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Strongly-ordered infrared limits for subtraction counterterms from factorisation

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After a brief introduction to the problem of subtraction of infrared divergences for high-order collider observables, we present a preliminary study of strongly-ordered soft and collinear multiple radiation from the point of view of factorisation. We show that the matrix elements of fields and Wilson lines that describe soft and collinear radiation in factorised scattering amplitudes can be re-factorised in strongly-ordered limits, providing a systematic method to compute them, to characterise their singularity structure, and to build local subtraction counterterms for strongly-ordered configurations. Our results provide tools for a detailed organisation of subtraction algorithms, in principle to all orders in perturbation theory.

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1. Setting the stage

High-order calculations of collider observables are of paramount importance in order to achieve the precision goals required by present and forthcoming experimental results, and possibly identify new physics signals. In this context, efficient and general algorithms to implement the cancellation of infrared divergences are a crucial tool. The problem first arises at next-to-leading order (NLO), and in that context efficient and completely general subtraction algorithms were developed in the past decades [1–4], and are currently implemented in the simulation codes employed by LHC experiments. The extension of these algorithms to next-to-next-to-leading order (NNLO) and beyond has been a major research goal for a number of years: several different methods have been proposed, and many have been successfully implemented to derive predictions for many LHC processes (methods are described, for example, in [5–17], and were recently reviewed in [18]).

All subtraction methods rely upon the infrared factorisation properties of scattering amplitudes, as applying to both virtual corrections and to soft and collinear real radiation (for a recent review, see [19]). It is natural to imagine that a detailed understanding of factorisation should play a crucial role in the ultimate organisation of subtraction algorithms, in principle to any order in perturbation theory. This viewpoint was pursued in [20] (see also [21]), where general expressions for soft and collinear counterterms were proposed, valid to all orders in perturbation theory.

A significant source of complexity in the organisation of infrared counterterms beyond NLO is the necessity to properly identify and account for strongly-ordered soft and collinear configurations, where clusters of particles become unresolved in a hierarchical pattern. While these configurations are in principle 'simple', and expected to correspond to products of lower-order radiative counterterms, they must be precisely controlled, since they serve to cancel singularities arising in mixed real-virtual contributions. Furthermore, the number of strongly-ordered configurations (and of the related counterterms) grows rapidly with the perturbative order.

In the present contribution, we will begin a discussion of strongly-ordered subtraction counterterms from the point of view of infrared factorisation. We will show in a few examples that the matrix elements of fields and Wilson lines that describe the factorised emission of soft and collinear particles can be re-factorised in strongly-ordered configurations. This provides formal expressions for strongly-ordered counterterms to all orders in perturbation theory, and allows to identify the patterns of cancellation between these counterterms and singularities involving mixed real and virtual corrections.

To set the stage, we begin by briefly reviewing the structure of infrared subtraction at NLO, in the case of final-state massless particles, and then we will show how the problem of strongly-ordered configurations arises at NNLO and beyond. While much of the discussion below is general, in concrete cases we will use the framework of Local Analytic Sector subtraction, developed and discussed in [16, 20, 22, 23].

At NLO, the subtraction problem can be succinctly summarised. For any infrared-safe observable X, the NLO distribution is computed by combining virtual and real corrections, as

$$\frac{d\sigma_{\rm NLO}}{dX} = \lim_{d \to 4} \left\{ \int d\Phi_n \, V_n \, \delta_n(X) + \int d\Phi_{n+1} \, R_{n+1} \, \delta_{n+1}(X) \right\}, \tag{1}$$

where we assumed that the Born contribution to the observable X involves n particles, V_n is the virtual correction to the n-particle process, R_{n+1} is the squared matrix element for single real radiation, and $\delta_p(X) = \delta(X - X_p)$ fixes the expression for the observable in the p-particle phase space to the prescribed value X. By general theorems (see [19]), explicit infrared poles in V_n are cancelled by the integration of phase space singularities for unresolved radiation in the second term of Eq. (1). In order to allow for an efficient numerical integration of R_{n+1}, the subtraction approach proposes to define a *local counterterm* $K^{(\mathbf{1})}_{n+1}$, and subsequently integrate it over the unresolved particle phase space, according to

$$K^{(\mathbf{1})}_{n+1} = \mathbf{L}^{(\mathbf{1})} \, R_{n+1} \, , \qquad I^{(\mathbf{1})}_n = \int d\Phi^{n+1}_{\mathrm{r},\,1} \, K^{(\mathbf{1})}_{n+1} \, , \tag{2}$$

where the operator $\mathbf{L}^{(\mathbf{1})}$ extracts the contributions of all singular regions to R_{n+1}, and one defines the radiative phase space by $d\Phi_{n+1} = d\Phi_n \, d\Phi^{n+1}_{\mathrm{r},\,1}$. One may then rewrite Eq. (1) identically as

$$\frac{d\sigma_{\mathrm{NLO}}}{dX} = \int d\Phi_n \Big(V_n + I^{(\mathbf{1})}_n \Big) \, \delta_n(X) + \int d\Phi_{n+1} \Big(R_{n+1} \, \delta_{n+1}(X) - K^{(\mathbf{1})}_{n+1} \delta_n(X) \Big) \, , \tag{3}$$

where use has been made of the IR safety of X. Now all IR poles cancel in the first integral, while phase-space singularities cancel in the second integral, which can then be performed directly in $d = 4$. Clearly, any actual implementation of this idea relies upon precise definitions of the *subtraction operator* $\mathbf{L}^{(\mathbf{1})}$, and of the *phase-space measure* $d\Phi^{n+1}_{\mathrm{r},\,1}$: indeed, the $(n+1)$-particle phase space $d\Phi_{n+1}$ does not naturally factorise as suggested, except in the soft limit. Subtraction methods differ in how these two problems are approached. In the context our method, the NLO subtraction operator is defined by partitioning the phase space into *sectors*, along the lines of [1], and a proper factorisation of the radiative phase space is achieved by employing *phase space mappings*, following [2] (for a detailed description, see [16]). A careful choice of different mappings for different contributions to the NLO local counterterm allows for a straightforward analytic integration, as discussed in [22].

At NNLO, the complexity of the subtraction problem grows very steeply in several directions. Phase-space mappings at NNLO and beyond are discussed in detail for example in Ref. [24], while the extension of our approach to NNLO was presented in [16, 22, 23]. Here, we will only discuss the general structure of counterterms, in order to highlight the appearance and relevance of strongly-ordered configurations. At NNLO, infrared cancellations require the combination of three terms: double virtual (VV), real-virtual (RV) and double real (RR). In analogy with Eq. (1), one writes

$$\frac{d\sigma_{\mathrm{NNLO}}}{dX} = \lim_{d \to 4} \left\{ \int d\Phi_n \, VV_n \, \delta_n(X) + \int d\Phi_{n+1} \, RV_{n+1} \, \delta_{n+1}(X) + \int d\Phi_{n+2} \, RR_{n+2} \, \delta_{n+2}(X) \right\} . \tag{4}$$

There are *four* kinds of singular phase-space configurations in Eq. (4), requiring four different local counterterms. We define

$$K^{(\mathbf{1})}_{n+2} = \mathbf{L}^{(\mathbf{1})} \, RR_{n+2} \, , \quad K^{(\mathbf{2})}_{n+2} = \mathbf{L}^{(\mathbf{2})} \, RR_{n+2} \, , \quad K^{(\mathbf{12})}_{n+2} = \mathbf{L}^{(\mathbf{1})} \, \mathbf{L}^{(\mathbf{2})} \, RR_{n+2} \, , \quad K^{(\mathbf{RV})}_{n+1} = \widetilde{\mathbf{L}}^{(\mathbf{1})} \, RV_{n+1} \, , \tag{5}$$

where $K^{(\mathbf{1})}_{n+2}$ collects configurations where a single particle becomes unresolved, $K^{(\mathbf{2})}_{n+2}$ those where two particles become unresolved, and $K^{(\mathbf{12})}_{n+2}$ defines their overlap: notice that the definition of $K^{(\mathbf{12})}_{n+2}$ implies that this counterterm collects precisely the strongly-ordered configurations that we are discussing. Finally, $K^{(\mathbf{RV})}_{n+1}$ collects configurations where the single radiated particle in the *RV* contribution becomes unresolved. The operator $\widetilde{\mathbf{L}}^{(\mathbf{1})}$ differs from $\mathbf{L}^{(\mathbf{1})}$ by the inclusion of a set of subleading-power contributions in the relevant normal variables, whose role will become apparent below (for details, see [23]). In analogy with Eq. (2), one must then integrate the local counterterms over their radiative phase spaces, defining

$$I^{(\mathbf{1})}_{n+1} = \int d\Phi^{n+2}_{\mathrm{r},\,1}\, K^{(\mathbf{1})}_{n+2}\,, \quad I^{(\mathbf{12})}_{n+1} = \int d\Phi^{n+2}_{\mathrm{r},\,1}\, K^{(\mathbf{12})}_{n+2}\,, \quad I^{(\mathbf{2})}_{n} = \int d\Phi^{n+2}_{\mathrm{r},\,2}\, K^{(\mathbf{2})}_{n+2}\,,$$
$$I^{(\mathbf{RV})}_{n} = \int d\Phi^{n+1}_{\mathrm{r},\,1}\, K^{(\mathbf{RV})}_{n+1}\,. \tag{6}$$

One finally constructs a subtracted expression for the NNLO distribution,

$$\begin{aligned}
\frac{d\sigma_{\mathrm{NNLO}}}{dX} \; = \; & \int d\Phi_n \Big[VV_n + I^{(\mathbf{2})}_n + I^{(\mathbf{RV})}_n \Big]\, \delta_n(X) \\
& + \int d\Phi_{n+1} \bigg[\Big(RV_{n+1} + I^{(\mathbf{1})}_{n+1} \Big)\, \delta_{n+1}(X) \, - \, \Big(K^{(\mathbf{RV})}_{n+1} + I^{(\mathbf{12})}_{n+1} \Big)\, \delta_n(X) \bigg] \\
& + \int d\Phi_{n+2} \bigg[RR_{n+2}\, \delta_{n+2}(X) - K^{(\mathbf{1})}_{n+2}\, \delta_{n+1}(X) - \Big(K^{(\mathbf{2})}_{n+2} - K^{(\mathbf{12})}_{n+2} \Big) \delta_n(X) \bigg]\,.
\end{aligned} \tag{7}$$

In Eq. (7), the last line can be integrated directly in $d = 4$, as all singular configurations have been subtracted from RR_{n+2} without double counting. In the second line, the first parenthesis is free of IR poles by standard arguments; also, the phase-space singularities of RV_{n+1} are subtracted, by construction, by $K^{(\mathbf{RV})}_{n+1}$, and those of $I^{(\mathbf{1})}_{n+1}$ by $I^{(\mathbf{12})}_{n+1}$; finally, the inclusion of non-minimal terms (which are non-singular in the radiative phase space) in $\widetilde{\mathbf{L}}^{(\mathbf{1})}$ allows to cancel the IR poles of $I^{(\mathbf{12})}_{n+1}$. When this delicate balance of cancellations is achieved in the second line, the first line is bound to be finite by the general theorems enforcing the cancellation of IR singularities. We wish to emphasise that the importance of strongly-ordered configurations grows steeply with the perturbative order: to this end, we write down the formal extension of Eq. (7) to N^{3}LO. Using notations that hopefully generalise transparently the NNLO case, one can write the subtracted distribution as

$$\begin{aligned}
\frac{d\sigma_{\mathrm{N^3LO}}}{dX} \; = \; & \int d\Phi_n \Big[VVV_n + I^{(\mathbf{3})}_n + I^{(\mathbf{RVV})}_n + I^{(\mathbf{RRV},\,\mathbf{2})}_n \Big]\, \delta_n(X) \\
& + \int d\Phi_{n+1} \Big(RVV_{n+1} + I^{(\mathbf{2})}_{n+1} + I^{(\mathbf{RRV},\,\mathbf{1})}_{n+1} \Big)\, \delta_{n+1}(X) \, - \, \Big(K^{(\mathbf{RVV})}_{n+1} + I^{(\mathbf{23})}_{n+1} + I^{(\mathbf{RRV},\,\mathbf{12})}_{n+1} \Big)\, \delta_n(X) \\
& + \int d\Phi_{n+2} \bigg\{ \Big(RRV_{n+2} + I^{(\mathbf{1})}_{n+2} \Big)\, \delta_{n+2}(X) - \Big(K^{(\mathbf{RRV},\mathbf{1})}_{n+2} + I^{(\mathbf{12})}_{n+2} \Big)\, \delta_{n+1}(X) \\
& \qquad - \bigg[\Big(K^{(\mathbf{RRV},\,\mathbf{2})}_{n+2} + I^{(\mathbf{13})}_{n+2} \Big) - \Big(K^{(\mathbf{RRV},\mathbf{12})}_{n+2} + I^{(\mathbf{123})}_{n+2} \Big) \bigg]\, \delta_n(X) \bigg\} \\
& + \int d\Phi_{n+3} \bigg[RRR_{n+3}\, \delta_{n+3}(X) - K^{(\mathbf{1})}_{n+3}\, \delta_{n+2}(X) - \Big(K^{(\mathbf{2})}_{n+3} - K^{(\mathbf{12})}_{n+3} \Big)\, \delta_{n+1}(X) \\
& \qquad - \Big(K^{(\mathbf{3})}_{n+3} - K^{(\mathbf{13})}_{n+3} - K^{(\mathbf{23})}_{n+3} + K^{(\mathbf{123})}_{n+3} \Big)\, \delta_n(X) \bigg]\,.
\end{aligned} \tag{8}$$

One sees that in order to achieve the cancellation of IR poles and the finiteness of the radiative phase space integrations in each line of Eq. (8) one needs a total of *eleven* local counterterms: of these, *five* display various kinds of strong ordering; for example, $K^{(\mathbf{23})}_{n+3}$ contains configurations with

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three unresolved particles, out of which two become unresolved faster than the third one. Clearly, a systematic analysis of strongly-ordered IR configurations is warranted, if one wishes to understand the subtraction process to all orders.

2. Infrared counterterms from factorisation

In Ref. [20], a systematic proposal was presented for the construction of soft and collinear local counterterms to all orders in perturbation theory. We display here some sample definitions, in order to introduce the appropriate language for analysing the strongly-ordered case. For multiple *soft* emissions, one can define *eikonal form factors*

$$\mathcal{S}_{n,\,m}\Big(\{\beta\},\{k\},\{\lambda\}\Big) \; = \; \langle k_1,\lambda_1;\ldots;k_m,\lambda_m|\; T\Bigg[\prod_{i=1}^{n}\Phi_{\beta_i}(\infty,0)\Bigg]\,|0\rangle \;, \tag{9}$$

where $\Phi_\beta(\infty,0)$ is a semi-infinite Wilson line stretching along the direction of the four-vector β, and T denotes time-ordering. Eikonal form factors mimic the emission of a system of m soft particles with momenta k_l from n hard emitters with four-velocities β_i, at leading power in the soft momenta. Squaring the form factors, one gets *eikonal transition probabilities*

$$S_{n,m}(\{\beta\},\{k\}) \;\; = \;\; \sum_{\lambda_i}\langle 0|\,\overline{T}\Bigg[\prod_{i=1}^{n}\Phi_{\beta_i}(0,\infty)\Bigg]|\{k\},\{\lambda\}\rangle\,\langle\{k\},\{\lambda\}|T\Bigg[\prod_{i=1}^{n}\Phi_{\beta_i}(\infty,0)\Bigg]|0\rangle \;, \tag{10}$$

which are natural candidates for soft local subtraction counterterms, since they retain all leading-power information on soft radiation. Note that the soft limit here is taken uniformly for all soft particles. The suitability of Eq. (10) as a soft counterterm is supported by integrating over the soft space and summing over the number of soft particles. Using *completeness*, this yields

$$\sum_{m=0}^{\infty}\int d\Phi_m\, S_{n,\,m}\Big(\{\beta\},\{k\}\Big) \; = \; \langle 0|\;\overline{T}\Bigg[\prod_{i=1}^{n}\Phi_{\beta_i}(0,\infty)\Bigg]\,T\,\Bigg[\prod_{i=1}^{n}\Phi_{\beta_i}(\infty,0)\Bigg]\,|0\rangle \;. \tag{11}$$

which is a total cross section in the presence of Wilson-line sources, and thus IR finite by general cancellation theorems. Note that Eq. (11) requires a uniform treatment of the UV region for different terms, since the phase-space integration is unbound; in the IR region, on the other hand, $d\Phi_m$ can be taken as the product of single-particle phase spaces, since the full phase space naturally factorises in the soft limit. A similar analysis can be performed for *collinear* radiation. In this case one can define *radiative jet functions*, mimicking collinear emission at leading power in transverse momenta. As an example, for final-state collinear radiation from a parent quark one defines

$$\overline{u}_s(p)\,\mathcal{J}_{q,\,m}^{\{\lambda\}}\big(\{k\};p,n\big) \;\equiv\; \langle p,s;k_1,\lambda_1;\ldots;k_m,\lambda_m|\,\overline{\psi}(0)\,\Phi_n(0,\infty)\,|0\rangle \;, \tag{12}$$

which in turn can be 'squared' to yield the *collinear transition probability*

$$\begin{aligned} J_{q,\,m}\big(\{k\};l,p,n\big) \;\; &\equiv \;\; \int d^d x\, \mathrm{e}^{\mathrm{i}l\cdot x}\sum_{\{\lambda\}}\langle 0|\,\overline{T}\Big[\Phi_n(\infty,x)\,\psi(x)\Big]\,|p,s;\{k\},\{\lambda\}\rangle \\ &\qquad\qquad\times\,\langle p,s;\{k\},\{\lambda\}|\,T\Big[\overline{\psi}(0)\,\Phi_n(0,\infty)\Big]\,|0\rangle \;, \end{aligned} \tag{13}$$

where the Furier transform has been introduced to fix the total final state momentum to l. Using completeness in this case yields the discontinuity of a two-point function,

$$\sum_{m=0}^{\infty} \int d\Phi_{m+1} \, J_{q,\,m}\big(\{k\}; l, p, n\big) = \mathrm{Disc}\Big[\int d^d x \, \mathrm{e}^{\mathrm{i} l \cdot x} \, \langle 0| \, T\Big[\Phi_n(\infty, x)\psi(x)\overline{\psi}(0)\Phi_n(0,\infty)\Big] \, |0\rangle\Big]. \quad (14)$$

which is also finite by power-counting. Note however that the collinear limit, $l^2 \to 0$, is not intrinsic to the definition in Eq. (12) and must be taken at a suitable stage in the calculation; furthermore, the collinear phase space does not simply factorise as in the soft limit: in order to effect the cancellation within the framework of subtraction, one needs to express the total (in general off-shell) momentum l^μ in terms of an on-shell parent momentum $\bar{l}^\mu$, to be associated with the Born process. This is how phase-space mappings make their appearance within the factorisation framework.

3. Strong ordering and refactorisation

Once a soft or collinear kernel for multiple radiation has been computed in a uniform limit (with all particles becoming unresolved at the same rate), for example by means of Eq. (10) or Eq. (13), it is straightforward in practice to extract strongly-ordered limits, by performing a further Taylor expansion in the normal variables of the most unresolved particle. We believe however that it is useful to understand these limits theoretically, in terms of factorisation: this will contribute to the automatic construction of finite combinations of real and mixed real-virtual corrections, by means of the *completeness* technique highlighted in Section 2. The basic idea is to treat eikonal form factors and jet functions as amplitudes in the presence of sources, and apply to them the same factorisation techniques that are used for the original scattering amplitudes. Here, as an example, we will focus on the soft sector: for related work in collinear limits, see [25–27].

3.1 Tree-level multiple soft emissions

Consider first the emission of two soft gluons at tree level. The corresponding current was computed in [28], and the result is easily reproduced by using Eq. (9) at tree level. In the strongly-ordered limit, with soft momenta $k_1^\mu \gg k_2^\mu$, the double soft-gluon current simplifies to

$$\Big[J_{\rm CG}^{(0),\,\rm s.o.}\Big]^{a_1 a_2}_{\mu_1 \mu_2}(k_1, k_2; \beta_i) \; = \; \bigg(J^{(0)\,a_2}_{\mu_2}(k_2)\,\delta^{a_1 a} + {\rm i} g_s \, f^{a_1 a_2 a} \, \frac{k_{1,\,\mu_2}}{k_1 \cdot k_2}\bigg) \, J^{(0)}_{\mu_1,\,a}(k_1) \, , \quad (15)$$

where the tree-level single soft-gluon current is given by the well-known expression

$$J^{(0)\,a}_{\mu}(k) \; = \; g_s \sum_{i=1}^{n} \frac{\beta_{i,\,\mu}}{\beta_i \cdot k} \, T_i^a \, . \quad (16)$$

The factorised structure of Eq. (15) suggests that the two-gluon eikonal form factor should also behave similarly at this order. Indeed, one finds the interesting factorisation

$$\begin{aligned}
\Big[\mathcal{S}^{(0)}_{n;\,1,\,1}\Big]^{a_1 a_2}_{\{d_i e_i\}}(k_1, k_2; \beta_i) \;\; & \equiv \;\; \langle k_2, a_2| \; \Phi_{\beta_{k_1}}^{a_1 b}(0,\infty) \prod_{i=1}^{n} \Phi_{\beta_i,\, d_i}^{\;\;\;\; c_i}(0,\infty) \; |0\rangle \\
& \qquad \times \langle k_1, b| \prod_{i=1}^{n} \Phi_{\beta_i,\, c_i e_i}(0,\infty) \, |0\rangle \Big|_{\rm tree} \\
& = \;\; \Big[\mathcal{S}^{(0)}_{n+1,1}\Big]^{a_2,\, a_1 b}_{\{d_i c_i\}} \big(k_2; \beta_{k_1}, \beta_i\big) \; \Big[\mathcal{S}^{(0)}_{n,1}\Big]_{b,\, \{c_i e_i\}}\big(k_1; \beta_i\big) \, , \qquad (17)
\end{aligned}$$

where for clarity we displayed all colour indices, and where we adopted the notation $\mathcal{S}^{(0)}_{n;\,m_1,\ldots m_p}$ for the strongly-ordered emission of p clusters of soft gluons, each containing m_k gluons, $k = 1, \ldots, p$. Notice the non-trivial colour structure of Eq. (17): the product on the *r.h.s.* is ordered, and the colour index of the k_1 Wilson line in the first factor is contracted with the colour index of the final-state gluon in the second factor. The physical interpretation is transparent: first, the original system of n Wilson lines corresponding to the hard particles radiates the harder gluon with momentum k_1, corresponding to the second factor on the *r.h.s.* of Eq. (17). This first gluon is much harder than the second one with momentum k_2, thus it turns into a Wilson line in the first factor (we could say that we now have a *Wilsonised* gluon). The augmented system of $n + 1$ Wilson lines then radiates the softer gluon. One easily checks that Eq. (17) reproduces Eq. (15), contracted with appropriate polarisation vectors. Clearly, Eq. (17) lends itself to a natural generalisation for the strongly-ordered emission of any number of gluons, organised in clusters of generic multiplicity. As an example, consider the emission of three gluons with strongly-ordered momenta $k_1^\mu \gg k_2^\mu \gg k_3^\mu$. Eq. (17) suggests the factorisation

$$\begin{aligned}
\Big[\mathcal{S}^{(0)}_{n;1,1,1}\Big]^{a_1 a_2 a_3}_{\{f_i e_i\}}(k_1, k_2, k_3; \beta_i) \equiv{}& \Big[\mathcal{S}^{(0)}_{n+2,1}\Big]^{a_3}_{\{f_i d_i\},\, a_1 b_1,\, a_2 b_2} \Big[\mathcal{S}^{(0)}_{n+1,1}\Big]^{b_2}_{\{d_i c_i\},\, b_1 g_1} \Big[\mathcal{S}^{(0)}_{n,1}\Big]^{g_1}_{\{c_i e_i\}} \\
= \langle k_3, a_3 |\, \Phi^{a_1 b_1}_{\beta_{k_1}}(0,\infty)\, \Phi^{a_2 b_2}_{\beta_{k_2}}(0,\infty) \prod_{i=1}^{n} \Phi^{f_i d_i}_{\beta_i}(0,\infty) |0\rangle \;& \langle k_2, b_2 |\, \Phi^{b_1 g_1}_{\beta_{k_1}}(0,\infty) \prod_{i=1}^{n} \Phi^{d_i c_i}_{\beta_i}(0,\infty) |0\rangle \\
& \times \langle k_1, g_1 | \prod_{i=1}^{n} \Phi^{c_i e_i}_{\beta_i}(0,\infty) |0\rangle \bigg|_{\rm tree} . \qquad (18)
\end{aligned}$$

One can readily verify that Eq. (18) reproduces the strongly-ordered limit of the triple soft-gluon current, derived and discussed in [29, 30]: an explicit calculation indeed gives the current

$$\begin{aligned}
\Big[J^{(0),\,\rm s.o.}_{\rm CCT}\Big]^{a_1 a_2 a_3}_{\mu_1 \mu_2 \mu_3}(k_1, k_2, k_3; \beta_i) \; = \; & \bigg[J^{\mu_3}_{a_3}(k_3)\, \delta^{a_1 b_1}\, \delta^{a_2 b_2} + {\rm i} g_s\, f^{a_1 a_3 b_1}\, \delta^{a_2 b_2}\, \frac{k_1^{\mu_3}}{k_1 \cdot k_3} \qquad (19) \\
& + {\rm i} g_s\, f^{a_2 a_3 b_2}\, \delta^{a_1 b_1}\, \frac{k_2^{\mu_3}}{k_2 \cdot k_3} \bigg] \bigg[J^{\mu_2}_{b_2}(k_2)\, \delta^{b_1 c_1} + {\rm i} g_s\, f^{b_1 b_2 c_1}\, \frac{k_1^{\mu_2}}{k_1 \cdot k_2} \bigg] J^{\mu_1}_{c_1}(k_1) \, .
\end{aligned}$$

It is not difficult to write down generalisations of Eqs. (17) and (18) to more soft particles and more general clusterings, although the resulting expressions are somewhat cumbersome: such expressions provide direct and straightforward ways to compute the corresponding tree-level currents. One should note also that the Wilson line matrix elements introduced above can of course be evaluated to any order in perturbation theory, and one may ask to what extent this tree-level factorisation generalises to higher orders. This question in general remains open, but we will now examine how a one-loop factorisation of this kind impacts the construction of subtraction counterterms.

3.2 One-loop refactorisation and subtraction counterterms

Within the framework of the factorisation approach to subtraction [20], using the definition of local soft counterterms given by Eq. (10), a natural candidate counterterm for real-virtual soft singularities is given by

$$K^{(\mathbf{RV}),\,\rm s}_{n+1} = \mathcal{A}^{(0)\dagger}_n\, S^{(1)}_{n,\,1}\, \mathcal{A}^{(0)}_n + \ldots , \qquad (20)$$

where $\mathcal{A}_n^{(0)}$ is the Born amplitude, and $S_{n,\,1}^{(1)}$ is the one-loop contribution to the single-radiative soft function, which is a colour matrix. The ellipsis contains terms that have no soft poles (but might still display soft phase-space singularities), or terms that are non-singular in phase space, but might still have soft poles. In words, $S_{n,\,1}^{(1)}$ accurately reproduces terms with joint soft poles and singular soft phase-space singularities of the complete squared matrix element. On the other hand, one sees from Eq. (7) that the soft poles of $K_{n+1}^{(\mathbf{RV}),\,\mathrm{s}}$ must cancel those arising from the integration of the strongly-ordered double-radiative soft counterterm $K_{n+2}^{(\mathbf{12}),\,\mathrm{s}}$ over the softest-particle phase space. The refactorisation of the double-radiative tree-level soft function in the strongly-ordered limit, discussed in Section 3.1 suggests an expression for the corresponding counterterm. One writes

$$
\begin{aligned}
K_{n+2}^{(\mathbf{12}),\,\mathrm{s}} &= \mathcal{A}_n^{(0)\,\dagger}\, S_{n,\,1,1}^{(0)}\, \mathcal{A}_n^{(0)} \\
&= \mathcal{A}_n^{(0)\,\dagger} \left[\mathcal{S}_{n,\,1}^{b,\,(0)}(\beta_i;k_1) \right]^\dagger \left[\mathcal{S}_{n+1,\,1}^{a_2,\,a_1 b\,(0)}(\beta_i,\beta_{k_1};k_2) \right]^\dagger \mathcal{S}_{n+1,\,1}^{a_2,\,a_1 c,(0)}(\beta_i,\beta_{k_1};k_2)\, \mathcal{S}_{n,\,1}^{c,\,(0)}(\beta_i;k_1)\, \mathcal{A}_n^{(0)} \\
&= \mathcal{A}_n^{(0)\,\dagger} \left[\mathcal{S}_{n,\,1}^{b,\,(0)}(\beta_i;k_1) \right]^\dagger S_{n+1,\,1}^{bc,\,(0)}(\beta_i,\beta_{k_1};k_2)\, \mathcal{S}_{n,\,1}^{c,\,(0)}(\beta_i;k_1)\, \mathcal{A}_n^{(0)}\,.
\end{aligned}
$$

Note now that the dependence on the softest momentum k_2 is confined to the innermost factor in the product on the last line. One can then use the finiteness of Eq. (11) at one loop,

$$
S_{n+1,\,0}^{bc,(1)}(\beta_i,\beta_{k_1}) + \int d\Phi_1(k_2)\, S_{n+1,\,1}^{bc,(0)}(\beta_i,\beta_{k_1};k_2) = [\text{finite in } d=4]\,, \tag{21}
$$

to propose what appears to be an alternative expression for the soft real virtual counterterm,

$$
K_{n+1}^{(\mathbf{RV}),\,\mathrm{s}} = \mathcal{A}_n^{(0)\,\dagger} \left[\mathcal{S}_{n,\,1}^{b,\,(0)}(\beta_i;k_1) \right]^\dagger S_{n+1,\,0}^{bc,(1)}(\beta_i,\beta_{k_1})\, \mathcal{S}_{n,\,1}^{c,\,(0)}(\beta_i;k_1)\, \mathcal{A}_n^{(0)} + \ldots\,. \tag{22}
$$

The obvious questions are whether Eq. (20) and Eq. (22) are compatible, and whether they match a direct calculation of soft singularities for the real-virtual squared matrix element. To answer these questions, we turn once again to the factorisation properties of the soft function, this time at the one-loop level. Treating the single-radiative eikonal form factor as a scattering amplitude in the presence of Wilson-line sources suggests that it can be factorised in the form

$$
\mathcal{S}_{n,1}(k;\beta_i) = \frac{\mathcal{J}_g(k,n)}{\mathcal{J}_{E,g}(\beta_k,n)}\, \mathcal{S}_{n+1,0}(\beta_k,\beta_i)\, \mathcal{S}_{n,1}^{\mathrm{fin}}(k;\beta_i)\,, \tag{23}
$$

which is reminiscent of the general form of infrared factorisation for virtual corrections to scattering amplitudes [19, 21, 31, 32]: $\mathcal{J}_g(k,n)$ is a gluon jet function responsible for collinear divergences associated with the radiated gluon, $\mathcal{J}_{E,g}(\beta_k,n)$ is its eikonal counterpart, responsible for the subtraction of soft-collinear poles, $\mathcal{S}_{n+1,0}$ is the virtual soft function for the full set of $(n+1)$ particles, and $\mathcal{S}_{n,1}^{\mathrm{fin}}(k;\beta_i)$ is IR finite. Expanding Eq. (23) to one loop, one finds

$$
\mathcal{S}_{n,1}^{(1)}(k;\beta_i) = \mathcal{S}_{n+1,0}^{(1)}(\beta_k,\beta_i)\, \mathcal{S}_{n,1}^{(0)}(k;\beta_i) + \left(\mathcal{J}_g^{(1)}(k,n) - \mathcal{J}_{E,g}^{(1)}(\beta_k,n) \right) \mathcal{S}_{n,1}^{(0)}(k;\beta_i)\,. \tag{24}
$$

Squaring Eq. (24), and retaining one-loop contributions, explains the relationship between the two candidate definitions of $K_{n+1}^{(\mathbf{RV}),\,\mathrm{s}}$, Eqs. (20) and (22): they differ by purely *hard collinear* contributions, arising from the soft-subtracted gluon jet. The two definitions are therefore consistent in the soft limit, as claimed. A non-trivial calculation of the soft limit of the real-virtual squared

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matrix element, and of the one-loop contribution to the single-radiative soft function, fully confirms these rather formal arguments. One finds [27]

$$S_{n,1}^{(1)}(k;\beta_i) = \mathbf{S}_k \, RV_{n+1} - \frac{\alpha_s^2 \mu^{2\epsilon}}{S_\epsilon} \sum_{i>j}^{n} \frac{\beta_i \cdot \beta_j}{\beta_i \cdot k \, \beta_j \cdot k} \mathbf{T}_i \cdot \mathbf{T}_j \left[\sum_{m=1}^{n} \frac{\gamma_m^{(1)}}{\epsilon} + \frac{b_0}{2\epsilon} \right], \tag{25}$$

where $\gamma_m^{(1)}$ is the one-loop contribution to the collinear anomalous dimension, responsible for hard collinear poles, $S_\epsilon = (4\pi e^{-\gamma_E})^\epsilon$ is the standard $\overline{MS}$ factor, and the calculation was performed for the bare soft function. Eq. (25) states that the one-loop contribution to the radiative soft function fully captures the soft singularities of the real-virtual squared matrix element, up to corrections that are proportional to purely collinear poles.

4. Outlook

Strongly-ordered infrared limits are important ingredients for subtraction algorithms, whose relevance and intricacy grow steeply at high orders, and they have an interesting structure from the point of view of factorisation. We have proposed tree-level factorisation formulas for strongly-ordered soft limits, which naturally generalise to an arbitrary number of soft emissions, and match existing results. At loop level, we have presented evidence that a refactorisation of soft functions provides insights in the structure of real-virtual soft subtraction counterterms: indeed, applying finiteness constraints which follow from completeness sums like Eq. (11) successfully links strongly-ordered double radiation to real-virtual corrections. With appropriate adjustments, our results generalise to collinear limits, where multiple radiative jet functions refactorise into products of lower-order ones in strongly-ordered collinear limits. In the collinear case, however, a detailed implementation must tackle the issue of phase-space mappings. Further details will be presented in a forthcoming publication [27].

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