Karlsruhe Institute of Technology

Wave phenomena
analysis and numerics

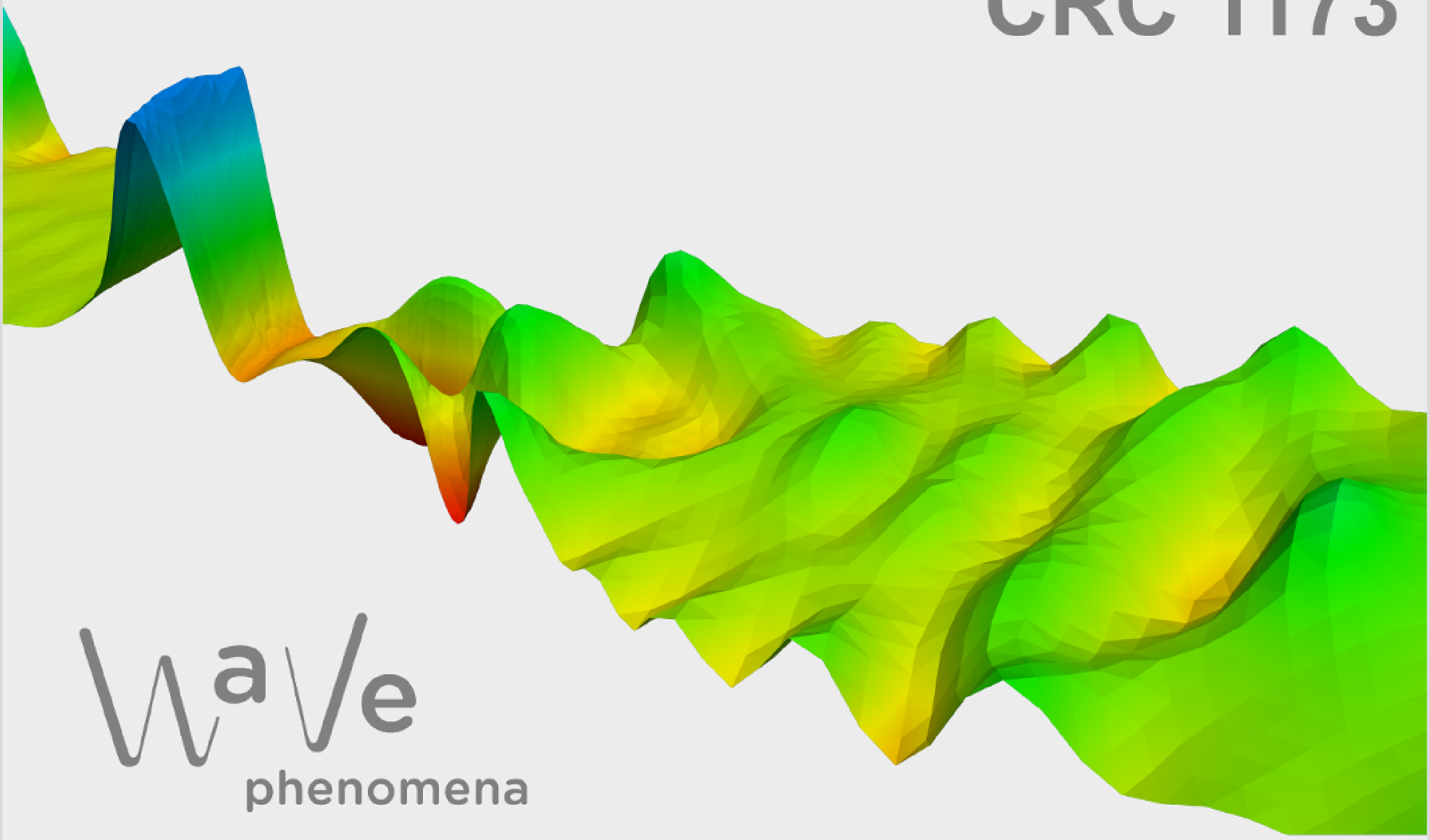
Recent developments in spectral theory for non-self-adjoint Hamiltonians

Lucrezia Cossetti, Luca Fanelli, Nico M. Schiavone

CRC Preprint 2022/57, November 2022

KARLSRUHE INSTITUTE OF TECHNOLOGY

CRC 1173



Wave phenomena

Participating universities

Universität Stuttgart

EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

Funded by

DFG

ISSN 2365-662X

Recent developments in spectral theory for non-self-adjoint Hamiltonians

Lucrezia Cossetti[1], Luca Fanelli[2] and Nico M. Schiavone[3]

[1] *Fakultät für Mathematik, Institut für Analysis, Karlsruher Institut für Technologie (KIT), Englerstraße 2, 76131 Karlsruhe, Germany; e-mail:* `lucrezia.cossetti@kit.edu`

[2] *Ikerbasque & Departamento de Matematicas, Universidad del País Vasco/Euskal Herriko Unibertsitatea (UPV/EHU), Aptdo. 644, 48080, Bilbao, Spain; e-mail:* `luca.fanelli@ehu.es`

[3] *Department of Mathematics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, Trojanova 13, 12000 Prague 2, Czechia; e-mail:* `schianic@fjfi.cvut.cz`

26 July 2022

Dedicated to Prof. Tohru Ozawa for his birthday, with deep gratitude for all he has done and sincere friendship

Abstract

The objective of this survey is to collect and elaborate on different tools, both well-established and more recent ones, which have been developed in the last decades to investigate spectral properties of non-self-adjoint operators of the form $H = H_0 + V$. More specifically, we will show how Hardy-type and Sobolev inequalities, together with Virial theorems and Birman-Schwinger principles enter into play in the analysis of the spectrum of these Hamiltonians.

Contents

2020 Mathematical Subject Classification. 47A55,47A25, 81Q15

Keywords. Spectrum, Self-Adjoint operators, Non Self-adjoint operators, Hardy-type inequalities, Mourre theory, Birman-Schwinger principle

1 Introduction

In this survey we consider the following Hamiltonian

$$H := H_0 + V.$$

This can be seen as a perturbation through the operator V of a reference operator H_0. Hereafter, unless explicitly mentioned, H_0 will represent a *general* self-adjoint operator in $L^2(\mathbb{R}^d)$, whereas V will denote a potential, *i.e.* a multiplication operator by a generating function $V\colon \mathbb{R}^d \to \mathbb{C}$. Notice that as V is possibly-complex valued, H does not need to be self-adjoint.[1]
If one is interested in the spectrum of the perturbed Hamiltonian H, then the following natural questions arise:

1. *Under which perturbations V the spectral properties of the free operator H_0 are preserved?*

Or, from a different point of view:

2. *How and to what extend spectral properties of H deviate from the ones of H_0?*

In this paper, without attempting to be complete, we collect several techniques that have been developed in the last decades to give a satisfactory answer to the questions posed above, both in a self-adjoint and non-self-adjoint context. As we will see in details below, some of the tools available in the self-adjoint framework do not carry over to the non-self-adjoint setting. This has required developing entirely new methods suitably adapted to the non-self-adjoint context.

To fix the notation we start recalling the following basic definitions. The spectrum $\sigma(H)$ of a closed linear operator H with dense domain in a Hilbert space $\mathfrak{H}$ is the set of complex numbers z for which the operator $H - z\colon \operatorname{dom}(H) \to \mathfrak{H}$ is not bijective. The resolvent set is the complement of the spectrum, namely $\rho(H) := \mathbb{C} \setminus \sigma(H)$. The set of all eigenvalues z of H of finite multiplicities which are isolated points of the spectrum and such that the range of $H - z$ is closed is the *discrete* spectrum $\sigma_d(H)$. The *essential* spectrum $\sigma_{\text{ess}}(H)$ of H is the set of complex numbers z such that $H - z$ is not Fredholm.[2] If H is a self-adjoint operator, $\sigma_{\text{ess}}(H)$ coincides with the complement set of $\sigma_d(H)$, *i.e.* $\sigma_{\text{ess}}(H) = \sigma(H) \setminus \sigma_d(H)$. In general, namely

[1]In classical quantum mechanics the notion of Hamiltonian is customarily connected to a purely self-adjoint setting (it is the observable which represents the total energy of the system). Nevertheless, it turned out that adopting a less fundamental approach one can provide a relaxation of the notion which is meaningful also for non-self-adjoint operators (see the introduction in [48] and references therein.)

[2]Actually, this is just *one* of the many possible definition for the essential spectrum, which are all equivalent only in the self-adjoint setting. In particular, the one we are using here is σ_{e_3} in the monograph [31] by Edmunds and Evans.

for non-self-adjoint operators, this property does not need to be true. Nevertheless, as we shall see, for the non-self-adjoint operators H considered in this paper, this will still be the case. The set of *all* eigenvalues is the *point* spectrum $\sigma_p(H)$.

Throughout the paper we make the standing assumption that the spectrum of the free Hamiltonian H_0 is purely essential and that there are no eigenvalues embedded in the essential spectrum, namely

$$\sigma(H_0) = \sigma_{\text{ess}}(H_0), \qquad \sigma_p(H_0) = \varnothing.$$

This assumption does not represent any loss of generality, as all the physically relevant examples we will give below enjoy this properties.

When available, the *disjoint* decomposition of the spectrum into discrete and essential part makes it possible to get information on the spectrum as *a whole*: locating the essential spectrum as a set excludes regions where the discrete spectrum can be located and vice-versa. This, as we will see in the first part of our analysis (Sec. 2), allows to find perturbations V such that the spectrum of the free Hamiltonian does not change: if V is *local* as dictated by the Weyl theorem then the essential spectrum remains stable; if, in addition, V is *small* as suggested by Hardy-type inequalities (when available) then no eigenvalues are created outside the essential spectrum and thus one has full spectral stability, *i.e.* $\sigma(H_0) = \sigma(H_0 + V)$.
This approach already provides useful information on the spectrum, but, as said, only on the spectrum in its entirety. However, not investigating the intrinsic nature of the single parts of the spectrum undermines the possibility of getting a satisfactory understanding of the system described by our Hamiltonian. For instance, it is a very crucial point in scattering theory to identify which part of the spectrum corresponds to the so-called *bound states* (states which remain localised near a scattering center) and which one corresponds to *scattering states* (states which move freely after the interaction with the potential). In atomic physics this distinction is identified with the separation between discrete and essential spectrum, respectively. However, according to the definition of the essential spectrum, this is only true if one can exclude a priori the presence of embedded eigenvalues. This is notoriously a very difficult problem especially because they have been constructed many examples of Hamiltonians which do admit embedded eigenvalues even though the physical intuitions behind the quantum mechanical tunnelling would like to exclude this possibility. Thus, in the second part of our analysis (Sec. 3), we will show how different, but also intertwined, techniques as Mourre theory, Birman-Schwinger principle and the method of multipliers can be used for the purpose of excluding embedded eigenvalues in the essential spectrum. In the case when the perturbation of the free Hamiltonian does create eigenvalues, it is an interesting problem understanding where these eigenvalues are located in the complex plane. A very powerful tool to address such a problem is represented by the Birman-Schwinger principle that we discuss in the third part of this survey.
We emphasise that throughout the paper we decided to keep almost always the reasoning on a formal level and to avoid all technical details. This to make the reader familiar with the ideas behind the techniques we will present and that could easily get lost within a too technical presentation. Nevertheless all the rigorous proofs of the results that we mention can be found in the related recalled references.

2 Qualitative spectral stability

As far as stability of the essential spectrum is concerned, it is well known from Weyl's theorem (see [58, Thm. XIII.14]) that perturbations V (possibly-complex valued) which decay sufficiently fast at infinity (or, more in general, which are relatively compact with respect to the free Hamiltonian H_0) do not change the essential spectrum, *i.e.* $\sigma_{\text{ess}}(H_0 + V) = \sigma_{\text{ess}}(H_0)$. On the other hand, Weyl's theorem alone does not exclude the possibility of these perturbations creating discrete eigenvalues outside the essential spectrum.

This can be explained easily for bounded from below H_0 and considering self-adjoint perturbations, namely for real-valued potentials $V : \mathbb{R}^d \to \mathbb{R}$. In this case, the min/max principle says that, in principle, if V is an attractive perturbation, *i.e.* a non-trivial and non-positive function, then

$$\inf \sigma(H_0 + V) \leq \inf \sigma(H_0). \tag{2.1}$$

If the inequality above is strict, then the attractive nature of the potential *does* affect the spectrum creating discrete eigenvalues below the essential spectrum. In passing, observe that if the potential is repulsive, namely V is non-trivial and non-negative, then $H_0 + V \geq 0$, thus $\inf \sigma(H_0 + V) = \inf \sigma(H_0)$ and no discrete eigenvalues can be created.

Nevertheless, it is a well known fact that the existence of Hardy-type inequalities for the free Hamiltonian H_0 represents a powerful source of sufficient conditions (intrinsically smallness) on the attractive perturbation to ensure that this strict bound does not occur. We elaborate more on this consequence of Hardy-inequalities in the next section.

2.1 Hardy-type inequalities

We start this subsection recalling the following definition (see [47]): we say that a self-adjoint and bounded from below operator H_0 on $L^2(\mathbb{R}^d)$ satisfies an Hardy-type inequality if there exists a non-trivial, measurable and almost everywhere positive function ρ such that

$$H_0 - \inf \sigma(H_0) \geq \rho. \tag{2.2}$$

We take $V := -\varepsilon|W|$, $\varepsilon > 0$ and assume, for simplicity, $W \in C_0^\infty(\mathbb{R}^d)$. From (2.2) one has

$$H_0 + V = H_0 - \varepsilon|W| = H_0 - \rho + \rho - \varepsilon|W| \geq \inf \sigma(H_0) + \rho - \varepsilon|W|.$$

If ε is sufficiently small, then $\rho - \varepsilon|W|$ is non-negative. Thus, from the variational characterisation of the spectrum and from (2.1), $\inf \sigma(H_0 + V) = \inf \sigma(H_0)$ and therefore no discrete eigenvalues are created. This simple argument shows the relevance of proving Hardy-type inequalities and, more importantly, of proving optimality of such inequalities: if inequality (2.2) is available, the "larger" (*i.e.* the sharper) is the right-hand-side, the "stronger" can be the attractive perturbation that still can be proven not affecting the free Hamiltonian. For reader's convenience we give below two concrete examples.

Example 2.1 (Schrödinger and Hardy inequality)**.** Let $H = -\Delta + V$ be the Schrödinger operator with $V : \mathbb{R}^d \to (-\infty, 0]$. It is well know that $\sigma(-\Delta) = \sigma_{\text{ess}}(-\Delta) = [0, \infty)$. In particular, $\inf \sigma(-\Delta) = 0$.

If $d \geq 3$, the operator $-\Delta$ satisfies the Hardy inequality

$$\int_{\mathbb{R}^d} |\nabla \psi(x)|^2 \, dx \geq \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2} \, dx$$

for any $\psi \in H^1(\mathbb{R}^d)$, which can be alternatively written as

$$-\Delta \geq \frac{(d-2)^2}{4} \frac{1}{|x|^2}, \tag{2.3}$$

(*cf.* (2.3) with (2.2) above), where (2.3) has to be interpreted in a form sense. From (2.3) and the general reasoning above one sees that if our attractive potential decays at least as fast as $|x|^{-2}$ at infinity and it is sufficiently small, then $-\Delta + V \geq 0$ which gives $\sigma(-\Delta + V) \geq 0$, thus no negative eigenvalues can occur.

Example 2.2 (Biharmonic operator and Rellich inequality)**.** Let $H = \Delta^2 + V$ be the perturbed biharmonic operator with $V : \mathbb{R}^d \to (-\infty, 0]$. It is well known that $\sigma(\Delta^2) = \sigma_{\mathrm{ess}}(\Delta^2) = [0, \infty)$. In particular, $\inf \sigma(\Delta^2) = 0$. Moreover, if $d \geq 5$, Δ^2 satisfies the Rellich inequality

$$\int_{\mathbb{R}^d} |\Delta \psi(x)|^2 \, dx \geq \frac{d^2(d-4)^2}{16} \int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^4} \, dx,$$

for any $\psi \in H^2(\mathbb{R}^d)$, which can be alternatively written as

$$\Delta^2 \geq \frac{d^2(d-4)^2}{16} \frac{1}{|x|^4},$$

(*cf.* again (2.2)). Reasoning as above one can see that for suitably small attractive potentials decaying at infinity at least as $|x|^{-4}$, the discrete spectrum of the biharmonic operator remains empty.

If one looks at the general argument presented above, it is clear that, due to the consistent use of the variational principle, this cannot be generalised to cover complex-valued potentials. Nevertheless, in specific situations, for instance in the case of Schrödinger operators, it has been shown that a suitable usage of the method of multipliers allows to still exploit Hardy-type inequalities to detect smallness/repulsivity conditions which ensure that the discrete spectrum remains empty (actually with this approach one can even prove that the entire point spectrum is empty) also in a non-self-adjoint setting. We briefly explain below the method taking the Schrödinger operator as toy model. Afterwards we will give references to similar results for other relevant models.

2.2 The method of multipliers: absence of discrete eigenvalues

As anticipated, to explain the method of multipliers we take as toy model the non-self-adjoint Schrödinger operator

$$H = -\Delta + V, \qquad V : \mathbb{R}^d \to \mathbb{C}, \qquad d \geq 3.$$

Here, for simplicity, we assume $V \in C_0^\infty(\mathbb{R}^d)$, nevertheless the method we shall illustrate below works under weaker assumption on the potential V (we refer to [17,36,37] where the method of multipliers for this model is developed in full details). We assume that there exists $\psi \in H^1(\mathbb{R}^d)$ which satisfies (weakly) the eigenvalue equation associated to H, namely

$$-\Delta\psi + V\psi = z\psi,$$

namely, for any $v \in H^1(\mathbb{R}^d)$, ψ satisfies

$$(v, -\Delta\psi) + (v, V\psi) = z(v, \psi). \tag{2.4}$$

In a nutshell, the method is based on producing several integral identities by choosing various test functions in (2.4) and later combining them in a way that one can ultimately detect conditions on V which imply $\psi = 0$. This would contradict ψ being an eigenfunction corresponding to z and thus $\sigma_p(H) = \varnothing$. Thus, in principle, the method can be used to totally exclude eigenvalues (not only discrete).

As a warm-up we start considering the self-adjoint case, namely the case of V being real-valued. Since we are interested in excluding discrete eigenvalues only, we assume that $z \in \mathbb{R}$ is *negative* (recall that since $V \in C_0^\infty(\mathbb{R}^d)$ then $\sigma_{\mathrm{ess}}(-\Delta + V) = \sigma_{\mathrm{ess}}(-\Delta) = [0, \infty)$). In this easy situation we consider as test function $v \in H^1(\mathbb{R}^d)$ in (2.4) the solution itself, namely $v := \psi$. Thus one easily obtains

$$\int_{\mathbb{R}^d} |\nabla\psi|^2 \, dx + \int_{\mathbb{R}^d} V|\psi|^2 \, dx = z \int_{\mathbb{R}^d} |\psi|^2 \, dx \leq 0,$$

where the last inequality follows from z being negative. Now one notices that in the term involving the potential it is the sign of the potential which gives the sign to the integral. This fact allows to easily detect repulsivity conditions on V which exclude discrete eigenvalues. Indeed, if $V_- = 0$ or the Hardy-type smallness condition

$$\int_{\mathbb{R}^d} V_-|\psi|^2 \, dx \leq a^2 \int_{\mathbb{R}^d} |\nabla\psi|^2 \, dx, \qquad a^2 < 1 \tag{2.5}$$

holds for any $\psi \in H^1(\mathbb{R}^d)$, then in both case one gets $\int_{\mathbb{R}^d} |\nabla\psi|^2 \, dx \leq 0$, which entails $\psi = 0$. Thus, no discrete (negative) eigenvalues can occur for repulsive ($V_- = 0$) potentials or for attractive but small (in the sense of (2.5)) potentials.

Now, we see how to handle the non-self-adjoint situation. Our purpose is to exclude discrete eigenvalues, *i.e.* we want to show that if z is an eigenvalue of H, then $z \notin \mathbb{C} \setminus [0, \infty)$. For simplicity, instead of considering this general case, we shall see how to exclude eigenvalues *only* outside a fixed "half-cone" containing $[0, \infty)$, namely in the set $\mathbb{C} \setminus \mathcal{C}$, where

$$\mathcal{C} := \{z \in \mathbb{C} \colon |\Im z| \leq \Re z, \Im z \neq 0\}. \tag{2.6}$$

The general situation will be considered in the next Subsection 3.2. By contradiction we assume $z \in \mathbb{C} \setminus \mathcal{C}$. Without loss of generality we assume also $\Im z \geq 0$, the case $\Im z < 0$ can be done similarly. Using as test function $v := \psi$ and taking the real and the imaginary part of the resulting identity one has

$$\int_{\mathbb{R}^d} |\nabla\psi|^2 \, dx + \int_{\mathbb{R}^d} \Re V|\psi|^2 \, dx = \Re z \int_{\mathbb{R}^d} |\psi|^2 \, dx, \tag{2.7}$$

and

$$\int_{\mathbb{R}^d} \Im V|\psi|^2\,dx = \Im z \int_{\mathbb{R}^d} |\psi|^2\,dx, \tag{2.8}$$

respectively. Subtracting (2.8) from (2.7) one gets

$$\int_{\mathbb{R}^d} |\nabla\psi|^2\,dx + \int_{\mathbb{R}^d} (\Re V - \Im V)|\psi|^2\,dx = (\Re z - \Im z)\int_{\mathbb{R}^d} |\psi|^2\,dx \leq 0,$$

where the last inequality follows from $\Im z > \Re z$ (*cf.* (2.6)). Now, if $\Re V$ is sufficiently repulsive (*i.e.* its negative part $[\Re V]_-$ is sufficiently small) and if $\Im V$ is sufficiently small, then a similar argument as in the self-adjoint case applies which entails $\psi = 0$. More precisely, the conditions that one assumes are

$$\int_{\mathbb{R}^d} [\Re V]_-|\psi|^2\,dx \leq b_1^2 \int_{\mathbb{R}^d} |\nabla\psi|^2\,dx, \qquad \int_{\mathbb{R}^d} |\Im V||\psi|^2\,dx \leq b_2^2 \int_{\mathbb{R}^d} |\nabla\psi|^2\,dx, \tag{2.9}$$

for $b_1^2 + b_2^2 < 1$.

3 Absence of embedded eigenvalues

In the previous section we have seen that the presence of Hardy-type inequalities for the free Hamiltonian H_0 provides sources of sufficient conditions on the potentials V which guarantee absence of *discrete* eigenvalues of H. The question of whether H can have eigenvalues *embedded* in the essential spectrum is in general more difficult and, as a matter of fact, the results that do exist require rather subtle tools. In the self-adjoint case, one successful theory is the so-called Mourre theory or positive commutator theory of Quantum Mechanics whose classical formulation we shortly recall below.

3.1 The Mourre theory and the Virial theorem

The starting point of the Mourre theory is the Virial theorem for self-adjoint operators. Here, for simplicity, we give just a formal statement of the result (we disregard domain issues, etc). We refer to [1, Prop.6.6], [25, Sec.4]) and also [58, Thm XIII.59 and notes on Sec. XIII.13] for the rigorous statements. See also [57].

Theorem 3.1 (Virial Theorem)**.** *Let H and A be self-adjoint operators on a Hilbert space $\mathcal{H}$. If ψ is an eigenfunction of H then*

$$(\psi, i[H,A]\psi) = 0. \tag{3.1}$$

Formally, the statement is obtained just expanding the commutator and using self-adjointness of the Hamiltonian. However, for unbounded operators a rigorous proof requires more care.

The following result is an immediate consequence of the Virial theorem 3.1.

Theorem 3.2 (Mourre Theorem)**.** *Let H and A be self-adjoint operators on on a Hilbert space $\mathcal{H}$. Assume that the commutator $i[H,A]$ between H and A is positive,* i.e. *there exists a positive number a such that*

$$i[H,A] \geq aI. \tag{3.2}$$

Then $\sigma_{\mathrm{p}}(H) = \varnothing$.

Proof. Even though the proof is trivial, we sketch it here for the sake of completeness. If ψ were an eigenfunction, then from 3.1 in the Virial theorem and from assumption (3.2) one would get the contradiction

$$0 = (\psi, i[H,A]\psi) \geq a > 0.$$

Then the point spectrum of H is empty. □

Remark 3.1. In 1980 Mourre [57] showed that the *global* positivity hypothesis (3.2) in Thm 3.2 can be relaxed. More precisely, he proved that *spectrally localized* positive commutator estimates (known as Mourre's inequalities) are sufficient to get the finiteness of embedded eigenvalues and the absence of singular continuous spectrum. In particular, Thm. 3.2 follows from his result.

Remark 3.2. Notice that Thm. 3.2 provides a condition to prove *total* absence of eigenvalues (not only embedded).

Applied to particular situations, the abstract positive-commutator condition (3.2) in Theorem 3.2 translates into natural repulsivity conditions on the potential V for Hamiltonians of the form $H = H_0 + V$. As above, we take as reference model the Schrödinger and perturbed bharmonic operators on $L^2(\mathbb{R}^d)$, namely we take $H_0 = -\Delta$ and $H_0 = \Delta^2$, respectively.

Example 3.1 (Schrödinger and perturbed biharmonic operators)**.** Let A be the generator of dilation, namely the self-adjoint operator

$$A := -\frac{i}{2}(x \cdot \nabla + \nabla \cdot x), \tag{3.3}$$

then the following commutation relations hold:

$$i[-\Delta, A] = -2\Delta, \qquad i[\Delta^2, A] = 4\Delta^2, \qquad i[V, A] = -x \cdot \nabla V.$$

Assuming the pointwise repulsivity condition $x \cdot \nabla V \leq 0$ gives

$$i[-\Delta + V, A] \geq -2\Delta, \qquad i[\Delta^2 + V, A] \geq 4\Delta^2. \tag{3.4}$$

Since $-\Delta$ and Δ^2 are just a non-negative operator, the inequalities above do not imply the validity of the positive commutator condition as stated in (3.2) and thus the Mourre theorem 3.2 does not directly apply. Nevertheless, since the right-hand-sides in (3.4) do have a sign, analogous reasoning as in Theorem 3.2 allow to conclude: Indeed, let $\psi \in H^1(\mathbb{R}^d)$ be an eigenfunction of $H = -\Delta + V$ and let $\phi \in H^2(\mathbb{R}^d)$ be an eigenfunction of $H = \Delta^2 + V$, then from Theorem 3.1 and from inequalities (3.4) one has

$$0 = (\psi, i[-\Delta + V, A]\psi) \geq \|\nabla\psi\|_2^2, \qquad 0 = (\phi, i[\Delta^2 + V, A]\phi) \geq 4\|\Delta\phi\|_2^2.$$

From these and from the decay assumptions on the eigenfunctions, it immediately follows that $\psi = \phi = 0$.

Remark 3.3. Notice that the pointwise repulsivity condition $x \cdot \nabla V \leq 0$ on the potential can be easily weakened asking the validity of a variational smallness condition (in the spirit of (2.5) and (2.9)) for the positive part of the radial derivative of the potential, namely

$$(\psi, [x \cdot \nabla V]_+ \psi) \leq c \|H_0^{1/2} \psi\|_2^2, \qquad \text{for all } \psi \in \operatorname{dom}(H_0^{1/2}). \tag{3.5}$$

If c in (3.5) is sufficiently small, then the absence of point spectrum still follows under the weaker assumption (3.5).

Notice that, as already mentioned, the possibility to exclude bound states in the previous two situations worked out only because both $-\Delta$ and Δ^2 are bounded below (actually non-negative) operators. For operators that are not bounded below, Mourre-type results still apply but they allow to exclude embedded eigenvalues only in part of the essential spectrum. As an explanatory example we consider below the perturbed Dirac operator.

Example 3.2 (Dirac operator)**.** Let $H = \mathcal{D}_m + V I_{\mathbb{C}^{N\times N}}$, where $\mathcal{D}_m$ is the free Dirac operator

$$\mathcal{D}_m := -i\boldsymbol{\alpha} \cdot \nabla + m\alpha_{d+1} = -i \sum_{k=1}^{d} \alpha_k \partial_k + m\alpha_{d+1}. \tag{3.6}$$

Here $m \geq 0$ is the mass, $N := 2^{\lceil d/2 \rceil}$ and $V \colon \mathbb{R}^d \to \mathbb{R}$. The Dirac matrices $\alpha_k \in \mathbb{C}^{N\times N}$, for $k \in \{1, \dots, d+1\}$ satisfy the anti-commutation relations

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_j^k$$

where δ_j^k is the Kronecker symbol.
If $m > 0$ the spectrum of the free Dirac operator $\mathcal{D}_m$ is the union of two disjoint semi-axis, namely

$$\sigma(\mathcal{D}_m) = \sigma_{\text{ess}}(\mathcal{D}_m) = (-\infty, -m] \cup [m, +\infty),$$

otherwise if $m = 0$ then its spectrum is the whole real line

$$\sigma(\mathcal{D}_0) = \sigma_{\text{ess}}(\mathcal{D}_0) = \mathbb{R}.$$

Computing the commutator with the dilation operator A defined in (3.3) one has

$$i[\mathcal{D}, A] = -i\boldsymbol{\alpha} \cdot \nabla, \qquad i[V, A] = -x \cdot \nabla V.$$

Differently from the previous case, since the commutator $i[\mathcal{D}, A] = -i\alpha \cdot \nabla$ does not have a sign, the full commutator $i[H, A] = -i\boldsymbol{\alpha} \cdot \nabla - x \cdot \nabla V I_{\mathbb{C}^{N\times N}}$ is also unsigned (even under the repulsivity condition $x \cdot \nabla V \leq 0$) and therefore the Mourre theorem does not apply as such. Nevertheless, a slight modification of the argument above gives the following analogous (but weaker) implications: if $V + x \cdot \nabla V \leq 0$, then $\sigma_p(H) \cap (m, \infty) = \varnothing$ (if $V + x \cdot \nabla V < 0$, then $\sigma_p(H) \cap [m, \infty) = \varnothing$). Analogously, if $V + x \cdot \nabla V \geq 0$, then $\sigma_p(H) \cap (-\infty, -m) = \varnothing$ (if $V + x \cdot \nabla V > 0$, then $\sigma_p(H) \cap (-\infty, m] = \varnothing$). If $V + x \cdot \nabla V = 0$ (as in the case of Coulomb potential or, more elementary, in the free case $V = 0$), then $\sigma_p(H) \cap [-m, m]^c = \varnothing$. Finally, if $M_1 \leq V + x \cdot \nabla V \leq M_2$, then $\sigma_p(H) \cap [-m - M_1, m + M_2]^c = \varnothing$. These are consequences of the following modified Virial theorem for the Dirac operator.

Theorem 3.3 (Virial for Dirac)**.** *Let $H = \mathcal{D}_m + V I_{\mathbb{C}^{N\times N}}$ be the perturbed Dirac operator, with $\mathcal{D}_m$ as in (3.6) and $V\colon \mathbb{R}^d \to \mathbb{R}$ and let A be the dilation operator defined in (3.3). If $\psi \in H^1(\mathbb{R}^d)^N$ is an eigenfunction of H corresponding to the eigenvalue $E \in \mathbb{R}$, then*

$$E\|\psi\|_2^2 = (\psi, (V + x\cdot\nabla V)\psi) + m(\psi, \alpha_{d+1}\psi). \tag{3.7}$$

Proof. Formally, the statement is obtained just observing that from Theorem 3.1 one has

$$0 = (\psi, i[H, A]\psi) = (\psi, -i\alpha\cdot\nabla\psi) - (\psi, x\cdot\nabla V\psi).$$

Then (3.7) is obtained just using the eigenvalue equation satisfied by ψ. □

Remark 3.4. Observe that the statement of Thm. (3.3) is formal (no hypotheses on the regularity/integrability of V are given). The rigorous result can be found, for instance, in [3, Thm. 4.1.1, Thm. 4.1.2].

Now if one chooses for α_{d+1} the following representation

$$\alpha_{d+1} = \begin{pmatrix} I_{\mathbb{C}^{N/2\times N/2}} & 0 \\ 0 & -I_{\mathbb{C}^{N/2\times N/2}} \end{pmatrix},$$

it is easy to see that

$$-m \leq \langle \alpha_{d+1} m\psi, \psi\rangle \leq m,$$

and thus the implications above follow easily from (3.7). See [3] and [62] for further details). To sum up, we have seen that in the case of the Dirac operator the lack of positivity of the commutator does not entail total absence of eigenvalues, nevertheless the underlying idea of the virial theorem can be still used to exclude eigenvalues embedded in (part of) the essential spectrum.

The beauty as well as the power of the Mourre theory relies on the fact that it provides a completely abstract condition (see (3.2)), *i.e.* not depending on the specific Hamiltonian of the system, to obtain information on eigenvalues embedded (and not) in the essential spectrum, which moreover, as shown in the examples above, is also easy to verify in the particular cases (at least on a formal level).
Nevertheless, if one considers non-self-adjoint Hamiltonians of the form $H = H_0 + V$, with $V\colon \mathbb{R}^d \to \mathbb{C}$ (complex-valued) several difficulties arise. As a matter of fact, already the Virial theorem 3.1 ceases to be valid: indeed if ψ is an eigenfunction of the non-self-adjoint Hamiltonian $H = H_0 + V$, *i.e.* ψ satisfies $H\psi = z\psi$, with $z \in \mathbb{C}$ (the spectrum is no more necessarily real), an easy computation gives

$$(\psi, i[H, A]\psi) = 2i\Im z(\psi, A\psi) - 2(\Im V\psi, A\psi). \tag{3.8}$$

Thus the mean value of the commutator over the state ψ is no longer zero and, as a matter of fact, it depends on the imaginary part of the eigenvalue and of the complex potential. Since the right-hand-side is no longer zero and it is not even defined in sign, finding conditions such that

$i[H, A] > 0$ would not lead to any contradiction. In other words, this shows that the standard Mourre theory does not easily carry over the non-self-adjoint framework.

Leaving aside for now the Mourre theory, it turned out in the last decade that in the non-self-adjoint case a successful technique to get absence of bound states (both discrete and embedded) is the method of multipliers which, so far we have seen at work only to show absence of discrete eigenvalues. In the next section we show how to modify the strategy developed in Sec 2.2 to obtain absence of embedded eigenvalues as well.

3.2 The method of multipliers: absence of embedded eigenvalues

We recall that the core of the method of multipliers already introduced in Sec 2.2 relies on finding suitable test functions in the weak formulation of the corresponding eigenvalue equation such that a consequent manipulation of the resulting identities leads to a contradiction. We consider again as toy model the Schrödinger operator. In Sec. 2.2 we have seen how to exclude eigenvalues outside the conical sector $\mathcal{C}$ defined in (2.6), *i.e.* far away from the essential spectrum of $-\Delta$. Now we consider the case of $z \in \mathcal{C}$ (this in particular covers embedded eigenvalues). Here, for simplicity, we consider $V \in C_0^\infty(\mathbb{R}^d)$. Nevertheless the result presented below holds under less restrictive hypotheses about V (see [17]). Considering as test function the skew-symmetric multiplier $v := [-\Delta, |x|^2]\psi$ and taking the real part of the resulting identity, one gets

$$\begin{aligned}\Re(H\psi, [-\Delta, |x|^2]\psi) &= \Re z(\psi, [-\Delta, |x|^2]\psi) \\ &= \Re z \Re(\psi, [-\Delta, |x|^2]\psi) - \Im z \Im(\psi, [-\Delta, |x|^2]\psi) \\ &= -\Im z \Im(\psi, [-\Delta, |x|^2]\psi),\end{aligned} \tag{3.9}$$

where the last identity follows from $(\psi, [-\Delta, |x|^2]\psi)$ being purely imaginary. Computing explicitly the L^2-products above gives

$$2\int_{\mathbb{R}^d} |\nabla\psi|^2\,dx + d\int_{\mathbb{R}^d} \Re V|\psi|^2\,dx + 2\Re\int_{\mathbb{R}^d} x\cdot V\psi\overline{\nabla\psi}\,dx = -2\Im z\Im\int_{\mathbb{R}^d} x\cdot\psi\overline{\nabla\psi}\,dx. \tag{3.10}$$

As a warm-up let suppose that V is real-valued, then $\Im z = 0$ and an integration by parts gives

$$2\int_{\mathbb{R}^d} |\nabla\psi|^2\,dx - \int_{\mathbb{R}^d} x\cdot\nabla V|\psi|^2\,dx = 0.$$

Notice that in the term involving V it is the sign of the radial derivative $x \cdot \nabla V$ that gives the sign to the integral. Thus, one can either assume the repulsivity condition $[x \cdot \nabla V]_+ = 0$ or the Hardy-type smallness condition

$$\int_{\mathbb{R}^d} [x\cdot\nabla V]_+|\psi|^2\,dx \le a^2\int_{\mathbb{R}^d} |\nabla\psi|^2\,dx, \qquad a^2 < 2,$$

valid for all $\psi \in H^1(\mathbb{R}^d)$. From both these conditions one has $\int_{\mathbb{R}^d} |\nabla\psi|^2\,dx \le 0$, which entails $\psi = 0$.

If V is complex-valued one immediately realizes that the sole identity (3.10) is not enough to obtain the contradiction as in the self-adjoint case. Indeed, the term on the right-hand-side which depends on the eigenvalue (or, better, on its imaginary part) does not have a sign and has to be controlled exploiting some suitable positive quantities. From this one understands that, in the non-self-adjoint case, more identities are needed to get positivity. More precisely, given $g\colon \mathbb{R}^d \to \mathbb{R}$ to be chosen later, one takes as test function $v = g\psi$ and then takes the real and the imaginary parts of the resulting identity. A suitable algebra of these three identities and suitable choices of g (see [17, 37] for details) give the following identity

$$\begin{aligned}\int_{\mathbb{R}^d} |\nabla\psi^-|^2\,dx + \frac{|\Im z|}{\sqrt{\Re z}}\int_{\mathbb{R}^d} |x||\nabla\psi^-|^2\,dx - \frac{d-1}{2}\frac{|\Im z|}{\sqrt{\Re z}}\int_{\mathbb{R}^d}\frac{|\psi|^2}{|x|}\,dx\\ -\int_{\mathbb{R}^d}\partial_r(r\Re V)|\psi|^2\,dx - 2\Im\int_{\mathbb{R}^d} x\Im V\psi\overline{\nabla\psi^-}\,dx + \frac{|\Im z|}{\sqrt{\Re z}}\int_{\mathbb{R}^d} r\Re V|\psi|^2\,dx = 0,\end{aligned} \tag{3.11}$$

where $\psi^-(x) := e^{-i\sqrt{\Re z}\,\mathrm{sgn}(\Im z)|x|}\psi(x)$. The auxiliary function ψ^- is related to the Sommerfeld radiation conditions

$$\lim_{r\to\infty}\int_{|x|=r}\Big|\nabla\psi - i\lambda^{1/2}\frac{x}{|x|}\psi\Big|^2\,d\sigma(r) = 0$$

for unique solvability of the Helmholtz equation $-\Delta\psi + V\psi = \lambda\psi$, $\lambda \geq 0$, once one observes that $|\nabla\psi^-|^2 = \big|\nabla\psi - i\lambda^{1/2}\frac{x}{|x|}\psi\big|^2$ (see [34] where the auxiliary function ψ^- was introduced for the first time). At this point, from identity (3.11), it is reasonable to believe that if $[\Re V]_-$, $[\partial_r(r\Re V)]_+$ and $\Im V$ are sufficiently small, then a similar argument as in the self-adjoint case applies which entails $\psi = 0$. Again, the smallness conditions are of Hardy-type, namely one asks

$$\int_{\mathbb{R}^d} r^2[\Re V]_-^2|\psi|^2\,dx \leq b_1^2\int_{\mathbb{R}^d}|\nabla\psi|^2\,dx, \qquad \int_{\mathbb{R}^d}[\partial_r(r\Re V)]_+|\psi|^2\,dx \leq b_2^2\int_{\mathbb{R}^d}|\nabla\psi|^2\,dx,$$

and

$$\int_{\mathbb{R}^d}|\Im V|^2|\psi|^2\,dx \leq b_3^2\int_{\mathbb{R}^d}|\nabla\psi|^2\,dx,$$

for suitably small b_1, b_2 and b_3. As for the negative term in the first line of (3.11), this can be estimated using the weighted Hardy inequality

$$\int_{\mathbb{R}^d}\frac{|\psi|^2}{|x|}\,dx \leq \frac{4}{(d-1)^2}\int_{\mathbb{R}^d}|x||\nabla\psi|^2\,dx, \qquad \psi \in H^1(\mathbb{R}^d), \quad d \geq 2,$$

and controlled in terms of the second one.

Remark 3.5 (Comparison between Mourre theory and method of multipliers)**.** In the two sections above we have seen that the abstract Mourre theory does not seem to be immediately applicable to non-self-adjoint Hamiltonian, instead, the method of multipliers, relying just on a direct manipulation of the eigenvalue equations does provide results in this more challenging setting. Nevertheless, one can easily show that when we restrict to the self-adjoint setting, the two approaches reduce one another. We clarify this fact again taking as toy model the Schrödinger operator $H = -\Delta + V$, with real potential $V\colon \mathbb{R}^d \to \mathbb{R}$.
As an outcome of the method of multipliers we had the following result.

Theorem 3.4. *Let $H = H_0 + V$, with $H_0 = -\Delta$ and $V\colon \mathbb{R}^d \to \mathbb{R}$. If $\psi \in H^1(\mathbb{R}^d)$ is an eigenfunction of H then*

$$(\psi, [H, [H_0, |x|^2]]\psi) = 0. \tag{3.12}$$

Proof. This follows from (3.9) with $\Im z = 0$ and using the simple fact

$$\begin{aligned} 2\Re(H\psi, [H_0, |x|^2]\psi) &= (H\psi, [H_0, |x|^2]\psi) + \overline{(H\psi, [H_0, |x|^2]\psi)} \\ &= (\psi, [H, [H_0, |x|^2]]\psi), \end{aligned}$$

where in the last identity we have used that H is self-adjoint and that $[H_0, |x|^2]$ is skew-symmetric. □

Then, the next step in the method was computing explicitly the double commutator and finding conditions on the potential such that the double commutator $(\psi, [H, [H_0, |x|^2]]\psi)$ had a sign. Looking at the Mourre theory, one notices that there it was crucial instead to give a sign to the commutator $(\psi, [H, A]\psi)$ of the Hamiltonian with the dilation operator A. On the other hand, an easy computation shows that

$$[H_0, |x|^2] = -4iA.$$

Thus the two methods are in fact the same method.

The ideas presented above related to the method of multipliers are developed in full details in a series of papers [15, 17, 18, 36, 37]. See also [8, 9] for some related results on Dirac operators. Refer to [15] for an adaptation of the method to the Lamé operator of elasticity. Instead for some recent result which relies on the positive commutator theory we refer to [4].

4 Spectral enclosures

So far we have considered only tools for ensuring spectral stability, in particular we have investigated only situations where the potentials perturbing the free Hamiltonian do not create eigenvalues (the point spectrum remains empty). Nevertheless, in the case when the perturbation does create eigenvalues, it is an interesting problem understanding where these eigenvalues are located in the complex plane. A very powerful tool to address such a problem is represented by the Birman-Schwinger principle that we discuss in the next section.

4.1 The Birman-Schwinger principle

The principle, originated in the work [5] by Birman and [61] by Schwinger, has become nowadays a fundamental and widely employed tool in spectral analysis. The easiest statement of this principle can be given as follows:

Theorem 4.1. *Let H be a perturbation of H_0 by a factorizable bounded potential $V = B^*A$,* i.e. *$H = H_0 + B^*A$ and let $z \in \rho(H_0)$. Then $z \in \sigma_p(H)$ if and only if $-1 \in \sigma_p(K_z)$, where $K_z := A(H_0 - z)^{-1}B^*$ denotes the so-called Birman-Schwinger operator.*

Proof. Let $z \in \sigma_p(H) \cap \rho(H_0)$. Then there exists a non-trivial $\psi \in \operatorname{dom}(H_0)$ such that $H\psi = z\psi$. Thus

$$H_0\psi + V\psi = z\psi \iff (H_0 - z)\psi = -V\psi \iff -\psi = (H_0 - z)^{-1}B^*A\psi$$

and so, denoting $\phi = A\psi$ and applying to both side the operator A we get

$$A(H_0 - z)^{-1}B^*\phi = -\phi,$$

so $-1 \in \sigma_p(K_z)$. The "if part" is achieved simply performing the previous few steps in reverse order. □

If -1 is an eigenvalue of K_z, this means that the norm of the Birman-Schwinger operator is at least 1, and consequently the eigenvalues of the perturbed operator H are confined in the complex region defined by $1 \leq \|K_z\|$; if in particular $\|K_z\| < 1$ uniformly with respect to z, then the point spectrum of H is empty. Thus the Birman-Schwinger principle allows either to locate the point spectrum in the complex-plane or to show its emptyness.
The advantage of studying the norm of the Birman-Schwinger operator is evident already from its definition: if we are able to bound the resolvent $(H_0 - z)^{-1}$, it is usually an easy matter setting A and B in a suitable normed space, and then obtain an estimate for K_z. This approach reduces to establishing suitable resolvent estimates for the unperturbed operator H_0. Each of them corresponds, via the Birman-Schwinger principle, to a localization estimate for the eigenvalues of the perturbed operator.
Before moving on to see the Birman-Schwinger principle at work in specific situations, we emphasise that Theorem 4.1 was stated under the convenient assumption of $V = B^*A$ being bounded (then H is well defined as operator sum of H_0 and V on $\operatorname{dom}(H_0)$) and considering only the case $z \in \rho(H_0)$ (so that $(H_0 - z)^{-1}$ is well defined). This simplified setting led to the trivial proof of the result given above. Nevertheless a more general and deeper result than Theorem 4.1 can be proved and it will be considered in the next sections. We emphasise that no proofs will be given there, as this would go beyond the scope of this survey, nevertheless all the detailed references will be provided to the reader.

4.1.1 Birman-Schwinger principle: Kato's approach

As we have observed before, requiring $V = B^*A$ to be a bounded operator ensured that $H = H_0 + B^*A$ is well-defined as a sum operator on $\operatorname{dom}(H) = \operatorname{dom}(H_0)$. For unbounded potentials, the Hamiltonian H is still well-defined on $\operatorname{dom}(H_0) \cap \operatorname{dom}(B^*A)$, but the latter intersection domain may consist of $\{0\}$ only. In this section, following the approach that dates back to Kato [43] and Konno and Kuroda [46], we recall how to rigorously construct an extension of this operator. Then, with this rigorous definition at hand, we give the general statement of the Birman-Schwinger principle.
Let $\mathfrak{H}, \mathfrak{H}'$ be Hilbert spaces and consider the densely defined, closed linear operators

$$H_0\colon \operatorname{dom}(H_0) \subseteq \mathfrak{H} \to \mathfrak{H}, \quad A\colon \operatorname{dom}(A) \subseteq \mathfrak{H} \to \mathfrak{H}', \quad B\colon \operatorname{dom}(B) \subseteq \mathfrak{H} \to \mathfrak{H}',$$

such that $\rho(H_0) \neq \varnothing$ and

$$\operatorname{dom}(H_0) \subseteq \operatorname{dom}(A), \quad \operatorname{dom}(H_0^*) \subseteq \operatorname{dom}(B).$$

Assume also, for simplicity, that $\sigma(H_0) \subset \mathbb{R}$. By $R_{H_0}(z) := (H_0 - z)^{-1}$, we denote the resolvent operator of H_0 for any $z \in \rho(H_0)$.
We assume the following set of assumptions.

Assumption A. For some, and hence for all, $z \in \rho(H_0)$, the operator $AR_{H_0}(z)B^*$, densely defined on $\mathrm{dom}(B^*)$, has a closed extension K_z in $\mathfrak{H}'$,

$$K_z = \overline{AR_{H_0}(z)B^*},$$

which we call the *Birman-Schwinger operator*, with norm bounded by

$$\|K_z\|_{\mathfrak{H}' \to \mathfrak{H}'} \leq \Lambda(z) \tag{4.1}$$

for some function $\Lambda \colon \rho(H_0) \to \mathbb{R}_+$.

Assumption B. There exists $z_0 \in \rho(H_0)$ such that $-1 \in \rho(K_{z_0})$.

The last assumption is a bit tricky to check in the application, but luckily for us is implied by the following, easier, one:

Assumption B′. There exists $z_0 \in \rho(H_0)$ such that $\Lambda(z_0) < 1$.

Indeed, assuming Assumptions A and B', we get that $\|K_{z_0}\|_{\mathfrak{H}' \to \mathfrak{H}'} < 1$. Thus, expanding in a Neumann series, we see that $(1 + K_{z_0})^{-1}$ exists and hence $-1 \in \rho(K_{z_0})$.
The extension of the perturbed operator $H_0 + B^*A$ is then constructed as follow.

Lemma 4.1 (Extension of operators with factorizable potential)**.** *Suppose Assumptions A and B. Let $z_0 \in \rho(H_0)$ such that $-1 \in \rho(K_{z_0})$. Then the operator*

$$R_H(z_0) = R_{H_0}(z_0) - \overline{R_{H_0}(z_0)B^*}(1 + K_{z_0})^{-1} AR_{H_0}(z_0) \tag{4.2}$$

defines a densely defined, closed, linear operator H in $\mathfrak{H}$ which has $R_H(z_0)$ as resolvent and which is an extension of $(H_0 + B^*A)|_{\mathrm{dom}(H_0) \cap \mathrm{dom}(B^*A)}$.

We can finally formulate the abstract Birman-Schwinger principle.

Lemma 4.2 (Birman-Schwinger principle)**.** *Suppose Assumptions A and B. Let $z_0 \in \rho(H_0)$ such that $-1 \in \rho(K_{z_0})$ and H be the extension of $H_0 + B^*A$ given by Lemma 4.1. Fix $z \in \sigma_p(H)$ with eigenfunction $0 \neq \psi \in \mathrm{dom}(H)$,* i.e. *$H\psi = z\psi$, and set $\phi := A\psi$.*
Then $\phi \neq 0$, and in addition

(i) if $z \in \rho(H_0)$ then

$$K_z \phi = -\phi$$

and in particular

$$1 \leq \|K_z\|_{\mathfrak{H}' \to \mathfrak{H}'} \leq \Lambda(z);$$

(ii) if $z \in \sigma(H_0) \setminus \sigma_p(H_0)$ and if H_0 is self-adjoint, then

$$\lim_{\varepsilon \to 0^\pm} (\varphi, K_{z+i\varepsilon}\phi)_{\mathfrak{H}'} = -(\varphi, \phi)_{\mathfrak{H}'} \tag{4.3}$$

for every $\varphi \in \mathfrak{H}'$. In particular

$$1 \leq \liminf_{\varepsilon \to 0^\pm} \|K_{z+i\varepsilon}\|_{\mathfrak{H}' \to \mathfrak{H}'} \leq \liminf_{\varepsilon \to 0^\pm} \Lambda(z + i\varepsilon). \tag{4.4}$$

Remark 4.1. Notice that differently from Theorem 4.1, Lemma 4.2 covers also the case of $z \notin \rho(H_0)$. (The proof of Lemma 4.2 in this case follows adapting the limiting argument introduced in [40, Thm. 8].)

4.1.2 Birman-Schwinger principle: abstract approach

In this subsection, we recall again the technicalities for the Birman-Schwinger principle and for properly define an operator perturbed by a factorizable potential, but with a different approach. This time we completely rely on the abstract analysis carried out by Hansmann and Krejčiřík in [40], to which we refer for more results and background. There, in addition to the point spectrum, appropriate versions of the principle are stated even for the residual, essential and continuous spectra.
Here we collect the set of hypotheses we need.

Assumption I. Let $\mathfrak{H}$ and $\mathfrak{H}'$ be complex separable Hilbert spaces, H_0 be a self-adjoint operator in $\mathfrak{H}$ and $|H_0| := (H_0^2)^{1/2}$ its absolute value. Also, let $A\colon \mathfrak{H} \supseteq \operatorname{dom}(A) \to \mathfrak{H}'$ and $B\colon \mathfrak{H} \supseteq \operatorname{dom}(B) \to \mathfrak{H}'$ be linear operators such that $\operatorname{dom}(|H_0|^{1/2}) \subseteq \operatorname{dom}(A) \cap \operatorname{dom}(B)$. We assume that for some (and hence for all) $b > 0$ the operators $A(|H_0| + b)^{-1/2}$ and $B(|H_0| + b)^{-1/2}$ are bounded and linear from $\mathfrak{H}$ to $\mathfrak{H}$.

At this point, defining $G_0 := |H_0|+1$, we can consider, for any $z \in \rho(H_0)$, the *Birman-Schwinger operator*

$$K_z := [AG_0^{-1/2}][G_0(H_0 - z)^{-1}][BG_0^{-1/2}]^*, \tag{4.5}$$

which is linear and bounded from $\mathfrak{H}'$ to $\mathfrak{H}'$.
The second assumption we need is stated below.

Assumption II. There exists $z_0 \in \rho(H_0)$ such that $-1 \notin \sigma(K_{z_0})$.

Again, recurring to Neumnn series, we can replace it with the following hypothesis, stronger but more manageable.

Assumption II′. There exists $z_0 \in \rho(H_0)$ such that $\|K_{z_0}\|_{\mathfrak{H}' \to \mathfrak{H}'} < 1$.

Before recalling the Birman-Schwinger principle, we properly define the formal perturbed operator $H_0 + V$ with $V = B^*A$.

Theorem 4.2. *Under Assumptions I and II, there exists a unique closed extension H of H_0+V such that* $\operatorname{dom}(H) \subseteq \operatorname{dom}(|H_0|^{1/2})$ *and the following representation formula holds true:*

$$(\phi, H\psi)_{\mathfrak{H}\to\mathfrak{H}} = (G_0^{1/2}\phi, (H_0 G_0^{-1} + [BG_0^{-1/2}]^* AG_0^{-1/2})G_0^{1/2}\psi)_{\mathfrak{H}\to\mathfrak{H}}$$

for $\phi \in \operatorname{dom}(|H_0|^{1/2})$, $\psi \in \operatorname{dom}(H)$.

This result correspond to Theorem 5 in [40], where the operator H is obtained via the pseudo-Friedrichs extension. Note that following the alternative approach by Kato [43], the extension of $H_0 + B^*A$ is not only closed, but also quasi-selfadjoint ([50]). We refer to the paper of Hansmann and Krejčiřík [40] for a cost-benefit comparison of the two methods, and for a list of cases when the two extensions coincide.
Finally, we can exhibit the abstract Birman-Schwinger principle, for the proof of which we refer to Theorem 6, 7, 8 and Corollary 4 of [40].

Theorem 4.3. *Under Assumption I and II, we have:*

(i) if $z \in \rho(H_0)$, *then* $z \in \sigma_p(H)$ *if and only if* $-1 \in \sigma_p(K_z)$;

(ii) if $z \in \sigma_c(H_0) \cap \sigma_p(H)$ *and* $H\psi = z\psi$ *for* $0 \neq \psi \in \operatorname{dom}(H)$, *then* $A\psi \neq 0$ *and*

$$\lim_{\varepsilon\to 0^\pm} (K_{z+i\varepsilon}A\psi, \phi)_{\mathfrak{H}'\to\mathfrak{H}'} = -(A\psi, \phi)_{\mathfrak{H}'\to\mathfrak{H}'}$$

for all $\phi \in \mathfrak{H}'$.

In particular

(i) if $z \in \sigma_p(H) \cap \rho(H_0)$, *then* $\|K_z\|_{\mathfrak{H}'\to\mathfrak{H}'} \geq 1$;

(ii) if $z \in \sigma_p(H) \cap \sigma_c(H_0)$, *then* $\liminf_{\varepsilon\to 0^\pm} \|K_{z+i\varepsilon}\|_{\mathfrak{H}\to\mathfrak{H}'} \geq 1$.

While from the "in particular" part of the previous theorem one could infer a localization for the eigenvalues of H, the principle can be employed in a "negative" way to prove their absence when the norm of the Birman-Schwinger operator is strictly less than 1 uniformly respect to $z \in \rho(H_0)$. This is precisely stated in the next concluding result, corresponding to Theorem 3 in [40], which is even richer: it not only gives information on the absence of the eigenvalues, but also on the invariance of the spectrum of the perturbed operator.

Theorem 4.4. *Suppose Assumption I and that* $\sup_{z\in\rho(H_0)} \|K_z\|_{\mathfrak{H}'\to\mathfrak{H}'} < 1$. *Then we have:*

(i) $\sigma(H_0) = \sigma(H)$;

(ii) $\sigma_p(H) \cup \sigma_r(H) \subseteq \sigma_p(H_0)$ *and* $\sigma_c(H_0) \subseteq \sigma_c(H)$.

In particular, if $\sigma(H_0) = \sigma_c(H_0)$, *then* $\sigma(H) = \sigma_c(H) = \sigma_c(H_0)$.

4.2 Case studies

We now specialize the situation from the abstract to more concrete settings.
Suppose that $\mathfrak{H} = \mathfrak{H}' = L^2(\mathbb{R}^d; \mathbb{C}^{N\times N})$, $N \in \mathbb{N}$, and V is the multiplication operator generated in $\mathfrak{H}$ by a matrix-valued (scalar-valued if $N = 1$) function $V : \mathbb{R}^d \to \mathbb{C}^{N\times N}$, with initial domain $\mathrm{dom}(V) = C_0^\infty(\mathbb{R}^d; \mathbb{C}^N)$. As customary, we consider the factorization of V given by the polar decomposition $V = UW$, where $W = \sqrt{V^*V}$ and the unitary matrix U is a partial isometry. Therefore we may set $A = \sqrt{W}$, $B = \sqrt{W}U^*$ and consider the corresponding multiplication operators generated by A and B^* in $\mathfrak{H}$ with initial domain $C_0^\infty(\mathbb{R}^d; \mathbb{C}^N)$, denoted by the same symbols. In the end, we can factorize the potential V in two closed operators A and B^*. Via the Closed Graph Theorem, Assumption I is verified.
Furthermore, in general the operator K_z defined in (4.5) is a bounded extension of the classical Birman-Schwinger operator $A(H_0 - z)^{-1}B^*$ defined on $\mathrm{dom}(B^*)$. Since in our case the initial domain of B^* is $C_0^\infty(\mathbb{R}^d; \mathbb{C}^N)$, hence dense in $\mathfrak{H}$, we get that K_z is exactly the closure of $A(H_0 - z)^{-1}B^*$.
In conclusion, everything reduces to the study of $\|A(H_0-z)^{-1}B^*\|_{\mathfrak{H}\to\mathfrak{H}}$: if there exists $z_0 \in \rho(H_0)$ such that this norm is strictly less than 1, then Theorem 4.3 holds; if this is true uniformly respect to $z \in \rho(H_0)$, then also Theorem 4.4 holds true.

Example 4.1 (Non-self-adjoint Schrödinger operator)**.** Consider the operator $H = -\Delta + V$ with $V : \mathbb{R}^d \to \mathbb{C}$. For V we use the decomposition

$$V(x) = \operatorname{sgn} V(x)|V(x)| = \operatorname{sgn} V(x)|V(x)|^{1/2}|V(x)|^{1/2}$$

for almost every $x \in \mathbb{R}^d$. Here $\operatorname{sgn} z := z/|z|$ if $z \neq 0$ and $\operatorname{sgn} z := 0$ if $z = 0$. Thus we choose

$$A(x) := |V(x)|^{1/2} \quad \text{and} \quad B(x) := \operatorname{sgn} \overline{V}(x)|V(x)|^{1/2}. \tag{4.6}$$

One has the following result on the bound of the norm of the Birman-Schwinger operator.

Lemma 4.3. *Let $H := -\Delta + V$, with $V : \mathbb{R}^d \to \mathbb{C}$. For V we use the decomposition $V = B^*A$ with A and B as in (4.6). Assume that $V \in L^{\gamma+d/2}(\mathbb{R}^d)$. Then there exists a positive constant $D_{\gamma,d}$ such that*

$$\|A(-\Delta - z)^{-1}B^*\|_{L^2\to L^2}^{\gamma+d/2} \leq D_{\gamma,d}|z|^{-\gamma}\|V\|_{L^{\gamma+d/2}}^{\gamma+d/2}, \qquad \textit{where} \begin{cases} \gamma = 1/2, & \textit{if } d = 1, \\ 0 < \gamma \leq 1/2, & \textit{if } d = 2, \\ 0 \leq \gamma \leq 1/2 & \textit{if } d \geq 3. \end{cases} \tag{4.7}$$

From the bound (4.7) and using the Birman-Schwinger principle (as stated in Theorem 4.3 or Theorem 4.4) one immediately has the following result on the localisation of the point spectrum of H.

Theorem 4.5. *Let $H = -\Delta + V$, with $V : \mathbb{R}^d \to \mathbb{C}$. Assume that $V \in L^{\gamma+d/2}(\mathbb{R}^d)$. Then, there exists a positive constant $D_{\gamma,d}$ such that*

$$|z|^\gamma \leq D_{\gamma,d}\|V\|_{L^{\gamma+d/2}}^{\gamma+d/2} \qquad \textit{where} \begin{cases} \gamma = 1/2, & \textit{if } d = 1, \\ 0 < \gamma \leq 1/2, & \textit{if } d \geq 2. \end{cases} \tag{4.8}$$

Moreover, when $d \geq 3$ there exists a positive constant $D_{0,d}$ such that if

$$\|V\|_{L^{d/2}} < D_{0,d}$$

then $\sigma_p(H) = \varnothing$.

The first contribution to the proof of Theorem 4.5 goes back to the work by Abramov, Aslanyan and Davies [2] covering the case $d = 1$, $\gamma = 1/2$, where they found the optimal constant $D_{1/2,1} = 1/2$. The higher dimensional framework was first addressed by Frank in his seminal paper [38]. Later, together with Simon [39], they extended Theorem 4.5 to cover long-range potentials, namely considering $1/2 < \gamma < d/2$, but under a radial symmetry assumption. Very recently Bögli and Cuenin [7] (see also [6]) proved that the result in [38] was optimal, thus ultimately disproving the celebrated conjecture by Laptev and Safronov [51] claiming that the bound (4.8) had to be true in $d \geq 2$ also for $1/2 < \gamma < d/2$.
In order to extend to higher dimensions the result in [2], based on a pointwise bound for the Green function of $-\frac{d^2}{dx^2} - z$, and clearly highly sensitive of the one dimensional framework, Frank in [38] introduces a new method which has permeated all the subsequent works on the topic and which relies on the combination of the Birman-Schwinger principle together with suitable resolvent estimates. More specifically, he used the Kenig-Ruiz-Sogge estimates on the conjugate line for the resolvent of the Laplacian from [45], namely

$$\|(-\Delta - z)^{-1}\|_{L^p \to L^{p'}} \leq C|z|^{-d/2+d/p-1}, \quad \frac{2}{d+1} \leq \frac{1}{p} - \frac{1}{p'} \leq \frac{2}{d}, \tag{4.9}$$

where $1/p + 1/p' = 1$ and $2/(d+1) \leq 1/p - 1/p' \leq 2/d$ if $d \geq 3$ and $2/3 \leq 1/p - 1/p' < 1$ if $d = 2$. With estimate (4.9) at hands, the proof of Lemma 4.3 is immediate.

Proof of Lemma 4.3. Let $d \geq 3$. From Hölder inequality and using (4.9) we have

$$\begin{aligned}\|A(H_0 - z)^{-1}B^*\|_{L^2 \to L^2} &\leq \|A\|_{L^{2p/(2-p)}}\|(H_0 - z)^{-1}\|_{L^p \to L^{p'}}\|B^*\|_{L^{2p/(2-p)}}\\ &\leq C|z|^{-d/2+d/p-1}\|V\|_{L^{p/(2-p)}};\end{aligned}$$

setting $p/(2-p) = \gamma + d/2$ and observing that

$$\frac{2}{d+1} \leq \frac{1}{p} - \frac{1}{p'} \leq \frac{2}{d} \Longleftrightarrow \frac{2d}{d+2} \leq p \leq \frac{2(d+1)}{d+3} \Longleftrightarrow 0 \leq \gamma \leq \frac{1}{2},$$

then bound (4.7) follows. The lower dimensional setting $d = 2$ can be shown similarly. □

Example 4.2 (Self-adjoint Schrödinger operator)**.** Now we consider the easier case of self-adjoint Schrödinger operators. Namely we consider $H = -\Delta + V$ with V real-valued. In this setting a much stronger result than Theorem 4.5 holds: this is represented by the so-called Lieb-Thirring inequalities, a fundamental tool developed to prove stability of matter (see, for instance, [54]).

Theorem 4.6 (Lieb-Thirring inequalities)**.** *Let $H = -\Delta + V$ be the Schrödinger operator where $V \colon \mathbb{R}^d \to \mathbb{R}$ is a real-valued potential. Assume that the negative part of the potential*

$V_- := -\min\{0, V\}$ lies in $L^{\gamma+d/2}(\mathbb{R}^d)$ for some non-negative γ to be specified below. Let $E_0 \leq E_1 \leq \cdots \leq 0$ the non-positive eigenvalues of H. Then, there exists a positive constant $L_{\gamma,d}$ such that

$$\sum_{j\geq 0} |E_j|^\gamma \leq L_{\gamma,d} \|V_-\|^{\gamma+d/2}_{L^{\gamma+d/2}}, \qquad \textit{where } \begin{cases} \gamma \geq 1/2 & \textit{if } d = 1, \\ \gamma > 0 & \textit{if } d = 2, \\ \gamma \geq 0 & \textit{if } d \geq 3. \end{cases} \tag{4.10}$$

The above estimates were proved in the non-endpoint case, namely for $\gamma > 1/2$ if $d = 1$ and for $\gamma > 0$ if $d \geq 2$, by Lieb and Thirring in the 1975 work [55]. The extension $\gamma = 1/2$ in the 1-dimensional case was obtained by Weidl [63], whereas the case $\gamma = 0$ for $d \geq 3$ was independently reached by Cwikel [24], Lieb [52] and Rozenbljum [59], and it is known as the CLR bound, which estimates from above the number of eigenvalues.

Remark 4.2. Some comments on this celebrated result follow. First of all the range of γ for which the inequalities (4.10) hold is optimal. In particular, one sees that $\gamma = 0$ is not allowed in lower dimensions $d = 1, 2$. This is in agreement with the criticality of $H_0 = -\Delta$ in lower dimensions or, what is the same, with the fact that any arbitrarily small non-trivial negative potential V always creates eigenvalues for the operator $H = -\Delta + V$. In light of this comment, inequalities (4.10) cannot hold for $\gamma = 0$ and $d = 1, 2$ as this would provide instead a smallness condition on the potential for excluding eigenvalues.
The main open question about the Lieb-Thirring inequalities regards the optimal constant $L_{\gamma,d}$. In particular it is still unknown its value in one of the most interesting physical case, namely when $d = 3$ and $\gamma = 1$. As a matter of fact, inequalities (4.10) for $\gamma = 1$ gives a bound for the sum of negative eigenvalues which was crucial for the proof of the stability of matter.
To conclude we just mention that, as in the non-self-adjoint setting, properties and structure of the Birman-Schwinger operator play a crucial role also in the proof of these inequalities. In particular, here the self-adjoint setting permits to employ the following stronger version of the principle: under suitable integrability conditions on the potential, the number N_z of negative eigenvalues of $-\Delta + V_-$ smaller than $-|z|$ equals the number B_z of eigenvalues greater than 1 of the Birman-Schwinger operator $K_z = \sqrt{V_-}(-\Delta + |z|)^{-1}\sqrt{V_-}$. In passing, note that in the self-adjoint framework only the negative part V_- of the potential counts indeed, via the variational min-max principle, a bound for the sum of eigenvalues powers of the full Hamiltonian $H = -\Delta + V$ is obtained from the analogous bound for the Hamiltonian $H = -\Delta + V_-$.

From the validity of Theorem 4.6 on has immediately the following bound for the single eigenvalue.

Theorem 4.7. *Let $H = -\Delta + V$ be the Schrödinger operator $V : \mathbb{R}^d \to \mathbb{R}$ a real-valued potential. Assume that the negative part of the potential $V_- := -\min\{0, V\}$ lies in $L^{\gamma+d/2}(\mathbb{R}^d)$ for some non-negative γ to be specified below. Then, there exists a positive constant $L_{\gamma,d}$ such that any non-positive eigenvalue E satisfies*

$$|E|^\gamma \leq L_{\gamma,d} \|V_-\|^{\gamma+d/2}_{L^{\gamma+d/2}}, \qquad \textit{where } \begin{cases} \gamma \geq 1/2 & \textit{if } d = 1, \\ \gamma > 0 & \textit{if } d \geq 2. \end{cases} \tag{4.11}$$

Moreover, when $d \geq 3$ there exists a positive constant $L_{0,d}$ such that if

$$\|V_-\|_{L^{d/2}} < L_{0,d}$$

there are no non-positive eigenvalue.

Estimates (4.11) for the single eigenvalue are known as Keller-type estimates as the first appearance of such bounds goes back to Keller [44] (see also [55] and [10] for more recent results). Even though inequalities (4.11) are immediate consequence of the Lieb-Thirring inequalities (4.10), one could give an alternative proof of (4.11) which is simply based on Sobolev inequalities. For the sake of completeness, we give a sketch of this proof below.

Proof of Theorem 4.7. We will see the proof only for $d \geq 3$, the lower dimensional case can be treated similarly. Since we are in the self-adjoint framework, the variational characterisation of the spectrum states that for any $u \in H^1(\mathbb{R}^d)$

$$\inf \sigma(-\Delta + V_-) = \inf_{\substack{\|u\|_{L^2}=1 \\ u \in H^1(\mathbb{R}^d)}} (u, (-\Delta - V_-)u).$$

Therefore, in order to get (4.11), it is sufficient to prove the following lower bound

$$(u, (-\Delta - V_-)u) \geq -L_{\gamma,d}^{1/\gamma} \|V_-\|_{\gamma+d/2}^{1+d/2\gamma} \tag{4.12}$$

for any $u \in H^1(\mathbb{R}^d)$ with $\|u\|_2 = 1$. Using Hölder and the interpolation inequality we have

$$\begin{aligned}
(u, V_- u) &\leq \|V_-\|_{\gamma+d/2} \|u\|_{2(2\gamma+d)/(2\gamma+d-2)}^2 \\
&\leq \|V_-\|_{\gamma+d/2} \|u\|_2^{4\gamma/(2\gamma+d)} \|u\|_{2d/(d-2)}^{2d/(2\gamma+d)} \\
&\leq \varepsilon^{-1-d/2\gamma} \tfrac{2\gamma}{2\gamma+d} \|V_-\|_{\gamma+d/2}^{1+d/2\gamma} + \varepsilon^{1+2\gamma/d} \tfrac{d}{2\gamma+d} \|u\|_{2d/(d-2)}^2
\end{aligned}$$

for some ε to be chosen later. In the last bound we have used the Young inequality $ab \leq a^p/p + b^q/q$ valid for all positive a, b and for $1/p + 1/q = 1$.
As for the kinetic term, using the Sobolev inequality one has

$$(u, -\Delta u) \geq S_d \|u\|_{2d/(d-2)}^2,$$

where S_d denotes the optimal Sobolev constant (see [53]). Plugging the two estimates together one has

$$(u, (-\Delta - V_-)u) \geq -\varepsilon^{-1-d/2\gamma} \tfrac{2\gamma}{2\gamma+d} \|V_-\|_{\gamma+d/2}^{1+d/2\gamma} + \big(S_d - \varepsilon^{1+2\gamma/d} \tfrac{d}{2\gamma+d}\big) \|u\|_{2d/(d-2)}^2,$$

which gives bound (4.12) after choosing a suitable small ε. □

Example 4.3 (Non-self-adjoint Dirac operator)**.** Let us consider now the non-self-adjoint Dirac operator $\mathcal{D}_{m,V} = \mathcal{D}_m + V$, where $\mathcal{D}_m$ is the free Dirac operator defined in (3.6) and V is a matrix-valued potential $V : \mathbb{R}^d \to \mathbb{C}^{N \times N}$, with $N = 2^{\lceil d/2 \rceil}$. The spectral studies for $\mathcal{D}_{m,V}$ were started by Cuenin, Laptev and Tretter in their celebrated work [22], for the 1-dimensional case. There they proved that if $V \in \mathbb{C}^{2\times 2}$ is a potential such that

$$\|V\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |V(x)|dx < 1,$$

where $|V(\cdot)|$ is the operator norm of $V(\cdot)$ in $\mathbb{C}^2$ with the Euclidean norm, then every non-embedded eigenvalue $z \in \mathbb{C} \setminus \{(-\infty, -m] \cup [m, +\infty)\}$ of $\mathcal{D}_{m,V}$ lies in the union

$$z \in \overline{B}_{R_0}(x_0^-) \cup \overline{B}_{R_0}(x_0^+) \tag{4.13}$$

of two disjoint closed disks in the complex plane, with centers and radius respectively

$$x_0^{\pm} = \pm m\sqrt{\frac{\|V\|_1^4 - 2\|V\|_1^2 + 2}{4(1-\|V\|_1^2)} + \frac{1}{2}}, \quad R_0 = m\sqrt{\frac{\|V\|_1^4 - 2\|V\|_1^2 + 2}{4(1-\|V\|_1^2)} - \frac{1}{2}}.$$

In particular, in the massless case the spectrum is $\sigma(\mathcal{D}_{0,V}) = \mathbb{R}$. Moreover, this inclusion is shown to be optimal. Again, the proof relies on the combination of the Birman-Schwinger principle with a resolvent estimate for the free Dirac operator, namely

$$\|(\mathcal{D}_m - z)^{-1}\|_{L^1(\mathbb{R})\to L^\infty(\mathbb{R})} \leq \sqrt{\frac{1}{2} + \frac{1}{4}\left|\frac{z+m}{z-m}\right| + \frac{1}{4}\left|\frac{z-m}{z+m}\right|}.$$

In some sense, this is the counterpart for the Dirac operator of the above-cited Abramov-Aslanyan-Davies inequality for the Schrödinger operator in 1-dimension.
One could ask if, in the same fashion of Frank's argument in [38], one can combine the Birman-Schwinger principle with $L^p - L^{p'}$ resolvent estimates for the free Dirac operator, to derive Keller-type inequalities for the perturbed Dirac operator. Unfortunately, these reasoning can not be straightforwardly applied, since such Kenig-Ruiz-Sogge-type estimates does not exists in the case of Dirac for dimension $d \geq 2$, as observed by Cuenin in [19]. Indeed, due to the Stein-Thomas restriction theorem and standard estimates for Bessel potentials, the resolvent $(\mathcal{D}_m - z)^{-1} \colon L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d)$ is bounded uniformly for $|z| > 1$ if and only if

$$\frac{2}{d+1} \leq \frac{1}{p} + \frac{1}{p'} \leq \frac{1}{d},$$

hence the only possible choice is $(d, p, p') = (1, 1, \infty)$. For the Schrödinger operator the situation is much better since the right-hand side of the above range is replaced by $2/d$, as per the Kenig-Ruiz-Sogge estimates.
For the higher dimensional case $d \geq 2$, we may refer among others to the works [20, 23, 28, 35] where the eigenvalues are localized in terms of L^p-norm of the potential, but, differently to the 1-dimensional case, the confinement region is unbounded around the spectrum of the free operator, namely $\sigma(\mathcal{D}_m) = (-\infty, -m] \cup [m, +\infty)$.
Recently, two of the authors of this survey in [30], together with D'Ancona, were able to get in higher dimension a bounded enclosure region asking smallness of the potential in a suitable mixed Lebesgue space. More precisely, introducing the functional space

$$Y \equiv Y(\mathbb{R}^d) := \bigcap_{j=1}^{d} L^1_{x_j} L^\infty_{\hat{x}_j}(\mathbb{R}^d), \qquad \hat{x}_j := (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d) \in \mathbb{R}^{d-1},$$

they showed that for suitably Y-small V, namely $\|V\|_Y < C_0$ for suitable positive constant C_0, then any eigenvalue $z \in \sigma_p(\mathcal{D}_{m,V})$ of $\mathcal{D}_{m,V}$ lies again in the union of two closed disks in $\mathbb{C}$ (*cf.* (4.13)) with centers and radius given by

$$x_0^\pm := \pm m\frac{\nu^2+1}{\nu^2-1}, \qquad R_0 := m\frac{2\nu}{\nu^2-1}, \qquad \nu \equiv \nu(V) := \left[\frac{(d+1)C_0}{\|V\|_Y} - d\right]^2 > 1.$$

In the massless case they proved spectral stability under the same smallness condition about V, namely $\sigma(\mathcal{D}_{0,V}) = \mathbb{R}$. As above, the proof comes again as a consequence of the machinery combining the Birman-Schwinger principle with uniform resolvent estimates. In particular, the following Agmon-Hörmander-type estimates were crucial for proving the result:

$$\|(\mathcal{D}_m - z)^{-1}\|_{X\to X^*} \leq \left[d + \left|\frac{z+m}{z-m}\right|^{\operatorname{sgn}(\Re z)/2}\right] \tag{4.14}$$

where the functional space X and its dual X^* are

$$X \equiv X(\mathbb{R}^d) := \bigcap_{j=1}^{d} L^1_{x_j} L^2_{\hat{x}_j}(\mathbb{R}^d), \qquad X^* \equiv X(\mathbb{R}^d) := \sum_{j=1}^{d} L^\infty_{x_j} L^2_{\hat{x}_j}(\mathbb{R}^d).$$

For previous resolvent estimates in the spirit of (4.14) we refer to [26, 27, 33]. In the direction of spectral stability result we should also mention the recent work [29] where the authors was able to prove spectral stability of perturbed Dirac and Klein-Gordon operators under suitable pointwise smallness/decay at infinity conditions for the potential. Last but not least in [56] the matricial structure of the Dirac operator was fully exploited to obtain Keller-type estimates dropping the smallness assumption on the potential. More precisely, the authors required the potentials to be of the form $V = vW$, where $v\colon \mathbb{R}^d \to \mathbb{C}$ is a scalar function with the desired integrability, whereas W is a constant matrix satisfying suitable rigidity conditions. Other remarkable results can be found in [14, 20, 21, 60]

This technique presented above is very robust and has produced interesting results also for other models than Schrödinger and Dirac operators. Among second order operators, we mention the Lamé operator of elasticity and we refer to [11, 12, 16] for the corresponding results in the spirit of the ones available for Schrodinger. For higher order operators, we refer to [41] for the application of the Birman-Schwinger principle to the bilaplacian in lower dimensions, see also [32]. We also mention that the same technique has been applied successfully to a discrete setting. In this respect we refer to [13, 42] and the recent preprint [49].

Acknowledgments

The research of the first author (L.C.) is Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173. The third author (N.M.S.) is member of the Gruppo Nazionale per L'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

References

[1] *Hilbert space methods in quantum mechanics.* EPFL Press, 2009.

[2] A. A. Abramov, A. Aslanyan, and E. B. Davies. Bounds on complex eigenvalues and resonances. *J. Phys. A*, 34(1):57–72, 2001.

[3] S. Albeverio. On bound states in the continuum of N-body systems and the virial theorem. *Ann. Physics*, 71:167–276, 1972.

[4] S. Avramska-Lukarska, D. Hundertmark, and H. Kovarik. Absence of positive eigenvalues of magnetic schrödinger operators. *arXiv preprint arXiv:2003.07294*, 2020.

[5] M. v. Birman. On the spectrum of singular boundary-value problems. *Mat. Sb. (N.S.)*, 55 (97):125–174, 1961.

[6] S. Bögli. Schrödinger operator with non-zero accumulation points of complex eigenvalues. *Comm. Math. Phys.*, 352(2):629–639, 2017.

[7] S. Bögli and J.-C. Cuenin. Counterexample to the Laptev–Safronov conjecture. *arXiv preprint arXiv:2109.06135*, 2021.

[8] N. Boussaid, P. D'Ancona, and L. Fanelli. Virial identity and weak dispersion for the magnetic Dirac equation. *J. Math. Pures Appl. (9)*, 95(2):137–150, 2011.

[9] F. Cacciafesta. Virial identity and dispersive estimates for the n-dimensional Dirac equation. *J. Math. Sci. Univ. Tokyo*, 18(4):441–463 (2012), 2011.

[10] E. A. Carlen, R. L. Frank, and E. H. Lieb. Stability estimates for the lowest eigenvalue of a Schrödinger operator. *Geom. Funct. Anal.*, 24(1):63–84, 2014.

[11] B. Cassano, L. Cossetti, and L. Fanelli. Eigenvalue bounds and spectral stability of lamé operators with complex potentials. *arXiv preprint arXiv:2101.09350*, 2021.

[12] B. Cassano, L. Cossetti, and L. Fanelli. Spectral enclosures for the damped elastic wave equation. *arXiv preprint arXiv:2108.07676*, 2021.

[13] B. Cassano, O. O. Ibrogimov, D. Krejčiřík, and F. Štampach. Location of eigenvalues of non-self-adjoint discrete Dirac operators. *Ann. Henri Poincaré*, 21(7):2193–2217, 2020.

[14] B. Cassano, F. Pizzichillo, and L. Vega. A Hardy-type inequality and some spectral characterizations for the Dirac-Coulomb operator. *Rev. Mat. Complut.*, 33(1):1–18, 2020.

[15] L. Cossetti. Uniform resolvent estimates and absence of eigenvalues for Lamé operators with complex potentials. *J. Math. Anal. Appl.*, 455(1):336–360, 2017.

[16] L. Cossetti. Bounds on eigenvalues of perturbed Lamé operators with complex potentials. *Math. Eng.*, 4(5):Paper No. 037, 29, 2022.

[17] L. Cossetti, L. Fanelli, and D. Krejčiřík. Absence of eigenvalues of Dirac and Pauli Hamiltonians via the method of multipliers. *Comm. Math. Phys.*, 379(2):633–691, 2020.

[18] L. Cossetti and D. Krejčiřík. Absence of eigenvalues of non-self-adjoint Robin Laplacians on the half-space. *Proc. Lond. Math. Soc.*, 121(3):584–616, 2020.

[19] J.-C. Cuenin. Estimates on complex eigenvalues for Dirac operators on the half-line. *Integral Equations Operator Theory*, 79(3):377–388, 2014.

[20] J.-C. Cuenin. Eigenvalue bounds for Dirac and fractional Schrödinger operators with complex potentials. *J. Funct. Anal.*, 272(7):2987–3018, 2017.

[21] J.-C. Cuenin. Eigenvalue estimates for bilayer graphene. *Ann. Henri Poincaré*, 20(5):1501–1516, 2019.

[22] J.-C. Cuenin, A. Laptev, and C. Tretter. Eigenvalue estimates for non-selfadjoint Dirac operators on the real line. *Ann. Henri Poincaré*, 15(4):707–736, 2014.

[23] J.-C. Cuenin and C. Tretter. Non-symmetric perturbations of self-adjoint operators. *J. Math. Anal. Appl.*, 441(1):235–258, 2016.

[24] M. Cwikel. Weak type estimates for singular values and the number of bound states of Schrödinger operators. *Ann. of Math. (2)*, 106(1):93–100, 1977.

[25] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon. *Schrödinger operators with application to quantum mechanics and global geometry.* Texts and Monographs in Physics. Springer-Verlag, Berlin, study edition, 1987.

[26] P. D'Ancona and L. Fanelli. Decay estimates for the wave and Dirac equations with a magnetic potential. *Comm. Pure Appl. Math.*, 60(3):357–392, 2007.

[27] P. D'Ancona and L. Fanelli. Strichartz and smoothing estimates of dispersive equations with magnetic potentials. *Comm. Partial Differential Equations*, 33(6):1082–1112, 2008.

[28] C. Dubuisson. On quantitative bounds on eigenvalues of a complex perturbation of a Dirac operator. *Integral Equations Operator Theory*, 78(2):249–269, 2014.

[29] P. D'Ancona, L. Fanelli, D. Krejčiřík, and N. M. Schiavone. Localization of eigenvalues for non-self-adjoint Dirac and Klein-Gordon operators. *Nonlinear Anal.*, 214:112565, 2022.

[30] P. D'Ancona, L. Fanelli, and N. M. Schiavone. Eigenvalue bounds for non-selfadjoint Dirac operators. *Math. Ann.*, pages 1–24, 2021.

[31] D. E. Edmunds and W. D. Evans. *Spectral theory and differential operators.* Oxford Mathematical Monographs. Oxford University Press, Oxford, 2018.

[32] A. Enblom. Estimates for eigenvalues of Schrödinger operators with complex-valued potentials. *Lett. Math. Phys.*, 106(2):197–220, 2016.

[33] M. B. Erdoğan, M. Goldberg, and W. R. Green. Limiting absorption principle and Strichartz estimates for Dirac operators in two and higher dimensions. *Comm. Math. Phys.*, 367(1):241–263, 2019.

[34] D. M. Èĭdus. On the principle of limiting absorption. *Mat. Sb. (N.S.)*, 57 (99):13–44, 1962.

[35] L. Fanelli and D. Krejčiřík. Location of eigenvalues of three-dimensional non-self-adjoint Dirac operators. *Lett. Math. Phys.*, 109(7):1473–1485, 2019.

[36] L. Fanelli, D. Krejčiřík, and L. Vega. Absence of eigenvalues of two-dimensional magnetic Schrödinger operators. *J. Funct. Anal.*, 275(9):2453–2472, 2018.

[37] L. Fanelli, D. Krejčiřík, and L. Vega. Spectral stability of Schrödinger operators with subordinated complex potentials. *J. Spectr. Theory*, 8(2):575–604, 2018.

[38] R. L. Frank. Eigenvalue bounds for Schrödinger operators with complex potentials. *Bull. Lond. Math. Soc.*, 43(4):745–750, 2011.

[39] R. L. Frank and B. Simon. Eigenvalue bounds for Schrödinger operators with complex potentials. II. *J. Spectr. Theory*, 7(3):633–658, 2017.

[40] M. Hansmann and D. Krejčiřík. The abstract Birman-Schwinger principle and spectral stability. *arXiv preprint arXiv:2010.15102*, 2020.

[41] O. O. Ibrogimov, D. Krejčiřík, and A. Laptev. Sharp bounds for eigenvalues of biharmonic operators with complex potentials in low dimensions. *Math. Nachr.*, 294(7):1333–1349, 2021.

[42] O. O. Ibrogimov and F. Štampach. Spectral enclosures for non-self-adjoint discrete Schrödinger operators. *Integral Equations and Operator Theory*, 91(6):1–15, 2019.

[43] T. Kato. Wave operators and similarity for some non-selfadjoint operators. *Math. Ann.*, 162:258–279, 1965/66.

[44] J. B. Keller. Lower bounds and isoperimetric inequalities for eigenvalues of the Schrödinger equation. *J. Mathematical Phys.*, 2:262–266, 1961.

[45] C. E. Kenig, A. Ruiz, and C. D. Sogge. Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators. *Duke Math. J.*, 55(2):329–347, 1987.

[46] R. Konno and S. T. Kuroda. On the finiteness of perturbed eigenvalues. *J. Fac. Sci. Univ. Tokyo Sect. I*, 13:55–63, 1966.

[47] D. Krejčiřík. Geometrical aspects of spectral theory. Available online on http://nsa.fjfi.cvut.cz/david/.

[48] D. Krejčiřík. Mathematical aspects of quantum mechanics with non-self-adjoint operators. Available online on http://nsa.fjfi.cvut.cz/david/.

[49] D. Krejčiřík, A. Laptev, and F. Stampach. Spectral enclosures and stability for non-self-adjoint discrete Schrödinger operators on the half-line. *arXiv preprint arXiv:2111.08265*, 2021.

[50] D. Krejčiřík and P. Siegl. Elements of spectral theory without the spectral theorem. *Non-selfadjoint operators in quantum physics: Mathematical aspects (432 pages), F. Bagarello, J.-P. Gazeau, F. H. Szafraniec, and M. Znojil, Eds., Wiley-Interscience, 2015*, 2015.

[51] A. Laptev and O. Safronov. Eigenvalue estimates for Schrödinger operators with complex potentials. *Comm. Math. Phys.*, 292(1):29–54, 2009.

[52] E. H. Lieb. The number of bound states of one-body Schroedinger operators and the Weyl problem. In *Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979)*, Proc. Sympos. Pure Math., XXXVI, pages 241–252. Amer. Math. Soc., Providence, R.I., 1980.

[53] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001.

[54] E. H. Lieb and R. Seiringer. *The stability of matter in quantum mechanics*. Cambridge University Press, Cambridge, 2010.

[55] E. H. Lieb and W. E. Thirring. Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities. In *The Stability of Matter: From Atoms to Stars*, pages 203–237. Springer, 1997.

[56] H. Mizutani and N. M. Schiavone. Keller-type bounds for Dirac operators perturbed by rigid potentials. *arXiv preprint arXiv:2108.12854*, 2021.

[57] E. Mourre. Absence of singular continuous spectrum for certain selfadjoint operators. *Comm. Math. Phys.*, 78(3):391–408, 1980/81.

[58] M. Reed and B. Simon. *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[59] G. V. Rozenbljum. Distribution of the discrete spectrum of singular differential operators. *Izv. Vysš. Učebn. Zaved. Matematika*, (1(164)):75–86, 1976.

[60] O. Safronov, A. Laptev, and F. Ferrulli. Eigenvalues of the bilayer graphene operator with a complex valued potential. *Anal. Math. Phys.*, 9(3):1535–1546, 2019.

[61] J. Schwinger. On the bound states of a given potential. *Proc. Nat. Acad. Sci. U.S.A.*, 47:122–129, 1961.

[62] B. Thaller. *The Dirac equation*. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1992.

[63] T. Weidl. On the Lieb-Thirring constants $L_{\gamma,1}$ for $\gamma \geq 1/2$. *Comm. Math. Phys.*, 178(1):135–146, 1996.