Massive form factors at $O(\alpha_s^3)$

Kay Schönwald[a,*]

[a] *Institut für Theoretische Teilchenphysik, Karlsruhe Institute of Technology (KIT),*
76128 Karlsruhe, Germany

E-mail: kay.schoenwald@kit.edu

We report on our recent calculation of massive quark form factors using a semi-numerical approach based on series expansions of the master integrals around singular and regular kinematic points and numerical matching. The methods allows to cover the whole kinematic range of negative and positive values of the virtuality *s* with at least seven significant digits accuracy.

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*Speaker

1. Introduction

Massive form factors are important objects in quantum field theory. They constitute the virtual corrections to many observables and processes like lepton pair production via the Drell-Yan process or the decay of the Higgs boson into heavy quarks. The more and more precise measurements of these processes make the inclusion of higher order corrections for the theory predictions necessary.

Up to $O(\alpha_s^2)$ the massive form factors are known in analytic form, see Refs. [1–7], and even higher orders in the dimensional regulator ϵ haven been considered in Refs. [8–12]. At $O(\alpha_s^3)$ only partial results are available. At this order the form factors have been considered in the large-N_C limit (with N_C the number of colors) in Refs. [9, 13–15], the light fermion contributions were calculated in Ref. [12] and in Ref. [16] all non-singlet contributions involving a closed heavy quark loop have been considered.

In these proceedings we report on our recent calculation of the massive form factors at $O(\alpha_s^3)$ in Refs. [17, 18], where we employed a semi-numerical method involving series expansions and numerical matching between them. In Section 2 we will summarize technical details, while in Section 3 we show some results. In Section 4 we conclude and give an outlook.

2. Massive form factors

To compute the massive form factors we consider the interaction of a massive quark with a vector, axial-vector, scalar or pseudo-scalar current, which are given by:

$$
\begin{aligned}
j_\mu^v &= \bar{\psi}\gamma_\mu\psi\,, \\
j_\mu^a &= \bar{\psi}\gamma_\mu\gamma_5\psi\,, \\
j^s &= m\,\bar{\psi}\psi\,, \\
j^p &= \mathrm{i}m\,\bar{\psi}\gamma_5\psi\,.
\end{aligned}
\tag{1}
$$

The vertex function can then be expressed through six scalar functions by

$$
\begin{aligned}
\Gamma_\mu^v(q_1,q_2) &= F_1^v(q^2)\gamma_\mu - \frac{\mathrm{i}}{2m}F_2^v(q^2)\sigma_{\mu\nu}q^\nu\,, \\
\Gamma_\mu^a(q_1,q_2) &= F_1^a(q^2)\gamma_\mu\gamma_5 - \frac{1}{2m}F_2^a(q^2)q_\mu\gamma_5\,, \\
\Gamma^s(q_1,q_2) &= mF^s(q^2)\,, \\
\Gamma^p(q_1,q_2) &= \mathrm{i}mF^p(q^2)\gamma_5\,.
\end{aligned}
\tag{2}
$$

Here the momentum q_1 (q_2) is incoming (outgoing), on-shell ($q_1^2 = q_2^2 = m^2$) and $q = q_1 - q_2$ is the outgoing momentum at the current j^δ with $q^2 = s$. The form factors have an expansion in the strong coupling constant α_s

$$
F_i^j = \sum_l \left(\frac{\alpha_s}{\pi}\right)^l F_i^{j,(l)}\,.
$$

We divide the form factor into non-singlet and singlet contributions, where the current couples to the heavy external quark line or an internal heavy quark loop, respectively. Some sample Feynman diagrams contributing to the form factors can be found in Fig. 1.
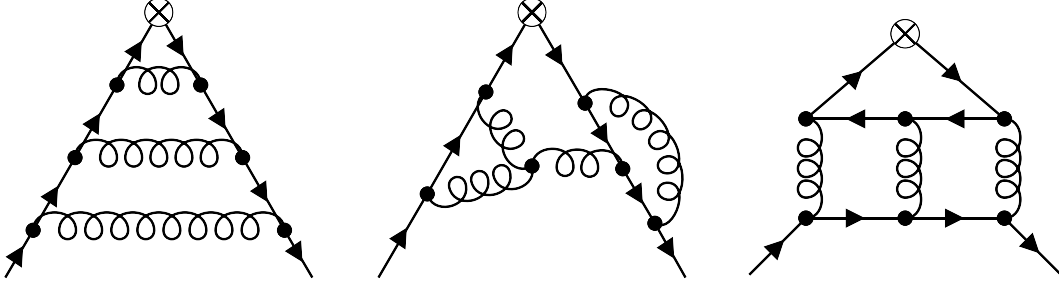
Figure 1: Sample Feynman diagrams contributing to the massive quark form factors. Solid lines represent massive quark, while curly lines represent gluons. The vertex with a cross refers to the coupling to one of the external currents. The diagram on the right is part of the singlet contributions.

The calculation of the form factors proceeds in the following way: We generate the diagrams with QGRAF [19] and use q2e [20, 21] to transform the output to FORM [22] input, where Dirac-, Lorenz- and color-algebra (with color [23]) is performed. The diagrams are mapped to predefined topologies using exp [20, 21]. The scalar integrals are reduced to master integrals with the help of Kira [24, 25] with Fermat [26] on a family-by-family basis. We make sure to reduce to a basis where the dependence on d and the kinematic variable s factorizes utilizing an improved version of ImproveMasters, first developed in Ref. [27]. Afterwards we symmetrize over all families and find 422 master integrals for the non-singlet contribution and 316 for the singlet diagrams. In a next step, we set up a systems of differential equations for the master integrals in the variable $\hat{s} = s/m^2$ by calculating the derivatives with the help of LiteRed [28, 29] and subsequent reduction with Kira.

Subsequently, the master integrals need to be solved. This is achieved using the semi-numerical approach presented in Ref. [30] and explained in more detail for the current problem in Ref. [18]. Let us summarize the main ideas of the approach:

1. We calculate boundary conditions for all master integrals at the special point $s = 0$. At this special point the master integrals in the non-singlet case reduce to three-loop on-shell propagators, which are well studied in the literature [31–33]. However, we needed to extend the depth of the ϵ expansion, since we encountered high spurious poles in the amplitude after reducing to master integrals. The results can be found in Ref. [18]. Since the singlet diagrams have massless cuts, we need to perform an asymptotic expansion around $s = 0$ to obtain their boundary conditions.

2. We calculate symbolic expansions around the point $s = 0$ by inserting a suitable ansatz into the system of differential equations. By comparing powers in ϵ, the expansion parameter s and possibly logarithms of the expansion parameter, we obtain a system of linear equations for the coefficients of the ansatz. We solve this system of equations with Kira and FireFly [34, 35] in terms of a small set of boundary conditions, which can be determined from step 1.

3. We calculate symbolic expansions around a new point s_1 and match the two expansions numerically at a point where both expansions converge, e.g. $s_1/2$.

4. Afterwards, we generate another symbolic expansion at s_2 and match it to the expansion around s_1 at a point where both expansions converge. This way we can map out the whole kinematics of the process.
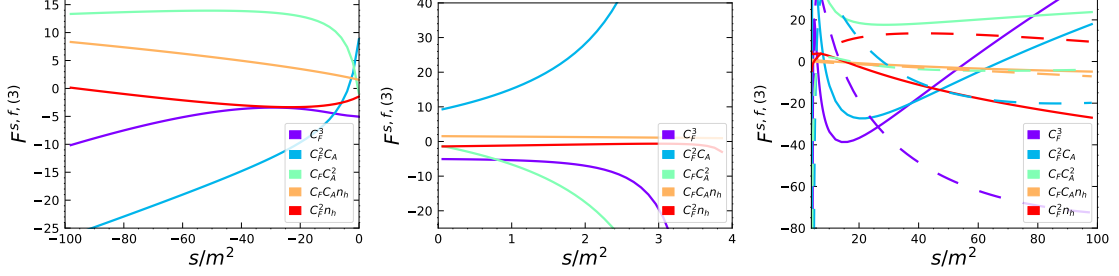
Figure 2: The UV-renormalized and infrared subtracted form factor $F^{s,f,(3)}$. Solid lines correspond to the real part, dashed lines to the imaginary part of the corresponding color factor. Taken from Ref. [18].
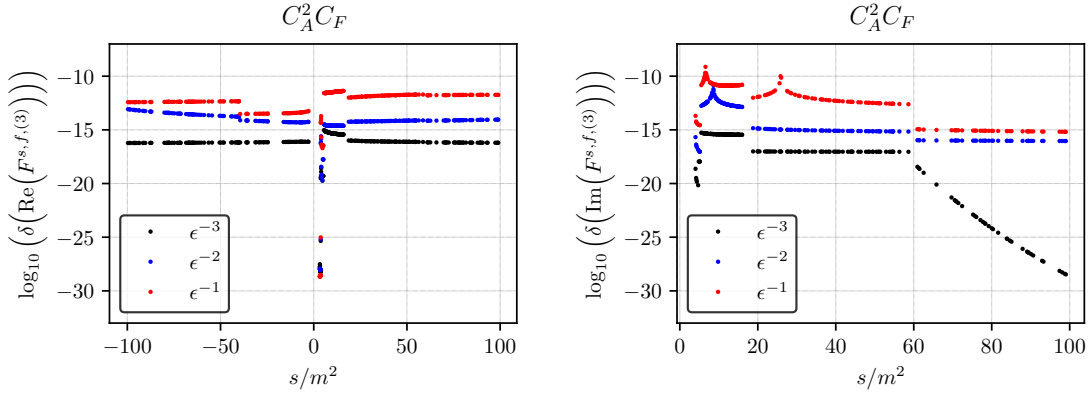
Figure 3: Relative cancellation of the real (left) and imaginary part (right) of the poles for the color factor $C_F C_A^2$ of $F^{s,f,(3)}$. Taken from Ref. [18].

3. Results

The main results of our method are overlapping series expansions which can be used to evaluate the massive form factors at any value of s. In Fig. 2 we show as an example the non-singlet contributions to the form factor F^s, where the masses and wave function are renormalized on-shell, the current in the $\overline{\text{MS}}$-scheme and the remaining infrared divergencies are subtracted by multiplying with a suitably defined Z-factor, which can be constructed from the cusp anomalous dimension [36–39] (see Refs. [9, 18] for the precise definition). The resulting finite form factors are labeled with an additional superscript f. Note, that our expansion around $s = 0$ is analytic, e.g. for $F^{s,f,(3)}$,

i.e. the non-singlet contribution to the scalar form factor, we find:

$$
\begin{aligned}
F^{s,f,(3)}\Big|_{s\to 0} = {} & C_A C_F^2 \left[\frac{4a_4}{3} + \frac{491\zeta_3}{96} + \frac{19\pi^2\zeta_3}{16} - \frac{45\zeta_5}{16} + \frac{26117}{4608} - \frac{1193\pi^2}{576} - \frac{65\pi^4}{432} + \frac{l_2^4}{18} \right. \\
& \left. + \frac{31}{36}\pi^2 l_2^2 + \frac{43}{18}\pi^2 l_2 \right] + C_A^2 C_F \left[-\frac{11a_4}{3} - \frac{947\zeta_3}{288} - \frac{51\pi^2\zeta_3}{64} + \frac{65\zeta_5}{32} - \frac{584447}{124416} + \frac{3011\pi^2}{3456} \right. \\
& \left. + \frac{179\pi^4}{3456} - \frac{11 l_2^4}{72} - \frac{11}{36}\pi^2 l_2^2 + \frac{49}{72}\pi^2 l_2 \right] + C_F^3 \left[12a_4 + \frac{87\zeta_3}{16} + \frac{\pi^2\zeta_3}{16} - \frac{5\zeta_5}{8} + \frac{55}{96} + \frac{643\pi^2}{192} \right. \\
& \left. + \frac{\pi^4}{48} + \frac{l_2^4}{2} - \frac{1}{2}\pi^2 l_2^2 - \frac{15}{2}\pi^2 l_2 \right] + C_F^2 T_F n_h \left[8a_4 + \frac{17\zeta_3}{18} - \frac{2083}{432} - \frac{52\pi^2}{81} - \frac{\pi^4}{720} + \frac{l_2^4}{3} \right. \\
& \left. - \frac{1}{3}\pi^2 l_2^2 + \frac{8}{9}\pi^2 l_2 \right] + C_F C_A T_F n_h \left[-6a_4 - \frac{199\zeta_3}{144} + \frac{\pi^2\zeta_3}{8} - \frac{5\zeta_5}{8} + \frac{209857}{15552} - \frac{4351\pi^2}{1296} \right. \\
& \left. - \frac{\pi^4}{288} - \frac{l_2^4}{4} + \frac{1}{4}\pi^2 l_2^2 + \frac{32}{9}\pi^2 l_2 \right] \\
& + \frac{s}{m^2} \left\{ C_A C_F^2 \left[-\frac{8a_4}{9} + \frac{2515\zeta_3}{2304} - \frac{29\pi^2\zeta_3}{144} - \frac{95\zeta_5}{48} + \frac{11191}{41472} - \frac{15101\pi^2}{62208} + \frac{37\pi^4}{4320} - \frac{l_2^4}{27} \right. \right. \\
& \left. + \frac{1259\pi^2 l_2^2}{2160} + \frac{1409\pi^2 l_2}{1440} \right] + C_A^2 C_F \left[\frac{5a_4}{4} + \frac{8675\zeta_3}{10368} - \frac{73\pi^2\zeta_3}{1152} + \frac{125\zeta_5}{384} - \frac{851465}{279936} \right. \\
& \left. + \frac{130417\pi^2}{186624} + \frac{689\pi^4}{103680} + \frac{5 l_2^4}{96} - \frac{1}{20}\pi^2 l_2^2 - \frac{12253\pi^2 l_2}{8640} \right] + C_F^3 \left[-\frac{29a_4}{9} - \frac{12401\zeta_3}{3456} \right. \\
& \left. - \frac{67\pi^2\zeta_3}{288} + \frac{85\zeta_5}{32} + \frac{22613}{41472} - \frac{69355\pi^2}{31104} + \frac{1727\pi^4}{25920} - \frac{29 l_2^4}{216} - \frac{1043\pi^2 l_2^2}{1080} + \frac{4013\pi^2 l_2}{1080} \right] \\
& + C_F^2 T_F n_h \left[\frac{8a_4}{9} + \frac{9889\zeta_3}{6912} - \frac{8059}{5184} - \frac{4261\pi^2}{25920} + \frac{7\pi^4}{1620} + \frac{\log^4(2)}{27} - \frac{1}{27}\pi^2 l_2^2 + \frac{4}{27}\pi^2 l_2 \right] \\
& + C_F C_A T_F n_h \left[-\frac{5a_4}{18} + \frac{1657\zeta_3}{1728} + \frac{\pi^2\zeta_3}{96} + \frac{5\zeta_5}{96} + \frac{2257}{3888} - \frac{13663\pi^2}{25920} - \frac{121\pi^4}{51840} - \frac{5 l_2^4}{432} \right. \\
& \left. \left. + \frac{5}{432}\pi^2 l_2^2 + \frac{55}{108}\pi^2 l_2 \right] \right\} + O\left(\frac{s^2}{m^4}\right) + n_l, \ n_l^2 \ \text{and} \ n_h^2 \ \text{terms,}
\end{aligned}
\tag{3}
$$

where $l_2 = \log(2)$, $a_4 = \text{Li}_4(1/2)$ and ζ_n is Riemann's zeta function evaluated at n and $C_F = T_F(N_C^2 - 1)/N_C$, $C_A = 2 T_F N_C$ are the quadratic Casimir operators of the SU(N_C) gauge group in the fundamental and adjoint representation, respectively, n_l is the number of massless quark flavors, n_h is the number of heavy quark flavors with mass m and $T_F = 1/2$.

There are several checks on our results. For example, the coefficient in front of the gauge parameter in the final result is smaller than 10^{-18} and we can reproduce the known analytic results in the planar limit, the contributions $\sim n_l$ and the n_h^2 contributions with at least 12 digits. Furthermore, the results are precise enough to calculate the leading and sub-leading logarithmic corrections in the high energy expansion for the first power suppressed contributions analytically.

These corrections have been obtained in Refs. [40–43] by considering an involved asymptotic expansion of the Feynman diagrams. We find agreement except of the quartic mass suppressed corrections to the form factor $F_2^{v,(3)}$. Our results have been confirmed by the authors of Ref. [43]. More details and analytic expressions for several expansion terms can be found in Ref. [18].

The precision of our final results can be estimated from the cancellation of the poles in the dimensional regulator ϵ, since they are known analytically and have to cancel in the final result. We use the logarithm to the base 10 of the relative pole cancellation (denoted by δ) as a measure of accurate digits. A plot of this measure for the form factor $F^{s,f,(3)}$ and the color factor $C_F C_A^2$, split into real and imaginary part, can be found in Fig. 3. We see that the precision for $s < 3$ and $s > 16$ is highest and decreases for the regions between the two thresholds at $s = 4m^2$ (two particle threshold) and $s = 16m^2$ (three particle threshold), which are not analytic. In total we estimate at least 7 significant digits over the whole kinematic range for all of our results. The results of the singlet contributions is significantly higher and estimated to be at least 10 digits.

A `Mathematica` package to evaluate the form factors in the non-singlet and singlet case numerically over the full kinematic range of s can be found at:

$$\text{https://gitlab.com/formfactors3l/formfactors3l}.$$

4. Conclusions and Outlook

We presented our recent calculation of massive quark form factors at $O(\alpha_s^3)$ which uses a semi-numerical method based on series expansions and numerical matching to obtain results for the form factors for the whole kinematic range of negative and positive values of the virtuality s. We obtain a precision of at least 7 significant digits in the non-singlet and 10 digits in the singlet case over the whole kinematic range, respectively. However, some kinematic regions are much more precise. Thus, it is for example possible to extract leading and sub-leading logarithmic contributions to the leading and first power suppressed terms in the high energy expansion analytically, confirming and correcting results in the literature. To complete the calculation of massive quark form factors the singlet diagrams where the external current couples to an internal light quark loop still need to be completed.

Acknowledgments

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