Complexity of Model Sets in Two-Step Nilpotent Lie Groups and Hyperbolic Space

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1 Introduction

In this thesis we study the structure of discrete point-sets in the absence of periodicity. There is a long history of interest in non-periodic structures and we will mention two branches from which this interest arose. First we address the theory of tilings, which is an inner mathematical field and also connected to art. And secondly the theory of solid state matter, which is inspired by physical observations.

Tilings

Periodic tilings are known since ancient times, even 4000BC the Sumerians used them for decoration. Another example are the ornaments on the friezes of Greek temples. The idea of non-periodic structure, or non-periodic tilings, is much newer. One of the oldest known references is in Harmonice Mundi, 1619, by Johannes Kepler in which he considers a tiling, which is a direct predecessor of the famous Penrose tiling. Since then the interest in such patterns has never broken up.
It is even present in the list of Hilbert's 23 problems he stated in 1900. A version of the 18th problem is the question if there exists a three dimensional polyhedron which tiles space and only in an aperiodic way. Such a tile was found in 1988 by Schmitt, [70], but for the plane this question is still unanswered.

Even before the discovery by Schmitt Hao Wang was intrigued to the study of aperiodic (Wang) tilings. A Wang tiling consists of squares with marked edges, such that only two edges which have the same mark can be placed side by side. Wang showed that there is a connection between Turing machines and his Wang tilings, in the sense that for any Turing machine a Wang tiling exists which models the Turing machine. Wang himself believed that there is no aperiodic set of Wang proto-tiles. He was proven wrong by his student Robert Berger who found an aperiodic set in 1966, [12]. This set consisted of 20426 proto-tiles. Afterwards examples with much less proto-tiles where found. Finally in 2021 the minimum of eleven proto-tiles was reached, by Jeandel and Rao, [43], following ideas of Karel, [20], who found a set of thirteen proto-tiles. That eleven proto-tiles are the minimal number also was shown by Jeandel and Rao, [43].

Besides the Wang case the question was to reduce the number of proto-tiles with which one can tile the plane but only in an aperiodic way. In 1974 Roger Penrose and Robert Ammann published a set consisting of two tiles with which it is possible to tile the plane but only in a non-periodic way, [63]. This record of two tiles stands until today, where the question if an aperiodic set of one proto-tile exists is still open.
This problem was also studied and is still studied in higher dimensions. One really recent result disproves the conjecture, that if one can tile $\mathbb{Z}^d$ with a finite set F, then one can also do this in a periodic way. This conjecture is due to Grünbaum and Shephard, [35], and Lagarias and Wang, [51]. The counterexample is by Greenfeld and Tao [31].

Solid-State Physics

Another branch of interest arises from solid-state physics. Here one is interested in the atomic structure of materials and this structure can be considered as a discrete point set in three dimensional Euclidean space. For a long time this field of research was occupied with considering lattices. In the 1930s Boris Nikolayevich Delone imposed the idea to reduce the assumed properties to two basic ones, namely discreteness and relative denseness, [21]. Sets which possess these two properties are named after him as Delone sets. It took a while until this idea had an impact on the physical side of research. In 1984 Dan Shechtman published a paper, [72], in which he found a non-periodic structure in the atom structure of an alloy of aluminium and manganese. The interesting observation Shechtman made is that the material he analysed had pure point diffraction. Pure point diffraction is observed experimentally by physicist by firing a laser beam on the solid under observation. Then one can measure the diffraction of this laser beam which gives information of the structure of the solid. If the diffraction pattern consists only of sharp spots one speaks of pure point diffraction. Mathematically speaking the picture one sees is the Fourier transformation of the autocorrelation of delta-functions sitting in the positions of the atoms, see [39, 40].
Clearly pure point diffraction indicates that the material is highly structured. The problem with the picture Shechtman observed was that the rotation symmetry he had seen can not appear in a crystal, i.e. lattice, in dimension three. This discovery let to the new terminology of quasi-crystals and to a transformation of the viewpoint in the whole field of Crystallography. After overcoming some resistance from within the community even the definition of what a crystal is has been changed by the International Union of Crystallography in 1992 to the following: A crystal is any solid with an essential discrete diffraction diagram.

1.1 Aperiodic Order in the Euclidean Case

On the mathematical side this new developments led to a higher interest in discrete structures with *pure point diffraction*. A first overview on the development can be found in the book by Marjorie Senechal, [71], which includes a historic perspective and an explanation of methods with which such sets can be analysed. The structures with pure point diffraction have been further studied since, see for example [6, 39, 40, 48, 77].

Another line of research is to characterise the structure of a discrete set by some local data, namely its *complexity* or its *repetitivity*, the one is concerned with the question how many different local configurations of given size there are and the other with how big the area is in which one sees all these local configurations, [47, 49, 57, 58].

One further question was, how to generate such sets? One possibility is *substitution*, from which one can also build for example the Penrose tiling mentioned in the beginning. In this method one starts with a legal configuration of tiles and then uses a dissection rule to fragment the tiles in to smaller tiles consisting of smaller versions of the proto-tiles. Afterwards the configuration is rescaled such that the new tiles have the same size as the tiles in the beginning. Iterating this process one gets a tiling in the limit.

Another method found by Meyer, [54, 55, 56], in the 1960s in a different context, is the so called *cut and project method*. Here one considers a lattice in the product of two spaces, called the physical space and the internal space. One wants to construct a set in the physical space and uses the internal space as a tool. The next step is to cut out a 'strip' in the product and project all the lattice points inside this strip to the physical space. Under certain assumptions the sets constructed in this way are called *model sets* and have the desired properties. For a formal definition see Section 2.2. This is the method we will consider in this thesis. To get an idea of what is going on consider Figure 1.1.

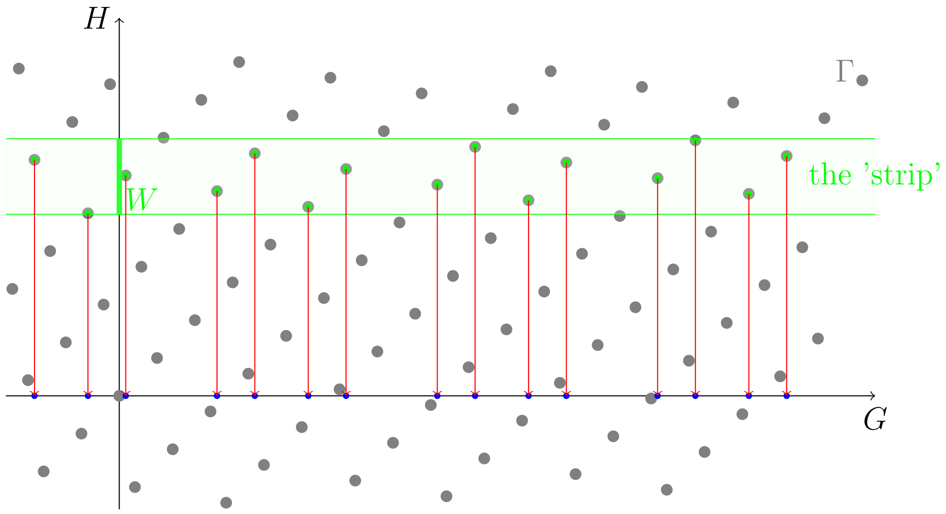


Figure 1.1: The picture indicates what the cut and project method does.

In the Euclidean case these sets have been studied in [5, 37, 41, 46, 57, 59]. For a comprehensive overview of model sets see the survey paper by Robert Moody, [58]. For an overview over the whole field see the books by Michael Baake and Uwe Grimm, [3, 4].

We want to especially highlight the complexity function $p(r)$ of a locally finite set. It counts the number of different local configurations of size r, see Definition 2.1.22. This function was first introduced by Lagarias and Pleasants in [50] and further studied in [49]. In combination with model sets it has also been studied in [2], for low dimensional cut and project sets, and in [9], for some special cases. A first general approach on this problem was done by Julien, [44], and further built upon in [45] by Koivusalo and Walton. Koivusalo and Walton did not only fix the problems with the approach by Julien they also eliminated some unnecessary conditions. They also formulated their theory in such a way that it is easier to generalize to the non-abelian case, while Julien's approach heavily depends of the Euclidean geometry.
This helps us in picking up their results and further generalising them to the non-Euclidean context. We will now cite the version of the Koivusalo-Walton Theorem which was the starting point for the progress made in this thesis.

Theorem [45, Informal version of Theorem 6.1]
Consider an aperiodic cut and project pattern with a convex polytopal window W. Then the complexity grows asymptotically as $p(r) \asymp r^{\alpha}$ for $\alpha \in \mathbb{N}$. The number α can be

derived from the ranks of the subgroups Γ^H of Γ and the dimensions of their $\mathbb{R}$-linear spans in $\mathbb{E}_<$.

In this theorem a model set with internal space $\mathbb{E}_< = \mathbb{R}^d$ and physical space $\mathbb{R}^n$ is considered. Further the α can be made explicit and is always bounded by $d \cdot n$. This maximum is only attained if the stabilizers of the hyperplanes which bound the window are trivial in Γ^H. That the complexity is determined only by the product of the two dimensions is really surprising.

Besides the Euclidean set-up the research extended to more general spaces like *locally compact abelian groups*. See the papers by Martin Schlottmann, [68, 69], which tell us that the approach by cut and project sets also works in this case without big changes in the methods. Also the diffraction of sets in the wider set-up has been studied for example by Lenz and Strungaru, [52].

1.2 The Non-Abelian Approach

More recently the whole approach could be generalized to an even more general set-up, namely *locally compact groups*, especially *non-abelian* ones. In this set-up the group structure takes the role of the translations, which where essential in the Euclidean case. For this approach the techniques from the euclidean case do not work any more so a whole bunch of new methods had to be invented. This process was initiated by Michael Björklund, Tobias Hartnick and Felix Pogorzelski, [14, 15, 16]. In this thesis we will use this viewpoint on model sets to develop our theory. Also the study of diffraction was transferred to the new context. This opens up a whole new field to study since there are many geometries to study besides the Euclidean/abelian-one. This program was further added to, see for example [11, 13, 53], and there is also progress which is not published yet.

This thesis wants to further contribute to this program by studying the complexity of model sets in this wider set-up. In this context Tobias Hartnick and Henna Koivusalo asked during the 2017 Oberwolfach workshop 'Spectral Structures and Topological Methods in Mathematical Quasicrystals' whether it is possible to compute the asymptotic behaviour of the complexity function for model sets in the Heisenberg group with respect to the Korányi norm using the method of Koivusalo and Walton, [45], and indicated some difficulties which have to be overcome to generalize the approach, [36]. So far no-one was able to solve these problems.

Tools of Analysis

The problem in this new set-up is that the Koivusalo-Walton approach highly depends on the fact that shifting hyperplanes in Euclidean spaces leaves them parallel. This is not the case in an Heisenberg geometry. In a three-dimensional Heisenberg group it turns out that the hyperplanes topple, if the group acts on them, but the good thing is that they only topple but do not bend. This happens because the Heisenberg group is two-step nilpotent. This is an important fact which we will use in the argumentation for the two-step nilpotent Lie group case.

In this thesis we solve the problem with the toppling by using the theory of Hyperplane Arrangements, which has been studied since the mid of the 20th century, see for example [67, 79].

A second tool which is needed for this approach is ergodic theory, which helps to estimate the number of points in certain sets. The approach used in this thesis is due to Amos Nevo and Alexander Gorodnik [29, 30].

Surprisingly the theory of hyperplane arrangements also helps if one wants to address the problem of studying the complexity of model sets in an hyperbolic geometry. The main trick here is to use the right viewpoint, namely the projective model of hyperbolic space. This model is in some sense really close to Euclidean geometry, if one is interested in intersections of hyperplanes.

1.3 Complexity - A Motivating Example

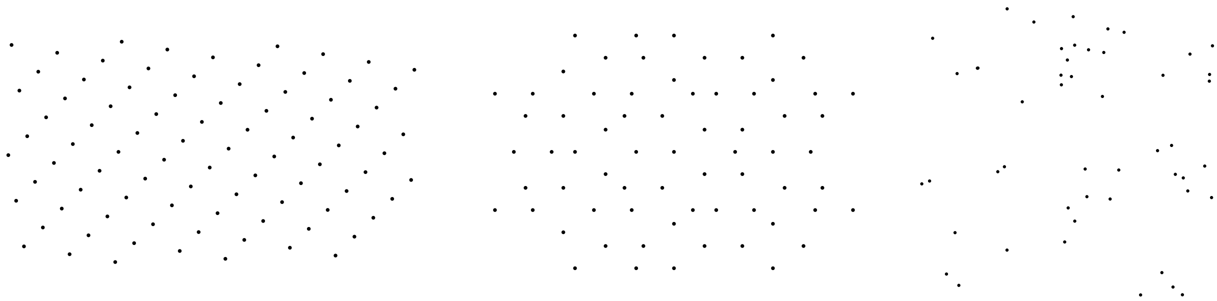
In this section we will motivate the effort to calculate the complexity function of a given discrete set. We will do this in a totally untechnical way just to give the reader an idea of what the aim of this analysis is and why the consideration of the complexity function is important.

At this point it is important to notice that the complexity highly depends on the choice of metric in a group, so we will work in the context of metric groups. We will later show that for two quasi-isometric metrics the asymptotic behaviour of the complexity function stays the same.

Theorem 2.1.27
Let G be a group, $\Lambda \subset G$ an FLC set and d and d' two quasi-isometric metrics on G. Let further p' be the complexity function with respect to d' and p the complexity function with respect to d. Then $p(r) \asymp p'(r)$.

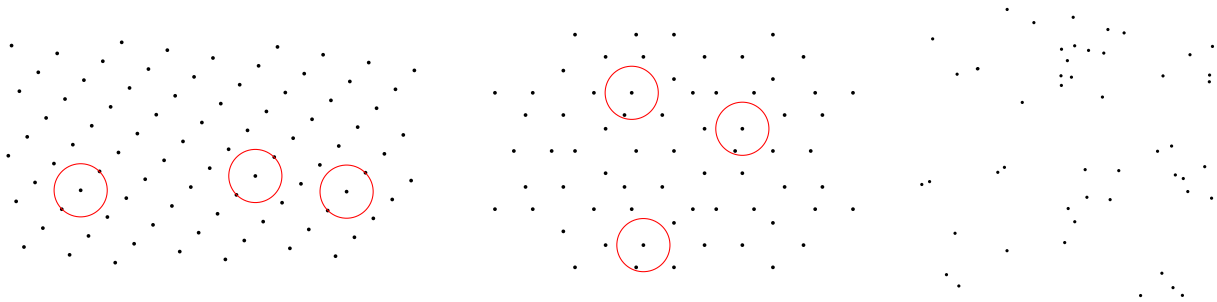
This result becomes really important in the case of homogeneous Lie groups since on them there exists a canonical class of metrics which are all quasi-isometric.

We consider the following pictures: On the left we see a part of a lattice, in the middle we see the vertices of a Penrose tiling and on the right we see some random point set. The images are just parts of the total point sets, which are spread across the whole plane.



Clearly the image on the left has more structure, than the other two images. On the other hand the picture on the right does not possess any structure, since it is a random point set. The interesting part is the picture in the middle, how much structure does it possess? Does it tend more to the left or more to the right side? And if we would consider another picture of this type, maybe of some other aperiodic tiling, do we have to sort it left or right of the Penrose tiling?

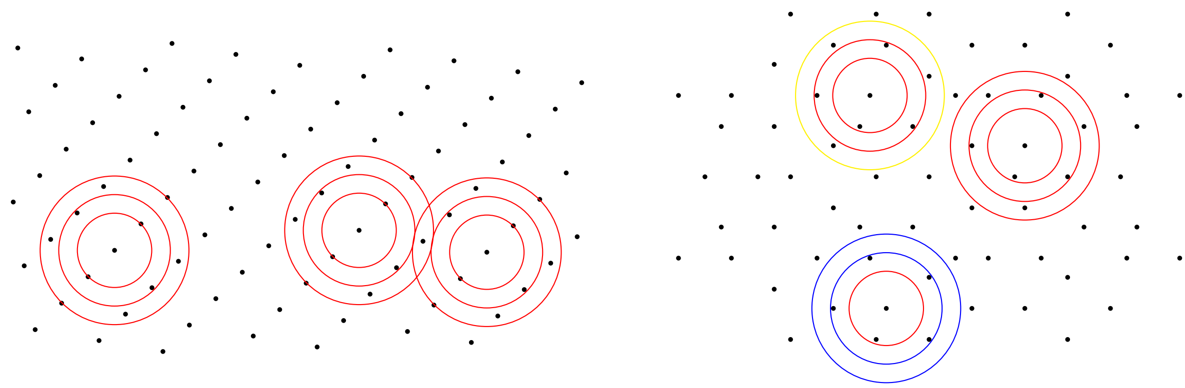
The complexity function gives an answer to this questions. In the following series of pictures we consider certain areas around some chosen points.



In the lattice case and the Penrose case the constellation of the points inside the red circles is the same, in the sense that we can translate the circles to each other and the points will also be translated on points in the other circle. In the random set there may be some points for which such circles exist but not in the part we see here. The idea now is to count how many options for the constellation inside such a circle exist.

In the lattice case clearly all the points have the same constellation around them, so the answer is that there is only one option. In the Penrose case there are more than one options, at least if the circle is big enough, but only finitely many. And in the random case there are infinitely many options for such constellations, for this reason we leave out the random case for the rest of the discussion.

The next step is to increase this circles and see how the number of different options behaves, we indicate this in the following pictures by the different colours.



We see that in the lattice case the number of options stays constant, but in the Penrose case the number of options increases. This is simply seen in the example since by increasing the circle the first time, the two areas in the top stay the same, while the one in the bottom is different. And in the next step even the two in the top are different, even if they could be mapped onto each other by a reflection, but not any more by a translation. So the behaviour of a lattice is clearly different from the Penrose case. But if we have two point sets from the aperiodic case, they will both have an increasing complexity function.

The main idea behind the complexity function is that the asymptotic behaviour of the counting of the number of options for this areas gives us a tool to sort the sets. This means that a discrete set for which the complexity function is constant is really structured, while a discrete set with slowly growing complexity function is still more structured as one with a faster growing complexity function.

Remark 1.3.1 For the asymptotic behaviour we use the common notation $g(t) \ll f(t)$ which means $\limsup\limits_{t\to\infty} \left|\frac{g(t)}{f(t)}\right| < \infty$. If both $g(t) \ll f(t)$ and $g(t) \gg f(t)$ holds we write $g(t) \asymp f(t)$.

For example the complexity function of the Penrose tiling grows quadratically, i.e. $p_{Penrose}(r) \asymp r^2$, this follows for example from [66, Theorem 7.17]. Therefore the Penrose tiling is less structured then a lattice. A discrete set which has cubic complexity growth, i.e. $p(r) \asymp r^3$, is then even less structured than the Penrose tiling and so forth.

In this thesis will establish some basic statements about complexity in metric groups. Especially we tried to translate some general statements from the euclidean case to the general case.

We can generalize a Theorem of Lagarias and Pleasant, [49, Theorem 2.1], which states that their is no sub-linear growth of the complexity function except the constant one.

Theorem 2.1.38
Let (G, d) be a lcsc group and $\Lambda \subseteq G$ be an R-relatively dense Delone set and $r_0 > 0$ and $r_1 > r_0 + 2R$ be two fixed radii. If $p(r_0) = p(r_1)$, then $p(r) = p(r_0)$ for all $r \geq r_0$.

Corollary 2.1.39
Let (G, d) be a lcsc group and $\Lambda \subseteq G$ be an R-relatively dense Delone set with complexity function $p(r)$. If there exists an $r' > 0$ such that $p(r') < \frac{r'}{2R}$, then $p(r)$ will become constant.

This result is easily established since the proof works exactly as in the euclidean case. Nevertheless it is interesting that this still holds.

Further we could also generalize another result by Lagarias and Pleasant, [49, Theorem 3.1], which characterises lattices by their complexity.

Theorem 2.1.32
Let (G, d) be a lcsc group and $\Lambda \subseteq G$ a Delone subset.

(a) If $p(r) = c$ for all $r \geq r_0$ for some constant $r_0 > 0$, then Λ is the union of c shifted cocompact lattices, i.e. there exists a cocompact lattice $\Gamma \subseteq G$ and a set $F \subseteq G$ with $|F| = c$ such that $\Lambda = \Gamma F$.

(b) If Λ is the union of c shifted versions of one cocompact lattices, then $p(r) \leq c$ for all $r > 0$.

This result fits in the viewpoint mentioned above that the complexity function measures how structured a discrete set is.

1.4 Complexity in Different Geometries

In this section we want to compare an FLC set in $\mathbb{R}^3$ and in the Heisenberg group $\mathbb{H}$, which is an example of a homogeneous Lie group. For some facts about the Heisenberg group see Section 5.3. Since the underlying set of $\mathbb{H}$ can be seen as $\mathbb{R}^3$ we can construct an FLC set by the same data.

We consider the lattice

$$\Gamma = \left\{ (x, x^*) \in \mathbb{R}^3 \times \mathbb{R}^3 \,\middle|\, x \in \mathbb{Z}[\sqrt{2}]^3 \right\}$$

this is also a lattice in $\mathbb{H} \times \mathbb{H}$. Further we choose a window W, how this exactly looks is not important at this point, but it should be precompact and have non-empty interior. We use this data to construct a model set in $\mathbb{R}^3$ respectively $\mathbb{H}$, see Figure 1.2.

Even if the two point sets look the same their complexity is different, in $\mathbb{R}^3$ this model set has the complexity $p(r) \asymp r^9$, but in the Heisenberg group $\mathbb{H}$ it has the complexity $p(r) \asymp r^{12}$. The reason for this behaviour is the different structure underlying the set $\mathbb{R}^3$. Since in the Heisenberg group we consider the group action instead of translations to compare the regions, the higher complexity indicates that matching two such regions is harder by this group action than by translations. Another affect certainly comes from the different behaviour of scaling balls in the two different geometries.

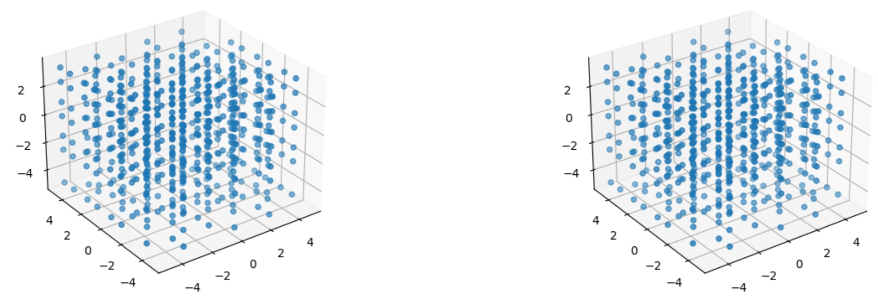


Figure 1.2: The two figures show two perspectives of the model set. We can view the point set as a subset of $\mathbb{R}^3$ or $\mathbb{H}$.

1.5 Main Results

For our theory we need the notion of finite local complexity. There are different definitions of this term in the literature but our first result proves the equivalence of these different notions. This holds in the generality of locally compact second countable (lcsc) groups equipped with a 'nice' metric. For a definition of 'nice' see Definition 2.1.3. The result was partially known before, see [14, Section 2], [13, Proposition 4.5] and also the unpublished notes on the seminar on approximate lattices held at KIT in summer semester 2022 by Tobias Hartnick. But the equivalence of all the notions in this generality has not appeared before.

Lemma 2.1.18 + Lemma 2.1.28 (Finite Local Complexity)
Let (G, d) be a lcsc group and $\Lambda \subseteq G$ a locally finite set, then the following are equivalent:

(a) For all bounded $B \subseteq G$ there exists a finite $F_B \subseteq G$ such that

$$\forall g \in G\ \exists h \in \Lambda^{-1}\Lambda\ \exists f \in F_B : \left(Bg^{-1} \cap \Lambda\right) h = Bf^{-1} \cap \Lambda.$$

(b) For all bounded $B \subseteq G$ there exists a finite $F_B \subseteq G$ such that

$$\forall g \in G\ \exists h \in G\ \exists f \in F_B : \left(Bg^{-1} \cap \Lambda\right) h = Bf^{-1} \cap \Lambda.$$

(c) $\Lambda\Lambda^{-1}$ is locally finite.

(d) For all $B \subseteq G$ bounded:

$$\left|\left\{B \cap \Lambda\lambda^{-1} \,\middle|\, \lambda \in \Lambda\right\}\right| < \infty.$$

(e) For the complexity function $p(r) < \infty$ for all $r \geq 0$.

This result gives different characterisations for what we will call *finite local complexity.* The formulas in (a) and (b) come from the perspective that one moves a fixed set B

over the subset Λ and realises that one can only see finitely many different patterns. In (d) the perspective is inverted in the sense that B is fixed and the Λ is shifted. So this two viewpoints are naturally close to each other. The item (e) just formalizes this viewpoints since the complexity function simply counts the different arising patterns if B is a ball of radius r. The item (c) is interesting because it is of a different nature, namely a finiteness condition.

The second preliminary result is that the ideas by Koivusalo and Walton, [45], translate one to one to the context of metric locally compact second countable groups.

Definition 1.5.1 Let $\Lambda(G, H, \Gamma, W)$ be a uniform model set, with $(G, d_G), (H, d_H)$ lcsc groups. Further let $\lambda \in \Lambda$, then

$$\left(\bigcap_{\mu \in \mathcal{S}_r(\lambda)} \mu \mathring{W} \right) \cap \left(\bigcap_{\mu \in \mathcal{S}_r^{\mathrm{C}}(\lambda)} \mu W^{\mathrm{C}} \right) =: W_r(\lambda)$$

is called the *r-acceptance domain of* λ.

The definition of $\mathcal{S}_r(\lambda)$ and $\mathcal{S}_r(\lambda)^{\mathrm{C}}$ is quite technical and postponed until Definition 3.0.8. In the following theorem we also use the terminology of model sets, which will be explained in Section 2.2.

Theorem 3.0.4 (Acceptance Domains)
Let $\Lambda(G, H, \Gamma, W)$ be a uniform model set, with $(G, d_G), (H, d_H)$ lcsc groups.

(a) The r-acceptance domains correspond one-to-one to the r-equivalence classes of Λ, i.e. let $\lambda, \lambda' \in \Lambda$ and $\tau(\lambda') \in W_r(\lambda)$, then $\lambda \sim_r \lambda'$.

(b) For $\lambda \not\sim_r \lambda'$ we have

$$W_r(\lambda) \cap W_r(\lambda') = \emptyset.$$

(c)

$$\overline{W} = \bigcup_{\lambda \in A_r^G} \overline{W_r(\lambda)}.$$

This result means that we can identify an equivalence class of a patch, see Definition 2.1.20, with a certain region of the window. The importance of this theorem is that it connects the number of equivalence classes of a subset of G to regions inside the window W, which lives in H. The idea which one should keep in mind is that points in $\Lambda \subset G$ which have the same pattern around them are close-by on the H side. This will be our main access to the complexity function.

For the first main result we have to restrict to two-step nilpotent homogeneous Lie groups. For a definition see the beginning of Chapter 5. The important thing is that on

homogeneous Lie groups there exists a canonical quasi-isometry class of metrics and as we have seen above quasi-isometric metrics deliver the same complexity. Furthermore in homogenous Lie groups we are able to scale things in a natural way. Homogeneous spaces in connection with aperiodic order are discussed in [16].
Another reason for this restriction is that in this context we can define hyperplanes in the Lie group and that the group acts on the space of the hyperplanes in the group.

Main Theorem A (Homogeneous Lie Group Case, 6.0.1)
Let (G, d_G) be a homogeneous Lie group and (H, d_H) a two-step nilpotent homogeneous Lie group. Let $\Lambda(G, H, \Gamma, W)$ be a polytopal model set, such that the P_i have trivial stabilizer and that $P_i \cap \Gamma_H = \emptyset$. Then for the complexity function $p(r)$ of Λ we have

$$p(r) \asymp r^{\mathrm{homdim}(G)\cdot\dim(H)}.$$

This is a generalisation of a theorem found by Koivusalo and Walton, see above. The interesting observation is that the exponent now does depend on the homogeneous dimension of the G side instead of its dimension. The homogeneous dimension is a measure for the growth rate of balls in a group, see Definition 5.1.9 and Proposition 5.1.19. This is an unexpected behaviour since one would naively assume that either $r^{\mathrm{homdim}(G)\,\mathrm{homdim}(H)}$ or $r^{\dim(G)\dim(H)}$ would be the asymptotic behaviour in the general case. This means that the important part on the G-side is the growth of balls, or more precisely the number of lattice points in growing sets. On the H-side the dimension, in the sense of the number of 'directions', is the more important part. This behaviour is not so easily seen in the euclidean case, since the growth rate of balls and the dimension are the same.

The second main result also generalizes the approach to hyperbolic spaces or respectively their isometry groups.

Main Theorem B (Hyperbolic Case, 7.2.10)
Let $\Lambda\left(\mathrm{Isom}(\mathcal{H}^n), \mathrm{Isom}(\mathcal{H}^d), \Gamma, W\right)$ be a uniform polytopal model set, then $p(r) \asymp e^{d(n-1)r}$.

This result is a first example in which a model set, which has exponential growth of the complexity function, can be constructed. This is not unexpected since balls grow exponentially in hyperbolic space and we have seen before that the growth of balls takes an important role for the asymptotic behaviour. The interesting note here is that all the results are of the same type, in the sense that the complexity function always behaves like

$$p(r) \asymp \mu_G(B_r(e))^{\dim(H)}.$$

Further it is noteworthy that the strategy with which the proof works is identical in all three cases. And this is really surprising since one would not suspect that the scaling argument one uses in the Lie group case works in the hyperbolic case. But we can do this by using the right model, namely the projective one. We discuss some basics of this model for hyperbolic space in Section 4.4.

Method of proof

Since in all three cases the strategy is the same we will now discuss the important steps in the proof and stress out the differences.

As already mentioned above the main tool of the proof is Theorem 3.0.4 since it connects the number of equivalence classes of patches to regions in the window, the acceptance domains. The geometry of the window plays an important role, we will only consider polytopal windows, therefore windows which are bounded by finitely many hyperplanes.

A first step in the proof is to see that we can estimate the number of acceptance domains, which are intersections of shifted versions of the window, by regions which are cut out of the window by shifted versions of the bounding hyperplanes. Technically speaking this means that we do not consider acceptance domains any more, but consider a finer partition. Here the major difference between the general case and the euclidean case is visible, since in the euclidean geometry the shifts of the hyperplanes stay parallel. In the two other cases the hyperplanes will rotate but will not bend, this is really important, since we are not able to treat the case of bending hyperplanes.

After establishing the connection from the regions to the complexity function we can start counting. Here the major tool is the Theorem of Beck, Theorem 4.3.14, in a version which is adapted to our case. The second ingredient which helps us to use the Theorem of Beck is that we can estimate the number of regions, in which hyperplanes cut an object, by the number of intersection points in the object. In this step we use that our geometries are close to the Euclidean one in some sense. At this step the dimension of the H-side appears in the exponent.

Since we now have a tool to count the regions if we know the hyperplanes we need to get more insight into their behaviour. To do this we need ergodic theoretic results which will give us the number of hyperplanes which we get by shifting. At this point the growth of balls on the G-side takes an important role and exactly here the homogeneous dimension comes into play. Or in the hyperbolic case we get the exponential behaviour at this step.

Also the angle in which the hyperplanes rotate is controllable, since we can always scale in without changing the asymptotic behaviour of the count. Here at this point the homogeneousness of the Lie groups is important and this surprisingly also works in the hyperbolic case.

Now the last step is to choose a suitable subset of the window which we want to cut with the hyperplanes. Here one has to carefully choose the parameters, since we want intersections inside the set but we do not want the hyperplanes to rotate to much. Also we want the hyperplanes to cut the set as if the edge of the window has cut the set. This part is really technical, but not necessary to understand the main idea.

The final step is to put all the pieces together. Since we carefully have built all the arguments together this is no problem and the proof is done.

1.6 Organisation of the Thesis

The thesis is organized in seven chapters, where the first six examine the theory and the last one gives some concrete examples. In Chapter 2 we state the basic definitions and ideas of the theory, we also prove the equivalence of the different notions of finite local complexity.

In Chapter 3 we extend the argumentation from Koivusalo and Walton to locally compact second countable groups and see, how the complexity can be computed in this general context. This leads to Theorem 3.0.4.

The following two chapters review the tools we need to prove our main theorem. In Chapter 4 we discuss the theory of hyperplane arrangements and prove the theorems we need for our counting argument, especially Theorem 4.3.14, which is an application of an approach by Beck to a new context. In Chapter 5 we review homogeneous Lie groups, which are the class of groups for which we can prove Theorem 6.0.1.

Then in the next chapter, Chapter 6, we will prove Theorem 6.0.1, which is one of the main results of this thesis. In this chapter we will also see how all the parts we discussed before fit together.

In Chapter 7 we extend our arguments to isometry groups of hyperbolic space. This chapter also builds upon ideas presented in the preceding chapter, and picks up the proof ideas and translates them to the new context.

Afterwards in Chapter 8 we will consider three explicit examples, two of which are Euclidean and well known while the third is a new example in the Heisenberg group.

In the end we will mention some ideas on how to further advance the research in this field.

2 Aperiodicity and Delone Sets

The aim of this chapter is to make the reader familiar with the different notions of discreteness we are working with. In general a discrete set does not naturally possess any specific structure. On the other hand lattices are highly structured discrete sets, since they are subgroups. We will explore the space between these two extrema.

Most of the information presented in this first chapter is well known, but often the statements are only formulated in the Euclidean set-up. For example see the book by Baake and Grimm, [3], or the papers by Lagarias, [46, 47, 49]. We also refer to the summary by Moody, [58], especially for Section 2.2. For some historic context one can consult the book by Senechal, [71]. The way we present this is aimed to motivate the questions we asked in the introduction.

2.1 Notions of Discreteness

Definition 2.1.1 Let X be a topological space and $S \subseteq X$ a subset. We call S *discrete* if for every $s \in S$ there exists an open subset $U \subseteq X$ such that $U \cap S = \{s\}$.
If X itself is discrete we call X a *discrete space*, this means that for every $x \in X$ the set $\{x\}$ is open.

Definition 2.1.2 Let (X, d) be a metric space. For $x \in X$ and $r > 0$, we denote the *r-ball around x* by

$$B_r(x) := \{y \in X \mid d(x, y) < r\}.$$

Remark that our balls are open. If we talk about the closed r-ball around x we mean

$$\{y \in X \mid d(x, y) \leq r\}.$$

Definition 2.1.3 Let G be a topological group and d a metric on G, then

(a) d is called *proper* if every closed ball is compact,

(b) d is called *right-invariant* if for all $g, h, l \in G$ holds $d(gl, hl) = d(g, h)$, *left-invariance* is defined analogously,

(c) d is called *compatible* if the topology induced by d coincides with the topology of G.

Definition 2.1.4 Let G be a topological group and $|\cdot| : G \to \mathbb{R}$ a function such that for all $g, h \in G$

(a) $|g| = 0$ if and only if $g = e$,

(b) $|g^{-1}| = |g|$,

(c) $|gh| \leq |g| + |h|$,

then $|\cdot|$ is called a norm on G.

Remark 2.1.5 If (G, d) is a metric group and d is right-invariant, then $|\cdot| := d(e, \cdot)$ is a norm.
Conversely if $|\cdot|$ is a norm, then $d(h, g) := |hg^{-1}|$ is a right-invariant metric.

Discrete sets can take many forms so we want to have more options to describe how they look. Two informations are naturally of interest: Firstly how close two points of a discrete set get and secondly how far are the points away from all the other points in the space? These questions lead to the following two definitions.

Definition 2.1.6 Let (X, d) be a metric space and $r > 0$ a parameter. We call a non-empty subset $S \subseteq X$ *r-uniformly discrete* if $d(x, y) \geq r$ for all $x, y \in S$, with $x \neq y$. If we are not interested in the parameter r we simply say S is *uniformly discrete*.

Definition 2.1.7 Let (X, d) be a metric space and $R > 0$ a parameter. We call a subset $S \subseteq X$ *R-relatively dense* if for every $x \in X$ we have $B_R(x) \cap S \neq \emptyset$. As before if we are not interested in the parameter we simply call S a *relatively dense* set.

A highly interesting family of discrete sets is uniformly discrete and relatively dense, therefore its points are spread across the whole space but never come close to each other in some sense.

Definition 2.1.8 Let (X, d) be a metric space and $r, R > 0$ be two parameters. We call a subset $S \subseteq X$ a *(r, R)-Delone set* if it is r-uniformly discrete and R-relatively dense. Again we drop the (r, R) if we do not care for the parameters.

Proposition 2.1.9 ([19, Proposition 3.C.3])
Every metric space (X, d) admits a Delone subset.

Proof. Since $\{x\}$ is uniformly discrete for every $x \in X$ we always find an r-uniformly discrete set. Now we assume that we are given an r-uniformly discrete set Λ. Consider the set

$$\mathcal{S} := \left\{ S \subseteq X \,\middle|\, \Lambda \subseteq S \text{ and } S \text{ is } r\text{-uniformly discrete} \right\}.$$

This set is not empty because it contains Λ. We can build a sequence of sets $S_i \in \mathcal{S}$ of the form $S_1 \subset S_2 \subset ...$, which is strictly increasing. Since all these chains are bounded from above we can use the Lemma of Zorn, which delivers us a maximal element $\tilde{S}$. Therefore there can be no $x \in X$ with $d(s,x) \geq r$ for all $s \in \tilde{S}$. So we conclude that $B_r(x) \cap \tilde{S} \neq \emptyset$ for all $x \in X$ and so $\mathcal{S}$ is an (r,r)-Delone set. □

Corollary 2.1.10 *Every r-uniformly discrete set can be extended to a (r,r)-Delone set.*

Example 2.1.11 (a) The easiest examples of Delone sets are lattices in $\mathbb{R}^n$. They even inherit a lot more structure than general Delone sets do.

(b) A lattice in which we randomly wiggle on the points a little is still a Delone set.

(c) Further examples arise by considering aperiodic sequences in $\mathbb{R}$ or aperiodic tilings in $\mathbb{R}^2$. For example the vertices of a Penrose tiling are a Delone set.

Definition 2.1.12 Let (X,d) be a metric space and $S \subseteq X$. We call S *locally finite* if for every bounded set $B \subseteq X$ we have that $B \cap S$ is a finite set.

Proposition 2.1.13 ([14, Page 6])
Let (X,d) be a proper metric space and $S \subseteq X$, then we have the following implications

$$S \textit{ uniformly discrete} \Rightarrow S \textit{ locally finite} \Leftrightarrow S \textit{ closed and discrete} \Rightarrow S \textit{ discrete.}$$

Proof. We start with an r-uniformly discrete set S. By the properness of the space we know that the closure of a bounded set B is compact. Therefore we can cover $\overline{B}$ with finitely many balls of radius $\frac{r}{4}$. The r-uniformly discreteness tells us that there is at most one element of S in each of these balls. Thus $S \cap \overline{B}$ is finite and therefore $S \cap B$ is finite, so S is locally finite.

Now let S be a locally finite set and $U \subseteq X$ be an open bounded set that contains $s \in S$. Then $U \cap S$ is finite and contains s, therefore we can find a minimal distance $r := \min \{d(x,s) \,|\, x \in U \cap S \setminus \{s\}\}$ and see that $S \cap B_{r/2}(s) = \{s\}$. Since s was arbitrary we get that S is discrete.
To see that S is closed we consider a sequence $(x_n)_{n\in\mathbb{N}}$ in S, which converges to $x \in X$. Here we use that the space X is complete, but this is implied by the properness. Then almost all x_n are contained in a bounded ball $B_r(x)$ around x. But $B_r(x) \cap S$ is finite so the sequence must be constant for all indices greater than some $n_0 \in \mathbb{N}$, so $x \in S$.

Now let S be closed and discrete and B be a bounded subset of X. Assume that $S \cap B$ is infinite, then we find at least one accumulation point in $\overline{B}$. Since S is closed this accumulation point belongs to S as well. But then S would have an accumulation point, which contradicts the discreteness. So $S \cap B$ must be finite and since B was arbitrary S is locally finite. □

All the implications of Proposition 2.1.13 are not reversible as the following two Euclidean examples show:

(a) The set $\left\{n + \frac{1}{n} \mid n \in \mathbb{N}\right\} \cup \mathbb{N} \subset \mathbb{R}$ is locally finite but not uniformly discrete.

(b) The set $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \subset \mathbb{R}$ is discrete but not locally finite.

From here on we leave the general set-up of metric spaces behind and consider locally compact second countable (lcsc) groups. The following theorem is due to Struble and guarantees the existence of a 'nice' metric on a lcsc group.

Theorem 2.1.14 (Strubles Theorem, [75])
A locally compact topological group G is metrizable with a proper, compatible, right-invariant metric, if and only if G is second countable.

The theorem also holds if we replace the right-invariance by left-invariance, this is just a choice of personal preference.

Convention 2.1.15 From now on if we talk about a lcsc group G we silently always assume that we have chosen a proper, compatible, right-invariant metric d, i.e. $G = (G, d)$. The choice of this metric is not canonical and equipping the same group with different metrics can result in a different behaviour.

Since we now work in the context of groups we can recall equivalent characterizations of uniform discreteness and relative denseness. Observe that one could also use this characterizations to define uniform discreteness and relative denseness without a metric.

Lemma 2.1.16 ([13, Proposition 2.2.])
Let G be a lcsc group. A subset $S \subseteq G$ is uniformly discrete if and only if there exists an open subset $U \subseteq G$ such that $SS^{-1} \cap U = \{e\}$.

Proof. Let us first assume that S is r-uniformly discrete, this means that for all different $x, y \in S$ we have $d(x, y) \geq r$, and by the right-invariance of d we get

$$r \leq d(x, y) = d\left(xy^{-1}, e\right).$$

We set $U = B_{r/2}(e)$ and are done.

For the other direction we argue similarly. Let $U \subseteq G$ such that $SS^{-1} \cap U = \{e\}$, then we can find an open ball centred at e of a given radius, say r, inside U. So

$SS^{-1} \cap B_r(e) = \{e\}$ and this means that for all $x, y \in S : d\left(xy^{-1}, e\right) \geq r$. And again by the right-invariance we get

$$r \leq d\left(xy^{-1}, e\right) = d(x, y).$$

$\square$

Lemma 2.1.17 ([13, Proposition 2.2.])
Let G be a lcsc group. A subset $S \subseteq G$ is relatively dense if and only if there exists a compact subset $K \subseteq G$ such that $KS = G$.

Proof. Lets first assume that S is relatively dense, so we find an $R > 0$ such that for all $g \in G$ holds $B_R(g) \cap S \neq \emptyset$. Let $K \supset B_R(e)$ be a compact set. Now let $g \in G$ be arbitrary. Then pick $s \in B_R(g) \cap S$ and $k \in G$ such that $g = ks$. We use the right-invariance of d and get

$$R > d(s, g) = d(s, ks) = d(e, k).$$

This shows that $k \in B_R(e) \subset K$.

For the other direction let K be such that $KS = G$. We can find a ball $B_R(e)$ such that $K \subset B_R(e)$. For $g \in G$, let $k \in K$ and $s \in S$ such that $g = ks$. Again by right-invariance we get

$$d(s, g) = d(s, ks) = d(e, k) < R.$$

Therefore $S \cap B_R(g) \neq \emptyset$ for all $g \in G$. $\square$

The following lemma is already partially known: In [13, Proposition 4.5] it is shown that c) implies d). In the seminar on approximate lattices held at KIT in summer semester 2022 by Tobias Hartnick the equivalence of (a), (b) and (c) was shown under the condition that Λ is a Delone set.

Lemma 2.1.18 *Let G be a lcsc group and $\Lambda \subseteq G$ a locally finite set, then the following are equivalent:*

(a) For all bounded $B \subseteq G$ there exists a finite $F_B \subseteq G$ such that

$$\forall g \in G \; \exists h \in \Lambda^{-1}\Lambda \; \exists f \in F_B : \left(Bg^{-1} \cap \Lambda\right) h = Bf^{-1} \cap \Lambda.$$

(b) For all bounded $B \subseteq G$ there exists a finite $F_B \subseteq G$ such that

$$\forall g \in G \; \exists h \in G \; \exists f \in F_B : \left(Bg^{-1} \cap \Lambda\right) h = Bf^{-1} \cap \Lambda.$$

(c) $\Lambda\Lambda^{-1}$ is locally finite.

(d) For all $B \subseteq G$ bounded:

$$\left|\left\{B \cap \Lambda\lambda^{-1} \,\middle|\, \lambda \in \Lambda\right\}\right| < \infty.$$

Proof. First we will show the equivalence of (a), (b) and (c). Afterwards we will show the equivalence of (c) and (d).

(a) $\Rightarrow$ (b): This step is obvious, since $\Lambda\Lambda^{-1} \subseteq G$.

(b) $\Rightarrow$ (c): We have to show that $\Lambda\Lambda^{-1} \cap B$ is finite for all bounded $B \subseteq G$. Without loss of generality we can assume that B is compact and contains the neutral element, otherwise we just simply enlarge B and notice that this would just increase the number of elements in the intersection. For this B we choose F_B such that (b) holds. Since F_B is finite and B is bounded we see that $B' := BF_B^{-1}$ is also bounded. Further we see, since Λ is locally finite, that $F := B' \cap \Lambda$ is finite.
Now let $\lambda_1, \lambda_2 \in \Lambda$ be arbitrary with $\lambda_1\lambda_2^{-1} \in B$. We get $\lambda_1 \in B\lambda_2 \cap \Lambda$ and since we assumed $e \in B$ we also get $\lambda_2 \in B\lambda_2 \cap \Lambda$. With our assumption we get that $h_1 \in G$ and $f_1 \in F_B$ exist with

$$(B\lambda_2 \cap \Lambda)h_1 = Bf_1^{-1} \cap \Lambda.$$

Putting the pieces together we obtain

$$\{\lambda_1, \lambda_2\}h_1 \subseteq (B\lambda_2 \cap \Lambda)h_1 = Bf_1^{-1} \cap \Lambda \subseteq BF_B^{-1} \cap \Lambda = B' \cap \Lambda = F.$$

So $\lambda_1\lambda_2^{-1} = (\lambda_1 h_1)(\lambda_2 h_1)^{-1} \in FF^{-1}$ and we get that $\Lambda\Lambda^{-1} \cap B \subseteq FF^{-1}$ is finite.

(c) $\Rightarrow$ (a): Let $B \subseteq G$ be bounded. Without loss of generality we can assume B to be symmetric, i.e. $B = B^{-1}$. Since B is bounded, BB is also bounded and $BB \cap \Lambda\Lambda^{-1}$ is finite by assumption. Then

$$BB \cap \Lambda\Lambda^{-1} = \bigcup_{b\in B} \bigcup_{\lambda\in\Lambda} Bb \cap \Lambda\lambda^{-1}.$$

As $Bb \cap \Lambda\lambda^{-1}$ is contained in the finite set $BB \cap \Lambda\Lambda^{-1}$ there are only finitely many possibilities for this set and there are $b_1, ..., b_s \in B$ and $\lambda_1, ..., \lambda_t \in \Lambda$ such that for arbitrary $b \in B$ and $\lambda \in \Lambda$ there exists a $n \in \{1, ..., s\}$ and a $m \in \{1, ..., t\}$ with

$$Bb \cap \Lambda\lambda^{-1} = Bb_n \cap \Lambda\lambda_m^{-1}.$$

Let $g \in G$ be arbitrary. Then the two following cases can appear:

Case 1: $Bg^{-1} \cap \Lambda = \emptyset$, to deal with this case we simply set $f_0 := g$, for one such g.

Case 2: $Bg^{-1} \cap \Lambda \neq \emptyset$, then there exists a $b' \in B$ such that $b'g^{-1} = \lambda \in \Lambda$. Set $b := b'^{-1}$, then $g^{-1} = b\lambda$ and, since B is symmetric, $b \in B$. Now choose n and m such that $Bb \cap \Lambda\lambda^{-1} = Bb_n \cap \Lambda\lambda_m^{-1}$ and set $h := \lambda^{-1}\lambda_m \in \Lambda^{-1}\Lambda$. Now we get

$$\left(Bg^{-1} \cap \Lambda\right) h = (Bb\lambda \cap \Lambda)\lambda^{-1}\lambda_m = \left(Bb \cap \Lambda\lambda^{-1}\right)\lambda_m = \left(Bb_n \cap \Lambda\lambda_m^{-1}\right)\lambda_m = Bb_n\lambda_m \cap \Lambda.$$

Finally we set $F_B' := \{\lambda_m^{-1}b_n^{-1} \mid n \in \{1, ..., s\}, m \in \{1, ..., t\}\}$, which is finite.

To combine both cases define $F_B := F_B' \cup \{f_0\}$.

(d) $\Rightarrow$ (c): Let $B \subseteq G$ be bounded. For $\{B \cap \Lambda\lambda^{-1} \mid \lambda \in \Lambda\}$ we use our assumption to find finitely many representatives $\lambda_1, ..., \lambda_k$ such that

$$\left\{B \cap \Lambda\lambda^{-1} \,\middle|\, \lambda \in \Lambda\right\} = \bigcup_{n=1}^{k} \left\{B \cap \Lambda\lambda_n^{-1}\right\}.$$

We get

$$B \cap \Lambda\Lambda^{-1} = \bigcup_{\lambda \in \Lambda} B \cap \Lambda\lambda^{-1} = \bigcup_{n=1}^{k} B \cap \Lambda\lambda_n^{-1}.$$

The sets $B \cap \Lambda\lambda_n^{-1} = (B\lambda_n \cap \Lambda)\lambda_n^{-1}$ are finite, since Λ is locally finite and therefore their union is also finite.

(c) $\Rightarrow$ (d): Let $B \subseteq G$ be bounded, then by our assumption $B \cap \Lambda\Lambda^{-1}$ is finite. Further

$$B \cap \Lambda\Lambda^{-1} = \bigcup_{\lambda \in \Lambda} B \cap \Lambda\lambda^{-1}.$$

Since the left hand side is finite this also holds for the right hand side. But this means that there can only be finitely many possibilities for the sets $B \cap \Lambda\lambda^{-1}$. So we get

$$\left|\left\{B \cap \Lambda\lambda^{-1} \mid \lambda \in \Lambda\right\}\right| < \infty.$$

□

Definition 2.1.19 Let G be a lcsc group. A locally finite set $\Lambda \subseteq G$ which fulfils the conditions in Lemma 2.1.18 is said to have *finite local complexity (FLC)* and is called an *FLC set*.

Definition 2.1.20 Let X be a metric space and $\Lambda \subseteq X$ a locally finite set. Let $r > 0$ and $\lambda \in \Lambda$. Then the *r-patch around* λ is the constellation of points from Λ around λ, which have distance at most r to λ, i.e. $P_r(\lambda) := B_r(\lambda) \cap \Lambda$.

If G is a lcsc group and $\Lambda \subseteq G$ a locally finite set, the set of patches of fixed radius $r > 0$ induces an equivalence relation on Λ: Let $\lambda, \mu \in \Lambda$, then

$$\lambda \sim_r \mu :\Leftrightarrow P_r(\lambda)\lambda^{-1} = P_r(\mu)\mu^{-1}.$$

Further we will denote the *r-equivalence class of* λ by $A_r(\lambda) := \{\mu \in \Lambda \mid \lambda \sim_r \mu\}$ and the set of all r-equivalence classes by $A_r := \{A_r(\lambda) \mid \lambda \in \Lambda\}$.

Remark 2.1.21 In the literature, there are also other names for what we call patches. In some sense we abuse the notation $P_r(\lambda)$ here, since we do not only think about it as the set $B_r(\lambda) \cap \Lambda$, but also remember the middle point λ of the patch. These patches are sometimes called *centred* or *marked patches* in the literature. An older name for patches is *r-star*, [71, Definition 1.6]. For more general patches also the name *cluster*, [3, Definition 5.4.], is common.

2.1.1 Complexity

Now we can define the main term with which this thesis is concerned: Complexity. Afterwards we will discuss some elementary properties of this term and formulate some basic results.

Definition 2.1.22 Let G be a lcsc group and $\Lambda \subseteq G$ a locally finite subset. Then the *complexity function* $p : \mathbb{R}^+ \to \mathbb{N}_0$ of Λ is defined as the function which counts the number of equivalence classes with respect to r, i.e.

$$p(r) := |A_r|.$$

Lemma 2.1.23 *Let G be a lcsc group and $\Lambda \subseteq G$ a locally finite subset. The number of r-equivalence classes of patches is given by*

$$p(r) = \left|\left\{ P_r(\lambda)\lambda^{-1} \,\middle|\, \lambda \in \Lambda \right\}\right| = \left|\left\{ B_r(e) \cap \Lambda\lambda^{-1} \,\middle|\, \lambda \in \Lambda \right\}\right|.$$

Proof. The first equation is by definition of the equivalence relation. The second follows from the identity $B_r(\lambda) = B_r(e)\lambda$, which holds because of the right-invariance of the metric. □

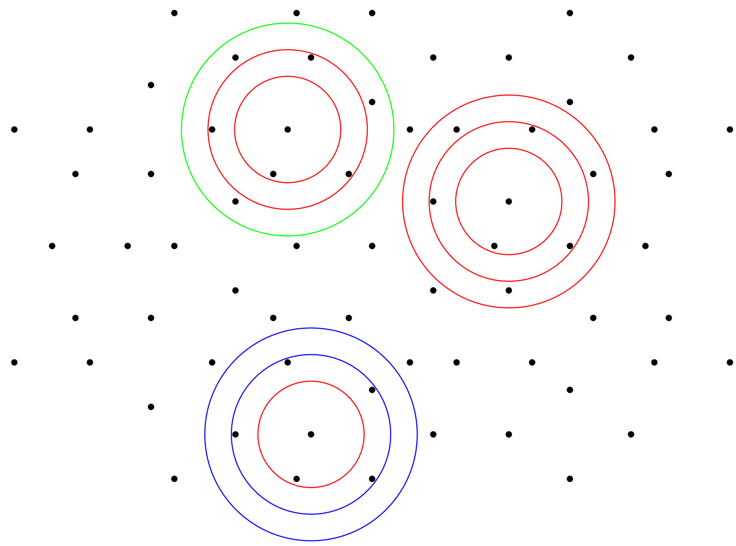


Figure 2.1: By increasing the radius r, points which are equivalent can get non-equivalent. To see this, compare the patches inside the circles of the same size, the different colours mark the equivalence class.

Remark 2.1.24 The complexity function is a good measure for how complicated an FLC set is. The faster the function grows the more complicated the set is. So we are mainly interested in the asymptotic behaviour of this function.

Remark 2.1.25 We always choose a metric on our group G and clearly this choice matters. But we will now show that at least two quasi-isometric groups G yield the same complexity for subsets.

Definition 2.1.26 Let G be a group and d and d' two metrics on G. We call d and d' quasi-isometric metrics on G if for all $g, h \in G$

$$\frac{1}{A} d'(g,h) - B \leq d(g,h) \leq A\, d'(g,h) + B$$

for some constants $A \geq 1$ and $B \geq 0$.

Theorem 2.1.27 *Let G be a group, $\Lambda \subset G$ an FLC set and d and d' two quasi-isometric metrics on G. Let further p' be the complexity function with respect to d' and p the complexity function with respect to d. Then $p(r) \asymp p'(r)$.*

Proof. Let B'_r denote a ball with respect to d' and p' the complexity function with respect to d'.
Since d' and d are quasi-isometric we have that

$$\begin{aligned} B'_{A(r-B)}(e) &= \{g \in G \mid A d_1(x,y) + B < r\} \\ &\subseteq B_r(e) \subseteq B'_{A(r+B)}(e) = \left\{ g \in G \,\middle|\, \frac{1}{A} d_1(x,y) - B < r \right\}. \end{aligned}$$

This yields that

$$p'(A(r-B)) \leq p(r) \leq p'\left(\frac{1}{A}(r-B)\right).$$

And therefore the asymptotic behaviour of p and p' is the same. $\square$

Now we will show that we can characterise FLC sets with the help of the complexity function.

Lemma 2.1.28 *Let G be a lcsc group and $\Lambda \subseteq G$ a locally finite subset. Then $p(r)$ is finite for all $r > 0$ if and only if Λ has FLC.*

Proof. If $p(r)$ is finite we have that $|\{B_r(e) \cap \Lambda\lambda^{-1} \mid \lambda \in \Lambda\}|$ is finite, and since each bounded $B \subseteq G$ is contained in some ball of the form $B_r(e)$ we see that condition (d) of Lemma 2.1.18 is fulfilled. On the other hand if $|\{B \cap \Lambda\lambda^{-1} \mid \lambda \in \Lambda\}| < \infty$ holds for all bounded $B \subseteq G$, then it clearly also holds for $B_r(e)$, $r > 0$. $\square$

Definition 2.1.29 Let G be a lcsc group and $\Lambda \subseteq G$ be a Delone set, then we define the *displacements* of $\lambda \in \Lambda$ as

$$\mathrm{Disp}(\lambda) := \{g \in G \mid g\lambda \in \Lambda\}.$$

Remark 2.1.30 Observe that in the definition of displacements the group acts from the left on λ. We will later see that this is the right choice, since we work with right-invariant metrics.

The equivalence class of an element is determined by its displacements to the other elements of the Delone set. So for each element, if we know what the displacements in distance at most $r > 0$ are, we can determine the r-equivalence class.

Proposition 2.1.31 *The complexity function p is monotonically increasing.*

Proof. Let $r_2 > r_1 > 0$ be two radii. We show that $p(r_2) \geq p(r_1)$. So if two patches $B_{r_2}(\lambda_1) \cap \Lambda$ and $B_{r_2}(\lambda_2) \cap \Lambda$ lie in the same equivalence class, this means that $(B_{r_2}(\lambda_1) \cap \Lambda)\lambda_1^{-1} = (B_{r_2}(\lambda_2) \cap \Lambda)\lambda_2^{-1}$. But then they are also in the same equivalence class for r_1, since $B_{r_1}(e) \subseteq B_{r_2}(e)$. □

The following theorem is due to Lagarias and Pleasants, [49, Theorem 3.1], in the case that $G = \mathbb{R}^n$, but we can generalise it to lcsc groups. It basically tells us that we can view FLC sets as a generalisation of lattices.

Theorem 2.1.32 *Let G be a lcsc group and $\Lambda \subseteq G$ a Delone subset.*

(a) If $p(r) = c$ for all $r \geq r_0$ for some constant $r_0 > 0$, then Λ is the union of c shifted cocompact lattices, i.e. there exists a cocompact lattice $\Gamma \subseteq G$ and a set $F \subseteq G$ with $|F| = c$ such that $\Lambda = \Gamma F$.

(b) If Λ is the union of c shifted shifted versions of one cocompact lattices, then $p(r) \leq c$ for all $r > 0$.

Proof. Assume Λ is Delone with the parameters (R, R').
First we assume that $p(r) = c$ for all $r \geq r_0$ for some constant r_0. Therefore there are c r-equivalence classes for all $r \geq r_0$, we will denote them by $A_1, ..., A_c$. Fix one $\lambda_i \in A_i$ for all $i \in \{1, ..., c\}$ and consider

$$\Gamma := \{g \in G \mid g\lambda_1 \in A_1\}.$$

We prove that Γ is a subgroup. Clearly $e \in \Gamma$. Now let $g_1, g_2 \in \Gamma$ then $\lambda_1 \sim_r g_1\lambda_1 \sim_r g_2\lambda_1$ for all $r > r_0$, i.e.

$$\Lambda\lambda_1^{-1} = \Lambda\lambda_1^{-1}g_1^{-1} = \Lambda\lambda_1^{-1}g_2^{-1}.$$

Multiplication with g_1 from the right yields

$$\Lambda\lambda_1^{-1}g_1 = \Lambda\lambda_1^{-1} = \Lambda\lambda_1^{-1}g_2^{-1}g_1.$$

Therefore since $\lambda_1 \in \Lambda$ we have $g_1^{-1}\lambda_1, g_1^{-1}g_2\lambda_1 \in \Lambda$ and further $g_1^{-1}\lambda_1 \sim_r \lambda_1 \sim_r g_1^{-1}g_2\lambda_1$, which means that Γ is a subgroup. Also observe that Γ is independent of the choice of λ_1, since if $\lambda_1' \in A_1$, then there exists an $h \in \Gamma$ such that $\lambda_1' = h\lambda_1$. If $g\lambda_1' \in A_1$, then $gh\lambda_1 \in A_1$, therefore $gh \in \Gamma$, which yields $g \in \Gamma$.
Let $\gamma_1 \neq \gamma_2 \in \Gamma$. Then

$$d(\gamma_1, \gamma_2) = d(\underbrace{\gamma_1\lambda_1}_{\in A_1 \subset \Lambda}, \underbrace{\gamma_2\lambda_1}_{\in A_1 \subset \Lambda}) > R,$$

therefore Γ is discrete. So Γ is a lattice, but we still have to show uniformity.
Next we show that Γ is independent of the choice of A_i, so let

$$\Gamma_2 := \{g \in G \mid g\lambda_2 \in A_2\}.$$

There exists a radius l such that $\lambda_2 \in B_l(\lambda_1)$, so λ_2 appears in all patches with radius greater than l around λ_1. Then there exists an element $g \in G$ such that $\lambda_1 g = \lambda_2 \in A_2$. Since all the patches around elements from one equivalence class look the same, we get that for all $\lambda \in A_1$ it holds that $\lambda g \in A_2$. Now let $\gamma \in \Gamma$. Then there exists a $\lambda \in A_1$ such that $\gamma\lambda_1 = \lambda$. This means that $\gamma\lambda_2 = \gamma\lambda_1 g = \lambda g \in A_2$.
So we have

$$\Lambda = \bigcup_{i=1}^{c} A_i = \bigcup_{i=1}^{c} \Gamma\lambda_i =: \Gamma F.$$

Now we show the uniformity of Γ. Since Λ is Delone, we have that for $\overline{B_{R'}(e)} \subseteq G$ it holds that

$$G = \Lambda\overline{B_{R'}(e)} = \Gamma F\overline{B_{R'}(e)}.$$

And $F\overline{B_{R'}(e)}$ is compact since F is finite.

Now assume that $\Lambda = \Gamma F$ with $|F| = c$ and Γ is a cocompact lattice. Clearly all the elements in Γf_i are in the same equivalence class for all $r > 0$, so there are at most c equivalence classes. □

Corollary 2.1.33 *Let G be a lcsc group and $\Gamma \subseteq G$ a Delone FLC subset, then Γ is a cocompact lattice if and only if $e \in \Lambda$ and $p(r) = 1$ for all $r > 0$.*

Example 2.1.34 (a) We give an example, why we do not have equality in Theorem 2.1.32 (b). Consider $\mathbb{Z}^2 \subset \mathbb{R}^2$ and shift it four times, by $(0,0)$, $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2})$ and take the union of these sets. By doing so we get the lattice $\left(\frac{1}{2}\mathbb{Z}\right)^2 \subset \mathbb{R}^2$, for which we have $p(r) = 1$.

(b) We give an example, why we need the Delone condition in the corollary, i.e. why FLC is not good enough. Consider $\mathbb{Z} \times \{0\} \subset \mathbb{R}^2$, this set is FLC and has $p(r) = 1$ for all $r > 0$, but it is no lattice in $\mathbb{R}^2$. The observation here is that FLC sets can be of 'lower dimension' or even finite.

Definition 2.1.35 A subset $\Lambda \subseteq G$ is called *aperiodic* if

$$\mathrm{per}(\Lambda) := \{g \in G \mid g\Lambda = \Lambda\}$$

is trivial. Here $\mathrm{per}(\Lambda)$ denotes the *group of periods of* Λ.

Remark 2.1.36 We use the terminology aperiodic in the sense that Lagarias and Pleasants do in [50]. In the newer literature the terminology is used in a different way.

Remark 2.1.37 By the theorem by Lagarias and Pleasants we see that a periodic behaviour in the Delone set only leads to a constant in the complexity function. So these sets are really tame and this fits into our view that lattices are highly structured. Since we want to explore some lighter ordered sets we are mainly interested in FLC sets which are aperiodic.

We can generalise another theorem by Lagarias and Pleasants [49, Theorem 2.1], which tells us that the complexity function of an aperiodic Delone set grows at least linearly.

Theorem 2.1.38 *Let $\Lambda \subseteq G$ be an R-relatively dense Delone set and $r_0 > 0$ and $r_1 > r_0 + 2R$ be two fixed radii. If $p(r_0) = p(r_1)$, then $p(r) = p(r_0)$ for all $r \geq r_0$.*

Proof. If $\lambda, \mu \in \Lambda$, then $\lambda \not\sim_{r_1} \mu$ if $\lambda \not\sim_{r_0} \mu$. And since $p(r_0) = p(r_1)$ we also have $\lambda \sim_{r_1} \mu$ if $\lambda \sim_{r_0} \mu$. Therefore $\lambda \sim_{r_1} \mu$ if and only if $\lambda \sim_{r_0} \mu$. This means that $P_{r_1}(\mu)$ is determined by $P_{r_0}(\mu)$.
Let $\mu \in \Lambda$ be arbitrary. We show that $P_{2r_1-r_0-2R}(\mu)$ is determined by $P_{r_0}(\mu)$. Fix one $\lambda_0 \in \Lambda$ and let $x \in B_{2r_1-r_0-2R}(\lambda_0)$. Then there exists a $y \in B_{r_1-r_0-R}(\lambda_0)$ such that $d(x,y) < r_1 - R$. And since Λ is R-relatively dense, there is a $\lambda \in B_R(y) \cap \Lambda$. Then $B_{r_0}(\lambda) \subseteq B_{r_1}(\lambda_0)$, since if $z \in B_{r_0}(\lambda)$ then

$$d(z, \lambda_0) \leq d(z, \lambda) + d(\lambda, y) + d(y, \lambda_0) < r_0 + R + r_1 - r_0 - R = r_1.$$

Further $x \in B_{r_1}(\lambda)$, since

$$d(x, \lambda) \leq d(x, y) + d(y, \lambda) < r_1 - R + R = r_1.$$

This means that $P_{r_1}(\lambda_0)$ determines $P_{r_0}(\lambda)$ which, by the first passage, determines $P_{r_1}(\lambda)$. Since x was an arbitrary point in $B_{2r_1-r_0-2R}(\lambda_0)$ we have

$$B_{2r_1-r_0-2R}(\lambda_0) \subseteq \bigcup_{\lambda \in P_{r_1-r_0}(\lambda_0)} B_{r_1}(\lambda).$$

Therefore $P_{2r_1-r_0-2R}(\lambda_0)$ is determined uniquely by $P_{r_1}(\lambda_0)$. And since λ_0 was arbitrary, this means that

$$p(2r_1 - r_0 - 2R) = p(r_1) = p(r_0).$$

Now we can iterate this process by considering $r_1' := 2r_1 - r_0 - 2R$ instead of r_1. $\square$

Corollary 2.1.39 *Let $\Lambda \subseteq G$ be an R-relatively dense Delone set with complexity function $p(r)$. If there exists an $r' > 0$ such that $p(r') < \frac{r'}{2R}$, then $p(r)$ will become constant.*

Proof. We know that $p(r)$ is monotonically increasing by Proposition 2.1.31. So assume that $p(r)$ is unbounded. Clearly $p(r_0) \geq 1$ for all $r_0 \in (0, 2R)$. Then by Theorem 2.1.38 we have

$$p(2R + r_0 + \varepsilon_1) > p(r_0) \geq 1,$$

for all $\varepsilon_1 > 0$, since otherwise p would become constant. We can repeat the argument iteratively, where $\varepsilon_i > 0$ for all $i \in \mathbb{N}$ and get

$$p(4R + r_0 + \varepsilon_1 + \varepsilon_2) \geq 3\,, \,\ldots\,, \,p(2mR + r_0 + \varepsilon_1 + \ldots + \varepsilon_m) \geq m + 1,$$

for any integer m. Set r_0 and the ε_i such that $2mR + r_0 + \varepsilon_1 + \ldots + \varepsilon_m = r'$, with $r_0 + \varepsilon_1 + \ldots + \varepsilon_m < 2R$, then

$$m + 1 \leq p(r') < \frac{2mR + r_0 + \varepsilon_1 + \ldots + \varepsilon_m}{2R} < m + 1.$$

This is a contradiction, so p can not be unbounded and therefore gets constant. $\square$

Remark 2.1.40 In [49] it is also shown that the constant $\frac{1}{2R}$ is optimal. Lagarias and Pleasants do this by constructing for any $\varepsilon > 0$ a non-crystalline Delone set such that there is a $r' > 0$ with $p(r') < (\frac{1}{2} + \varepsilon)\frac{r'}{R}$. (Non-crystalline means that the Delone set does not have a full lattice as its translation symmetry.)
Since there are counterexamples in the Euclidean case the constant $\frac{1}{2R}$ is optimal for the more general case, too.

This result means that there can be no sublinear growth of the complexity function besides a constant one.

2.2 The Cut and Project Method

In this section we will consider the possibly most important method to construct FLC sets, the *cut and project method*. This method is due to Meyer, [54, 55, 56], in its original form. The generalisation to locally compact groups is due to Michael Björklund, Tobias Hartnick and Felix Pogorzelski. We are only considering this method since it is the one with which we will work in this thesis. Another important class of FLC sets is built by substitution methods, but we will not study them. These two big classes do overlap but in general they are not the same.

Definition 2.2.1 A *cut and project scheme* is a triple (G, H, Γ), where G and H are lcsc groups and $\Gamma \subseteq G \times H$ is a lattice, which projects injectively to G and densely to H. We call the cut and project scheme *uniform* if the lattice Γ is uniform.

Convention 2.2.2 We will put G or H in the index if we consider the projection from an object of $G \times H$ to this respective component. See for example Figure 2.2. The projections to the components are denoted by π_G and π_H.

$$
\begin{tikzcd}
& G\text{-side} & & H\text{-side} & \\
G & & G \times H \arrow[ll, "\pi_G"'] \arrow[rr, "\pi_H"] & & H \\
\Gamma_G \arrow[u, hook] \arrow[rrrru, "\tau"', bend right] & & \Gamma \arrow[u, hook] \arrow[ll, two heads, "\cong", "\pi_G|_\Gamma"'] \arrow[rr, two heads, "\pi_H|_\Gamma"'] & & \Gamma_H \arrow[u, hook]
\end{tikzcd}
$$

Figure 2.2: Diagram to visualize a cut and project set.

Definition 2.2.3 Let (G, H, Γ) be a cut and project scheme and $W \subseteq H$. Then the set

$$\Lambda(G, H, \Gamma, W) := \pi_G((G \times W) \cap \Gamma)$$

is called a *cut and project set (CPS)*. If the data is clear, we will denote the CPS by Λ.

The set W is called the *window* of the CPS.

If the lattice Γ is uniform we will also call the CPS *uniform*.

Further we define $\tau : \Gamma_G \to H, \gamma \mapsto \left(\pi_H \circ \pi_G|_\Gamma^{-1}\right)(\gamma)$, which is called the *starmap*.

Remark 2.2.4 The name starmap for τ comes from the case, where the lattice sits in such a way inside the product $G \times H$ that the map is simply Galois conjugation. For example $\left\{(a, a^*) \in \mathbb{R}^2 \mid a \in \mathbb{Z}[\sqrt{2}]\right\}$, where a^* is the Galois conjugation of a. The oldest examples for CPS are of this form and so this name was established.

Remark 2.2.5 We can also characterize the CPS by the starmap, since

$$\pi_G((G \times W) \cap \Gamma) = \tau^{-1}(W).$$

Since we now formulated the construction method, the next step is to see in which cases a CPS is an FLC set. For the Euclidean version of the following proposition see [3, Lemma 7.4.].

Proposition 2.2.6 *Let G and H be lcsc groups and $\Gamma \subseteq G \times H$ a uniform lattice, such that $\pi_H(\Gamma)$ is dense in H. Further let $U \subseteq H$ be an open non-empty set. Then there exists a compact set $K \subseteq G$ such that*

$$G \times H = (K \times U)\Gamma.$$

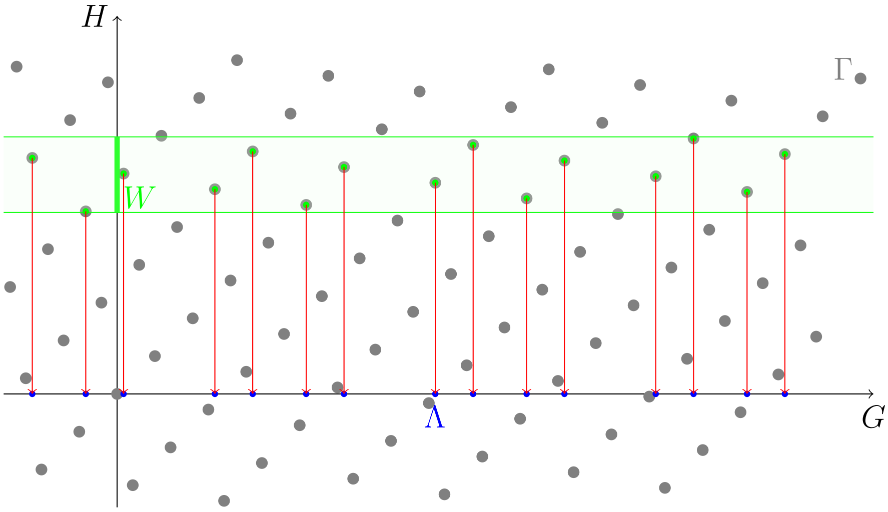


Figure 2.3: This figure shows a CPS, where $G = \mathbb{R}$, $H = \mathbb{R}$ and W is chosen to be an interval.

Proof. Since Γ is a uniform lattice in $G \times H$, we find a compact set $C \subset G \times H$, such that $G \times H = C\Gamma$. By projecting C to G and H we get compact sets $C_G := \pi_G(C)$ and $C_H := \pi_H(C)$. We know that the product of two compact sets is compact, therefore $C_G \times C_H$ is compact in $G \times H$. It is clear that $C \subseteq C_G \times C_H$, so

$$G \times H = (C_G \times C_H)\Gamma.$$

By the density of Γ_H in H we get a covering

$$\bigcup_{\gamma \in \Gamma} U\pi_H(\gamma) = H \supset C_H.$$

Since C_H is compact, we can choose a finite subcovering with finite $F \subseteq \Gamma$ such that

$$\bigcup_{\gamma \in F} U\pi_H(\gamma) \supset C_H.$$

Now let $z \in G \times H$ be arbitrary. By the choice of C we find a $\gamma \in \Gamma$ such that $z\gamma^{-1} \in C \subseteq C_G \times C_H$. By our covering argument we find an $f \in F$ such that $\pi_H(z\gamma^{-1}) \in U\pi_H(f)$ and therefore $\pi_H(z\gamma^{-1}f^{-1}) \in U$. If we project the same element to G we get

$$\pi_G\left(z\gamma^{-1}f^{-1}\right) \in C_G\pi_G\left(F^{-1}\right) =: K.$$

Now K is compact since C_G is compact and $\pi_G(F^{-1})$ is finite. Putting things together we realise

$$z = \left(z\gamma^{-1}f^{-1}\right)(f\gamma) \in (K \times U)\Gamma.$$

□

The following proposition is well known, a version in the Euclidean set-up can be found in [3, Proposition 7.5.].

Proposition 2.2.7 ([13, Proposition 2.13.])
Let $\Lambda(G, H, \Gamma, W)$ be a uniform CPS.

(a) If $W^\circ \neq \emptyset$, then Λ is relatively dense.

(b) If W is relatively compact, then Λ is uniformly discrete.

(c) If W is relatively compact and $W^\circ \neq \emptyset$, then Λ has FLC.

Proof. (a) We are using Proposition 2.2.6. Since $W^\circ \neq \emptyset$ this also holds for the inverse $(W^{-1})^\circ \neq \emptyset$ and we can choose an open subset $\emptyset \neq U \subseteq W^{-1}$. By Proposition 2.2.6 we find a compact set K such that $G \times H = (K \times U)\Gamma$. Let $g \in G$ be arbitrary. We can find $u \in U, k \in K$ and $\gamma \in \Gamma$ such that

$$(g, e_H) = (k, u)(\gamma_G, \gamma_H).$$

This tells us, that $u\gamma_H = e_H$ and therefore $\gamma_H = u^{-1} \in (W^{-1})^{-1} = W$, so $\gamma_G \in \Lambda$. Therefore $g = k\gamma_G \in K\Lambda$. This shows the claim.

(b) Let us assume Λ is not uniformly discrete, then for all $r > 0$ there exists $x, y \in \Lambda$ such that $d(x, y) < r$. By the right-invariance of d this is equivalent to $d(e, yx^{-1}) < r$. We can lift x and y to elements in the product and get

$$\pi_G|_\Gamma^{-1}(x) =: (x_G, x_H),\ \pi_G|_\Gamma^{-1}(y) =: (y_G, y_H) \in \Gamma \cap (G \times W).$$

Since Γ and G are groups, we can deduce $(y_G, y_H)\left(x_G^{-1}, x_H^{-1}\right) \in \Gamma \cap (G \times WW^{-1})$. And since we know that $yx^{-1} \in B_r(e_G)$, we get

$$(y_G, y_H)\left(x_G^{-1}, x_H^{-1}\right) \in \Gamma \cap \left(B_r(e_G) \times WW^{-1}\right).$$

Since W is relatively compact, WW^{-1} is relatively compact and therefore bounded. Also $B_r(e_G)$ is bounded and therefore the product $B_r(e_G) \times WW^{-1}$ is bounded. Since Γ is a lattice, we get that $\Gamma \cap (B_r(e_G) \times WW^{-1})$ is finite. By the injectivity of π_G, we know that $d(a_G, b_G) \neq 0$ for $a \neq b \in \Gamma$. So we get that $d(a_G, b_G) > 0$ for $a, b \in \Gamma \cap (B_r(e_G) \times WW^{-1})$ and by finiteness there is a minimal distance $\tilde{d}$. Now set $\tilde{r} < \tilde{d}$ and conclude $\Gamma \cap (B_{\tilde{r}}(e_G) \times WW^{-1}) = \{(e_G, e_H)\}$. This is a contradiction to the assumption, since we do not find two elements, which are this close together. Therefore Λ has to be uniformly discrete for $\tilde{r}$.

(c) By (a) and (b) we know that Λ is a Delone set. We want to use the characterisation (c) of Lemma 2.1.18, so we show that $B \cap \Lambda\Lambda^{-1}$ is finite for a bounded set $B \subset H$. It is enough to show, that the preimage of this set is finite. Since intersection and projection commutes it holds that

$$\pi_G^{-1}\left(B \cap \Lambda\Lambda^{-1}\right) = \pi_G^{-1}(B) \cap \pi_G^{-1}\left(\Lambda\Lambda^{-1}\right).$$

Now we can consider the two parts separately and then intersect them, so the preimage of B is obviously $\pi_G^{-1}(B) = B \times H$.
For the second part, we need to remember the definition of Λ. This was given by $\Lambda = \pi_G((G \times W) \cap \Gamma)$, so

$$\pi_G^{-1}\left(\Lambda\Lambda^{-1}\right) = \pi_G^{-1}\left(\pi_G((G \times W) \cap \Gamma)\pi_G((G \times W) \cap \Gamma)^{-1}\right).$$

We want to show, that this is a subset of $\Gamma \cap (G \times WW^{-1})$. So let $\lambda_1, \lambda_2 \in \Lambda$. Then they are both in Γ_G and therefore $\lambda_1\lambda_2^{-1} \in \Gamma_G$ and there exists a unique preimage inside Γ which we name $(\lambda_1\lambda_2^{-1}, x)$. On the other hand, the preimage of λ_i, $i \in \{1,2\}$, is $(\lambda_i, w_i) \in \Gamma \cap (G \times W)$. Therefore

$$(\lambda_1, w_1)(\lambda_2, w_2)^{-1} = (\lambda_1, w_1)\left(\lambda_2^{-1}, w_2^{-1}\right) = \left(\lambda_1\lambda_2^{-1}, w_1w_2^{-1}\right) \in \Gamma \cap \left(G \times WW^{-1}\right).$$

Since the preimage was unique and $\pi_G(\lambda_1\lambda_2^{-1}, w_1w_2^{-1}) = \lambda_1\lambda_2^{-1}$ holds, we get that $\pi_G^{-1}\left(\lambda_1\lambda_2^{-1}\right) \in \Gamma \cap (G \times WW^{-1})$.

Combining the two arguments, we get

$$\pi_G^{-1}\left(B \cap \Lambda\Lambda^{-1}\right) = (B \times H) \cap \Gamma \cap \left(G \times WW^{-1}\right) = \Gamma \cap \left(B \times WW^{-1}\right).$$

Since W is relatively compact, we get that $\overline{W}$ is compact. Since $W \subseteq \overline{W}$ we see $W \subseteq H$ is bounded. Hence there exists an $r > 0$ such that $r > d(w_1, w_2)$ for all $w_1, w_2 \in W$. And once more by right-invariance of the metric, we get $r > d\left(w_1w_2^{-1}, e\right)$. This tells us that $WW^{-1} \subset B_r(e)$ and therefore it is bounded. Further $B \subseteq G$ was a bounded set. We see that $B \times WW^{-1} \subset G \times H$ is bounded in the product. Since Γ is a lattice it has FLC and therefore $(B \times WW^{-1}) \cap \Gamma$ is finite.

□

Example 2.2.8 (a) We start by a counterexample which shows how the non-empty interior property influences the behaviour. Let $G = H = \mathbb{R}^2$ and

$$\Gamma = \left|\left\{(a, b, a^*, b^*)^\top \in \mathbb{R}^4 \,\middle|\, a, b \in \mathbb{Z}[\sqrt{2}]\right\}\right|,$$

where a^* is image of a under the non-trivial Galois-automorphism of $\mathbb{Q}[\sqrt{2}]$. Further we set the window as $W := [0,1] \times \{0\}$, then

$$\Lambda(G, H, \Gamma, W) = \left\{(a, 0)^\mathrm{T} \in \mathbb{R}^2 \,\middle|\, a^* \in (0,1) \text{ and } a \in \mathbb{Z}[\sqrt{2}]\right\}$$

and this is not relatively dense in G.

(b) The next counterexample shows, why the relative compactness, especially the boundedness, is important. Let $G = H = \mathbb{R}$ and $\Gamma = \left\{(a, a^*)^\mathrm{T} \,\middle|\, a \in \mathbb{Z}[\sqrt{2}]\right\}$. Now set $W := \bigcup_{k \in \mathbb{Z}}[2k, 2k+1]$, which is unbounded and therefore not relatively compact. Then

$$\Lambda(G, H, \Gamma, W) = \left\{a \in \mathbb{Z}[\sqrt{2}] \mid a^* \in [2k, 2k+1]\right\},$$

which is not discrete in $\mathbb{R}$.

For detailed examples see Chapter 8.

Definition 2.2.9 A CPS $\Lambda(G, H, \Gamma, W)$ is called a *model set* if W is relatively compact and $W^\circ \neq \emptyset$. Again if Γ is a uniform lattice we call the model set *uniform*.

Definition 2.2.10 A model set $\Lambda(G, H, \Gamma, W)$ is called:

(a) *Regular* if $\Gamma_H \cap \partial W = \emptyset$,

(b) *of polytopal type* if the window W is a polytope (this obviously only makes sense, when the notion of a polytope makes sense in the space H).

Remark 2.2.11 The condition of regularity is useful since the lattice points which project onto the border of W behave differently as the ones which project into the interior. We will use this condition many times, but it is not really restrictive, because we can always shift the window a little to assure that the model set is regular. In the literature the term is also called Γ-regular, to highlight the dependence on the chosen lattice.

Proposition 2.2.12 ([13, Proposition 2.13.])
A model set is Delone if and only if it is uniform.

Proof. We already have seen in Proposition 2.2.7 that uniform model sets are Delone. Now consider the model set $\Lambda(G, H, \Gamma, W)$ and assume it to be Delone. Then Λ is relatively dense and therefore we find a compact $K \subseteq G$ such that $G = K\Lambda$. Since $\pi_H(\Gamma)$ is dense in H we have that $\overline{B_r(e)}\Gamma_H = H$ for some arbitrary fixed r. Further we know that $\overline{W}$ is compact.
Consider $(g, h) \in G \times H$, then there exists $u \in \overline{B_r(e)}$ and $\gamma_1 \in \Gamma_G$ such that $h = u\tau(\gamma_1)$. Further there exists a $k \in K$ and a $\lambda \in \Lambda$, such that $g\gamma_1^{-1} = k\lambda$. Combining everything yields

$$\begin{aligned}(g, h) &= \left(g\gamma_1^{-1}, u\right)(\gamma_1, \tau(\gamma_1)) = (k\lambda, u)(\gamma_1, \tau(\gamma_1)) \\ &= \left(k, u\tau(\lambda)^{-1}\right)\underbrace{(\lambda, \tau(\lambda))(\gamma_1, \tau(\gamma_1))}_{\in\Gamma} \in \left(K \times \overline{B_r(e)}W^{-1}\right)\Gamma.\end{aligned}$$

Since $\overline{W}$ is compact, $\overline{W}^{-1}$ is compact and therefore $\overline{B_r(e)}\,\overline{W}^{-1}$ is compact. Further $\left(K \times \overline{B_r(e)}\,\overline{W}^{-1}\right)$ is compact in the product and since (g, h) was arbitrary this proves that Γ is uniform. $\square$

There is one question which we did not address in this chapter namely if a CPS exists for all choices of G and H. The problem hereby is that we have to find a lattice $\Gamma \subseteq G \times H$ with the desired properties, but the existence of such a lattice is non trivial, especially in this general context. So if one wants to construct an FLC set in a group G with the cut and project method, the task is to find a suitable group H and a suitable lattice.

We want to further analyse the behaviour of CPS, we will do this in Chapter 3. But before that we collect some further basic knowledge in the rest of this chapter.

2.3 Counting Lattice Points

For CPS the lattice points in $G \times W$ play an important role, since they are the preimage of the CPS. We will not only need this for the window but for more general sets $G \times B_t^H(h)$, $h \in H$ and $t > 0$, and we want to know how the lattice points are distributed in this set. To be more precise we ask the question, how does the number of lattice points in $B_r^G(e) \times B_t^H(h)$ grow as r goes to infinity. The answer will be given by the following proposition.

Proposition 2.3.1 (Growth Lemma)
Let G and H be lcsc groups and μ_G a Haar measure on G. For a uniform CPS and a bounded open set $\emptyset \neq A \subseteq H$. The asymptotic growth of the number of lattice points inside $B_r^G(e) \times A$ is bounded by

$$\mu_G\left(B_{r-k_1}^G(e)\right) \ll \left|\left(B_r^G(e) \times A\right) \cap \Gamma\right| \ll \mu_G\left(B_{r+k_2}^G(e)\right),$$

for some constants $k_1, k_2 > 0$ as $r \to \infty$.

The proof consists of the following two lemmas.

Lemma 2.3.2 *Let G and H be lcsc groups and μ_G a Haar measure on G. For a uniform CPS and a bounded open set $\emptyset \neq A \subseteq H$ the growth of the number of lattice points inside $B_r^G(e) \times A$ is asymptotically bounded from above by*

$$\left|\left(B_r^G(e) \times A\right) \cap \Gamma\right| \ll \mu_G\left(B_{r+k_2}^G(e)\right),$$

where k_2 is some constant as $r \to \infty$.

Proof. Since Γ is a lattice it is uniformly discrete, therefore we find a constant c_1 such that for all $\gamma_1 \neq \gamma_2 \in \Gamma$ holds that $d(\gamma_1, \gamma_2) > c_1$. If we halve the constant we get that $B_{\frac{c_1}{2}}^{G\times H}(\gamma_1) \cap B_{\frac{c_1}{2}}^{G\times H}(\gamma_2) = \emptyset$.

Since A is bounded we find a second constant c_2 such that $A \subseteq B_{c_2}^H(e)$ and that for every $x \in G \times A$ we have $B_{c_1}^{G\times H}(x) \subseteq G \times B_{c_2}^H(e)$. The norm in the product is given by the maximum of the norm of the components.

The idea is that we build a set which contains not only the points of $\left(B_r^G(e) \times A\right) \cap \Gamma$ but also the balls around them. Then we can obtain an upper bound for the number of points in $\left(B_r^G(e) \times A\right) \cap \Gamma$ by estimating how often the thickened set of points could fit in this set via a volume estimate. Since by our choice of $\frac{c_1}{2}$ the balls do not overlap. We obtain that

$$\sum_{\gamma \in (B_r^G(e) \times A) \cap \Gamma} \mu_{G\times H}\left(B_{\frac{c_1}{2}}^{G\times H}(\gamma)\right) \leq \mu_{G\times H}\left(B_{r+c_1}^G(e) \times B_{c_2}^H(e)\right).$$

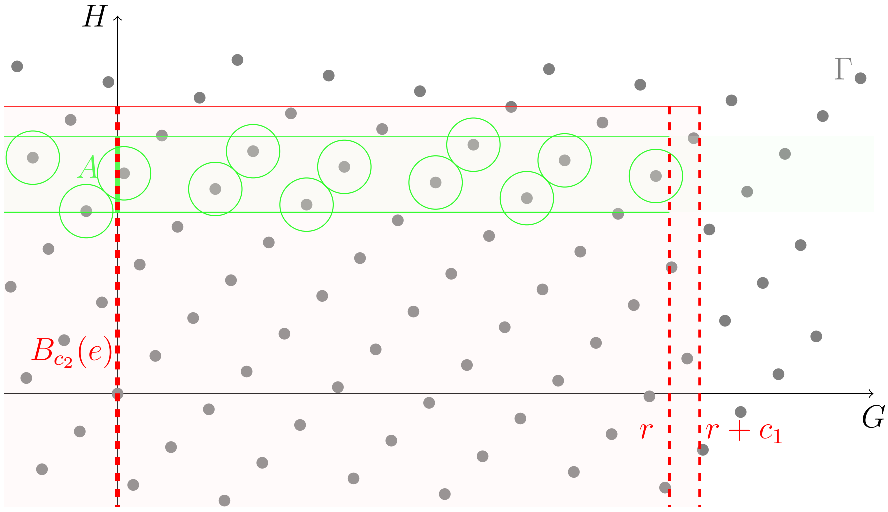


Figure 2.4: In this figure the situation in the proof of Lemma 2.3.2 is shown. The idea is to compare the volume of the union of the green balls with the volume of the indicated red rectangle.

We need the "$+c_1$" in the index to make sure that the set also contains all the balls whose center lie close to the border of $B_r^G(e)$. The volume of a ball is independent of its center point, since the metric and the Haar measure are right-invariant. Therefore we can write the inequality as

$$\left|\left(B_r^G(e) \times A\right) \cap \Gamma\right| \cdot \mu_{G\times H}\left(B_{\frac{c_1}{2}}^{G\times H}(e)\right) \leq \mu_{G\times H}\left(B_{r+c_1}^G(e) \times B_{c_2}^H(e)\right).$$

We will now divide this inequality by $\mu_{G\times H}\left(B_{\frac{c_1}{2}}^{G\times H}(e)\right)$, which is just a constant dependent on c_1, which we will denote by c_1', and the constant $\mu_H\left(B_{c_2}^H(e)\right)$ will be denoted by c_2'.

$$\begin{aligned}\left|\left(B_r^G(e) \times A\right) \cap \Gamma\right| &\leq \frac{\mu_{G\times H}\left(B_{r+c_1}^G(e) \times B_{c_2}^H(e)\right)}{c_1'} = \frac{\mu_G\left(B_{r+c_1}^G(e)\right) \cdot \mu_H\left(B_{c_2}^H(e)\right)}{c_1'} \\ &= \frac{c_2'}{c_1'}\mu_G\left(B_{r+c_1}^G(e)\right).\end{aligned}$$

□

Lemma 2.3.3 *Let G and H be lcsc groups and μ_G a Haar measure on G. For a uniform CPS and a bounded open set $\emptyset \neq A \subseteq H$ the growth of the number of lattice points inside $B_r^G(e) \times A$ is asymptotically bounded from below by*

$$\left|\left(B_r^G(e) \times A\right) \cap \Gamma\right| \gg \mu_G\left(B_{r-k_1}^G(e)\right),$$

where k_1 is some constant as $r \to \infty$.

Proof. We choose an open ball $B^H_\varepsilon(\gamma_H) \subseteq A$ with $\gamma_H \in \Gamma_H$, this can be done since Γ_H is dense in H and A is open and therefore $\Gamma_H \cap A$ is dense in A. Let ε be fixed.
Assume that $\gamma_H = e$, the argument also works for general γ_H, but this just complicates the notation. By Proposition 2.2.6 we find a compact set $K \subseteq G$ such that $G \times H = (K \times B^H_\varepsilon(e))\Gamma$. Since K is compact it is bounded and we can consider $K = \overline{B^G_c}(e)$ with c large enough. Then for all $z \in G \times H$: $\left(B^G_c(e) \times B^H_\varepsilon(e)\right) z \cap \Gamma \neq \emptyset$. This holds true, since we can write $z = (k_z, u_z)(\gamma_{zG}, \gamma_{zH})$ with $(\gamma_{zG}, \gamma_{zH}) \in \Gamma$, $k_z \in B^G_c(e)$ and $u_z \in B^H_\varepsilon(e)$. But then

$$(\gamma_{zG}, \gamma_{zH}) = \left(k_z^{-1}, u_z^{-1}\right) z \in \left(B^G_c(e) \times B^H_\varepsilon(e)\right) z \cap \Gamma,$$

since $k_z^{-1} \in B^G_c(e)$ and $u_z^{-1} \in B^H_\varepsilon(e)$.
We can find a lower bound for the growth if we can fit enough of the sets of type $\left(B^G_c(e) \times B^H_\varepsilon(e)\right) z$ into $B^G_r(e) \times A$ in a disjoint way. This comes down to

$$\begin{aligned}\left|\left(B^G_r(e) \times A\right) \cap \Gamma\right| &> \left|\left(B^G_r(e) \times B^H_\varepsilon(e)\right) \cap \Gamma\right| \\ &\geq \max\left\{|X| \,\middle|\, X \subseteq G, \text{ such that } \forall x \in X : B^G_c(x) \subset B^G_r(e) \right.\\ &\qquad \left. \text{and } B^G_c(x) \cap B^G_c(y) = \emptyset \ \forall x \neq y \in X\right\}. \\ &\geq \max\left\{|X| \,\middle|\, X \subset B^G_{r-c}(e) \text{ and } X \text{ is } 2c\text{-uniformly discrete}\right\}.\end{aligned}$$

We can extend every $2c$-uniformly discrete set to a $(2c, 2c)$-Delone set by Corollary 2.1.10. Thus

$$\left|\left(B^G_r(e) \times A\right) \cap \Gamma\right| \geq \max\left\{|X| \,\middle|\, X \subset B^G_{r-c}(e) \text{ and } X \text{ is a } (2c, 2c)\text{-Delone subset of } B^G_r(e)\right\}.$$

For every such Delone set we can cover $B_{r-c}(e)$ with balls $B_{2c}(x)$ for $x \in X$, so that

$$\bigcup_{x \in X} B^G_{2c}(x) \supset B^G_{r-c}(e) \Rightarrow \sum_{x \in X} \mu_G\left(B^G_{2c}(x)\right) \geq \mu_G\left(B^G_{r-c}(e)\right).$$

Since the metric and the Haar measure are right-invariant all these balls have the same measure and we get

$$|X| \cdot \mu_G\left(B^G_{2c}(e)\right) \geq \mu_G\left(B^G_{r-c}(e)\right) \Leftrightarrow |X| \geq \frac{\mu_G\left(B^G_{r-c}(e)\right)}{\mu_G\left(B^G_{2c}(e)\right)}.$$

Summing up we have

$$\left|\left(B^G_r(e) \times A\right) \cap \Gamma\right| > \frac{\mu_G\left(B^G_{r-c}(e)\right)}{\mu_G\left(B^G_{2c}(e)\right)}.$$

□

Definition 2.3.4 Let G be a locally compact group and let d be a right-invariant metric on G compatible with the topology on G. Then (G, d) is a metric group with *exact polynomial growth of degree* κ if there exists a constant $c > 0$ such that

$$\lim_{r \to \infty} \frac{\mu_G(B_r(e))}{c r^\kappa} = 1.$$

Corollary 2.3.5 *Under the conditions of Proposition 2.3.1 if (G,d) is a lcsc metric group with exact polynomial growth of degree κ with respect to d, then*

$$\left|\left(B_r^G(e) \times A\right) \cap \Gamma\right| \asymp r^\kappa.$$

Remark 2.3.6 The degree in the polynomial growth depends on the metric.

The lattice point counting argument will help us later on to understand the asymptotic behaviour of the complexity function. But by itself this counting argument is of some interest. One could clearly establish the same result by ergodic theory but we have chosen to not use this here. We will see an ergodic theoretic argument later which is based on the theory built by Nevo and Gorodnik, [30]. Using this theory a benefit is that one can estimate the constants involved in the asymptotic behaviour. But we will see that we can not control the constants anyway. So by this more down-to-earth approach we get a better understanding of the influence of the components in a CPS.

3 Complexity of Model Sets

In this chapter we generalise the idea that the complexity function of a CPS can be understood by considering a certain decomposition of the window. This idea first appeared in the paper by Julien, [44], and then is extended in the paper by Koivusalo and Walton, [45]. We will stick close to the strategy of Koivusalo and Walton and freely use their notation most of the time. Nevertheless we have to remark that they only work in the Euclidean set-up so this chapter expands their ideas. In this chapter we will only introduce the main tool of this analysis, Theorem 3.0.4. At the end of the chapter we will see what we need to understand to determine the asymptotic behaviour of the complexity function. The whole rest of the thesis is than devoted to solving this problem in certain cases.

Remark 3.0.1 In the following discussion one should always think of connected groups. If one considers non-connected groups the statements will not fail, but counting connected components in a totally disconnected space is somewhat senseless.

Definition 3.0.2 (Pre-Acceptance domains)
Let $\Lambda(G, H, \Gamma, W)$ be a CPS. The image of $A_r^G(\lambda)$ under the star-map is called the *r-pre-acceptance domain of λ*

$$A_r^H(\lambda) := \tau\left(A_r^G(\lambda)\right) \subseteq H.$$

We denote the set of all pre-acceptance domains by A_r^H. And since τ is injective we have $\left|A_r^H\right| = p(r)$.

Definition 3.0.3 Let $\Lambda(G, H, \Gamma, W)$ be a uniform model set, with $(G, d_G), (H, d_H)$ lcsc groups. Further let $\lambda \in \Lambda$, then

$$\left(\bigcap_{\mu \in \mathcal{S}_r(\lambda)} \mu \mathring{W}\right) \cap \left(\bigcap_{\mu \in \mathcal{S}_r^{\mathrm{C}}(\lambda)} \mu W^{\mathrm{C}}\right) =: W_r(\lambda),$$

where $\mathcal{S}_r(\lambda)$ will be defined in Definition 3.0.8, is called the *r-acceptance domain of λ*.

Theorem 3.0.4 (Acceptance domains)
Let $\Lambda(G, H, \Gamma, W)$ be a uniform model set, with $(G, d_G), (H, d_H)$ lcsc groups.

(a) $A_r^H(\lambda) \subset W_r(\lambda)$.

(b) *The r-acceptance domains correspond one-to-one to the r-equivalence classes of* Λ*, i.e. let* $\lambda, \lambda' \in \Lambda$ *and* $\tau(\lambda') \in W_r(\lambda)$*, then* $\lambda \sim_r \lambda'$.

(c) *For* $\lambda \not\sim_r \lambda'$ *we have*

$$W_r(\lambda) \cap W_r(\lambda') = \emptyset.$$

(d)

$$\overline{W} = \bigcup_{\lambda \in A_r^G} \overline{W_r(\lambda)}.$$

Remark 3.0.5 That we have to take the closure of the window in the theorem does not make a big difference, since by Γ-regularity there are no projected lattice points on the boundary. This also holds for the shifted window, since if $\gamma_1, \gamma_2 \in \Gamma_H$ with $\gamma_1 \in \gamma_2 \partial W$, then $\gamma_2^{-1}\gamma_1 \in \partial W$ in contradiction to the regularity of the model set. So for all acceptance domains $\partial W_r(\lambda) \cap \Gamma_H = \emptyset$.

Corollary 3.0.6 $p(r) = |\{W_r(\lambda) \mid \lambda \in \Lambda\}|$.

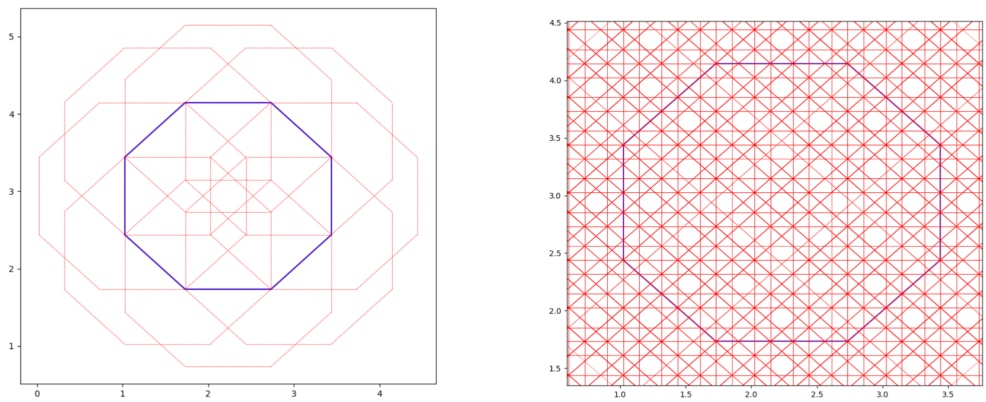


Figure 3.1: The decomposition of an octagonal window in blue, by shifted versions in red, on the left for $r = 1$ and on the right for $r = 5$. In this example $H = \mathbb{R}^2$.

The following Lemma is an extension from the Euclidean case, [45, Lemma 2.1]. It enables us to localize the position of the displacements after mapping with the starmap.

Lemma 3.0.7 *Let Λ be a regular model set and $\lambda \in \Lambda$, and $\mu \in G$. If $\mu\lambda \in \Lambda$ then $\mu \in \Gamma_G$. On the other hand if $\mu \in \Gamma_G$:*

$$\mu\lambda \in \Lambda \;\Leftrightarrow\; \tau(\lambda) \in \tau(\mu)^{-1}\mathring{W} \;\Leftrightarrow\; \tau(\mu) \in \mathring{W}\tau(\lambda)^{-1}.$$

In particular $\tau(\mathrm{Disp}(\lambda)) \subset \mathring{W}\mathring{W}^{-1}$.

Proof. Since $\lambda, \mu\lambda \in \Lambda$ we find elements $\gamma, \delta \in \Gamma$ such that $\pi_G(\gamma) = \lambda$, $\pi_G(\delta) = (\mu\lambda)^{-1}$. Then $\gamma\delta \in \Gamma$ and $\pi_G(\gamma\delta) = \lambda(\mu\lambda)^{-1} = \mu^{-1} \in \Gamma_G$ and therefore $\mu \in \Gamma_G$.
Now let $\mu \in \Gamma_G$. By definition $\mu\lambda \in \Lambda$ if and only if $\tau(\mu\lambda) \in \mathring{W}$ and since τ is a homomorphism this is equivalent to $\tau(\mu) \in \mathring{W}\tau(\lambda)^{-1}$ and $\tau(\lambda) \in \tau(\mu)^{-1}\mathring{W}$. □

Definition 3.0.8 (r-slab)
Let $\Lambda(G, H, \Gamma, W)$ be a model set. We define the *r-slab* as

$$\mathcal{S}_r := \pi_H\left(\left\{(\gamma, \mu) \in \Gamma \,\middle|\, |\gamma| < r \text{ and } \mu \in WW^{-1}\right\}\right).$$

Further if we only are interested in the displacements of a certain equivalence class we define the *r-slab of* λ as

$$\mathcal{S}_r(\lambda) := \pi_H\left(\left\{(\gamma, \mu) \in \Gamma \,\middle|\, |\gamma| < r \text{ and } \mu \in WW^{-1} \text{ and } \gamma^{-1} \in \mathrm{Disp}(\lambda)\right\}\right)$$

and

$$\mathcal{S}_r^{\mathrm{C}}(\lambda) := \pi_H\left(\left\{(\gamma, \mu) \in \Gamma \,\middle|\, |\gamma| < r \text{ and } \mu \in WW^{-1} \text{ and } \gamma^{-1} \notin \mathrm{Disp}(\lambda)\right\}\right).$$

Remark 3.0.9 In the paper of Koivusalo and Walton the sets $S_r(\lambda)$ and $S_r^{\mathrm{C}}(\lambda)$ are called P_{in} and P_{out}, where P is the patch around λ. We changed the notation to highlight the connection to the slab.

Lemma 3.0.10 *Let Λ be a model set and $\lambda, \mu \in \Lambda$. Then*

$$\lambda \sim_r \mu \Leftrightarrow \mathcal{S}_r(\lambda) = \mathcal{S}_r(\mu).$$

Proof. Assume $\lambda \sim_r \mu$, then $(B_r(\lambda) \cap \Lambda)\lambda^{-1} = (B_r(\mu) \cap \Lambda)\mu^{-1}$. Let $x \in S_r(\lambda)$ then there exists a $(\gamma, x) \in \Gamma$ such that

$$\begin{aligned} &\gamma^{-1}\lambda \in B_r(\lambda) \cap \Lambda \\ &\qquad \Leftrightarrow \gamma^{-1} \in (B_r(\lambda) \cap \Lambda)\lambda^{-1} = (B_r(\mu) \cap \Lambda)\mu^{-1} \\ &\qquad \Leftrightarrow \gamma^{-1}\mu \in B_r(\mu) \cap \Lambda. \end{aligned}$$

Therefore $x \in S_r(\mu)$.
Now assume $\mathcal{S}_r(\lambda) = \mathcal{S}_r(\mu)$ and let $x \in (B_r(\lambda) \cap \Lambda)\lambda^{-1}$ then $\tau(x^{-1}) \in \mathcal{S}_r(\lambda) = \mathcal{S}_r(\mu)$ and this implies $x \in (B_r(\mu) \cap \Lambda)\mu^{-1}$. □

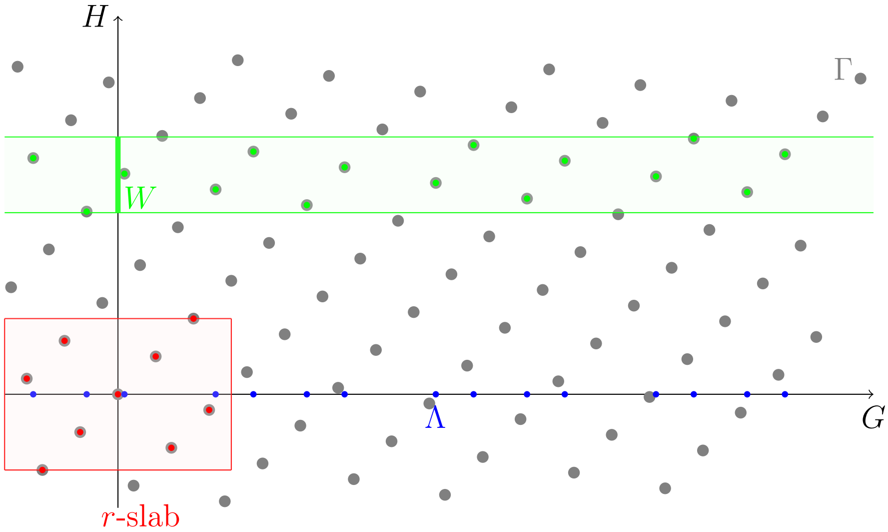


Figure 3.2: This figure shows the preimage of the slab for a fixed r in the setting of a $\mathbb{R} \times \mathbb{R}$ CPS.

Proof of Theorem 3.0.4. Lemma 3.0.10 tells us that for all $\lambda' \in A_r^G(\lambda)$ the set

$$W_r(\lambda') := \left(\bigcap_{\mu \in \mathcal{S}_r(\lambda')} \mu \mathring{W} \right) \cap \left(\bigcap_{\mu \in \mathcal{S}_r(\lambda')^{\mathrm{C}}} \mu W^{\mathrm{C}} \right)$$

is the same. So to prove (a) and (b) it is enough to show $\tau(\lambda) \in W_r(\lambda)$. By the definition of the r-slab of λ we have for all $\mu \in \mathcal{S}_r(\lambda)$ that there is a $\mu_G \in \Gamma_G$ with $\tau(\mu_G) = \mu$ and $\mu_G^{-1}\lambda \in \Lambda$. Further Lemma 3.0.7 tells us that $\tau(\lambda) \in \mu\mathring{W}$. For $\mu \in \mathcal{S}_r^{\mathrm{C}}(\lambda)$ Lemma 3.0.7 tells us that $\tau(\lambda) \notin \mu W$, but this means that $\tau(\lambda) \in \mu W^{\mathrm{C}}$. So it follows that $\tau(\lambda) \in W_r(\lambda)$.

Now let $\lambda \not\sim_r \lambda'$, so by Lemma 3.0.10 $S_r(\lambda) \neq S_r(\mu)$ and the disjointness of $W_r(\lambda)$ and $W_r(\lambda')$ follows by the same argument.

Finally we show that the $\overline{W_r(\lambda)}$ tile the closure of the window $\overline{W}$. The inclusion $\overline{W_r(\lambda)} \subseteq \overline{W}$ is clear since $e_H \in \mathcal{S}_r(\lambda)$ for all $\lambda \in \Lambda$ and all $r > 0$. Since Γ_H is dense in W and $W_r(\lambda)$ is open we know that Γ_H is dense in $W_r(\lambda)$. Since $A_r^H(\lambda) = \Gamma_H \cap W_r(\lambda)$ we know that $A_r^H(\lambda)$ is dense in $W_r(\lambda)$. Therefore the completion by sequences $\overline{A_r^H(\lambda)}^{seq}$ is the topological closure $\overline{W_r(\lambda)}$. Further since every $\gamma \in \Gamma_H \cap W$ has to belong to some $W_r(\lambda)$ we get that

$$W \cap \Gamma_H = \bigcup_{\lambda \in A_r^G} A_r^H(\lambda).$$

Completion by sequences on both sides delivers

$$\overline{W} = \bigcup_{\lambda \in A_r^G} \overline{A_r^H(\lambda)}^{seq} = \bigcup_{\lambda \in A_r^G} \overline{W_r(\lambda)}.$$

$\square$

Now we have a tool with which we can determine the complexity function by counting the number of acceptance domains inside the window. We can use this to find an upper bound to the growth.

Remark 3.0.11 Observe that an acceptance domain does not have to be connected, see Figure 3.3.

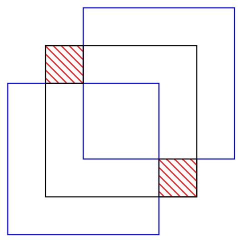


Figure 3.3: The figure shows that the acceptance domains are not necessarily connected, the two marked connected components belong to the same acceptance domain.

Lemma 3.0.12 *For a CPS* $\Lambda(G, H, \Gamma, W)$ *we have*

$$\left|A_r^H\right| \leq \#\pi_0\left(W \setminus \bigcup_{\mu \in \mathcal{S}_r} \mu \partial W\right).$$

Proof. By Theorem 3.0.4 we know that the acceptance domains $W_r(\lambda)$ tile the window W and that they are disjoint. Further for every $A_r(\lambda)$ we know that

$$A_r(\lambda) \subset W_r(\lambda).$$

So we get for the boundary of an acceptance domain

$$\begin{aligned}
\partial W_r(\lambda) =& \partial\left(\left(\bigcap_{\mu \in \mathcal{S}_r(\lambda)} \mu \mathring{W}\right) \cap \left(\bigcap_{\mu \in \mathcal{S}_r(\lambda)^{\mathrm{C}}} \mu W^{\mathrm{C}}\right)\right) \\
\subseteq& \left(\bigcup_{\mu \in \mathcal{S}_r(\lambda)} \mu \partial \mathring{W}\right) \cup \left(\bigcup_{\mu \in \mathcal{S}_r(\lambda)^{\mathrm{C}}} \mu \partial W^{\mathrm{C}}\right) = \bigcup_{\mu \in \mathcal{S}_r} \mu \partial W.
\end{aligned}$$

Therefore every connected component of $W \setminus \bigcup_{\mu \in \mathcal{S}_r} \mu \partial W$ is contained in some $W_r(\lambda)$, thus

$$\left|A_r^H\right| = |W_r| \leq \#\pi_0\left(W \setminus \bigcup_{\mu \in \mathcal{S}_r} \mu \partial W\right).$$

□

The problem is now reduced to a counting problem for which in the general case we do not know an answer.
In the Euclidean case Koivusalo and Walton, [45], have shown that for polytopal windows it is possible to determine the asymptotic growth of the complexity function by calculating the stabilizers of the hyperplanes, which bound the polytopal window. They do this by using the nice behaviour of shifted hyperplanes, namely that they stay parallel. What Walton and Koivusalo do not use is the extensive literature on so called *hyperplane arrangements*. We will in the following chapter recall these arrangements and collect some properties, which will help counting the number of acceptance domains in the polytopal case. Afterwards we will extend the term of a polytopal window beyond the Euclidean case. Combining these two things will give us the growth of the complexity function in a more general set-up.

4 Hyperplane Arrangements

In this chapter we will consider hyperplane arrangements in $\mathbb{R}^d$ and learn how to treat certain combinatorial questions. Most of the work in this chapter is already known, but we extend the ideas to convex subsets of $\mathbb{R}^d$. For an introduction to the topic of hyperplane arrangements we refer to the book of Dimca, [22], and the lecture notes from Stanley, [74], from which we import the notation and definitions. We also mention the work by Grünbaum, [32, 33, 34].
After recalling the general set-up we will state some well known formulas for counting flats in an arrangement. In the second section we will associate to each hyperplane a characteristic polynomial, this is a common strategy to analyse hyperplane arrangements. We are mainly interested in how the arrangement behaves in a bounded region of $\mathbb{R}^n$, which will later on be the window of the model set or a small subset of the window. To analyse this local behaviour we introduce a new version of the characteristic polynomial which depends on such a bounded set. In the next section we will consider the Erdős-Beck Theorem and prove a dual version of the statement, which then gives us the result we were looking for.
In the final section we will see that our argumentation also holds in hyperbolic space.

4.1 Basics about Hyperplane Arrangements

Definition 4.1.1 A finite set of affine hyperplanes $\mathcal{H} = \{P_1, ..., P_n\}$ in $\mathbb{R}^d$ is called a *hyperplane arrangement.*

Definition 4.1.2 An arrangement of n hyperplanes $\mathcal{H}$ in $\mathbb{R}^d$ is called:

(a) *Simple* if the hyperplanes are in general position, therefore for all $I \subseteq \mathcal{H}$ with $|I| = i$, $i \leq d$, we have

$$\dim \left(\bigcap_{H \in I} H \right) = d - i,$$

and for $i > d$ the intersection is the empty set.

(b) *Central* if $\bigcap_{H \in \mathcal{H}} H \neq \emptyset$.

(c) *Central with respect to B*, for $B \subseteq \mathbb{R}^d$, if $B \cap \bigcap_{H \in \mathcal{H}} H \neq \emptyset$.

(d) *Essential* if the dimension of the space spanned by the normals of the hyperplanes is d, the dimension of this space is called the *rank* of the arrangement and is denoted by $\mathrm{rk}(\mathcal{H})$.

Remark 4.1.3 Instead of writing hyperplane arrangement all the time we will simply write arrangement most of the time.

Definition 4.1.4 Let $\mathcal{H}$ be an arrangement in $\mathbb{R}^d$.

(a) A non-empty intersection of hyperplanes from $\mathcal{H}$ is called a *flat* of $\mathcal{H}$. The set of all flats is denoted by $F(\mathcal{H})$, the set of all flats of dimension k is denoted by $F_k(\mathcal{H})$. Remark that the intersection over the empty set is also a flat, so $\mathbb{R}^d \in F(\mathcal{H})$. Further remark that the flats of dimension $d-1$ are exactly the hyperplanes, i.e. $F_{d-1}(\mathcal{H}) = \mathcal{H}$.

(b) $F(\mathcal{H})$ is partially ordered by reverse inclusion, therefore for $X, Y \in F(\mathcal{H})$ it is $X \leq Y$ if $Y \subseteq X$. The set $F(\mathcal{H})$ together with the partial order is called the *intersection poset of* $\mathcal{H}$.

(c) Two arrangements $\mathcal{A}$ and $\mathcal{B}$ are *combinatorially equivalent* if they have isomorphic intersection posets sets $F(\mathcal{A})$ and $F(\mathcal{B})$.

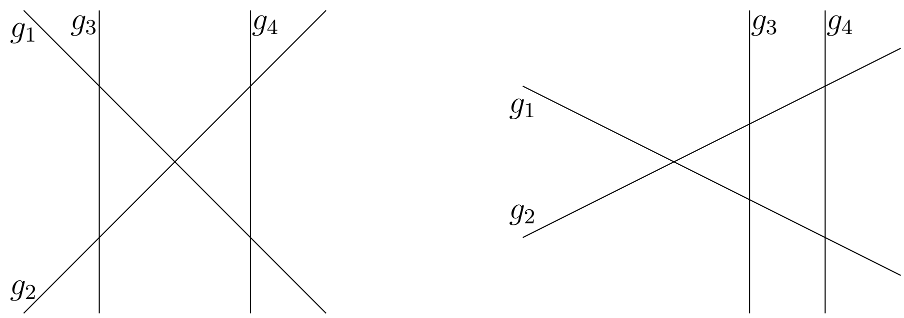


Figure 4.1: Two combinatorially equivalent arrangements.

Example 4.1.5 Consider the two arrangements shown in Figure 4.1. Both arrangements are combinatorially equivalent, because they both have the intersection poset shown in Figure 4.2. We see that there are ten flats, namely $\mathbb{R}^2$, the lines $g_1, ..., g_4$ and the five intersection points. Further notice that the plane is divided into ten connected components.

Definition 4.1.6 For an arrangement $\mathcal{H}$ and $f \in F(\mathcal{H})$ we define the following arrangements:

(a) $S(f) := \{H \in \mathcal{H} \mid f \subseteq H\}$ and $a(f) := |S(f)|$.

(b) $\mathcal{H}^f := \{H \cap f \neq \emptyset \mid H \in \mathcal{H} \backslash S(f)\}$, this is a hyperplane arrangement in f.

(c) For $H \in \mathcal{H}$ we denote $\mathcal{H}_H := \mathcal{H} \backslash \{H\}$.

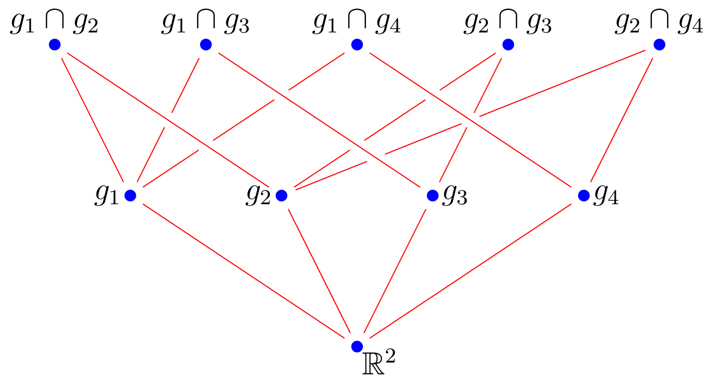


Figure 4.2: The intersection poset of the two arrangements from Figure 4.1. The red lines mark the order of reverse inclusion from bottom to top.

Definition 4.1.7 Let $\mathcal{H}$ be an arrangement in $\mathbb{R}^d$.

(a) The connected components of

$$\mathbb{R}^d \setminus \bigcup_{H \in \mathcal{H}} H$$

are called *regions* of the arrangement. The set of all regions is denoted by $f_d(\mathcal{H})$. The number of these regions is denoted by $r(\mathcal{H}) := |f_d(\mathcal{H})|$.

(b) A *face* of the arrangement is a set $\emptyset \neq A = \overline{R} \cap f$, where R is a region and $f \in F(\mathcal{H})$. If $\dim(f) = i$ we talk about an i-face of the arrangement. The set of all faces of dimension i is denoted by $f_i(\mathcal{H})$.

Example 4.1.8 We consider Example 4.1.5 again. In both the arrangements there are 10 regions, which we already have seen above. Further there are fourteen 1-faces, where four belong to g_1 and g_2 and three to g_3 and g_4. And finally there are five 0-faces, the intersection points of the lines.

There is a well known upper bound for the number of k-faces of an arrangement in $\mathbb{R}^n$, first found by Schläfli, [67]. There are two exact expressions of this bound appearing in the paper by Buck, [17], and the one by Zaslavsky, [79].

Proposition 4.1.9 ([26, Theorem 1.1])
The maximal number of k-faces in an arrangement of n hyperplanes in $\mathbb{R}^d$ is given by

$$\left|f_k^d(n)\right| = \max_{|\mathcal{H}|=n} |f_k(\mathcal{H})| = \sum_{i=d-k}^{d} \binom{n}{i} \binom{i}{d-k} \qquad (\textit{Buck's formula}),$$

$$\left|f_k^d(n)\right| = \binom{n}{d-k} \sum_{i=0}^{k} \binom{n-d+k}{i} \qquad (\textit{Zaslavsky's formula}).$$

Remark 4.1.10 These maxima are attained if and only if the arrangement is simple. For a general arrangement there is no simple formula to calculate the number of k-faces.

Following the argumentation of Fukuda, Saito and Tamura, [26], we establish inequalities, which help us bound the number of k-faces in a general arrangement. To do so we will show the following theorem from which the inequalities follow as a corollary.

Theorem 4.1.11 ([26, Theorem 1.2.])
For an arbitrary arrangement $\mathcal{H}$ of hyperplanes in $\mathbb{R}^d$, the mean number of $(k-1)$-faces of a k-face is less than $2k$ for $k \in \{1, ..., d\}$.

Proof. For an arbitrary hyperplane $H \in \mathcal{H}$ we can distinguish the following three types of faces:

(a) Type 1: Faces of $\mathcal{H}_H$.

(b) Type 2: Faces of $\mathcal{H}^H$.

(c) Type 3: Faces from $\mathcal{H}_H$, which are separated by H.

Here type 3 is a subtype of type 1, but it is important to count the faces of type 3 twice, so once as faces of type 1 and a second time as faces of type 3.
The k-faces of type 3 have a $k-1$-face lying in H. For the d-faces, the regions, we see directly that

$$|f_d(\mathcal{H})| = \left|f_d\left(\mathcal{H}_H\right)\right| + \left|f_{d-1}\left(\mathcal{H}^H\right)\right|, \tag{i}$$

since we can only have d-faces of type 1 and 3. For all $k \in \{1, ..., d-1\}$ we only get an inequality

$$|f_k(\mathcal{H})| \leq \left|f_k\left(\mathcal{H}_H\right)\right| + \left|f_k\left(\mathcal{H}^H\right)\right| + \left|f_{k-1}\left(\mathcal{H}^H\right)\right|. \tag{ii}$$

Only inequality holds in (ii) since multiple intersections can happen in H. As an easy example consider three lines intersecting in one point, we have six 1-faces. Using the formula to count we get a bound of seven 1-faces. Equality holds in the case of a simple arrangement.

Claim: The mean number of the $d-1$ faces of a d-face in an arrangement in $\mathbb{R}^d$ is less than $2d$.

We know that a $(d-1)$-face lies in the boundary of exactly two d-faces, therefore the claim is equivalent to $2d \cdot |f_d(\mathcal{H})| \geq 2|f_{d-1}(\mathcal{H})|$. We can show this by induction over $n := |\,\mathcal{H}\,|$ and d.
For $n = 1$ the statement is obvious since there are exactly two regions which share one $(d-1)$-face. For $d = 1$ the statement also holds, since on a line all intervals have two endpoints, which are the hyperplanes in $d = 1$, except for the two unbounded intervals at the end, which have only one endpoint.

We show that if the statement holds for some n in dimension d and all n in all the dimensions smaller than d then it also holds for $n+1$ in dimension d:

$$\begin{aligned}\frac{1}{d}|f_{d-1}(\mathcal{H})| &\overset{\text{(ii)}}{\leq} \frac{1}{d}\left|f_{d-1}\left(\mathcal{H}_H\right)\right| + \frac{1}{d}\left|f_{d-1}\left(\mathcal{H}^H\right)\right| + \frac{1}{d}\left|f_{d-2}\left(\mathcal{H}^H\right)\right| \\ &\leq \left|f_d\left(\mathcal{H}_H\right)\right| + \frac{1}{d}\left|f_{d-1}\left(\mathcal{H}^H\right)\right| + \frac{1}{d}(d-1)\left|f_{d-1}\left(\mathcal{H}^H\right)\right| \overset{\text{(i)}}{=} |f_d(\mathcal{H})|.\end{aligned}$$

From the claim we can deduce that in each k-dimensional flat the mean number of $k-1$-faces of a k-face is less than $2k$. Since each k-face is contained in a unique k-flat we can consider the average over all flats which concludes the proof. □

Corollary 4.1.12 ([26, Corollary 3.1 and 3.4.])
The following relations hold for an arrangement $\mathcal{H}$ of n hyperplanes in $\mathbb{R}^d$:

(a) $|f_k(\mathcal{H})| \geq \frac{d-k+1}{k}|f_{k-1}(\mathcal{H})|$ *for* $1 \leq k \leq d$, *where equality holds if and only if* $|f_k(\mathcal{H})| = 0$.

(b) $\binom{d-j}{d-k}|f_j(\mathcal{H})| \leq \binom{k}{j}|f_k(\mathcal{H})|$ *for* $0 \leq j \leq k \leq d$.

(c) $|f_k(\mathcal{H})| \leq |f_{d-k}(\mathcal{H})|$ *for* $0 \leq k \leq \lfloor \frac{d}{2} \rfloor$.

(d) $|f_k(\mathcal{H})| \leq \binom{d}{k}|f_d(\mathcal{H})|$ *for* $0 \leq k \leq d$.

Proof. (a) A $(k-1)$-face bounds at least $2(d-k+1)$ k-faces. Together with Theorem 4.1.11 this implies that $2k|f_k(\mathcal{H})| \geq 2(d-k+1)|f_{k-1}(\mathcal{H})|$.

(b) This follows by applying (a) multiple times,

$$\begin{aligned}|f_k(\mathcal{H})| &\geq \frac{d-k+1}{k}|f_{k-1}(\mathcal{H})| \geq \frac{d-k+1}{k}\frac{d-(k-2)}{k-1}|f_{k-2}(\mathcal{H})| \\ &\geq \ldots \geq \frac{(d-j)!}{(d-k)!}\frac{j!}{k!}|f_j(\mathcal{H})|.\end{aligned}$$

Therefore

$$\binom{d-j}{d-k}|f_j(\mathcal{H})| = \frac{(d-j)!}{(d-k)!(k-j)!}|f_j(\mathcal{H})| \leq \frac{k!}{j!(k-j)!}|f_k(\mathcal{H})| = \binom{k}{j}|f_k(\mathcal{H})|.$$

(c) Follows directly from (b).

(d) Follows directly from (b).

□

These are some basic inequalities and give an idea on how to work in the set-up of hyperplane arrangements. Observe that we were able to show an upper bound for the number of regions but giving a non-trivial lower bound is much harder.

4.2 Characteristic Polynomial

In this section we will define the characteristic polynomial of a hyperplane arrangement, which is a well known term, [74, Definition 1.3.]. We will also define a new version of this polynomial which depends on a subset $B \subseteq \mathbb{R}^n$. Then we will see how the evaluation of the polynomial in -1 is connected to the regions of the arrangement or respectively the regions inside B. Afterwards we will calculate the evaluation under certain restrictions on the intersections of the hyperplanes. All this is known for the characteristic polynomial, we transfer all the arguments to the subset case.

Definition 4.2.1 Let $B \subseteq \mathbb{R}^d$ and $\mathcal{H}$ a hyperplane arrangement in $\mathbb{R}^d$, then we denote the *number of regions with respect to* B by

$$r_B(\mathcal{H}) := \#\pi_0\left(B \setminus \bigcup_{H\in\mathcal{H}} H\right).$$

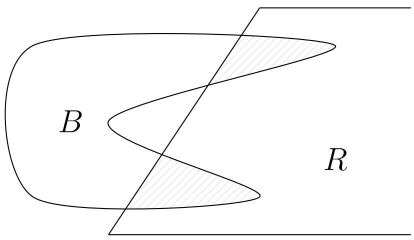


Figure 4.3: If B is not convex a region R of $\mathcal{H}$ can induce multiple regions in B, indicated by the shaded areas.

Remark 4.2.2 In general the number of regions with respect to some B is different from $|\{f \in f_d(\mathcal{H}) \mid f \cap B \neq \emptyset\}|$, which is the number of regions which intersect B, see Figure 4.3. But we will show in Proposition 4.2.3 that for convex B the two terms are equal.

Proposition 4.2.3 *Let $\mathcal{H}$ be a hyperplane arrangement in $\mathbb{R}^d$ and $B \subseteq \mathbb{R}^d$ be convex, then*

$$r_B(\mathcal{H}) = |\{f \in f_d(\mathcal{H}) \mid f \cap B \neq \emptyset\}|.$$

Proof. The regions of a hyperplane arrangement are convex, since B is also convex we have for all $R \in f_d(\mathcal{H})$ that $R \cap B$ is convex and especially connected. So each region of the arrangement either intersects B and therefore contributes exactly one to $r_B(\mathcal{H})$ or it does not intersect at all. $\square$

Definition 4.2.4 Let $\mathcal{H}$ be a hyperplane arrangement in $\mathbb{R}^d$ and $B \subseteq \mathbb{R}^d$ then the *flats with respect to* B are

$$F_B(\mathcal{H}) := \{f \in F(\mathcal{H}) \mid f \cap B \neq \emptyset\}.$$

If we are only interested in the flats of dimension k we write $F_{k,B}(\mathcal{H})$. Also we define the arrangement

$$\mathcal{H}_B := \{H \in \mathcal{H} \mid H \cap B \neq \emptyset\}.$$

Remark 4.2.5 Observe that the number of regions with respect to B only depends on $\mathcal{H}_B$.

If we only consider the area inside B we lose some properties, which hold in the Euclidean space. For example there can be hyperplanes which are non-parallel but do not intersect, so the parallel postulate does not hold. The behaviour here is the same as in hyperbolic space, where for each hyperplane H and each point P outside of the hyperplanes there are at least two hyperplanes through P, which do not intersect H. Especially observe that this is a model for hyperbolic space, if B is a ball, the so called projective model or in dimension two the Beltrami-Klein model.

Definition 4.2.6 Let $B \subseteq H$ be a bounded region and P_1, P_2 two hyperplanes in H. We call P_1 and P_2 *almost parallel with respect to* B if $P_1 \cap P_2 \cap B = \emptyset$.

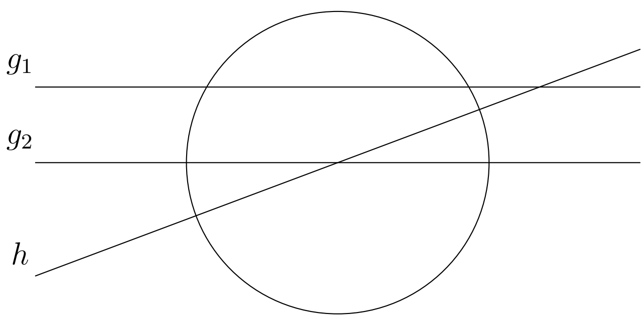


Figure 4.4: Two parallel lines g_1 and g_2 where one is almost parallel to h and the other is not. Further observe that the plane is decomposed into 6 regions and the ball only in 5.

There are multiple ways to define the characteristic polynomial of a hyperplane arrangement. We will first start with the definition we will mostly use and then give an alternative characterisation by Whitney's theorem, [22, Theorem 2.6].

Definition 4.2.7 The *characteristic polynomial* of a hyperplane arrangement $\mathcal{H}$ in $\mathbb{R}^n$ is defined by

$$\chi_{\mathcal{H}}(t) = \sum_{\substack{\mathcal{A}\subseteq\mathcal{H} \\ \mathcal{A}\text{ central}}} (-1)^{|\mathcal{A}|} t^{\dim\left(\bigcap_{H\in\mathcal{A}} H\right)}.$$

If the arrangement is obvious we omit the index. Let $B \subseteq \mathbb{R}^n$ then

$$\chi_{\mathcal{H},B}(t) = \sum_{\substack{\mathcal{A}\subseteq\mathcal{H} \\ \mathcal{A}\text{ central with respect to } B}} (-1)^{|\mathcal{A}|} t^{\dim\left(\bigcap_{H\in\mathcal{A}} H\right)}.$$

is the *characteristic polynomial with respect to B*.

Definition 4.2.8 The *Möbius function of the intersection poset of $\mathcal{H}$* is the unique function $\mu : F(\mathcal{H}) \times F(\mathcal{H}) \to \mathbb{Z}$ such that

(a) $\mu(f,f) = 1$ for any $f \in F(\mathcal{H})$,

(b) $\sum_{f\leq g\leq h} \mu(f,g) = 0$ for all $f,h \in F(\mathcal{H})$ with $f < h$,

(c) $\mu(f,g) = 0$ for $f \not\leq g$.

We further set $\mu(f) := \mu\left(\mathbb{R}^d, f\right)$ if $\mathcal{H}$ is an arrangement in $\mathbb{R}^d$.

In the same way we define the *Möbius function* of an arbitrary partially ordered set.

Remark 4.2.9 We will now cite some theorems from the book of Dimca, [22], who uses the term of a lattice in the sense of a meet-join-lattice. For us the term of a lattice is reserved for discrete subgroups. So we will talk about partially ordered sets with minimal and maximal elements, in the sense of a meet-join-lattice.
Further we will not state the proofs which are already in the book, we will only add the parts for the subset case.

Theorem 4.2.10 (Cross-Cut Theorem, [22, Theorem 2.5])
Let L be a finite partially ordered set with a minimal element $\hat{0}$ and a maximal element $\hat{1}$. Let $T \subset L$ such that $\hat{0} \notin T$ and suppose that for any $l \in L$, $l \neq \hat{0}$, there is an element $t \in T$ such that $t \leq l$.
Let N_k denote the number of k-element subsets of T with join $\hat{1}$. Then

$$\mu(\hat{0},\hat{1}) = \sum_{k\geq 0} (-1)^k N_k.$$

The following theorem is due to Hassler Whitney, [78], for arrangements, where each hyperplane contains the origin. Its extension to arbitrary arrangements appears in the book by Orlik and Terao [61, Lemma 2.3.8].

Theorem 4.2.11 (Whitney's Theorem, [22, Theorem 2.6])
If $\mathcal{H}$ is an arrangement in $\mathbb{R}^d$, then

$$\chi_{\mathcal{H}}(t) = \sum_{f \in F(\mathcal{H})} \mu(f) t^{\dim(f)}$$

and

$$\chi_{\mathcal{H},B}(t) = \sum_{f \in F_B(\mathcal{H})} \mu(f) t^{\dim(f)}.$$

Proof. We only prove the second formula, since the first one appears in the book by Dimca. Let $f \in F_B(\mathcal{H})$ then $L(f) := \{g \in F(\mathcal{H}) \mid g \leq f\}$ is a partially ordered set with maximal element f and minimal element $\mathbb{R}^d$. We see that $S(f) := \{H \in \mathcal{H}_B \mid f \subseteq H\}$ fulfils the conditions of Theorem 4.2.10, thus

$$\mu(f) = \sum_{k \geq 0} (-1)^k N_k(f).$$

Here we write $N_k(f)$ to highlight the dependence on the flat f. We use this to show the claim:

$$\begin{aligned}
\chi_{\mathcal{H},B}(t) &= \sum_{\substack{\mathcal{A} \subseteq \mathcal{H} \\ \mathcal{A} \text{ central with respect to } B}} (-1)^{|\mathcal{A}|} t^{\dim\left(\bigcap_{H \in \mathcal{A}} H\right)} \\
&= \sum_{f \in F_B(\mathcal{H})} t^{\dim(f)} \sum_{\substack{\mathcal{A} \subseteq S(f) \\ \cap_{H \in \mathcal{A}} H = f \\ |\mathcal{A}| = i}} (-1)^i = \sum_{f \in F_B(\mathcal{H})} t^{\dim(f)} \sum_{i \geq 0} (-1)^i N_i(f) \\
&= \sum_{f \in F_B(\mathcal{H})} \mu(f) t^{\dim(f)}.
\end{aligned}$$

$\square$

Theorem 4.2.12 (Deletion-Restriction Theorem, [22, Theorem 2.7])
Let $\mathcal{H}$ be a hyperplane arrangement in $\mathbb{R}^d$ and $H_0 \in \mathcal{H}$, then for the triple $(\mathcal{H}, \mathcal{H}^{H_0}, \mathcal{H}_{H_0})$ we have

$$\chi_{\mathcal{H}}(t) = \chi_{\mathcal{H}_{H_0}}(t) - \chi_{\mathcal{H}^{H_0}}(t)$$

and for every convex subset $B \subseteq \mathbb{R}^n$

$$\chi_{\mathcal{H},B}(t) = \chi_{\mathcal{H}_{H_0},B}(t) - \chi_{\mathcal{H}^{H_0},B}(t).$$

Proof. The proofs of the two equations are similar, we will only prove the second one. We can split up the sum which defines $\chi_{\mathcal{H},B}$

$$\chi_{\mathcal{H},B}(t) = \sum_{\substack{H_0 \in \mathcal{A} \subseteq \mathcal{H} \\ \mathcal{A} \text{ central with respect to } B}} (-1)^{|\mathcal{A}|} t^{\dim\left(\bigcap_{H \in \mathcal{A}} H\right)} + \sum_{\substack{H_0 \notin \mathcal{A} \subseteq \mathcal{H} \\ \mathcal{A} \text{ central with respect to } B}} (-1)^{|\mathcal{A}|} t^{\dim\left(\bigcap_{H \in \mathcal{A}} H\right)}.$$

The second sum is $\chi_{\mathcal{H}_{H_0},B}(t)$, that is clear. Thus we have to show that the first sum is $-1 \cdot \chi_{\mathcal{H}^{H_0},B}(t)$. For a subarrangement $\mathcal{A} \subseteq \mathcal{H}$, which contains H_0 and is central, consider $\mathcal{B} := \{H \cap H_0 \mid H \in \mathcal{A} \backslash \{H_0\}\}$, which is an arrangement in $\mathcal{H}^{H_0}$. Note that $H \cap H_0$ can be equal for different H, then we keep them both in the set, i.e. $\mathcal{B}$ is a multiset. Then clearly we have that $|\mathcal{A}| - 1 = |\mathcal{B}|$ and also $\dim\left(\bigcap_{H \in \mathcal{A}} H\right) = \dim\left(\bigcap_{H \in \mathcal{B}} H\right)$. This means that

$$\sum_{\substack{H_0 \in \mathcal{A} \subseteq \mathcal{H} \\ \mathcal{A} \text{ central with respect to } B}} (-1)^{|\mathcal{A}|} t^{\dim\left(\bigcap_{H \in \mathcal{A}} H\right)} = \sum_{\substack{\mathcal{B} \subseteq \mathcal{H}^{H_0} \\ \mathcal{B} \text{ central with respect to } B \cap H_0}} (-1) \cdot (-1)^{|\mathcal{B}|} t^{\dim\left(\bigcap_{H \in \mathcal{B}} H\right)} = -1 \cdot \chi_{\mathcal{H}^{H_0},B}(t).$$

□

Lemma 4.2.13 ([74, Lemma 2.1.])
Let $\mathcal{H}$ be a hyperplane arrangement in $\mathbb{R}^d$ and $H_0 \in \mathcal{H}$. For the triple $(\mathcal{H}, \mathcal{H}^{H_0}, \mathcal{H}_{H_0})$ we have $r(\mathcal{H}) = r(\mathcal{H}^{H_0}) + r(\mathcal{H}_{H_0})$. Let further $B \subseteq \mathbb{R}^d$ be a convex subset, then $r_B(\mathcal{H}) = r_B(\mathcal{H}^{H_0}) + r_B(\mathcal{H}_{H_0})$.

Proof. For the proof of the first recurrence we cite the proof from Stanley, [74]:
Note that $r(\mathcal{H})$ equals $r(\mathcal{H}_{H_0})$ plus the number of regions which are cut in half by H_0, we already have seen this in the proof of Theorem 4.1.11. Let R be such a region in $\mathcal{H}_{H_0}$, then $R \cap H_0$ is a region in $\mathcal{H}^{H_0}$. Now let S be a region of $\mathcal{H}^{H_0}$, then points in $\mathbb{R}^d$ near S on either side of H_0 belong to the same region R' of $\mathcal{H}_{H_0}$, since any $H \in \mathcal{H}$ separating them would intersect S. Thus R' is cut in two by H_0. We have established a bijection between regions of $\mathcal{H}_{H_0}$ cut into two by H_0 and regions of $\mathcal{H}^{H_0}$, establishing the first recurrence.

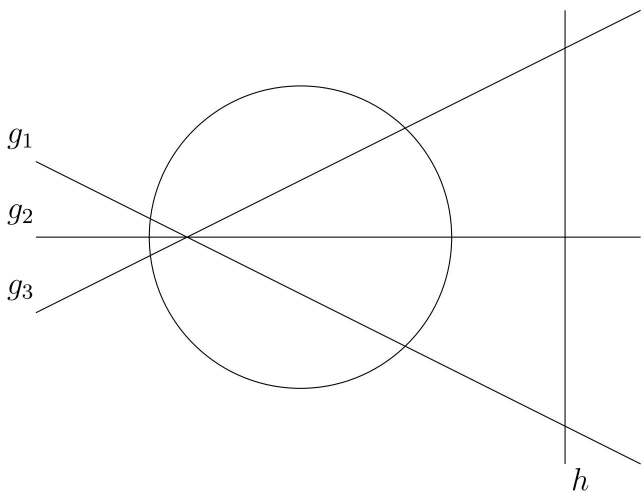


Figure 4.5: The plane is divided into 10 regions by the lines. If $g_2 = H_0$, we have $r(\mathcal{H}_{g_2}) = 7$ and $r(\mathcal{H}^{g_2}) = 3$. If $h = H_0$, we have $r(\mathcal{H}_h) = 6$ and $r(\mathcal{H}^h) = 4$.
The ball is divided into 6 regions by the lines. If $g_2 = H_0$, we have $r_B(\mathcal{H}_{g_2}) = 4$ and $r_B(\mathcal{H}^{g_2}) = 2$. If $h = H_0$, we have $r_B(\mathcal{H}_h) = 6$ and $r_B(\mathcal{H}^h) = 0$.

A similar argument also shows the second recurrence. Note that $r_B(\mathcal{H})$ equals $r_B(\mathcal{H}_{H_0})$ plus the number of regions which is cut in half by H_0, in this case this can also be zero regions if $H_0 \cap B = \emptyset$.
If $H_0 \cap B = \emptyset$ we have $r_B(\mathcal{H}) = r_B(\mathcal{H}_{H_0})$ and $r_B(\mathcal{H}^{H_0}) = 0$ so the recurrence holds.
Now assume $H_0 \cap B \neq \emptyset$. Let R be a region of $\mathcal{H}$ with $R \cap B \neq \emptyset$ and set $R' := R \cap B$. Further assume $R' \cap H_0 \neq \emptyset$, then $R' \cap H_0$ is part of a region of $\mathcal{H}^{H_0}$ which intersects B. Conversely assume S is a region of $\mathcal{H}^{H_0}$, which intersects B, and $S' := S \cap B$. Then points in $\mathbb{R}^d \cap B$ near S' on either side of H_0 belong to the same region R'' of $\mathcal{H}_{H_0}$, which also intersects B. Thus R'' is cut in two by H_0 and we again established a bijection between the regions of $\mathcal{H}_{H_0}$ in B which are cut in two parts by H_0 and regions of $\mathcal{H}^{H_0}$ in B. This shows the second recurrence. □

Remark 4.2.14 In Lemma 4.2.13 the first recurrence is a corollary of Theorem 4.1.11, but for the second we need to put in some extra efforts. The two arguments differ because one uses the structure of flats and the other uses the faces of the arrangement. Both are somewhat similar but one has to be careful not to mix up the terms.

Theorem 4.2.15 ([22, Theorem 2.8])
Let $\mathcal{H}$ be a hyperplane arrangement in $\mathbb{R}^d$, then $r(\mathcal{H}) = (-1)^d \chi_{\mathcal{H}}(-1)$. Let further $B \subseteq \mathbb{R}^d$ be a convex subset, then $r_B(\mathcal{H}) = (-1)^d \chi_{\mathcal{H},B}(-1)$.

Proof. Set $s(\mathcal{H}) := (-1)^d \chi_{\mathcal{H}}(-1)$, we show that $r(\mathcal{H}) = s(\mathcal{H})$ by showing that they possess the same recurrence and the same initial case. For the recurrence we show that $s(\mathcal{H}) = s\left(\mathcal{H}^{H_0}\right) + s\left(\mathcal{H}_{H_0}\right)$ holds for $H_0 \in \mathcal{H}$. By Theorem 4.2.12

$$\begin{aligned} s(\mathcal{H}) &= (-1)^d \chi_{\mathcal{H}}(-1) = (-1)^d \left(\chi_{\mathcal{H}_{H_0}}(-1) - \chi_{\mathcal{H}^{H_0}}(-1)\right) \\ &= (-1)^d \chi_{\mathcal{H}_{H_0}}(-1) + (-1)^{d-1} \chi_{\mathcal{H}^{H_0}}(-1) = s\left(\mathcal{H}_{H_0}\right) + s\left(\mathcal{H}^{H_0}\right). \end{aligned}$$

Therefore $r(\mathcal{H})$ and $s(\mathcal{H})$ fulfil the same recurrence by Lemma 4.2.13 and we can deduce the claim by induction other the elements in $\mathcal{H}$. The initial case is $\mathcal{H} = \emptyset$, then $r(\emptyset) = 1$ and $\chi_{\emptyset}(t) = t^d$ and therefore $s(\emptyset) = 1$.
The same argument shows the subset case, since we established both Theorem 4.2.12 and Lemma 4.2.13 also for this case. □

Now we are able to calculate the number of regions by evaluating the characteristic polynomial. The problem now is that the characteristic polynomial is quite complicated, but since we are only interested in a bound on the number of regions it is good enough.

Proposition 4.2.16 ([74, Proposition 2.4.])
Let $\mathcal{H}$ be a simple hyperplane arrangement in $\mathbb{R}^d$. Let $m := |\,\mathcal{H}\,|$, then

$$\chi_{\mathcal{H}}(t) = \sum_{i=0}^{d} (-1)^i t^{n-i} \binom{m}{i}.$$

Proof. Every $\mathcal{A} \subseteq \mathcal{H}$ with $|\mathcal{A}| \leq d$ defines a flat of dimension $d - |\mathcal{A}|$. For all $\mathcal{A} \subseteq \mathcal{H}$ with $|\mathcal{A}| > d$ we have $\bigcap_{H \in \mathcal{A}} H = \emptyset$, because the arrangement is in general position. So the coefficients in $\chi_{\mathcal{H}}(t)$ simply count the number of possible choices of $\mathcal{A} \subseteq \mathcal{H}$ with $|\mathcal{A}| \leq d$, therefore

$$\chi_{\mathcal{H}}(t) = \sum_{i=0}^{d} (-1)^i t^{n-i} \binom{m}{i}.$$

□

Corollary 4.2.17 *Let $\mathcal{H}$ be a simple hyperplane arrangement in $\mathbb{R}^d$. Let $m := |\mathcal{H}|$, then*

$$r(\mathcal{H}) = \sum_{i=0}^{d} \binom{m}{i}.$$

Remark 4.2.18 The corollary shows that in the case of a simple hyperplane arrangement the maximal number of regions is attained, compare Proposition 4.1.9 with $k = d$.

Proposition 4.2.19 (Corollary of [74, Theorem 3.10.])
Let $\mathcal{H}$ be a hyperplane arrangement in $\mathbb{R}^d$ and $B \subseteq \mathbb{R}^d$ convex, then

$$r_B(\mathcal{H}) \geq |F_B(\mathcal{H})| \geq |F_{0,B}(\mathcal{H})|.$$

Proof. We know by Theorem 4.2.11 and Theorem 4.2.15 that

$$r_B(\mathcal{H}) = (-1)^d \chi_{\mathcal{H},B}(-1) = (-1)^d \sum_{f \in F_B(\mathcal{H})} \mu(f)(-1)^{\dim(f)}.$$

And by [74, Theorem 3.10.] we know $\mu(f)(-1)^{d-\dim(f)} > 0$, so

$$r_B(\mathcal{H}) = \sum_{f \in F_B(\mathcal{H})} \mu(f)(-1)^{d+\dim(f)} = \sum_{f \in F_B(\mathcal{H})} \mu(f)(-1)^{d-\dim(f)} \geq \sum_{f \in F_B(\mathcal{H})} 1 = |F_B(\mathcal{H})|.$$

The second inequality of the claim is trivial. □

This proposition tells us that in order to determine a lower bound for the number of regions in a convex set B it is enough to count the number of intersection points, i.e. zero-dimensional flats, in B.

4.3 Versions of Beck's Theorem

In this section we will consider different versions of the Theorem of Beck, [10]. This theorem also appears in the literature under the name Erdős-Beck Theorem, since a formulation of the theorem was conjectured by Erdős. Another reason clearly is to distinguish the Erdős-Beck Theorem from Beck's monadicity Theorem. But we will simply call it Beck's Theorem.

We use some ideas of Beck's paper and translate them to our problem. In Beck's Theorem one considers a set of points in Euclidean space and asks how many hyperplanes are spanned by these points. We consider the dual problem, in which we have hyperplanes and ask how many intersection points are there. Since we work in Euclidean space we can not simply use a dualization argument from projective space. But it turns out that the argumentation is quite similar except for some details.

We start by addressing the problem in dimension two, in which we will also prove the original version of Beck's Theorem in our language and by using the Szemerédi-Trotter Theorem. Then we translate the argumentation of Beck to our problem and afterwards extend it to higher dimensions.

4.3.1 Beck's Theorem in Dimension Two

There are multiple equivalent versions of the Szemerédi-Trotter Theorem, the equivalence is not hard to prove so we leave this to the reader, and state three of them, which we use later. The first version of the theorem was proved in [76].

Theorem 4.3.1 (Szemerédi-Trotter Theorem)

i) Let P be a set of n points and L a set of m lines in $\mathbb{R}^2$, then

$$|\{(p,l) \in P \times L \mid p \in l\}| \ll n^{\frac{2}{3}} m^{\frac{2}{3}} + n + m.$$

A suitable constant is given by $c = 2.5$.

ii) Let P be a set of n points and L a set of m lines in $\mathbb{R}^2$, if $\sqrt{n} \leq m \leq \binom{n}{2}$ holds, then

$$|\{(p,l) \in P \times L \mid p \in l\}| \ll n^{\frac{2}{3}} m^{\frac{2}{3}}.$$

A suitable constant is given by $c = 2.5$.

iii) Let P be a set of n points in $\mathbb{R}^2$ and $k \geq 2$ an integer, let further $\mathcal{L}$ be the set of all lines in $\mathbb{R}^2$, then

$$\left|\left\{l \in \mathcal{L} \;\middle|\; |P \cap l| \geq k\right\}\right| \ll \frac{n^2}{k^3} + \frac{n}{k}.$$

We use the Szemerédi-Trotter Theorem to prove the first version of Beck's Theorem, [10].

Theorem 4.3.2 (Beck's Theorem version 1, [10, Theorem 3.1.])
There exist constants $\beta, \gamma \in (0,1)$ such that for every set P of n points in $\mathbb{R}^2$, one of the two alternatives hold:

(a) There is a line that contains at least βn points of P,

(b) there are at least γn^2 lines spanned by P.

Proof. Let $L_1, ..., L_t$ denote the lines which are spanned by P, further let $l_i = |L_i \cap P|$ for $i \in \{1, ..., t\}$. We consider the number of all pairs of points from P, we have two possibilities to count them, since every pair lies on exactly one line:

$$\binom{n}{2} = \sum_{i=1}^{t} \binom{l_i}{2}. \tag{i}$$

This approach is a standard double counting argument. We will use these two ways of counting the number of points. We will consider lines, which contain at least 2^k points and less than 2^{k+1} points, this partitions the set of all lines. We denote the set of all such lines by

$$A_k := \left\{ l \in \{L_1, ..., L_t\} \,\middle|\, 2^k \leq |l \cap P| < 2^{k+1} \right\}.$$

We fix a large enough constant u, and split up the sum (i) into three parts:

$$S_1 := \sum_{\substack{i \in \{1,...,t\} \\ 2^u \leq l_i < \sqrt{2n}}} \binom{l_i}{2},$$

$$S_2 := \sum_{\substack{i \in \{1,...,t\} \\ \sqrt{2n} \leq l_i < \beta n}} \binom{l_i}{2},$$

$$S_3 := \sum_{\substack{i \in \{1,...,t\} \\ 2 \leq l_i < 2^u}} \binom{l_i}{2} + \sum_{\substack{i \in \{1,...,t\} \\ \beta n \leq l_i < n}} \binom{l_i}{2}.$$

First we will bound the first sum by partitioning the interval

$$S_1 = \sum_{u \leq j} \sum_{\substack{i \in \{1,...,t\} \\ 2^j \leq l_i < 2^{j+1} \\ l_i < \sqrt{2n}}} \binom{l_i}{2} \leq \sum_{\substack{u \leq j \\ 2^j < \sqrt{2n}}} |A_j| \binom{2^{j+1}}{2}.$$

Now we can use part iii) of Theorem 4.3.1 and get

$$S_1 \ll \sum_{\substack{u \leq j \\ 2^j < \sqrt{2n}}} \left(\frac{n^2}{2^{3j}} + \frac{n}{2^j} \right) 2^j \left(2^j - 1\right).$$

Since $2^j < \sqrt{2n}$ the first term in the bracket dominates the second one asymptotically, therefore we find a suitable constant c such that

$$S_1 \leq \sum_{\substack{u \leq j \\ 2^j < \sqrt{2n}}} c \frac{n^2}{2^{2j}} \left(2^j - 1\right) \leq cn^2 \sum_{\substack{u \leq j \\ 2^j < \sqrt{2n}}} \frac{1}{2^j} < \frac{1}{4} \binom{n}{2},$$

where the last inequality holds if u is large enough. Now we also bound the second sum by the same method

$$S_2 \ll \sum_{\substack{j\in\mathbb{N} \\ \sqrt{2n}\le 2^j<\beta n}} \left(\frac{n^2}{2^{3j}}+\frac{n}{2^j}\right) 2^j\left(2^j-1\right).$$

Here the second term in the bracket dominates the first one asymptotically, so we can find a constant c such that

$$S_2 \le \sum_{\substack{j\in\mathbb{N} \\ \sqrt{2n}\le 2^j<\beta n}} cn\left(2^j-1\right) < 2n\left(2\beta n-\sqrt{2}\sqrt{n}\right) < \frac{1}{4}\binom{n}{2}.$$

The last inequality holds if β is small enough.
So we see that the third sum has to be bigger than $\frac{1}{2}\binom{n}{2}$, since

$$\binom{n}{2} = S_1+S_2+S_3 \le \frac{1}{2}\binom{n}{2}+S_3.$$

If we have a line which contains more than βn points of P we are done, since then alternative (a) holds. If we have no such line, we get

$$\frac{1}{2}\binom{n}{2} \le S_3 = \sum_{\substack{i\in\{1,\dots,t\} \\ 2\le l_i<2^u}} \binom{l_i}{2}.$$

Let A denote the set of lines which contain less than 2^u points, then

$$|A| = \sum_{\substack{i\in\{1,\dots,t\} \\ 2\le l_i<2^u}} 1 \ge \binom{2^u}{2}^{-1} \sum_{\substack{i\in\{1,\dots,t\} \\ 2\le l_i<2^u}} \binom{l_i}{2} \ge \binom{2^u}{2}^{-1}\frac{1}{2}\binom{n}{2} \gg n^2.$$

And we see that alternative (b) holds. □

Remark 4.3.3 In Beck's original proof he uses $\beta = \frac{1}{100}$, which is not the optimal choice, but should give the reader an intuition of the size of β.

The following theorem is also proved in the paper by Beck, [10]. It turns out that it is equivalent to Theorem 4.3.2 and we will therefore call it Beck's Theorem version 2 and prove it by showing the equivalence.

Theorem 4.3.4 (Beck's Theorem version 2, [10, Theorem 1.2.])
Let P be a set of n points in the affine plane $\mathbb{R}^2$. Let further $\{L_1,\dots,L_t\}$ be the set of lines spanned by P. If $\max_{1\le i\le t}|L_i\cap P| = n-x$ for some $0\le x\le n-2$ then $t\ge c\cdot n\cdot x$ for some constant c.

Equivalence of Theorem 4.3.2 and Theorem 4.3.4.

(a) We show first how Theorem 4.3.4 follows from Theorem 4.3.2.

If $n - x < \beta n$, the second alternative of Theorem 4.3.2 holds. Then there are at least γn^2 lines spanned by P. And since $n > x$, we get $t \geq \gamma n^2 > \gamma n x$.

If $n - x \geq \beta n$, the first alternative holds. Let L_i be the line which contains $n - x$ points, then there are x points outside of L_i and we choose $\lceil \beta x \rceil$ of them and call that set Z. Then we have $\lceil \beta x \rceil (n - x)$ tuples of points (p, q) with $p \in Z, q \in L_i$. Every pair yields a line but some lines can fall together, therefore we have to subtract the number of lines which could be generated by the points outside of L_i. So we get a lower bound for the number of lines by

$$\beta x \cdot (n - x) - \binom{\beta x}{2} = \beta x \left(n - x - \frac{1}{2}(\beta x - 1) \right) \geq \beta^2 x (n - x) > \beta^3 x n.$$

Here we could choose $c = \beta^3$, and see that the claim holds.

So in total we can set $c = \min\{\gamma, \beta^3\}$.

(b) Now we show the other implication. If $n - x < \beta n$ we get $t \geq cnx \geq cn^2(1 - \beta)$ and setting $\gamma := c(1 - \beta)$ shows that we are in alternative (b) of Theorem 4.3.2.

Now let $n - x \geq \beta n$. Then clearly alternative (a) of Theorem 4.3.2 holds and we are done.

□

We have seen the strategy, with which Beck proved his theorem. We should remark, that Beck did not use the Szemerédi-Trotter Theorem, but proved the bounds he needed directly, this took the most part of his paper. It is an interesting coincidence that Beck's paper and the one from Szemerédi and Trotter were published in exactly the same journal at the same time.

4.3.2 Dual Versions of Beck's Theorem in Dimension Two

After we have seen Beck's original approach we will now translate his arguments to our case. We call it the local dual of Beck's Theorem, since we are only interested in the behaviour inside a bounded region and we dualize the statement, in the sense that we interchange points with lines and lines with points. We will now state the theorem and prove it at the end of the subsection.

Theorem 4.3.5 (Local dual of Beck's Theorem)
There exists a constant c such that for all hyperplane arrangements $\mathcal{H} = \{H_1, ..., H_n\}$ in $\mathbb{R}^2$ and $B \subseteq \mathbb{R}^2$, where $\mathcal{H}$ consists of two families $\mathcal{A}_1$ and $\mathcal{A}_2$ such that each pair $(f, g) \in \mathcal{A}_1 \times \mathcal{A}_2$ intersects inside B and $|\mathcal{A}_1| = |\mathcal{A}_2| = \frac{n}{2}$, one of the following two cases holds:

(a) There is a point $p \in B$ such that $|\{H \in \mathcal{H} \mid p \in H\}| > \frac{n}{100}$.

(b) The number of intersection points in B, i.e. $|F_{0,B}(\mathcal{H})|$, is bigger than $c \cdot n^2$.

Definition 4.3.6 For a hyperplane arrangement $\mathcal{H}$ let

$$t(\mathcal{H}, k) := |\{p \in F_0(\mathcal{H}) \mid a(p) \geq k\}|,$$
$$t^*(\mathcal{H}, k) := |\{p \in F_0(\mathcal{H}) \mid k \leq a(p) < 2k\}|.$$

Further we can maximize t and t^* over all possible arrangements:

$$t(n, k) := \max_{|\mathcal{H}|=n} t(\mathcal{H}, k),$$
$$t^*(n, k) := \max_{|\mathcal{H}|=n} t^*(\mathcal{H}, k).$$

Clearly $t^*(n, k) \leq t(n, k)$.

Now we establish the bounds we need for Theorem 4.3.5. The ones we show in Lemma 4.3.7 are the ones Beck also used and the lemma is the dual of [10, Lemma 2.1.]. The upper bound shown in Lemma 4.3.8 is used instead of the one Beck shows in [10, Theorem 1.5.], he only states that Szemerédi and Trotter have shown a similar bound.

Lemma 4.3.7 *For a hyperplane arrangement $\mathcal{H}$ in $\mathbb{R}^2$ we have*

$$t(n, k) \leq \frac{n(n-1)}{k(k-1)}, \quad \text{for all } 2 \leq k \leq n, \tag{1}$$
$$t(n, k) < \frac{2n}{k}, \quad \text{for all } \sqrt{2n} < k \leq n. \tag{2}$$

Proof. For the first formula we consider the number of pairs of lines. On the one side we consider all possible pairs of lines and on the other side the pairs of lines through points in which at least k lines intersect:

$$t(n, k) \cdot \binom{k}{2} \leq \binom{n}{2}.$$

For the second inequality the points in which at least k lines intersect are denoted by $p_1, ..., p_t$. Assume there is an $l \in \{0, ..., k-1\}$ such that $t = \frac{2n+l}{k} \in \mathbb{N}$. Then $t < \sqrt{2n} + \frac{l}{k}$, since $\sqrt{2n} < k$. Remember that $S(p_i)$ is the set of all lines through the point p_i. Notice that $|S(p_i)| \geq k$ and $|S(p_i) \cap S(p_j)| \leq 1$ for $i \neq j$, since two points are connected by exactly one line.

$$
\begin{aligned}
n = |\,\mathcal{H}\,| &\geq \left|\bigcup_{i=1}^{t} S(p_i)\right| \geq \sum_{i=1}^{t} |S(p_i)| - \sum_{1\leq i<j\leq t} |S(p_i)\cap S(p_j)| \\
&\geq \sum_{i=1}^{t} k - \sum_{1\leq i<j\leq t} 1 = tk - \frac{1}{2}t(t-1) > 2n + l - \frac{1}{2}\left(\sqrt{2n}+\frac{l}{k}\right)\left(\sqrt{2n}+\underbrace{\frac{l}{k}-1}_{<0}\right) \\
&> 2n + l - n - \sqrt{\frac{n}{2}\frac{l}{k}} = n + l\left(1-\frac{\sqrt{n}}{\sqrt{2}k}\right) > n + l\left(1-\frac{1}{2}\right) \geq n.
\end{aligned}
$$

This is a contradiction, so $t = \lceil \frac{2n}{k} \rceil$ cannot hold and also $t > \frac{2n}{k}$ is not possible since we can simply ignore some points to get the same contradiction. □

The next Lemma is a corollary of Theorem 4.3.1.

Lemma 4.3.8 ([76, Theorem 2])
There is some constant $\beta > 0$ such that for every hyperplane arrangement $\mathcal{H}$ in $\mathbb{R}^2$ with $|\,\mathcal{H}\,| = n$:

$$
t(n,k) < \beta\frac{n^2}{k^3}, \quad \forall 3 \leq k \leq \sqrt{n}.
$$

Proof. Let $t = \left\lceil \frac{c^3n^2}{k^3} \right\rceil = \frac{c^3n^2+l}{k^3} \in \mathbb{N}$, where $l \in [0, k^3)$ and $c = 2.5$. Assume that there are t points with $a(p) \geq k$.
Then

$$
\sqrt{t} = n\sqrt{\frac{c^3}{k^3}+\frac{l}{n^2k^3}} \leq n\sqrt{\frac{c^3}{k^3}+\frac{k^3}{n^2k^3}} < n\sqrt{\frac{2.5^3}{3^3}+\frac{1}{n^2}} < n,
$$

since $n > 3$. Further

$$
\begin{aligned}
\binom{t}{2} &= \frac{1}{2}\frac{c^3n^2+l}{k^3}\left(\frac{c^3n^2+l}{k^3}-1\right) \geq \frac{1}{2}\left(c^3\sqrt{n}+\frac{l}{n^{\frac{3}{2}}}\right)\left(c^3\sqrt{n}+\frac{l}{n^{\frac{3}{2}}}-1\right) \\
&\geq \frac{1}{2}c^3\sqrt{n}(c^3\sqrt{n}-1) > n,
\end{aligned}
$$

since $c = 2.5$ and $n > 3$. Therefore we can use version ii) of the Szemerédi-Trotter Theorem: There is a constant c such that the number of incidences is lower than $cn^{2/3}m^{2/3}$ and we know that $c = 2.5$ works. The t points induce $t \cdot k$ incidences, but

$$
t \cdot k = \frac{c^3n^2+l}{k^2} \not< cn^{2/3}t^{2/3},
$$

a contradiction to the theorem. Therefore $t < \lceil \frac{c^3n^2}{k^3} \rceil$. We see that $\beta = 2.5^3$ is a possible choice. □

Proof of Theorem 4.3.5. We count the number of pairs of lines, we get

$$\binom{n}{2} \geq \sum_{p \in F_{0,B}(\mathcal{H})} \binom{a(p)}{2} \geq |\,\mathcal{A}_1\,| \cdot |\,\mathcal{A}_2\,| = \frac{1}{4}n^2.$$

On the left side we counted all the possible options, in the middle we counted the pairs of lines which intersect inside B and on the right we counted the pairs of lines from the two families, since we know that they intersect in B. We will split up the sum into three parts:

$$S_1 := \sum_{\substack{p \in F_{0,B}(\mathcal{H}) \\ 2^k \leq a(p) < \sqrt{n}}} \binom{a(p)}{2}, \quad S_2 := \sum_{\substack{p \in F_{0,B}(\mathcal{H}) \\ \sqrt{n} \leq a(p) < \frac{n}{100}}} \binom{a(p)}{2},$$

$$S_3 := \sum_{\substack{p \in F_{0,B}(\mathcal{H}) \\ 2 \leq a(p) < 2^k}} \binom{a(p)}{2} + \sum_{\substack{p \in F_{0,B}(\mathcal{H}) \\ \frac{n}{100} \leq a(p) \leq n}} \binom{a(p)}{2},$$

where k is constant depending on β which we is chosen so that the following bound on S_1 holds. Now we will bound S_1 and S_2. We start with S_1 using Lemma 4.3.8:

$$S_1 = \sum_{l \geq k} \sum_{\substack{2^l \leq a(p) < 2^{l+1} \\ a(p) < \sqrt{n}}} \binom{a(p)}{2} \leq \sum_{\substack{l \geq k \\ 2^l < \sqrt{n}}} t^*\left(n, 2^l\right) \binom{2^{l+1}}{2} = \sum_{\substack{l \geq k \\ 2^l < \sqrt{n}}} t^*\left(n, 2^l\right) 2^l \left(2^{l+1} - 1\right)$$

$$\leq \sum_{\substack{l \geq k \\ 2^l < \sqrt{n}}} \beta \frac{n^2}{2^{3l}} 2^l \left(2^{l+1} - 1\right) \leq 2\beta n^2 \sum_{l \geq k} \frac{1}{2^l} = \frac{4\beta}{2^k} n^2 \leq \frac{1}{8} \binom{n}{2}.$$

Here we have chosen k in dependence on β, since we know that one possible choice for β is 2.5^3 we get $k = 10$ as an option if $n \geq 2$.
For the next sum we use Lemma 4.3.7 and Lemma 4.3.8:

$$\begin{aligned}
S_2 &= \sum_{l \geq 0} \sum_{\substack{2^l \sqrt{n} \leq a(p) \leq 2^{l+1} \sqrt{n} \\ a(p) < \frac{n}{100}}} \binom{a(p)}{2} \leq \sum_{\substack{l \geq 0 \\ 2^{l+1} \sqrt{n} \leq \frac{n}{100}}} t\left(n, 2^l \sqrt{n}\right) \binom{2^{l+1}\sqrt{n}}{2} \\
&= t\left(n, \sqrt{n}\right) \binom{2\sqrt{n}}{2} + \sum_{\substack{l \geq 1 \\ 2^{l+1} \sqrt{n} \leq \frac{n}{100}}} t\big(n, \underbrace{2^l \sqrt{n}}_{> \sqrt{2n}}\big) \binom{2^{l+1}\sqrt{n}}{2} \\
&< \beta \frac{n^2}{n^{3/2}} \sqrt{n} \left(2\sqrt{n} - 1\right) + \sum_{\substack{l \geq 1 \\ 2^{l+1} \sqrt{n} \leq \frac{n}{100}}} \frac{2n}{2^l \sqrt{n}} 2^l \sqrt{n} \left(2^{l+1}\sqrt{n} - 1\right) \\
&< 2\beta n^{3/2} + 4n^{3/2} \sum_{\substack{l \geq 1 \\ 2^{l+1} \leq \frac{\sqrt{n}}{100}}} 2^l = 2\beta n^{3/2} + 4n^{3/2} \left(\frac{\sqrt{n}}{50} - 2\right) \\
&= \frac{2}{25} n^2 + (2\beta - 8) n^{3/2} < \frac{1}{4} \binom{n}{2}
\end{aligned}$$

Combining the two results we get a lower bound for S_3

$$S_3 \geq |\mathcal{A}_1| \cdot |\mathcal{A}_2| - \frac{1}{4}\binom{n}{2} - \frac{1}{8}\binom{n}{2} \geq \frac{1}{16}n^2.$$

So now assume that condition (a) of the theorem does not hold, then

$$|F_{0,B}(\mathcal{H})| \geq \sum_{\substack{p \in F_{0,B} \\ 2 \leq a(p) < 2^k}} 1 \geq \binom{2^k}{2}^{-1} \sum_{\substack{p \in F_{0,B} \\ 2 \leq a(p) < 2^k}} \binom{a(p)}{2} \geq \binom{2^k}{2}^{-1} \frac{1}{16}n^2.$$

Since we have seen that $k = 10$ is a possible choice the constant would be $c = \frac{1}{8380416}$. □

Remark 4.3.9 In condition (a) the constant $\frac{1}{100}$ is by no means optimal, but since for us the constants play an insignificant role we stick to the original constant used by Beck.

We have even proved a stronger theorem which we will state as a corollary.

Corollary 4.3.10 (Strong version of the local dual of Beck's Theorem)
There exists a constant c such that for all hyperplane arrangements $\mathcal{H} = \{H_1, ..., H_n\}$ in $\mathbb{R}^2$ and $B \subseteq \mathbb{R}^2$, where $\mathcal{H}$ consists of two families $\mathcal{A}_1$ and $\mathcal{A}_2$, such that each pair $(f, g) \in \mathcal{A}_1 \times \mathcal{A}_2$ intersects inside B and $|\mathcal{A}_1| = |\mathcal{A}_2| = \frac{n}{2}$, one of the following two cases holds:

(a) There is a point $p \in B$ such that $|\{H \in \mathcal{H} \mid p \in H\}| > \frac{n}{100}$.

(b) There exists a constant $k \geq 1$ such that the number of intersection points in B in which at most 2^k lines intersect is bigger than $c \cdot n^2$. Thus

$$\left|\left\{p \in F_{0,B}(\mathcal{H}) \,\middle|\, 2 \leq a(p) \leq 2^k\right\}\right| \geq c \cdot n^2.$$

4.3.3 Beck's Theorem in Higher Dimensions

We want to generalise the argument to higher dimensions, unfortunately the Szemerédi-Trotter Theorem does not generalise nicely. The higher dimensional version, which was proved by Agarwal and Aronov, [1], is given by:

Theorem 4.3.11 (Higher dimensional Szemerédi-Trotter Theorem, [1])
Given a set of n points and m hyperplanes, which are spanned by the points, in $\mathbb{R}^d$, the number of incidences is bounded from above by

$$\mathcal{O}\left(m^{2/3}n^{d/3} + n^{d-1}\right).$$

Or equivalently, the number of hyperplanes which contain k or more points is bounded from above by

$$\mathcal{O}\left(\frac{n^d}{k^3}+\frac{n^{d-1}}{k}\right).$$

But this is not good enough for our proof, since in terms of the proof of Theorem 4.3.5, the S_1 part of the sum is not controllable any more.

Another generalization of the Szemerédi-Trotter Theorem is given by Elekes and Tóth, [23], but this version uses the terminology of γ-saturated flats, which we will now recall to state the theorem.

Definition 4.3.12 Let $\mathcal{H}$ be a hyperplane arrangement in $\mathbb{R}^d$. Then a flat $f \in F_r(\mathcal{H})$ is called *γ-saturated*, $\gamma > 0$, if $\left(\bigcup_{p \in F_0(\mathcal{H})} p\right) \cap \left(\bigcup_{f \in F_r(\mathcal{H})} f\right)$ span at least $\gamma \cdot \left|f \cap \left(\bigcup_{p \in F_0(\mathcal{H})} p\right)\right|^r$ distinct $r-1$ dimensional flats in f.

Theorem 4.3.13 ([23, Theorem 2.3])
For any $d \geq 2$ and $\gamma > 0$, there is a constant $c(d,\gamma)$ such that: For every point set P of n points in $\mathbb{R}^d$, the number of hyperplanes which contain more than k points of P and are γ-saturated is at most

$$c(d,\gamma)\left(\frac{n^d}{k^{d+1}}+\frac{n^{d-1}}{k^{d-1}}\right),$$

where the hyperplane arrangement is given by all hyperplanes spanned by P.

It is possible to prove our desired theorem with this bound, but we had to assume that our hyperplanes are γ-saturated for some γ. But this condition is hard to show.

What we are able to do is to give an inductive proof of a higher dimensional dual version of Beck's Theorem, this is again quite similar to the approach Beck took to generalise his own theorem to higher dimensions, [10, Theorem 5.4]

Theorem 4.3.14 (Higher dimensional local dual of Beck's Theorem)
There exists a constant c_d such that for all hyperplane arrangements $\mathcal{H}$ in $\mathbb{R}^d$ and every $B \subseteq \mathbb{R}^d$ convex, where $\mathcal{H}$ consist of d families $\mathcal{A}_1, ..., \mathcal{A}_d$ with $|\,\mathcal{A}_i\,| = \frac{n}{d}$ and such that for all $(f_1, ..., f_d) \in \mathcal{A}_1 \times ... \times \mathcal{A}_d$ we have $B \cap \bigcap_{i=1}^d f_i = \{p\}$ for some point $p \in B$ and additionally at most $c_d \cdot |\,\mathcal{A}_i\,|$ hyperplanes from $\mathcal{A}_i$ can intersect in one point, with $0 < c_d < \frac{1}{100}$, the number of intersection points in B, i.e. $|F_{0,B}(\mathcal{H})|$, exceeds $c_d \cdot n^d$.

Proof. The idea of the proof is that for one fixed family $\mathcal{A}_i$ of hyperplanes, the other families induce a hyperplane arrangement in all $H \in \mathcal{A}_i$, thus we can conclude by induction, since the case $d = 2$ is already completed by Theorem 4.3.5. Notice that in Corollary 4.3.10 we even proved the stronger statement that

$$\left|\left\{p \in F_{0,B}(\mathcal{H}) \,\middle|\, 2 \le a(p) < 2^k\right\}\right| \ge c_2 \cdot n^2$$

if for all $p \in B$ we have $a(p) < \frac{n}{100}$. It is guaranteed by the assumption that at most $c \cdot |\,\mathcal{A}_i\,|$ hyperplanes from $\mathcal{A}_i$ can intersect in one point and $c < \frac{1}{100}$ that $a(p) < \frac{n}{100}$.

Now consider the family $\mathcal{A}_1$, we are interested in the $d-1$ dimensional arrangement, which is induced on the hyperplanes $H \in \mathcal{A}_1$. Notice that for $H_i \in \mathcal{A}_i$, $H_j \in \mathcal{A}_j$ and $H_k \in \mathcal{A}_k$, with $i \ne j \ne k \ne i$, we have $H_i \cap H_j \ne H_i \cap H_k$ since otherwise we get a contradiction to the assumption that $B \cap \bigcap_{i=1}^{d} f_i = \{p\}$ for all $(f_1, ..., f_d) \in \mathcal{A}_1 \times ... \times \mathcal{A}_d$. So the different families induce different $(d-2)$-hyperplanes on the hyperplanes of $\mathcal{A}_1$. For $H \in \mathcal{A}_1$ set

$$\mathcal{H}^{H*} := \{H \cap f \mid f \in \mathcal{A}_2 \cup ... \cup \mathcal{A}_d\},$$

here $H \cap f \ne \emptyset$ holds for all f by the assumption on the intersection behaviour. Now we prove the following claim.

Claim: $\left|\mathcal{H}^{H*}\right| > \delta \cdot n$, for some constant δ, and at least $\varepsilon \cdot \frac{n}{d}$ hyperplanes $H \in \mathcal{A}_1$, for some constant $\varepsilon > 0$.

We show this by considering two families and show that the one induces enough $(d-2)$-dimensional planes on the hyperplanes of the second one. In order to do so let P be a generic 2-dimensional plane in $\mathbb{R}^d$, i.e. $P \cap H$ is one-dimensional for all $H \in \mathcal{A}_1 \cup \mathcal{A}_2$ and all the $P \cap H$ are distinct for different H. And $P \cap f$ is a point for all $f = H \cap K$ with $H \in \mathcal{A}_1$, $K \in \mathcal{A}_2$, which are also distinct if the $K \cap H$ are distinct. So each hyperplane corresponds to a line and each $d-2$ dimensional flat corresponds to a point. The intersection behaviour for the lines clearly fulfils the assumptions in Theorem 4.3.5. So we can apply the theorem and get $c \cdot \left(\frac{n}{d}\right)^2$ intersection points and therefore $c \cdot \left(\frac{n}{d}\right)^2$ induced flats. Since each hyperplane can carry at most $\frac{n}{d}$ induced flats we see that the flats have to spread out such that the claim holds.

Now we can do the induction. Denote by $\tilde{\mathcal{A}}_1$ the set of hyperplanes from $\mathcal{A}_1$ for which the claim holds. It is clear that the intersection behaviour of the different families also holds in the $(d-1)$ dimensional arrangement induced on the hyperplanes in $\tilde{\mathcal{A}}_1$. Also assume that the stronger statement from Corollary 4.3.10 is proved for all dimensions $l \ge 2$ up to $d-1$, where $k \ge 1$ is a constant, i.e.

$$\left|\left\{p \in F_{0,B}(\mathcal{H}) \,\middle|\, l \le a(p) < l^k\right\}\right| \ge c_l \cdot n^l.$$

Then we get the following inequality

$$|F_{0,B}(\mathcal{H})| \cdot \binom{d^k}{d} > \sum_{\substack{p \in F_{0,B}(\mathcal{H}) \\ d \le a(p) < d^k}} \binom{a(p)}{d} \ge \sum_{H \in \mathcal{A}_1} \sum_{\substack{p \in F_{0,B}(\mathcal{H}^{H*}) \\ d-1 \le a_*(p) < (d-1)^k}} \binom{a_*(p)}{d-1}.$$

For the last inequality notice that $a_*(p)$ now only counts the hyperplanes in $\mathcal{H}^{H*}$ and we only have to take $d-1$ out of them since we fixed the choice $H \in \mathcal{A}_1$. Now further by the induction assumption

$$\begin{aligned}
\sum_{H\in\mathcal{A}_1} \sum_{\substack{p\in F_{0,B}\left(\mathcal{H}^{H*}\right)\\ d-1\le a_*(p)<(d-1)^k}} \binom{a_*(p)}{d-1} &\geq \sum_{H\in\tilde{\mathcal{A}}_1} \sum_{\substack{p\in F_{0,B}(\mathcal{H}^{H*})\\ d-1\le a_*(p)<(d-1)^k}} \binom{a_*(p)}{d-1}\\
&\geq \sum_{H\in\tilde{\mathcal{A}}_1} \left|\left\{p\in F_{0,B}\left(\mathcal{H}^{H*}\right)\,\middle|\, d-1\le a_*(p)<(d-1)^k\right\}\right|\\
&\geq \sum_{H\in\tilde{\mathcal{A}}_1} c_{d-1}\cdot\delta^{d-1}n^{d-1} \geq \varepsilon\frac{n}{d}c_{d-1}\delta^{d-1}n^{d-1} = \frac{\varepsilon c_{d-1}}{d}\cdot\delta^{d-1}n^d.
\end{aligned}$$

This finally yields

$$\left|\left\{F_{0,B}(\mathcal{H})\,\middle|\, d\le a(p)<d^k\right\}\right| \geq \binom{d^k}{d}^{-1}\frac{\varepsilon c_{d-1}}{d}\cdot\delta^{d-1}n^d =: c_d n^d.$$

□

What we have seen is that under the assumptions of Theorem 4.3.14 we get at least $c_d\cdot n^d$ intersection points inside B. Further by Proposition 4.2.19 we know that this is a lower bound for the number of regions with respect to B. This will enable us to bound the number of the acceptance domains, if we consider a ball B inside the window W of the model set.

4.4 Extension to Hyperbolic Hyperplanes

In this section we want to generalise our combinatorial arguments to hyperbolic n-space $\mathcal{H}^n$. This is easy since we can consider the projective model, which in dimension two is also called the Beltrami–Klein model. In this model $\mathcal{H}^n$ is to be considered as an n-dimensional unit ball in $\mathbb{R}^n$ and the hyperplanes in $\mathcal{H}^n$ are intersections of the unit ball with hyperplanes from $\mathbb{R}^n$, [65, Theorem 6.1.4.]. This is exactly the set-up which we considered above. See also Figure 4.6 for a visualisation of the two-dimensional case.

Further since the geodesics are straight lines in this model we see that sets in $\mathcal{H}^n$ are convex if and only if they are convex in $\mathbb{R}^n$, so we can also consider convex subsets of $\mathcal{H}^n$ in the way we did above.
This shows that our combinatorial arguments from above hold in the hyperbolic setting, especially Theorem 4.3.14. This does not change if we consider another model of hyperbolic space, for example the Poincaré upper half-plane model or the Poincaré disc model, which may be the more common models for hyperbolic space. We will give a short introduction to hyperbolic spaces in Section 7.1.

For our argument we could also choose the hyperboloid model, since also in this model a hyperbolic hyperplane is just an intersection of the hyperboloid with a hyperplane from

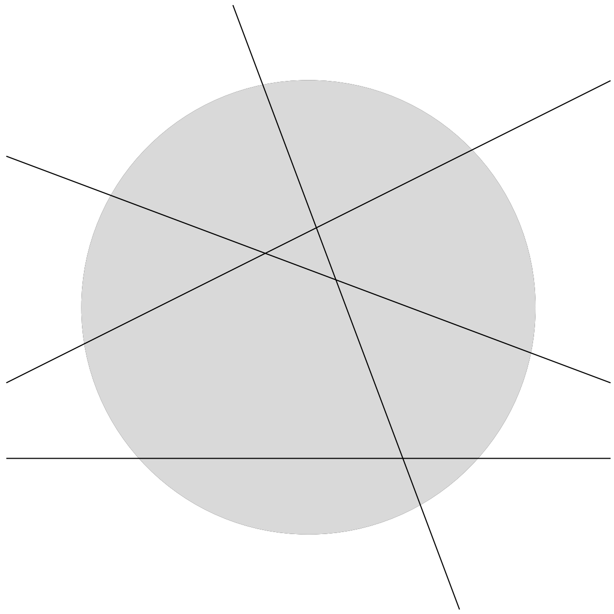


Figure 4.6: In this picture we see the projective model of the hyperbolic plane embedded in the euclidean plane. The hyperbolic plane is the inside of the circle, the hyperbolic lines are the intersection of this interior with the euclidean lines, therefore the intersection behaviour can be described by the euclidean intersection behaviour of straight lines.

the Euclidean space containing the hyperboloid. For a connection of these to models see Figure 4.7 and the correspond paragraph in the book by Ratcliffe, [65, §6.1].

We will use the projective perspective in Chapter 7 to establish results on the complexity function of model sets in hyperbolic space.

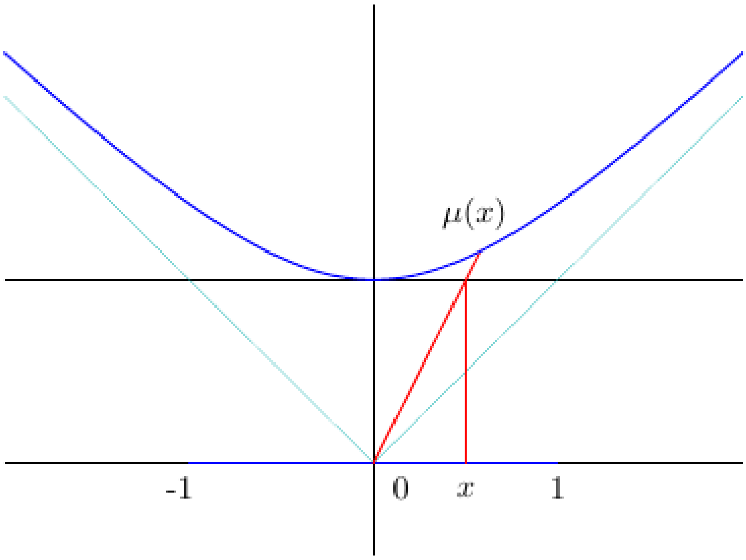


Figure 4.7: This figures shows the connection between the hyperbola and the projective model via gnomic projection. This picture is taken from [65, Figure 6.1.1].

5 Homogeneous Lie Groups

In this chapter we review the basic concepts of homogeneous Lie groups, which will be the spaces in which our discrete structures will live. We will state some known properties, see for example the book by Fischer and Ruzhansky, [24], or the preprint by Beckus, Hartnick and Pogorzelski, [11], which is closer to our situation, for a short introduction.

There are essentially two reasons why Homogeneous Lie groups are suitable for our purpose. Firstly their exists a natural class of metric on them, which are all quasi isometric and therefore deliver the same complexity. And Secondly we can scale things, in this scaling the homogeneous dimension plays an important role.

In the first section we will explain the basic properties of homogeneous Lie groups, the underlying structure and fix some notations. In the second section we will further analyse the group law of homogeneous Lie groups and consider their action on the Lie algebra. In the third section we will give a detailed example. In the fourth section we will briefly analyse lattices in homogeneous groups, which will mainly tell us that we do not have to worry about uniformity. And in the fifth section we review some Ergodic theoretic results, which we will need later.

5.1 Basic Properties

In the following and the rest of the thesis, except stated otherwise, all Lie groups will be assumed to be connected, simply connected, real and finite dimensional.

Definition 5.1.1 ([24, Definition 1.6.1.])

(a) Let $\mathfrak{g}$ be a Lie algebra. If one of the following two conditions hold we say $\mathfrak{g}$ is *nilpotent*.

  (i) There is a $k \in \mathbb{N}$ such that for all $X \in \mathfrak{g}$ we have $\mathrm{ad}_X^k = 0$.

  (ii) The lower central series of $\mathfrak{g}$ terminates at 0 in finitely many steps, i.e. if $\mathfrak{g}_{(1)} := \mathfrak{g}$ and $\mathfrak{g}_{(j)} := [\mathfrak{g}, \mathfrak{g}_{(j-1)}]$, there exists a k such that $\mathfrak{g}_{(k)} = \{0\}$.

(b) A Lie group G is said to be nilpotent if its Lie algebra is *nilpotent*.

(c) A Lie algebra is *s-step nilpotent* if $\mathfrak{g}_{(s)} = \{0\}$ and $\mathfrak{g}_{(s-1)} \neq \{0\}$.

(d) A Lie group is *s-step nilpotent* if its Lie algebra is s-step nilpotent.

Proposition 5.1.2 ([24, Proposition 1.6.6.])
Let G be a connected, simply connected, nilpotent Lie group with Lie algebra $\mathfrak{g}$, then:

(a) *The exponential map,* exp, *is a global diffeomorphism from* $\mathfrak{g}$ *to* G.

(b) *If* G *is identified with* $\mathfrak{g}$ *via* exp, *the group law* $(x, y) \mapsto xy$ *is a polynomial map.*

(c) *If* $d\lambda_{\mathfrak{g}}$ *denotes a Lebesgue measure on the vector space* $\mathfrak{g}$, *then* $d\lambda_{\mathfrak{g}} \circ \exp^{-1}$ *is a bi-invariant Haar measure on* G.

Since our groups are always real and finite dimensional we can view the set underlying G as an n-dimensional vector space, by part (a) of the proposition. Here n is the dimension of G, which we will denote by $\dim(G)$.

Definition 5.1.3 ([24, Definition 3.1.7.])
A family of *dilations* of a Lie algebra $\mathfrak{g}$ is a family of linear mappings $\{D_r, r > 0\}$ from $\mathfrak{g}$ to itself which satisfies:

(a) The mappings are of the form

$$D_r = \exp\big(A \ln(r)\big) = \sum_{l=0}^{\infty} \frac{1}{l!} \big(\ln(r)A\big)^l,$$

where A is a diagonalisable linear operator on $\mathfrak{g}$ with positive eigenvalues, exp denotes the exponential map of operators and $\ln(r)$ the natural logarithm of $r > 0$.

(b) Each D_r is a morphism of the Lie algebra $\mathfrak{g}$, that is, a linear mapping from $\mathfrak{g}$ to itself, which respects the Lie bracket, i.e.

$$\forall X, Y \in \mathfrak{g}, r > 0 : [D_r X, D_r Y] = D_r[X, Y].$$

Definition 5.1.4 A *homogeneous* Lie group is a connected simply connected Lie group, whose Lie algebra is equipped with a fixed family of dilations.

Remark 5.1.5 The eigenvalues of the operator A in Definition 5.1.3 are called *weights* and are denoted by $\nu_1, ..., \nu_n$. Using a basis of eigenvectors for A the matrices describing A and D_r have the following form:

$$A = \begin{pmatrix} \nu_1 & & & \\ & \nu_2 & & \\ & & \ddots & \\ & & & \nu_n \end{pmatrix} \text{ and } D_r = \begin{pmatrix} r^{\nu_1} & & & \\ & r^{\nu_2} & & \\ & & \ddots & \\ & & & r^{\nu_n} \end{pmatrix}.$$

Proposition 5.1.6 ([24, Proposition 3.1.10]) *A homogeneous Lie group is nilpotent.*

Proof. Let D_r be the family of dilations. We denote by $W_{\nu_i} \subseteq \mathfrak{g}$ the eigenspace of A corresponding to the eigenvalue ν_i. If $\nu \in \mathbb{R}$ but ν is no eigenvalue of A we set $W_\nu := \{0\}$.

So we have that $D_r X = r^\nu X$ for $X \in W_\nu$. And if $X \in W_\nu$ and $Y \in W_{\nu'}$ we have

$$D_r[X,Y] = [D_r X, D_r Y] = r^{\nu+\nu'}[X,Y]$$

and hence

$$[W_\nu, W_{\nu'}] \subseteq W_{\nu+\nu'}.$$

This means that the lower central series will terminate, since there are only finitely many different weights. $\square$

Remark 5.1.7 By a theorem of Siebert, [73], there is an even stronger version of Proposition 5.1.6, which tells us that every connected lcsc group, which is equipped with dilations, is a simply connected nilpotent Lie group. So our restriction to connected simply connected Lie groups in the definition of homogeneity is not a serious restriction and follows from the desire to have a dilation structure.

Remark 5.1.8 By Proposition 5.1.2 and Proposition 5.1.6 we are able to transport the dilations from $\mathfrak{g}$ to G in the following way: The maps $\exp \circ D_r \circ \exp^{-1}$, $r > 0$, are automorphisms of the group G. These maps are also called dilations and we will also denote them by D_r, this fits together with our notation later on, which ignores the exponential map.

Definition 5.1.9 Let G be a homogeneous Lie group with a fixed family of dilations given by $D_r = \exp(A \ln r)$, then the *homogeneous dimension* of G is defined as $\operatorname{homdim}(G) = \operatorname{tr}(A)$. The eigenvalues of A are denoted by ν_i and they are sorted by size, i.e. $0 < \nu_1 \leq \nu_2 \leq ... \leq \nu_n$.

Remark 5.1.10 The dilation family of a homogeneous Lie group G is not unique. For a given family of dilations D_r we can define a new family $\tilde{D}(r) := D_{r^\alpha} = \exp(\alpha A \ln r)$ for any $\alpha > 0$.
Changing the dilation family also changes the weights and therefore the homogeneous dimension of G. There is per se no canonical choice, a rather natural assumption would be to normalize $\nu_1 = 1$.

Convention 5.1.11 We will always assume to have a fixed family of dilations on our homogeneous Lie group, i.e. $G = (G, D_r)$. This is meant additionally to Convention 2.1.15, so $G = (G, d, D_r)$.

We have seen that we can identify G with $\mathfrak{g}$ via exp. We want to further explain how we do this and what this means.

Convention 5.1.12 (Exponential coordinates)
Let G be a connected simply connected nilpotent Lie group, then by Proposition 5.1.2 we know that exp is a global diffeomorphism. This means that we can identify the sets underlying G and $\mathfrak{g}$. On $\mathfrak{g}$ we define the group operation

$$X * Y := \log(\exp(X)\exp(Y)).$$

This can be evaluated with the Baker-Campbell-Hausdorff (BCH) Formula, which tells us that

$$X * Y = X + \sum_{\substack{k,m\geq 0\\ p_i+q_i\geq 0\\ i\in\{0,\dots,k\}}} (-1)^k \frac{\mathrm{ad}_X{}^{p_1}\circ \mathrm{ad}_Y{}^{q_1}\circ \dots \circ \mathrm{ad}_X{}^{p_k}\circ \mathrm{ad}_Y{}^{q_k}\circ \mathrm{ad}_X{}^{m}}{(k+1)(q_1+\dots+q_k+1)\cdot p_1!\cdot q_1!\cdot\dots\cdot p_k!\cdot q_k!\cdot m!}(Y).$$

This sum looks complicated at first sight, but the nilpotency tells us it is a finite sum. Since if G is n-step nilpotent all summands with $\sum p_i + \sum q_j + m > n$ are zero. The first few terms of the sum look like

$$X * Y = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\left([X,[X,Y]] - [Y,[X,Y]]\right) - \frac{1}{24}[Y,[X,[X,Y]]]\dots .$$

The group $(\mathfrak{g}, *)$ is isomorphic to the original group G, so we choose to work with this group instead of the original one. The group law $*$ is called the *Baker-Campbell-Hausdorff multiplication*. The whole approach is called *working in exponential coordinates.*
We use the convention that we do not distinguish the elements of the Lie group and the Lie algebra in our notation, so $X = \exp(X)$, but it should be clear whether we work in the group or the algebra.

What will be important for us is the interplay between the dilation structure, the metric on G and the Haar measure. We can find a right-invariant metric on G by Strubles theorem, [75], therefore we have a metric $d : G\times G \to \mathbb{R}_{\geq 0}$ such that

$$\forall x, y, z \in G : d(xz, yz) = d(x, y).$$

Such a metric induces a norm by $|\cdot| := d(e,\cdot)$. We will now address the problem from the other side, if we are given a norm we can define a right-invariant metric by $d(x, y) := |xy^{-1}|$.

Definition 5.1.13 ([24, Definition 3.1.33.])
Let G be a group with a family of dilations. A *homogeneous quasi-norm* is a continuous non-negative function $G \to [0,\infty], x \mapsto |x|$ satisfying

(a) $|x^{-1}| = |x|$,

(b) $|D_r(x)| = r\cdot|x|$ for all $r > 0$,

(c) $|x| = 0$ if and only if $x = e$.

It is called a *homogeneous norm* if additionally

(d) for all $x, y \in G$ it is $|xy| \leq |x| + |y|$.

The following theorem by Hebisch and Sikora, [38], tells us that we can always find a homogeneous norm on a homogeneous Lie group. See also [24, Theorem 3.1.39.] or [11, Proposition 3.27].

Lemma 5.1.14 *If G is a homogeneous Lie group,*

(a) then there exists a homogeneous norm $|\cdot|$ on G.

(b) If $|\cdot|$ is a homogeneous quasi-norm on G, then the open balls with respect to $|\cdot|$ are precompact and generate the topology, i.e. $|\cdot|$ is compatible.

Lemma 5.1.15 ([24, Proposition 3.1.35.])
Any two homogeneous quasi-norms $|\cdot|$ and $|\cdot|'$ on G are mutually equivalent, in the sense that there exists $a, b > 0$ such that for all $g \in G$ it is $a|g|' \leq |g| \leq b|g|'$.

By Theorem 2.1.27 we get the following.

Corollary 5.1.16 *Any two homogeneous quasi-norms on G yield the same asymptotic behaviour of the complexity function.*

Convention 5.1.17 A homogeneous norm is associated to a right-invariant metric $d(x, y) := |xy^{-1}|$. From now on we assume that we always have a fixed homogeneous norm on our homogeneous Lie group and an associated metric. We see by Lemma 5.1.14 (b) that our space is proper and that the metric is compatible with the topology. So we get a nice metric in the sense of Convention 2.1.15.
Further we have a Haar measure on the group which we denote by μ_G.
So in total our group G carries the data $D_r, |\cdot|, \mu_G$, i.e. $G = (G, D_r, |\cdot|, \mu_G)$.

An important fact with respect to the induced topology on our group is given by the following corollary of Lemma 5.1.14.

Corollary 5.1.18 *If $|\cdot|$ is a homogeneous quasi-norm on a homogeneous Lie group G of dimension n, then the topology induced by the norm coincides with the Euclidean topology on the underlying vector space.*

The next proposition collects some useful properties of the behaviour of balls in a homogeneous Lie group.

Proposition 5.1.19 ([24, Section 3.1.3 and 3.1.6])
Let G be a homogeneous Lie group and $|\cdot|$ a homogeneous norm on G with the associated right-invariant metric d, then for $x, y \in G$ and $r, s > 0$:

(a) $B_r(x) = B_r(e) \cdot x$,

(b) $B_r(e)B_s(x) \subseteq B_{r+s}(x)$,

(c) $D_r(xy) = D_r(x)D_r(y)$,

(d) $D_r(B_s(e)) = B_{r\cdot s}(e)$,

(e) $D_r(B_s(x)) = B_{r\cdot s}(D_r(x))$,

(f) $\mu_G(B_r(x)) = r^{\operatorname{homdim}(G)} \cdot \mu_G(B_1(e))$.

Proof. Most of the proof follows from direct computation.

(a)
$$\begin{aligned} B_r(x) &= \{y \in G \,|\, d(x,y) < r\} = \{y \in G \,|\, d\left(e, yx^{-1}\right) < r\} \\ &= \{yx \in G \,|\, d(e,y) < r\} = \{y \in G \,|\, d(e,y) < r\}x \\ &= B_r(e)x. \end{aligned}$$

(b) This follows from the triangle inequality, since for $x, y, z \in G$:
$$\left|yzx^{-1}\right| \leq |y| + \left|zx^{-1}\right|.$$
So for $y \in B_r(e)$ and $z \in B_s(x)$ we get $yz \in B_{r+s}(x)$.

(c) This is a direct consequence of the definition of D_r.

(d)
$$\begin{aligned} D_r(B_s(e)) &= \{D_r(y) \in G \,|\, |y| < s\} = \{y \in G \,|\, |D_{1/r}(y)| < s\} \\ &= \{y \in G \,|\, |y| < rs\} = B_{r\cdot s}(e). \end{aligned}$$

(e)
$$D_r(B_s(x)) = D_r(B_s(e)x) = B_{rs}(e)D_r(x) = B_{rs}(D_r(x)).$$

(f) We use that the Haar measure in nilpotent Lie groups is unimodular, i.e. left- and right-invariant
$$\mu_G(B_r(x)) = \mu_G(B_r(e)x) = \mu_G(B_r(e)) = \mu_G(D_r(B_1(e))) = r^{\operatorname{homdim}(G)}\mu_G(B_1(e)).$$

□

Remark 5.1.20 Part (f) of Proposition 5.1.19 tells us that G has exact polynomial growth with exponent $\operatorname{homdim}(G)$, see Definition 2.3.4. Since all homogeneous quasi-norms on G are mutually equivalent the homogeneous dimension is independent of the choice of a metric, but clearly dependent on the choice of dilation structure.

Lemma 5.1.21 *Let G be a homogeneous Lie group and $x \in G$ fixed. Then for all $\varepsilon > 0$ there exists $\delta(x) > 0$ such that for all $u \in B_{\delta(x)}(e)$:*

$$xux^{-1} \in B_\varepsilon(e).$$

Proof. This follows by a direct calculation using the Baker-Campbell-Hausdorff formula

$$\begin{aligned}
xux^{-1} &= \left(x + u + \frac{1}{2}[x,u] + \frac{1}{12}([x[x,u]] - [u[x,u]]) - ...\right) x^{-1} \\
&= \left(x + u + \frac{1}{2}[x,u] + \frac{1}{12}([x[x,u]] - [u[x,u]]) - ...\right) \\
&\quad - x + \frac{1}{2}\left[\left(x + u + \frac{1}{2}[x,u] + \frac{1}{12}([x[x,u]] - [x[x,u]]) - ...\right), -x\right] + ... \\
&= u + B(u,x),
\end{aligned}$$

where $B(x,u)$ only contains terms which include $[u,x]$, so $B(x,u) \to 0$ as $u \to 0$, by the continuity of the Lie bracket. Observe that if x were not fixed, then also $B(x,u) \to 0$ as $x \to 0$ would hold. $\square$

Corollary 5.1.22 *Let G be a homogeneous Lie group. Further let $\varepsilon > 0$ and $x \in B_\varepsilon(e)$ then there exists $\delta(\varepsilon) > 0$ such that for all $u \in B_{\delta(\varepsilon)}(e)$ there exists a $c(\varepsilon) > 0$ such that*

$$xux^{-1} \in B_{\delta(\varepsilon)+c(\varepsilon)}(e),$$

where $\delta(\varepsilon) \to \infty$ and $c(\varepsilon) \to 0$ as $\varepsilon \to 0$.

5.2 Polynomial Law

In Proposition 5.1.2 we have stated that the group law is a polynomial map, we will now explain what this means and what additional properties the polynomials have. First we fix some notation for this section. Let G be group with a polynomial group law and underlying n-dimensional vector space. Let $x, y \in G$, then

$$xy = (P_1(x,y), P_2(x,y), ..., P_n(x,y)),$$

where the P_i are polynomials in $2n$ variables. Further we use the multi-index notation for $\alpha \in \mathbb{N}_0^n$: $x^\alpha := x_1^{\alpha_1}...x_n^{\alpha_n}$, $[\alpha] = \alpha_1\nu_1 + ... + \alpha_n\nu_n$ and $|\alpha| = \alpha_1 + ... + \alpha_n$. Remember that we assumed $0 < \nu_1 \leq \nu_2 \leq ... \leq \nu_n$.
The existence of a dilation family leads to the following restriction of the form of the polynomials P_i.

Proposition 5.2.1 ([24, Proposition 3.1.24])
For any $i \in \{1, ..., n\}$ we have

$$(xy)_i = x_i + y_i + \sum_{\substack{\alpha,\beta \in \mathbb{N}_0^n \setminus \{0\} \\ [\alpha]+[\beta]=\nu_i}} c_{i,\alpha,\beta} x^\alpha y^\beta.$$

Proof. By BCH formula we get for the i-th entry that

$$(xy)_i = x_i + y_i + \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\setminus\{0\}\\ |\alpha|+|\beta|\geq 2}} c_{i,\alpha,\beta}x^\alpha y^\beta =: x_i + y_i + R_i(x,y).$$

And from the dilation restriction we get

$$r^{\nu_i}(x_i + y_i + R_i(x,y)) = r^{\nu_i}(xy)_i = (D_r(x)D_r(y))_i = r^{\nu_i}x_i + r^{\nu_i}y_i + R_i(D_r(x), D_r(y)),$$

therefore $r^{\nu_i}R_i(x,y) = R_i(D_r(x), D_r(y))$ and this forces all the coefficients $c_{j,\alpha,\beta}$ with $[\alpha]+[\beta]\neq\nu_i$ to be zero. $\square$

Since the weights are ordered by size this proposition means that the polynomial P_i can only involve coordinates of x and y, which correspond to lower weights. This means that P_i is of the form

$$P_i(x,y) = x_i + y_i + R_i(x_1, ..., x_{i-1}, y_1, ..., y_{i-1}),$$

so the number of relevant variables decreases radically for small i.

We now will introduce a new kind of Lie group, since we are interested in the action of the Lie group G on the space of hyperplanes in the $\dim(G)$-dimensional vector space underlying $\mathfrak{g}$. This action is given by the group law. This yields another restriction on the polynomial law. Let H be a hyperplane given by

$$H = \left\{ a + \sum_{i=1}^{n-1} t_i v_i \,\middle|\, t_i \in \mathbb{R} \right\}$$

with $a, v_i \in \mathfrak{g}$. Thus xH for some $x \in G$ is given by

$$xH = \left\{ \left(P_1(x, a + \sum_{i=1}^{n-1} t_i v_i), ..., P_n(x, a + \sum_{i=1}^{n-1} t_i v_i) \right) \,\middle|\, t_i \in \mathbb{R} \right\}.$$

We want this to be a hyperplane again, but this means that in all the P_k we can never have a product of two t_i. This means in the notation from above that $\beta = (\beta_1, ..., \beta_n)$ is of the form that for some $l \in \{1, ..., n\}$ it is $\beta_l = 1$ and for all $j \neq l$ it is $\beta_j = 0$, if $c_{i,\alpha,\beta} \neq 0$. Adding this new restriction to the already found restriction from Proposition 5.2.1 we get the following form of the group law:

$$(xy)_i = x_i + y_i + \sum_{j=1}^{i-1} \sum_{\substack{\alpha\in\mathbb{N}_0^n\setminus\{0\}\\ [\alpha]+\nu_j=\nu_i}} c_{i,\alpha,j}x^\alpha y_j. \tag{5.2.1}$$

Definition 5.2.2 A *hyperplane* in a homogeneous Lie group G is the image of a hyperplane in the Lie algebra $\mathfrak{g}$ under the exponential map. The set of all hyperplanes in G is denoted by $\mathcal{H}(G)$.
A *half-space* in a homogeneous Lie group G is the image of a half-space in the Lie algebra $\mathfrak{g}$ under the exponential map.

Definition 5.2.3 A group G is *non-crooked* if $\mathcal{H}(G) \subset \mathcal{P}(G)$ is G left-invariant.

The name is motivated by the idea that the action of crooked homogeneous Lie groups bend the hyperplanes into more general hypersurfaces. To understand the combinatorics from Chapter 4 for crooked groups is a lot harder then handling the non-crooked ones, we will discuss this at the end of the thesis in the open questions section.

Definition 5.2.4 A Lie group G is called *locally k-step nilpotent* if for all $X, Y \in \mathfrak{g}$ we have $\mathrm{ad}_X^k(Y) = 0$.

Theorem 5.2.5 *Let G be a homogeneous Lie group, then the following are equivalent*

(a) G is non-crooked,

(b) G is 2-step nilpotent or abelian,

(c) G is locally 2-step nilpotent.

Remark 5.2.6 The term locally k-step nilpotent sometimes appears in the literature under the name k-Engel group. We prove that the properties locally two-step nilpotent and two-step nilpotent are equivalent, for bigger k the two notions are different.

In the rest of the section we prove the theorem.

Lemma 5.2.7 *Every locally two-step nilpotent homogeneous Lie group G is a non-crooked homogeneous Lie group.*

Proof. We consider the BCH-multiplication, which for locally two-step nilpotent Lie groups yields for i-th coordinate

$$(xy)_i = x_i + y_i + \frac{1}{2}[x, y]_i = x_i + y_i + \sum_{\substack{\alpha,\beta \in \mathbb{N}_0^n \setminus \{0\} \\ [\alpha]+[\beta]=\nu_i}} c_{i,\alpha,\beta} x^\alpha y^\beta .$$

Since the bracket is bi-linear, with respect to the scalar multiplication in $\mathbb{R}^n$, we see that the polynomials have to be of the form of equation (5.2.1). $\square$

At first sight the condition of locally two-step nilpotent seems weaker than the condition of being two-step nilpotent, but in fact the two are equivalent by the following proposition.

Proposition 5.2.8 *Let G be a Lie group. If G is locally two-step nilpotent then G is two-step nilpotent or abelian. If G is two-step nilpotent it is locally two-step nilpotent.*

Proof. The conclusion from two-step nilpotent to locally two-step nilpotent is trivial, so two-step nilpotent Lie groups are locally two-step nilpotent Lie groups.
So now assume that we have a locally two-step nilpotent Lie group and arbitrary $X, Y, Z \in \mathfrak{g}$, then

$$\begin{aligned} 0 &= [X+Y,[X+Y,Z]] = [X,[X,Z]] + [X,[Y,Z]] + [Y,[X,Z]] + [Y,[Y,Z]] \\ &= [X,[Y,Z]] + [Y,[X,Z]] \,. \end{aligned}$$

This means that $[X,[Y,Z]] = -[Y,[X,Z]] = [Y,[Z,X]]$. By using Jacobi's identity we have

$$0 = [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]].$$

And therefore by using the equality we found before we have $2[Y,[Z,X]] = [Z,[Y,X]]$. Since X, Y and Z are arbitrary, we can switch the roles of Y and Z. Thus $2[Z,[Y,X]] = [Y,[Z,X]]$. So in total this means

$$[Z,[Y,X]] = 2[Y,[Z,X]] = 4[Z,[Y,X]],$$

and therefore $[Z,[Y,X]] = 0$. So G is two-step nilpotent or abelian. □

We have seen that the class of non-crooked homogeneous Lie groups contains the abelian and the two-step nilpotent homogeneous Lie groups. We will now see that a higher nilpotency degree always implies crookedness.

Proposition 5.2.9 *A three-step homogeneous Lie group is crooked.*

Proof. Let H be a hyperplane given by $H = v_0 + \sum_{i=1}^n t_i v_i$, with $t_i \in \mathbb{R}$, $v_i \in \mathfrak{g}$ and $n+1$ the dimension of the Lie group. We get for all $X \in \mathfrak{g}$

$$\begin{aligned} X * H &= X + H + \frac{1}{2}[X,H] + \frac{1}{12}\big([X,[X,H]] - [H,[X,H]]\big) \\ &= X + v_0 + \frac{1}{2}[X,v_0] + \frac{1}{12}\big([X,X,v_0] + [v_0,[X,v_0]]\big) \\ &\quad + \sum_{i=1}^n t_i \cdot \left(v_i + \frac{1}{2}[X,v_i] + \frac{1}{12}\big([X,[X,v_i]] - [v_0,[X,v_i]] - [v_i,[X,v_0]]\big)\right) \\ &\quad - \frac{1}{12}\sum_{i=1}^n\sum_{j=1}^n t_i t_j [v_i,[X,v_j]]. \end{aligned}$$

So for this to be non-crooked the last sum has to disappear, i.e.

$$\sum_{i=1}^n\sum_{j=1}^n t_i t_j [v_i,[X,v_j]] = 0.$$

But since the t_i, t_j are parameters we can only compare the summands with the same coefficients, so for all $i, j \in \{1, ..., n\}$, $i \neq j$,

$$[v_i, [X, v_j]] + [v_j, [X, v_i]] = 0$$

and for the diagonal, i.e. $i = j$,

$$[v_i, [X, v_i]] = 0.$$

But this last conditions is the locally two-step nilpotency condition, which as we have seen implies two-step nilpotency. Which is a contradiction to the assumption of three-step nilpotency. □

Corollary 5.2.10 *All nilpotent homogeneous Lie groups, with nilpotency degree greater than two, are crooked.*

For a full list of all nilpotent Lie groups up to dimension seven see the thesis by Gong, [28].

5.3 Example: The Heisenberg Group

We present an example of an easily understood homogeneous Lie group. One can keep this example in mind in the following parts of the thesis. For this example we choose the three dimensional Heisenberg group $\mathbb{H}$. For more on this group and the different models in which one can view it see the paper by Balogh, Fässler and Sobrino, [8]. The most common model of this group is as upper triangular matrices with ones on the diagonal. So the group action is given by

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+x & c+z+ay \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{pmatrix}.$$

The associated Lie algebra $\mathfrak{h}$ has the form of upper triangular matrices with zeros on the diagonal, a basis for $\mathfrak{h}$ is given by the following elements

$$X := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, Z := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Lie bracket is the standard matrix bracket given by

$$[A, B] := AB - BA.$$

So we get for the basis vectors $[X, Y] = Z$, $[X, Z] = 0$ and $[Y, Z] = 0$. This tells us that the Heisenberg group is 2-step nilpotent. Now we consider the isomorphic model

$\mathbb{H}' := (\mathbb{R}^3, *)$, where we identify $\alpha X + \beta Y + \gamma Z$ with $(\alpha, \beta, \gamma) \in \mathbb{R}^3$. So we will work in exponential coordinates again. The group action we get from the BCH-multiplication is

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ ay - xb \end{pmatrix}.$$

We can read of the polynomials P_1, P_2 and P_3 from this form and see that they fulfil the required form for non-crooked Lie groups.

A next step is to set the dilation structure on $\mathbb{H}$. Since we set $\nu_1 = 1$, as a natural choice, we get

$$D_r(\begin{pmatrix} a \\ b \\ c \end{pmatrix}) = \begin{pmatrix} ra \\ rb \\ r^2 c \end{pmatrix}.$$

Lets check if this is compatible with the Lie bracket:

$$\begin{aligned} \left[D_r(\begin{pmatrix} a \\ b \\ c \end{pmatrix}), D_r(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) \right] &= \left[\begin{pmatrix} ra \\ rb \\ r^2 c \end{pmatrix}, \begin{pmatrix} rx \\ ry \\ r^2 z \end{pmatrix} \right] = \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ r^2 ay - r^2 xb \end{pmatrix} \\ &= D_r(\frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ ay - xb \end{pmatrix}) = D_r(\left[\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right]). \end{aligned}$$

Further we set the norm on $\mathbb{H}$, which then gives us also an associated right-invariant metric d,

$$\left| \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right| := \max \left\{ |a|, |b|, \sqrt{c} \right\}.$$

That this is a homogeneous norm is clear, one can check all the conditions from Definition 5.1.13, the only one which is not trivial is the triangle inequality. For a proof of this see for example [8, Example 5.10.]. There are other choices for homogeneous norms but as we have seen in Lemma 5.1.15 they are all mutually equivalent.

Finally we need a Haar measure on $\mathbb{H}$, this is simply given by the Lebesgue measure on $\mathbb{R}^3$, by Proposition 5.1.2. So in total we consider $(\mathbb{H}, D_r, |\cdot|, \mu_{\mathbb{H}})$. And by considering the weights we see that the homogeneous dimension of $\mathbb{H}$ is four, which is different from its dimension, which is three.

We further want to see how the action moves the hyperplanes in $\mathbb{R}^3$. So let

$$H = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t_1 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + t_2 \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix},$$

be a hyperplane and $(x, y, z) \in \mathbb{H}$, then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} H = \begin{pmatrix} x + a_1 + t_1 v_1 + t_2 w_1 \\ y + a_2 + t_1 v_2 + t_2 w_2 \\ z + a_3 + t_1 v_3 + t_2 w_3 + x(a_2 + t_1 v_2 + t_2 w_2) - y(a_1 + t_1 v_1 + t_2 w_1) \end{pmatrix}.$$

First we see that this is again a hyperplane, but we also see that the new direction vectors can have a new direction, we simplify and get:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} H = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t_1 \begin{pmatrix} v_1 \\ v_2 \\ v_3 + xv_2 - yv_1 \end{pmatrix} + t_2 \begin{pmatrix} w_1 \\ w_2 \\ w_3 + xw_2 - yw_1 \end{pmatrix}.$$

This example should give an idea, where the difference to the Euclidean case is: The Euclidean action on the space of hyperplanes always gives a parallel plane, hence we have two options either the planes do not intersect or they stay the same. This changes in the more general set-up, there we can have intersections of H and xH.

5.4 Lattices in Homogeneous Lie Groups

In this short section we will briefly talk about lattices in homogeneous Lie groups, the main property which we will use is that these groups are nilpotent. The main source for this section is the book by Raghunathan, [64], and especially chapter 2 about lattices in nilpotent Lie groups.
We start by a reminder of the definition of a lattice in a lcsc group. We have already seen that for our purpose we need uniform lattices, but luckily we get the uniformity for free in nilpotent Lie groups. Afterwards we will consider the influence of the Lie algebra $\mathfrak{g}$ on the question of existence of lattices in G.

Definition 5.4.1 A *lattice* in a lcsc group G is a discrete subgroup Γ such that G/Γ has finite measure. A lattice is called *uniform* if G/Γ is compact.

Lemma 5.4.2 ([64, Theorem 2.1.])
Let G be a nilpotent Lie group, then all lattices $\Gamma \subseteq G$ are uniform.

Lemma 5.4.3 ([64, Corollary of Theorem 2.10.])
Let G be a nilpotent Lie group and $\Gamma \subseteq G$ a lattice, then Γ is finitely generated and the minimal number of generators is bounded from above by $\dim(G)$.

Lemma 5.4.4 ([64, Theorem 2.1.])
Let G be a nilpotent, connected and simply connected Lie group and $\Gamma \subseteq G$ a discrete subgroup, then Γ is a lattice if and only if it is not contained in a proper connected subgroup.

Theorem 5.4.5 ([64, Theorem 2.12])
Let G be a simply connected nilpotent Lie group and let $\mathfrak{g}$ be its Lie algebra. Then G

admits a lattice if and only if $\mathfrak{g}$ admits a basis with respect to which the constants of structure are rational.

Here the constants of structure are the constants which appear in the bracket by combining generators. Say $X_1, ..., X_n$ are a basis of the Lie algebra, then

$$[X_i, X_j] = \sum_{l=1}^{n} c_{ijl} X_l$$

and the constants of structure are the c_{ijl}.

These structure constants can not be chosen arbitrarily, since our considered Lie group is nilpotent. There is a complete list of all nilpotent Lie algebra up to dimension seven, which can be found in the thesis by Gong, [28], from which one can also read the possible structure constants. Almost all the examples in the list of Gong have rational structure constants. So the existence of a lattice is not uncommon in Lie groups, at least for small dimensions.

5.5 Ergodic Theorems for Homogeneous Lie Groups

In this section we use the theory built by Gorodnik and Nevo, [29, 30, 60]. The aim is to establish the counting argument Theorem 5.5.4 and give criteria for its usage in our case.

Definition 5.5.1 ([30, Definition 1.1.])
Let O_ε, $\varepsilon > 0$ be a family of symmetric neighbourhoods of the identity in a lcsc group G, which are decreasing in ε and $\lim_{\varepsilon\to 0} O_\varepsilon = \{e\}$. Then a family of bounded Borel subsets of finite Haar measure $(B_t)_{t>0}$ is *well-rounded w.r.t* O_ε if for every $\delta > 0$ there exists $\varepsilon, t_1 > 0$ such that for all $t \geq t_1$

$$\mu_G(O_\varepsilon B_t O_\varepsilon) \leq (1+\delta)\mu_G\left(\bigcap_{u,v\in O_\varepsilon} uB_t v\right).$$

In our set-up we will always fix O_ε as $B_\varepsilon(e)$, this does not make a difference by [42, Remark 2.3.].

Definition 5.5.2 ([30, Definition 1.4 and 1.5])
Let G be a lcsc group G and B_t a family of bounded Borel subsets of finite Haar measure. And let β_{G,B_t} be the operator

$$\beta_{G,B_t} f(x) := \frac{1}{\mu_G(B_t)} \int_{B_t} f\left(g^{-1}x\right) d\mu_G(g)$$

for $f \in L^2(G)$. We say that the *mean ergodic theorem in* $L^2(G)$ holds if

$$\left\| \beta_{G,B_t} f - \int_G f d\mu \right\|_{L^2(G)} \to 0 \text{ as } t \to \infty$$

for all $f \in L^2(G)$. We say that the *stable mean ergodic theorem in* $L^2(G)$ holds if the mean ergodic theorem in $L^2(G)$ holds for the sets

$$B_t^+(\varepsilon) = O_\varepsilon B_t O_\varepsilon \text{ and } B_t^-(\varepsilon) = \bigcap_{u,v \in O_\varepsilon} u B_t v,$$

for all $\varepsilon \in (0, \varepsilon_1)$ with $\varepsilon_1 > 0$.

Remark 5.5.3 From now on we fix a Haar measure $\mu_{G \times H}$, which we assume to be normalized by $\mu_{G\times H/\Gamma}\left(G \times H/\Gamma\right) = 1$.

Theorem 5.5.4 ([30, Theorem 1.7])
Let G be a lcsc group, $\Gamma \subseteq G$ a lattice and $(B_t)_{t>0}$ a well-rounded family of subsets of G. Assume that the averages $\beta_{G/\Gamma,B_t}$ supported on B_t satisfy the stable mean ergodic theorem in $L^2(G/\Gamma)$. Then

$$\lim_{t \to \infty} \frac{|\Gamma \cap B_t|}{\mu_G(B_t)} = 1.$$

To apply this theorem we have to show that the sets we will consider are well-rounded and they satisfy the stable mean ergodic theorem. We now give criteria which ensure this.

Lemma 5.5.5 *Let G be a homogeneous Lie group and $(B_t(x))_{t>0}$ a family of balls in G. Then this family is well-rounded.*

Proof. We have to show that for every $\delta > 0$ there exists some $\varepsilon, t_1 > 0$ such that for all $t \geq t_1$ holds

$$\mu_G(B_\varepsilon(e) B_t(x) B_\varepsilon(e)) \leq (1+\delta)\mu_G\left(\bigcap_{u,v \in B_\varepsilon(e)} u B_t(x) v\right).$$

We first show that we can choose ε such that for a any constant $k \in (0,t)$ we have $B_{t-k}(x) \subseteq \bigcap_{u,v\in B_\varepsilon(e)} u B_t(x) v$. So let $g \in B_{t-k}(x)$, then we can write g as $uu^{-1}gv^{-1}v$ with $u, v \in B_\varepsilon(e)$ and we have to show that $u^{-1}gv^{-1} \in B_t(x)$.

$$d\left(x, u^{-1}gv^{-1}\right) = \left|xvg^{-1}u\right|_G = \left|xvx^{-1}xg^{-1}u\right|_G \leq \left|xvx^{-1}\right|_G + \left|xg^{-1}\right|_G + |u|_G \leq c_x(\varepsilon) + t - k + \varepsilon.$$

Here the last inequality holds by Lemma 5.1.21. And we have to choose ε so small that $k > \varepsilon + c_x(\varepsilon)$, which is possible since $c(\varepsilon) \to 0$ as $\varepsilon \to 0$.

On the other hand $B_\varepsilon(e)B_t(x)B_\varepsilon(e) \subseteq B_{\varepsilon+t}(x)B_\varepsilon(e)$. Let $y \in B_{\varepsilon+t}(x)$ and $u \in B_\varepsilon(e)$ then

$$d(x, yu) = \left|xu^{-1}y^{-1}\right|_G = \left|xu^{-1}x^{-1}xy^{-1}\right|_G \leq \left|xu^{-1}x^{-1}\right|_G + \left|xy^{-1}\right|_G \leq c_x(\varepsilon) + \varepsilon + t.$$

Therefore $B_\varepsilon(e)B_t(x)B_\varepsilon(e) \subseteq B_{\varepsilon+t+c_x(\varepsilon)}(x)$. Therefore we can choose $\varepsilon > 0$ and $k \in (0, t)$ such that

$$B_\varepsilon(e)B_t(x)B_\varepsilon(e) \subseteq B_{\varepsilon+t+c_x(\varepsilon)}(x) \text{ and } B_{t-k}(x) \subseteq \bigcap_{u,v\in B_\varepsilon(e)} uB_t(x)v$$

hold simultaneously. Now we can use that we can calculate the measure of balls in homogeneous Lie groups by Proposition 5.1.19:

$$\mu_G(B_\varepsilon(e)B_t(x)B_\varepsilon(e)) \leq \mu_G(B_{\varepsilon+t+c_x(\varepsilon)}(x)) = (t + \varepsilon + c_x(\varepsilon))^{\mathrm{homdim}(G)}\mu_G(B_1(e))$$

and

$$(t-k)^{\mathrm{homdim}(G)}\mu_G\left(B_1(e)\right) \leq \mu_G\left(B_{t-k}(x)\right) \leq \mu_G\left(\bigcap_{u,v\in B_\varepsilon(e)} uB_t(x)v\right).$$

Combining the arguments we see that we have to choose ε, k and t_1 such that for all $t > t_1$

$$\left(\frac{(t+\varepsilon+c_x(\varepsilon))}{(t-k)}\right)^{\mathrm{homdim}(G)} \leq (1+\delta).$$

□

Lemma 5.5.6 *Let G be a homogeneous Lie group and $(B_r(x))_{t>0}$ a constant family of balls in G. Then this family is well-rounded.*

Proof. We have already seen in the proof of Lemma 5.5.5 that $\bigcap_{u,v\in B_\varepsilon(e)} uB_r(x)v$ contains a ball of the form $B_{r-k}(x)$ for any $k \in (0, t)$ if we choose ε accordingly. On the other hand $B_\varepsilon(e)B_r(x)B_\varepsilon(e)$ is contained in a ball $B_{r+\varepsilon+c_x(\varepsilon)}(x)$ and therefore has finite measure. Choosing ε accordingly we are done. □

Now what is left to show is that the stable mean ergodic theorem holds for our families of sets. To do so we use [27, Theorem 3.33.] which tells us that we have to check that our families are Følner sequences. Alternatively see the work by Nevo, [60], especially step I in the proof of Theorem 5.1 is exactly what we need.

Definition 5.5.7 ([27, Definition 3.31])
Let G be a lcsc group acting on a measure space (X, μ). A sequence $F_1, F_2, ...$ of subsets of finite, non-zero measure is called a *(right) Følner sequence* if for all $g \in G$

$$\lim_{i\to\infty} \frac{\mu_G\left(F_i g \triangle F_i\right)}{\mu_G\left(F_i\right)} = 0,$$

where $\triangle$ denotes the symmetric difference.

Lemma 5.5.8 *Let $G \times H$ be a product of homogeneous Lie groups, $\Gamma \subseteq G \times H$ a lattice, $(B_t^G(e))_{t>0}$ a family of balls in G and $(B_r^H(x))_{t>0}$ a ball in H. The family $(B_t^G(e) \times B_r^H(x))_{t>0}/\Gamma$ is a Følner sequence in $(G \times H)/\Gamma$.*

Proof. We use Proposition 2.2.6, which tells us that for every $r > 0$ and every $x \in H$ we find a compact $K \subseteq G$ such that $\left(K \times B_r^H(x)\right)\Gamma = G \times H$. And since every compact set K is contained in some $B_t^G(e)$, for t large enough, we get that $\left(B_t^G(e) \times B_r^H(x)\right)\Gamma = G \times H$. And therefore $\mu_{G\times H/\Gamma}\left(\left(B_t^G(e) \times B_r^H(x)\right)_{G\times H/\Gamma}\right) = 1$. This also holds if we consider balls $B_t^G(g)$ instead of $B_t^G(e)$. Therefore we have for every $(g,h) \in G \times H$ that there exists a $t_{g,h} > 0$ such that $\left(B_t^G(g) \times B_r^H(xh)\right)\Gamma = G \times H$ and so

$$\lim_{t\to\infty} \mu_{G\times H/\Gamma}\left(\left(B_t^G(g) \times B_r^H(xh)\right)_{G\times H/\Gamma} \triangle \left(B_t^G(e) \times B_r^H(x)\right)_{G\times H/\Gamma}\right) = 0$$

for all $(g,h) \in G \times H$. □

6 Complexity of Model Sets in Homogeneous Lie Groups

In this chapter we address the problem of determining the asymptotic behaviour of the complexity function $p(r)$ of a model set $\Lambda(G, H, \Gamma, W)$, where G and H are homogeneous Lie groups and H is non-crooked. Observe, since all lattices in nilpotent groups are uniform, that Λ is a uniform model set. Also W will be assumed to be a polytopal window. We will use the methods, which we established in Chapter 4 and will see that the upper bound is almost trivial to show. But the lower bound for the growth rate needs a lot more effort. The idea of the proof is similar to the Euclidean case done by Koivusalo and Walton, [45]. But we have to overcome some additional obstacles, namely that our groups are not abelian and that our action on the space of hyperplanes in H does not preserve parallelity.

Since W is assumed to be a polytopal window, we fix some notation for the hyperplanes which bound W:

$$W = \bigcap_{i=1}^{N} P_i^+ \subset H.$$

Here $P_1, ..., P_N$ denote the hyperplanes which bound W and $P_1^+, ..., P_N^+$ the half-spaces associated to the hyperplanes in which W lies. The other half-space corresponding to P_i is then denoted by P_i^-. We assume the set $\{P_1, ..., P_N\}$ to be irredundant, i.e. for each $i \in \{1, ..., N\}$ it is $\cap_{j=1, j\neq i}^{N} P_j^+ \neq W$. For more information on polytopes in Euclidean space see the book by Grünbaum, [34]. We will use some facts from the book, for example that $N > \dim(H)$ holds.

The main theorem, which we will prove in this chapter is the following.

Theorem 6.0.1 *Let G be a homogeneous Lie group and H a non-crooked homogeneous Lie group. Let $\Lambda(G, H, \Gamma, W)$ be a polytopal model set, such that the P_i have trivial stabilizer and that $P_i \cap \Gamma_H = \emptyset$. Then for the complexity function $p(r)$ of Λ we have*

$$p(r) \asymp r^{\mathrm{homdim}(G)\cdot\dim(H)}.$$

Remark 6.0.2 Notice that the assumption of Γ regularity is replaced by the stronger assumption that for all $i \in \{1, ..., N\}$ we have $P_i \cap \Gamma_H = \emptyset$.

Since a non-trivial stabilizer can only lower the growth of the complexity function we get the following corollary.

Corollary 6.0.3 *Let G be a homogeneous Lie group and H a non-crooked homogeneous Lie group. Let $\Lambda(G, H, \Gamma, W)$ be a regular polytopal model set. Then for the complexity function $p(r)$ of Λ we have*

$$p(r) \ll r^{\mathrm{homdim}(G)\cdot \dim(H)}.$$

In the proof of Theorem 6.0.1 we will use the language from Chapter 4 and Chapter 5.

6.1 Upper Bound

Remember that we have shown in Lemma 3.0.12 that

$$p(r) = |A_r^H| \leq \#\pi_0\left(W \setminus \bigcup_{\mu \in \mathcal{S}_r} \mu\partial W\right).$$

Now we will use our assumption that W is polytopal and consider the hyperplane arrangement given by $\mathcal{H} = \{\mu P_i \mid \mu \in \mathcal{S}_r, i \in \{1, ..., N\}\}$.

Lemma 6.1.1 *For a polytopal window $W \subset H$ we have*

$$\#\pi_0\left(W \setminus \bigcup_{\mu \in \mathcal{S}_r} \mu\partial W\right) \leq \#\pi_0\left(H \setminus \bigcup_{\mu \in \mathcal{S}_r} \bigcup_{i=1}^{N} \mu P_i\right).$$

Proof. Since $\bigcup_{\mu \in \mathcal{S}_r} \mu\partial W \subset \bigcup_{\mu \in \mathcal{S}_r} \bigcup_{i=1}^{N} \mu P_i$ we have that

$$\#\pi_0\left(W \setminus \bigcup_{\mu \in \mathcal{S}_r} \mu\partial W\right) \leq \#\pi_0\left(W \setminus \bigcup_{\mu \in \mathcal{S}_r} \bigcup_{i=1}^{N} \mu P_i\right).$$

Since $e \in \mathcal{S}_r$ for all r we have $\partial W \subset \bigcup_{\mu \in \mathcal{S}_r} \bigcup_{i=1}^{N} \mu P_i$ such that all regions inside W stay the same if we increase W to H. If we add the regions outside of W we therefore get

$$\#\pi_0\left(W \setminus \bigcup_{\mu \in \mathcal{S}_r} \bigcup_{i=1}^{N} \mu P_i\right) \leq \#\pi_0\left(H \setminus \bigcup_{\mu \in \mathcal{S}_r} \bigcup_{i=1}^{N} \mu P_i\right).$$

□

Now we are able to use Proposition 4.1.9, which gives us an upper bound on the number of regions in an arrangement.

Proposition 6.1.2 (Upper bound)
With the assumptions of Theorem 6.0.1 we have

$$p(r) \ll r^{\mathrm{homdim}(G)\cdot \dim(H)}.$$

Proof. By Lemma 3.0.12, Lemma 6.1.1 and Proposition 4.1.9 we have

$$p(r) \leq \#\pi_0 \left(H \setminus \bigcup_{\mu \in \mathcal{S}_r} \bigcup_{i=1}^{N} \mu P_i \right) \leq \sum_{i=0}^{\dim(H)} \binom{N \cdot |\, \mathcal{S}_r \,|}{i} \ll (N \cdot |\, \mathcal{S}_r \,|)^{\dim(H)}.$$

And further by Proposition 5.1.19 G is a group with exact polynomial growth of degree $\mathrm{homdim}(G)$, so we can use Corollary 2.3.5 to get

$$p(r) \ll (N \cdot |\, \mathcal{S}_r \,|)^{\dim(H)} \ll r^{\mathrm{homdim}(G) \cdot \dim(H)}.$$

□

6.2 Lower Bound

For the lower bound our strategy is to find a small ball inside W for which it does not make a difference if we intersect it with μW or one of the μP_i^+, where μ is an element out of a subset $U_i(r) \subset \mathcal{S}_r$. If we have found such a ball we will construct $U_i(r)$ as a subset of the slab. The sets $U_i(r)$ induce a hyperplane arrangement by acting on the set of the P_i, which fulfils the assumptions in Theorem 4.3.14.

Definition 6.2.1 For a given polytopal window W we fix the following parameters:

(a) A *centre of the window* $c_W \in W$ such that $\sup\{r \in \mathbb{R} \mid B_r(c_W) \subseteq W\}$ is maximal,

(b) the *inner radius* of the window $I_W := \sup\{r \in \mathbb{R} \mid B_r(c_W) \subseteq W\}$,

(c) the *outer radius* of the window $O_W := \inf\{r \in \mathbb{R} \mid W \subseteq B_r(c_W)\}$,

(d) the *size of* $\partial_i W$

$$F_i := \sup\{r \in \mathbb{R} \mid \exists p \in P_i : B_r(p) \cap \partial_i W = B_r(p) \cap P_i\}$$

and the minimum of all the sizes of the faces

$$F_W := \min\{F_i \mid i \in \{1, ..., N\}\},$$

(e) for each face $\partial_i W$ a *face centre* $p_i \in \partial_i W$ such that $B_{F_W}(p_i) \cap \partial_i W = B_{F_W}(p_i) \cap P_i$.

Remark 6.2.2 The centres may not be unique but we fix a choice for the rest of the argument.

Definition 6.2.3 Let $B \subseteq H$ be a bounded region. For $s \in H$ we say $s\partial_i W$ *cuts* B *fully* if

$$(s\partial_i W) \cap B = (sP_i) \cap B \neq \emptyset.$$

If additionally

$$(sP_i^+) \cap B = (sW) \cap B \neq \emptyset,$$

we say $s\partial_i W$ *cuts* B *all-round*.

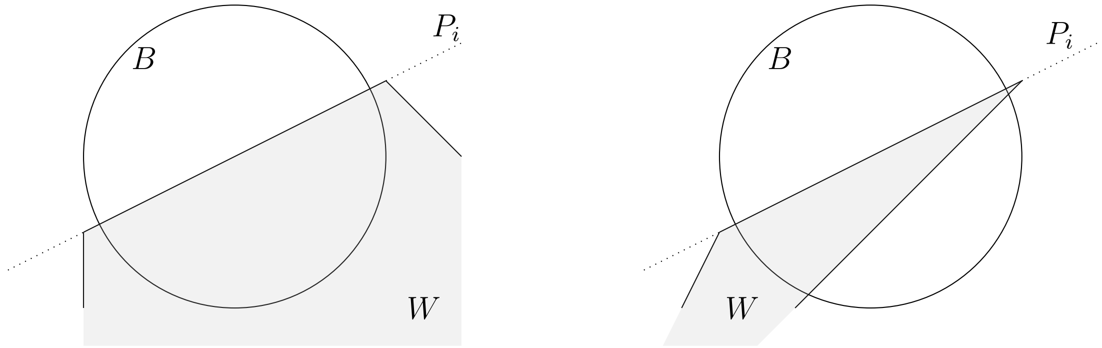


Figure 6.1: On the left $\partial_i W$ cuts B fully and all-round on the right $\partial_i W$ cuts B fully but not all-round.

Remark 6.2.4 An all-round cut is always a full cut, but the converse is false, see Figure 6.1.

We will now consider a small ball inside the window, additionally we consider subsets of the r-slab, we will require some properties on both these sets, which will be listed in the definition. At first sight they may look arbitrary but it will become clear why we need them.

Definition 6.2.5 Let $(k, h) \in \mathbb{R}^2$. The region we will consider is $B_h(c_W)$ and the set from which we operate is $U_i := B_k(c_W p_i^{-1})$, for all $i \in \{1, ..., N\}$. If the following conditions are fulfilled we call (k, h) a *good pair*:

(a) $0 < k < h$,

(b) $h < I_W$, therefore $B_h(c_W) \subset W$,

(c) $\forall a \in B_{O_W}(e)$, $x \in B_{2h}(e)$: $|axa^{-1}|_H \leq F_W$,

(d) $\forall i \in \{1, ..., N\}$: $\forall s \in U_i$: $(sP_i^+) \cap B_h(c_W) = (sW) \cap B_h(c_W)$.

Proposition 6.2.6 *A good pair exists.*

For the proof we need some preparation. It will be given after Corollary 6.2.12.

Lemma 6.2.7 *Let (k,h) fulfil conditions (a) and (c) of Definition 6.2.5, then for any $i \in \{1,...,N\}$ and for every $s \in U_i$ holds $s\partial_i W$ cuts $B_h(c_W)$ fully.*

Proof. First we show that for all $s \in U_i$ we get $s\partial_i W \cap B_h(c_W) \neq \emptyset$. We can write $s = a \cdot c_W \cdot p_i^{-1}$ with $a \in B_k(e)$. Then

$$d(s \cdot p_i, c_W) = \left|a \cdot c_W \cdot p_i^{-1} \cdot p_i \cdot c_W^{-1}\right| = |a| < k < h.$$

Now we need to show that $s\partial_i W \cap B_h(c_W) = sP_i \cap B_h(c_W)$. This is equivalent to

$$\partial_i W \cap s^{-1}B_h(c_W) = P_i \cap s^{-1}B_h(c_W).$$

The inclusion $\subseteq$ is obvious since $\partial_i W \subset P_i$. We show that $s^{-1}B_h(c_W) \subseteq B_{F_W}(p_i)$, then the claim follows from the definition of p_i and F_W. Let $x \cdot c_W \in B_h(c_W)$ be an arbitrary element and $s = a \cdot c_W \cdot p_i^{-1}$ as above

$$d(s^{-1}xc_W, p_i) = \Big| \underbrace{p_i c_W^{-1}}_{=:y \in B_{O_W}(e)} \cdot \underbrace{a^{-1}x}_{\in B_{h+k}(e)} \cdot \underbrace{c_W p_i^{-1}}_{=y^{-1}} \Big| \leq F_W.$$

The inequality follows by condition (c) of Definition 6.2.5. □

From the proof we can extract the following corollary.

Corollary 6.2.8 *For every $i \in \{1,...,N\}$ and every $s \in U_i$ the sets sP_i and $B_k(c_W)$ intersect non-trivially.*

We will also need the definition of an intersection angle between two hyperplanes, since we will show that by acting with a small element, we can only rotate a plane a bit.

Definition 6.2.9 The angle between two hyperplanes P and Q in $\mathbb{R}^d$ with normals n_P and n_Q, both normalized, is given by

$$\sphericalangle(P,Q) := \cos^{-1}\left(|\langle n_P, n_Q\rangle|\right),$$

where $\cos^{-1}$ maps to $[0, \frac{\pi}{2}]$.
For $i,j \in \{1,...,N\}$ with $i \neq j$ we denote by α_{ij} the angle between $c_W p_i^{-1} P_i$ and $c_W p_j^{-1} P_j$, i.e.

$$\alpha_{ij} := \sphericalangle\left(c_W p_i^{-1} P_i, c_W p_j^{-1} P_j\right).$$

Remark 6.2.10 In the definition we use $c_W p_i^{-1} P_i$ instead of P_i since this plane is sort of the prototype for the family $U_i P_i$, all the other planes from this family then result from an action with a small element since every $u \in U_i$ is of the form $ac_W p_i^{-1}$ with $a \in B_k(e)$.

Lemma 6.2.11 *For all $r > 0$ there exists $\beta(r)$, with $\beta(r) \to 0$ for $r \to 0$, such that for all $x \in B_r(e) \subseteq H$ and any hyperplane P we have $\sphericalangle(xP, P) \leq \beta(r)$.*

Proof. Since H is a non-crooked homogeneous Lie group we know that xP is again a hyperplane. So let

$$P = \left\{ a + \sum_{i=1}^{n} t_i v_i \mid t_i \in \mathbb{R} \right\}$$

where $a, v_i \in \mathbb{R}^n$. By the form of the group action, which we discussed in Section 5.2, we know that xP is of the form

$$xP = \big(f_1(x,P), ..., f_n(x,P)\big)^{\mathrm{T}}$$

with f_i polynomials of a special form, namely

$$f_i(x,P) = x_i + P_i + \sum_{k=1}^{n} \sum_{\substack{\alpha_1,...,\alpha_n \in \mathbb{N} \\ \sum \alpha_i \neq 0}} c_{k,\alpha_1,...,\alpha_n} P_k x_1^{\alpha_1} ... x_n^{\alpha_n}.$$

The direction vectors are the ones from P plus some deviation which depends on x. If x gets smaller the two planes are getting closer to being parallel. □

Corollary 6.2.12 *For all $r > 0$ there exists $\beta(r)$, with $\beta(r) \to 0$ for $r \to 0$, such that for all $x, y \in B_r(e) \subseteq H$ and any hyperplane P we have $\sphericalangle(xP, yP) \leq 2\beta(r)$.*

Proof of Proposition 6.2.6. By Corollary 5.1.22 there exists an upper bound b_1 on h such that for all $a \in B_{O_W}(e)$, $x \in B_{2h}(e)$: $|axa^{-1}|_H \leq F_W$. Set $h' := \min\{b_1, \frac{I_W}{2}\}$ and set $k' := \frac{h'}{2}$ then the conditions (a), (b) and (c) are fulfilled.
By Lemma 6.2.7 we know that for any $i \in \{1, ..., N\}$ and all $s \in B_{k'}(c_W p_i^{-1})$ it holds that sP_i cuts $B_{h'}(c_W)$ fully, which then also holds for all $h \leq h'$.
Now assume that there is a cut which is full but not all-round, therefore

$$sP_i^+ \cap B_{h'}(c_W) \neq sW \cap B_{h'}(c_W).$$

To be more precise we have $sW \cap B_{h'}(c_W) \subsetneq sP_i^+ \cap B_{h'}(c_W)$ since $sW \subset sP_i^+$. Let

$$b_s^i := \inf \left\{ r \in \mathbb{R} \mid \exists x \in B_r(c_W) : x \in sP_i^+, x \notin sW \right\}.$$

We see that $b_s^i \neq 0$ since sW is a polytope with non-empty interior. Now set

$$h := \min \left\{ h', \inf_{s \in B_{k'}(c_W p_i^{-1})} \{b_s^i\} \right\}$$

and $k = \frac{h}{2}$. The last thing to observe now is that the infimum over the b_s^i is not zero. If it were zero this would mean that the polytope sW could become arbitrarily thin such that only an even smaller ball would fit in. But this can not be the case since we have seen that we always rotate the bounding hyperplanes only by a small amount. □

Convention 6.2.13 From now on let (k,h) be a good pair.

Observe that $U_i \subset WW^{-1}$ for all $i \in \{1,...,N\}$. Further notice that we operate differently on the different hyperplanes which bound W, the U_i may overlap but they are not equal. Additionally we have chosen $B_h(c_W)$ so that for each of the hyperplanes it does not make a difference if we operate on the face $\partial_i W$ or on the hyperplane P_i.

Now we will reconsider the dependence on the growing parameter r and the lattice Γ.

Definition 6.2.14 Set $U_i(r) := \pi_H((B_r^G(e) \times U_i) \cap \Gamma)$ which is a finite subset of U_i.

Remark 6.2.15 Observe that $U_i(r)$ is a subset of the r-slab $\mathcal{S}_r$, since $U_i \subset WW^{-1}$.

Proposition 6.2.16 *The number of connected components of*

$$B_h(c_W) \setminus \bigcup_{i=1}^{N} \bigcup_{s \in U_i(r)} s\partial_i W$$

is a lower bound of the number of acceptance domains $\left|A_r^H\right|$, i.e.

$$\#\pi_0 \left(B_h(c_W) \setminus \bigcup_{i=1}^{N} \bigcup_{s \in U_i(r)} s\partial_i W \right) \leq \left|A_r^H\right|$$

Proof. Recall that a pre-acceptance domain $A_r^H(\lambda)$ is contained in an acceptance domain $W_r(\lambda)$. Let C be a connected component of $B_h(c_W) \setminus \bigcup_{i=1}^N \bigcup_{s\in U_i(r)} s\partial_i W$. By Lemma 6.2.7 we can replace the faces by the hyperplanes without changing the connected components in $B_h(c_W)$, so we consider $B_h(c_W) \setminus \bigcup_{i=1}^N \bigcup_{s\in U_i(r)} sP_i$.
We will show that if an acceptance domain intersects a connected component of $B_h(c_W) \setminus \bigcup_{i=1}^N \bigcup_{s\in U_i(r)} sP_i$ it is already fully contained in it. Let C' be another connected component of $B_h(c_W) \setminus \bigcup_{i=1}^N \bigcup_{s\in U_i(r)} sP_i$ and assume that $C \cap W_r(\lambda) \neq \emptyset \neq C' \cap W_r(\lambda)$. Between C and C' is a hyperplane sP_i for some $i \in \{1,...,N\}$ and $s \in U_i(r)$. Therefore $C \subset s\mathring{P}^+$ and $C' \subset s\mathring{P}^-$ or the other way round. And since the cut $s\partial_i W$ is all-round we get that $C \subset s\mathring{W}$ and $C' \subset sW^{\mathrm{C}}$ or the other way round. But either $W_r(\lambda) \subset s\mathring{W}$ or $W_r(\lambda) \subset sW^{\mathrm{C}}$, a contradiction. □

Proposition 6.2.17 *There exists a good pair (k,h) such that:*

(a) For all $I \subseteq \{1,....,N\}$ with $|I| = \dim(H)$ and all $u_1 \in U_{i_1},...,u_{\dim(H)} \in U_{i_{\dim(H)}}$ we get

$$u_1P_{i_1} \cap ... \cap u_{\dim(H)}P_{i_{\dim(H)}} = \{s\}, \text{ where } s \in B_h(c_W).$$

(b) *For every constant $c > 0$ and all $s \in H$, there is a r_0 such that for all $r > r_0$ we get that*

$$|\{u \in U_i(r) \mid s \in uP_i\}| \leq c|U_i(r)|.$$

We postpone the proof of the proposition in order to first prove the lower bound.

Proposition 6.2.18 (Lower bound) *With the assumptions of Theorem 6.0.1 we have*

$$p(r) \gg r^{\mathrm{homdim}(G)\cdot\dim(H)}.$$

Proof. Let (h, k) be a good pair that fulfils Proposition 6.2.17. And let $U_i(r)$ and $B_h(c_W)$ be chosen as above. Pick out one $I \subseteq \{1, ..., N\}$ with $|I| = \dim(H)$, then the hyperplane arrangement $\mathcal{H} = \{U_i(r)P_i \mid i \in I\}$ fulfils the assumptions of Theorem 4.3.14. So by Proposition 6.2.16 we get

$$p(r) = \left|A_r^H\right| \geq \#\pi_0\left(B_h(c_W) \setminus \bigcup_{i=1}^{N} \bigcup_{s \in U_i(r)} s\partial_i W\right) \gg |U_i(r)|^{\dim(H)}.$$

And by Corollary 2.3.5 we know the growth of the $U_i(r)$ so

$$p(r) \gg r^{\mathrm{homdim}(G)\cdot\dim(H)}.$$

□

At this point we have proved Theorem 6.0.1, the only thing missing is the proof of Proposition 6.2.17, which will be done now.

Lemma 6.2.19 *For $i, j \in \{1, ..., \dim(H)\}$, $i \neq j$, there exists a good pair (k, h) such that $\forall u \in U_i = B_k(c_W p_i^{-1}), v \in U_j$ holds uP_i and vP_j are not almost parallel with respect to $B_h(c_W)$.*

Proof. Fix some $i, j \in \{1, ..., N\}$ with $i \neq j$. By Corollary 6.2.8 all the uP_i, vP_j with $u \in U_i$, $v \in U_j$ intersect $B_k(c_W)$.
Further we can control the angle between the two hyperplanes by Lemma 6.2.11 so that for all $u \in U_i, v \in U_j$:

$$\sphericalangle(uP_i, vP_j) \geq \sphericalangle(P_i, P_j) - \sphericalangle(uP_i, P_j) - \sphericalangle(P_i, vP_j) \geq \alpha_{ij} - 2\beta(k),$$

where $\beta(k)$ is from Lemma 6.2.11. We can choose k small enough such that we have $0 < \alpha_{ij} - 2\beta(k) < \frac{\pi}{2}$, this means that the hyperplanes can not be parallel so they intersect somewhere. For two hyperplanes which intersect the same ball of radius k and which intersect in at least a given angle there is a bound for the distance of the $(d-1)$-dimensional intersection and the center point of the ball

$$c(k) := k\left(1 + \frac{1}{\tan\left(\frac{\alpha_{ij} - 2\beta(k)}{2}\right)}\right).$$

The idea how to establish this bound is to consider the space which is orthogonal to the intersection of uP_i and vP_j and contains c_W. Than one can argue in a two-dimensional plane.
The bound $c(k)$ goes to zero if k goes to zero, so we can choose k so small that $c(k) < h$. Therefore the two planes intersect inside $B_h(c_W)$. □

Convention 6.2.20 We choose $i_1, ..., i_{\dim(H)}$ such that $\bigcap_{l=1}^{\dim(H)} c_W p_{i_l}^{-1} P_{i_l} = \{c_W\}$. So this is a set of hyperplanes in which each intersection of k hyperplanes has dimension $\dim(H) - k$. From now on we fix such a family and denote it by $\mathcal{F}$. Without loss of generality $\mathcal{F} = \{P_1, ..., P_{\dim(H)}\}$.

Corollary 6.2.21 *There exists a good pair* $(k, h) \in \mathbb{R}^2$ *such that for all* $u_i \in U_i$ *and* $i \in \{1, ..., \dim(H)\}$ *it holds that*

$$\bigcap_{i=1}^{\dim(H)} u_i P_i = \{x\} \in B_h(c_W).$$

Proof. By the choice of the family $\mathcal{F}$ we know that $\bigcap_{i=1}^{\dim(H)} c_W p_i^{-1} P_i = \{c_W\}$, we will first show that there is a k_0 such that for all $0 < k \leq k_0$ this also holds if we replace $c_W p_i^{-1}$ by $u_i \in U_i = B_k(c_W p_i^{-1})$.
This intersection behaviour means that if we choose some vector $v \parallel c_W p_i^{-1} P_i$ then $v \parallel c_W p_j^{-1} P_j$ can at most hold for all but one j, since otherwise the intersection of all hyperplanes would end up in a line instead of a point. We have to choose k_0 such that for all $i \in \{1, ..., \dim(H)\}$ and all $v \parallel u_i H_i$, $u_i \in U_i$, there exists a $j \in \{1, ..., \dim(H)\}$ and a $u_j \in U_j$ such that $v \nparallel u_j P_j$. Since operating with an element from U_i only rotates the hyperplane a little it is possible to find such a k_0 and than the property also holds for all k smaller than k_0.
Now we have to check that the intersection point also lies inside of $B_h(c_W)$. We do this stepwise. It is clear that $\bigcap_{i=1}^{\dim(H)} c_W p_i^{-1} P_i = \{c_W\}$ and $c_W \in B_h(c_W)$. Now we change $c_W p_1^{-1}$ to some $u_1 \in U_1$ and consider $u_1 P_1 \cap \bigcap_{i=2}^{\dim(H)} c_W p_i^{-1} P_i = \{x_1\}$. We already know that $\bigcap_{i=2}^{\dim(H)} c_W p_i^{-1} P_i$ is a subspace of dimension 1 and that $u_1 P_1$ intersects this subspace. Since $u_1 = a_1 c_W p_1^{-1}$ with $a_1 \in B_k(e)$ the hyperplane $u_1 P_1$ is just a small shift, this follows from the form of the group action, and a small rotation away from $c_W p_1^{-1} P_1$, this follows from Lemma 6.2.11. Therefore $d(x_1, c_W) < \varepsilon_1(k)$, where ε_1 depends on k and goes to zero if k goes to zero. We can iterate this process and get a new solution on

each step until we end at x_d, $d = \dim(H)$, where we have

$$d(x_d, c_W) < d(x_d, x_{d-1}) + d(x_{d-1}, x_{d-2}) + ... + d(x_2, x_1) + d(x_1, c_W) < \sum_{i=1}^{d} \varepsilon_i(k) =: \varepsilon(k).$$

So by choosing k such that $\varepsilon(k) < h$ we get the claim. This is possible since $\varepsilon(k) \to 0$ for $k \to 0$. □

The corollary tells us that all intersections result in a single point in $B_h(c_W)$, but it is not clear that different choices of u_j result in different intersection points. This is a major difference to the Euclidean case, since here the action is just translation, so by acting on a hyperplane we get a parallel hyperplane, which then either is still the same hyperplane or does not intersect the original hyperplane at all.

For the second part of Proposition 6.2.17 we use the ergodic theory we recalled in Section 5.5.

Lemma 6.2.22 *Consider a family $U_i(r) \cdot P_i$. For any constant $c > 0$ and all $s \in H$ there is a $r_0 > 0$ such that for all $r \geq r_0$ we get that*

$$\left|\{u \in U_i(r) \mid s \in uP_i\}\right| \leq c \cdot \left|U_i(r)\right|.$$

Proof. Let $u \in U_i(r)$ such that $s \in uP_i$. This implies that $u^{-1} \in P_i s^{-1}$, thus $u \in U_i(r) \cap (P_i s^{-1})^{-1}$. So the question is how many elements are in $U_i(r) \cap (P_i s^{-1})^{-1}$ compared to the number of elements in $U_i(r)$. To get an estimate via the Haar measure we have to thicken $(P_i s^{-1})^{-1}$ since it is a subset of lower dimension. We consider an ε-strip around the set, so we choose a finite set $A(\varepsilon) \subset (P_i s^{-1})^{-1} \cap U_i$ such that

$$U_i \cap (P_i s^{-1})^{-1} \subset \bigcup_{p \in A(\varepsilon)} B_\varepsilon(p)$$

and further let $S_\varepsilon := U_i(r) \cap \bigcup_{p \in A(\varepsilon)} B_\varepsilon(p)$. We have seen that we can use Theorem 5.5.4 so for every $\delta > 0$ and r large enough

$$\begin{aligned}\delta &\geq \left||U_i(r)| - \mu_{G\times H}(B_r(e) \times U_i)\right| = \left||U_i(r)| - \mu_G(B_r(e))\mu_H(U_i)\right| \\ &= \left||U_i(r)| - r^{\mathrm{homdim}(G)}\mu_G(B_1(e))\mu_H(U_i)\right|.\end{aligned}$$

Since S_ε is a finite union of balls we can use the same argument for all the balls simultaneously and get that

$$\begin{aligned}\lim_{r\to\infty} \frac{\left|U_i(r) \cap (P_i s^{-1})^{-1}\right|}{\left|U_i(r)\right|} &< \lim_{r\to\infty} \frac{\left|S_\varepsilon\right|}{\left|U_i(r)\right|} = \frac{\sum_{p\in A(\varepsilon)} r^{\mathrm{homdim}(G)}\mu_G(B_1(e))\mu_H(B_\varepsilon(p))}{r^{\mathrm{homdim}(G)}\mu_G(B_1(e))\mu_H(U_i)} \\ &= \frac{|A(\varepsilon)| \cdot (\mu_H(B_\varepsilon(e)))}{\mu_H(U_i)} \xrightarrow{\varepsilon\to 0} 0.\end{aligned}$$

□

7 Complexity of Model Sets in Hyperbolic Space

In this chapter we extend our arguments to hyperbolic space or more precisely to its isometry group. We will consider a model set where the window W is a lift of a set $\widetilde{W}$, which lives in hyperbolic space. The window will be lifted to the isometry group. Then we will show that the decomposition into acceptance domains of W is in one to one correspondence with a decomposition of $\widetilde{W}$. By this argument and Section 4.4 we can use the same techniques as in Euclidean space, namely counting regions in hyperplane arrangements. The result we get is not surprising, the complexity function of such a model set will grow exponentially since the volume of balls in hyperbolic space grows exponentially.

7.1 Basics on Hyperbolic Space

We denote by $\mathcal{H}^n$ the hyperbolic space of dimension n without assuming any model to be chosen. Further we consider two models of hyperbolic space. First the projective model of hyperbolic space, given by

$$D^n = \{x \in \mathbb{R}^n \mid |x| < 1\}$$

equipped with the metric

$$d_D(x,y) = \cosh^{-1}\left(\frac{1 - xy}{\sqrt{1-|x|^2}\sqrt{1-|y|^2}}\right).$$

Be aware that this model is not conformal, this means that the angles one sees on a picture, the Euclidean angles, are not the same as the hyperbolic angles. The advantage in considering this model for our purpose is that the planes of different dimensions are merely planes in $\mathbb{R}^n$ intersected with D^n, see Theorem 7.1.8 below. Another advantage is that convexity in this model is the same as convexity in the corresponding Euclidean space, since hyperbolic geodesics are straight, in the Euclidean sense.

The second model we will use is the hyperboloid model. So consider $\mathbb{R}^{n+1}$ as a Lorentzian space and define

$$H^n := \left\{x \in \mathbb{R}^{n+1} \,\middle|\, ||x||_L^2 = -1 \text{ and } x_{n+1} > 0\right\}.$$

Here $||x||_L$ denotes the Lorentzian norm of x, i.e. $||x||_L^2 = |(x_1, ..., x_n)|^2 - x_{n+1}^2$. The distance of $x, y \in H^n$ is given by

$$d_H(x,y) = \cosh^{-1}\left(\sum_{i=1}^{n} x_i y_i - x_{n+1}y_{n+1}\right).$$

The reason why we consider this model is that the isometry group is easily understood.

Lemma 7.1.1 ([65, Theorem 3.2.3. + Corollary])
Every positive Lorentz transformation of $\mathbb{R}^{n+1}$ restricts to an isometry of H^n, and every isometry of H^n extends to a unique positive Lorentz transformation of $\mathbb{R}^{n+1}$.
Moreover the restriction induces an isomorphism from the positive Lorentz group, $O^+(n,1)$, to the group of hyperbolic isometries $\mathrm{Isom}(H^n)$.

By this lemma we see that the isometry group is simply the matrix group $O^+(n,1)$ which acts on H^n in the usual way, i.e. matrix multiplication. This is something we understand, which makes considering the hyperboloid model preferable for questions revolving around the action of the isometry group. The group $O^+(n,1)$ is given by the Lorentzian matrices with positive entry at the $(n+1,n+1)$-th position. A matrix A is called *Lorentzian* if

$$A^{\mathrm{T}}JA = J, \text{ where } J = \mathrm{diag}(1,...,1,-1).$$

Further we will need to switch between the two models, this can be done by the isometries

$$\mu : D^n \to H^n, \quad x \mapsto \frac{x+e_{n+1}}{|||x+e_{n+1}||_L|},$$
$$\mu^{-1} : H^n \to D^n, \quad x \mapsto \left(\frac{x_1}{x_{n+1}},...,\frac{x_n}{x_{n+1}}\right).$$

We can also push the action of the isometry group from one model to the other by this isomorphisms, i.e. if $f \in O^+(1,n)$ and $x \in D^n$ then $f(x) := \mu^{-1}(f(\mu(x)))$.
We will only consider the action of the isometry group $O^+(n,1)$. There are more ways in which we can make a group act on hyperbolic space, by changing the model of hyperbolic space. But this will always provide an isomorphic group, so our results hold for 'the' isometry group of $\mathcal{H}^n$.

7.1.1 Volume Growth

It is commonly known that the volume of a hyperbolic ball $B_r(h)$, $h \in \mathcal{H}^n$, grows exponentially, see for example [65, §3.4]. The explicit formula for the volume of a ball $B_r(h) \subset \mathcal{H}^n$ is given by

$$\mu_{\mathcal{H}}(B_r(h)) = c_n \int_0^r \sinh^{n-1}(t)dt \asymp e^{(n-1)r},$$

where c_n is a constant, which is the volume of the sphere of dimension $n-1$.

Following Ratcliffe, [65, §11.6 Haar Measure], we will normalize the Haar measure on the isometry group in the following way.

Definition 7.1.2 We denote by $\mathrm{Isom}(\mathcal{H}^n)$ the *isometry group of* $\mathcal{H}^n$. Let further K be the compact subgroup that fixes the origin in D^n, then the Haar integral can be expressed as

$$\int_G \phi(g) d\mu_{\mathrm{Isom}(\mathcal{H}^n)} = \int_{\mathrm{Isom}(\mathcal{H}^n)/K} \left(\int_K \phi(gh) d\mu_K \right) d\mu_{gK},$$

where $d\mu_K$ is the left-invariant Haar measure on K and μ_{gK} the left-invariant measure on $\mathrm{Isom}(\mathcal{H}^n)/K$. We normalize the Haar measure $\mu_{\mathrm{Isom}(\mathcal{H}^n)}$ by

$$\int_K d\mu_K = 1.$$

Lemma 7.1.3 ([65, Lemma 4, page 558])
Let $x_0 \in \mathcal{H}^n$ and let $R \subseteq \mathcal{H}^n$ be an open(closed) subset. Further let

$$S = \{ g \in \mathrm{Isom}(\mathcal{H}^n) \,|\, g.x_0 \in R \}.$$

Then S is open(closed) and the Haar measure of S is the volume of the set R.

We can use the Growth Lemma, Proposition 2.3.1, in this set-up and get

Lemma 7.1.4 *Let $(\mathrm{Isom}(\mathcal{H}^n), \mathrm{Isom}(\mathcal{H}^d), \Gamma)$ be a cut and project scheme and let $\emptyset \neq A \subset \mathrm{Isom}(\mathcal{H}^d)$ be a bounded open set. The asymptotic growth of the number of lattice points inside $B_r^{\mathrm{Isom}(\mathcal{H}^n)}(e) \times A$ is given by*

$$\left| \left(B_r^{\mathrm{Isom}(\mathcal{H}^n)}(e) \times A \right) \cap \Gamma \right| \asymp \mu_G \left(B_r^{\mathrm{Isom}(\mathcal{H}^n)}(e) \right) \asymp e^{(n-1)r},$$

as $r \to \infty$.

7.1.2 Metrics on Isometry Groups

The following Proposition is stated in the book by Cornulier and de la Harpe, [19], for a left-invariant metric, see also [18, Section 4]. In [16] it is stated in an example on page 8 and motivated by [62, Proposition 4.4.6.]

Proposition 7.1.5 *Let (X, d_X) be a proper metric space with basepoint $x_0 \in X$. The function d defined on $\mathrm{Isom}(X) \times \mathrm{Isom}(X)$ by*

$$d(f, g) = \sup_{x \in X} d_X \left(f^{-1}(x), g^{-1}(x) \right) e^{-d_X(x_0, x)}$$

is a right-invariant proper compatible metric on $\mathrm{Isom}(X)$.

Lemma 7.1.6 ([16, Proposition 2.1])
Let (X, d_X) be a proper metric space with basepoint $x_0 \in X$ and d defined as in Proposition 7.1.5. Let $f, g \in \operatorname{Isom}(X)$ with $d(f,g) < r$, then

$$d_X\left(f^{-1}(x_0), g^{-1}(x_0)\right) < r.$$

Contrary if $d_X(z,y) < r$ and $f, g \in \operatorname{Isom}(X)$ such that $z = f^{-1}(x_0)$ and $y = g^{-1}(x_0)$, then $d(f,g) < r + \frac{2}{e}$.

Proof. This is a direct computation:

$$\begin{aligned} d_X\left(f^{-1}(x_0), g^{-1}(x_0)\right) &= d_X\left(f^{-1}(x_0), g^{-1}(x_0)\right) e^{-d_X(x_0,x_0)} \\ &\leq \sup_{x \in X} d_X\left(f^{-1}(x), g^{-1}(x)\right) e^{-d_X(x_0,x)} = d(f,g) < r. \end{aligned}$$

And for the second conclusion:

$$\begin{aligned} &d(f,g) = \sup_{x \in X} d_X\left(f^{-1}(x), g^{-1}(x)\right) e^{-d_X(x_0,x)} \\ &\leq \sup_{x \in X} \left(d_X\left(f^{-1}(x), f^{-1}(x_0)\right) + d_X\left(f^{-1}(x_0), g^{-1}(x_0)\right) + d_X\left(g^{-1}(x_0), g^{-1}(x)\right)\right) e^{-d_X(x_0,x)} \\ &= \sup_{x \in X} (d_X(z,y) + 2d_X(x,x_0)) e^{-d_X(x_0,x)} < r + \frac{2}{e}. \end{aligned}$$

□

7.1.3 Hyperbolic Polytopes

Since our whole argument relies on our window being polytopal we discuss how polytopes in hyperbolic space look. We import some properties from the book by Ratcliffe, [65].

Definition 7.1.7 A *side* of a convex subset C of $\mathcal{H}^n$ is a non-empty, maximal, convex subset of ∂C.

Theorem 7.1.8 ([65, Theorem 6.1.4.])
A subset $P \subset D^n$ is a hyperbolic m-plane of D^n if and only if P is the non-empty intersection of D^n with an m-plane of $\mathbb{R}^n$.

Remark 7.1.9 Ratcliffe denotes by m-planes the planes of dimension m, i.e. a $(d-1)$-plane is a hyperplane, a 1-plane is a line and a 0-plane is a point.

Theorem 7.1.10 ([65, Theorem 6.2.4.])
If S is a side of a convex subset C of $\mathcal{H}^n$, then $\overline{C} \cap \langle S \rangle = S$, where $\langle S \rangle$ denotes the plane spanned by S.

Definition 7.1.11 Let P be a hyperplane in $\mathcal{H}^n$. An *open half-space* of $\mathcal{H}^n$ is a connected component of $\mathcal{H}^n \setminus P$.

Lemma 7.1.12 ([65, Exercise 6.2.4])
Let C be a closed proper subset of $\mathcal{H}^n$. Then C is convex if and only if C is the intersection of all closed half-spaces of $\mathcal{H}^n$ which contain C.

Definition 7.1.13 A *convex polyhedron* P in $\mathcal{H}^n$ is a non-empty, closed, convex subset of $\mathcal{H}^n$ such that the collection of its sides is locally finite in $\mathcal{H}^n$.

Theorem 7.1.14 ([65, Theorem 6.3.1])
Every side of an m-dimensional convex polyhedron P in $\mathcal{H}^n$ has dimension $m-1$.

Theorem 7.1.15 ([65, Theorem 6.3.6.])
An m-dimensional convex polyhedron P in $\mathcal{H}^n$, with $m > 0$, is compact if and only if

(a) the polyhedron P has at least $m+1$ sides,

(b) the polyhedron P has only finitely many sides,

(c) each side of P is compact.

Corollary 7.1.16 *Let P be a n-dimensional compact convex polyhedron, then P is the intersection of finitely many half-spaces.*

Definition 7.1.17 A *polytope* in $\mathcal{H}^n$ is a convex polyhedron P in $\mathcal{H}^n$ such that

(a) P has only finitely many vertices,

(b) P is the convex hull of its vertices.

Theorem 7.1.18 ([65, Theorem 6.5.1.])
A convex polyhedron P in $\mathcal{H}^n$ is a polytope in $\mathcal{H}^n$ if and only if P is compact.

We thus have seen that compact hyperbolic polytopes are similar to Euclidean ones in the sense that they are given by a finite intersection of half-spaces.

7.2 Hyperbolic Model Sets

We fix an arbitrary basepoint $x_0 \in \mathcal{H}^n$.

Definition 7.2.1 We call $\Lambda\left(\mathrm{Isom}(\mathcal{H}^n), \mathrm{Isom}(\mathcal{H}^d), \Gamma, W\right)$ a *model set with lifted window*, if it is a model set and W is of the form

$$W := \left\{ g \in \mathrm{Isom}\left(\mathcal{H}^d\right) \,\middle|\, g.x_0 \in \widetilde{W} \right\},$$

where $\widetilde{W} \subseteq \mathcal{H}^d$.

Definition 7.2.2 Let $\widetilde{P}$ be a hyperplane in $\mathcal{H}^d$, then we call

$$P := \left\{ f \in \mathrm{Isom}(\mathcal{H}^d) \,\middle|\, f.x_0 \in \widetilde{P} \right\}$$

a *lifted hyperplane* in the isometry group. We also use the notation for the induced half-spaces, i.e. let $\widetilde{P^+}$ be a half-space in $\mathcal{H}^d$ with bounding hyperplane $\widetilde{P}$, then we call $P^+ := \left\{ f \in \mathrm{Isom}(\mathcal{H}^d) \,\middle|\, f.x_0 \in \widetilde{P^+} \right\}$ a *lifted half-space* in the isometry group with bounding hyperplane P.

Lemma 7.2.3 ([65, Page 62 Corollary 4.])
The group of hyperbolic isometries $\mathrm{Isom}\left(\mathcal{H}^d\right)$ *acts transitively on the set of hyperbolic* m*-planes of* $\mathcal{H}^d$ *for each dimension* m*.*

Definition 7.2.4 A subset $W \subset \mathrm{Isom}\left(\mathcal{H}^d\right)$ is called a *polytope* if it is the intersection of finitely many lifted half-spaces and it has finite volume. The intersection of the corresponding half-spaces in $\mathcal{H}^d$ is then denoted by $\widetilde{W}$.

Convention 7.2.5 We use the same notation as in the nilpotent case, i.e.

$$W = \bigcap_{i=1}^{N} P_i^+.$$

Definition 7.2.6 We call $\Lambda\left(\mathrm{Isom}(\mathcal{H}^n), \mathrm{Isom}(\mathcal{H}^d), \Gamma, W\right)$ a *polytopal model set*, if it is a model set and W is a (lifted) polytope.

Proposition 7.2.7 *Let* $\Lambda\left(\mathrm{Isom}\left(\mathcal{H}^n\right), \mathrm{Isom}\left(\mathcal{H}^d\right), \Gamma, W\right)$ *be a polytopal model set and* $r > 0$*. Then the* r*-acceptance domains are in one to one correspondence with the subsets of* $\widetilde{W}$ *of the form*

$$\widetilde{W_r(\lambda)} := \left(\bigcap_{\mu \in \mathcal{S}_r(\lambda)} \mu \overset{\circ}{\widetilde{W}} \right) \cap \left(\bigcap_{\mu \in \mathcal{S}_r(\lambda)^{\mathrm{C}}} \mu \widetilde{W}^{\mathrm{C}} \right).$$

Proof. Let $\lambda \in \Lambda$. The r-acceptance domain of λ is given by

$$W_r(\lambda) = \left(\bigcap_{\mu \in \mathcal{S}_r(\lambda)} \mu \mathring{W} \right) \cap \left(\bigcap_{\mu \in \mathcal{S}_r(\lambda)^{\mathrm{C}}} \mu W^{\mathrm{C}} \right).$$

Consider μW, this is nothing else than

$$\begin{aligned} \mu W &= \left\{ \mu g \in \mathrm{Isom}(\mathcal{H}^d) \,\middle|\, g.x_0 \in \widetilde{W} \right\} \\ &= \left\{ g \in \mathrm{Isom}(\mathcal{H}^d) \,\middle|\, \left(\mu^{-1} g\right).x_0 \in \widetilde{W} \right\} \\ &= \left\{ g \in \mathrm{Isom}(\mathcal{H}^d) \,\middle|\, g.x_0 \in \mu\left(\widetilde{W}\right) \right\}. \end{aligned}$$

This also works if we consider μW^{C}. So this means

$$W_r(\lambda) = \left\{ g \in \mathrm{Isom}(\mathcal{H}^d) \,\middle|\, g.x_0 \in \left(\bigcap_{\mu \in \mathcal{S}_r(\lambda)} \mu \mathring{\widetilde{W}} \right) \cap \left(\bigcap_{\mu \in \mathcal{S}_r(\lambda)^{\mathrm{C}}} \mu \widetilde{W}^{\mathrm{C}} \right) \right\}.$$

□

Definition 7.2.8 Let $\Lambda\left(\mathrm{Isom}(\mathcal{H}^n), \mathrm{Isom}(\mathcal{H}^d), \Gamma, W\right)$ be a polytopal model set. The sets $\widetilde{W_r(\lambda)}$ are called the *induced r-acceptance domains of* Λ. The set of all induced r-acceptance domains of Λ is denoted by $\widetilde{W_r}$.

Corollary 7.2.9 *Let $\Lambda\left(\mathrm{Isom}(\mathcal{H}^n), \mathrm{Isom}(\mathcal{H}^d), \Gamma, W\right)$ be a polytopal model set, then $p(r) = \left|\widetilde{W_r}\right|$.*

Theorem 7.2.10 *Let $\Lambda\left(\mathrm{Isom}(\mathcal{H}^n), \mathrm{Isom}(\mathcal{H}^d), \Gamma, W\right)$ be a uniform polytopal model set, then $p(r) \asymp e^{d(n-1)r}$.*

7.2.1 Upper Bound

Lemma 7.2.11 *Let $\Lambda\left(\mathrm{Isom}(\mathcal{H}^n), \mathrm{Isom}(\mathcal{H}^d), \Gamma, W\right)$ be a polytopal model set, then*

$$p(r) = \left|\widetilde{W_r}\right| \leq \#\pi_0 \left(\widetilde{W} \setminus \bigcup_{\mu \in \mathcal{S}_r} \mu \partial \widetilde{W} \right).$$

Proof. The the proof follows the lines of the proof of Lemma 3.0.12. □

Lemma 7.2.12 *Let $\Lambda\left(\mathrm{Isom}(\mathcal{H}^n), \mathrm{Isom}(\mathcal{H}^d), \Gamma, W\right)$ be a polytopal model set, then*

$$\#\pi_0\left(\widetilde{W} \setminus \bigcup_{\mu\in\mathcal{S}_r} \mu\partial\widetilde{W}\right) \leq \#\pi_0\left(\mathcal{H}^d \setminus \bigcup_{\mu\in\mathcal{S}_r}\bigcup_{i=1}^{N} \mu\widetilde{P}_i\right).$$

Proof. This proof is similar to Lemma 6.1.1. Since $\mu\partial\widetilde{W} \subset \bigcup_{i=1}^N \mu P_i$ we clearly have

$$\#\pi_0\left(\widetilde{W} \setminus \bigcup_{\mu\in\mathcal{S}_r} \mu\partial\widetilde{W}\right) \leq \#\pi_0\left(W \setminus \bigcup_{\mu\in\mathcal{S}_r}\bigcup_{i=1}^{N} \mu\widetilde{P}_i\right).$$

And since $e \in \mathcal{S}_r$ for all $r > 0$ we have $\partial\widetilde{W} \subset \bigcup_{\mu\in\mathcal{S}_r}\bigcup_{i=1}^N \mu\widetilde{P}_i$. Therefore if we replace $\widetilde{W}$ by $\mathcal{H}^d$ we do not change the regions inside W and add the regions outside of W. Therefore

$$\#\pi_0\left(W \setminus \bigcup_{\mu\in\mathcal{S}_r}\bigcup_{i=1}^{N} \mu\widetilde{P}_i\right) \leq \#\pi_0\left(\mathcal{H}^d \setminus \bigcup_{\mu\in\mathcal{S}_r}\bigcup_{i=1}^{N} \mu\widetilde{P}_i\right).$$

□

Proposition 7.2.13 (Upper bound)
Let $\Lambda\left(\mathrm{Isom}(\mathcal{H}^n), \mathrm{Isom}(\mathcal{H}^d), \Gamma, W\right)$ be a polytopal model set, then

$$p(r) \ll e^{d(n-1)r}.$$

Proof. By Lemma 7.2.11 and Lemma 7.2.12 it is

$$p(r) \leq \#\pi_0\left(\mathcal{H}^d \setminus \bigcup_{\mu\in\mathcal{S}_r}\bigcup_{i=1}^{N} \mu P_i\right).$$

By considering $\mathcal{H}^d$ in the projective model this is nothing else than a hyperplane arrangement in $\mathbb{R}^d$ intersected with a ball. Hereby the hyperplanes are the Euclidean hyperplanes $\{\mu P_i \mid \mu \in \mathcal{S}_r\}$. For this we know an upper bound by Proposition 4.1.9

$$p(r) \leq \#\pi_0\left(\mathcal{H}^d \setminus \bigcup_{\mu\in\mathcal{S}_r}\bigcup_{i=1}^{N} \mu P_i\right) \leq \sum_{i=1}^{d}\binom{N\cdot|\,\mathcal{S}_r\,|}{i} \ll (N\cdot|\,\mathcal{S}_r\,|)^d \asymp e^{d(n-1)r}.$$

□

7.2.2 Lower Bound

We use the same notation as in Definition 6.2.1 for the window $\widetilde{W} \subset \mathcal{H}^d$ and we also use the terms 'cuts fully' and 'cuts all-round' as defined in Definition 6.2.3. Similar to the nilpotent case we will consider two 'small' balls

$$B_k(c_W) \subset B_h(c_W) \subset W$$

with fixed but variable radii k and h. Further we also choose a new basepoint, we will see that the center of the window c_W is a good choice.

Definition 7.2.14 We fix a choice of $q_i \in \text{Isom}(\mathcal{H}^d)$ with $q_i(p_i) = c_W$ for every $i \in \{1, ..., N\}$ and such that $q_i\left(\widetilde{P}_i\right) \neq q_j\left(\widetilde{P}_j\right)$ for $i \neq j$ and

$$\bigcap_{i=1}^{N} q_i\left(\widetilde{P}_i\right) = \{c_W\}.$$

Further set $U_i := B_k(q_i) \subset WW^{-1}$.

Remark 7.2.15 The choice $U_i := B_k(q_i) \subset WW^{-1}$ is possible since $q_i \in W^{-1}$, because $q_i^{-1}(c_W) = p_i \in W$. And also $Id \in W$, since we have chosen the basepoint c_W, which lies inside the window.

Lemma 7.2.16 *If $k + h < F_W$ and $f \in B_k(q_i)$, then $f\left(\widetilde{P}_i\right)$ cuts $B_h(c_W)$ fully.*

Proof. Since $f \in B_k(q_i)$ we can write $f = sq_i$ with $s \in B_k(\text{Id})$.

$$\begin{aligned} d_{\mathcal{H}}(f(p_i), c_W) &= d_{\mathcal{H}}(sq_i(p_i), c_W) = d_{\mathcal{H}}(s(c_W), c_W)e^{-d_{\mathcal{H}}(c_W, c_W)} \\ &\leq \sup_{x \in X} d_{\mathcal{H}}(s(x), x)e^{-d_{\mathcal{H}}(c_W, x)} = d(s, \text{Id}) < k \end{aligned}$$

Therefore $f\left(\widetilde{P}_i\right)$ intersects $B_k(c_W)$. Further we have to show that

$$f\left(\widetilde{\partial_i W}\right) \cap B_h(c_W) = f\left(\widetilde{P}_i\right) \cap B_h(c_W),$$

but this is equivalent to

$$\widetilde{\partial_i W} \cap B_h\left(f^{-1}(c_W)\right) = \widetilde{P}_i \cap B_h\left(f^{-1}(c_W)\right), \tag{7.2.1}$$

since f^{-1} is an isometry. The inclusion '$\subseteq$' in equation (7.2.1) is trivial, since $\widetilde{\partial_i W} \subset \widetilde{P}_i$. For the other inclusion we have to show that $B_h\left(f^{-1}(c_W)\right) \subset B_{F_W}(p_i)$.

$$d_{\mathcal{H}}\left(f^{-1}(c_W), p_i\right) = d_{\mathcal{H}}\left(f^{-1}(c_W), q_i^{-1}(c_W)\right) < k$$

by Lemma 7.1.6, since $d(f, q_i) < k$. And by the assumption $k + h < F_W$ we get the claim. □

Corollary 7.2.17 *For all $f \in B_k(q_i)$ we have $f\left(\widetilde{P}_i\right)$ intersects $B_k(c_W)$.*

Definition 7.2.18 For a given polytopal window W we fix a constant ι_W such that for all $i \in \{1, ..., N\}$

$$B_{\iota_W}(p_i) \cap \widetilde{W} = B_{\iota_W}(p_i) \cap \widetilde{P_i^+}.$$

Remark 7.2.19 Clearly $\iota_W \leq F_W$.

Lemma 7.2.20 *If $k + h < \iota_W$ and $f \in B_k(q_i)$, then $f\left(\widetilde{P_i}\right)$ cuts $B_h(c_W)$ all-round.*

Proof. We have seen in Lemma 7.2.16 that $f\left(\widetilde{P_i}\right)$ cuts $B_h(c_W)$ fully, so we only have to show that

$$f\left(\widetilde{P_i^+}\right) \cap B_h(c_W) = f\left(\widetilde{W}\right) \cap B_h(c_W).$$

Since $\widetilde{W} \subset \widetilde{P_i^+}$ the inclusion '$\supseteq$' is clear. An equivalent formulation of the condition is

$$\widetilde{P_i^+} \cap B_h\left(f^{-1}(c_W)\right) = \widetilde{W} \cap B_h\left(f^{-1}(c_W)\right).$$

We observe that, by writing $f = sq_i$ with $s \in B_k(\mathrm{Id})$,

$$d_{\mathcal{H}^n}\left(f^{-1}(c_W), p_i\right) = d_{\mathcal{H}^n}\left(q_i^{-1}s^{-1}(c_W), q_i^{-1}(c_W)\right) = d_{\mathcal{H}^n}\left(s^{-1}(c_W), c_W\right) < k,$$

by Lemma 7.1.6. Then $B_h\left(f^{-1}(c_W)\right) \subset B_{\iota_W}(p_i)$ and therefore

$$\widetilde{P_i^+} \cap B_h\left(f^{-1}(c_W)\right) = \widetilde{W} \cap B_h\left(f^{-1}(c_W)\right).$$

□

Definition 7.2.21 Set $U_i(r) := \pi_{\mathrm{Isom}(\mathcal{H}^d)}\left((B_r(\mathrm{Id}) \times B_k(q_i)) \cap \Gamma\right) \subset \mathcal{S}_r$.

Proposition 7.2.22 *For $k + h < \iota_W$ the number of connected components of $B_h(c_W) \setminus \bigcup_{i=1}^N \bigcup_{s \in U_i(r)} s\widetilde{\partial_i W}$ is a lower bound for the number of acceptance domains $\left|\widetilde{W_r}\right|$, i.e.*

$$\#\pi_0\left(B_h(c_W) \setminus \bigcup_{i=1}^{N} \bigcup_{f \in U_i(r)} f\left(\widetilde{\partial_i W}\right)\right) = \#\pi_0\left(B_h(c_W) \setminus \bigcup_{i=1}^{N} \bigcup_{f \in U_i(r)} f\left(\widetilde{P_i}\right)\right) \leq \left|\widetilde{W_r}\right|.$$

Proof. The proof follows as in Proposition 6.2.16. Let C be a connected component of $B_h(c_W) \setminus \bigcup_{i=1}^N \bigcup_{f \in U_i(r)} f\left(\widetilde{\partial_i W}\right)$. By Lemma 7.2.20 we can replace the faces by the hyperplanes without changing the connected components in $B_h(c_W)$, so we consider $B_h(c_W) \setminus \bigcup_{i=1}^N \bigcup_{f \in U_i(r)} f\left(\widetilde{P_i}\right)$.
We will show that if an acceptance domain intersects a connected component of $B_h(c_W) \setminus \bigcup_{i=1}^N \bigcup_{f \in U_i(r)} f\left(\widetilde{P_i}\right)$ it is already fully contained in it. Let C' be another, but different, connected component of $B_h(c_W) \setminus \bigcup_{i=1}^N \bigcup_{f \in U_i(r)} f\left(\widetilde{P_i}\right)$ and assume that $C \cap \widetilde{W_r(\lambda)} \neq \emptyset \neq C' \cap \widetilde{W_r(\lambda)}$. Between C and C' is a hyperplane $f\left(\widetilde{P_i}\right)$ for some

$i \in \{1, ..., N\}$ and $f \in U_i(r)$. Therefore $C \subset f\left(\overset{\circ}{\widetilde{P^+}}\right)$ and $C' \subset f\left(\overset{\circ}{\widetilde{P^-}}\right)$ or the other way round. And since the cut $f\left(\widetilde{\partial_i W}\right)$ is all-round we get that $C \subset f\left(\overset{\circ}{\widetilde{W}}\right)$ and $C' \subset f\left(\widetilde{W^{\mathrm{C}}}\right)$ or the other way round. But either $\widetilde{W_r(\lambda)} \subset f\left(\overset{\circ}{\widetilde{W}}\right)$ or $\widetilde{W_r(\lambda)} \subset f\left(\widetilde{W^{\mathrm{C}}}\right)$, a contradiction. $\square$

To use Theorem 4.3.14 we have to bound the number of hyperplanes from one family $U_i(r)\left(\widetilde{P_i}\right)$ which go through one point. And we further have to show that for all $(u_{i_1}, ..., u_{i_d}) \in U_{i_1}(r) \times ... \times U_{i_d}(r)$ we get that

$$u_{i_1}\left(\widetilde{P_1}\right) \cap ... \cap u_{i_d}\left(\widetilde{P_d}\right) = \{s\} \in B_h(c_W).$$

As in the nilpotent case we are done if we can control the Euclidean angle between $q_i\left(\widetilde{P}\right)$ and $f\left(\widetilde{P}\right)$, where $f \in B_k(q_i)$. By the Euclidean angle we mean the angle between the two hyperplanes considered as hyperplanes of $\mathbb{R}^n$, while the hyperbolic angle is the angle of the hyperbolic hyperplanes which live inside D^n. Especially note that the Euclidean angle is different from the hyperbolic angle.

Proposition 7.2.23 *Let P, Q be hyperplanes in $\mathbb{R}^n$ with normals n_P and n_Q and Euclidean angle $\cos^{-1}(|\langle n_P \cdot n_Q\rangle|)$. Then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $f \in B_\delta(Id) \subset \mathrm{Isom}(\mathcal{H}^n)$ it is*

$$\left|\cos^{-1}\left(|\langle n_P, n_Q\rangle|\right) - \cos^{-1}\left(|\langle n_{f(P)}, n_{f(Q)}\rangle|\right)\right| < \varepsilon.$$

Proof. The critical part of the proof is the action of the isometry group and how it affects the Euclidean angle. Let P and Q be given by

$$P = \left\{p_0 + \sum_{i=1}^{n-1} t_i v_i \,\middle|\, t_i \in \mathbb{R}\right\} \quad \text{and} \quad Q = \left\{q_0 + \sum_{i=1}^{n-1} l_i w_i \,\middle|\, l_i \in \mathbb{R}\right\},$$

where $v_i, w_i \in \mathbb{R}^n$ are linearly independent vectors and $p_0, q_0 \in \mathbb{R}^n$ the base points. Further assume that these two Euclidean hyperplanes intersect D^n, then $P \cap D^n$, resp. $Q \cap D^n$, are hyperbolic hyperplanes in the projective model by Theorem 7.1.8.

By Lemma 7.2.3 we know that $f(P) = P'$ and $f(Q) = Q'$ are hyperplanes again. Now consider the action of $f \in O^+(n,1)$, with associated matrix A, on the hyperbolic hyperplanes

$$\begin{aligned} f(P) &= \mu^{-1}(f(\mu(P))) = \mu^{-1}\left(A \cdot \frac{P+e_{n+1}}{|||P+e_{n+1}||_L|}\right) \\ &= \mu^{-1}\left(\frac{1}{|||P+e_{n+1}||_L|}\left(Ap_0 + Ae_{n+1} + \sum_{i=1}^{n-1} t_i A v_i\right)\right) \\ &\overset{(a)}{=} \mu^{-1}\left(Ap_0 + Ae_{n+1} + \sum_{i=1}^{n-1} t_i A v_i\right) \\ &= p_0' + \sum_{i=1}^{n-1} t_i' v_i' = P'. \end{aligned}$$

Where equality at (a) holds by the definition of μ^{-1}. Further the map μ^{-1} simply cuts off the $(n+1)$-th entry of the vector and scales all the others by its size. Since matrix multiplication is continuous in each entry we see that the directions v_i' are not far away from the v_i. This means that the hyperplane is rotated only by a small amount controlled by the norm of A, resp. f. And since $f \in B_\delta(Id)$ we can choose δ such that

$$\left|\cos^{-1}\left(|\langle n_P, n_Q\rangle|\right) - \cos^{-1}\left(|\langle n_{f(P)}, n_{f(Q)}\rangle|\right)\right| < \varepsilon.$$

□

Definition 7.2.24 If k, h fulfil the conditions of Lemma 7.2.20, Proposition 7.2.22 and Proposition 7.2.23, i.e. $k < \delta$ and $k + h < \iota_W$, we call them a *good pair.*

Corollary 7.2.25 *For a good pair $(k,h) \in \mathbb{R}^2$ and all $f_i \in U_i$, where $I \subseteq \{1,...,N\}$, $|I| = d$, it is*

$$\bigcap_{i \in I} f_i\left(\widetilde{P}_i\right) = \{s\} \in B_h(c_W).$$

Corollary 7.2.26 *Let (k,h) be a good pair. For any constant $c > 0$ and all $s \in B_h(c_W)$ there is an r_0 such that for all $r \geq r_0$ we get that*

$$\left|\left\{f \in U_i(r) \,\middle|\, s \in f\left(\widetilde{P}_i\right)\right\}\right| \leq c \cdot |U_i(r)|.$$

Proof. The proof works as the proof of Lemma 6.2.22 since in the argument we only use Theorem 5.5.4 and that balls in homogeneous Lie groups are well-rounded. But clearly $\mathbb{R}^n$ is a homogeneous Lie group and considering the projective model of hyperbolic space we can use the same techniques. □

Proposition 7.2.27 (Lower bound)
Let $\Lambda\left(\mathrm{Isom}\left(\mathcal{H}^n\right), \mathrm{Isom}\left(\mathcal{H}^d\right), \Gamma, W\right)$ be a polytopal model set, then

$$p(r) \gg e^{d(n-1)r}.$$

Proof. By Proposition 7.2.22 the inequality

$$\#\pi_0\left(B_h(c_W)\setminus\bigcup_{i=1}^{N}\bigcup_{f\in U_i(r)} f\left(\widetilde{P}_i\right)\right)\leq p(r)$$

holds. By considering $\mathcal{H}^d$ in the projective model this is nothing else than a hyperplane arrangement in $\mathbb{R}^d$ intersected with a ball. Hereby the hyperplanes are the Euclidean hyperplanes $\{\mu P_i \mid \mu \in \mathcal{S}_r\}$. And we have further seen that by the choice of $U_i(r)$ the assumptions of Theorem 4.3.14 are fulfilled, therefore

$$|U_i(r)|^d \ll \#\pi_0\left(B_h(c_W)\setminus\bigcup_{i=1}^{N}\bigcup_{f\in U_i(r)} f\left(\widetilde{P}_i\right)\right).$$

And by Lemma 7.1.4 we get

$$|U_i(r)|^d \asymp e^{r(n-1)d}.$$

This yields $p(r) \gg e^{r(n-1)d}$. $\square$

8 Examples

In this chapter we will consider some examples, in which we determine the asymptotic growth rate of the complexity function $p(r)$ of explicit model sets. We will start with two examples in the Euclidean set-up, for which we will use the theorem established by Koivusalo and Walton, [45]. Especially in these two examples we will see how the stabilizers of the bounding hyperplanes have an influence on the complexity function. Afterwards we will give an example in the Heisenberg Group, which we already discussed in Section 5.3. In this new example we will see that the stabilizers do not play an important role in the wider set-up.

To treat the first two examples we cite the theorem for the Euclidean case.

Theorem 8.0.1 (Koivusalo and Walton, [45, Theorem 6.1.])
Let $\Lambda(\mathbb{R}^k, \mathbb{R}^d, \Gamma, W)$ be a polytopal model set. Let further

$$\mathcal{F} := \left\{ I \subset \{1, ..., N\} \,\middle|\, |I| = d, \bigcap_{i \in I} P_i \text{ is a point} \right\}$$

and set

$$\alpha := \max_{f \in \mathcal{F}} \sum_{i \in f} \left(d - \operatorname{rk}\left(\Gamma^{P_i}\right) + \dim\left(\left\langle \left(\Gamma^{P_i}\right)_H \right\rangle\right)\right),$$

where Γ^{P_i} is the preimage of the stabilizer of P_i in Γ_H and $\operatorname{rk}\left(\Gamma^{P_i}\right)$ denotes the rank of this free abelian group. Then

$$p(r) \asymp r^{\alpha}.$$

A direct corollary of this theorem is seen since $\dim\left(\left\langle \left(\Gamma^{P_i}\right)_H \right\rangle\right) \leq \operatorname{rk}\left(\Gamma^{P_i}\right)$.

Corollary 8.0.2 ([45, Corollary 6.2.])
For any regular polytopal model set $\Lambda(\mathbb{R}^k, \mathbb{R}^d, \Gamma, W)$ it is $p(r) \ll r^{d \cdot k}$.

And further we see that in the absence of a stabilizer we get the maximal complexity.

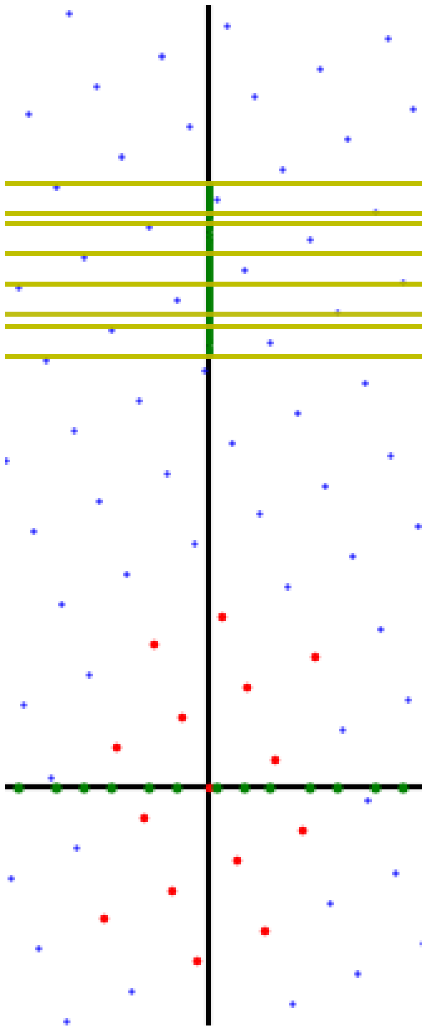
Corollary 8.0.3 ([45, Corollary 6.3.])
For a generic regular polytopal model set $\Lambda(\mathbb{R}^k, \mathbb{R}^d, \Gamma, W)$ we have $p(r) \asymp r^{d \cdot k}$.

8.1 Euclidean Example 1: Silver-Mean Chain

As a first example we choose the simplest possible one, we set $G = \mathbb{R}$ and $H = \mathbb{R}$ and the lattice Γ is spanned by

$$b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad b_2 = \begin{pmatrix} -\sqrt{2} \\ \sqrt{2} \end{pmatrix}.$$

This is the standard datum for a cut and project set to construct the silver-mean chain, see for example [3, Section 7.1.]. We do not really construct the silver-mean chain here since we chose a different window, but the argumentation stays the same.



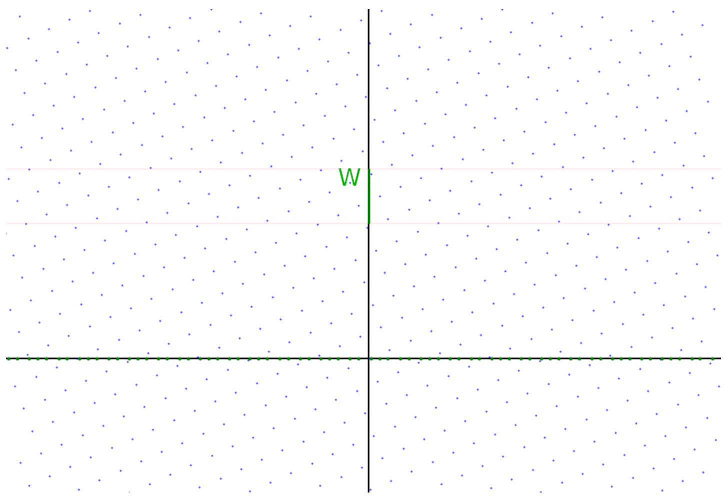


Figure 8.1: In the top picture we see the resulting set-up in which the window W is highlighted, the horizontal axis is G and the vertical one is H. In the picture to the left we see a part of the top picture in which we added additional information. The window is decomposed in 7 acceptance domains. The red dots are the preimage of the slab, the green points on the horizontal axis are the points in Λ.

As a window we choose the interval $[2.5, 3.5] \subset H$. This window is polytopal in a trivial sense, the hyperplanes which bound W are just the two points 2.5 and 3.5. We see that this yields a regular model set since $a + b\sqrt{2} = \frac{1}{2}$ has no integer solution. Also the rank of these hyperplanes is 0, since there is no non-trivial integer solution to $a + b\sqrt{2} = 0$. So putting things together we get

$$p(r) \asymp r^{\alpha}; \quad \alpha = 1.$$

So for the resulting regular polytopal model set $\Lambda(\mathbb{R}, \mathbb{R}, \Gamma, W)$ we get linear growth of the complexity function. The whole setting is shown in Figure 8.1.

Lets briefly talk about the decomposition of the window, it is interesting to see that it seems to be symmetric. On the other hand we notice that the different acceptance domains have different sizes. There is unpublished work in progress by Tobias Hartnick and Maximilian Wackenhuth that connects the size of an acceptance domain with the frequency of the appearance of a corresponding patch in the model set, in the general set-up of lcsc groups.

For the next example we increase the dimension of both G and H. Some phenomena can only be observed if the dimension of H is bigger than one, for example that the acceptance domains are not connected. The advantage when G and H have dimension one is that one can draw the whole cut and project scheme in one two dimensional picture. For higher dimensions one always has to split up the picture into multiple parts, which makes it harder to perceive which point is projected where.

8.2 Euclidean Example 2: Ammann-Beenker Tiling

We consider the Ammann-Beenker tiling, see Figure 8.2, or to be more precise the vertex set of the tiling. In this example we can see what role the stabilizers play and how much work it can be to determine them. It would get even more complicated if we could not use the symmetry of the window and therefore would not be able to treat all the hyperplanes simultaneously.

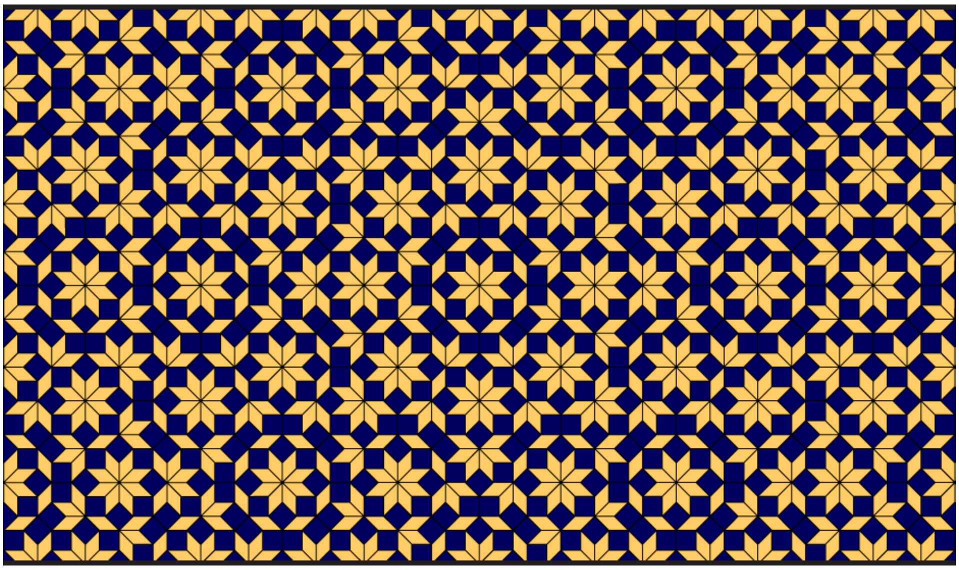


Figure 8.2: A part of an Ammann-Beenker tiling, see [25]. Observe the local 8-fold rotation symmetry.

The Ammann-Beenker tiling is a standard example for a model set and the data with which we can construct it is known. For more information to this we refer to [3, Example 7.8]. Now we state the data for the CPS $\Lambda(\mathbb{R}^2, \mathbb{R}^2, \Gamma, W)$.

The lattice Γ is given by

$$\Gamma = \left\langle b_1 = \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, b_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\-1\\1\end{pmatrix}, b_3 = \begin{pmatrix}0\\1\\0\\-1\end{pmatrix}, b_4 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\1\\1\end{pmatrix}\right\rangle_{\mathbb{Z}}.$$

As the window we consider the image of a translated fundamental cell of Γ,

$$W = \pi_H\left(\left\{\sum_{i=1}^{4}\lambda_i b_i \,\middle|\, \lambda_i \in [0,1]\right\} + \left(0.5, \sqrt{3}, 0, 0\right)^{\mathrm{T}}\right).$$

The shift, $x = (0.5, \sqrt{3}, 0, 0)^{\mathrm{T}}$, is chosen so that the model set is regular. The resulting window is an octagon, shown in Figure 8.3. Clearly this window is polytopal.

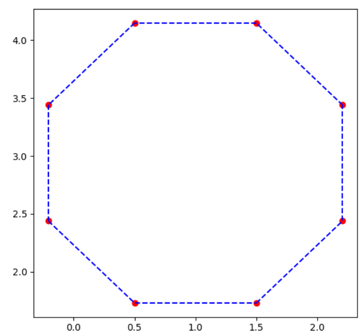


Figure 8.3: The window of an Ammann-Beenker-tiling. The picture is made by a python code which uses the cut and project method to construct the vertex set of the tiling.

We will now calculate the half-spaces, which define W. In a first step we compute the eight hyperplanes, which come in pairs of parallels. For $t \in \mathbb{R}$ we get the hyperplanes

$$\begin{aligned}
P_{1a} &= t \cdot b_{1H} + x_H + b_{2H} + b_{3H} + b_{4H}, & P_{1b} &= t \cdot b_{1H} + x_H,\\
P_{2a} &= t \cdot b_{2H} + x_H + b_{3H} + b_{4H}, & P_{2b} &= t \cdot b_{2H} + x_H + b_{1H},\\
P_{3a} &= t \cdot b_{3H} + x_H + b_{4H}, & P_{3b} &= t \cdot b_{3H} + x_H + b_{1H} + b_{2H},\\
P_{4a} &= t \cdot b_{4H} + x_H, & P_{4b} &= t \cdot b_{1H} + x_H + b_{1H} + b_{2H} + b_{3H}.
\end{aligned}$$

Since for every projected basis-vector we find one which is orthogonal to this one we can state the half-spaces as follows. For $u > 0$

$$\begin{aligned}
P_{1a}^+ &= P_{1a} - u \cdot b_{3H}, & P_{1b}^+ &= P_{1b} + u \cdot b_{3H},\\
P_{2a}^+ &= P_{2a} - u \cdot b_{4H}, & P_{2b}^+ &= P_{2b} + u \cdot b_{4H},\\
P_{3a}^+ &= P_{3a} + u \cdot b_{1H}, & P_{3b}^+ &= P_{3b} - u \cdot b_{1H},\\
P_{4a}^+ &= P_{4a} + u \cdot b_{2H}, & P_{4b}^+ &= P_{4b} - u \cdot b_{2H}.
\end{aligned}$$

So that $W = P_{1a}^+ \cap P_{1b}^+ \cap P_{2a}^+ \cap P_{2b}^+ \cap P_{3a}^+ \cap P_{3b}^+ \cap P_{4a}^+ \cap P_{4b}^+$. The next step is to calculate the ranks of the stabilisers of the hyperplanes. This is an additional exercise which will result in

$$\begin{aligned}
\Gamma^{P_{1a}} &= \langle b_1, b_2 - b_4 \rangle_{\mathbb{Z}}, & \Gamma^{P_{1b}} &= \langle b_1, b_2 - b_4 \rangle_{\mathbb{Z}}, \\
\Gamma^{P_{2a}} &= \langle b_2, b_1 + b_3 \rangle_{\mathbb{Z}}, & \Gamma^{P_{2b}} &= \langle b_2, b_1 + b_3 \rangle_{\mathbb{Z}}, \\
\Gamma^{P_{3a}} &= \langle b_3, b_2 + b_4 \rangle_{\mathbb{Z}}, & \Gamma^{P_{3b}} &= \langle b_3, b_2 + b_4 \rangle_{\mathbb{Z}}, \\
\Gamma^{P_{4a}} &= \langle b_4, b_3 - b_1 \rangle_{\mathbb{Z}}, & \Gamma^{P_{4b}} &= \langle b_4, b_3 - b_1 \rangle_{\mathbb{Z}}.
\end{aligned}$$

We see that the stabilizers of all the hyperplanes have rank 2. This will make the following calculation really easy and the reason for this is the symmetry of the window. We still need to calculate $\dim(\langle(\Gamma^{P_i})_H\rangle)$. It is clear that $\dim(\langle(\Gamma^{P_i})_H\rangle) \geq 1$, since it is not trivial. On the other hand the stabilizer of P_i contains $\langle(\Gamma^{P_i})_H\rangle$ and the stabilizer of P_i has dimension 1 for all i in this example. So we see that $\dim(\langle(\Gamma^{P_i})_H\rangle) \leq 1$. And we conclude $\dim(\langle(\Gamma^{P_i})_H\rangle) = 1$ and by this get $\dim\left(\left\langle\left(\Gamma^{P_i}\right)_H\right\rangle\right) - \operatorname{rk}\left(\Gamma^{P_i}\right) = 2 - 1 = 1$ for all the hyperplanes.

Now we can use Theorem 8.0.1 and get $p(r) \asymp r^2$.

This result is known, since the Ammann-Beenker tiling is linear repetitive. But the interesting point here is that it is the least possible complexity we could get. So the boundary of the window is really stable and this is sort of obvious in this example, because as the window we picked the image of the translated fundamental cell of the lattice. And it is clear that the fundamental cells bounding hyperplanes are stabilized by the lattice vectors which span it.
What we can learn from that is: If we want low complexity the window and the lattice should fit together in some sense. On the other hand if we want the complexity to be as large as possible the window should be completely irrational to the lattice, in the sense that all the stabilizers are trivial.

As a final part of this example we consider the acceptance domains. We again observe the symmetric behaviour of the decomposition, see Figure 8.4. And again the sizes of the acceptance domains vary. Here another problem arises, namely we do not know if the acceptance domains are connected, so multiple connected components could correspond to the same acceptance domain. Therefore the pictures are nice to get a feeling for what happens but counting in the pictures does not lead anywhere.

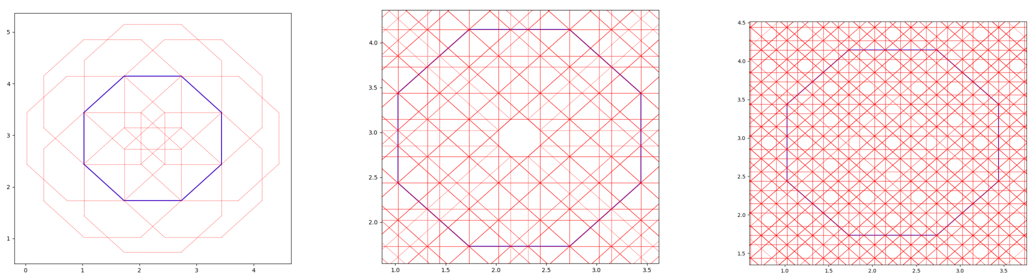


Figure 8.4: The decomposition of the window for increasing radius of the considered patches.

8.3 Heisenberg Example

As our last example we consider a model set in $\mathbb{H} \times \mathbb{H}$. We already considered the Heisenberg group $\mathbb{H}$ in Section 5.3. So we will consider $\Lambda(\mathbb{H}, \mathbb{H}, \Gamma, W)$. We fix the lattice as

$$\Gamma = \left\{ (h, h^*) \in \mathbb{H} \times \mathbb{H} \,\middle|\, h \in \mathbb{H}\left(\mathbb{Z}\left[\sqrt{2}\right]\right) \right\},$$

where $\mathbb{H}(\mathbb{Z}[\sqrt{2}])$ is the group of upper triangle matrices with entries from $\mathbb{Z}[\sqrt{2}]$ and ones on the diagonal. Further h^* is the element, where all the entries of h are conjugated, in the sense of the Galois conjugation in $\mathbb{Z}[\sqrt{2}]$. This is clearly a lattice and its projection is dense and injective on both components. As a window we choose something simple, a cube, therefore we will take the standard cube and rotate it and afterwards translate it so that the model set is regular, i.e. $W = R_{n,\alpha}\left([0,2]^3\right) + v$. Where $R_{n,\alpha}$ is a rotation in three dimensional space, which can be parametrised by a vector $n \in \mathbb{R}^3$, which gives the rotation axis, and an angle α. Then the rotation is given by the matrix

$$R_{n,\alpha} = \begin{pmatrix} n_1^2(1-\cos(\alpha)) + \cos(\alpha) & n_1 n_2(1-\cos(\alpha)) - n_3\sin(\alpha) & n_1 n_3(1-\cos(\alpha)) + n_2\sin(\alpha) \\ n_1 n_2(1-\cos(\alpha)) + n_3\sin(\alpha) & n_2^2(1-\cos(\alpha)) + \cos(\alpha) & n_2 n_3(1-\cos(\alpha)) + n_1\sin(\alpha) \\ n_1 n_3(1-\cos(\alpha)) - n_2\sin(\alpha) & n_2 n_3(1-\cos(\alpha)) - n_1\sin(\alpha) & n_3^2(1-\cos(\alpha)) + \cos(\alpha) \end{pmatrix}.$$

We can describe the cube as an intersection of six half-spaces, which we will then rotate by the matrix. For parameters $t, s \in \mathbb{R}$ we set:

$$P_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \cdot t + \begin{pmatrix}0\\1\\0\end{pmatrix} \cdot s, \qquad P_2 = \begin{pmatrix}1\\0\\0\end{pmatrix} \cdot t + \begin{pmatrix}0\\1\\0\end{pmatrix} \cdot s + \begin{pmatrix}0\\0\\2\end{pmatrix},$$

$$P_3 = \begin{pmatrix}1\\0\\0\end{pmatrix} \cdot t + \begin{pmatrix}0\\0\\1\end{pmatrix} \cdot s, \qquad P_4 = \begin{pmatrix}1\\0\\0\end{pmatrix} \cdot t + \begin{pmatrix}0\\0\\1\end{pmatrix} \cdot s + \begin{pmatrix}0\\2\\0\end{pmatrix},$$

$$P_5 = \begin{pmatrix}0\\1\\0\end{pmatrix} \cdot t + \begin{pmatrix}0\\0\\1\end{pmatrix} \cdot s, \qquad P_6 = \begin{pmatrix}0\\1\\0\end{pmatrix} \cdot t + \begin{pmatrix}0\\0\\1\end{pmatrix} \cdot s + \begin{pmatrix}2\\0\\0\end{pmatrix}.$$

We fix the rotation axis as $n_1 = n_2 = n_3 = \frac{1}{\sqrt{3}}$, the angle as $\alpha = \frac{\pi}{2}$ and the translation vector $v = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^{\mathrm{T}}$. Now we can calculate the rotated hyperplanes

$$R_{n,\alpha}P_1 = \frac{1}{3}\left(\begin{pmatrix}1\\1+\sqrt{3}\\1-\sqrt{3}\end{pmatrix} \cdot t + \begin{pmatrix}1-\sqrt{3}\\1\\1+\sqrt{3}\end{pmatrix} \cdot s\right), \quad R_{n,\alpha}P_2 = \frac{1}{3}\left(\begin{pmatrix}1\\1+\sqrt{3}\\1-\sqrt{3}\end{pmatrix} \cdot t + \begin{pmatrix}1-\sqrt{3}\\1\\1+\sqrt{3}\end{pmatrix} \cdot s\right) + \frac{2}{3}\begin{pmatrix}1+\sqrt{3}\\1-\sqrt{3}\\1\end{pmatrix},$$

$$R_{n,\alpha}P_3 = \frac{1}{3}\left(\begin{pmatrix}1\\1+\sqrt{3}\\1-\sqrt{3}\end{pmatrix} \cdot t + \begin{pmatrix}1+\sqrt{3}\\1-\sqrt{3}\\1\end{pmatrix} \cdot s\right), \quad R_{n,\alpha}P_4 = \frac{1}{3}\left(\begin{pmatrix}1\\1+\sqrt{3}\\1-\sqrt{3}\end{pmatrix} \cdot t + \begin{pmatrix}1+\sqrt{3}\\1-\sqrt{3}\\1\end{pmatrix} \cdot s\right) + \frac{2}{3}\begin{pmatrix}1-\sqrt{3}\\1\\1+\sqrt{3}\end{pmatrix},$$

$$R_{n,\alpha}P_5 = \frac{1}{3}\left(\begin{pmatrix}1-\sqrt{3}\\1\\1+\sqrt{3}\end{pmatrix} \cdot t + \begin{pmatrix}1+\sqrt{3}\\1-\sqrt{3}\\1\end{pmatrix} \cdot s\right), \quad R_{n,\alpha}P_6 = \frac{1}{3}\left(\begin{pmatrix}1-\sqrt{3}\\1\\1+\sqrt{3}\end{pmatrix} \cdot t + \begin{pmatrix}1+\sqrt{3}\\1-\sqrt{3}\\1\end{pmatrix} \cdot s\right) + \frac{2}{3}\begin{pmatrix}1\\1+\sqrt{3}\\1-\sqrt{3}\end{pmatrix}.$$

The constructed point set can be seen in Figure 8.5.

To use our theory we have to check the condition that on every hyperplane $R_{n,\alpha}P_i + v$ there is no projected lattice point, i.e. $\mathbb{Z}[\sqrt{2}]^3 \cap R_{n,\alpha}P_i + v = \emptyset$. This just yields a system of linear equations for each hyperplane and one can see that the condition holds.[1]

[1]We checked the condition using a computer algebra system.

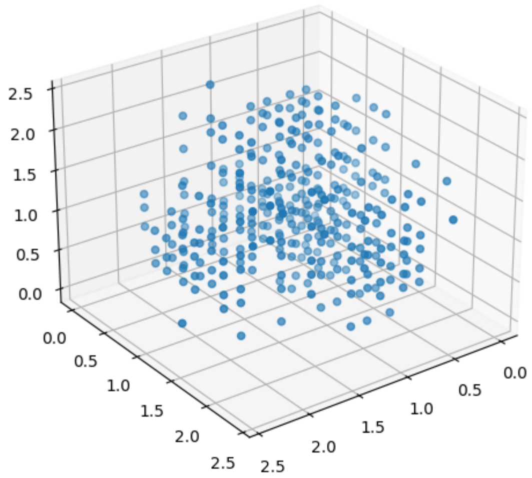


Figure 8.5: This figure shows a part of the points resulting from the model set construction. Here we view the set underlying the Heisenberg group as $\mathbb{R}^3$. Not all points in the region are shown, but one should notice that the points are aligned in lines.

The next thing to check is that the stabilizers of these hyperplanes are trivial. But this is also easy to see, since the only hyperplanes in the Heisenberg group which have a non-trivial stabilizer contain the center. And all our hyperplanes clearly do not contain the center.

So we can apply Theorem 6.0.1 and see that the complexity function in our example asymptotically grows like $r^{3\cdot 4} = r^{12}$.

We see that the calculation and effort is similar to the Euclidean case. One thing which can increase the effort is to understand how the stabilizers of the planes look, but in this example this was really tame.

Now consider Figure 8.5 again. Here we view the set underlying the Heisenberg group as $\mathbb{R}^3$. One can think of this set as a triple silver mean chain. Our theory tells us that this set has a complexity of $p(r) \asymp r^{12}$, but if we would view the same set as an Euclidean set we would get a complexity of $p(r) \asymp r^9 = r^{3\cdot 3}$. This stresses out the fact that the complexity of a set is determined by the structure of the space in which the set is located. A big part of this difference comes form the different metric on the space, but also the comparison of the areas around the points is different. Since in the euclidean setting the comparison is by translations, but in the Heisenberg case it is by the group operation. In some sense this means that mapping a patch in the Euclidean case to another patch is easier as in the Heisenberg case.

8.4 Hyperbolic Example

We abstain from giving a hyperbolic example. It is possible to calculate all the needed data, i.e. a polytope in $\mathcal{H}^n$ and a lattice in $\mathrm{Isom}(\mathcal{H}^n) \times \mathrm{Isom}(\mathcal{H}^d)$ and then do the cut and project. But the computation of this data is quite challenging with limited computing power. And one needs to find many lattice points to get a picture which really gives an intuition of what is going on. This problem becomes even more critical since we can only compute the lattice in the isometry group and then get the points by acting on a base point, since a point in hyperbolic space has a non-trivial stabilizer multiple points in the lattice can result in the same point in the picture. This means that the number of computed lattice points has to be even higher.
The data for a CPS in the hyperbolic plane can be found in the introduction of [16], there is also a picture of an aperiodic Voronoi tiling of the Poincaré disc.

9 Open Questions

There are a number of questions, which we could not address in this thesis. We want to summarize them here, which should also give an idea on possibilities how to extend the work done in this thesis. Clearly this list is not exhaustive and there are many more ways to further extend the research in this field.

(a) The first possible generalisation is to enlarge the set from which we can pick G and H in the model set $\Lambda(G, H, \Gamma, W)$. There are different options to do so:

  (i) We could extend the theory to higher nilpotency degree by understanding the combinatorics of the intersection of hypersurfaces. This certainly involves some understanding of algebraic geometry.

  (ii) Another direction of progress would be to consider p-adic spaces, like in the paper of Baake, Moody and Schlottmann, [7]. But in this set-up the combinatoric is not manageable yet.

  (iii) The most general idea would be to not use any combinatorics and work directly with the results from Chapter 3. Here one has to invent some new tools to count the acceptance domains, but if this is done one can make statements in a most general set-up.

  (iv) Another idea which follows the lines of Chapter 7 is to consider cut and project sets in groups acting on metric spaces. Then one can use the information from the metric space in the group, as we did with hyperbolic space. There are many options, and a first idea would be to consider proper homogeneous metric spaces, since their isometry groups are lcsc groups, some ideas in this direction can be found in [16].

(b) A second possible generalisation is to not only consider polytopal windows. A first possible extension would be to consider windows which can be approximated by polytopes and state results for them. If one wants to use general windows we again will need algebraic geometry to handle the intersection behaviour.

There is also some further work in progress, in which it is shown that the relative size of an acceptance domain inside the window is exactly the frequency of the patch inside the model set. This is some unpublished work by Tobias Hartnick and Maximilian Wackenhuth.

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