' and integrating over $(0,\infty)$ yields

$$
\begin{aligned}
(\phi_k')^2(0_+;\lambda) &= -k^2 \int_0^\infty (1-\lambda V_0 - W)(\phi_k^2)' \, dy \\
&= -2k \int_0^\infty k\sqrt{1-\lambda V_0 - W}\,\phi_k \phi_k' \sqrt{1-\lambda V_0 - W}\, dy \\
&\leq k\|\sqrt{1-\lambda V_0 - W}\|_\infty \left(\int_0^\infty |\phi_k'|^2 \, dy + k^2 \int_0^\infty (1-\lambda V_0 - W)\phi_k^2 \, dy \right) \\
&= -k\|\sqrt{1-\lambda V_0 - W}\|_\infty \phi_k'(0_+;\lambda),
\end{aligned}
$$

where we used relation (6.9) in the last equality. We deduce that $\phi'(0_+;\lambda) < 0$ and

$$
|\phi_k'(0_+;\lambda)| \leq k\|\sqrt{1-\lambda V_0 - W}\|_\infty. \tag{6.10}
$$

Estimating the L^2-norm of $\phi_k(\cdot;\lambda)$, we obtain that

$$
\|\phi_k(\cdot;\lambda)\|_2^2 = \frac{1}{\bar v}\|\sqrt{\bar v}\phi_k(\cdot;\lambda)\|_2^2 \leq \frac{1}{\bar v}\int_0^\infty (1-\lambda V_0 - W)\phi_k^2 \, dy \leq \frac{1}{\bar v k^2}|\phi_k'(0_+;\lambda)| \leq \frac{1}{\bar v k}\|\sqrt{1-\lambda V_0 - W}\|_\infty,
$$

where we used (6.9) and (6.10). In particular, we find that $\|\phi_k(\cdot;\lambda)\|_2 \lesssim 1$ as claimed in $(\tilde L 3)$.

For $s \geq 0$ denote the linearization of (6.6) around $a = 0$ by

$$
A_\lambda : h_\sharp^s(\mathbb{R}) \subset h_\sharp^{s-2}(\mathbb{R}) \to h_\sharp^{s-2}(\mathbb{R}), \qquad (A_\lambda a)_k := A_\lambda^k a_k \text{ for } k \in \mathbb{Z},
$$

where

$$
A_\lambda^k := 2\phi_k'(0_+;\lambda) + k^2\alpha \text{ for } k \in \mathbb{Z}.
$$

Then (6.6) can be written as

$$
A_\lambda a - n(a) = 0, \qquad \text{where} \qquad n(a)_k = \frac{\gamma k^2}{4}(a*a*a)_k. \tag{6.11}
$$

For $m \in \mathbb{Z}$ let us denote by $e^m \in l_\sharp^2(\mathbb{R})$ the sequence, which satisfies $e_k^m = 0$ for $k \neq \pm m$ and $e_m^m = -e_{-m}^m = \frac{1}{\sqrt 2}$.

Lemma 6.3. *Assume $(\tilde L 0)$–$(\tilde L 3)$. Then*

$$
\|\phi_k(\cdot;\lambda)\|_{L^\infty(\mathbb{R})} \lesssim k^{\frac{1}{2}}, \quad \|\phi_k'(\cdot;\lambda)\|_{L^2(\mathbb{R})} \lesssim k, \quad \|\phi_k'(\cdot;\lambda)\|_{L^\infty(\mathbb{R})} \lesssim k^{\frac{3}{2}} \tag{6.12}
$$

uniformly for $\lambda \in I_{\lambda_*}$. In particular, $|\phi_k'(0;\lambda)| \lesssim |k|^{\frac{3}{2}}$ and consequently $A_\lambda^k = \alpha k^2 + O(k^{\frac{3}{2}})$ as $k \to \pm\infty$.

Proof. By a result of Komornik, cf. [13], the estimate

$$
\|u\|_\infty \leq C\|u\|_{L^2}
$$

holds true for every solution u of $-u''+q(y)u = 0$ on (a,b) with the constant $C = \max\left\{6\sqrt{\|q\|_{L^1(a,b)}}, \frac{12}{\sqrt{b-a}}\right\}$. We apply this result to the solutions $\phi_k(\cdot;\lambda)$ of (6.5) with $q = k^2(1-\lambda V_0 - W)$, $a \geq 0$ and $b = a + c$ with $c := 2(\sqrt{\|1-\lambda V_0 - W\|_\infty}k)^{-1}$. Then

$$
6\sqrt{\|q\|_{L^1(a,b)}} \leq 6k\sqrt{\|1-\lambda V_0 - W\|_\infty}\sqrt c = 6\sqrt 2\sqrt k \sqrt[4]{\|1-\lambda V_0 - W\|_\infty} = \frac{12}{\sqrt{b-a}}
$$

and thus for a constant $\tilde{C}$ only depending on $\|1-\lambda V_0 - W\|_\infty$ we have $\|\phi_k(\cdot;\lambda)\|_{L^\infty(a,b)} \leq k^{\frac{1}{2}}\tilde{C}\|\phi_k(\cdot;\lambda)\|_{L^2(a,b)}$
$\lesssim k^{\frac{1}{2}}$ by $(\tilde{L}3)$. Since $a \geq 0$ was arbitrary we obtain the first part of (6.12).

Multiplying (6.5) with u, u' and integrating from $a \geq 0$ to ∞ we get

$$\int_a^\infty k^2(1 - \lambda V_0(y) - W(y))\phi_k(y;\lambda)^2 + \phi_k'(y;\lambda)^2\, dy = -\phi_k(a;\lambda)\phi_k'(a;\lambda), \tag{6.13}$$

$$\int_a^\infty 2k^2(1 - \lambda V_0(y) - W(y))\phi_k(y;\lambda)\phi_k'(y;\lambda)\, dy = -\phi_k'(a;\lambda)^2, \tag{6.14}$$

respectively. Using $(\tilde{L}3)$ and applying the Cauchy–Schwarz inequality to (6.14) we find

$$\|\phi_k'(\cdot;\lambda)\|_{L^\infty}^2 \lesssim k^2 \|\phi_k'(\cdot;\lambda)\|_{L^2(0,\infty)} \tag{6.15}$$

and from (6.13), (6.15) we get

$$\|\phi_k'(\cdot;\lambda)\|_{L^2(0,\infty)}^2 \lesssim k^2 + \|\phi_k(\cdot;\lambda)\|_{L^\infty(0,\infty)}\|\phi_k'(\cdot;\lambda)\|_{L^\infty(0,\infty)}$$
$$\lesssim k^2 + \|\phi_k(\cdot;\lambda)\|_{L^\infty(0,\infty)}k\|\phi_k'(\cdot;\lambda)\|_{L^2(0,\infty)}^{\frac{1}{2}}.$$

The L^∞–estimate from the first part of the lemma leads to

$$\|\phi_k'(\cdot;\lambda)\|_{L^2(0,\infty)}^2 \lesssim k^2 + k^{\frac{3}{2}}\|\phi_k'(\cdot;\lambda)\|_{L^2(0,\infty)}^{\frac{1}{2}} \leq k^2 + C_\epsilon k^2 + \epsilon\|\phi_k'(\cdot;\lambda)\|_{L^2(0,\infty)}^2,$$

where we have used Young's inequality with exponents $4/3$ and 4. This implies the second inequality in (6.12). Inserting this into (6.15) we obtain the third inequality in (6.12). $\square$

Lemma 6.4. *Assume $(\tilde{L}0)$–$(\tilde{L}3)$ and let $s \geq 0$. Then the operator $A_\lambda : h_\sharp^s(\mathbb{R}) \subset h_\sharp^{s-2}(\mathbb{R}) \to h_\sharp^{s-2}(\mathbb{R})$ is self-adjoint. Its spectrum is discrete and consist of the values $(A_\lambda^k)_{k\in\mathbb{N}}$. Moreover $\ker A_{\lambda_*} = \operatorname{span}\{e^{k_*}\}$ and $\ker A_\lambda = \{0\}$ for $\lambda \in I_{\lambda_*} \setminus \{\lambda_*\}$.*

Proof. Due to Lemma 6.3 and since $\phi_{-k} = \phi_k$ for all $k \in \mathbb{Z}$ one can verify that $A_\lambda : h_\sharp^s(\mathbb{R}) \subset h_\sharp^{s-2}(\mathbb{R}) \to h_\sharp^{s-2}(\mathbb{R})$ acting like an infinite dimensional diagonal matrix is self-adjoint. Using the characterization of the spectrum via Weyl-sequences one sees that A_λ has the spectrum $\sigma(A_\lambda) = \operatorname{clos}\{A_\lambda^k : k \in \mathbb{N}\}$. Due to Lemma 6.3 the set $\{A_\lambda^k : k \in \mathbb{N}\}$ is discrete and hence $\sigma(A_\lambda^k)$ consists of the set of eigenvalues $\{A_\lambda^k : k \in \mathbb{N}\}$. Finally, let us determine the kernel of A_λ. On the one hand, $a \in \ker A_\lambda$ if and only if there exists $k \in \mathbb{N}$ such that $A_\lambda^k = 0$, and in this case $a = e^k$ (here we use that $A_\lambda^k = A_\lambda^{-k}$). On the other hand, using the characterization of the domain of L_k^λ from Section 5.1 we know that $L_k^\lambda\phi = 0$ if and only if $\phi(x) = \phi_k(|x|;\lambda)$ and $A_\lambda^k = 0$. Thus, bringing both facts together and using assumption $(\tilde{L}2)$ we obtain the final claim of the lemma. $\square$

Similarly as in (4.3) we define the operator

$$\tilde{A}_\lambda := A_\lambda + P^*,$$

where $P^*a = a_{k_*}e^{k_*}$.

Lemma 6.5. *Assume $(\tilde{L}0)$–$(\tilde{L}3)$ and $s \geq 0$. Then we have that $0 \in \rho(\tilde{A}_\lambda)$ for all $\lambda \in I_{\lambda_*}$ and hence $\tilde{A}_\lambda^{-1} : h_\sharp^{s-2}(\mathbb{R}) \to h_\sharp^s(\mathbb{R})$ is a bounded linear operator. Moreover, if $f \in h_\sharp^s(\mathbb{R})$, $s \geq \frac{5}{2}$ is given and $a \in h_\sharp^s(\mathbb{R})$ solves*

$$a = \tilde{A}_\lambda^{-1}(-\gamma M(f) + P^*a) \qquad where \qquad M(f)_k := k^2 f_k \tag{6.16}$$

then $\Phi(x,y) := \sum_{k\in\mathbb{N}} a_k\phi_k(x;\lambda)\sin(kx)$ *satisfies* $\Phi \in X_s$ *and solves*

$$L_\lambda \Phi + \gamma\delta_0(y)F_{xx} = 0$$

for $F(x,y) = \sum_{k\in\mathbb{N}} f_k\phi_k(y;\lambda)\sin(kx)$ *in the weak sense, i.e.,*

$$\int_{\mathbb{T}}\int_{\mathbb{R}} \Phi_y\Psi_y - (1 - \lambda V_0(y) - W(y))\,\Phi_{xx}\Psi\,dy\,dx + \int_{\mathbb{T}}(\alpha\Phi_{xx}(x,0) + \gamma F_{xx}(x,0))\,\Psi(x,0)\,dx = 0$$

for any $\Psi \in H^1(\mathbb{T}; H^1(\mathbb{R}))$.

Proof. Lemma 6.4 says that $\ker A_{\lambda_*} = \mathrm{span}\{e^{k_*}\}$ and $\ker A_\lambda = \{0\}$ for any $\lambda \in I_{\lambda_*} \setminus \{\lambda_*\}$. We need to show that $0 \in \rho(\tilde{A}_\lambda)$ for any $\lambda \in I_{\lambda_*}$. Let $b \in h_\sharp^{s-2}(\mathbb{R})$ be arbitrary, then $\tilde{A}_\lambda a = b$ if and only if

$$\begin{cases} A_\lambda^k a_k = b_k, & \text{if } k \neq k_*, \\ A_\lambda^k a_k + a_k = b_k, & \text{if } k = k_*, \end{cases}$$

which is equivalent to

$$a_k = \frac{1}{A_\lambda^k}b_k \quad \text{if} \quad k \neq k_* \qquad \text{and} \qquad a_{k_*} = \frac{1}{A_\lambda^{k_*}+1}b_{k_*}. \tag{6.17}$$

Due to Lemma 6.3 we obtain that for any $b \in h_\sharp^{s-2}(\mathbb{R})$ the sequence a defined by (6.17) belongs to $h_\sharp^s(\mathbb{R})$ and solve $\tilde{A}_\lambda a = b$; whence $0 \in \rho(\tilde{A}_\lambda)$.

Now suppose that $f \in h_\sharp^s(\mathbb{R})$ with $s \geq \frac{5}{2}$ and that $a \in h_\sharp^s(\mathbb{R})$ solves (6.16). The regularity of Φ follows from Lemma A.3. Moreover, $(A_\lambda a)_k = -\gamma k^2 f_k$ and hence

$$2\phi_k'(0_+;\lambda)a_k + \alpha k^2 a_k = -\gamma k^2 f_k. \tag{6.18}$$

Using that

$$L_{0,\lambda}^k\phi_k = 0 \text{ on } \mathbb{R}\setminus\{0\} \text{ for all } k\in\mathbb{N},$$

we deduce by testing with $a_k\psi_k \in H^1(\mathbb{R})$ and summing for $1 \leq k \leq K_0$ that

$$0 = \sum_{k=1}^{K_0}\int_{\mathbb{R}} a_k\phi_k'\psi_k' + k^2(1 - \lambda V_0(y) - W(y))\phi_k\psi_k\,dy + a_k(\phi_k'(0_+;\lambda) - \phi_k'(0_-;\lambda))\psi_k(0).$$

Since $\phi_k(\cdot;\lambda)$ is even with respect to y we obtain by (6.18)

$$0 = \sum_{k=1}^{K_0}\int_{\mathbb{R}} a_k\phi_k'\psi_k' + k^2(1 - \lambda V_0(y) - W(y))\phi_k\psi_k\,dy - \alpha k^2 a_k \underbrace{\phi_k(0;\lambda)}_{=1}\psi_k(0) - \gamma k^2 f_k\psi_k(0).$$

Taking the limit $K_0 \to \infty$ in the previous equation will lead to

$$0 = \int_{\mathbb{T}}\int_{\mathbb{R}} \Phi_y\Psi_y - (1 - \lambda V_0(y) - W(y))\,\Phi_{xx}\Psi + \int_{\mathbb{T}}(\alpha\Phi_{xx}(x,0) + \gamma F_{xx}(x,0))\,\Psi(x,0)\,dx$$

for any $\Psi \in H^1(\mathbb{T}; H^1(\mathbb{R}))$ due to the following estimates:

$$\int_{\mathbb{T}}\int_{\mathbb{R}}|\Phi_y\Psi_y|\,dy\,dx \leq \sum_{k\in\mathbb{N}}\|a_k\phi_k'\|_{L^2(\mathbb{R})}\|\psi_k'\|_{L^2(\mathbb{R})} \leq \|\Phi\|_{L^2(\mathbb{T};H^1(\mathbb{R}))}\|\Psi\|_{L^2(\mathbb{T};H^1(\mathbb{R}))}$$

$$\leq C\|a\|_{h^1(\mathbb{R})}\|\Psi\|_{L^2(\mathbb{T};H^1(\mathbb{R}))},$$

$$\int_{\mathbb{T}}\int_{\mathbb{R}}|(1 - \lambda V_0(y) - W(y))\,\Phi_{xx}\Psi|\,dy\,dx \leq \sum_{k\in\mathbb{N}}\|1 - \lambda V_0 - W\|_{L^\infty(\mathbb{R})}k^2\|\phi_k\|_{L^2(\mathbb{R})}\|\psi_k\|_{L^2(\mathbb{R})}$$

$$\leq \|1 - \lambda V_0 - W\|_{L^\infty(\mathbb{R})} \|\Phi\|_{H^2(\mathbb{T};L^2(\mathbb{R}))} \|\Psi\|_{L^2(\mathbb{T};L^2(\mathbb{R}))}$$

$$\leq C\|a\|_{h^2(\mathbb{R})} \|\Psi\|_{L^2(\mathbb{T};L^2(\mathbb{R}))}$$

$$\int_{\mathbb{T}} |\Phi_{xx}(x,0)\Psi(x,0)| \, dx \leq \sup_y \|\Phi_{xx}(\cdot,y)\|_{L^2(\mathbb{T})} \|\Psi(\cdot,y)\|_{L^2(\mathbb{T})}$$

$$\leq \|\Phi\|_{C(\mathbb{R};H^2(\mathbb{T}))} \|\Psi\|_{C(\mathbb{R};L^2(\mathbb{T}))}$$

$$\leq C\|a\|_{h^{\frac{5}{2}}(\mathbb{R})} \|\Psi\|_{C(\mathbb{R};L^2(\mathbb{T}))},$$

$$\int_{\mathbb{T}} |F_{xx}(x,0)\Psi(x,0)| \, dx \leq \|f\|_{h^{\frac{5}{2}}(\mathbb{R})} \|\Psi\|_{C(\mathbb{R};L^2(\mathbb{T}))}$$

together with the continuous embeddings $H^1(\mathbb{T};H^1(\mathbb{R})) \subset L^2(\mathbb{T};H^1(\mathbb{R})) \cap C(\mathbb{R};L^2(\mathbb{T}))$ and $h^s_\sharp(\mathbb{R}) \subset h^{5/2}_\sharp(\mathbb{R}) \subset h^2_\sharp(\mathbb{R}) \subset h^1_\sharp(\mathbb{R})$ since $s \geq \frac{5}{2}$. $\square$

In the same spirit as in Section 4, let us reformulate our problem (6.11) in a way suitable for applying the Crandall–Rabinowitz theorem. Using the above lemma, Eq. (6.11) is equivalent to

$$G(a,\lambda) = 0,$$

where the function $G : h^s_\sharp(\mathbb{R}) \times I_{\lambda_*} \to h^s_\sharp(\mathbb{R})$, $s \geq \frac{5}{2}$, is defined by

$$G(a,\lambda) := a + \tilde{A}_\lambda^{-1}\left(-\frac{\gamma}{4}M(a*a*a) - P^*a\right) \qquad \text{and} \qquad M(f)_k := k^2 f_k \quad \text{for} \quad f \in h^s_\sharp(\mathbb{R}). \tag{6.19}$$

Remark 6.6. Notice that $h^s_\sharp(\mathbb{R})$, $s \geq 1$ is a Banach algebra, cf. Lemma A.4. Thus, for $a \in h^s_\sharp(\mathbb{R})$ the nonlinearity $a*a*a$ stays in $h^s_\sharp(\mathbb{R})$ and $M(a*a*a) \in h^{s-2}_\sharp(\mathbb{R})$. Hence, in order to control the nonlinearity in $G(a,\lambda)$, it is necessary that $\tilde{A}_\lambda^{-1}$ is a bounded operator from $h^{s-2}_\sharp(\mathbb{R})$ to $h^s_\sharp(\mathbb{R})$. Otherwise, assume that we would only have that $\tilde{A}_\lambda^{-1}$ is bounded from $h^{s-2}_\sharp(\mathbb{R})$ to $h^{s'}_\sharp(\mathbb{R})$ where $s' < s$, then the mapping G is merely bounded from $h^s_\sharp(\mathbb{R}) \times I_\lambda \to h^{s'}_\sharp(\mathbb{R})$. In this case, the Fréchet derivative has the property that $D_a G(0,\lambda) : h^s_\sharp(\mathbb{R}) \to h^s_\sharp(\mathbb{R})$ (cf. Lemma 6.7(ii)) but is no longer a Fredholm operator from $h^s_\sharp(\mathbb{R}) \to h^{s'}_\sharp(\mathbb{R})$ since the co-dimension of its image is infinite. The Fredholm property at $\lambda = \lambda_*$, however, is important for applying the Crandall–Rabinowitz theorem for bifurcation.

The following lemma provides the necessary preparations to apply bifurcation theory to $G(a,\lambda) = 0$.

Lemma 6.7. *Let $s \geq \frac{5}{2}$. The map $G : h^s_\sharp(\mathbb{R}) \times I_{\lambda_*} \to h^s_\sharp(\mathbb{R})$ is a C^∞-map. Moreover the following holds:*

(i) *The function ϕ_k is continuously differentiable with respect to λ and $\psi_k(y;\lambda) := \partial_\lambda \phi_k(y;\lambda)$ satisfies*

$$L^\lambda_{0,k}\psi_k = k^2 V_0(y)\phi_k \text{ on } (0,\infty), \quad \psi_k(0;\lambda) = 0 \tag{6.20}$$

and $\psi_k'(0;\lambda) = k^2 \int_0^\infty V_0\phi_k^2 \, dy = O(k^2)$.

(ii) *The linearization of G about $a = 0$, given by*

$$D_a G(0,\lambda) = \mathrm{Id} - \tilde{A}_\lambda^{-1}P^* : h^s_\sharp(\mathbb{R}) \to h^s_\sharp(\mathbb{R})$$

is a Fredholm operator of index zero. Its kernel is trivial for $\lambda \in I_{\lambda_*}$, $\lambda \neq \lambda_*$ and it is given by $\mathrm{span}\{e^{k_*}\}$ if $\lambda = \lambda_*$.

(iii) *The mixed second derivative of G about $a = 0$ is given by*

$$D^2_{a\lambda}G(0,\lambda) = \tilde{A}_\lambda^{-1}B\tilde{A}_\lambda^{-1}P^* : h^s_\sharp(\mathbb{R}) \to h^{s+2}_\sharp(\mathbb{R}) \subset h^s_\sharp(\mathbb{R}),$$

where $B : h^{s+2}_\sharp(\mathbb{R}) \to h^s_\sharp(\mathbb{R})$ is the pointwise multiplication with $2\psi_k'(0;\lambda)$.

Proof. (i) We are only interested in $\psi_k(\cdot,\lambda) = \partial_\lambda \phi_k(\cdot,\lambda)$ on $[0,\infty)$. To find $\psi_k(\cdot,\lambda)$ we differentiate $L_{0,k}^\lambda \phi_k = 0$ on $(0,\infty)$, $\phi_k(0;\lambda)=1$ with respect to $\lambda \in I_{\lambda_*}$ and obtain (6.20). If we define $Q\phi_k(\cdot,\lambda): \mathbb{R} \to \mathbb{R}$ as the odd extension around $y=0$ of $\phi_k(\cdot,\lambda): (0,\infty) \to \mathbb{R}$ then we see that $\psi_k(\cdot,\lambda)$ is given by $\psi_k(\cdot,\lambda) = (L_{0,k}^\lambda)^{-1}(k^2 V_0 Q\phi_k(\cdot,\lambda))|_{[0,\infty)}$. Testing the differential equation in (6.20) with ϕ_k and noting that $L_{0,k}^\lambda \phi_k = 0$ on $(0,\infty)$ we find

$$\langle L_{0,k}^\lambda \psi_k, \phi_k\rangle_{L^2(0,\infty)} = (-\psi_k'\phi_k + \psi_k\phi_k')\big|_0^\infty + \langle \psi_k, L_{0,k}^\lambda \phi_k\rangle_{L^2(0,\infty)} = \psi_k'(0_+;\lambda) = O(k^2),$$

as claimed, in view of $\langle L_{0,k}^\lambda \psi_k, \phi_k\rangle_{L^2(0,\infty)} = \int_0^\infty k^2 V_0 \phi_k^2\, dy = O(k^2)$ because $V_0 \in L^\infty(\mathbb{R})$ and $(\tilde{L}3)$.

(ii) The mapping properties of G follow from Remark 6.6. The differentiability of the cubic nonlinearity $a*a*a$ with respect to a is also a straightforward property of the Banach algebra property of $h_\sharp^s(\mathbb{R})$. The differentiability property of G with respect to λ follows from differentiability of $\lambda \mapsto \phi_k'(0_+;\lambda)$ as given in (i). As in Lemma 4.4 the Fredholm property of $D_a(G(0,\lambda))$ is satisfied since it is a compact perturbation of the identity and the characterization of its kernel can be seen in a similar way using Lemma 6.4.

(iii) Note that $\frac{d}{d\lambda}\tilde{A}_\lambda^k = \frac{d}{d\lambda}A_\lambda^k = 2\psi_k'(0;\lambda)$. Since $\psi_k'(0;\lambda) = O(k^2)$ by (i) we have the mapping property $B: h_\sharp^{s+2}(\mathbb{R}) \to h_\sharp^s(\mathbb{R})$. $\square$

We are now in a position to apply the Crandall–Rabinowitz theorem for $G(a,\lambda): h_\sharp^s(\mathbb{R}) \times I_{\lambda_*} \to X_s$ for $s \geq \frac{5}{2}$ in order to proof Theorem 2.10 provided that the *transversality condition* in (2.6) is satisfied.

Proof of Theorem 2.10. The existence result follows from the Crandall–Rabinowitz theorem applied to $G(a,\lambda)=0$. Successfully applied, it provides an interval $I_{\lambda_*} \subset \mathbb{R}$ containing λ_*, and a smooth curve through $(0,\lambda_*)$ of the form

$$\{(a(\varepsilon),\lambda(\varepsilon)) \mid |\varepsilon| < \varepsilon_0\} \subset h^s(\mathbb{R}) \times I_{\lambda_*}$$

of nontrivial solutions of (6.11) with $\lambda(0)=\lambda_*$ and $D_\varepsilon a(0) = e^{k_*}$. The curve $(a(\varepsilon),\lambda(\varepsilon)) \subset h^s(\mathbb{R}) \times I_{\lambda_*}$ then translates via $\Phi(\varepsilon)(x,y) = \sum_{k\in\mathbb{N}} a_k(\varepsilon)\phi_k(y;\lambda)\sin(kx)$ and by Lemma 6.5 into the curve $\{(\Phi(\varepsilon),\lambda(\varepsilon)) \mid |\varepsilon| < \varepsilon_0\} \subset X_s \times I_{\lambda_*}$ of nontrivial solutions of (6.1) with the stated property. The Crandall–Rabinowitz theorem requires that the linearization

$$D_a G(0,\lambda_*) = \mathrm{Id} - \tilde{A}_{\lambda_*}^{-1} P^*: h_\sharp^s(\mathbb{R}) \to h_\sharp^s(\mathbb{R})$$

is a Fredholm operator of index zero with $\dim \ker D_a G(0,\lambda_*) = 1$ and the transversality condition

$$D_{a\lambda}^2 G(0,\lambda_*)e^{k_*} \notin \mathrm{range}\, D_a G(0,\lambda_*) \tag{6.21}$$

is satisfied. The Fredholm property is already shown in Lemma 6.7 (ii) and the kernel of $D_a G(0,\lambda_*)$ is one dimensional and spanned by e^{k_*}, that is

$$\ker D_a G(0,\omega_*) = \mathrm{span}\{e^{k_*}\}.$$

Concerning the transversality condition (6.21), assume on the contrary that there exists $b \in h^s(\mathbb{R})$ such that

$$D_{a\lambda}^2 G(0,\lambda_*)e^{k_*} = D_a G(0,\lambda_*)b.$$

Then,

$$\langle D_{a\lambda}^2 G(0,\lambda_*)e^{k_*}, e^{k_*}\rangle_{l^2(\mathbb{R})} = \langle D_a G(0,\lambda_*)b, e^{k_*}\rangle_{l^2(\mathbb{R})}.$$

Using the formulas from Lemma 6.7 (ii) and (iii) and the fact that $\tilde{A}_{\lambda_*}^{-1} P^* = P^*$ together with the symmetry of $\tilde{A}_{\lambda_*}^{-1}$ we obtain that

$$\begin{aligned}2\psi_{k_*}'(0;\lambda_*) &= \langle Be^{k_*}, e^{k_*}\rangle_{l^2} = \langle D_{a\lambda}^2 G(0,\lambda_*)e^{k_*}, e^{k_*}\rangle_{l^2(\mathbb{R})} = \langle D_a G(0,\lambda_*)b, e^{k_*}\rangle_{l^2(\mathbb{R})}\\ &= \langle b - P^* b, e^{k_*}\rangle_{l^2(\mathbb{R})} = 0.\end{aligned} \tag{6.22}$$

But due to Lemma 6.7(i) this is a contradiction to (2.6). This finishes the proof of Theorem 2.10.

Similarly as in the previous section, we determine the bifurcation formulas. The Fréchet derivatives of G with respect to a are given by

$$D_a G(a, \lambda)e^{k_*} = e^{k_*} - \tilde{A}_\lambda^{-1}\left(\frac{3}{4}\gamma M(a * a * e^{k_*}) + e^{k_*}\right)$$

$$D_{aa}^2 G(a, \lambda)[e^{k_*}, e^{k_*}] = -\frac{3}{2}\gamma \tilde{A}_\lambda^{-1}\left(M(a * e^{k_*} * e^{k_*})\right)$$

$$D_{aaa}^3 G(a, \lambda)[e^{k_*}, e^{k_*}, e^{k_*}] = -\frac{3}{2}\gamma \tilde{A}_\lambda^{-1}\left(M(e^{k_*} * e^{k_*} * e^{k_*})\right)$$

where M is defined as in (6.19).

Proposition 6.8. *Let $\{(a(\varepsilon), \omega(\varepsilon)) \mid |\varepsilon| < \varepsilon_0\} \subset h^s(\mathbb{R}) \times I_{\lambda_*}$ be the local bifurcation curve found in Theorem 2.10 corresponding to the bifurcation point $(0, \lambda_*)$. Then*

$$\dot{\lambda}(0) = 0 \qquad and \qquad \ddot{\lambda}(0) = -\frac{3\gamma}{4\int_0^\infty V_0(y)\phi_{k_*}^2\,dy}.$$

Proof. The proof follows essentially the lines of the proof of Proposition 4.7. We obtain that $\dot{\lambda}(0) = 0$, which is due to the cubic character of the nonlinearity and

$$\ddot{\lambda}(0) = -\frac{1}{3}\frac{\langle D_{aaa}^3 G(0, \lambda_*)[e^{k_*}, e^{k_*}, e^{k_*}], e^{k_*}\rangle}{\langle D_{a\lambda}^2 G(0, \lambda_*)e^{k_*}, e^{k_*}\rangle}.$$

Due to Lemma 6.7(i) and (6.22) the denominator is given by

$$\langle D_{a\lambda}^2 G(0, \lambda_*)e^{k_*}, e^{k_*}\rangle = 2\int_0^\infty k^2 V_0(y)\psi_{k_*}^2\,dy,$$

and the numerator reads

$$\langle D_{aaa}^3 G(0, \lambda_*)[e^{k_*}, e^{k_*}, e^{k_*}], e^{k_*}\rangle = -\frac{3}{2}\gamma k_*^2(e^{k_*} * e^{k_*} * e^{k_*})_{k_*}.$$

Since $(e^{k_*} * e^{k_*} * e^{k_*})_{k_*} = -3$, as shown in Lemma A.2, the statement follows. $\square$

7. Examples for distributional Γ

In what follows we prove Corollaries 2.13 and 2.14, which state the existence of traveling waves for (1.4) in the specific cases, when the potentials are given as in (P1) and (P2), respectively.

7.1. (P1) V a δ-potential on a constant background

We consider the particular case when $V_0 = 1$, $W = 0$ so that we have a positive constant background potential with a multiple of a delta potential on top, i.e.,

$$V(\lambda, y) = \lambda + \alpha\delta_0(y).$$

We verify the conditions $(\tilde{L}0) - (\tilde{L}3)$ of Theorem 2.10; thereby proving part one of Corollary 2.13. Let us fix a wavenumber $k_* \in \mathbb{N}$ and a value $\lambda_* < 1$. We determine $\alpha > 0$ from

$$\alpha = \frac{2\sqrt{1 - \lambda_*}}{k_*}.$$

Notice that the transversality condition and ($\tilde{L}0$) are trivially satisfied. Moreover, the validity of ($\tilde{L}1$) and ($\tilde{L}3$) follow immediately from Remark 6.2, since $1 - \lambda V_0 - W = 1 - \lambda > 0$. Condition ($\tilde{L}2$) is exactly the same as ($L2$) since our operator L_λ^k is the same as the one considered in Corollary 2.5 of Case (P1). Since the choice of α, k_*, λ_* is the same as in Corollary 2.5 condition ($\tilde{L}1$) holds and we are finished with treating this example.

Now, Corollary 2.13 follows from Theorem 2.10 and the bifurcation formulas are an immediate consequence of Proposition 6.8.

7.2. (P2) V a δ-potential on a step background

Now, we consider the case when $V_0 = \mathbf{1}_{|y|\geq b}$, $W = \beta\mathbf{1}_{|y|<b}$ for some $b > 0$ so that the potential V is given by

$$V(\lambda, y) = \lambda\mathbf{1}_{|y|\geq b} + \beta\mathbf{1}_{|y|<b} + \alpha\delta_0(y),$$

Again we verify the conditions ($\tilde{L}0$) − ($\tilde{L}3$) of Theorem 2.10; thereby proving Corollary 2.14. First we fix a wavenumber $k_* \in \mathbb{N}$ and a value $\lambda_* < 1$. According to Corollary 2.14 we have to distinguish between the case $\beta < 1$, $\beta > 1$, and $\beta = 1$. Notice that the transversality condition and ($\tilde{L}0$) are trivially satisfied for all $\beta \in \mathbb{R}$.

Let us begin with the case $\beta < 1$. The validity of ($\tilde{L}1$) and ($\tilde{L}3$) follow immediately from Remark 6.2, since $1 - \lambda V_0 - W = (1 - \lambda)\mathbf{1}_{|y|\geq b} + (1 - \beta\mathbf{1}_{|y|<b}) > 0$. It remains to consider ($\tilde{L}2$). But again the operator L_λ^k is the same as the one considered in Corollary 2.5 of Case (P1) and the choice of α in Corollary 2.14 is exactly the same as in Corollary 2.6 of Case (P1). Hence ($\tilde{L}1$) holds and this example is complete.

Next we consider the case $\beta > 1$. Here we have made the choices

$$\sqrt{\beta - 1}b = \pi$$

and

$$\alpha = \frac{2\sqrt{1 - \lambda_*}}{k_*}.$$

We are left to verify ($\tilde{L}1$) − ($\tilde{L}3$) of Theorem 2.10. For ($\tilde{L}1$) we need to consider the operator $L_{0,\lambda}^k = -\frac{d^2}{dy^2} + k^2(1 - \lambda\mathbf{1}_{|y|\geq b} - \beta\mathbf{1}_{|y|<b}) : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ which is self-adjoint with $\sigma_{ess}(L_{0,\lambda}^k) \subset [k^2(1 - \lambda), \infty)$. Thus $0 \in \rho(L_{0,\lambda}^k)$ if and only if $L_{0,\lambda}^k\phi = 0$ for some $\phi \in H^2(\mathbb{R})$ implies that $\phi = 0$. In other words: we need to rule out that $L_{0,\lambda}^k$ has a zero eigenvalue. This can be seen from Lemma A.6 in the Appendix if we set $\alpha = 0$ (no delta potential in the equation) and $\mu = 0$, i.e., $\tilde{\lambda} = \lambda$ and $\tilde{\beta} = \beta$. Moreover, we need to make the obvious changes $\sqrt{1 - \beta} = i\sqrt{\beta - 1}$ and $\sinh(ix) = i\sin(x)$, $\cosh(ix) = \cos(x)$. Following the ansatz (A.1) for the eigenfunction we obtain $c_0 = d_0$ and $c_1 = d_1$ due to the C^1-matching at $x = 0$. Moreover, the choice of $\sqrt{\beta - 1}b = \pi$ results in the invertible matrices

$$M_\pm = \begin{pmatrix} 0 & -e^{-k\sqrt{1-\tilde{\lambda}}b} \\ \sqrt{1 - \tilde{\beta}}(-1)^k & \pm\sqrt{1 - \tilde{\lambda}}e^{-k\sqrt{1-\tilde{\lambda}}b} \end{pmatrix}.$$

Hence the conclusion $c_1 = -d_1$ from Lemma A.6 holds and leads to $c_1 = d_1 = 0$. An inspection of the C^1-compatibility at $y = \pm b$ then yields $c_2 = d_2 = c_0 = d_0 = 0$. Therefore, there is no zero-eigenvalue of $L_{0,\lambda}^k$ for any $k \in \mathbb{N}$ and any $\lambda \in (-\infty, 1)$ and ($\tilde{L}1$) holds.

Concerning ($\tilde{L}2$) we need to study a zero-eigenvalue of L_λ^k. The answer is again given by Lemma A.6 in the Appendix since we already know the invertibility of the matrices $M_\pm$. Hence the eigenvalue condition is given by (5.2) with the obvious changes from the hyperbolic functions to the trigonometric function and reads

$$\frac{k\alpha}{\sqrt{\beta - 1}} = \frac{-\sqrt{\beta - 1}\sin(k\sqrt{\beta - 1}b) + \sqrt{1 - \lambda}\cos(k\sqrt{\beta - 1}b)}{\sqrt{\beta - 1}\cos(k\sqrt{\beta - 1}b) + \sqrt{1 - \lambda}\sin(k\sqrt{\beta - 1}b)}.$$

In view of $\sqrt{\beta - 1}b = \pi$ this reduces to

$$\alpha = \frac{2\sqrt{1-\lambda}}{k}$$

and hence L_λ^k has a zero-eigenvalue if and only if $k = k_*$ and $\lambda = \lambda_*$. Thus $(\tilde{L}2)$ holds. Finally, in order to verify $(\tilde{L}3)$, we compute the function ϕ_k which solves $L_\lambda^k \phi_k = 0$ on $(0, \infty)$ with $\phi_k(0) = 1$. From [Lemma A.6](#) we obtain

$$\begin{cases} \phi_k(y, \lambda) = \cos(k\sqrt{\beta-1}y) + ic_1 \sin(k\sqrt{\beta-1}y), & y \in [0, b], \\ \phi_k(y, \lambda) = c_2 e^{-k\sqrt{1-\lambda}y}, & y \geq b \end{cases}$$

with $c_1 = \frac{-\sqrt{1-\lambda}}{i\sqrt{\beta-1}}$ and $c_2 = e^{k\sqrt{1-\lambda}b}(-1)^k$. Computing the L^2-norm of ϕ_k we find that

$$\frac{1}{2}\|\phi_k(\cdot, \lambda)\|_{L^2(\mathbb{R})}^2 = \int_0^b \left(\cos(k\sqrt{\beta-1}y) - \frac{\sqrt{1-\lambda}}{\sqrt{\beta-1}}\sin(k\sqrt{\beta-1}y) \right)^2 dy + \int_b^\infty e^{2k\sqrt{1-\lambda}(b-y)}\, dy$$

$$= \frac{1}{2k\sqrt{1-\lambda}} + \frac{1}{2}\left(\frac{1-\lambda}{\beta-1} + 1 \right) b \leq C \left(1 + \frac{1}{k} \right),$$

where the constant $C > 0$ is independent of k and can be chosen uniformly for λ sufficiently close to λ_*. This shows the validity of $(\tilde{L}3)$.

The last case to be considered is $\beta = 1$. Also here, we are left to verify conditions $(\tilde{L}1)$, $(\tilde{L}2)$, and $(\tilde{L}3)$. First we find that in this case with $\tilde{\lambda} = \lambda$ and $\tilde{\beta} = \beta = 1$ condition [(5.2)](#) is replaced by

$$\frac{k\alpha}{2} = \frac{\sqrt{1-\lambda}}{1 + \sqrt{1-\lambda}kb}, \tag{7.1}$$

which follows from a suitable adaptation of [Lemma A.6](#). A zero eigenvalue of $L_{0,\lambda}^k$ correspond to values k, λ satisfying [(7.1)](#) with $\alpha = 0$ which is impossible. Since $\sigma_{ess}(L_{0,\lambda}^k) = [k^2(1-\lambda), \infty)$ this shows that $(\tilde{L}1)$ holds. If we recall the definition of α, i.e.,

$$\alpha = \frac{2\sqrt{1-\lambda_*}}{k_*(1 + \sqrt{1-\lambda_*}k_*b)}$$

and compare with the 0-eigenvalue condition [(7.1)](#) we see that this ensures that 0 is a (simple) eigenvalue of L_λ^k if and only if $\lambda = \lambda_*$ and $k = k_*$. Hence, $(\tilde{L}2)$ holds. To see $(\tilde{L}3)$ we compute (also with the help of an adaptation of [Lemma A.6](#)) that the functions ϕ_k solving $L_\lambda^k \phi_k = 0$ on $(0, \infty)$ with $\phi_k(0) = 1$ are given by

$$\begin{cases} \phi_k(y, \lambda) = 1 + c_1 y, & y \in [0, b], \\ \phi_k(y, \lambda) = c_2 e^{-k\sqrt{1-\lambda}y}, & y \geq b \end{cases}$$

with $c_1 = \frac{-\sqrt{1-\lambda}k}{kb\sqrt{1-\lambda}+1}$ and $c_2 = \frac{e^{k\sqrt{1-\lambda}b}}{kb\sqrt{1-\lambda}+1}$. From this we directly calculate that $\|\phi_k(\cdot, \lambda)\|_{L^2(\mathbb{R})}^2 = O(1)$ as $k \to \infty$ uniformly for λ sufficiently close to λ_*. Hence, $(\tilde{L}3)$ holds.

Now, [Corollary 2.14](#) follows from [Theorem 2.10](#).

Acknowledgments

Appendix. Auxiliary results

Lemma A.1. *Let $A(x) = \sum_{k\in\mathbb{N}} a_k \sin(kx)$, then*

$$A^3(x) = -\frac{1}{4}\sum_{k\in\mathbb{N}}(a * a * a)_k \sin(kx),$$

where $a = (a_k)_{k \in \mathbb{Z}}$ is an infinite sequence with $a_k = -a_{-k}$ for all $k \in \mathbb{Z}$. The notation $(a * a * a)_k$ is used to denote the kth entry in the sequence obtained by convolution $a * a * a$.

Proof. If a is a sequence as above then using $a_k = -a_{-k}$ for all k we find that

$$\sum_{k \in \mathbb{Z}} \left(-\frac{1}{2} i a_k \right) e^{ikx} = \sum_{k \in \mathbb{Z}} \frac{1}{2} a_k \sin(kx) = \sum_{k \in \mathbb{N}} a_k \sin(kx) = A(x),$$

and

$$A^3(x) = \sum_{k \in \mathbb{Z}} \frac{1}{8} i (a * a * a)_k e^{ikx}.$$

We are going to show that the Fourier coefficients $(a * a * a)_k$ are odd with respect to k. Notice first that

$$(a * a * a)_k = \sum_{j \in \mathbb{Z}} \left(\sum_{l \in \mathbb{Z}} a_{k-j-l} a_l \right) a_j.$$

We also have that

$$(a * a * a)_{-k} = \sum_{j \in \mathbb{Z}} \left(\sum_{l \in \mathbb{Z}} a_{-k-j-l} a_l \right) a_j = -\sum_{j \in \mathbb{Z}} \left(\sum_{l \in \mathbb{Z}} a_{k+j+l} a_l \right) a_j$$

$$= -\sum_{j \in \mathbb{Z}} \left(\sum_{l \in \mathbb{Z}} a_{k-j-l} a_{-l} \right) a_{-j} = -\sum_{j \in \mathbb{Z}} \left(\sum_{l \in \mathbb{Z}} a_{k-j-l} a_l \right) a_j = -(a * a * a)_k.$$

From this we deduce that

$$A^3(x) = \sum_{k \in \mathbb{Z}} \frac{1}{8} i (a * a * a)_k e^{ikx} = -\frac{1}{8} \sum_{k \in \mathbb{Z}} (a * a * a)_k \sin(xk) = -\frac{1}{4} \sum_{k \in \mathbb{N}} (a * a * a)_k \sin(xk). \quad \square$$

Lemma A.2. *Let $k \in \mathbb{N}$ and e^{k_*} be a sequence such that $e_k^{k_*} = 0$ if $k \neq \pm k_*$, $e_{k_*}^{k_*} = -e_{-k_*}^{k_*} = 1$. Then,*

$$(e^{k_*} * e^{k_*} * e^{k_*})_k = \begin{cases} 1, & \text{if} \quad k = 3k_*, \\ -1, & \text{if} \quad k = -3k_*, \\ -3, & \text{if} \quad k = k_*, \\ 3, & \text{if} \quad k = -k_*. \end{cases}$$

Proof. The convolution $e^{k_*} * e^{k_*} * e^{k_*}$ is given by

$$(e^{k_*} * e^{k_*} * e^{k_*})_k = \sum_{j \in \mathbb{Z}} \left(\sum_{l \in \mathbb{Z}} e_{k-j-l}^{k_*} e_l^{k_*} \right) e_j^{k_*} = \sum_{l \in \mathbb{Z}} e_{k-k_*-l}^{k_*} e_l^{k_*} - \sum_{l \in \mathbb{Z}} e_{k+k_*-l}^{k_*} e_l^{k_*}$$

$$= e_{k-2k_*}^{k_*} - 2 e_k^{k^*} + e_{k+2k_*}^{k_*}$$

and the claim follows. $\quad \square$

Lemma A.3. *Let $a \in h^s(\mathbb{R})$ for some $s \geq 0$ and define $\Phi(x, y) = \sum_{k \in \mathbb{N}} a_k \phi_k(y; \lambda) \sin(kx)$ for $x \in \mathbb{T}$ and $y \in \mathbb{R}$. Then*

  (i) $\Phi \in H^s(\mathbb{T}; L^2(\mathbb{R}))$
  (ii) $\Phi \in H^{s-1}(\mathbb{T}; H^1(\mathbb{R}))$
 (iii) $\Phi \in H^{s-2}(\mathbb{T}; H^2(0, \infty))$
 (iv) $\Phi \in C(\mathbb{R}; H^{s-\frac{1}{2}}(\mathbb{T}))$
  (v) $\Phi \in C^1(\mathbb{R}; H^{s-\frac{3}{2}}(\mathbb{T}))$

Proof. We verify that

$$\|\Phi\|^2_{H^s(\mathbb{T};L^2(\mathbb{R}))} \le C \sum_{k\in\mathbb{N}} a_k^2 k^{2s} \|\phi_k\|^2_{L^2(\mathbb{R})} \le C\|a\|^2_{h^s(\mathbb{R})},$$

$$\|\Phi\|^2_{H^{s-1}(\mathbb{T};H^1(\mathbb{R}))} \le C \sum_{k\in\mathbb{N}} a_k^2 k^{2s-2} \|\phi_k'\|^2_{L^2(\mathbb{R})} \le C\|a\|^2_{h^s(\mathbb{R})},$$

$$\|\Phi\|^2_{H^{s-2}(\mathbb{T};H^2(0,\infty))} \le C \sum_{k\in\mathbb{N}} a_k^2 k^{2s-4} \|\phi_k''\|^2_{L^2(0,\infty)}$$

$$\le C1 + \|1 - \lambda V_0 - W\|_{L^\infty(\mathbb{R})} \sum_{k\in\mathbb{N}} a_k^2 k^{2s} \|\phi_k\|^2_{L^2(0,\infty)} \le C\|a\|^2_{h^s(\mathbb{R})},$$

$$\|\Phi\|^2_{C(\mathbb{R};H^{s-\frac{1}{2}}(\mathbb{T}))} \le C \sum_{k\in\mathbb{N}} a_k^2 k^{2s-1} \|\phi_k\|^2_{L^\infty(\mathbb{R})} \le C\|a\|^2_{h^s(\mathbb{R})},$$

$$\|\Phi\|^2_{C^1(\mathbb{R};H^{s-\frac{3}{2}}(\mathbb{T}))} \le C \sum_{k\in\mathbb{N}} a_k^2 k^{2s-3} \|\phi_k'\|^2_{L^\infty(\mathbb{R})} \le C\|a\|^2_{h^s(\mathbb{R})}. \quad \square$$

Lemma A.4. *For $s \ge 1$ the space $h^s(\mathbb{R})$ is a Banach algebra with respect to convolution.*

Proof. In this proof we use the l^1-norm $\|a\|_{l^1(\mathbb{R})} = \sum_{k\in\mathbb{Z}} |a_k|$ for a sequence $a = (a_k)_{k\in\mathbb{Z}} \in l^1(\mathbb{R})$, i.e., the Banach space of all real sequences with finite l^1-norm. Due to convexity we have the inequality

$$|k|^s \le 2^{s-1}(|k-l|^s + |l|^s).$$

Therefore, if $a, b \in h^s(\mathbb{R})$ then

$$|k|^s(a*b)_k = |k|^s \left| \sum_{l\in\mathbb{Z}} a_{k-l} b_l \right| \le 2^{s-1} \sum_{l\in\mathbb{Z}} |k-l|^s |a_{k-l}||b_l| + |a_{k-l}||l|^s|b_l|.$$

Using the convolution inequality $\|\tilde{a} * \tilde{b}\|_{l^2} \le \|\tilde{a}\|_{l^2}\|\tilde{b}\|_{l^1}$ once for $(\tilde{a})_k = |k|^s|a_k|$, $(\tilde{b})_k = |b_k|$ and once for $(\tilde{a})_k = |a_k|$, $(\tilde{b})_k = |k|^s|b_k|$ we get

$$\|a*b\|_{h^s(\mathbb{R})} \le 2^{s-1}(\|a\|_{h^s(\mathbb{R})}\|b\|_{l^1(\mathbb{R})} + \|a\|_{l^1(\mathbb{R})}\|b\|_{h^s(\mathbb{R})}).$$

Finally, $a \in h^s(\mathbb{R})$ implies $a \in l^1(\mathbb{R})$ due to

$$\sum_{k\in\mathbb{Z}} |a_k| = \sum_{k\in\mathbb{Z}} |a_k|(|k|+1)\frac{1}{|k|+1} \le C\|a\|_{h^1(\mathbb{R})} \le C\|a\|_{h^s(\mathbb{R})}. \quad \square$$

Lemma A.5. *Let $L = -\frac{d^2}{dy^2} + q(y)$ with $q \in L^\infty(\mathbb{R})$ be a self-adjoint operator on $L^2(\mathbb{R})$ with domain $D(L) = H^2(\mathbb{R})$. Then, for any $\alpha \in \mathbb{R}$, we have that $L_\alpha := L + \alpha\delta_0$ is self-adjoint with domain $D(L_\alpha) = \{u \in H^1(\mathbb{R}) \cap (H^2(0,\infty) \cup H^2(-\infty,0)) \mid u'(0_+) - u'(0_-) = -\alpha u(0)\}$. Moreover for any $\alpha \in \mathbb{R}$ the following holds:*

(i) For sufficiently large $\mu > 0$ we have that $(L_\alpha + \mu)^{-1} : H^{-1}(\mathbb{R}) \to H^1(\mathbb{R})$ is bounded.
(ii) $\sigma_{ess}(L_\alpha) = \sigma_{ess}(L)$.

Proof. A proof of the self-adjointness of L_α for any $\alpha \in \mathbb{R}$ is given in [4]. For (i) we first note that L_α is a semi-bounded self-adjoint operator so that $L_\alpha + \mu$ is a positive operator for $\mu > 0$ sufficiently large. Its bilinear form $b_{L_\alpha+\mu} : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{R}$ is coercive and equivalent to the standard $H^1(\mathbb{R})$-inner product. Therefore, any $f \in H^{-1}(\mathbb{R})$ can be represented by a unique $u \in H^1(\mathbb{R})$ such that $b_{L_\alpha+\mu}(u,\phi) = f(\phi)$ for any $\phi \in H^1(\mathbb{R})$ and $b_{L_\alpha+\mu}(u,u) = \|\phi\|^2_{H^{-1}(\mathbb{R})}$. This proves (i).

For (ii) we may take $\lambda < 0$ sufficiently negative such that $\lambda \in \rho(L) \cap \rho(L_\alpha) \cap \mathbb{R}$ since both L, L_α are semi-bounded from below. Using (i) we may also assume λ sufficiently negative that $(\lambda - L_\alpha)^{-1} : H^{-1}(\mathbb{R}) \to$

$H^1(\mathbb{R})$ is bounded. By Weyl's criterion it is sufficient to show that the operator $W_\lambda := (\lambda - L)^{-1} - (\lambda - L_\alpha)^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is compact in order to prove the statement. Since

$$W_\lambda = (\lambda - L_\alpha)^{-1} \circ ((\lambda - L_\alpha)(\lambda - L)^{-1} - \mathrm{Id}) = \underbrace{(\lambda - L_\alpha)^{-1}}_{H^{-1}(\mathbb{R}) \to H^1(\mathbb{R}) \subset L^2(\mathbb{R})} \circ \underbrace{(L - L_\alpha)}_{H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})} \circ \underbrace{(\lambda - L)^{-1}}_{L^2(\mathbb{R}) \to H^1(\mathbb{R})}$$

and since $L - L_\alpha = -\alpha \delta_0 : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})$ is a bounded operator with 1-dimensional range spanned by δ_0 we see that W_λ is indeed compact. This finishes the proof. $\square$

Lemma A.6. *Let $\tilde{\lambda}, \tilde{\beta} < 1$. Then the eigenvalue problem* (5.1) *is solvable for $\phi \in D(L_\lambda^k)$ if and only if* (5.2) *holds. In this case the eigenspace is one-dimensional.*

Proof. Solutions of the differential equation in (5.1) have to be of the form

$$\phi(y, \lambda) = \begin{cases} c_2 e^{-k\sqrt{1-\tilde{\lambda}}y}, & y \geq b, \\ c_0 \cosh(k\sqrt{1-\tilde{\beta}}y) + c_1 \sinh(k\sqrt{1-\tilde{\beta}}y), & y \in [0, b], \\ d_0 \cosh(k\sqrt{1-\tilde{\beta}}y) + d_1 \sinh(k\sqrt{1-\tilde{\beta}}y), & y \in [-b, 0], \\ d_2 e^{k\sqrt{1-\tilde{\lambda}}y}, & y \leq -b \end{cases} \tag{A.1}$$

with C^1-compatibility conditions at $x = \pm b$ and continuity at $x = 0$. The latter implies $c_0 = d_0$ and the condition $\phi'(0+) - \phi'(0-) + k^2 \alpha \phi(0) = 0$ at $x = 0$ translates into

$$k\sqrt{1-\tilde{\beta}}(c_1 - d_1) + k^2 \alpha c_0 = 0. \tag{A.2}$$

The C^1-compatibility leads to the following set of four equations

$$\begin{cases} c_0 \cosh(k\sqrt{1-\tilde{\beta}}b) + c_1 \sinh(k\sqrt{1-\tilde{\beta}}b) & = c_2 e^{-k\sqrt{1-\tilde{\lambda}}b}, \\ \sqrt{1-\tilde{\beta}}\left(c_0 \sinh(k\sqrt{1-\tilde{\beta}}b) + c_1 \cosh(k\sqrt{1-\tilde{\beta}}b)\right) & = -\sqrt{1-\tilde{\lambda}}c_2 e^{-k\sqrt{1-\tilde{\lambda}}b}, \\ d_0 \cosh(k\sqrt{1-\tilde{\beta}}b) - d_1 \sinh(k\sqrt{1-\tilde{\beta}}b) & = d_2 e^{-k\sqrt{1-\tilde{\lambda}}b}, \\ \sqrt{1-\tilde{\beta}}\left(-d_0 \sinh(k\sqrt{1-\tilde{\beta}}b) + d_1 \cosh(k\sqrt{1-\tilde{\beta}}b)\right) & = \sqrt{1-\tilde{\lambda}}d_2 e^{-k\sqrt{1-\tilde{\lambda}}b}. \end{cases}$$

These four equations can be written as

$$\begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} -c_0 \cosh(k\sqrt{1-\tilde{\beta}}b) \\ -\sqrt{1-\tilde{\beta}}c_0 \sinh(k\sqrt{1-\tilde{\beta}}b) \\ -c_0 \cosh(k\sqrt{1-\tilde{\beta}}b) \\ \sqrt{1-\tilde{\beta}}c_0 \sinh(k\sqrt{1-\tilde{\beta}}b) \end{pmatrix}$$

with

$$M_\pm = \begin{pmatrix} \pm \sinh(k\sqrt{1-\tilde{\beta}}b) & -e^{-k\sqrt{1-\tilde{\lambda}}b} \\ \sqrt{1-\tilde{\beta}}\cosh(k\sqrt{1-\tilde{\beta}}b) & \pm\sqrt{1-\tilde{\lambda}}e^{-k\sqrt{1-\tilde{\lambda}}b} \end{pmatrix}.$$

Since both M_+ and M_- are invertible we see that w.l.o.g. we can choose $c_0 = 1$. Moreover, the structure of the linear systems yields that $c_1 = -d_1$ and $c_2 = d_2$. Finally, solving for c_1, c_2 we get

$$c_1 = -\frac{\sqrt{1-\tilde{\beta}}\sinh(k\sqrt{1-\tilde{\beta}}b) + \sqrt{1-\tilde{\lambda}}\cosh(k\sqrt{1-\tilde{\beta}}b)}{\sqrt{1-\tilde{\beta}}\cosh(k\sqrt{1-\tilde{\beta}}b) + \sqrt{1-\tilde{\lambda}}\sinh(k\sqrt{1-\tilde{\beta}}b)},$$

$$c_2 = e^{k\sqrt{1-\tilde{\lambda}}b}(c_1 \sinh(k\sqrt{1-\tilde{\beta}}b) + \cosh(k\sqrt{1-\tilde{\beta}}b)).$$

Inserting c_1, $d_1 = -c_1$ and $c_0 = 1$ into (A.2) yields the condition (5.2) as claimed. It also shows that the eigenspace is one-dimensional (the only degree of freedom is the choice of c_0 which we took to be 1 w.l.o.g.). $\square$

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