ANNALES MATHÉMATIQUES

BLAISE PASCAL

Takeshi Katsura, Masahito Ogawa & Airi Takeuchi

On the magic square C*-algebra of size 4

Volume 29, nº 1 (2022), p. 99-148.

https://doi.org/10.5802/ambp.408

Publication éditée par le laboratoire de mathématiques Blaise Pascal de l'université Clermont Auvergne, UMR 6620 du CNRS Clermont-Ferrand — France

CENTRE MERSENNE

Publication membre du Centre Mersenne pour l'édition scientifique ouverte
http://www.centre-mersenne.org/
e-ISSN : 2118-7436

Annales mathématiques Blaise Pascal **29**, 99-148 (2022)

On the magic square C*-algebra of size 4

Takeshi Katsura
Masahito Ogawa
Airi Takeuchi

Abstract

In this paper, we investigate the structure of the magic square C*-algebra $A(4)$ of size 4. We show that a certain twisted crossed product of $A(4)$ is isomorphic to the homogeneous C*-algebra $M_4(C(\mathbb{R}P^3))$. Using this result, we show that $A(4)$ is isomorphic to the fixed point algebra of $M_4(C(\mathbb{R}P^3))$ by a certain action. From this concrete realization of $A(4)$, we compute the K-groups of $A(4)$ and their generators.

Introduction

Let $n = 1, 2, \dots$. The magic square C*-algebra $A(n)$ of size n is the underlying C*-algebra of the quantum group $A_s(n)$ defined by Wang in [9] as a free analogue of the symmetric group $\mathfrak{S}_n$. In [2, Proposition 1.1], it is claimed that for $n = 1, 2, 3$, $A(n)$ is isomorphic to $\mathbb{C}^{n!}$, and hence commutative and finite dimensional. We give the proof of this fact in Proposition 2.1. In [3, Proposition 1.2] it is proved that for $n \geq 4$, $A(n)$ is non-commutative and infinite dimensional. We see that for $n \geq 5$, $A(n)$ is not exact (Proposition 2.5). Something interesting happens for $A(4)$ (see [1, 2, 3]). In [3], Banica and Moroianu constructed a $*$-homomorphism from $A(4)$ to $M_4(C(SU(2)))$ by using the Pauli matrices, and showed that it is faithful in some weak sense. In [2], Banica and Collins showed that the $*$-homomorphism above is in fact faithful by using integration techniques. We reprove this fact in Corollary 7.9. Our method uses a twisted crossed product. The following is the first main result.

Theorem A (Theorem 3.6)**.** *The twisted crossed product* $A(4) \rtimes_\alpha^{\mathrm{tw}} (K \times K)$ *is isomorphic to* $M_4(C(\mathbb{R}P^3))$.

The notation in this theorem is explained in Section 3. From this theorem, we see that the magic square C*-algebra $A(4)$ of size 4 is isomorphic to a C^*-subalgebra of the homogeneous C^*-algebra $M_4(C(\mathbb{R}P^3))$. The next theorem, which is the second main result, expresses this C^*-subalgebra as a fixed point algebra of $M_4(C(\mathbb{R}P^3))$.

The first named author is supported by JSPS KAKENHI Grant Number JP18K03345. The third named author is supported by Masason Foundation.
Keywords: C*-algebra, magic square C*-algebra, twisted crossed product, K-theory.
2020 *Mathematics Subject Classification*: 46L05, 46L55, 46L80.

Theorem B (Theorem 8.2)**.** *The fixed point algebra $M_4(C(\mathbb{R}P^3))^\beta$ of the action β is isomorphic to $A(4)$.*

See Section 8 for the definition of the action β. We remark that Theorem B can be also obtained by combining [1, Theorem 3.1, Theorem 5.1] and [4, Proposition 3.3]. Our proof of Theorem B uses a twisted crossed product instead of quantum groups used in [1, 4], and gives an explicit and straightforward isomorphism.

Since β is concrete, we can analyze $M_4(C(\mathbb{R}P^3))^\beta$ very explicitly. In particular, we can compute the K-groups of $M_4(C(\mathbb{R}P^3))^\beta$ explicitly. As a corollary we get the following which is the third main result.

Theorem C (Theorem 15.16)**.** *We have $K_0(A(4)) \cong \mathbb{Z}^{10}$ and $K_1(A(4)) \cong \mathbb{Z}$. More specifically, $K_0(A(4))$ is generated by $\{[p_{i,j}]_0\}_{i,j=1}^4$, and $K_1(A(4))$ is generated by $[u]_1$.*

The positive cone $K_0(A(4))_+$ of $K_0(A(4))$ is generated by $\{[p_{i,j}]_0\}_{i,j=1}^4$ as a monoid.

Note that $\{p_{i,j}\}_{i,j=1}^4$ is the generating set of $A(4)$ consisting of projections, and u is the defining unitary (see Definition 15.15). We should remark that the computation $K_0(A(4)) \cong \mathbb{Z}^{10}$ and $K_1(A(4)) \cong \mathbb{Z}$ and that $K_0(A(4))$ is generated by $\{[p_{i,j}]_0\}_{i,j=1}^4$ were already obtained by Voigt in [8] by using Baum–Connes conjecture for quantum groups. In fact, Voigt got the corresponding results for $A(n)$ with $n \geq 4$. Theorem C gives totally different proofs for the results by Voigt in [8] by analyzing the structure of $A(4)$ directly which seems not to be applied to $A(n)$ for $n > 4$. That $K_1(A(4))$ is generated by $[u]_1$ was not obtained in [8], and is a new result. Combining this result with the computation that $K_1(A(n)) \cong \mathbb{Z}$ for $n \geq 4$ in [8] and the easy fact that the surjection $A(n) \to A(4)$ in Corollary 2.4 for $n \geq 4$ sends the defining unitary to the direct sum of the defining unitary and the units, we obtain that $K_1(A(n)) \cong \mathbb{Z}$ is generated by the K_1 class of the defining unitary for $n \geq 4$. We would like to thank Christian Voigt for the discussion about this observation.

This paper is organized as follows. In Section 1, we define magic square C*-algebras $A(n)$ and their abelianizations $A^{\mathrm{ab}}(n)$. In Section 2, we investigate $A(n)$ for $n \neq 4$. From Section 3, we study $A(4)$. In Section 3, we introduce the twisted crossed product $A(4) \rtimes_\alpha^{\mathrm{tw}} (K \times K)$, and state Theorem A. We give the proof of Theorem A from Section 4 to Section 7. In Section 8, we state and prove Theorem B. From Section 9 to Section 15, we prove Theorem C.

Acknowledgments

The first named author thanks Junko Muramatsu for helping the research in the beginning of the research. The authors are grateful to Makoto Yamashita for calling attention to [8], and to Christian Voigt for the discussion on the results in [8]. The authors are also grateful to the anonymous referees for the evaluation of our paper and for the constructive critics.

1. Definitions of and basic facts on magic square C*-algebras

Definition 1.1. Let $n = 1, 2, \ldots$. The *magic square C*-algebra of size n* is the universal unital C*-algebra $A(n)$ generated by $n \times n$ projections $\{p_{i,j}\}_{i,j=1}^n$ satisfying

$$\sum_{i=1}^n p_{i,j} = 1 \quad (j = 1, 2, \ldots, n), \qquad \sum_{j=1}^n p_{i,j} = 1 \quad (i = 1, 2, \ldots, n).$$

Remark 1.2. The magic square C*-algebra $A(n)$ is the underlying C*-algebra of the quantum group $A_s(n)$ defined by Wang in [9] as a free analogue of the symmetric group $\mathfrak{S}_n$.

We fix a positive integer n. Let $\mathfrak{S}_n$ be the symmetric group of degree n whose element is considered to be a bijection on the set $\{1, 2, \ldots, n\}$.

Definition 1.3. By the universality of $A(n)$, there exists an action $\alpha\colon \mathfrak{S}_n \times \mathfrak{S}_n \curvearrowright A(n)$ defined by

$$\alpha_{(\sigma,\mu)}(p_{i,j}) = p_{\sigma(i),\mu(j)}$$

for $(\sigma, \mu) \in \mathfrak{S}_n \times \mathfrak{S}_n$ and $i, j = 1, 2, \ldots, n$.

Definition 1.4. Let $A^{\mathrm{ab}}(n)$ be the universal unital C*-algebra generated by $n\times n$ projections $\{p_{i,j}\}_{i,j=1}^n$ satisfying the relations in Definition 1.1 and

$$p_{i,j}p_{k,l} = p_{k,l}p_{i,j} \qquad (i, j, k, l = 1, 2, \ldots, n).$$

The following lemma follows immediately from the definitions.

Lemma 1.5. *The C*-algebra $A^{\mathrm{ab}}(n)$ is the abelianization of $A(n)$. More specifically, there exists a natural surjection $A(n) \twoheadrightarrow A^{\mathrm{ab}}(n)$ sending each projection $p_{i,j}$ to $p_{i,j}$, and every $*$-homomorphism from $A(n)$ to an abelian C*-algebra factors through this surjection.*

Proposition 1.6. *The abelian C*-algebra $A^{\mathrm{ab}}(n)$ is isomorphic to the C*-algebra $C(\mathfrak{S}_n)$ of continuous functions on the discrete set $\mathfrak{S}_n$.*

Proof. For each $\sigma \in \mathfrak{S}_n$, we define a character χ_σ of $A^{\mathrm{ab}}(n)$ by

$$\chi_\sigma(p_{i,j}) = \begin{cases} 1 & (i = \sigma(j)) \\ 0 & (i \neq \sigma(j)). \end{cases}$$

Note that such a character χ_σ uniquely exists by the universality of $A^{\mathrm{ab}}(n)$. It is easy to see that any character of $A^{\mathrm{ab}}(n)$ is in the form of χ_σ for some $\sigma \in \mathfrak{S}_n$. This shows that $A^{\mathrm{ab}}(n)$ is isomorphic to $C(\mathfrak{S}_n)$ by the Gelfand theorem. □

We can compute minimal projections of $A^{\mathrm{ab}}(n)$ as follows.

Proposition 1.7. *For $\sigma \in \mathfrak{S}_n$, we set*

$$p_\sigma \coloneqq p_{\sigma(1),1}p_{\sigma(2),2}\cdots p_{\sigma(n),n} \in A^{\mathrm{ab}}(n).$$

Then $\{p_\sigma\}_{\sigma\in\mathfrak{S}_n}$ is the set of minimal projections of $A^{\mathrm{ab}}(n)$.

Proof. Since $A^{\mathrm{ab}}(n)$ is commutative, p_σ is a projection for every $\sigma \in \mathfrak{S}_n$. For $\sigma \in \mathfrak{S}_n$, let χ_σ be the character defined in the proof of Proposition 1.6. Then we have

$$\chi_{\sigma'}(p_\sigma) = \begin{cases} 1 & (\sigma' = \sigma) \\ 0 & (\sigma' \neq \sigma) \end{cases}$$

for $\sigma, \sigma' \in \mathfrak{S}_n$. This shows that $\{p_\sigma\}_{\sigma\in\mathfrak{S}_n}$ is the set of minimal projections of $A^{\mathrm{ab}}(n)$. □

For each $\sigma \in \mathfrak{S}_n$, we can define a character χ_σ of $A(n)$ by the same formula as in the proof of Proposition 1.6 (or to be the composition of the character χ_σ in the proof of Proposition 1.6 and the natural surjection $A(n) \twoheadrightarrow A^{\mathrm{ab}}(n)$). With these characters we have the following as a corollary of Proposition 1.6 (It is easy to show it directly).

Corollary 1.8. *The set of all characters of the magic square C*-algebra $A(n)$ is $\{\chi_\sigma \mid \sigma \in \mathfrak{S}_n\}$ whose cardinality is $n!$.*

2. General results on magic square C*-algebras

In this section, we investigate $A(n)$ for $n \neq 4$. The results in this section are known to specialists.

Proposition 2.1. *For $n = 1, 2, 3$, $A(n)$ is commutative. Hence the surjection $A(n) \twoheadrightarrow A^{\mathrm{ab}}(n)$ is an isomorphism for $n = 1, 2, 3$.*

Proof. For $n = 1$ and $n = 2$, it is easy to see $A(1) \cong \mathbb{C}$ and $A(2) \cong \mathbb{C}^2$. To show that $A(3)$ is commutative, it suffices to show $p_{1,1}$ commutes with $p_{2,2}$. In fact if $p_{1,1}$ commutes with $p_{2,2}$, we can see that $p_{1,1}$ commutes with $p_{2,3}$, $p_{3,2}$ and $p_{3,3}$ using the action α defined in Definition 1.3. Then $p_{1,1}$ commutes with every generators because $p_{1,1}$ is orthogonal to and hence commutes with $p_{1,2}$, $p_{1,3}$, $p_{2,1}$ and $p_{3,1}$. Using the action α again, we see that every generators commutes with every generators.

Now we are going to show that $p_{1,1}$ commutes with $p_{2,2}$. We have

$$\begin{aligned} p_{1,1}p_{2,2} = (1 - p_{1,2} - p_{1,3})p_{2,2} &= p_{2,2} - p_{1,3}p_{2,2} \\ &= p_{2,2} - (1 - p_{2,3} - p_{3,3})p_{2,2} = p_{3,3}p_{2,2}. \end{aligned}$$

By symmetry, we have $p_{2,2}p_{3,3} = p_{1,1}p_{3,3}$ and $p_{3,3}p_{1,1} = p_{2,2}p_{1,1}$. Hence we get

$$p_{1,1}p_{2,2} = p_{3,3}p_{2,2} = (p_{2,2}p_{3,3})^* = (p_{1,1}p_{3,3})^* = p_{3,3}p_{1,1} = p_{2,2}p_{1,1}.$$

This completes the proof. □

Proposition 2.2. *Let $n_1, n_2, \ldots, n_k$ be positive integers, and set $n = \sum_{j=1}^{k} n_j$. There exists a surjection from $A(n)$ to the unital free product $\ast_{j=1}^{k} A(n_j)$.*

Proof. The desired surjection is obtained by sending the generators $\{p_{i,j}\}_{i,j=1}^{n_1}$ of $A(n)$ to the generators of $A(n_1) \subset \ast_{j=1}^{k} A(n_j)$, the generators $\{p_{i,j}\}_{i,j=n_1+1}^{n_1+n_2}$ of $A(n)$ to the generators of $A(n_2) \subset \ast_{j=1}^{k} A(n_j)$ and so on, and by sending the other generators of $A(n)$ to 0. □

Corollary 2.3. *Let n be a positive integer. There exists a surjection from $A(n+1)$ to $A(n)$.*

Proof. This follows from Proposition 2.2 because $A(n) * A(1) \cong A(n) * \mathbb{C} \cong A(n)$. □

Corollary 2.4. *Let n, m be positive integers with $n \geq m$. There exists a surjection from $A(n)$ to $A(m)$.*

Proof. This follows from Corollary 2.3. □

Proposition 2.5. *For $n \geq 5$, $A(n)$ is not exact.*

Proof. Note that an image of an exact C*-algebra is exact (see [5, Corollary 9.4.3]). By Corollary 2.4, it suffices to show that $A(5)$ is not exact. By Proposition 2.2, there exists a surjection from $A(5)$ to $A(2) * A(3) \cong \mathbb{C}^2 * \mathbb{C}^6$ which is not exact (see [5, Proposition 3.7.11]). This completes the proof. □

The C^*-algebra $A(4)$ is not commutative, but is exact, in fact is subhomogeneous (Corollary 7.9). From the next section, we investigate the structure of $A(4)$.

3. Twisted crossed product

We denote elements $\sigma \in \mathfrak{S}_4$ by $(\sigma(1)\sigma(2)\sigma(3)\sigma(4))$. We define the Klein (four) group K by

$$K := \{t_1, t_2, t_3, t_4\} \subset \mathfrak{S}_4$$

where t_1 is the identity (1234) of $\mathfrak{S}_4$, $t_2 = (2143)$, $t_3 = (3412)$ and $t_4 = (4321)$. The group K is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

We choose the indices so that we have $t_i t_j = t_{t_i(j)}$ for $i, j = 1, 2, 3, 4$. Note that we have $t_i(j) = t_j(i)$ for $i, j = 1, 2, 3, 4$.

Definition 3.1. Define unitaries c_1, c_2, c_3, c_4 in $M_2(\mathbb{C})$ by

$$c_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad c_2 := \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad c_3 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad c_4 := \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

The unitaries c_1, c_2, c_3, c_4 are called the Pauli matrices.

Definition 3.2. Put $\omega = (1342) \in \mathfrak{S}_4$. Define a map $\varepsilon\colon \{1, 2, 3, 4\}^2 \to \{1, -1\}$ by

$$\varepsilon(i, j) := \begin{cases} 1 & \text{if } i = 1 \text{ or } j = 1 \text{ or } \omega(i) = j \\ -1 & \text{otherwise,} \end{cases}$$

for each $i, j = 1, 2, 3, 4$.

Table 3.1. Values of $\varepsilon(i, j)$

$i \backslash j$	1	2	3	4
1	1	1	1	1
2	1	−1	1	−1
3	1	−1	−1	1
4	1	1	−1	−1

We have the following calculation which can be proved straightforwardly.

Lemma 3.3. *For $i, j = 1, 2, 3, 4$, we have $c_i c_j = \varepsilon(i, j) c_{t_i(j)}$.*

From this lemma and the computation $t_i t_j = t_{t_i(j)}$, we have the following lemma which means that $K^2 \ni (t_i, t_j) \mapsto \varepsilon(i, j) \in \{1, -1\}$ becomes a cocycle of K.

Lemma 3.4. *For $i, j, k = 1, 2, 3, 4$, we have $\varepsilon(i, j)\varepsilon(t_i(j), k) = \varepsilon(i, t_j(k))\varepsilon(j, k)$.*

Proof. Compute $c_i c_j c_k$ in the two ways, namely $(c_i c_j) c_k$ and $c_i (c_j c_k)$. □

Hence the following definition makes sense. Let us denote by the same symbol α the restriction of the action $\alpha\colon \mathfrak{S}_4 \times \mathfrak{S}_4 \curvearrowright A(4)$ to $K \times K \subset \mathfrak{S}_4 \times \mathfrak{S}_4$.

Definition 3.5. Let $A(4) \rtimes_\alpha^{\mathrm{tw}} (K \times K)$ be the twisted crossed product of the action α and the cocycle

$$(K \times K)^2 \ni ((t_i, t_j), (t_k, t_l)) \longmapsto \varepsilon(i, k)\varepsilon(j, l) \in \{1, -1\}.$$

By definition, $A(4) \rtimes_\alpha^{\mathrm{tw}} (K \times K)$ is the universal C^*-algebra generated by the unital subalgebra $A(4)$ and unitaries $\{u_{i,j}\}_{i,j=1}^4$ such that

$$u_{i,j} x u_{i,j}^* = \alpha_{(t_i,t_j)}(x) \qquad \text{for all } i,j \text{ and all } x \in A(4)$$

and

$$u_{i,j} u_{k,l} = \varepsilon(i,k)\varepsilon(j,l) u_{t_i(k),t_j(l)} \qquad \text{for all } i,j,k,l.$$

We denote by $\mathcal{R}_{\mathrm{u}}$ the latter relation. The former relation is equivalent to the relation

$$u_{i,j} p_{k,l} = p_{t_i(k),t_j(l)} u_{i,j} \qquad \text{for all } i,j,k,l$$

which is denoted by $\mathcal{R}_{\mathrm{up}}$.

Recall that $A(4)$ is the universal unital C^*-algebra generated by the set $\{p_{i,j}\}_{i,j=1}^4$ of projections satisfying the following relation denoted by $\mathcal{R}_{\mathrm{p}}$

$$\sum_{i=1}^4 p_{i,j} = 1 \quad (j = 1,2,3,4), \qquad \sum_{j=1}^4 p_{i,j} = 1 \quad (i = 1,2,3,4).$$

The following is the first main theorem.

Theorem 3.6. *The twisted crossed product $A(4) \rtimes_\alpha^{\mathrm{tw}} (K \times K)$ is isomorphic to $M_4(C(\mathbb{R}P^3))$.*

We finish the proof of this theorem in the end of Section 7.

To prove this theorem, we start with finite presentation of the C^*-algebra $C(\mathbb{R}P^3)$ in the next section.

4. Real projective space $\mathbb{R}P^3$

Definition 4.1. We set an equivalence relation $\sim$ on the manifold

$$S^3 := \left\{ a = (a_1, a_2, a_3, a_4) \in \mathbb{R}^4 \,\middle|\, \sum_{i=1}^4 a_i^2 = 1 \right\}$$

so that $a \sim b$ if and only if $a = b$ or $a = -b$. The quotient space $S^3/\sim$ is the real projective space $\mathbb{R}P^3$ of dimension 3. The equivalence class of $(a_1, a_2, a_3, a_4) \in S^3$ is denoted as $[a_1, a_2, a_3, a_4] \in \mathbb{R}P^3$.

Definition 4.2. For $i, j = 1,2,3,4$, we define a continuous function $f_{i,j}$ on $\mathbb{R}P^3$ by $f_{i,j}([a_1, a_2, a_3, a_4]) = a_i a_j$ for $[a_1, a_2, a_3, a_4] \in \mathbb{R}P^3$.

Note that $f_{i,j}$ is a well-defined continuous function.

Lemma 4.3. *The functions* $\{f_{i,j}\}_{i,j=1}^4$ *satisfy the following relation*

$$\begin{aligned} &f_{i,j} = f_{i,j}^* = f_{j,i} \quad \text{for all } i, j, \\ &f_{i,j} f_{k,l} = f_{i,k} f_{j,l} \quad \text{for all } i, j, k, l, \\ &\sum_{i=1}^{4} f_{i,i} = 1. \end{aligned}$$

Proof. This follows from easy computation. □

Definition 4.4. We denote by $\mathcal{R}_{\mathrm{f}}$ the relation in Lemma 4.3.

Proposition 4.5. *The* C^**-algebra* $C(\mathbb{R}P^3)$ *is the universal unital* C^**-algebra generated by elements* $\{f_{i,j}\}_{i,j=1}^4$ *satisfying* $\mathcal{R}_{\mathrm{f}}$.

Proof. Let A be the universal unital C^*-algebra generated by elements $\{f_{i,j}\}_{i,j=1}^4$ satisfying $\mathcal{R}_{\mathrm{f}}$. For $i, j, k, l = 1, 2, 3, 4$, we have

$$f_{i,j} f_{k,l} = f_{i,k} f_{j,l} = f_{k,i} f_{l,j} = f_{k,l} f_{i,j}.$$

Hence A is commutative. Thus there exists a compact set X such that $A \cong C(X)$.

By Lemma 4.3, we have a unital $*$-homomorphism $A \to C(\mathbb{R}P^3)$. This induces a continuous map $\varphi\colon \mathbb{R}P^3 \to X$. It suffices to show that this continuous map is homeomorphic.

We first show that φ is injective. Take $[a_1, a_2, a_3, a_4]$ and $[b_1, b_2, b_3, b_4] \in \mathbb{R}P^3$ with $\varphi([a_1, a_2, a_3, a_4]) = \varphi([b_1, b_2, b_3, b_4])$. Then, for $i, j = 1, 2, 3, 4$, we have $a_i a_j = b_i b_j$. Since $\sum_{i=1}^4 a_i^2 = 1$, there exists i_0 such that $a_{i_0} \neq 0$. Set $\sigma = b_{i_0}/a_{i_0} \in \mathbb{R}$. Since $a_i a_{i_0} = b_i b_{i_0}$, we have $a_i = \sigma b_i$ for $i = 1, 2, 3, 4$. Since $\sum_{i=1}^4 a_i^2 = \sum_{i=1}^4 b_i^2 = 1$, we get $\sigma = \pm 1$. Hence $[a_1, a_2, a_3, a_4] = [b_1, b_2, b_3, b_4]$. This shows that φ is injective.

Next we show that φ is surjective. Take a unital character $\chi\colon A \to \mathbb{C}$ of A. To show that φ is surjective, it suffices to find $[a_1, a_2, a_3, a_4] \in \mathbb{R}P^3$ such that $\chi(f_{i,j}) = a_i a_j$ for all $i, j = 1, 2, 3, 4$. Since $\sum_{i=1}^4 \chi(f_{i,i}) = \chi\big(\sum_{i=1}^4 f_{i,i}\big) = 1$, there exists i_0 such that $\chi(f_{i_0,i_0}) \neq 0$. Since

$$f_{i_0,i_0} = f_{i_0,i_0} \sum_{i=1}^{4} f_{i,i} = \sum_{i=1}^{4} f_{i_0,i_0} f_{i,i} = \sum_{i=1}^{4} f_{i_0,i} f_{i_0,i} = \sum_{i=1}^{4} f_{i_0,i} f_{i_0,i}^*.$$

we have $\chi(f_{i_0,i_0}) > 0$. Put $a_i := \frac{\chi(f_{i_0,i})}{\sqrt{\chi(f_{i_0,i_0})}}$. We have

$$\sum_{i=1}^{4} a_i^2 = \sum_{i=1}^{4} \frac{\chi(f_{i_0,i})^2}{\chi(f_{i_0,i_0})} = \sum_{i=1}^{4} \frac{\chi(f_{i_0,i_0})\chi(f_{i,i})}{\chi(f_{i_0,i_0})} = \sum_{i=1}^{4} \chi(f_{i,i}) = 1.$$

We also have

$$\chi(f_{i,j}) = \frac{\chi(f_{i_0,i})\chi(f_{i_0,j})}{\chi(f_{i_0,i_0})} = a_i a_j,$$

for $i, j = 1, 2, 3, 4$. This shows that φ is surjective.

Since $\mathbb{R}P^3$ is compact and X is Hausdorff, $\varphi\colon \mathbb{R}P^3 \to X$ is a homeomorphism. Thus we have shown that A is isomorphic to $C(\mathbb{R}P^3)$. □

Let $\{e_{i,j}\}_{i,j=1}^4$ be the matrix unit of $M_4(\mathbb{C})$. Then $\{e_{i,j}\}_{i,j=1}^4$ satisfies the following relation denoted by $\mathcal{R}_{\mathrm{e}}$;

$$e_{i,j} = e_{j,i}^* \quad \text{for all } i, j,$$
$$e_{i,j}e_{k,l} = \delta_{j,k}e_{i,l} \quad \text{for all } i, j, k, l,$$
$$\sum_{i=1}^4 e_{i,i} = 1,$$

here $\delta_{j,k}$ is the Kronecker delta. It is well-known, and easy to see, that $M_4(\mathbb{C})$ is the universal unital C*-algebra generated by $\{e_{i,j}\}_{i,j=1}^4$ satisfying $\mathcal{R}_{\mathrm{e}}$.

The C^*-algebra $M_4(C(\mathbb{R}P^3)) = C(\mathbb{R}P^3, M_4(\mathbb{C})) = C(\mathbb{R}P^3) \otimes M_4(\mathbb{C})$ is the universal unital C^*-algebra generated by $\{f_{i,j}\}_{i,j=1}^4$ and $\{e_{i,j}\}_{i,j=1}^4$ satisfying $\mathcal{R}_{\mathrm{f}}$, $\mathcal{R}_{\mathrm{e}}$ and the following relation denoted by $\mathcal{R}_{\mathrm{fe}}$;

$$f_{i,j}e_{k,l} = e_{k,l}f_{i,j} \quad \text{for all } i, j, k, l.$$

5. Unitaries

Definition 5.1. For $i, j = 1, 2, 3, 4$, we define a unitary $U_{i,j} \in M_4(\mathbb{C}) \subset M_4(C(\mathbb{R}P^3))$ by

$$U_{i,j} := \sum_{k=1}^4 \varepsilon(i,k)\varepsilon(k,j)e_{t_i(k),t_j(k)}$$

From a direct calculation, we have

$$U_{1,1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad U_{1,2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$U_{1,3} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad U_{1,4} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$U_{2,1} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad U_{2,2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$U_{2,3} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad U_{2,4} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$U_{3,1} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad U_{3,2} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$U_{3,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad U_{3,4} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$U_{4,1} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad U_{4,2} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$U_{4,3} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad U_{4,4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have the following. We denote the transpose matrix of a matrix M by M^{T}.

Proposition 5.2. *For* $(a_1, a_2, a_3, a_4) \in \mathbb{C}^4$,

$$(b_1, b_2, b_3, b_4)^{\mathrm{T}} \coloneqq U_{i,j}(a_1, a_2, a_3, a_4)^{\mathrm{T}},$$

satisfies $\sum_{k=1}^{4} b_k c_k = c_i \big(\sum_{k=1}^{4} a_k c_k\big) c_j^*$.

Proof. For $i, j, k = 1, 2, 3, 4$, we have

$$c_i c_{t_j(k)} = \varepsilon(i, t_j(k)) c_{t_i(t_j(k))} \qquad c_{t_i(k)} c_j = \varepsilon(t_i(k), j) c_{t_j(t_i(k))}.$$

Hence $c_i c_{t_j(k)} c_j^* = \varepsilon(i, t_j(k)) \varepsilon(t_i(k), j)^{-1} c_{t_i(k)}$. Since

$$\varepsilon(i, t_j(k)) \varepsilon(k, j) = \varepsilon(i, k) \varepsilon(t_i(k), j),$$

we have

$$\varepsilon(i, t_j(k)) \varepsilon(t_i(k), j)^{-1} = \varepsilon(i, k) \varepsilon(k, j)^{-1} = \varepsilon(i, k) \varepsilon(k, j)$$

This shows that $U_{i,j} = \sum_{k=1}^4 \varepsilon(i, k) \varepsilon(k, j) e_{t_i(k), t_j(k)}$ satisfies the desired property. □

Proposition 5.3. *For $i, j, k, l = 1, 2, 3, 4$, we have*

$$U_{i,j} U_{k,l} = \varepsilon(i, k) \varepsilon(j, l) U_{t_i(k), t_j(l)}.$$

Proof. We have

$$\begin{aligned}
U_{i,j} U_{k,l} &= \left(\sum_{m=1}^4 \varepsilon(i, m) \varepsilon(m, j) e_{t_i(m), t_j(m)} \right) \left(\sum_{n=1}^4 \varepsilon(k, n) \varepsilon(n, l) e_{t_k(n), t_l(n)} \right) \\
&= \left(\sum_{m=1}^4 \varepsilon(i, t_k(m)) \varepsilon(t_k(m), j) e_{t_i(t_k(m)), t_j(t_k(m))} \right) \\
&\qquad\qquad \times \left(\sum_{n=1}^4 \varepsilon(k, t_j(n)) \varepsilon(t_j(n), l) e_{t_k(t_j(n)), t_l(t_j(n))} \right) \\
&= \sum_{m=1}^4 \varepsilon(i, t_k(m)) \varepsilon(t_k(m), j) \varepsilon(k, t_j(m)) \varepsilon(t_j(m), l) e_{t_i(t_k(m)), t_l(t_j(m))}
\end{aligned}$$

Since we have

$$\varepsilon(i, t_k(m)) \varepsilon(k, m) = \varepsilon(i, k) \varepsilon(t_i(k), m), \quad \varepsilon(k, t_j(m)) \varepsilon(m, j) = \varepsilon(k, m) \varepsilon(t_k(m), j),$$
$$\varepsilon(m, j) \varepsilon(t_j(m), l) = \varepsilon(m, t_j(l)) \varepsilon(j, l),$$

we get

$$\varepsilon(i, t_k(m)) \varepsilon(t_k(m), j) \varepsilon(k, t_j(m)) \varepsilon(t_j(m), l) = \varepsilon(i, k) \varepsilon(j, l) \varepsilon(t_i(k), m) \varepsilon(m, t_j(l)).$$

Hence we obtain

$$\begin{aligned}
U_{i,j} U_{k,l} &= \sum_{m=1}^4 \varepsilon(i, k) \varepsilon(j, l) \varepsilon(t_i(k), m) \varepsilon(m, t_j(l)) e_{t_i(t_k(m)), t_j(t_l(m))} \\
&= \varepsilon(i, k) \varepsilon(j, l) U_{t_i(k), t_j(l)}.
\end{aligned}$$

□

One can also prove this proposition using Proposition 5.2.

6. Projections

Definition 6.1. We define $P_{1,1} \coloneqq \sum_{i,j=1}^{4} f_{i,j} e_{i,j} \in M_4(C(\mathbb{R}P^3))$. For $i, j = 1, 2, 3, 4$, we define $P_{i,j} \in M_4(C(\mathbb{R}P^3))$ by

$$P_{i,j} \coloneqq U_{i,j} P_{1,1} U_{i,j}^*.$$

Note that $U_{1,1} = 1$.

Proposition 6.2. *For each $i, j = 1, 2, 3, 4$, $P_{i,j}$ is a projection.*

Proof. It suffices to show that $P_{1,1}$ is a projection. We have

$$P_{1,1}^* = \sum_{i,j=1}^{4} f_{i,j}^* e_{i,j}^* = \sum_{i,j=1}^{4} f_{j,i} e_{j,i} = P_{1,1},$$

and

$$\begin{aligned} P_{1,1}^2 &= \sum_{i,j=1}^{4} f_{i,j} e_{i,j} \sum_{k,l=1}^{4} f_{k,l} e_{k,l} = \sum_{i,j,k,l=1}^{4} f_{i,j} e_{i,j} f_{k,l} e_{k,l} \\ &= \sum_{i,j,l=1}^{4} f_{i,j} f_{j,l} e_{i,l} = \sum_{i,j,l=1}^{4} f_{i,l} f_{j,j} e_{i,l} = \sum_{i,l=1}^{4} f_{i,l} e_{i,l} = P_{1,1}. \end{aligned}$$

Hence $P_{1,1}$ is a projection. □

Proposition 6.3. *The set $\{P_{i,j}\}_{i,j=1}^4$ of projections and the set $\{U_{i,j}\}_{i,j=1}^4$ of unitaries satisfy $\mathcal{R}_{\mathrm{up}}$.*

Proof. This follows from the computation

$$\begin{aligned} U_{i,j} P_{k,l} U_{i,j}^* &= U_{i,j} U_{k,l} P_{1,1} U_{k,l}^* U_{i,j}^* \\ &= (\varepsilon(i,k)\varepsilon(j,l))^2 U_{t_i(k),t_j(l)} P_{1,1} U_{t_i(k),t_j(l)}^* = P_{t_i(k),t_j(l)} \end{aligned}$$

using Proposition 5.3. □

Proposition 6.4. *The set $\{P_{i,j}\}_{i,j=1}^4$ of projections satisfies $\mathcal{R}_{\mathrm{p}}$.*

Proof. From Proposition 6.3, it suffices to show

$$P_{1,1} + P_{1,2} + P_{1,3} + P_{1,4} = 1, \qquad P_{1,1} + P_{2,1} + P_{3,1} + P_{4,1} = 1.$$

This follows from the following direct computations

$$P_{1,1} = \begin{pmatrix} f_{1,1} & f_{1,2} & f_{1,3} & f_{1,4} \\ f_{2,1} & f_{2,2} & f_{2,3} & f_{2,4} \\ f_{3,1} & f_{3,2} & f_{3,3} & f_{3,4} \\ f_{4,1} & f_{4,2} & f_{4,3} & f_{4,4} \end{pmatrix},$$

$$P_{1,2} = \begin{pmatrix} f_{2,2} & -f_{2,1} & -f_{2,4} & f_{2,3} \\ -f_{1,2} & f_{1,1} & f_{1,4} & -f_{1,3} \\ -f_{4,2} & f_{4,1} & f_{4,4} & -f_{4,3} \\ f_{3,2} & -f_{3,1} & -f_{3,4} & f_{3,3} \end{pmatrix}, \quad P_{2,1} = \begin{pmatrix} f_{2,2} & -f_{2,1} & f_{2,4} & -f_{2,3} \\ -f_{1,2} & f_{1,1} & -f_{1,4} & f_{1,3} \\ f_{4,2} & -f_{4,1} & f_{4,4} & -f_{4,3} \\ -f_{3,2} & f_{3,1} & -f_{3,4} & f_{3,3} \end{pmatrix},$$

$$P_{1,3} = \begin{pmatrix} f_{3,3} & f_{3,4} & -f_{3,1} & -f_{3,2} \\ f_{4,3} & f_{4,4} & -f_{4,1} & -f_{4,2} \\ -f_{1,3} & -f_{1,4} & f_{1,1} & f_{1,2} \\ -f_{2,3} & -f_{2,4} & f_{2,1} & f_{2,2} \end{pmatrix}, \quad P_{3,1} = \begin{pmatrix} f_{3,3} & -f_{3,4} & -f_{3,1} & f_{3,2} \\ -f_{4,3} & f_{4,4} & f_{4,1} & -f_{4,2} \\ -f_{1,3} & f_{1,4} & f_{1,1} & -f_{1,2} \\ f_{2,3} & -f_{2,4} & -f_{2,1} & f_{2,2} \end{pmatrix},$$

$$P_{1,4} = \begin{pmatrix} f_{4,4} & -f_{4,3} & f_{4,2} & -f_{4,1} \\ -f_{3,4} & f_{3,3} & -f_{3,2} & f_{3,1} \\ f_{2,4} & -f_{2,3} & f_{2,2} & -f_{2,1} \\ -f_{1,4} & f_{1,3} & -f_{1,2} & f_{1,1} \end{pmatrix}, \quad P_{4,1} = \begin{pmatrix} f_{4,4} & f_{4,3} & -f_{4,2} & -f_{4,1} \\ f_{3,4} & f_{3,3} & -f_{3,2} & -f_{3,1} \\ -f_{2,4} & -f_{2,3} & f_{2,2} & f_{2,1} \\ -f_{1,4} & -f_{1,3} & f_{1,2} & f_{1,1} \end{pmatrix}.$$

□

By Proposition 5.3, Proposition 6.2, Proposition 6.3 and Proposition 6.4, we have a $*$-homomorphism $\Phi\colon A(4) \rtimes_{\alpha}^{\mathrm{tw}} (K \times K) \to M_4(C(\mathbb{R}P^3))$ sending $p_{i,j}$ to $P_{i,j}$ and $u_{i,j}$ to $U_{i,j}$. In the next section, we construct the inverse map of Φ.

7. The inverse map

Definition 7.1. For $i, j = 1, 2, 3, 4$, we set

$$E_{i,j} \coloneqq \frac{1}{4} \sum_{k=1}^{4} \varepsilon(i,k)\varepsilon(k,j) u_{t_i(k), t_j(k)} \in A(4) \rtimes_{\alpha}^{\mathrm{tw}} (K \times K)$$

Definition 7.2. For $i, j = 1, 2, 3, 4$, we set

$$F_{i,j} \coloneqq \sum_{k=1}^{4} E_{k,i} p_{1,1} E_{j,k} \in A(4) \rtimes_{\alpha}^{\mathrm{tw}} (K \times K).$$

Lemma 7.3. *For $i, j = 1, 2, 3, 4$, we have $u_{i,1}E_{1,1}u_{1,j} = E_{i,j}$. For $i = 1, 2, 3, 4$, we have $u_{i,i}E_{1,1} = E_{1,1}u_{i,i} = E_{1,1}$. We also have $E_{1,1}^2 = E_{1,1}$.*

Proof. We have $E_{1,1} = \frac{1}{4}\sum_{k=1}^{4} u_{k,k}$. For $i, j = 1, 2, 3, 4$, we have

$$u_{i,1}E_{1,1}u_{1,j} = \frac{1}{4}\sum_{k=1}^{4} u_{i,1}u_{k,k}u_{1,j} = \frac{1}{4}\sum_{k=1}^{4} \varepsilon(i,k)\varepsilon(k,j)u_{t_i(k),t_j(k)} = E_{i,j}.$$

For $i = 1, 2, 3, 4$, we have

$$u_{i,i}E_{1,1} = \frac{1}{4}\sum_{k=1}^{4} u_{i,i}u_{k,k} = \frac{1}{4}\sum_{k=1}^{4} \varepsilon(i,k)^2 u_{t_i(k),t_i(k)} = \frac{1}{4}\sum_{k=1}^{4} u_{k,k} = E_{1,1}.$$

Similarly, we get $E_{1,1}u_{i,i} = E_{1,1}$. Finally, we have $E_{1,1}^2 = \frac{1}{4}\sum_{k=1}^{4} u_{k,k}E_{1,1} = E_{1,1}$. □

Proposition 7.4. *The set* $\{E_{i,j}\}_{i,j=1}^{4}$ *satisfies* $\mathcal{R}_{\mathrm{e}}$.

Proof. We have $E_{1,1} = \frac{1}{4}\sum_{k=1}^{4} u_{k,k}$. We also have

$$\begin{aligned} E_{2,2} &= \frac{1}{4}(u_{1,1} + u_{2,2} - u_{3,3} - u_{4,4}) \\ E_{3,3} &= \frac{1}{4}(u_{1,1} - u_{2,2} + u_{3,3} - u_{4,4}) \\ E_{4,4} &= \frac{1}{4}(u_{1,1} - u_{2,2} - u_{3,3} + u_{4,4}). \end{aligned}$$

Hence $\sum_{i=1}^{4} E_{i,i} = u_{1,1} = 1$.

It is easy to see $E_{1,1}^* = E_{1,1}$. For $i = 1, 2, 3, 4$, we have

$$E_{1,1}u_{i,1}^* = E_{1,1}u_{i,i}u_{i,1}^* = E_{1,1}u_{1,i}u_{i,1}u_{i,1}^* = E_{1,1}u_{1,i}$$

and $u_{1,i}^*E_{1,1} = u_{i,1}E_{1,1}$ similarly. Hence by Lemma 7.3, we obtain

$$E_{i,j}^* = (u_{i,1}E_{1,1}u_{1,j})^* = u_{1,j}^*E_{1,1}u_{i,1}^* = u_{j,1}E_{1,1}u_{1,i} = E_{j,i}$$

for $i, j = 1, 2, 3, 4$.

By Lemma 7.3, we obtain

$$\begin{aligned} E_{i,j}E_{j,k} = u_{i,1}E_{1,1}u_{1,j}u_{j,1}E_{1,1}u_{1,k} &= u_{i,1}E_{1,1}u_{j,j}E_{1,1}u_{1,k} \\ &= u_{i,1}E_{1,1}^2u_{1,k} = u_{i,1}E_{1,1}u_{1,k} = E_{i,k} \end{aligned}$$

for $i, j, k = 1, 2, 3, 4$. The proof ends if we show $E_{i,j}E_{k,l} = 0$ for $i, j, k, l = 1, 2, 3, 4$ with $j \neq k$. It suffices to show $E_{1,1}u_{1,j}u_{k,1}E_{1,1} = 0$ for $j, k = 1, 2, 3, 4$ with $j \neq k$. Since $u_{1,j}u_{k,1} = u_{k,j} = \varepsilon(k, t_k(j))u_{k,k}u_{1,t_k(j)}$, it suffices to show $E_{1,1}u_{1,j}E_{1,1} = 0$ for

$j = 2, 3, 4$. For $j = 2$, we get

$$\begin{aligned} 4E_{1,1}u_{1,2}E_{1,1} &= \sum_{k=1}^{4} u_{k,k}u_{1,2}E_{1,1} \\ &= u_{1,2}E_{1,1} + u_{1,2}u_{2,2}E_{1,1} - u_{1,2}u_{3,3}E_{1,1} - u_{1,2}u_{4,4}E_{1,1} \\ &= 0 \end{aligned}$$

By similar computations, we get $E_{1,1}u_{1,3}E_{1,1} = E_{1,1}u_{1,4}E_{1,1} = 0$. This completes the proof. □

Proposition 7.5. *The set* $\{F_{i,j}\}_{i,j=1}^4$ *satisfy* $\mathcal{R}_\mathrm{f}$.

Proof. For $i, j = 1, 2, 3, 4$, Proposition 7.4 shows

$$\begin{aligned} F_{i,j}^* = \left(\sum_{k=1}^{4} E_{k,i}p_{1,1}E_{j,k} \right)^* &= \sum_{k=1}^{4} E_{j,k}^* p_{1,1}^* E_{k,i}^* \\ &= \sum_{k=1}^{4} E_{k,j}p_{1,1}E_{i,k} = F_{j,i}. \end{aligned}$$

Next, we show $F_{i,j} = F_{j,i}$ for $i, j = 1, 2, 3, 4$. We are going to prove $F_{2,4} = F_{4,2}$. The other 5 cases can be proved similarly. To show that $F_{2,4} = F_{4,2}$, it suffices to show $E_{1,2}p_{1,1}E_{4,1} = E_{1,4}p_{1,1}E_{2,1}$ because it implies $E_{k,2}p_{1,1}E_{4,k} = E_{k,4}p_{1,1}E_{2,k}$ for $k = 1, 2, 3, 4$ by multiplying $E_{k,1}$ from left and $E_{1,k}$ from right. By Lemma 7.3, we have

$$\begin{aligned} 4E_{1,2}p_{1,1}E_{4,1} &= (u_{1,2} - u_{2,1} - u_{3,4} + u_{4,3})p_{1,1}u_{4,1}E_{1,1} \\ &= (p_{1,2}u_{1,2} - p_{2,1}u_{2,1} - p_{3,4}u_{3,4} + p_{4,3}u_{4,3})u_{4,1}E_{1,1} \\ &= (p_{1,2}u_{4,2} + p_{2,1}u_{3,1} - p_{3,4}u_{2,4} - p_{4,3}u_{1,3})E_{1,1} \\ &= (p_{1,2}u_{1,3}u_{4,4} - p_{2,1}u_{1,3}u_{3,3} + p_{3,4}u_{1,3}u_{2,2} - p_{4,3}u_{1,3})E_{1,1} \\ &= (p_{1,2} - p_{2,1} + p_{3,4} - p_{4,3})u_{1,3}E_{1,1} \end{aligned}$$

$$\begin{aligned} 4E_{1,4}p_{1,1}E_{2,1} &= (u_{1,4} - u_{2,3} + u_{3,2} - u_{4,1})p_{1,1}u_{2,1}E_{1,1} \\ &= (p_{1,4}u_{1,4} - p_{2,3}u_{2,3} + p_{3,2}u_{3,2} - p_{4,1}u_{4,1})u_{2,1}E_{1,1} \\ &= (p_{1,4}u_{2,4} + p_{2,3}u_{1,3} - p_{3,2}u_{4,2} - p_{4,1}u_{3,1})E_{1,1} \\ &= (-p_{1,4}u_{1,3}u_{2,2} + p_{2,3}u_{1,3} - p_{3,2}u_{1,3}u_{4,4} + p_{4,1}u_{1,3}u_{3,3})E_{1,1} \\ &= (-p_{1,4} + p_{2,3} - p_{3,2} + p_{4,1})u_{1,3}E_{1,1}. \end{aligned}$$

Since

$$\begin{aligned}p_{1,1}+p_{1,2}+p_{1,3}+p_{1,4}+p_{3,1}+p_{3,2}+p_{3,3}+p_{3,4}\\ =2=p_{1,1}+p_{2,1}+p_{3,1}+p_{4,1}+p_{1,3}+p_{2,3}+p_{3,3}+p_{4,3},\end{aligned}$$

we have

$$p_{1,2}-p_{2,1}+p_{3,4}-p_{4,3}=-p_{1,4}+p_{2,3}-p_{3,2}+p_{4,1}.$$

Therefore, we obtain $E_{1,2}p_{1,1}E_{4,1}=E_{1,4}p_{1,1}E_{2,1}$. Thus we have proved $F_{2,4}=F_{4,2}$.

Next we show $F_{i,j}F_{k,l}=F_{i,k}F_{j,l}$ for $i,j,k,l=1,2,3,4$, To show this, it suffices to show $p_{1,1}E_{j,k}p_{1,1}=p_{1,1}E_{k,j}p_{1,1}$ for $j,k=1,2,3,4$. We are going to prove $p_{1,1}E_{3,4}p_{1,1}=p_{1,1}E_{4,3}p_{1,1}$. The other 5 cases can be proved similarly. This follows from the following computation

$$\begin{aligned}4p_{1,1}E_{3,4}p_{1,1}&=p_{1,1}(u_{3,4}+u_{4,3}-u_{1,2}-u_{2,1})p_{1,1}\\&=p_{1,1}(u_{3,4}+u_{4,3})p_{1,1}-p_{1,1}p_{1,2}u_{1,2}-p_{1,1}p_{2,1}u_{2,1}\\&=p_{1,1}(u_{3,4}+u_{4,3})p_{1,1},\end{aligned}$$

$$\begin{aligned}4p_{1,1}E_{4,3}p_{1,1}&=p_{1,1}(u_{4,3}+u_{3,4}+u_{2,1}+u_{1,2})p_{1,1}\\&=p_{1,1}(u_{3,4}+u_{4,3})p_{1,1}+p_{1,1}p_{2,1}u_{2,1}+p_{1,1}p_{1,2}u_{1,2}\\&=p_{1,1}(u_{3,4}+u_{4,3})p_{1,1}.\end{aligned}$$

Finally we show $\sum_{i=1}^{4}F_{i,i}=1$. For $i=1,2,3,4$, we have

$$\begin{aligned}F_{i,i}&=\sum_{k=1}^{4}E_{k,i}p_{1,1}E_{i,k}=\sum_{k=1}^{4}u_{k,1}E_{1,1}u_{1,i}p_{1,1}u_{i,1}E_{1,1}u_{1,k}\\&=\sum_{k=1}^{4}u_{k,1}E_{1,1}p_{1,i}u_{1,i}u_{i,1}E_{1,1}u_{1,k}=\sum_{k=1}^{4}u_{k,1}E_{1,1}p_{1,i}u_{i,i}E_{1,1}u_{1,k}\\&=\sum_{k=1}^{4}u_{k,1}E_{1,1}p_{1,i}E_{1,1}u_{1,k}.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\sum_{i=1}^{4}F_{i,i}&=\sum_{i=1}^{4}\sum_{k=1}^{4}u_{k,1}E_{1,1}p_{1,i}E_{1,1}u_{1,k}\\&=\sum_{k=1}^{4}u_{k,1}E_{1,1}^{2}u_{1,k}=\sum_{k=1}^{4}u_{k,1}E_{1,1}u_{1,k}=\sum_{k=1}^{4}E_{k,k}=1\end{aligned}$$

by Lemma 7.3 and Proposition 7.4. We are done. □

Proposition 7.6. *The sets* $\{E_{i,j}\}_{i,j=1}^4$ *and* $\{F_{i,j}\}_{i,j=1}^4$ *satisfy* $\mathcal{R}_{\mathrm{fe}}$.

Proof. For $i,j,k,l = 1,2,3,4$, we have $E_{i,j}F_{k,l} = F_{k,l}E_{i,j}$ because

$$E_{i,j}F_{k,l} = E_{i,j}\sum_{m=1}^4 E_{m,k}p_{1,1}E_{l,m} = E_{i,k}p_{1,1}E_{l,j},$$

$$F_{k,l}E_{i,j} = \sum_{m=1}^4 E_{m,k}p_{1,1}E_{l,m}E_{i,j} = E_{i,k}p_{1,1}E_{l,j}$$

by Proposition 7.4. □

By Proposition 7.4, Proposition 7.5 and Proposition 7.6, we have a $*$-homomorphism $\Psi\colon M_4(C(\mathbb{R}P^3)) \to A(4) \rtimes_\alpha^{\mathrm{tw}} (K\times K)$ sending $f_{i,j}$ to $F_{i,j}$ and $e_{i,j}$ to $E_{i,j}$.

We are going to see that this map Ψ is the inverse of Φ. We first show $\Psi\circ\Phi = \mathrm{id}_{A(4)\rtimes_\alpha^{\mathrm{tw}}(K\times K)}$.

Proposition 7.7. *For* $x \in A(4) \rtimes_\alpha^{\mathrm{tw}} (K\times K)$*, we have* $\Psi(\Phi(x)) = x$.

Proof. For $i,j = 1,2,3,4$, we have

$$\begin{aligned}
\Psi(\Phi(u_{i,j})) = \Psi(U_{i,j}) &= \sum_{k=1}^4 \varepsilon(i,k)\varepsilon(k,j)\Psi(e_{t_i(k),t_j(k)}) \\
&= \sum_{k=1}^4 \varepsilon(i,k)\varepsilon(k,j)E_{t_i(k),t_j(k)} \\
&= \frac{1}{4}\sum_{k=1}^4 \varepsilon(i,k)\varepsilon(k,j)\sum_{m=1}^4 \varepsilon(t_i(k),m)\varepsilon(m,t_j(k))u_{t_i(t_k(m)),t_j(t_k(m))} \\
&= \frac{1}{4}\sum_{k=1}^4\sum_{l=1}^4 \varepsilon(i,k)\varepsilon(k,j)\varepsilon(t_i(k),t_k(l))\varepsilon(t_k(l),t_j(k))u_{t_i(l),t_j(l)}.
\end{aligned}$$

Since we have

$$\begin{aligned}
&\frac{1}{4}\sum_{k=1}^4 \varepsilon(i,k)\varepsilon(k,j)\varepsilon(t_i(k),t_k(l))\varepsilon(t_k(l),t_j(k)) \\
&\qquad = \frac{1}{4}\sum_{k=1}^4 \varepsilon(i,k)\varepsilon(t_i(k),t_k(l))\varepsilon(t_k(l),t_j(k))\varepsilon(k,j) \\
&\qquad = \frac{1}{4}\sum_{k=1}^4 \varepsilon(i,l)\varepsilon(k,t_k(l))\varepsilon(t_k(l),k)\varepsilon(l,j) = \delta_{l,1},
\end{aligned}$$

we obtain $\Psi(\Phi(u_{i,j})) = u_{i,j}$. By the computation in the proof of Proposition 7.6, we have

$$\Psi(P_{1,1}) = \Psi\left(\sum_{i,j=1}^{4} f_{i,j}e_{i,j}\right) = \sum_{i,j=1}^{4} F_{i,j}E_{i,j} = \sum_{i,j=1}^{4} E_{i,i}p_{1,1}E_{j,j} = p_{1,1}.$$

For $i, j = 1, 2, 3, 4$, we have

$$\Psi(\Phi(p_{i,j})) = \Psi(P_{i,j}) = \Psi(U_{i,j})\Psi(P_{1,1})\Psi(U_{i,j})^* = u_{i,j}p_{1,1}u_{i,j}^* = p_{i,j}.$$

These show that $\Psi(\Phi(x)) = x$ for all $x \in A(4) \rtimes_{\alpha}^{\mathrm{tw}} (K \times K)$. □

Next, we show $\Phi \circ \Psi = \mathrm{id}_{M_4(C(\mathbb{R}P^3))}$.

Proposition 7.8. *For $x \in M_4(C(\mathbb{R}P^3))$, we have $\Phi(\Psi(x)) = x$.*

Proof. For $i, j = 1, 2, 3, 4$, we have

$$\begin{aligned}
\Phi(\Psi(e_{i,j})) = \Phi(E_{i,j}) &= \frac{1}{4}\sum_{k=1}^{4} \varepsilon(i,k)\varepsilon(k,j)\Phi(u_{t_i(k),t_j(k)}) \\
&= \frac{1}{4}\sum_{k=1}^{4} \varepsilon(i,k)\varepsilon(k,j)U_{t_i(k),t_j(k)} \\
&= \frac{1}{4}\sum_{k=1}^{4} \varepsilon(i,k)\varepsilon(k,j)\sum_{m=1}^{4} \varepsilon(t_i(k),m)\varepsilon(m,t_j(k))e_{t_i(t_k(m)),t_j(t_k(m))} \\
&= \frac{1}{4}\sum_{k=1}^{4}\sum_{l=1}^{4} \varepsilon(i,k)\varepsilon(k,j)\varepsilon(t_i(k),t_k(l))\varepsilon(t_k(l),t_j(k))e_{t_i(l),t_j(l)} \\
&= e_{i,j}
\end{aligned}$$

as in the proof of Proposition 7.7. For $i, j = 1, 2, 3, 4$, we have

$$\begin{aligned}
\Phi(\Psi(f_{i,j})) = \Phi(F_{i,j}) &= \sum_{k=1}^{4} \Phi(E_{k,i})\Phi(p_{1,1})\Phi(E_{j,k}) \\
&= \sum_{k=1}^{4} e_{k,i}P_{1,1}e_{j,k} \\
&= \sum_{k=1}^{4} e_{k,i}\left(\sum_{l,m=1}^{4} f_{l,m}e_{l,m}\right)e_{j,k} \\
&= \sum_{k=1}^{4} f_{i,j}e_{k,k} = f_{i,j}.
\end{aligned}$$

These show that $\Phi(\Psi(x)) = x$ for all $x \in M_4(C(\mathbb{R}P^3))$. □

By these two propositions, we get Theorem 3.6. As its corollary, we have the following.

Corollary 7.9 (cf. [2, Theorem 4.1]). *There is an injective $*$-homomorphism $A(4) \to M_4(C(\mathbb{R}P^3))$.*

Proof. This follows from Theorem 3.6 because the $*$-homomorphism $A(4) \to A(4) \rtimes_\alpha^{\mathrm{tw}} (K \times K)$ is injective. □

One can see that the injective $*$-homomorphism constructed in this corollary is nothing but the Pauli representation constructed in [3] and considered in [2]. Note that Banica and Collins remarked after [2, Definition 2.1] that the target of the Pauli representation can be replaced by $M_4(C(SO_3))$ instead of $M_4(C(SU_2))$. Here SO_3 is homeomorphic to $\mathbb{R}P^3$ whereas SU_2 is homeomorphic to S^3.

8. Action

One can see that the dual group of $K \times K$ is isomorphic to $K \times K$ using the product of the cocycle ε (see below).

Table 8.1. Values of $\varepsilon(i, j)\varepsilon(j, i)$

$i \backslash j$	1	2	3	4
1	1	1	1	1
2	1	1	−1	−1
3	1	−1	1	−1
4	1	−1	−1	1

Let $\widehat{\alpha}\colon K \times K \curvearrowright A(4) \rtimes_\alpha^{\mathrm{tw}} (K \times K)$ be the dual action of α. Namely $\widehat{\alpha}$ is determined by the following equation for all i, j, k, l

$$\widehat{\alpha}_{i,j}(p_{k,l}) = p_{k,l}, \qquad \widehat{\alpha}_{i,j}(u_{k,l}) = \varepsilon(i, k)\varepsilon(k, i)\varepsilon(j, l)\varepsilon(l, j)u_{k,l},$$

where we write $\widehat{\alpha}_{(t_i,t_j)}$ as $\widehat{\alpha}_{i,j}$.

For $i, j = 1, 2, 3, 4$, define $\sigma_{i,j}\colon \mathbb{R}P^3 \to \mathbb{R}P^3$ by $\sigma_{i,j}([a_1, a_2, a_3, a_4]) = [b_1, b_2, b_3, b_4]$ for $[a_1, a_2, a_3, a_4] \in \mathbb{R}P^3$ where $(b_1, b_2, b_3, b_4) \in S^3$ is determined by

$$(b_1, b_2, b_3, b_4)^{\mathrm{T}} = U_{i,j}(a_1, a_2, a_3, a_4)^{\mathrm{T}},$$

in other words $\sum_{k=1}^4 b_k c_k = c_i\big(\sum_{k=1}^4 a_k c_k\big)c_j^*$ by Proposition 5.2. Let $\beta\colon K \times K \curvearrowright M_4(C(\mathbb{R}P^3))$ be the action determined by $\beta_{i,j}(F) = \operatorname{Ad} U_{i,j} \circ F \circ \sigma_{i,j}$ for $F \in M_4(C(\mathbb{R}P^3)) = C(\mathbb{R}P^3, M_4(\mathbb{C}))$ where we write $\beta_{(t_i,t_j)}$ as $\beta_{i,j}$.

Proposition 8.1. *The $*$-homomorphism $\Phi\colon A(4) \rtimes_{\alpha}^{\mathrm{tw}} (K \times K) \to M_4(C(\mathbb{R}P^3))$ is equivariant with respect to $\widehat{\alpha}$ and β.*

Proof. For $i, j = 1, 2, 3, 4$, we have $P_{1,1} \circ \sigma_{i,j} = \mathrm{Ad}\, U_{i,j} \circ P_{1,1}$. In fact for $[a_1, a_2, a_3, a_4] \in \mathbb{R}P^3$, on one hand we have

$$(P_{1,1} \circ \sigma_{i,j})([a_1, a_2, a_3, a_4]) = (b_1, b_2, b_3, b_4)^{\mathrm{T}}(b_1, b_2, b_3, b_4),$$

where

$$(b_1, b_2, b_3, b_4)^{\mathrm{T}} = U_{i,j}(a_1, a_2, a_3, a_4)^{\mathrm{T}},$$

and on the other hand we have

$$(\mathrm{Ad}\, U_{i,j} \circ P_{1,1})([a_1, a_2, a_3, a_4]) = U_{i,j}(a_1, a_2, a_3, a_4)^{\mathrm{T}}(a_1, a_2, a_3, a_4)U_{i,j}^*$$

here note $U_{i,j}^* = U_{i,j}^{\mathrm{T}}$ because the entries of $U_{i,j}$ are -1, 0 or 1. For $i, j, k, l = 1, 2, 3, 4$, we have

$$\begin{aligned}
\beta_{i,j}(P_{k,l}) &= \mathrm{Ad}\, U_{i,j} \circ (\mathrm{Ad}\, U_{k,l} \circ P_{1,1}) \circ \sigma_{i,j} \\
&= \mathrm{Ad}\, U_{i,j} \circ \mathrm{Ad}\, U_{k,l} \circ \mathrm{Ad}\, U_{i,j} \circ P_{1,1} \\
&= \mathrm{Ad}(U_{i,j} U_{k,l} U_{i,j}) \circ P_{1,1} \\
&= \mathrm{Ad}\, U_{k,l} \circ P_{1,1} = P_{k,l}.
\end{aligned}$$

For $i, j, k, l = 1, 2, 3, 4$, we also have

$$\begin{aligned}
\beta_{i,j}(U_{k,l}) &= \mathrm{Ad}\, U_{i,j} \circ U_{k,l} \circ \sigma_{i,j} \\
&= U_{i,j} U_{k,l} U_{i,j}^* \\
&= \varepsilon(i,k)\varepsilon(j,l) U_{t_i(k), t_j(l)} U_{i,j}^* \\
&= \varepsilon(i,k)\varepsilon(j,l)\varepsilon(k,i)^{-1}\varepsilon(l,j)^{-1} U_{k,l} U_{i,j} U_{i,j}^* \\
&= \varepsilon(i,k)\varepsilon(j,l)\varepsilon(k,i)\varepsilon(l,j) U_{k,l}
\end{aligned}$$

here note that $U_{k,l} \in M_4(C(\mathbb{R}P^3)) = C(\mathbb{R}P^3, M_4(\mathbb{C}))$ is a constant function. These complete the proof. □

The following is the second main theorem.

Theorem 8.2. *The fixed point algebra $M_4(C(\mathbb{R}P^3))^{\beta}$ of the action β is isomorphic to $A(4)$.*

Proof. This follows from Theorem 3.6 and Proposition 8.1 because the fixed point algebra $\big(A(4) \rtimes_{\alpha}^{\mathrm{tw}} (K \times K)\big)^{\widehat{\alpha}}$ of $\widehat{\alpha}$ is $A(4)$. □

As we remark in Introduction, this theorem can be also obtained by combining [1, Theorem 3.1, Theorem 5.1] and [4, Proposition 3.3]. Compared with this method, our proof is explicit and straightforward.

9. Quotient Space $\mathbb{R}P^3/(K \times K)$

Definition 9.1. We set $A := M_4(C(\mathbb{R}P^3))^\beta$.

By Theorem 8.2, the C^*-algebra $A(4)$ is isomorphic to A. From this section, we compute the structure of A and its K-groups.

In this section, we study the quotient Space $\mathbb{R}P^3/(K \times K)$ of $\mathbb{R}P^3$ by the action σ of $K \times K$. In [6], it is proved that this quotient space $\mathbb{R}P^3/(K \times K)$ is homeomorphic to S^3.

Definition 9.2. We denote by X the quotient space $\mathbb{R}P^3/(K \times K)$ of the action σ of $K \times K$. We denote by $\pi\colon \mathbb{R}P^3 \to X$ the quotient map.

We use the following lemma later.

Lemma 9.3. *For $i, j = 2, 3, 4$ and $[a_1, a_2, a_3, a_4] \in \mathbb{R}P^3$ with $\sigma_{i,j}([a_1, a_2, a_3, a_4]) = [a_1, a_2, a_3, a_4]$, we have $P_{k,l}([a_1, a_2, a_3, a_4]) = P_{t_i(k),t_j(l)}([a_1, a_2, a_3, a_4])$ for $k, l = 1, 2, 3, 4$.*

Proof. This follows from

$$\begin{aligned}
P_{k,l}([a_1, a_2, a_3, a_4]) &= \beta_{i,j}(P_{k,l})([a_1, a_2, a_3, a_4]) \\
&= \operatorname{Ad} U_{i,j}\big(P_{k,l}(\sigma_{i,j}([a_1, a_2, a_3, a_4]))\big) \\
&= \operatorname{Ad} U_{i,j}\big(P_{k,l}([a_1, a_2, a_3, a_4])\big) \\
&= \big(\operatorname{Ad} U_{i,j}(P_{k,l})\big)([a_1, a_2, a_3, a_4]) \\
&= P_{t_i(k),t_j(l)}([a_1, a_2, a_3, a_4]).
\end{aligned}$$

□

Definition 9.4. For each $i, j = 2, 3, 4$, define

$$\widetilde{F}_{i,j} := \big\{[a_1, a_2, a_3, a_4] \in \mathbb{R}P^3 \,\big|\, \sigma_{i,j}([a_1, a_2, a_3, a_4]) = [a_1, a_2, a_3, a_4]\big\} \subset \mathbb{R}P^3$$

to be the set of fixed points of $\sigma_{i,j}$, and define $F_{i,j} \subset X$ to be the image $\pi(\widetilde{F}_{i,j})$.

We have $\widetilde{F}_{i,j} = \pi^{-1}(F_{i,j})$. The following two propositions can be proved by direct computation using the computation of $U_{i,j}$ after Definition 5.1

Proposition 9.5. *For each $i = 2, 3, 4$, $\sigma_{1,i}$ and $\sigma_{i,1}$ have no fixed points.*

Proposition 9.6. *For each $i, j = 2, 3, 4$, $\widetilde{F}_{i,j}$ is homeomorphic to a disjoint union of two circles. More precisely, we have*

$$\widetilde{F}_{2,2} = \{[a, b, 0, 0], [0, 0, a, b] \in \mathbb{R}P^3 \,|\, a, b \in \mathbb{R},\ a^2 + b^2 = 1\}$$
$$\widetilde{F}_{2,3} = \{[a, b, -b, a], [a, b, b, -a] \in \mathbb{R}P^3 \,|\, a, b \in \mathbb{R},\ 2(a^2 + b^2) = 1\}$$
$$\widetilde{F}_{2,4} = \{[a, b, a, b], [a, b, -a, -b] \in \mathbb{R}P^3 \,|\, a, b \in \mathbb{R},\ 2(a^2 + b^2) = 1\}$$
$$\widetilde{F}_{3,2} = \{[a, b, b, a], [a, b, -b, -a] \in \mathbb{R}P^3 \,|\, a, b \in \mathbb{R},\ 2(a^2 + b^2) = 1\}$$
$$\widetilde{F}_{3,3} = \{[a, 0, b, 0], [0, a, 0, b] \in \mathbb{R}P^3 \,|\, a, b \in \mathbb{R},\ a^2 + b^2 = 1\}$$
$$\widetilde{F}_{3,4} = \{[a, a, b, -b], [a, -a, b, b] \in \mathbb{R}P^3 \,|\, a, b \in \mathbb{R},\ 2(a^2 + b^2) = 1\}$$
$$\widetilde{F}_{4,2} = \{[a, b, a, -b], [a, b, -a, b] \in \mathbb{R}P^3 \,|\, a, b \in \mathbb{R},\ 2(a^2 + b^2) = 1\}$$
$$\widetilde{F}_{4,3} = \{[a, a, b, b], [a, -a, b, -b] \in \mathbb{R}P^3 \,|\, a, b \in \mathbb{R},\ 2(a^2 + b^2) = 1\}$$
$$\widetilde{F}_{4,4} = \{[a, 0, 0, b], [0, a, b, 0] \in \mathbb{R}P^3 \,|\, a, b \in \mathbb{R},\ a^2 + b^2 = 1\}$$

Definition 9.7. We set $\widetilde{F} := \bigcup_{i,j=2}^{4} \widetilde{F}_{i,j}$ and $F := \bigcup_{i,j=2}^{4} F_{i,j}$. We also set $\widetilde{O} := \mathbb{R}P^3 \setminus \widetilde{F}$ and $O := X \setminus F$.

We have $\widetilde{F} = \pi^{-1}(F)$ and hence $\widetilde{O} = \pi^{-1}(O)$. Note that $\widetilde{O}$ is the set of points $[a_1, a_2, a_3, a_4] \in \mathbb{R}P^3$ such that $\sigma_{i,j}([a_1, a_2, a_3, a_4]) \neq [a_1, a_2, a_3, a_4]$ for all $i, j = 1, 2, 3, 4$ other than $(i, j) = (1, 1)$. Note also that $\widetilde{F}$ and F are closed, and hence $\widetilde{O}$ and O are open.

Definition 9.8. For each i_2, i_3, i_4 with $\{i_2, i_3, i_4\} = \{2, 3, 4\}$, define $\widetilde{F}_{(i_2 i_3 i_4)} \subset \mathbb{R}P^3$ by

$$\widetilde{F}_{(i_2 i_3 i_4)} := \widetilde{F}_{i_2,2} \cap \widetilde{F}_{i_3,3} \cap \widetilde{F}_{i_4,4},$$

and define $F_{(i_2 i_3 i_4)} \subset X$ to be the image $\pi(\widetilde{F}_{(i_2 i_3 i_4)})$.

Proposition 9.9. *For each i_2, i_3, i_4 with $\{i_2, i_3, i_4\} = \{2, 3, 4\}$, we have*

$$\widetilde{F}_{(i_2 i_3 i_4)} = \widetilde{F}_{i_2,2} \cap \widetilde{F}_{i_3,3} = \widetilde{F}_{i_2,2} \cap \widetilde{F}_{i_4,4} = \widetilde{F}_{i_3,3} \cap \widetilde{F}_{i_4,4}.$$

We also have

$$\widetilde{F}_{(234)} = \{[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]\},$$
$$\widetilde{F}_{(342)} = \left\{\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right], \left[\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right], \left[\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right]\right\},$$
$$\widetilde{F}_{(423)} = \left\{\left[-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], \left[\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right]\right\},$$
$$\widetilde{F}_{(243)} = \left\{\left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right], \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0\right], \left[0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right], \left[0, 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]\right\},$$

$$\widetilde{F}_{(432)} = \left\{\left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right], \left[\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 0\right], \left[0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right], \left[0, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right]\right\},$$
$$\widetilde{F}_{(324)} = \left\{\left[\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right], \left[\frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}}\right], \left[0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right], \left[0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right]\right\}.$$

Proof. This follows from Proposition 9.6. □

Proposition 9.10. *For each i_2, i_3, i_4 with $\{i_2, i_3, i_4\} = \{2, 3, 4\}$, $F_{(i_2 i_3 i_4)}$ consists of one point.*

Proof. This follows from Proposition 9.9. □

Definition 9.11. For each i_2, i_3, i_4 with $\{i_2, i_3, i_4\} = \{2, 3, 4\}$, we set $x_{(i_2 i_3 i_4)} \in X$ by $F_{(i_2 i_3 i_4)} = \{x_{(i_2 i_3 i_4)}\}$.

Proposition 9.12. *For each $i, j = 2, 3, 4$, $F_{i,j}$ is homeomorphic to a closed interval whose endpoints are $x_{(i_2 i_3 i_4)}$ with $i_j = i$,*

Proof. This follows from Proposition 9.6. See also Figure 13.2 and the remarks around it. □

Note that $F \subset X$ is the complete bipartite graph between $\{x_{(234)}, x_{(342)}, x_{(423)}\}$ and $\{x_{(243)}, x_{(432)}, x_{(324)}\}$. See Figure 13.2.

Definition 9.13. For $i, j = 2, 3, 4$, we define

$$F^\circ_{i,j} := F_{i,j} \setminus \{x_{(i_2 i_3 i_4)} \mid i_j = i\},$$

and define

$$F^\circ := \bigcup_{i,j=2}^{4} F^\circ_{i,j}, \qquad F^\bullet := \{x_{(234)}, x_{(342)}, x_{(423)}, x_{(243)}, x_{(432)}, x_{(324)}\}.$$

Definition 9.14. We set $\widetilde{F}^\circ_{i,j} := \pi^{-1}(F^\circ_{i,j})$ for $i, j = 2, 3, 4$, $\widetilde{F}^\circ := \pi^{-1}(F^\circ)$ and $\widetilde{F}^\bullet := \pi^{-1}(F^\bullet)$.

10. **Exact sequences**

For a locally compact subset Y of $\mathbb{R}P^3$ which is invariant under the action σ, the action $\beta \colon K \times K \curvearrowright M_4(C(\mathbb{R}P^3))$ induces the action $K \times K \curvearrowright M_4(C_0(Y))$ which is also denoted by β. We use the following lemma many times.

Lemma 10.1. *Let Y be a locally compact subset of $\mathbb{R}P^3$ which is invariant under the action σ. Let Z be a closed subset of Y which is invariant under the action σ. Then we have a a short exact sequence*

$$0 \longrightarrow M_4(C_0(Y \setminus Z))^\beta \longrightarrow M_4(C_0(Y))^\beta \longrightarrow M_4(C_0(Z))^\beta \longrightarrow 0$$

Proof. It suffices to show that $M_4(C_0(Y))^\beta \to M_4(C_0(Z))^\beta$ is surjective. The other assertions are easy to see.

Take $f \in M_4(C_0(Z))^\beta$. Since $M_4(C_0(Y)) \to M_4(C_0(Z))$ is surjective, there exists $g \in M_4(C_0(Y))$ with $g|_Z = f$. Set $g_0 \in M_4(C_0(Y))$ by

$$g_0 := \frac{1}{16} \sum_{i,j=1}^{4} \beta_{i,j}(g).$$

Then $g_0 \in M_4(C_0(Y))^\beta$ and $g_0|_Z = f$. This completes the proof. □

We also use the following lemma many times.

Lemma 10.2. *Let Y be a locally compact subset of $\mathbb{R}P^3$ which is invariant under the action σ. Let Z be a closed subset of Y such that $Y = \bigcup_{i,j=1}^4 \sigma_{i,j}(Z)$ and that $\sigma_{i,j}(Z) \cap Z = \emptyset$ for $i, j = 1, 2, 3, 4$ with $(i, j) \neq (1, 1)$. Then we have $M_4(C_0(Y))^\beta \cong M_4(C_0(Z))$.*

Proof. The restriction map $M_4(C_0(Y))^\beta \to M_4(C_0(Z))$ is an isomorphism because its inverse is given by

$$M_4(C_0(Z)) \ni f \longmapsto \sum_{i,j=1}^{4} \beta_{i,j}(f) \in M_4(C_0(Y))^\beta.$$ □

Under the situation of the lemma above, $\pi\colon Z \to \pi(Z) = \pi(Y)$ is a homeomorphism. Hence we have $M_4(C_0(Y))^\beta \cong M_4(C_0(Z)) \cong M_4\big(C_0(\pi(Z))\big) = M_4\big(C_0(\pi(Y))\big)$.

The following lemma generalize Lemma 10.2.

Lemma 10.3. *Let G be a subgroup of $K \times K$. Let Y be a locally compact subset of $\mathbb{R}P^3$ which is invariant under the action σ. Suppose that each point of Y is fixed by $\sigma_{i,j}$ for all $(t_i, t_j) \in G$. Let Z be a closed subset of Y such that $Y = \bigcup_{i,j=1}^4 \sigma_{i,j}(Z)$ and that $\sigma_{i,j}(Z) \cap Z = \emptyset$ for $i, j = 1, 2, 3, 4$ with $(t_i, t_j) \notin G$. Then we have $M_4(C_0(Y))^\beta \cong C_0(Z, D)$ where*

$$D := \{T \in M_4(\mathbb{C}) \mid \operatorname{Ad} U_{i,j}(T) = T \text{ for all } (t_i, t_j) \in G\}.$$

Proof. We have a restriction map $M_4(C_0(Y))^\beta \to C_0(Z, D)$ which is an isomorphism because its inverse is given by

$$C_0(Z, D) \ni f \longmapsto \sum_{(i,j) \in I} \beta_{i,j}(f) \in M_4(C_0(Y))^\beta,$$

where an index set I is chosen so that $\{(t_i, t_j) \in K \times K \mid (i, j) \in I\}$ becomes a complete representative of the quotient $(K \times K)/G$. $\square$

Under the situation of the lemma above, $\pi\colon Z \to \pi(Z) = \pi(Y)$ is a homeomorphism. Hence we have $M_4(C_0(Y))^\beta \cong C_0(Z, D) \cong C_0(\pi(Z), D) = C_0(\pi(Y), D)$.

Definition 10.4. We set $I \coloneqq M_4(C_0(\widetilde{O}))^\beta$ and $B \coloneqq M_4(C(\widetilde{F}))^\beta$.

By Lemma 10.1 we get a short exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0.$$

From this sequence, we get a six-term exact sequence

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(B) \\ \delta_1 \uparrow & & & & \downarrow \delta_0 \\ K_1(B) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I). \end{array}$$

From next section, we compute $K_i(B)$, $K_i(I)$ and δ_i for $i = 0, 1$. Consult [7] for basics of K-theory.

11. The Structure of the Quotient B

Definition 11.1. For $i, j = 2, 3, 4$, let $D_{i,j}$ be the fixed algebra of $\operatorname{Ad} U_{i,j}$ on $M_4(\mathbb{C})$.

From the direct computation, we have the following.

Proposition 11.2. *For each $i, j = 2, 3, 4$, $D_{i,j}$ is isomorphic to $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$. More precisely, we have*

$$D_{2,2} = \left\{ \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{pmatrix} \right\}, \qquad D_{2,3} = \left\{ \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ -h & g & f & -e \\ d & -c & -b & a \end{pmatrix} \right\},$$

$$D_{2,4} = \left\{ \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ c & d & a & b \\ g & h & e & f \end{pmatrix} \right\}, \qquad D_{3,2} = \left\{ \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ h & g & f & e \\ d & c & b & a \end{pmatrix} \right\},$$

$$D_{3,3} = \left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & c & 0 & d \\ e & 0 & f & 0 \\ 0 & g & 0 & h \end{pmatrix} \right\}, \qquad D_{3,4} = \left\{ \begin{pmatrix} a & b & c & d \\ b & a & -d & -c \\ e & f & g & h \\ -f & -e & h & g \end{pmatrix} \right\},$$

$$D_{4,2} = \left\{ \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ c & -d & a & -b \\ -g & h & -e & f \end{pmatrix} \right\}, \qquad D_{4,3} = \left\{ \begin{pmatrix} a & b & c & d \\ b & a & d & c \\ e & f & g & h \\ f & e & h & g \end{pmatrix} \right\},$$

$$D_{4,4} = \left\{ \begin{pmatrix} a & 0 & 0 & b \\ 0 & c & d & 0 \\ 0 & e & f & 0 \\ g & 0 & 0 & h \end{pmatrix} \right\},$$

where a, b, c, d, e, f, g, h run through $\mathbb{C}$.

Definition 11.3. For each i_2, i_3, i_4 with $\{i_2, i_3, i_4\} = \{2, 3, 4\}$, define $D_{(i_2 i_3 i_4)} \subset \mathbb{R}P^3$ by

$$D_{(i_2 i_3 i_4)} \coloneqq D_{i_2,2} \cap D_{i_3,3} \cap D_{i_4,4}.$$

Proposition 11.4. *For each i_2, i_3, i_4 with $\{i_2, i_3, i_4\} = \{2, 3, 4\}$, we have*

$$D_{(i_2 i_3 i_4)} = D_{i_2,2} \cap D_{i_3,3} = D_{i_2,2} \cap D_{i_4,4} = D_{i_3,3} \cap D_{i_4,4},$$

and $D_{(i_2 i_3 i_4)}$ is isomorphic to $\mathbb{C}^4$. More precisely, we have

$$D_{(234)} = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \right\} \qquad D_{(423)} = \left\{ \begin{pmatrix} a & b & c & d \\ b & a & -d & -c \\ c & -d & a & -b \\ d & -c & -b & a \end{pmatrix} \right\}$$

$$D_{(342)} = \left\{ \begin{pmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{pmatrix} \right\} \qquad D_{(243)} = \left\{ \begin{pmatrix} a & b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & d & c \end{pmatrix} \right\}$$

$$D_{(432)} = \left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & c & 0 & d \\ b & 0 & a & 0 \\ 0 & d & 0 & c \end{pmatrix} \right\} \qquad D_{(324)} = \left\{ \begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix} \right\}$$

where a, b, c, d run through $\mathbb{C}$.

Definition 11.5. We set $B^\circ \coloneqq M_4(C_0(\widetilde{F}^\circ))^\beta$ and $B^\bullet \coloneqq M_4(C(\widetilde{F}^\bullet))^\beta$. We also set $B^\circ_{i,j} \coloneqq M_4(C_0(\widetilde{F}^\circ_{i,j}))^\beta$ for $i, j = 2, 3, 4$ and $B_{(i_2 i_3 i_4)} \coloneqq M_4(C_0(\widetilde{F}_{(i_2 i_3 i_4)}))^\beta$ for i_2, i_3, i_4 with $\{i_2, i_3, i_4\} = \{2, 3, 4\}$.

From the discussion up to here, we have the following proposition.

Proposition 11.6. *We have*

$$B^\circ \cong \bigoplus_{i,j=2}^{4} B^\circ_{i,j}, \qquad\qquad B^\bullet \cong \bigoplus_{\{i_2,i_3,i_4\}=\{2,3,4\}} B_{(i_2 i_3 i_4)}.$$

We also have

$$B^\circ_{i,j} \cong C_0(F^\circ_{i,j}, D_{i,j}) \cong C_0\big((0,1), M_2(\mathbb{C}) \oplus M_2(\mathbb{C})\big),$$

for $i, j = 2, 3, 4$ *and*

$$B_{(i_2 i_3 i_4)} \cong C(F_{(i_2 i_3 i_4)}, D_{(i_2 i_3 i_4)}) \cong \mathbb{C}^4$$

for i_2, i_3, i_4 *with* $\{i_2, i_3, i_4\} = \{2, 3, 4\}$.

From this proposition, we get

$$B^\circ \cong C_0\big((0,1), M_2(\mathbb{C}) \oplus M_2(\mathbb{C})\big)^9 \cong C_0\big((0,1), M_2(\mathbb{C})\big)^{18}, \quad B^\bullet \cong (\mathbb{C}^4)^6 \cong \mathbb{C}^{24}.$$

12. K-groups of the quotient B

From the short exact sequence

$$0 \longrightarrow B^\circ \longrightarrow B \longrightarrow B^\bullet \longrightarrow 0,$$

we get a six-term exact sequence

$$\begin{array}{ccccc} 0 = K_0(B^\circ) & \longrightarrow & K_0(B) & \longrightarrow & K_0(B^\bullet) \cong \mathbb{Z}^{24} \\ \uparrow & & & & \downarrow \delta \\ 0 = K_1(B^\bullet) & \longleftarrow & K_1(B) & \longleftarrow & K_1(B^\circ) \cong \mathbb{Z}^{18}. \end{array}$$

From this sequence, we have $K_0(B) \cong \ker \delta$ and $K_1(B) \cong \operatorname{coker} \delta$. Next we compute $\delta \colon K_0(B^\bullet) \to K_1(B^\circ)$.

Proposition 12.1. *Under the isomorphism* $\Phi \colon A(4) \to A$*, the* C^**-algebra* $A^{\mathrm{ab}}(4)$ *is canonically isomorphic to* $B^\bullet$*.*

Proof. Since $B^\bullet \cong \mathbb{C}^{24}$ is commutative, the surjection $A(4) \cong A \twoheadrightarrow B \twoheadrightarrow B^\bullet$ factors through the surjection $A(4) \twoheadrightarrow A^{\mathrm{ab}}(4)$. The induced surjection $A^{\mathrm{ab}}(4) \twoheadrightarrow B^\bullet$ is an isomorphism because $A^{\mathrm{ab}}(4) \cong \mathbb{C}^{24}$. □

For $i, j = 1, 2, 3, 4$, the image of $P_{i,j} \in A$ under a surjection is denoted by the same symbol $P_{i,j}$. By Proposition 1.7 and Proposition 12.1, the 24 minimal projections of $B^\bullet$ are

$$P_{(i_1 i_2 i_3 i_4)} \coloneqq P_{i_1,1} P_{i_2,2} P_{i_3,3} P_{i_4,4} \in B^\bullet$$

for $(i_1 i_2 i_3 i_4) \in \mathfrak{S}_4$.

Definition 12.2. For $\sigma \in \mathfrak{S}_4$, we define $q_\sigma \coloneqq [P_\sigma]_0 \in K_0(B^\bullet)$.

Note that $\{q_\sigma\}_{\sigma \in \mathfrak{S}_4}$ is a basis of $K_0(B^\bullet) \cong \mathbb{Z}^{24}$.

Proposition 12.3. *For each i_2, i_3, i_4 with $\{i_2, i_3, i_4\} = \{2, 3, 4\}$, the 4 minimal projections of $\mathbb{C}^4 \cong B_{(i_2 i_3 i_4)} \subset B^\bullet$ are $P_{\sigma t_k}$ for $k = 1, 2, 3, 4$ where $\sigma \coloneqq (1 i_2 i_3 i_4) \in \mathfrak{S}_4$.*

Proof. Take i_2, i_3, i_4 with $\{i_2, i_3, i_4\} = \{2, 3, 4\}$. Since the 4 points in $\widetilde{F}_{(i_2 i_3 i_4)}$ are fixed by $\sigma_{i_2,2}$, $\sigma_{i_3,3}$ and $\sigma_{i_4,4}$, we have $P_{k,l} = P_{t_{i_j}(k), t_j(l)}$ in $B_{(i_2 i_3 i_4)}$ for $k, l = 1, 2, 3, 4$ and $j = 2, 3, 4$ by Lemma 9.3. More concretely we have

$$\begin{aligned} P_{1,1} &= P_{i_2,2} = P_{i_3,3} = P_{i_4,4}, \\ P_{i_2,1} &= P_{1,2} = P_{i_4,3} = P_{i_3,4}, \\ P_{i_3,1} &= P_{i_4,2} = P_{1,3} = P_{i_2,4}, \\ P_{i_4,1} &= P_{i_3,2} = P_{i_2,3} = P_{1,4} \end{aligned}$$

in $B_{(i_2 i_3 i_4)}$. These four projections are mutually orthogonal, and their sum equals to 1. Thus the 4 minimal projections of $B_{(i_2 i_3 i_4)}$ are $P_{(1 i_2 i_3 i_4)}$, $P_{(i_2 1 i_4 i_3)}$, $P_{(i_3 i_4 1 i_2)}$ and $P_{(i_4 i_3 i_2 1)}$. □

Take $i, j = 2, 3, 4$, and fix them for a while. Let $(1 m_2 m_3 m_4) \in \mathfrak{S}_4$ be the unique even permutation with $m_j = i$, and $(1 n_2 n_3 n_4) \in \mathfrak{S}_4$ be the unique odd permutation with $n_j = i$. We set $\sigma = (1 m_2 m_3 m_4)$ and $\tau = (1 n_2 n_3 n_4)$. Then we have the following commutative diagram with exact rows;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B^\circ & \longrightarrow & B & \longrightarrow & B^\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B^\circ_{i,j} & \longrightarrow & B_{i,j} & \longrightarrow & B_{(m_2 m_3 m_4)} \oplus B_{(n_2 n_3 n_4)} & \longrightarrow & 0. \end{array}$$

By Lemma 9.3, we have $P_{k,l} = P_{t_i(k), t_j(l)}$ in $B_{i,j}$ for $k, l = 1, 2, 3, 4$. Let $\omega = (1342) \in \mathfrak{S}_4$. Note that we have $t_i(\omega(i)) = \omega^2(i)$ and $t_i(\omega^2(i)) = \omega(i)$. One can see that $B_{i,j}$ is a direct sum of two C^*-subalgebras $B^\cap_{i,j}$ and $B^\cup_{i,j}$ where $B^\cap_{i,j}$ is generated by

$$P_{1,1} = P_{i,j}, \quad P_{1,j} = P_{i,1}, \quad P_{\omega(i),\omega(j)} = P_{\omega^2(i),\omega^2(j)}, \quad P_{\omega(i),\omega^2(j)} = P_{\omega^2(i),\omega(j)}$$

and $B^\cup_{i,j}$ is generated by

$$P_{1,\omega(j)} = P_{i,\omega^2(j)}, \quad P_{1,\omega^2(j)} = P_{i,\omega(j)}, \quad P_{\omega(i),1} = P_{\omega^2(i),j}, \quad P_{\omega(i),j} = P_{\omega^2(i),1}.$$

Note that $P_{1,1}+P_{1,j} = P_{\omega(i),\omega(j)}+P_{\omega(i),\omega^2(j)}$ is the unit of $B^{\cap}_{i,j}$, and $P_{1,\omega(j)}+P_{1,\omega^2(j)} = P_{\omega(i),1} + P_{\omega(i),j}$ is the unit of $B^{\cup}_{i,j}$. It turns out that both $B^{\cap}_{i,j}$ and $B^{\cup}_{i,j}$ are isomorphic to the universal unital C^*-algebra generated by two projections, which is isomorphic to

$$\left\{ f \in C([0,1], M_2(\mathbb{C})) \;\middle|\; f(0) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, f(1) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}.$$

This fact can be proved directly, but we do not prove it here because we do not need it. The image of $B^{\cap}_{i,j}$ under the surjection $B_{i,j} \to B_{(m_2m_3m_4)} \oplus B_{(n_2n_3n_4)}$ is $(\mathbb{C}p_\sigma + \mathbb{C}p_{\sigma t_j}) \oplus (\mathbb{C}p_\tau + \mathbb{C}p_{\tau t_j})$. Therefore, the image of $B^{\cup}_{i,j}$ under the surjection $B_{i,j} \to B_{(m_2m_3m_4)} \oplus B_{(n_2n_3n_4)}$ is $(\mathbb{C}p_{\sigma t_{\omega(j)}} + \mathbb{C}p_{\sigma t_{\omega^2(j)}}) \oplus (\mathbb{C}p_{\tau t_{\omega(j)}} + \mathbb{C}p_{\tau t_{\omega^2(j)}})$. We set $v^{\cap}_{i,j}, v^{\cup}_{i,j} \in K_1(B^{\circ}_{i,j})$ by $v^{\cap}_{i,j} := \delta'(q_\sigma)$ and $v^{\cup}_{i,j} := \delta'(q_{\sigma t_{\omega(j)}})$ where

$$\delta' \colon K_0(B_{(m_2m_3m_4)} \oplus B_{(n_2n_3n_4)}) \to K_1(B^{\circ}_{i,j})$$

is the exponential map. Then we have the following.

Lemma 12.4. *The set $\{v^{\cap}_{i,j}, v^{\cup}_{i,j}\}$ is a generator of $K_1(B^{\circ}_{i,j}) \cong \mathbb{Z}^2$, and we have*

$$\delta'(q_\sigma) = \delta'(q_{\sigma t_j}) = v^{\cap}_{i,j}, \qquad \delta'(q_{\sigma t_{\omega(j)}}) = \delta'(q_{\sigma t_{\omega^2(j)}}) = v^{\cup}_{i,j},$$
$$\delta'(q_\tau) = \delta'(q_{\tau t_j}) = -v^{\cap}_{i,j}, \qquad \delta'(q_{\tau t_{\omega(j)}}) = \delta'(q_{\tau t_{\omega^2(j)}}) = -v^{\cup}_{i,j}.$$

Proof. Choose a closed interval $Z \subset \mathbb{R}P^3$ such that $\pi \colon Z \to F_{i,j}$ is a homeomorphism (see Figure 13.2 and the remarks around it for an example of such a space). Let $z_0, z_1 \in Z$ be the point such that $\pi(z_0) = v_{(m_2m_3m_4)}$ and $\pi(z_1) = v_{(n_2n_3n_4)}$. Then we have $B^{\circ}_{i,j} \cong C_0(Z \setminus \{z_0, z_1\}, D_{i,j})$. Let $B'_{i,j}$ be the inverse image of $B_{(m_2m_3m_4)}$ under the surjection $B_{i,j} \to B_{(m_2m_3m_4)} \oplus B_{(n_2n_3n_4)}$. Then we have the following commutative diagram with exact rows;

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B^{\circ}_{i,j} & \longrightarrow & B'_{i,j} & \longrightarrow & B_{(m_2m_3m_4)} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B^{\circ}_{i,j} & \longrightarrow & C_0(Z \setminus \{z_0\}, D_{i,j}) & \longrightarrow & D_{i,j} & \longrightarrow & 0.
\end{array}$$

Let us denote by φ the homomorphism from $K_0(B_{(m_2m_3m_4)})$ to $K_0(D_{i,j})$ induced by the vertical map from $B_{(m_2m_3m_4)} \cong D_{(m_2m_3m_4)}$ to $D_{i,j}$. Then $K_0(D_{i,j}) \cong \mathbb{Z}^2$ is spanned by $\varphi(q_\sigma) = \varphi(q_{\sigma t_j})$ and $\varphi(q_{\sigma t_{\omega(j)}}) = \varphi(q_{\sigma t_{\omega^2(j)}})$. Since $K_l(C_0(Z \setminus \{z_0\}, D_{i,j})) = 0$ for $l = 0, 1$, $K_0(D_{i,j}) \to K_1(B^{\circ}_{i,j})$ is an isomorphism. This shows that $\{v^{\cap}_{i,j}, v^{\cup}_{i,j}\}$ is a generator of $K_1(B^{\circ}_{i,j}) \cong \mathbb{Z}^2$. We also have $\delta'(q_\sigma) = \delta'(q_{\sigma t_j})$ and $\delta'(q_{\sigma t_{\omega(j)}}) = \delta'(q_{\sigma t_{\omega^2(j)}})$. Similarly, we have $\delta'(q_\tau) = \delta'(q_{\tau t_j})$ and $\delta'(q_{\tau t_{\omega(j)}}) = \delta'(q_{\tau t_{\omega^2(j)}})$.

Since the image of the projection $P_{1,1} \in B_{i,j}$ under the surjection $B_{i,j} \to B_{(m_2m_3m_4)} \oplus B_{(n_2n_3n_4)}$ is $P_\sigma + P_\tau$, we have $\delta'(q_\sigma + q_\tau) = 0$. Hence $\delta'(q_\tau) = -v^{\cap}_{i,j}$. Similarly we have $\delta'(q_{\sigma t_{\omega(j)}} + q_{\tau t_{\omega(j)}}) = 0$ because the image of $P_{1,\omega(j)} \in B_{i,j}$ under the surjection $B_{i,j} \to B_{(m_2m_3m_4)} \oplus B_{(n_2n_3n_4)}$ is $P_{\sigma t_{\omega(j)}} + P_{\tau t_{\omega(j)}}$. We are done. □

From these computation, we get the following proposition.

Proposition 12.5. *The exponential map $\delta\colon K_0(B^\bullet) \to K_1(B^\circ)$ is as Table 12.1.*

We will see that $K_1(B) \cong \operatorname{coker}\delta$ is isomorphic to $\mathbb{Z}^4 \oplus \mathbb{Z}/2\mathbb{Z}$ in Proposition 15.5. This implies $K_0(B) \cong \ker\delta$ is isomorphic to $\mathbb{Z}^{10}$ because $\ker\delta$ is a free abelian group with dimension $24 - 18 + 4 = 10$. Below, we examine the generator of $K_0(B) \cong \ker\delta$.

For $i, j = 1, 2, 3, 4$, we have

$$P_{i,j} = P_{i,j} \sum_{k \neq i} \sum_{l=1}^{n} P_{k,l} = \sum_{i=\sigma(j)} P_\sigma$$

in $B^\bullet$. Hence $[P_{i,j}]_0 = \sum_{i=\sigma(j)} q_\sigma$ in $K_0(B^\bullet)$.

Proposition 12.6. *The group* $\ker\delta$ *is generated by* $\{[P_{i,j}]_0 \mid i, j = 1, 2, 3, 4\}$.

Proof. It is straightforward to check that $[P_{i,j}]_0$ is in $\ker\delta$ for $i, j = 1, 2, 3, 4$.

Take $x \in \ker\delta$, and we will show that x is in the subgroup generated by $\{[P_{i,j}]_0 \mid i, j = 1, 2, 3, 4\}$. Write $x = \sum_{\sigma\in\mathfrak{S}_4} n_\sigma q_\sigma$ with $n_\sigma \in \mathbb{Z}$. Subtracting $n_{(4213)}[P_{2,2}]_0 + n_{(4132)}[P_{1,2}]_0$ from x, we may assume $n_{(4213)} = n_{(4132)} = 0$ without loss of generality. Subtracting $n_{(4312)}[P_{3,2}]_0 + n_{(4123)}[P_{2,3}]_0 + n_{(4231)}[P_{1,4}]_0$ from x, we may further assume $n_{(4312)} = n_{(4123)} = n_{(4231)} = 0$ without loss of generality. Subtracting $n_{(2341)}[P_{2,1}]_0 + n_{(3142)}[P_{3,1}]_0$ from x, we may further assume $n_{(2341)} = n_{(3142)} = 0$ without loss of generality. Subtracting $n_{(2413)}[P_{4,2}]_0 + n_{(3214)}[P_{4,4}]_0 + n_{(1324)}[P_{1,1}]_0$ from x, we may further assume $n_{(2413)} = n_{(3214)} = n_{(1324)} = 0$ without loss of generality. Now we will show $x = 0$ using $x \in \ker\delta$.

Since $n_{(3241)} + n_{(4132)} = n_{(3142)} + n_{(4231)}$, we have $n_{(3241)} = 0$.
Since $n_{(2314)} + n_{(3241)} = n_{(2341)} + n_{(3214)}$, we have $n_{(2314)} = 0$.
Since $n_{(1423)} + n_{(2314)} = n_{(1324)} + n_{(2413)}$, we have $n_{(1423)} = 0$.
Since $n_{(1423)} + n_{(4132)} = n_{(1432)} + n_{(4123)}$, we have $n_{(1432)} = 0$.
Since $n_{(3124)} + n_{(4213)} = n_{(3214)} + n_{(4123)}$, we have $n_{(3124)} = 0$.
Since $n_{(2431)} + n_{(4213)} = n_{(2413)} + n_{(4231)}$, we have $n_{(2431)} = 0$.
Since $n_{(1342)} + n_{(2431)} = n_{(1432)} + n_{(2341)}$, we have $n_{(1342)} = 0$.
Since $n_{(2314)} + n_{(4132)} = n_{(2134)} + n_{(4312)}$, we have $n_{(2134)} = 0$.
Since $n_{(2431)} + n_{(3124)} = n_{(2134)} + n_{(3421)}$, we have $n_{(3421)} = 0$.
Since $n_{(1423)} + n_{(3241)} = n_{(1243)} + n_{(3421)}$, we have $n_{(1243)} = 0$.

TABLE 12.1. Computation of the exponential map δ

	2,2		3,3		4,4		4,3		2,4		3,2		3,4		4,2		2,3	
q \ v	$\cap$	$\cup$	$\cap$	$\cup$	$\cap$	$\cup$	$\cap$	$\cup$	$\cap$	$\cup$	$\cap$	$\cup$	$\cap$	$\cup$	$\cap$	$\cup$	$\cap$	$\cup$
(1234)	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
(2143)	1	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
(3412)	0	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
(4321)	0	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
(1342)	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0	0	0	0
(2431)	0	0	0	0	0	0	0	1	1	0	0	1	0	0	0	0	0	0
(3124)	0	0	0	0	0	0	0	1	0	1	1	0	0	0	0	0	0	0
(4213)	0	0	0	0	0	0	1	0	0	1	0	1	0	0	0	0	0	0
(1423)	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0
(2314)	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	0
(3241)	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	1
(4132)	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	1
(1243)	−1	0	0	0	0	0	−1	0	0	0	0	0	−1	0	0	0	0	0
(2134)	−1	0	0	0	0	0	0	−1	0	0	0	0	0	−1	0	0	0	0
(3421)	0	−1	0	0	0	0	0	−1	0	0	0	0	−1	0	0	0	0	0
(4312)	0	−1	0	0	0	0	−1	0	0	0	0	0	0	−1	0	0	0	0
(1432)	0	0	−1	0	0	0	0	0	−1	0	0	0	0	0	−1	0	0	0
(2341)	0	0	0	−1	0	0	0	0	−1	0	0	0	0	0	0	−1	0	0
(3214)	0	0	−1	0	0	0	0	0	0	−1	0	0	0	0	0	−1	0	0
(4123)	0	0	0	−1	0	0	0	0	0	−1	0	0	0	0	−1	0	0	0
(1324)	0	0	0	0	−1	0	0	0	0	0	−1	0	0	0	0	0	−1	0
(2413)	0	0	0	0	0	−1	0	0	0	0	0	−1	0	0	0	0	−1	0
(3142)	0	0	0	0	0	−1	0	0	0	0	−1	0	0	0	0	0	0	−1
(4231)	0	0	0	0	−1	0	0	0	0	0	0	−1	0	0	0	0	0	−1

Since $n_{(1234)} + n_{(2143)} = n_{(1243)} + n_{(2134)} = 0$, $n_{(1234)} + n_{(3412)} = n_{(1432)} + n_{(3214)} = 0$ and $n_{(2143)} + n_{(3412)} = n_{(2413)} + n_{(3142)} = 0$, we have $2n_{(1234)} = 0$. Hence $n_{(1234)} = 0$. This implies $n_{(2143)} = n_{(3412)} = 0$. Finally, since $n_{(1234)} + n_{(4321)} = n_{(1324)} + n_{(4231)}$, we have $n_{(4321)} = 0$. We have shown that $x = 0$. This completes the proof. □

From Proposition 12.6 (or its proof), we see that $K_0(B) \cong \ker\delta$ is isomorphic to $\mathbb{Z}^n$ with $n \leq 10$. Note that the group generated by $\{[P_{i,j}]_0 \mid i, j = 1, 2, 3, 4\}$ is in fact generated by 10 elements

$$[P_{1,1}]_0, [P_{1,2}]_0, [P_{1,3}]_0, [P_{1,4}]_0, [P_{2,1}]_0, [P_{2,2}]_0, [P_{2,3}]_0, [P_{3,1}]_0, [P_{3,2}]_0, [P_{3,3}]_0.$$

We will show that $K_0(B) \cong \ker \delta$ is isomorphic to $\mathbb{Z}^{10}$ in Proposition 15.5.

TABLE 12.2. Computation of $[P_{i,j}]_0$

i	1				2				3				4			
q \ j	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
(1234)	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1
(2143)	0	1	0	0	1	0	0	0	0	0	0	1	0	0	1	0
(3412)	0	0	1	0	0	0	0	1	1	0	0	0	0	1	0	0
(4321)	0	0	0	1	0	0	1	0	0	1	0	0	1	0	0	0
(1342)	1	0	0	0	0	0	0	1	0	1	0	0	0	0	1	0
(2431)	0	0	0	1	1	0	0	0	0	0	1	0	0	1	0	0
(3124)	0	1	0	0	0	0	1	0	1	0	0	0	0	0	0	1
(4213)	0	0	1	0	0	1	0	0	0	0	0	1	1	0	0	0
(1423)	1	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0
(2314)	0	0	1	0	1	0	0	0	0	1	0	0	0	0	0	1
(3241)	0	0	0	1	0	1	0	0	1	0	0	0	0	0	1	0
(4132)	0	1	0	0	0	0	0	1	0	0	1	0	1	0	0	0
(1243)	1	0	0	0	0	1	0	0	0	0	0	1	0	0	1	0
(2134)	0	1	0	0	1	0	0	0	0	0	1	0	0	0	0	1
(3421)	0	0	0	1	0	0	1	0	1	0	0	0	0	1	0	0
(4312)	0	0	1	0	0	0	0	1	0	1	0	0	1	0	0	0
(1432)	1	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0
(2341)	0	0	0	1	1	0	0	0	0	1	0	0	0	0	1	0
(3214)	0	0	1	0	0	1	0	0	1	0	0	0	0	0	0	1
(4123)	0	1	0	0	0	0	1	0	0	0	0	1	1	0	0	0
(1324)	1	0	0	0	0	0	1	0	0	1	0	0	0	0	0	1
(2413)	0	0	1	0	1	0	0	0	0	0	0	1	0	1	0	0
(3142)	0	1	0	0	0	0	0	1	1	0	0	0	0	0	1	0
(4231)	0	0	0	1	0	1	0	0	0	0	1	0	1	0	0	0

The positive cone $K_0(B^\bullet)_+$ of $K_0(B^\bullet)$ is the set of sums of q_σ's. In other words, we have

$$K_0(B^\bullet)_+ = \left\{ \sum_{\sigma\in\mathfrak{S}_4} n_\sigma q_\sigma \;\middle|\; n_\sigma = 0,1,2,\ldots \right\}$$

Proposition 12.7. *The intersection $K_0(B^\bullet)_+ \cap \ker\delta$ is the set of sums of $[P_{i,j}]_0$'s.*

Proof. It is clear that $[P_{i,j}]_0$ is in $K_0(B^\bullet)_+ \cap \ker\delta$ for $i,j = 1,2,3,4$. Thus the set of sums of $[P_{i,j}]_0$'s is contained in $K_0(B^\bullet)_+ \cap \ker\delta$.

Take $x \in K_0(B^\bullet)_+ \cap \ker\delta$. By Proposition 12.6, there exist $n_{i,j} \in \mathbb{Z}$ for $i,j = 1,2,3,4$ such that $x = \sum_{i,j=1}^4 n_{i,j}[P_{i,j}]_0$. We set $n := \sum_{n_{i,j}<0}(-n_{i,j})$. If $n = 0$, then x is in the set of sums of $[P_{i,j}]_0$'s. If $n > 0$, then we will show that there exist $n'_{i,j} \in \mathbb{Z}$ for $i,j = 1,2,3,4$ such that $x = \sum_{i,j=1}^4 n'_{i,j}[P_{i,j}]_0$ and that $n' := \sum_{n'_{i,j}<0}(-n'_{i,j})$ satisfies $0 \le n' < n$. Repeating this argument at most n times, we will find $n''_{i,j} \in \mathbb{Z}$ for $i,j = 1,2,3,4$ such that $x = \sum_{i,j=1}^4 n''_{i,j}[P_{i,j}]_0$ and that $n'' := \sum_{n''_{i,j}<0}(-n''_{i,j})$ satisfies $n'' = 0$. This shows that x is in the set of sums of $[P_{i,j}]_0$'s.

Since $n > 0$ we have $i_0, j_0 \in \{1,2,3,4\}$ such that $n_{i_0,j_0} < 0$. To simplify the notation, we assume $i_0 = 3$ and $j_0 = 1$. The other 15 cases can be shown similarly. Since $x \in K_0(B^\bullet)_+$, the coefficient of v_σ in x is non-negative for all $\sigma \in \mathfrak{S}_4$. In particular, so is for $\sigma \in \mathfrak{S}_4$ with $i_0 = \sigma(j_0)$. Since the coefficient of $v_{(3,1,2,4)}$ in x is non-negative we have $n_{3,1} + n_{1,2} + n_{2,3} + n_{4,4} \ge 0$. Since $n_{3,1} < 0$, we have $n_{1,2} + n_{2,3} + n_{4,4} > 0$. Hence either $n_{1,2}$, $n_{2,3}$ or $n_{4,4}$ is positive. Similarly, since the coefficients of

$$v_{(3,1,4,2)}, v_{(3,2,1,4)}, v_{(3,2,4,1)}, v_{(3,4,1,2)}, v_{(3,4,2,1)}$$

in x are non-negative, we obtain that either $n_{1,2}$, $n_{4,3}$ or $n_{2,4}$ is positive etc. Then by Lemma 12.8 below we have either

(i) $n_{i_1,2}$ $n_{i_1,3}$ and $n_{i_1,4}$ are positive for some $i_1 \in \{1,2,4\}$,

(ii) n_{1,j_1} n_{2,j_1} and n_{4,j_1} are positive for some $j_1 \in \{2,3,4\}$, or

(iii) $n_{i_1,j_1}, n_{i_1,j_2}, n_{i_2,j_1}$ and n_{i_2,j_2} are positive for some distinct $i_1, i_2 \in \{1,2,4\}$ and distinct $j_1, j_2 \in \{2,3,4\}$.

In the case (i), we set $n'_{i,j}$ by

$$n'_{i,j} = \begin{cases} n_{i,j} + 1 & \text{for } i \in \{1,2,3,4\} \setminus \{i_1\} \text{ and } j = 1, \\ n_{i,j} - 1 & \text{for } i = i_1 \text{ and } j = 2,3,4 \\ n_{i,j} & \text{otherwise.} \end{cases}$$

Then since $n'_{3,1} = n_{3,1} + 1$, $n' := \sum_{n'_{i,j}<0}(-n'_{i,j})$ satisfies $0 \leq n' < n$. We also have $x = \sum_{i,j=1}^{4} n'_{i,j}[P_{i,j}]_0$ because $\sum_{i=1}^{4}[P_{i,1}]_0 = \sum_{j=1}^{4}[P_{i_1,j}]_0$. In the case (ii), we get the same conclusion for $n'_{i,j}$ defined by

$$n'_{i,j} = \begin{cases} n_{i,j} + 1 & \text{for } i = 3 \text{ and } j \in \{1,2,3,4\} \setminus \{j_1\}, \\ n_{i,j} - 1 & \text{for } i = 1,2,4 \text{ and } j = j_1 \\ n_{i,j} & \text{otherwise.} \end{cases}$$

In the case (iii), we define $n'_{i,j}$ by

$$n'_{i,j} = \begin{cases} n_{i,j} + 1 & \text{for } i \in \{1,2,3,4\} \setminus \{i_1, i_2\} \text{ and } j \in \{1,2,3,4\} \setminus \{j_1, j_2\}, \\ n_{i,j} - 1 & \text{for } i = i_1, i_2 \text{ and } j = j_1, j_2 \\ n_{i,j} & \text{otherwise.} \end{cases}$$

Since $n'_{3,1} = n_{3,1} + 1$, $n' := \sum_{n'_{i,j}<0}(-n'_{i,j})$ satisfies $0 \leq n' < n$. We also have $x = \sum_{i,j=1}^{4} n'_{i,j}[P_{i,j}]_0$ because

$$\sum_{i=1}^{4}[P_{i,j_1}]_0 + \sum_{i=1}^{4}[P_{i,j_2}]_0 = \sum_{j=1}^{4}[P_{i_3,j}]_0 + \sum_{j=1}^{4}[P_{i_4,j}]_0.$$

where $\{i_3, i_4\} = \{1,2,3,4\} \setminus \{i_1, i_2\}$. This completes the proof. □

Lemma 12.8. *Let a, b, c and d, e, f are distinct three numbers, respectively. Suppose $n_{i,j} \in \mathbb{Z}$ for $i = a, b, c$ and $j = d, e, f$ satisfy that either $n_{\omega(d),d}$, $n_{\omega(e),e}$ or $n_{\omega(f),f}$ is positive for all bijection $\omega\colon \{d, e, f\} \to \{a, b, c\}$. Then we have either*

(i) *$n_{i_1,d}$ $n_{i_1,e}$ and $n_{i_1,f}$ are positive for some $i_1 \in \{a, b, c\}$,*

(ii) *n_{a,j_1} n_{b,j_1} and n_{c,j_1} are positive for some $j_1 \in \{d, e, f\}$, or*

(iii) *n_{i_1,j_1}, n_{i_1,j_2}, n_{i_2,j_1} and n_{i_2,j_2} are positive for some distinct $i_1, i_2 \in \{a, b, c\}$ and distinct $j_1, j_2 \in \{d, e, f\}$.*

Proof. To the contrary, assume that the conclusion does not hold. Then for $j = d, e, f$, either $n_{a,j}$, $n_{b,j}$ or $n_{c,j}$ is non-positive. Thus we obtain a map $\omega\colon \{d, e, f\} \to \{a, b, c\}$ such that $n_{\omega(j),j}$ is non-positive for $j = d, e, f$. If the cardinality of the image of ω is three, then ω is a bijection and it contradicts the assumption. If the cardinality of the image of ω is two, let i_1 be the element in $\{a, b, c\}$ which is not in the image of ω. Then we have either $n_{i_1,d}$ $n_{i_1,e}$ or $n_{i_1,f}$ is non-positive. Let $j_1 \in \{d, e, f\}$ be an element such that n_{i_1,j_1} is non-positive. If the cardinality of $\omega^{-1}(\omega(j_1))$ is two, we get a bijection $\omega'\colon \{d, e, f\} \to \{a, b, c\}$ such that $n_{\omega(d),d}$, $n_{\omega(e),e}$ and $n_{\omega(f),f}$ are non-positive. This is a contradiction. If the cardinality of $\omega^{-1}(\omega(j_1))$ is one, we have either n_{i_1,j_2}, n_{i_1,j_3}, n_{i_2,j_2} or n_{i_2,j_3} is non-positive where $i_2 = \omega(j_1)$ and $\{j_2, j_3\} = \{d, e, f\} \setminus \{j_1\}$. In this case, we can find a bijection $\omega' \colon \{d, e, f\} \to \{a, b, c\}$ such that $n_{\omega(d),d}$, $n_{\omega(e),e}$ and $n_{\omega(f),f}$ are non-positive. This is a contradiction. Finally, if the cardinality of the image of ω is one, let i_1 be the unique element of the image of ω, and i_2 and i_3 be the other two elements in $\{a, b, c\}$. We have $j_2, j_3 \in \{d, e, f\}$ such that n_{i_2,j_2} and n_{i_3,j_3} are non-positive. If $j_2 \neq j_3$, then we can find a bijection $\omega' \colon \{d, e, f\} \to \{a, b, c\}$ such that $n_{\omega(d),d}$, $n_{\omega(e),e}$ and $n_{\omega(f),f}$ are non-positive. This is a contradiction. If $j_2 = j_3$, then we have either n_{i_2,j_1}, $n_{i_2,j_1'}$, $n_{i_3,j_1'}$ or n_{i_3,j_1} is non-positive where $\{j_1, j_1'\} = \{d, e, f\} \setminus \{j_2\}$. In this case, we can find a bijection $\omega' \colon \{d, e, f\} \to \{a, b, c\}$ such that $n_{\omega(d),d}$, $n_{\omega(e),e}$ and $n_{\omega(f),f}$ are non-positive. This is a contradiction. We are done. □

13. The Structure of the Ideal I

Definition 13.1. Define a subspace V of $\mathbb{R}P^3$ by

$$V := \{[a_1, a_2, a_3, a_4] \in \mathbb{R}P^3 \mid a_1, a_2, a_3 > |a_4|\}.$$

The next proposition gives us a motivation to compute the subspace V and its closure $\overline{V}$ in $\mathbb{R}P^3$.

Proposition 13.2. *We have the following facts.*

(i) *For each $i, j = 1, 2, 3, 4$ with $(i, j) \neq (1, 1)$, we have $\sigma_{i,j}(V) \cap V = \emptyset$*

(ii) *The restriction of π to V is a homeomorphism onto $\pi(V) \subset X$.*

(iii) $\overline{V} = \{[a_1, a_2, a_3, a_4] \in \mathbb{R}P^3 \mid a_1, a_2, a_3 \geq |a_4|\}$ *and* $\pi(\overline{V}) = X$.

Proof. (i) and (iii) can be checked directly, and (ii) follows from (i). □

In the next proposition, when we write $[a_1, a_2, a_3, a_4] \in \overline{V}$, we mean (a_1, a_2, a_3, a_4) satisfies $a_1, a_2, a_3 \geq |a_4|$.

Proposition 13.3. *The map*

$$h \colon \overline{V} \ni [a_1, a_2, a_3, a_4] \longmapsto \big(3a_1^2 + a_4^2 + 4a_4|a_4|, 3a_2^2 + a_4^2 + 4a_4|a_4|, 3a_3^2 + a_4^2 + 4a_4|a_4|\big) \in \mathbb{R}^3$$

is a homeomorphism onto the hexahedron whose 6 faces are isosceles right triangles and whose vertices are $(0, 0, 0)$*,* $(3, 0, 0)$*,* $(0, 3, 0)$*,* $(0, 0, 3)$ *and* $(2, 2, 2)$*. This map sends V onto the interior of the hexahedron.*

Proof. First note that we have $|a_4| \leq 1/2$ for $[a_1, a_2, a_3, a_4] \in \overline{V}$. When $|a_4| = 1/2$, we have $a_1 = a_2 = a_3 = 1/2$. We have $h([1/2, 1/2, 1/2, 1/2]) = (2, 2, 2)$ and $h([1/2, 1/2, 1/2, -1/2]) = (0, 0, 0)$. When $|a_4| = 0$, we have $a_1, a_2, a_3 \geq 0$ and $a_1^2 + a_2^2 + a_3^2 = 1$. Thus

$$\{h([a_1, a_2, a_3, 0]) \,|\, [a_1, a_2, a_3, 0] \in \overline{V}\} = \{(x, y, z) \in \mathbb{R}^3 \,|\, x, y, z \geq 0,\ x + y + z = 3\}$$

which is the equilateral triangle whose vertices are $(3, 0, 0)$, $(0, 3, 0)$ and $(0, 0, 3)$. For each t with $-1/2 < t < 0$, we have

$$\{h([a_1, a_2, a_3, t]) \,|\, [a_1, a_2, a_3, t] \in \overline{V}\} = \{(x, y, z) \in \mathbb{R}^3 \,|\, x, y, z \geq 0,\ x + y + z = 3(1 - 4t^2)\}$$

which is the equilateral triangle whose vertices are $(3(1 - 4t^2), 0, 0)$, $(0, 3(1 - 4t^2), 0)$ and $(0, 0, 3(1 - 4t^2))$. Thus

$$\{h([a_1, a_2, a_3, a_4]) \,|\, [a_1, a_2, a_3, a_4] \in \overline{V}, a_4 \leq 0\}$$

is the tetrahedron whose vertices are $(0, 0, 0)$, $(3, 0, 0)$, $(0, 3, 0)$ and $(0, 0, 3)$. Note that for each $[a_1, a_2, a_3, a_4] \in \overline{V}$ with $a_4 \geq 0$, the point $h([a_1, a_2, a_3, a_4])$ is the reflection point of $h([a_1, a_2, a_3, -a_4])$ with respect to the plane $x + y + z = 3$ because the vector $(8a_4^2, 8a_4^2, 8a_4^2)$ is orthogonal to the plane $x + y + z = 3$ and the point $(3a_1^2 + a_4^2, 3a_2^2 + a_4^2, 3a_3^2 + a_4^2)$ is on the plane $x + y + z = 3$. Thus

$$\{h([a_1, a_2, a_3, a_4]) \,|\, [a_1, a_2, a_3, a_4] \in \overline{V}, a_4 \geq 0\}$$

is the reflection of the tetrahedron above with respect to the plane $x + y + z = 3$, which in turn is the tetrahedron whose vertices are $(3, 0, 0)$, $(0, 3, 0)$, $(0, 0, 3)$ and $(2, 2, 2)$. From the discussion above, we see that h is injective. Therefore we see that h is a homeomorphism from $\overline{V}$ onto the hexahedron whose vertices are $(0, 0, 0)$, $(3, 0, 0)$, $(0, 3, 0)$, $(0, 0, 3)$ and $(2, 2, 2)$. We can also see that the map h sends V onto the interior of the hexahedron. □

Definition 13.4. Define $O_0 := \pi(V) \subset O$.

By Proposition 13.2 (ii) and Proposition 13.3, $O_0 \cong V$ is homeomorphic to $\mathbb{R}^3$.

Definition 13.5. We set $E := \widetilde{F} \cap \overline{V}$ and $E_{i,j} := \widetilde{F}_{i,j} \cap \overline{V}$ for $i, j = 2, 3, 4$.

We have $E = \bigcup_{i,j=2}^{4} E_{i,j}$. For $i, j = 2, 3, 4$ with $i \neq j$, the map $\pi\colon E_{i,j} \to F_{i,j}$ is a homeomorphism. For $i = 2, 3, 4$ the map $\pi\colon E_{i,i} \to F_{i,i}$ is a 2-to-1 map except the middle point.

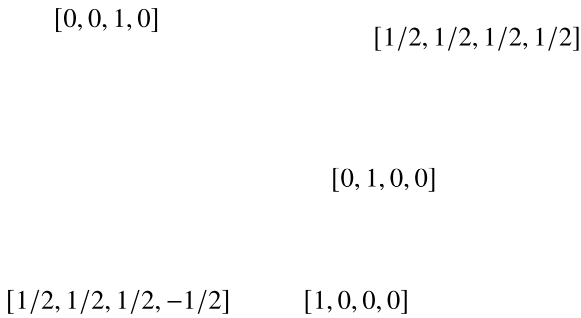

FIGURE 13.1. $\overline{V}$

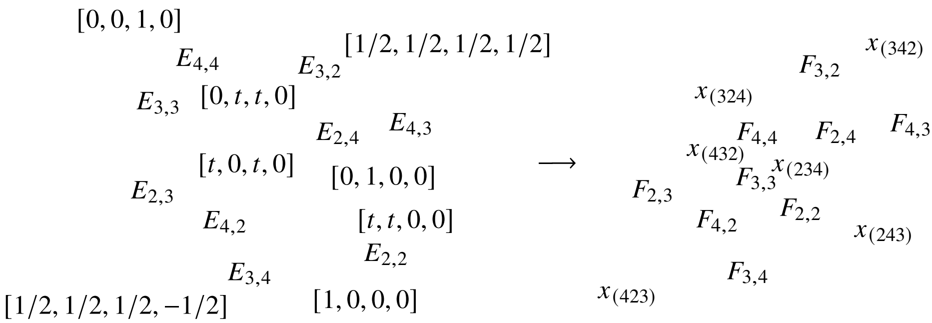

FIGURE 13.2. $\pi\colon E \to F$ $(t = 1/\sqrt{2})$

We have

$$\begin{aligned}
E_{2,2} &= \left\{[a,b,0,0] \in \overline{V} \,\middle|\, a,b \geq 0,\ a^2+b^2=1\right\},\\
E_{2,3} &= \left\{[a,b,b,-a] \in \overline{V} \,\middle|\, 0 \leq a \leq b,\ 2(a^2+b^2)=1\right\},\\
E_{2,4} &= \left\{[a,b,a,b] \in \overline{V} \,\middle|\, 0 \leq b \leq a,\ 2(a^2+b^2)=1\right\},\\
E_{3,2} &= \left\{[a,b,b,a] \in \overline{V} \,\middle|\, 0 \leq a \leq b,\ 2(a^2+b^2)=1\right\},\\
E_{3,3} &= \left\{[a,0,b,0] \in \overline{V} \,\middle|\, a,b \geq 0,\ a^2+b^2=1\right\},\\
E_{3,4} &= \left\{[a,a,b,-b] \in \overline{V} \,\middle|\, 0 \leq b \leq a,\ 2(a^2+b^2)=1\right\},\\
E_{4,2} &= \left\{[a,b,a,-b] \in \overline{V} \,\middle|\, 0 \leq b \leq a,\ 2(a^2+b^2)=1\right\},
\end{aligned}$$

$$E_{4,3} = \left\{[a,a,b,b] \in \overline{V} \,\middle|\, 0 \le b \le a,\ 2(a^2+b^2) = 1\right\},$$
$$E_{4,4} = \left\{[0,a,b,0] \in \overline{V} \,\middle|\, a,b \ge 0,\ a^2+b^2 = 1\right\}.$$

Definition 13.6. We set $R_x^+, R_y^+, R_z^+, R_x^-, R_y^-, R_z^- \subset \overline{V}$ by

$$R_x^\pm := \left\{\left[\sqrt{1-3t^2}, t, t, \pm t\right] \in \overline{V} \,\middle|\, 0 < t < 1/2\right\}$$
$$R_y^\pm := \left\{\left[t, \sqrt{1-3t^2}, t, \pm t\right] \in \overline{V} \,\middle|\, 0 < t < 1/2\right\}$$
$$R_z^\pm := \left\{\left[t, t, \sqrt{1-3t^2}, \pm t\right] \in \overline{V} \,\middle|\, 0 < t < 1/2\right\}$$

We see that $R_x^+ \cup R_y^+ \cup R_z^+ \cup R_x^- \cup R_y^- \cup R_z^-$ is the space obtained by subtracting E from the "edges" of $\overline{V}$.

Definition 13.7. We set $R^+, R^- \subset O$ by

$$R^\pm := \pi(R_x^\pm) = \pi(R_y^\pm) = \pi(R_z^\pm)$$

Note that π induces a homeomorphism from $R_x^\pm$ (or $R_y^\pm$, $R_z^\pm$) to $R^\pm$. Hence both R^+ and R^- are homeomorphic to $\mathbb{R}$.

Definition 13.8. We set

$$\widehat{T}_{2,3} := \left\{[t,a,b,-t] \in \overline{V} \,\middle|\, 0 < t < 1/2,\ a,b > t,\ a^2+b^2 = 1-2t^2\right\},$$
$$\widehat{T}_{3,4} := \left\{[a,b,t,-t] \in \overline{V} \,\middle|\, 0 < t < 1/2,\ a,b > t,\ a^2+b^2 = 1-2t^2\right\},$$
$$\widehat{T}_{4,2} := \left\{[b,t,a,-t] \in \overline{V} \,\middle|\, 0 < t < 1/2,\ a,b > t,\ a^2+b^2 = 1-2t^2\right\},$$
$$\widehat{T}_{3,2} := \left\{[t,a,b,t] \in \overline{V} \,\middle|\, 0 < t < 1/2,\ a,b > t,\ a^2+b^2 = 1-2t^2\right\},$$
$$\widehat{T}_{4,3} := \left\{[a,b,t,t] \in \overline{V} \,\middle|\, 0 < t < 1/2,\ a,b > t,\ a^2+b^2 = 1-2t^2\right\},$$
$$\widehat{T}_{2,4} := \left\{[b,t,a,t] \in \overline{V} \,\middle|\, 0 < t < 1/2,\ a,b > t,\ a^2+b^2 = 1-2t^2\right\}.$$

These 6 spaces are the interiors of the 6 "faces" of $\overline{V}$.

Definition 13.9. We set

$$\widehat{T}^r_{2,3} := \left\{[t,a,b,-t] \in \widehat{T}_{2,3} \,\middle|\, a > b\right\}, \qquad \widehat{T}^l_{2,3} := \left\{[t,a,b,-t] \in \widehat{T}_{2,3} \,\middle|\, a < b\right\}$$
$$\widehat{T}^r_{3,4} := \left\{[a,b,t,-t] \in \widehat{T}_{3,4} \,\middle|\, a > b\right\}, \qquad \widehat{T}^l_{3,4} := \left\{[a,b,t,-t] \in \widehat{T}_{3,4} \,\middle|\, a < b\right\}$$
$$\widehat{T}^r_{4,2} := \left\{[b,t,a,-t] \in \widehat{T}_{4,2} \,\middle|\, a > b\right\}, \qquad \widehat{T}^l_{4,2} := \left\{[b,t,a,-t] \in \widehat{T}_{4,2} \,\middle|\, a < b\right\}$$
$$\widehat{T}^r_{3,2} := \left\{[t,a,b,t] \in \widehat{T}_{3,2} \,\middle|\, a > b\right\}, \qquad \widehat{T}^l_{3,2} := \left\{[t,a,b,t] \in \widehat{T}_{3,2} \,\middle|\, a < b\right\}$$
$$\widehat{T}^r_{4,3} := \left\{[a,b,t,t] \in \widehat{T}_{4,3} \,\middle|\, a > b\right\}, \qquad \widehat{T}^l_{4,3} := \left\{[a,b,t,t] \in \widehat{T}_{4,3} \,\middle|\, a < b\right\}$$
$$\widehat{T}^r_{2,4} := \left\{[b,t,a,t] \in \widehat{T}_{2,4} \,\middle|\, a > b\right\}, \qquad \widehat{T}^l_{2,4} := \left\{[b,t,a,t] \in \widehat{T}_{2,4} \,\middle|\, a < b\right\}.$$

For $i, j = 2, 3, 4$ with $i \neq j$, the set $\widehat{T}_{i,j} \setminus (\widehat{T}^r_{i,j} \cup \widehat{T}^l_{i,j})$ is the interior of $E_{i,j}$.

Definition 13.10. For $i, j = 2, 3, 4$ with $i \neq j$, we set

$$T_{i,j} := \pi(\widehat{T}^r_{i,j}) = \pi(\widehat{T}^l_{i,j}).$$

Note that π induces a homeomorphism from $\widehat{T}^r_{i,j}$ (or $\widehat{T}^l_{i,j}$) to $T_{i,j}$. Hence $T_{i,j}$ is homeomorphic to $\mathbb{R}^2$.

The space O is a disjoint union (as a set) of

$$O_0,\ T_{2,3},\ T_{3,4},\ T_{4,2},\ R^-,\ T_{3,2},\ T_{4,3},\ T_{2,4},\ R^+.$$

We use these spaces to compute the K-groups of $I = M_4(C_0(\widetilde{O}))^\beta$.

14. K-groups of the ideal I

Definition 14.1. We set $I_0 := M_4\big(C_0(\pi^{-1}(O_0))\big)^\beta$ and $I^\star := M_4\big(C_0(\pi^{-1}(O \setminus O_0))\big)^\beta$.

We have a short exact sequence

$$0 \longrightarrow I_0 \longrightarrow I \longrightarrow I^\star \longrightarrow 0.$$

We have $I_0 \cong M_4(C_0(V)) \cong M_4(C_0(O_0)) \cong M_4(C_0(\mathbb{R}^3))$.

Definition 14.2. We set $T := T_{2,3} \cup T_{3,4} \cup T_{4,2} \cup T_{3,2} \cup T_{4,3} \cup T_{2,4}$ and $R := R^- \cup R^+$. We set $I^\circ := M_4\big(C_0(\pi^{-1}(T))\big)^\beta$ and $I^\bullet := M_4\big(C_0(\pi^{-1}(R))\big)^\beta$.

We have $I^\circ \cong M_4(C_0(T)) \cong \bigoplus_{i,j} M_4(C_0(T_{i,j})) \cong M_4(C_0(\mathbb{R}^2))^6$ and

$$I^\bullet \cong M_4(C_0(R)) \cong M_4(C_0(R^-)) \oplus M_4(C_0(R^+)) \cong M_4(C_0(\mathbb{R}))^2.$$

We have a short exact sequence

$$0 \longrightarrow I^\circ \longrightarrow I^\star \longrightarrow I^\bullet \longrightarrow 0.$$

This induces a six-term exact sequence

$$\begin{array}{ccccccc}
\mathbb{Z}^6 \cong K_0(I^\circ) & \longrightarrow & K_0(I^\star) & \longrightarrow & K_0(I^\bullet) & = & 0 \\
\uparrow & & & & \downarrow & & \\
\mathbb{Z}^2 \cong K_1(I^\bullet) & \longleftarrow & K_1(I^\star) & \longleftarrow & K_1(I^\circ) & = & 0.
\end{array}$$

We set $r^- \in K_1\big(M_4(C_0(R^-))\big)$ and $r^+ \in K_1\big(M_4(C_0(R^+))\big)$ to be the images of $v_{(1234)} \in K_0(B_{(234)}) \subset K_0(B^\bullet)$ under the exponential maps coming from the exact sequences

$$0 \longrightarrow M_4(C_0(R^\pm)) \longrightarrow M_4\big(C_0(\pi^{-1}(R^\pm \cup \{x_{(234)}\}))\big)^\beta \longrightarrow B_{(234)} \longrightarrow 0.$$

Then similarly as the proof of Lemma 12.4, we see that r^- and r^+ are the generators of $K_1\big(M_4(C_0(R^-))\big) \cong \mathbb{Z}$ and $K_1\big(M_4(C_0(R^+))\big) \cong \mathbb{Z}$, respectively.

Let $\omega = (1342) \in \mathfrak{S}_4$. For $i = 2, 3, 4$, we set $w_{i,\omega(i)} \in K_0\big(M_4(C_0(T_{i,\omega(i)}))\big)$ to be the image of the generator r^- of $K_1\big(M_4(C_0(R^-))\big)$ under the index map coming from the exact sequences

$$0 \longrightarrow M_4(C_0(T_{i,\omega(i)})) \longrightarrow M_4\big(C_0(\pi^{-1}(T_{i,\omega(i)} \cup R^-))\big)^{\beta} \longrightarrow M_4(C_0(R^-)) \longrightarrow 0.$$

Since

$$M_4\big(C_0(\pi^{-1}(T_{2,3} \cup R^-))\big)^{\beta} \cong M_4\big(C_0(\widehat{T}^r_{2,3} \cup R^-_y)\big) \cong M_4\big(C_0((0,1) \times (0,1])\big)$$

whose K-groups are 0, $w_{2,3}$ is a generator of $K_0\big(M_4(C_0(T_{2,3}))\big) \cong \mathbb{Z}$. Similarly, $w_{3,4}$ and $w_{4,2}$ are generators of $K_0\big(M_4(C_0(T_{3,4}))\big) \cong \mathbb{Z}$ and $K_0\big(M_4(C_0(T_{4,2}))\big) \cong \mathbb{Z}$, respectively.

Similarly for $i = 2, 3, 4$, we set the generator $w_{\omega(i),i}$ of $K_0\big(M_4(C_0(T_{\omega(i),i}))\big) \cong \mathbb{Z}$ to be the image of the generator r^+ of $K_1\big(M_4(C_0(R^+))\big)$ under the index map coming from the exact sequences

$$0 \longrightarrow M_4(C_0(T_{\omega(i),i})) \longrightarrow M_4\big(C_0(\pi^{-1}(T_{\omega(i),i} \cup R^+))\big)^{\beta} \longrightarrow M_4(C_0(R^+)) \longrightarrow 0.$$

Then the index map from

$$K_1(I^\bullet) \cong K_1\big(M_4(C_0(R^-))\big) \oplus K_1\big(M_4(C_0(R^+))\big) \cong \mathbb{Z}^2$$

to

$$\begin{aligned} K_0(I^\circ) \cong K_0\big(M_4(C_0(T_{2,3}))\big) &\oplus K_0\big(M_4(C_0(T_{3,4}))\big) \oplus K_0\big(M_4(C_0(T_{4,2}))\big) \\ &\oplus K_0\big(M_4(C_0(T_{3,2}))\big) \oplus K_0\big(M_4(C_0(T_{4,3}))\big) \oplus K_0\big(M_4(C_0(T_{2,4}))\big) \cong \mathbb{Z}^6 \end{aligned}$$

becomes $\mathbb{Z}^2 \ni (a, b) \mapsto (a, a, a, b, b, b) \in \mathbb{Z}^6$. Thus we have the following.

Proposition 14.3. *We have $K_0(I^\star) \cong \mathbb{Z}^4$ and $K_1(I^\star) = 0$.*

We denote by $s_1, s_2, s_3, s_4 \in K_0(I^\star)$ the images of $w_{2,3}, w_{3,4}, w_{3,2}, w_{4,3} \in K_0(I^\circ)$. Then $\{s_1, s_2, s_3, s_4\}$ becomes a basis of $K_0(I^\star) \cong \mathbb{Z}^4$. Note that the images of $w_{4,2}, w_{2,4} \in K_0(I^\circ)$ are $-s_1 - s_2 \in K_0(I^\star)$ and $-s_3 - s_4 \in K_0(I^\star)$, respectively.

We have a six-term exact sequence

$$\begin{array}{ccccc} 0 = K_0(I_0) & \longrightarrow & K_0(I) & \longrightarrow & K_0(I^\star) \cong \mathbb{Z}^4 \\ \uparrow & & & & \downarrow \\ 0 = K_1(I^\star) & \longleftarrow & K_1(I) & \longleftarrow & K_1(I_0) \cong \mathbb{Z}. \end{array} \tag{14.1}$$

To compute the index map $K_0(I^\star) \to K_1(I_0)$, we need the following lemma.

Lemma 14.4. *The index map from $K_0(I^\circ) \cong \mathbb{Z}^6$ to $K_1(I_0) \cong \mathbb{Z}$ coming from the short exact sequence*

$$0 \longrightarrow I_0 \longrightarrow M_4\big(C_0(\pi^{-1}(O_0 \cup T))\big)^\beta \longrightarrow I^\circ \longrightarrow 0.$$

is 0.

Proof. We set $\widehat{T} := \bigcup_{i,j}(\widehat{T}^r_{i,j} \cup \widehat{T}^l_{i,j})$ where i, j run $2, 3, 4$ with $i \neq j$. We have the following commutative diagram with exact rows;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_0 & \longrightarrow & M_4\big(C_0(\pi^{-1}(O_0 \cup T))\big)^\beta & \longrightarrow & I^\circ & \longrightarrow & 0 \\ & & \downarrow{\cong} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_4(C_0(V)) & \longrightarrow & M_4(C_0(V \cup \widehat{T}))\big) & \longrightarrow & M_4(C_0(\widehat{T})) & \longrightarrow & 0. \end{array}$$

Note that $V \cup \widehat{T} = \pi^{-1}(O_0 \cup T) \cap \overline{V}$. From this diagram, we see that the index map $K_0(I^\circ) \to K_1(I_0)$ factors through $K_0(M_4(C_0(\widehat{T})))$.

Take $i, j = 2, 3, 4$ with $i \neq j$. Let $a^r_{i,j} \in K_0\big(M_4(C_0(\widehat{T}^r_{i,j}))\big)$ and $a^l_{i,j} \in K_0\big(M_4(C_0(\widehat{T}^l_{i,j}))\big)$ be the images of the generator $w_{i,j}$ of $K_0\big(M_4(C_0(T_{i,j}))\big)$ under the homomorphism induced by π. Under the map $K_0(I^\circ) \to K_0(M_4(C_0(\widehat{T})))$, the generator $w_{i,j}$ of $K_0\big(M_4(C_0(T_{i,j}))\big)$ goes to $a^r_{i,j} + a^l_{i,j}$. Under the index map $K_0(M_4(C_0(\widehat{T}))) \to K_1\big(M_4(C_0(V))\big)$ the element $a^r_{i,j} + a^l_{i,j}$ goes to 0 because the side to V from $\widehat{T}^r_{i,j}$ and the one from $\widehat{T}^l_{i,j}$ differ if $\widehat{T}^r_{i,j}$ and $\widehat{T}^l_{i,j}$ are identified through the map π to $T_{i,j}$. Thus we see that the map $K_0(I^\circ) \to K_1\big(M_4(C_0(V))\big) \cong K_1(I_0)$ is 0. □

By this lemma, the composition of the map $K_0(I^\circ) \to K_0(I^\star)$ and the index map $K_0(I^\star) \to K_1(I_0)$ is 0. Since the map $\mathbb{Z}^6 \cong K_0(I^\circ) \to K_0(I^\star) \cong \mathbb{Z}^4$ is a surjection, we see that the index map $K_0(I^\star) \to K_1(I_0)$ is 0. Thus we have the following.

Proposition 14.5. *We have $K_0(I) \cong K_0(I^\star) \cong \mathbb{Z}^4$ and $K_1(I) \cong K_1(I_0) \cong \mathbb{Z}$.*

15. K-groups of A

Recall the six-term exact sequence

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(B) \\ \delta_1 \uparrow & & & & \downarrow \delta_0 \\ K_1(B) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I). \end{array}$$

In this section, we calculate the exponential map $\delta_0\colon K_0(B) \to K_1(I)$ and the index map $\delta_1\colon K_1(B) \to K_0(I)$.

Proposition 15.1. *The exponential map $\delta_0\colon K_0(B) \to K_1(I)$ is* 0.

Proof. Since $K_0(B)$ is generated by 16 elements $\{[P_{i,j}]_0\}_{i,j=1}^4$, the map $K_0(A) \to K_0(B)$ is surjective. Hence the exponential map $\delta_0\colon K_0(B) \to K_1(I)$ is 0. □

By the definitions of the generators of K-groups we did so far, we have the following. (See Figure 13.2 for the relation between T and F.)

Proposition 15.2. *The index map $\delta''\colon K_1(B^\circ) \cong \mathbb{Z}^{18} \to K_0(I^\circ) \cong \mathbb{Z}^6$ coming from the short exact sequence*

$$0 \longrightarrow I^\circ \longrightarrow M_4\big(C_0(\pi^{-1}(T \cup F^\circ))\big)^\beta \longrightarrow B^\circ \longrightarrow 0.$$

is as Table 15.1.

Table 15.1. Computation of the index map δ''

	2,2		3,3		4,4		2,3		3,4		4,2		3,2		4,3		2,4	
w \ v	∩	∪	∩	∪	∩	∪	∩	∪	∩	∪	∩	∪	∩	∪	∩	∪	∩	∪
2,3	0	0	0	0	−1	−1	1	1	0	0	0	0	0	0	0	0	0	0
3,4	−1	−1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0
4,2	0	0	−1	−1	0	0	0	0	0	0	1	1	0	0	0	0	0	0
3,2	0	0	0	0	−1	−1	0	0	0	0	0	0	1	1	0	0	0	0
4,3	−1	−1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0
2,4	0	0	−1	−1	0	0	0	0	0	0	0	0	0	0	0	0	1	1

Definition 15.3. The composition of the index map $\delta''\colon K_1(B^\circ) \to K_0(I^\circ)$ and the map $K_0(I^\circ) \to K_0(I^\star)$ is denoted by $\eta\colon K_1(B^\circ) \to K_0(I^\star)$

We set $\widetilde{\eta}\colon K_1(B^\circ) \to K_0(I^\star) \oplus \mathbb{Z}/2\mathbb{Z}$ by $\widetilde{\eta}(w^\cap_{i,j}) = (\eta(w^\cap_{i,j}), 0)$ and $\widetilde{\eta}(w^\cup_{i,j}) = (\eta(w^\cup_{i,j}), 1)$ for $i, j = 2, 3, 4$.

We denote the generator of $\mathbb{Z}/2\mathbb{Z}$ in $K_0(I^\star) \oplus \mathbb{Z}/2\mathbb{Z}$ by s_5.

Proposition 15.4. *The map $\widetilde{\eta}\colon K_1(B^\circ) \to K_0(I^\star) \oplus \mathbb{Z}/2\mathbb{Z}$ is surjective, and its kernel coincides with the image of $\delta\colon K_0(B^\bullet) \to K_1(B^\circ)$.*

Proof. Since

$$\widetilde{\eta}(w^\cap_{2,3}) = s_1, \quad \widetilde{\eta}(w^\cap_{3,4}) = s_2, \quad \widetilde{\eta}(w^\cap_{3,2}) = s_3, \quad \widetilde{\eta}(w^\cap_{4,3}) = s_4,$$

s_1, s_2, s_3, s_4 are in the image of $\widetilde{\eta}$. Since $\widetilde{\eta}(w^\cup_{2,2} + w^\cup_{3,3} + w^\cup_{4,4}) = s_5$, s_5 is also in the image of $\widetilde{\eta}$. Thus $\widetilde{\eta}$ is surjective.

TABLE 15.2. Computation of $\widetilde{\eta}$

	2,2		3,3		4,4		2,3		3,4		4,2		3,2		4,3		2,4	
s \ v	$\cap$	$\cup$	$\cap$	$\cup$	$\cap$	$\cup$	$\cap$	$\cup$	$\cap$	$\cup$	$\cap$	$\cup$	$\cap$	$\cup$	$\cap$	$\cup$	$\cap$	$\cup$
1	0	0	1	1	−1	−1	1	1	0	0	−1	−1	0	0	0	0	0	0
2	−1	−1	1	1	0	0	0	0	1	1	−1	−1	0	0	0	0	0	0
3	0	0	1	1	−1	−1	0	0	0	0	0	0	1	1	0	0	−1	−1
4	−1	−1	1	1	0	0	0	0	0	0	0	0	0	0	1	1	−1	−1
5	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

It is straightforward to check $\widetilde{\eta} \circ \delta = 0$ Hence the image of δ is contained in the kernel of $\widetilde{\eta}$. Suppose

$$x = \sum_{i,j=2}^{4} n_{i,j}^{\cap} w_{i,j}^{\cap} + \sum_{i,j=2}^{4} n_{i,j}^{\cup} w_{i,j}^{\cup}$$

is in the kernel of $\widetilde{\eta}$ where $n_{i,j}^{\cap}, n_{i,j}^{\cup} \in \mathbb{Z}$ for $i, j = 2, 3, 4$. We will show that x is in the image of δ. By adding

$$n_{2,3}^{\cup}\delta(q_{(3142)}) + n_{3,4}^{\cup}\delta(q_{(4312)}) + n_{4,2}^{\cup}\delta(q_{(2341)}) + n_{3,2}^{\cup}\delta(q_{(2413)}) + n_{4,3}^{\cup}\delta(q_{(3421)}) + n_{2,4}^{\cup}\delta(q_{(4123)})$$

we may assume

$$n_{2,3}^{\cup} = n_{3,4}^{\cup} = n_{4,2}^{\cup} = n_{3,2}^{\cup} = n_{4,3}^{\cup} = n_{2,4}^{\cup} = 0$$

without loss of generality. By subtracting $n_{3,3}^{\cup}\delta(q_{(4321)}) + n_{4,4}^{\cup}\delta(q_{(3412)})$, we may further assume $n_{3,3}^{\cup} = n_{4,4}^{\cup} = 0$ without loss of generality. Then $n_{2,2}^{\cup}$ is even since the coefficient of c_5 in $\widetilde{\eta}(x)$ is 0. Hence by adding

$$\frac{n_{2,2}^{\cup}}{2}\big(\delta(q_{(2143)}) - \delta(q_{(3412)}) - \delta(q_{(4321)})\big)$$

we may further assume $n_{2,2}^{\cup} = 0$ without loss of generality. Thus we may assume $x = \sum_{i,j=2}^{4} n_{i,j}^{\cap} w_{i,j}^{\cap}$. By adding $n_{2,2}^{\cap}\delta(q_{(1243)}) + n_{3,3}^{\cap}\delta(q_{(1432)}) + n_{4,4}^{\cap}\delta(q_{(1324)})$, we may further assume $n_{2,2}^{\cap} = n_{3,3}^{\cap} = n_{4,4}^{\cap} = 0$ without loss of generality. By subtracting $n_{4,2}^{\cap}\delta(q_{(1423)}) + n_{2,4}^{\cap}\delta(q_{(1342)})$, we may further assume $n_{4,2}^{\cap} = n_{2,4}^{\cap} = 0$ without loss of generality. Thus we may assume

$$x = n_{2,3}^{\cap} w_{2,3}^{\cap} + n_{3,4}^{\cap} w_{3,4}^{\cap} + n_{3,2}^{\cap} w_{3,2}^{\cap} + n_{4,3}^{\cap} w_{4,3}^{\cap}.$$

Then we have $n_{2,3}^{\cap} = n_{3,4}^{\cap} = n_{3,2}^{\cap} = n_{4,3}^{\cap} = 0$ because

$$\widetilde{\eta}(x) = n_{2,3}^{\cap} s_1 + n_{3,4}^{\cap} s_2 + n_{3,2}^{\cap} s_3 + n_{4,3}^{\cap} s_4.$$

Thus $x = 0$. We have shown that x is in the image of δ. Hence the image of δ coincides with the kernel of $\widetilde{\eta}$. □

As a corollary of this proposition, we have the following as predicted.

Proposition 15.5. *We have $K_0(B) \cong \mathbb{Z}^{10}$ and $K_1(B) \cong \mathbb{Z}^4 \oplus \mathbb{Z}/2\mathbb{Z}$.*

Proof. By Proposition 15.4, we see that $K_1(B) \cong \operatorname{coker}\delta$ is isomorphic to $\mathbb{Z}^4 \oplus \mathbb{Z}/2\mathbb{Z}$. This implies $K_0(B) \cong \ker\delta$ is isomorphic to $\mathbb{Z}^{10}$ because $\ker\delta$ is a free abelian group with dimension $24 - 18 + 4 = 10$. □

We also have the following.

Proposition 15.6. *The index map $\delta_1\colon K_1(B) \to K_0(I)$ is as $K_1(B) \cong \mathbb{Z}^4 \oplus \mathbb{Z}/2\mathbb{Z} \ni (n, m) \mapsto n \in \mathbb{Z}^4 \cong K_0(I)$.*

Proof. From the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & I^\star & \longrightarrow & M_4\big(C_0(\pi^{-1}((O \setminus O_0) \cup F))\big)^\beta & \longrightarrow & B & \longrightarrow & 0, \end{array}$$

the index map $\delta_1\colon K_1(B) \to K_0(I)$ coincides with the map $K_1(B) \to K_0(I^\star)$ if we identify $K_0(I) \cong K_0(I^\star)$ as we did in Proposition 14.5.

From the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^\circ & \longrightarrow & M_4\big(C_0(\pi^{-1}(T \cup F^\circ))\big)^\beta & \longrightarrow & B^\circ & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^\star & \longrightarrow & M_4\big(C_0(\pi^{-1}((O \setminus O_0) \cup F))\big)^\beta & \longrightarrow & B & \longrightarrow & 0, \end{array}$$

we have the commutative diagram

$$\begin{array}{ccc} K_1(B^\circ) & \longrightarrow & K_0(I^\circ) \\ \downarrow & & \downarrow \\ K_1(B) & \longrightarrow & K_0(I^\star). \end{array}$$

From this diagram, we see that the map $K_1(B) \to K_0(I^\star)$ is as $K_1(B) \cong \mathbb{Z}^4 \oplus \mathbb{Z}/2\mathbb{Z} \ni (n, m) \mapsto n \in \mathbb{Z}^4 \cong K_0(I^\star)$. This completes the proof. □

Definition 15.7. Define a unitary $w \in C(S^3, M_2(\mathbb{C}))$ by

$$\begin{aligned} w(a_1, a_2, a_3, a_4) &= a_1c_1 + a_2c_2 + a_3c_3 + a_4c_4 \\ &= \begin{pmatrix} a_1 + a_2\sqrt{-1} & a_3 + a_4\sqrt{-1} \\ -a_3 + a_4\sqrt{-1} & a_1 - a_2\sqrt{-1} \end{pmatrix} \end{aligned}$$

for $(a_1, a_2, a_3, a_4) \in S^3$.

Then $[w]_1$ is the generator of $K_1\big(C(S^3, M_2(\mathbb{C}))\big) \cong K_1\big(M_4(C(S^3))\big) \cong \mathbb{Z}$.

Let $\varphi\colon A \to M_4(C(S^3))$ be the composition of the embedding $A \to M_4(C(\mathbb{R}P^3))$ and the map $M_4(C(\mathbb{R}P^3)) \to M_4(C(S^3))$ induced by $[\,\cdot\,]\colon S^3 \to \mathbb{R}P^3$. Let $\widetilde{\pi}\colon S^3 \to X$ be the composition of $[\,\cdot\,]\colon S^3 \to \mathbb{R}P^3$ and $\pi\colon \mathbb{R}P^3 \to X$. We set V' of S^3 by

$$V' := \left\{(a_1, a_2, a_3, a_4) \in S^3 \,\middle|\, a_1, a_2, a_3 > |a_4|\right\}.$$

Then V' is homeomorphic to V via $[\,\cdot\,]$, and hence to O_0 via $\widetilde{\pi}$. Note that the map $M_4(C_0(V')) \hookrightarrow M_4(C(S^3))$ induces the isomorphism

$$K_1\big(M_4(C_0(V'))\big) \to K_1\big(M_4(C(S^3))\big).$$

Since $I_0 \cong M_4(C_0(O_0)) \cong M_4(C_0(V'))$ canonically, we set a generator y of $K_1(I_0)$ which corresponds to the generator $[w]_1$ of $K_1\big(M_4(C(S^3))\big)$ via the isomorphism $K_1\big(M_4(C_0(V'))\big) \to K_1\big(M_4(C(S^3))\big)$. We denote by the same symbol y the generator of $K_1(I) \cong K_1(I_0)$ corresponding to $y \in K_1(I_0)$.

Proposition 15.8. *The image of $y \in K_1(I)$ under the map $K_1(I) \to K_1(A) \to K_1\big(M_4(C(S^3))\big)$ is $32[w]_1$.*

Proof. The map $I_0 \to I \to A \to M_4(C(S^3))$ is induced by $\widetilde{\pi}\colon \widetilde{\pi}^{-1}(O_0) \to O_0$ when we identify I_0 with $M_4(C_0(O_0))$. We have

$$\widetilde{\pi}^{-1}(O_0) = \coprod_{i,j=1}^{4} \sigma^+_{i,j}(V') \amalg \coprod_{i,j=1}^{4} \sigma^-_{i,j}(V')$$

where $\sigma^{\pm}_{i,j}\colon S^3 \to S^3$ is induced by the unitary $\pm U_{i,j}$ similarly as $\sigma_{i,j}\colon \mathbb{R}P^3 \to \mathbb{R}P^3$ for $i, j = 1, 2, 3, 4$. These 32 homeomorphisms preserve the orientation of S^3. Therefore, the image of $y \in K_1(I_0)$, and hence the one of $y \in K_1(I)$, in $K_1\big(M_4(C(S^3))\big)$ is $32[w]_1$. □

Definition 15.9. Define the linear map $\xi : M_2(\mathbb{C}) \to \mathbb{C}^4$ by

$$\xi\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = \frac{1}{\sqrt{2}}(a_{11}, a_{12}, a_{21}, a_{22}).$$

Definition 15.10. Define unital $*$-homomorphisms $\iota, \iota' \colon M_2(\mathbb{C}) \to M_4(\mathbb{C})$ by

$$\iota\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{11} & a_{21} \\ 0 & 0 & a_{21} & a_{22} \end{pmatrix},$$

$$\iota'\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = \begin{pmatrix} a_{11} & 0 & a_{12} & 0 \\ 0 & a_{11} & 0 & a_{12} \\ a_{21} & 0 & a_{22} & 0 \\ 0 & a_{21} & 0 & a_{22} \end{pmatrix}.$$

Lemma 15.11. *For each $M, N \in M_2(\mathbb{C})$, we have*

$$\xi(M)\iota(N) = \xi(MN), \qquad \iota'(M)\xi(N)^{\mathrm{T}} = \xi(MN)^{\mathrm{T}}.$$

Proof. It follows from a direct computation. □

Definition 15.12. Define $U \in M_4(A)$ by

$$U = \begin{pmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \end{pmatrix}.$$

It can be easily checked that U is a unitary.

Proposition 15.13. *The image of $[U]_1 \in K_1(A)$ under the map $K_1(A) \to K_1\big(M_4(C(S^3))\big)$ is $16[w]_1$.*

Proof. Let $\varphi_4 \colon M_4(A) \to M_4\big(M_4(C(S^3))\big)$ be the $*$-homomorphism induced by φ. Set $\mathbb{U} := \varphi_4(U)$. For $i, j = 1, 2, 3, 4$, the (i, j)-entry $\mathbb{U}_{i,j} \in C(S^3, M_4(\mathbb{C}))$ of $\mathbb{U}$ is given by

$$\mathbb{U}_{i,j}(a_1, a_2, a_3, a_4) = U_{i,j}(a_1, a_2, a_3, a_4)^{\mathrm{T}}(a_1, a_2, a_3, a_4)U_{i,j}^*$$

for each $(a_1, a_2, a_3, a_4) \in S^3$.

Let $W \in M_4(\mathbb{C})$ be

$$W = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\sqrt{-1} & 0 & 0 \\ 0 & 0 & 1 & -\sqrt{-1} \\ 0 & 0 & -1 & -\sqrt{-1} \\ 1 & \sqrt{-1} & 0 & 0 \end{pmatrix}.$$

Then W is a unitary.

Take $(a_1, a_2, a_3, a_4) \in S^3$ and $i, j = 1, 2, 3, 4$. We set

$$(b_1, b_2, b_3, b_4) = (a_1, a_2, a_3, a_4)U_{i,j}^*.$$

By Proposition 5.2, we have $\sum_{k=1}^4 b_k c_k = c_i\Big(\sum_{k=1}^4 a_k c_k\Big)c_j^*$. We also have

$$\begin{aligned}\xi\left(\sum_{k=1}^4 b_k c_k\right)W &= \frac{1}{\sqrt{2}}(b_1 + b_2\sqrt{-1}, b_3 + b_4\sqrt{-1}, -b_3 + b_4\sqrt{-1}, b_1 - b_2\sqrt{-1})W \\ &= (b_1, b_2, b_3, b_4)\end{aligned}$$

Hence we get

$$\begin{aligned}(a_1, a_2, a_3, a_4)U_{i,j}^* &= \xi\left(c_i\left(\sum_{k=1}^4 a_k c_k\right)c_j^*\right)W \\ &= \xi(c_i)\iota\left(\left(\sum_{k=1}^4 a_k c_k\right)c_j^*\right)W \\ &= \xi(c_i)\iota(w(a_1, a_2, a_3, a_4))\iota(c_j^*)W\end{aligned}$$

by Lemma 15.11. Similarly, we get

$$\begin{aligned}U_{i,j}(a_1, a_2, a_3, a_4)^{\mathrm{T}} &= W^{\mathrm{T}}\xi\left(c_i\left(\sum_{k=1}^4 a_k c_k\right)c_j^*\right)^{\mathrm{T}} \\ &= W^{\mathrm{T}}\iota'\left(c_i\left(\sum_{k=1}^4 a_k c_k\right)\right)\xi(c_j^*)^{\mathrm{T}} \\ &= W^{\mathrm{T}}\iota'(c_i)\iota'(w(a_1, a_2, a_3, a_4))\xi(c_j^*)^{\mathrm{T}}\end{aligned}$$

by Lemma 15.11. Define $\mathbb{V}, \mathbb{W}, \mathbb{W}' \in M_4(M_4(\mathbb{C}))$ by

$$\mathbb{V} = (\xi(c_j^*)^{\mathrm{T}}\xi(c_i))_{i,j=1}^4,$$

$$\mathbb{W} = \begin{pmatrix} \iota(c_1^*)W & 0 & 0 & 0 \\ 0 & \iota(c_2^*)W & 0 & 0 \\ 0 & 0 & \iota(c_3^*)W & 0 \\ 0 & 0 & 0 & \iota(c_4^*)W \end{pmatrix},$$

$$\mathbb{W}' = \begin{pmatrix} W^{\mathrm{T}}\iota'(c_1) & 0 & 0 & 0 \\ 0 & W^{\mathrm{T}}\iota'(c_2) & 0 & 0 \\ 0 & 0 & W^{\mathrm{T}}\iota'(c_3) & 0 \\ 0 & 0 & 0 & W^{\mathrm{T}}\iota'(c_4) \end{pmatrix}.$$

One can check that these are unitaries. If we consider these as constant functions in $M_4\big(C(S^3, M_4(\mathbb{C}))\big)$, we have

$$\mathbb{U} = \mathbb{W}'\iota_4'(w)\mathbb{V}\iota_4(w)\mathbb{W},$$

where $\iota_4(w), \iota'_4(w) \in M_4\big(C(S^3, M_4(\mathbb{C}))\big)$ are defined as

$$\iota_4(w) = \begin{pmatrix} \iota(w(\cdot)) & 0 & 0 & 0 \\ 0 & \iota(w(\cdot)) & 0 & 0 \\ 0 & 0 & \iota(w(\cdot)) & 0 \\ 0 & 0 & 0 & \iota(w(\cdot)) \end{pmatrix},$$

$$\iota'_4(w) = \begin{pmatrix} \iota'(w(\cdot)) & 0 & 0 & 0 \\ 0 & \iota'(w(\cdot)) & 0 & 0 \\ 0 & 0 & \iota'(w(\cdot)) & 0 \\ 0 & 0 & 0 & \iota'(w(\cdot)) \end{pmatrix}.$$

Since $[\iota_4(w)]_1 = [\iota'_4(w)]_1 = 8[w]_1$, we obtain $[\mathbb{U}]_1 = 16[w]_1$. □

Proposition 15.14. *We have $K_0(A) \cong \mathbb{Z}^{10}$ and $K_1(A) \cong \mathbb{Z}$. More specifically, $K_0(A)$ is generated by $\{[P_{i,j}]_0\}_{i,j=1}^4$, and $K_1(A)$ is generated by $[U]_1$. Moreover, the positive cone $K_0(A)_+$ of $K_0(A)$ is generated by $\{[P_{i,j}]_0\}_{i,j=1}^4$ as a monoid.*

Proof. We have already seen that $K_0(A) \to K_0(B)$ is isomorphic, and we have a short exact sequence

$$0 \longrightarrow K_1(I) \longrightarrow K_1(A) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

From this, we see that $K_1(A)$ is isomorphic to either $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}$. If $K_1(A)$ is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$, one can choose an isomorphism so that $y \in K_1(I)$ goes to $(1,0) \in \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. Then the image of the map $K_1(A) \to K_1\big(M_4(C(S^3))\big) \cong \mathbb{Z}$ is $32\mathbb{Z}$ by Proposition 15.8. This is a contradiction because the image of $[U]_1 \in K_1(A)$ is 16 by Proposition 15.13. Hence $K_1(A)$ is isomorphic to $\mathbb{Z}$ so that $y \in K_1(I)$ goes to 2. By Proposition 15.8 and Proposition 15.13, $[U]_1 \in K_1(A)$ corresponds to $1 \in \mathbb{Z}$. Thus $[U]_1$ is a generator of $K_1(A) \cong \mathbb{Z}$.

It is clear that the monoid generated by $\{[P_{i,j}]_0\}_{i,j=1}^4$ is contained in the positive cone $K_0(A)_+$. The positive cone $K_0(A)_+$ maps into the positive cone $K_0(B^\bullet)_+$ under the surjection $A \to B^\bullet$. Hence by Proposition 12.7, $K_0(A)_+$ is contained in the monoid generated by $\{[P_{i,j}]_0\}_{i,j=1}^4$. Thus $K_0(A)_+$ is the monoid generated by $\{[P_{i,j}]_0\}_{i,j=1}^4$. □

Definition 15.15. Define $u \in M_4(A(4))$ by

$$u = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{pmatrix}.$$

It can be easily checked that u is a unitary. This unitary u is called the *defining unitary* of the magic square C*-algebra $A(4)$.

By Proposition 15.14, we get the third main theorem.

Theorem 15.16. *We have* $K_0(A(4)) \cong \mathbb{Z}^{10}$ *and* $K_1(A(4)) \cong \mathbb{Z}$*. More specifically,* $K_0(A(4))$ *is generated by* $\{[p_{i,j}]_0\}_{i,j=1}^4$*, and* $K_1(A(4))$ *is generated by* $[u]_1$*.*

The positive cone $K_0(A(4))_+$ *of* $K_0(A(4))$ *is generated by* $\{[p_{i,j}]_0\}_{i,j=1}^4$ *as a monoid.*

As mentioned in the introduction, the computation $K_0(A(4)) \cong \mathbb{Z}^{10}$ and $K_1(A(4)) \cong \mathbb{Z}$ and that $K_0(A(4))$ is generated by $\{[p_{i,j}]_0\}_{i,j=1}^4$ were already obtained by Voigt in [8]. We give totally different proofs of these facts. That $K_1(A(4))$ is generated by $[u]_1$ and the computation of the positive cone $K_0(A(4))_+$ of $K_0(A(4))$ are new.

References

[1] Teodor Banica and Julien Bichon. Quantum groups acting on 4 points. *J. Reine Angew. Math.*, 626:75–114, 2009.

[2] Teodor Banica and Benoît Collins. Integration over the Pauli quantum group. *J. Geom. Phys.*, 58(8):942–961, 2008.

[3] Teodor Banica and Sergiu Moroianu. On the structure of quantum permutation groups. *Proc. Am. Math. Soc.*, 135(1):21–29, 2007.

[4] Julien Bichon and Robert Yuncken. Quantum subgroups of the compact quantum group $\mathrm{SU}_{-1}(3)$. *Bull. Lond. Math. Soc.*, 46(2):315–328, 2014.

[5] Nathanial P. Brown and Narutaka Ozawa. *C^*-algebras and finite-dimensional approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, 2008.

[6] S. Osugi. The homology theory of quotient spaces of the spheres by the action of the finite groups, 2018. Master thesis, Keio University.

[7] Mikael Rørdam, F. Larsen, and Niels Laustsen. *An introduction to K-theory for C^*-algebras*, volume 49 of *London Mathematical Society Student Texts*. Cambridge University Press, 2000.

[8] Christian Voigt. On the structure of quantum automorphism groups. *J. Reine Angew. Math.*, 732:255–273, 2017.

[9] Shuzhou Wang. Quantum symmetry groups of finite spaces. *Commun. Math. Phys.*, 195(1):195–211, 1998.

TAKESHI KATSURA
Department of Mathematics
Faculty of Science and Technology
Keio University
3-14-1 Hiyoshi, Kouhoku-ku, Yokohama
223-8522 JAPAN
katsura@math.keio.ac.jp

MASAHITO OGAWA
Library & Information center
Yokohama City University
22-2 Seto, Kanazawa-ku, Yokohama
236-0027 JAPAN

AIRI TAKEUCHI
Karlsruhe Institute of Technology
Department of Mathematics
76128 Karlsruhe
GERMANY
airi.takeuchi@partner.kit.edu