SUPPORTING CALCULATION FOR THE PAPER: *ANALYSIS OF A PEACEMAN-RACHFORD ADI SCHEME FOR MAXWELL EQUATIONS IN HETEROGENEOUS MEDIA*

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Abstract. The first eigenvalue of a one-dimensional transmission problem is calculated. The eigenvalue problem originates from the analysis of electromagnetic fields near an interior edge in a heterogeneous medium.

This note is inspired by [6]. Our goal is a formula for the first eigenvalue of the system

$$\begin{aligned} (\psi^{(i)})''(\varphi) &= -\kappa^2\psi^{(i)}(\varphi) \qquad \text{for } \varphi\in I_i,\ i\in\{1,\dots,4\},\\ \varepsilon^{(1)}\psi^{(1)}(0) &= \varepsilon^{(4)}\psi^{(4)}(2\pi), \quad (\psi^{(1)})'(0)=(\psi^{(4)})'(2\pi),\\ \psi^{(1)}(\tfrac{\pi}{2}) &= \psi^{(2)}(\tfrac{\pi}{2}), \quad (\psi^{(1)})'(\tfrac{\pi}{2}) = (\psi^{(2)})'(\tfrac{\pi}{2}),\\ \psi^{(2)}(\pi) &= \psi^{(3)}(\pi), \quad (\psi^{(2)})'(\pi)=(\psi^{(3)})'(\pi),\\ \psi^{(3)}(\tfrac{3}{2}\pi) &= \psi^{(4)}(\tfrac{3}{2}\pi), \quad \varepsilon^{(1)}(\psi^{(3)})'(\tfrac{3}{2}\pi) = \varepsilon^{(4)}(\psi^{(4)})'(\tfrac{3}{2}\pi), \end{aligned} \tag{1}$$

where

$$I_1 := (0,\tfrac{\pi}{2}), \quad I_2 := (\tfrac{\pi}{2},\pi), \quad I_3 := (\pi,\tfrac{3}{2}\pi), \quad I_4 := (\tfrac{3}{2}\pi, 2\pi),$$

and

$$\varepsilon^{(1)} \neq \varepsilon^{(4)} \tag{2}$$

are two positive numbers. The notation $f^{(i)}$ further refers to the restriction $f|_{I_i}$ of a function $f\in L^2(0,2\pi)$. Inspired by [2, 3, 1, 4, 5], we use (1) in [7] during the study of a two-dimensional Laplacian on the disk with transmission conditions. Note that the transmission conditions are different from the ones in [1, 2, 3, 6, 5]. The following representation formula for the first eigenvalue of (1) is essential to decompose the domain of the Laplacian into a regular and a singular space, see Section 3.3 in [7]. Eventually, this uses to analyze certain components of the electric field near interior edges of a heterogeneous material, see Lemma 5.10 therein.

The main result of this note is formula (3) for the first eigenvalue of (1). We establish the relation within the following three lemmas. The statement and the proofs are to some extend in analogy to Lemmas 1–2 in [6].

Lemma 1. *Let $\varepsilon^{(4)} > \varepsilon^{(1)}$. The first eigenvalue κ_1^2 of (1) satisfies the formula*

$$\frac{(\varepsilon^{(4)}-\varepsilon^{(1)})^2}{\varepsilon^{(4)}\varepsilon^{(1)}} = \frac{4\sin^2(\kappa_1\pi)}{\cos(\frac{3}{2}\kappa_1\pi)\cos(\frac{1}{2}\kappa_1\pi)}. \tag{3}$$

The associated eigenspace is one-dimensional, and the next eigenvalue is greater or equal than 1.

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Proof. 1) Throughout, we make multiple times use of basic trigonometric formulas. In the first three parts of this proof, we take an eigenfunction $\psi \neq 0$ of (1) with eigenvalue $\lambda := \kappa_1^2 \in (0,1)$ for granted. This uses to derive the asserted estimates. In the fourth step, we prove the existence of this eigenvalue.

We first conclude the representation

$$\psi^{(i)}(\varphi) = a^{(i)} \cos(\sqrt{\lambda}\varphi) + b^{(i)} \sin(\sqrt{\lambda}\varphi), \qquad i \in \{1, \dots, 4\}, \tag{4}$$

with real numbers $a^{(i)}, b^{(i)}$. The transmission conditions then imply the relations $a^{(1)} = a^{(2)} = a^{(3)}$ and $b^{(1)} = b^{(2)} = b^{(3)}$. The fact $\lambda \neq 0$ and the second line of (1) lead to the expressions

$$\begin{aligned} a^{(1)} &= \frac{\varepsilon^{(4)}}{\varepsilon^{(1)}}(a^{(4)} \cos(\sqrt{\lambda}2\pi) + b^{(4)} \sin(\sqrt{\lambda}2\pi)), \\ b^{(1)} &= -a^{(4)} \sin(\sqrt{\lambda}2\pi) + b^{(4)} \cos(\sqrt{\lambda}2\pi). \end{aligned} \tag{5}$$

Combining the last line of (1) with (5) then results in the identities

$$\begin{aligned} a^{(4)} \cos(\sqrt{\lambda}\tfrac{3}{2}\pi) + b^{(4)} \sin(\sqrt{\lambda}\tfrac{3}{2}\pi) &= a^{(1)} \cos(\sqrt{\lambda}\tfrac{3}{2}\pi) + b^{(1)} \sin(\sqrt{\lambda}\tfrac{3}{2}\pi) \\ = \frac{\varepsilon^{(4)}}{\varepsilon^{(1)}}(a^{(4)} \cos(\sqrt{\lambda}2\pi) &+ b^{(4)} \sin(\sqrt{\lambda}2\pi)) \cos(\sqrt{\lambda}\tfrac{3}{2}\pi) \\ &+ (-a^{(4)} \sin(\sqrt{\lambda}2\pi) + b^{(4)} \cos(\sqrt{\lambda}2\pi)) \sin(\sqrt{\lambda}\tfrac{3}{2}\pi), \end{aligned}$$

being equivalent to the statement

$$\begin{aligned} &a^{(4)}\Big(\cos(\sqrt{\lambda}\tfrac{3}{2}\pi) - \tfrac{\varepsilon^{(4)}}{\varepsilon^{(1)}} \cos(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\tfrac{3}{2}\pi) + \sin(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\tfrac{3}{2}\pi)\Big) \\ &\quad = b^{(4)}\Big(- \sin(\sqrt{\lambda}\tfrac{3}{2}\pi) + \tfrac{\varepsilon^{(4)}}{\varepsilon^{(1)}} \sin(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\tfrac{3}{2}\pi) + \cos(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\tfrac{3}{2}\pi)\Big) \\ &\quad =: b^{(4)} A_1(\lambda). \end{aligned} \tag{6}$$

Relating the derivative condition in the last line of (1) to (5), we further infer the equation

$$\begin{aligned} &\varepsilon^{(4)}\sqrt{\lambda}(-a^{(4)} \sin(\sqrt{\lambda}\tfrac{3}{2}\pi) + b^{(4)} \cos(\sqrt{\lambda}\tfrac{3}{2}\pi)) \\ &\quad = \varepsilon^{(1)}\sqrt{\lambda}\Big(- \tfrac{\varepsilon^{(4)}}{\varepsilon^{(1)}} a^{(4)} \cos(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\tfrac{3}{2}\pi) - \tfrac{\varepsilon^{(4)}}{\varepsilon^{(1)}} b^{(4)} \sin(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\tfrac{3}{2}\pi) \\ &\qquad\qquad - a^{(4)} \sin(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\tfrac{3}{2}\pi) + b^{(4)} \cos(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\Big), \end{aligned}$$

implying with $\lambda > 0$ the identity

$$\begin{aligned} &a^{(4)}\Big(- \varepsilon^{(4)} \sin(\sqrt{\lambda}\tfrac{3}{2}\pi) + \varepsilon^{(4)} \cos(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\tfrac{3}{2}\pi) + \varepsilon^{(1)} \sin(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\Big) \\ &\quad = b^{(4)}\Big(- \varepsilon^{(4)} \cos(\sqrt{\lambda}\tfrac{3}{2}\pi) - \varepsilon^{(4)} \sin(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\tfrac{3}{2}\pi) \\ &\qquad\qquad + \varepsilon^{(1)} \cos(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\Big) \\ &\quad =: b^{(4)} A_2(\lambda). \end{aligned} \tag{7}$$

We now distinguish two cases for $a^{(4)}$, the first one leading to a contradiction.

2) Assume in the following that $a^{(4)} = 0$. In view of (5), $b^{(4)}$ is then different from zero, and $A_1(\lambda) = 0$. Introduce next the numbers

$$\delta := \frac{\varepsilon^{(4)}}{\varepsilon^{(1)}} - 1, \qquad \xi := \varepsilon^{(4)} - \varepsilon^{(1)}.$$

Employing (6) and trigonometric calculus, we arrive at the equations

$$\begin{aligned} 0 = A_1(\lambda) &= - \sin(\sqrt{\lambda}\tfrac{3}{2}\pi) + \delta \sin(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\tfrac{3}{2}\pi) + \sin(\sqrt{\lambda}\tfrac{7}{2}\pi) \\ &= \delta \sin(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\tfrac{3}{2}\pi) + 2 \cos(\sqrt{\lambda}\tfrac{5}{2}\pi) \sin(\sqrt{\lambda}\pi). \end{aligned} \tag{8}$$

Similar reasoning leads from (7), meaning $A_2(\lambda) = 0$, to the expressions

$$\begin{aligned} 0 &= -\frac{\varepsilon^{(4)}}{\varepsilon^{(1)}} \cos(\sqrt{\lambda}\tfrac{3}{2}\pi) - \delta \sin(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\tfrac{3}{2}\pi) + \cos(\sqrt{\lambda}\tfrac{7}{2}\pi) \\ &= -\delta \cos(\sqrt{\lambda}\tfrac{3}{2}\pi) - \delta \sin(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\tfrac{3}{2}\pi) - 2 \sin(\sqrt{\lambda}\tfrac{5}{2}\pi) \sin(\sqrt{\lambda}\pi). \end{aligned} \tag{9}$$

Multiplying the last identity by $\cos(\sqrt{\lambda}\frac{3}{2}\pi) \sin(\sqrt{\lambda}2\pi)$ and inserting the formula for δ from (8), we obtain the relations

$$\begin{aligned} 0 &= -\delta \cos^2(\sqrt{\lambda}\tfrac{3}{2}\pi) \sin(\sqrt{\lambda}2\pi) - \delta \sin^2(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\tfrac{3}{2}\pi) \cos(\sqrt{\lambda}\tfrac{3}{2}\pi) \\ &\quad - 2 \sin(\sqrt{\lambda}\tfrac{5}{2}\pi) \sin(\sqrt{\lambda}\pi) \sin(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\tfrac{3}{2}\pi) \\ &= 2 \cos(\sqrt{\lambda}\tfrac{5}{2}\pi) \sin(\sqrt{\lambda}\pi) \cos(\sqrt{\lambda}\tfrac{3}{2}\pi) \\ &\quad + 2 \cos(\sqrt{\lambda}\tfrac{5}{2}\pi) \sin(\sqrt{\lambda}\pi) \sin(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\tfrac{3}{2}\pi) \\ &\quad - 2 \sin(\sqrt{\lambda}\tfrac{5}{2}\pi) \sin(\sqrt{\lambda}\pi) \sin(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\tfrac{3}{2}\pi). \end{aligned} \tag{10}$$

By assumption $\sin(\sqrt{\lambda}\pi) \neq 0$, and we conclude the statement

$$\begin{aligned} 0 &= 2\Big(\cos(\sqrt{\lambda}\tfrac{5}{2}\pi) \cos(\sqrt{\lambda}\tfrac{3}{2}\pi) - \sin(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\pi)\Big) \\ &= 2\Big(\cos(\sqrt{\lambda}\tfrac{4+1}{2}\pi) \cos(\sqrt{\lambda}\tfrac{4-1}{2}\pi) - \sin(\sqrt{\lambda}\tfrac{3-1}{2}\pi) \sin(\sqrt{\lambda}\tfrac{3+1}{2}\pi)\Big) \\ &= \cos(\sqrt{\lambda}\pi) + \cos(\sqrt{\lambda}4\pi) - \cos(\sqrt{\lambda}\pi) + \cos(\sqrt{\lambda}3\pi) \\ &= 2 \cos(\sqrt{\lambda}\tfrac{7}{2}\pi) \cos(\sqrt{\lambda}\tfrac{1}{2}\pi). \end{aligned}$$

As a result, λ is an element of the set $\{\frac{1}{49}, \frac{9}{49}, \frac{25}{49}\}$. Inserting $\lambda = 1/49$ into the formula for δ from (8), we arrive at a contradiction to the assumption on $\varepsilon^{(4)}/\varepsilon^{(1)}$. The two other options for λ lead to $\delta < -1$, meaning $\varepsilon^{(4)}/\varepsilon^{(1)} < 0$. This is a contradiction to the positivity of $\varepsilon^{(1)}$ and $\varepsilon^{(4)}$. Altogether, $a^{(4)}$ has to be nonzero.

3) Let $a^{(4)} \neq 0$. After scaling, we assume $a^{(4)} = 1$. We distinguish two cases.

3.i) Assume $A_2(\lambda) \neq 0$. Equation (7), the definition $\xi = \varepsilon^{(4)} - \varepsilon^{(1)}$, and trigonometric manipulations then lead to the result

$$\begin{aligned} b^{(4)} &= \frac{-\varepsilon^{(4)} \sin(\sqrt{\lambda}\frac{3}{2}\pi) + \varepsilon^{(4)} \cos(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) + \varepsilon^{(1)} \sin(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi)}{-\varepsilon^{(4)} \cos(\sqrt{\lambda}\frac{3}{2}\pi) - \varepsilon^{(4)} \sin(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) + \varepsilon^{(1)} \cos(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi)} \\ &= \frac{-\varepsilon^{(4)} \sin(\sqrt{\lambda}\frac{3}{2}\pi) + \varepsilon^{(4)} \sin(\sqrt{\lambda}\frac{7}{2}\pi) - \xi \sin(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi)}{-\varepsilon^{(4)} \cos(\sqrt{\lambda}\frac{3}{2}\pi) + \varepsilon^{(4)} \cos(\sqrt{\lambda}\frac{7}{2}\pi) - \xi \cos(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi)} \\ &= -\frac{2\varepsilon^{(4)} \cos(\sqrt{\lambda}\frac{5}{2}\pi) \sin(\sqrt{\lambda}\pi) - \xi \sin(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi)}{2\varepsilon^{(4)} \sin(\sqrt{\lambda}\frac{5}{2}\pi) \sin(\sqrt{\lambda}\pi) + \xi \cos(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi)}. \end{aligned} \tag{11}$$

Inserting this representation for $b^{(4)}$ into (6), we deduce the equations

$$\begin{aligned} 0 &= \cos(\sqrt{\lambda}\tfrac{3}{2}\pi) - \frac{\varepsilon^{(4)}}{\varepsilon^{(1)}} \cos(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\tfrac{3}{2}\pi) + \sin(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\tfrac{3}{2}\pi) \\ &\quad + \Big(- \sin(\sqrt{\lambda}\tfrac{3}{2}\pi) + \tfrac{\varepsilon^{(4)}}{\varepsilon^{(1)}} \sin(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\tfrac{3}{2}\pi) + \cos(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\tfrac{3}{2}\pi)\Big) \\ &\quad \cdot \frac{2\varepsilon^{(4)} \cos(\sqrt{\lambda}\frac{5}{2}\pi) \sin(\sqrt{\lambda}\pi) - \xi \sin(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi)}{2\varepsilon^{(4)} \sin(\sqrt{\lambda}\frac{5}{2}\pi) \sin(\sqrt{\lambda}\pi) + \xi \cos(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi)} \\ &= 2 \sin(\sqrt{\lambda}\tfrac{5}{2}\pi) \sin(\sqrt{\lambda}\pi) - \delta \cos(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\tfrac{3}{2}\pi) \\ &\quad + \Big(2 \cos(\sqrt{\lambda}\tfrac{5}{2}\pi) \sin(\sqrt{\lambda}\pi) + \delta \sin(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\Big) \\ &\quad \cdot \frac{2\varepsilon^{(4)} \cos(\sqrt{\lambda}\frac{5}{2}\pi) \sin(\sqrt{\lambda}\pi) - \xi \sin(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi)}{2\varepsilon^{(4)} \sin(\sqrt{\lambda}\frac{5}{2}\pi) \sin(\sqrt{\lambda}\pi) + \xi \cos(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi)}. \end{aligned} \tag{12}$$

Algebraic and trigonometric manipulations then lead to the identities

$$\begin{aligned}0 &= 4\varepsilon^{(4)}\sin^2(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin^2(\sqrt{\lambda}\pi) + 4\varepsilon^{(4)}\cos^2(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin^2(\sqrt{\lambda}\pi)\\
&\quad + 2\xi\sin(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi)\cos(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\\
&\quad - 2\xi\cos(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi)\sin(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\\
&\quad - 2\delta\varepsilon^{(4)}\cos(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\sin(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi)\\
&\quad + 2\delta\varepsilon^{(4)}\sin(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\cos(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi)\\
&\quad - \delta\xi\cos^2(\sqrt{\lambda}2\pi)\cos^2(\sqrt{\lambda}\tfrac{3}{2}\pi) - \delta\xi\sin^2(\sqrt{\lambda}2\pi)\cos^2(\sqrt{\lambda}\tfrac{3}{2}\pi)\\
&= 4\varepsilon^{(4)}\sin^2(\sqrt{\lambda}\pi) + 2\xi\sin(\sqrt{\lambda}\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\sin(\sqrt{\lambda}\tfrac{1}{2}\pi)\\
&\quad - 2\delta\varepsilon^{(4)}\sin(\sqrt{\lambda}\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\sin(\sqrt{\lambda}\tfrac{1}{2}\pi) - \delta\xi\cos^2(\sqrt{\lambda}\tfrac{3}{2}\pi). \qquad (13)\end{aligned}$$

Taking additionally the relation $\xi - \delta\varepsilon^{(4)} = -\delta\xi$ into account, we conclude the equation

$$0 = 4\varepsilon^{(4)}\sin^2(\sqrt{\lambda}\pi) - 2\delta\xi\big(\sin(\sqrt{\lambda}\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\sin(\sqrt{\lambda}\tfrac{1}{2}\pi) + \tfrac{1}{2}\cos^2(\sqrt{\lambda}\tfrac{3}{2}\pi)\big).$$

Combining the fact $\sin^2(\sqrt{\lambda}\pi) \neq 0$ and

$$\sin(\sqrt{\lambda}\pi)\sin(\sqrt{\lambda}\tfrac{1}{2}\pi) = -\tfrac{1}{2}(\cos(\sqrt{\lambda}\tfrac{3}{2}\pi) - \cos(\sqrt{\lambda}\tfrac{1}{2}\pi)),$$

we altogether arrive at the result

$$\begin{aligned}\frac{(\varepsilon^{(4)} - \varepsilon^{(1)})^2}{\varepsilon^{(1)}\varepsilon^{(4)}} = \frac{\delta\xi}{\varepsilon^{(4)}} &= \frac{2\sin^2(\sqrt{\lambda}\pi)}{\sin(\sqrt{\lambda}\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\sin(\sqrt{\lambda}\tfrac{1}{2}\pi) + \tfrac{1}{2}\cos^2(\sqrt{\lambda}\tfrac{3}{2}\pi)}\\
&= \frac{4\sin^2(\sqrt{\lambda}\pi)}{\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\cos(\sqrt{\lambda}\tfrac{1}{2}\pi)}. \qquad (14)\end{aligned}$$

As the last expression on the right hand side is only positive on $(0, 1/9)$ and strictly monotonically increasing there, $\lambda \in (0, 1/9)$ is characterized by (14).

3.ii) Assume $A_2(\lambda) = 0$. We want to derive a contradiction for this option. Using trigonometric identities, we first infer the relations

$$\begin{aligned}0 &= -\varepsilon^{(4)}\cos(\sqrt{\lambda}\tfrac{3}{2}\pi) - \varepsilon^{(4)}\sin(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) + \varepsilon^{(1)}\cos(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\\
&= \varepsilon^{(4)}(\cos(\sqrt{\lambda}\tfrac{7}{2}\pi) - \cos(\sqrt{\lambda}\tfrac{3}{2}\pi)) - \xi\cos(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\\
&= -2\varepsilon^{(4)}\sin(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi) - \xi\cos(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi). \qquad (15)\end{aligned}$$

As a result of $a^{(4)} = 1$ and (7), the identities

$$\begin{aligned}0 &= -\varepsilon^{(4)}\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) + \varepsilon^{(4)}\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) + \varepsilon^{(1)}\sin(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\\
&= \varepsilon^{(4)}(\sin(\sqrt{\lambda}\tfrac{7}{2}\pi) - \sin(\sqrt{\lambda}\tfrac{3}{2}\pi)) - \xi\sin(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\\
&= 2\varepsilon^{(4)}\cos(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi) - 2\xi\sin(\sqrt{\lambda}\pi)\cos(\sqrt{\lambda}\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\end{aligned}$$

also follow. The last equation is equivalent to the formula

$$\begin{aligned}0 &= \varepsilon^{(4)}\cos(\sqrt{\lambda}\tfrac{5}{2}\pi) - \xi\cos(\sqrt{\lambda}\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\\
&= (\varepsilon^{(4)} - \tfrac{1}{2}\xi)\cos(\sqrt{\lambda}\tfrac{5}{2}\pi) - \tfrac{1}{2}\xi\cos(\sqrt{\lambda}\tfrac{1}{2}\pi). \qquad (16)\end{aligned}$$

This implies $\cos(\sqrt{\lambda}\tfrac{5}{2}\pi) > 0$, meaning $\lambda \in (0, \tfrac{1}{25}) \cup (\tfrac{9}{25}, 1)$.

We next return to (15). Using $\xi = \varepsilon^{(4)} - \varepsilon^{(1)}$, it is equivalent to the relation

$$1 - \frac{\varepsilon^{(1)}}{\varepsilon^{(4)}} = -2\frac{\sin(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi)}{\cos(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)}. \qquad (17)$$

Considering the sign of the right hand side and the precondition $\varepsilon^{(4)} > \varepsilon^{(1)}$, we infer that $\lambda \in (9/25, 1)$. Combining (17) with the inequalities

$$-2\frac{\sin(\sqrt{x}\frac{5}{2}\pi)\sin(\sqrt{x}\pi)}{\cos(\sqrt{x}2\pi)\cos(\sqrt{x}\frac{3}{2}\pi)} > -2\sin(\sqrt{x}\tfrac{5}{2}\pi)\sin(\sqrt{x}\pi)$$
$$> -2\sin(\tfrac{7}{4}\pi)\sin(\tfrac{7}{10}\pi) > 1, \qquad x \in (\tfrac{9}{25}, \tfrac{49}{100}],$$

we further conclude that $\lambda \in (\frac{49}{100}, 1)$. In view of (16) and the relations

$$(\varepsilon^{(4)} - \tfrac{\xi}{2})\cos(\sqrt{y}\tfrac{5}{2}\pi) - \tfrac{\xi}{2}\cos(\sqrt{y}\tfrac{1}{2}\pi) > (\varepsilon^{(4)} - \tfrac{\xi}{2})\tfrac{1}{\sqrt{2}} - \tfrac{\xi}{2}\cos(\tfrac{7}{20}\pi)$$
$$> \tfrac{\varepsilon^{(4)}}{\sqrt{2}} - \tfrac{\varepsilon^{(4)}}{2}(\tfrac{1}{\sqrt{2}} + \cos(\tfrac{7}{20}\pi)) > 0, \qquad y \in (\tfrac{49}{100}, \tfrac{81}{100}),$$

we obtain that $\lambda \in [\frac{81}{100}, 1)$. Monotonicity and (17) then lead to the contradiction

$$1 - \frac{\varepsilon^{(1)}}{\varepsilon^{(4)}} = -2\frac{\sin(\sqrt{\lambda}\pi)}{\cos(\sqrt{\lambda}\frac{3}{2}\pi)}\frac{\sin(\sqrt{\lambda}\frac{5}{2}\pi)}{\cos(\sqrt{\lambda}2\pi)} > \frac{4}{3}\frac{\sin(\frac{9}{4}\pi)}{\cos(\frac{9}{5}\pi)} > 1.$$

Altogether, $A_2(\lambda)$ has to be zero, and λ satisfies (14).

4) Let $\lambda \in (0,1)$ satisfy (14). In consideration of parts 1)–3), it now suffices to show that λ is indeed an eigenvalue. Let ψ be given by (4) with numbers $a^{(i)}, b^{(i)}$ that are to be determined.

4.i) We first consider the case $A_2(\lambda) = 0$. By definition of λ, identity (13) is then still true. Reversing the reasoning in part 3.i), we infer the formula

$$\begin{aligned}0 &= (2\cos(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi) + \delta\sin(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi))\\ &\quad\cdot(2\varepsilon^{(4)}\cos(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi) - \xi\sin(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi))\\ &= A_1(\lambda)(2\varepsilon^{(4)}\cos(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi) - \xi\sin(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)), \qquad (18)\end{aligned}$$

involving the term $A_1(\lambda)$ from (6). If $A_1(\lambda)$ was also zero, the arguments in part 2) would apply again, and would lead to a contradiction. As a result, the second factor in (18) is zero. Employing again the trigonometric manipulations in (11), we infer the identity

$$0 = -\varepsilon^{(4)}\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) + \varepsilon^{(4)}\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) + \varepsilon^{(1)}\sin(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi).$$

In presence of $A_2(\lambda) = 0$, formula (7) is consequently valid. We then choose $a^{(4)} = 1$, and define

$$b^{(4)} := \frac{\cos(\sqrt{\lambda}\frac{3}{2}\pi) - \frac{\varepsilon^{(4)}}{\varepsilon^{(1)}}\cos(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\frac{3}{2}\pi) + \sin(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\frac{3}{2}\pi)}{A_1(\lambda)}.$$

(Note that $A_1(\lambda) \neq 0$). This means that also (6) is valid. Choosing now $a^{(1)}$ and $b^{(1)}$ according to (5) as well as $a^{(1)} = a^{(2)} = a^{(3)}$, $b^{(1)} = b^{(2)} = b^{(3)}$, the function ψ is an eigenfunction of (1) to the eigenvalue λ.

4.ii) Consider finally the option $A_2(\lambda) \neq 0$. We then define $b^{(4)}$ by (11), set $a^{(4)} = 1$, define $a^{(1)} = a^{(2)} = a^{(3)}$ and $b^{(1)} = b^{(2)} = b^{(3)}$ by (5), and choose ψ as in (4). It then remains to check that the transmission conditions in (1) are fulfilled. By definition of $a^{(1)}$ and $b^{(1)}$, the transmission conditions in lines 2–4 of (1) are true. Taking also the choice of $b^{(4)}$ into account, we infer that (7) and equivalently the derivative transmission condition in the last line of (1) are valid. The definition of λ finally yields that (13) is satisfied. This is equivalent to the validity of (12). As the latter formula holds if and only if (6) is true, we have shown that ψ fulfills all transmission conditions in (1). □

Lemma 2. *Let $\varepsilon^{(1)} > \varepsilon^{(4)} > 0$ with*

$$\frac{\varepsilon^{(4)}}{\varepsilon^{(1)}} \neq 1 - 2\frac{\cos(\frac{5}{14}\pi)\sin(\frac{1}{7}\pi)}{\sin(\frac{2}{7}\pi)\cos(\frac{3}{14}\pi)}.$$

Then the statements of Lemma 1 are valid.

Proof. 1) We proceed as in the proof of Lemma 1, and assume first that there is an eigenvalue $\lambda := \kappa_1^2 \in (0,1)$ of (1) with associated eigenfunction $\psi \neq 0$. The representation formula $\psi^{(i)}(\varphi) = a^{(i)}\cos(\sqrt{\lambda}\phi) + b^{(i)}\sin(\sqrt{\lambda}\phi)$ is again assumed with $a^{(i)}, b^{(i)} \in \mathbb{R}$. Then all statements and formulas in part 1) of the proof of Lemma 1 are still valid.

2) The reasoning in part 2) of the proof for Lemma 1 shows also here that $a^{(4)} \neq 0$.

3.i) We can assume that $a^{(4)} = 1$. In case the expression $A_2(\lambda)$ from (7) is nonzero, the arguments in part 3.i) of the proof for Lemma 1 imply also here that λ satisfies the desired formula (3).

3.ii) Next we want to show by contradiction that $A_2(\lambda) \neq 0$. So assume that $A_2(\lambda) = 0$. Note then that the formulas up to (16) are still true in the current setting $\varepsilon^{(1)} > \varepsilon^{(4)}$. Then (16) implies that $\cos(\sqrt{\lambda}\frac{5}{2}\pi) < 0$ and $\cos(\sqrt{\lambda}\frac{1}{2}\pi) + \cos(\sqrt{\lambda}\frac{5}{2}\pi) > 0$. Hence $\lambda \in (\frac{1}{25}, \frac{1}{9}) \cup (\frac{1}{4}, \frac{9}{25})$. Reformulating (15), we infer

$$1 - \frac{\varepsilon^{(1)}}{\varepsilon^{(4)}} = -2\frac{\sin(\sqrt{\lambda}\frac{5}{2}\pi)\sin(\sqrt{\lambda}\pi)}{\cos(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\frac{3}{2}\pi)}. \tag{19}$$

The sign of the expression on the right hand side further yields $\lambda \in (\frac{1}{25}, \frac{1}{16})$. Combining $\xi = \varepsilon^{(4)} - \varepsilon^{(1)}$ with (16), we additionally conclude

$$1 - \frac{\varepsilon^{(1)}}{\varepsilon^{(4)}} = \frac{\cos(\sqrt{\lambda}\frac{5}{2}\pi)}{\cos(\sqrt{\lambda}\frac{3}{2}\pi)\cos(\sqrt{\lambda}\pi)}.$$

In view of the relations

$$\frac{\varepsilon^{(1)}}{\varepsilon^{(4)}} - 1 = -\frac{\cos(\sqrt{\lambda}\frac{5}{2}\pi)}{\cos(\sqrt{\lambda}\frac{3}{2}\pi)\cos(\sqrt{\lambda}\pi)} < -\sqrt{2}\frac{\cos(\frac{5}{8}\pi)}{\cos(\frac{3}{8}\pi)} = \sqrt{2},$$

$$2\frac{\sin(\sqrt{\lambda}\frac{5}{2}\pi)\sin(\sqrt{\lambda}\pi)}{\cos(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\frac{3}{2}\pi)} > 2\frac{\sin(\frac{5}{8}\pi)\sin(\frac{1}{8}\pi)}{\cos(\frac{2}{5}\pi)} > 2,$$

for $\lambda \in (\frac{1}{25}, \frac{1}{16})$, we arrive at a contradiction to (19). This shows that $A_2(\lambda) \neq 0$, and hence λ satisfies (3).

4) The arguments in parts 4.i)–4.ii) apply also here and show that $\lambda \in (0,1)$ satisfying (3) is indeed an eigenvalue of (1) with one-dimensional eigenspace. □

For the proof of the following result, we use the strategy of the proof for Lemma 2 in [6].

Lemma 3. *Let ε satisfy (2). Then the results of Lemma 1 remain true.*

Proof. 1) In presence of Lemmas 1 and 2, we only treat the case

$$\frac{\varepsilon^{(4)}}{\varepsilon^{(1)}} = 1 - 2\frac{\cos(\frac{5}{14}\pi)\sin(\frac{1}{7}\pi)}{\sin(\frac{2}{7}\pi)\cos(\frac{3}{14}\pi)}. \tag{20}$$

As in the proof of Lemma 1, we first assume that $\lambda \in (0,1)$ is an eigenvalue of (1) with eigenfunction $\psi \neq 0$. As the arguments in part 1) of the proof for Lemma 1 are also valid in the current setting, we use the definitions and formulas from there.

2) The goal is to show that $a^{(4)} = 0$. To that end, we prove by contradiction that the second factor on the left hand side of (7) is nonzero. So assume, it was zero. Then,

$$\frac{\varepsilon^{(4)}}{\varepsilon^{(1)}} = -\frac{\sin(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\frac{3}{2}\pi)}{\sin(\sqrt{\lambda}\frac{3}{2}\pi)(\cos(\sqrt{\lambda}2\pi)-1)}. \tag{21}$$

Formula (7) additionally implies that $b^{(4)}$ or $A_2(\lambda)$ has to be zero.

2.i) Let first $b^{(4)} = 0$. Then $a^{(4)} \neq 0$. Here (6) implies

$$\frac{\varepsilon^{(4)}}{\varepsilon^{(1)}} = \frac{\sin(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\frac{3}{2}\pi)+\cos(\sqrt{\lambda}\frac{3}{2}\pi)}{\cos(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\frac{3}{2}\pi)}. \tag{22}$$

Studying the sign of the terms on the right hand side of (21)–(22), we obtain $\lambda \in (0, \frac{1}{16})$. Then, the right hand side of (22) is, however, greater than 1, contradicting (20). As a result, $b^{(4)} \neq 0$.

2.ii) Let $A_2(\lambda) = 0$. Then, (9) is again valid and yields

$$\frac{\varepsilon^{(4)}}{\varepsilon^{(1)}} - 1 = -2\frac{\sin(\sqrt{\lambda}\frac{5}{2}\pi)\sin(\sqrt{\lambda}\pi)}{\cos(\sqrt{\lambda}\frac{3}{2}\pi)+\sin(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\frac{3}{2}\pi)}. \tag{23}$$

As $\varepsilon^{(4)}/\varepsilon^{(1)} > 0$, we infer from (21) that $\lambda \in (0, \frac{1}{9}) \cup (\frac{1}{4}, \frac{4}{9})$. We next employ that $-\cos(\sqrt{y}\frac{7}{2}\pi) > \cos(\sqrt{y}\frac{\pi}{2})$ for $y \in (\frac{1}{16}, \frac{1}{9})$, and that $\cos(\sqrt{y}\frac{7}{2}\pi) > 0$ for $y \in (\frac{1}{4}, \frac{4}{9})$. Then we infer the relations

$$\begin{aligned}
\frac{\sin(\sqrt{y}2\pi)\cos(\sqrt{y}\frac{3}{2}\pi)}{\sin(\sqrt{y}\frac{3}{2}\pi)(1-\cos(\sqrt{y}2\pi))} &= \frac{2\sin(\sqrt{y}\pi)\cos(\sqrt{y}\pi)\cos(\sqrt{y}\frac{3}{2}\pi)}{2\sin^2(\sqrt{y}\pi)\sin(\sqrt{y}\frac{3}{2}\pi)}\\
&= \frac{1}{\tan(\sqrt{y}\pi)\tan(\sqrt{y}\frac{3}{2}\pi)} \geq \frac{1}{\tan(\frac{1}{4}\pi)\tan(\frac{3}{8}\pi)} = \frac{1}{\tan(\frac{3}{8}\pi)}, \quad y \in (0, \tfrac{1}{16}],\\
\frac{2\sin(\sqrt{y}\frac{5}{2}\pi)\sin(\sqrt{y}\pi)}{\cos(\sqrt{y}\frac{3}{2}\pi)+\sin(\sqrt{y}2\pi)\sin(\sqrt{y}\frac{3}{2}\pi)} &= \frac{2\sin(\sqrt{y}\frac{5}{2}\pi)\sin(\sqrt{y}\pi)}{\cos(\sqrt{y}\frac{3}{2}\pi)+\frac{1}{2}(\cos(\sqrt{y}\frac{1}{2}\pi)-\cos(\sqrt{y}\frac{7}{2}\pi))}\\
&= \frac{\sin(\sqrt{y}\frac{5}{2}\pi)\sin(\sqrt{y}\pi)+\frac{1}{2}(\cos(\sqrt{y}\frac{3}{2}\pi)-\cos(\sqrt{y}\frac{7}{2}\pi))}{\frac{1}{2}(\cos(\sqrt{y}\frac{3}{2}\pi)+\cos(\sqrt{y}\frac{1}{2}\pi))+\sin(\sqrt{y}\frac{5}{2}\pi)\sin(\sqrt{y}\pi)} \geq 1, \quad y \in (\tfrac{1}{16}, \tfrac{1}{9}) \cup (\tfrac{1}{4}, \tfrac{4}{9}).
\end{aligned}$$

These contradict (20). Altogether, the second factor on the left hand side of (7) is nonzero.

3) We next show by contradiction that $A_2(\lambda) = 0$, see (7). Assume hence that $A_2(\lambda) \neq 0$. Combining the result of part 2) with (7), we conclude $a^{(4)}, b^{(4)} \neq 0$. We hence choose $a^{(4)} = 1$. The reasoning and formulas in part 3.i) of the proof for Lemma 1 are then again true and we conclude (14) in this way. Assumption (20) yields $\lambda = 1/49$ as well as (8). Repeating the arguments in part 2) of the proof for Lemma 1 in reverse order, we furthermore conclude that $A_2(\lambda) = 0$. This leads to a contradiction. Altogether, $A_2(\lambda) = 0$, and part 2) with (7) imply that $a^{(4)} = 0$.

4) Repeating the arguments in part 2) of the proof for Lemma 1, we now conclude that $\lambda = 1/49$. In view of (20), $\kappa_1 = \sqrt{\lambda}$ satisfies formula (3).

5) It now suffices to show that $\lambda = 1/49$ is an eigenvalue of (1). The above reasoning then shows that the associated eigenspace is only one-dimensional. Introduce a potential eigenfunction ψ via (4) with $a^{(4)} = 0$, $b^{(4)} = 1$ and $a^{(1)} = a^{(2)} = a^{(3)}$, $b^{(1)} = b^{(2)} = b^{(3)}$ satisfying (5). By construction, ψ satisfies the transmission conditions in lines 2-4 of (1).

We next show that ψ also satisfies the remaining transmission relations in line 5 of (1). This is equivalent to establishing (6) and (7). Assumption (20) implies that (8) is valid, meaning that $A_1(\lambda) = 0$. In presence of the choice $a^{(4)} = 0$, we conclude that (6) is true. Using the reasoning part 2) of the proof for Lemma 1 in reverse order, we finally conclude that (9) is valid, implying $A_2(\lambda) = 0$. As a result, also (7) is proven. □

References

[1] M. Costabel and M. Dauge: Singularities of electromagnetic fields in polyhedral domains, Arch. Ration. Mech. Anal. 151 (3) (2000), 221–276.

[2] R.B. Kellogg: Singularities in interface problems, in: Numerical solution of partial differential equations-II. SYNSPADE 1970 (ed. B. Hubbard), Academic Press, New York (1971), 351–400.

[3] R.B. Kellogg: On the Poisson equation with intersecting interfaces, Appl. Anal. 4 (1974/75), 101–129.

[4] H. Triebel: Higher analysis, Johann Ambrosius Barth, Leipzig, 1992.

[5] K. Zerulla: Analysis of a dimension splitting scheme for Maxwell equations with low regularity in heterogeneous media, J. Evol. Equ. 22, 90 (2022).

[6] K. Zerulla: A formula for the first positive eigenvalue of a one-dimensional transmission problem, Karlsruhe Institute of Technology, 2022. URL: `https://doi.org/10.5445/IR/1000144767`

[7] K. Zerulla: Analysis of a Peaceman-Rachford ADI scheme for Maxwell equations in heterogeneous media, Preprint of CRC 1173, Karlsruhe Institute of Technology, 2022.

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