



# A unified error analysis for nonlinear wave-type equations with application to acoustic boundary conditions

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## Abstract

In this work we present a unified error analysis for abstract space discretizations of wave-type equations with nonlinear quasi-monotone operators. This yields an error bound in terms of discretization and interpolation errors that can be applied to various equations and space discretizations fitting in the abstract setting. We use the unified error analysis to prove novel convergence rates for a non-conforming finite element space discretization of wave equations with nonlinear acoustic boundary conditions and illustrate the error bound by a numerical experiment.

**Mathematics Subject Classification** Primary 65M12 · 65M15; Secondary 65M60 · 65J15

## 1 Introduction

In this paper we present a unified error analysis for abstract non-conforming space discretizations of nonlinear wave-type equations with quasi-monotone operators. The unified error analysis was introduced in [15, 16] for linear wave equations and extended in [18] to semilinear problems. It is an abstract framework in which wave equations as well as a variety of spatial discretizations are considered as evolution equations in Hilbert spaces. Using such an abstract framework allows to derive an abstract error bound in terms of approximation properties of the space discretization method. This error bound can then be used to prove convergence rates for all space discretizations of wave-type equations which fit into the abstract setting. This was demonstrated in [16]

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for isoparametric finite element discretizations of wave equations with linear acoustic boundary conditions and a dG discretization of Maxwell equations.

The aim of this paper is to extend the unified error analysis to nonlinear evolution equations with quasi-monotone operators. As a specific application, we use this theory to prove error bounds for a non-conforming finite element discretization of the wave equation with nonlinear acoustic boundary conditions. This is a generalization of the results in the thesis [20].

Acoustic boundary conditions were first mentioned in [5]. Since then, many papers studied their properties, wellposedness, and stability, and they are still in the focus of current research, cf. [4, 10, 11, 21, 23, 26] and references therein.

However, there are only very few numerical papers considering these boundary conditions. We are aware of [16] and [17]. In these papers space discretizations for wave equations with linear acoustic boundary conditions were derived and analyzed in the energy and the  $L^2$ -norm, respectively. In the present paper, we now consider the space discretization of nonlinear acoustic boundary conditions as proposed in [13, 14, 28], and extend the results from [16] to this case.

Since acoustic boundary conditions include derivatives on the boundary, they are usually posed on domains with smooth boundaries. A common choice to discretize such problems are isoparametric finite elements. Since this involves to approximate the boundary of the domain, the discretization becomes non-conforming. Unfortunately, this makes the error analysis much more involved since the exact and the numerical solution are not defined on the same domain which causes errors in the bilinear form of the weak form.

We derive an order  $p$  isoparametric finite element discretization of the wave equation with nonlinear acoustic boundary conditions and show that it fits into the setting of the unified error analysis. Using the abstract error bound we prove order  $p$  convergence of the method in the energy norm, where we tackle the appearing approximation errors stemming from the domain approximation or interpolation by known error bounds from [7, 8]. A major difficulty lies in the discretization of the nonlinearities, since this must be done in such a way that it preserves the quasi-monotonicity to ensure the stability of the numerical scheme. Furthermore, the discretization errors of the nonlinearities have to be bounded. While both is straightforward for conforming discretizations, it turns out to be much more involved in the non-conforming case.

We are not aware of any other results in this direction, neither of such a general error analysis for non-conforming space discretizations of nonlinear wave-type equations, nor of results concerning the discretization of wave equations with nonlinear acoustic boundary conditions. Nevertheless, we mention the following works going in the same direction. In [9], a full discretization in an abstract framework similar to the one used in this paper was considered. But only a conforming space discretization was analyzed and no error bounds but only weak convergence of the discretization was shown. For quasilinear equations, a related framework was introduced in [19, 22], covering quasilinear wave and Maxwell equations. However, the error analysis in this work relies on properties of quasilinear operators that cannot be used for nonlinear acoustic boundary conditions and in general for equations with maximal quasi-monotone operators.

This paper is structured as follows. In Sect. 2 we introduce the wave equation with nonlinear acoustic boundary conditions with a corresponding finite element space

discretization and state an error bound of the spatial discretization. We then present in Sect. 3 the unified error analysis for nonlinear first-order evolution equations and use the results in Sect. 4 to analyze nonlinear second-order wave-type equations. As main results we derive abstract error bounds for the space discretizations. Finally, in Sect. 5 we use these abstract bounds of the unified error analysis to prove the space discretization error bound for the wave equations with nonlinear acoustic boundary conditions and illustrate it with some numerical experiments.

## 2 The wave equation with nonlinear acoustic boundary conditions

In this section we present the analytical framework for the wave equation with acoustic boundary conditions and a suitable finite element space discretization. Additionally, we present a space discretization error bound which we will prove by application of the unified error analysis in Sect. 5.

### 2.1 Problem statement and analytical framework

Let  $\Omega \subset \mathbb{R}^n$ ,  $n = 2, 3$ , be a bounded domain with  $C^2$ -boundary  $\Gamma$  and outer normal vector  $\mathbf{n}$ . We consider the acoustic wave equation with non-local reacting acoustic boundary conditions in the following form: seek  $u: [0, T] \times \Omega \rightarrow \mathbb{R}$ ,  $\delta: [0, T] \times \Gamma \rightarrow \mathbb{R}$  satisfying

$$u_{tt} + k_{\Omega}u - c_{\Omega}\Delta u = f_{\Omega}, \quad t \geq 0, \mathbf{x} \in \Omega, \quad (1a)$$

$$\mu\delta_{tt} + d\delta_t + k_{\Gamma}\delta + \rho u_t - c_{\Gamma}\Delta_{\Gamma}\delta = f_{\Gamma}, \quad t \geq 0, \mathbf{x} \in \partial\Omega, \quad (1b)$$

$$\eta(\delta_t) = \partial_{\mathbf{n}}u + \theta(u_t), \quad t \geq 0, \mathbf{x} \in \partial\Omega, \quad (1c)$$

$$u(0) = u^0, \quad u_t(0) = v^0, \quad \delta(0) = \delta^0, \quad \delta_t(0) = \vartheta^0. \quad (1d)$$

Here  $\Delta_{\Gamma}$  denotes the Laplace–Beltrami operator on  $\Gamma$ .

**Remark 2.1** It is possible to include nonlinear forcing terms  $F_{\Omega}(\mathbf{x}, u)$  and  $F_{\Gamma}(\mathbf{x}, \delta)$  at the right-hand side of (1a) and (1b), respectively. This was considered in [20] for the wave equation with kinetic boundary conditions and such terms can be treated similarly for the acoustic boundary conditions. We omit this here for the sake of a clearer presentation.

We make the following assumptions on the coefficients and nonlinearities in (1).

**Assumption 2.2** a) The constants satisfy  $c_{\Omega}, c_{\Gamma}, \mu > 0$ ,  $k_{\Omega}, k_{\Gamma} \geq 0$ ,  $d, \rho \in \mathbb{R}$ .  
 b) The function  $\theta \in C(\mathbb{R}; \mathbb{R})$  satisfies  $\theta(0) = 0$  and is strictly monotonically increasing with

$$(\theta(\xi_1) - \theta(\xi_2))(\xi_1 - \xi_2) \geq \theta_0 |\xi_1 - \xi_2|^2, \quad \xi_1, \xi_2 \in \mathbb{R},$$

for some  $\theta_0 > 0$ . Further, there exist

$$1 \leq \zeta \begin{cases} < \infty, & n = 2, \\ \leq 3, & n = 3, \end{cases} \tag{2}$$

and a constant  $C > 0$  such that for all  $\xi \in \mathbb{R}$

$$|\theta(\xi)| \leq C(1 + |\xi|^\zeta). \tag{3}$$

- c) The function  $\eta: \mathbb{R} \rightarrow \mathbb{R}$  is globally Lipschitz continuous and satisfies  $\eta(0) = 0$ . We then have that  $\tilde{\eta}$  defined via  $\tilde{\eta}(\xi) = \eta(\xi) - \frac{\rho}{c_\Omega} \xi$  is also Lipschitz continuous and denote the Lipschitz constant of  $\tilde{\eta}$  by  $L_\eta$ .
- d) The inhomogeneities satisfy  $f_\Omega \in W_{loc}^{1,1}([0, \infty); C(\overline{\Omega}))$  and  $f_\Gamma \in W_{loc}^{1,1}([0, \infty); C(\Gamma))$ .

*Weak formulation* To prove wellposedness and derive a finite element discretization, we now present a weak formulation of the wave equation with acoustic boundary conditions (1). We make use of the densely embedded Hilbert spaces

$$V = \mathbb{H}^1 \hookrightarrow H = \mathbb{H}^0,$$

where

$$\mathbb{H}^0 := L^2(\Omega) \times L^2(\Gamma), \quad \mathbb{H}^k := H^k(\Omega) \times H^k(\Gamma), \quad k \geq 1.$$

Note that for  $k \geq 3$ , the spaces  $H^k(\Gamma)$  require more boundary regularity to be well-defined, e.g.,  $\Gamma \in C^k$ , which denotes that  $\Gamma$  is a  $C^k$  boundary.

By multiplying (1a) and (1b) with test functions defined on  $\Omega$  and  $\Gamma$ , respectively, applying integration by parts and inserting the nonlinear coupling (1c), we obtain the weak formulation of (1): seek  $\mathbf{u} = [u, \delta]^\top \in C^2([0, T]; H) \cap C^1([0, T]; V)$  satisfying

$$m(\mathbf{u}'', \boldsymbol{\varphi}) + \langle \mathcal{D}(\mathbf{u}'), \boldsymbol{\varphi} \rangle_{V^* \times V} + a(\mathbf{u}, \boldsymbol{\varphi}) = m(\mathbf{f}, \boldsymbol{\varphi}), \quad \text{for } t \geq 0 \text{ and all } \boldsymbol{\varphi} \in V, \tag{4}$$

$$\mathbf{u}(0) = \mathbf{u}^0, \quad \mathbf{u}'(0) = \mathbf{v}^0,$$

where for  $\mathbf{v} = [v, z]^\top$ ,  $\boldsymbol{\varphi} = [\varphi, \psi]^\top \in V$  we have

$$m(\mathbf{v}, \boldsymbol{\varphi}) = \int_\Omega v \varphi \, dx + \mu \int_\Gamma z \psi \, ds, \tag{5a}$$

$$a(\mathbf{v}, \boldsymbol{\varphi}) = c_\Omega \int_\Omega \nabla v \cdot \nabla \varphi \, dx + k_\Omega \int_\Omega v \varphi \, dx + c_\Gamma \int_\Gamma \nabla_\Gamma z \cdot \nabla_\Gamma \psi \, ds + k_\Gamma \int_\Gamma z \psi \, ds, \tag{5b}$$

$$\langle \mathcal{D}(\mathbf{v}), \boldsymbol{\varphi} \rangle_{V^* \times V} = \int_\Gamma c_\Omega (\theta(v) - \eta(z)) \varphi + (dz + \rho v) \psi \, ds, \tag{5c}$$

$$\mathbf{f} = \left[ f_\Omega, \frac{1}{\mu} f_\Gamma \right]^\top, \tag{5d}$$

$$\mathbf{u}^0 = \left[ u^0, \delta^0 \right]^\top, \quad \mathbf{v}^0 = \left[ v^0, \vartheta^0 \right]^\top. \tag{5e}$$

Note that  $m$  is an inner product on  $H$  and  $\tilde{a} := a + m$  is an inner product on  $V$ .

**Remark 2.3** Assumption 2.2 ensures that (4) is globally wellposed, we comment on this in Sect. 4.1, cf. Corollary 4.4, and Sect. 5.

### 2.2 Finite element space discretization

For the space discretization of (1) we consider the bulk-surface finite element method from [7] which was also used in [16] to discretize the wave equation with linear acoustic boundary conditions. We give a brief introduction of the finite element spaces and refer to [7] for further details on the bulk-surface finite element method.

*The bulk-surface finite element method* Let  $\Gamma \in C^{p+1}$  for some  $p \geq 1$  and let  $\mathcal{T}_h^\Omega$  be a consistent and quasi-uniform mesh consisting of isoparametric elements  $K$  of degree  $p$  which discretizes  $\Omega$ . By  $h$  we denote the maximal mesh width of  $\mathcal{T}_h^\Omega$ . The discretized domain is then given by

$$\Omega_h = \bigcup_{K \in \mathcal{T}_h^\Omega} K$$

and its boundary by  $\Gamma_h = \partial\Omega_h$ . The bulk and the surface finite element space of order  $p$  are then defined by

$$V_{h,p}^\Omega := \{v_h \in C(\Omega_h) \mid v_h|_K = \widehat{v}_h \circ (F_K)^{-1} \text{ with } \widehat{v}_h \in \mathbb{P}_p(\widehat{K}) \text{ for all } K \in \mathcal{T}_h^\Omega\},$$

$$V_{h,p}^\Gamma := \{\vartheta_h \in C(\Gamma_h) \mid \vartheta_h = v_h|_{\Gamma_h} \text{ with } v_h \in V_{h,p}^\Omega\},$$

respectively. Here,  $\widehat{K}$  denotes the reference triangle with corresponding polynomial space  $\mathbb{P}_p(\widehat{K})$  of order  $p$ , and  $F_K$  is the transformation from  $\widehat{K}$  to  $K$ . Note that by construction we have  $v_h|_{\Gamma_h} \in V_{h,p}^\Gamma$  for all  $v_h \in V_{h,p}^\Omega$ .

As approximation space for  $V$  we set  $V_h = V_{h,p}^\Omega \times V_{h,p}^\Gamma$ . Note that, since  $\Omega_h$  is only an approximation of  $\Omega$ , we have  $V_h \not\subseteq V$ , i.e., the discretization is non-conforming. Hence, to relate functions in  $V_h$  with functions in  $V$ , in [7], for  $\mathbf{v}_h = [v_h, \vartheta_h]^\top \in V_h$  and  $h < h_0$  sufficiently small, a lifted version

$$\mathbf{v}_h^\ell = \left[ v_h^\ell, \vartheta_h^\ell \right]^\top \in V \cap (C(\Omega) \times C(\Gamma)) \tag{6}$$

was constructed. By  $I_{h,\Omega}: C(\overline{\Omega}) \rightarrow V_{h,p}^\Omega$  and  $I_{h,\Gamma}: C(\Gamma) \rightarrow V_{h,p}^\Gamma$  we denote the order  $p$  nodal interpolation operators in  $\Omega$  and on  $\Gamma$ , respectively, and set for  $\mathbf{v} = [v, \vartheta]^\top \in V$

$$I_h \mathbf{v} = [I_{h,\Omega} v, I_{h,\Gamma} \vartheta]^\top \in V_h.$$

*The spatially discretized equation* We now state the finite element discretization of (1). For this, let

$$\sum_{\Gamma_h} \cdot \Delta \mathbf{s} : C(\Gamma_h) \rightarrow \mathbb{R}$$

be an elementwise defined quadrature formula that approximates the integral  $\int_{\Gamma_h} \cdot \, ds$ . We require that the quadrature formula has positive weights and is of order greater than  $2p$ , s.t. polynomials up to degree  $2p$  are integrated exactly and we have for all  $z_h, \psi_h \in V_{h,p}^{\Gamma}$

$$\int_{\Gamma_h} z_h \psi_h \, ds = \sum_{\Gamma_h} z_h \psi_h \Delta \mathbf{s}. \tag{7}$$

For  $\mathbf{v}_h = [v_h, z_h]^T, \mathbf{v}'_h = [\varphi_h, \psi_h]^T \in V_h$  we define

$$m_h(\mathbf{v}_h, \boldsymbol{\varphi}_h) = \int_{\Omega_h} v_h \varphi_h \, d\mathbf{x} + \mu \int_{\Gamma_h} z_h \psi_h \, ds, \tag{8a}$$

$$\begin{aligned} a_h(\mathbf{v}_h, \boldsymbol{\varphi}_h) &= c_{\Omega} \int_{\Omega_h} \nabla v_h \cdot \nabla \varphi_h \, d\mathbf{x} + k_{\Omega} \int_{\Omega_h} v_h \varphi_h \, d\mathbf{x} \\ &+ c_{\Gamma} \int_{\Gamma_h} \nabla_{\Gamma} z_h \cdot \nabla_{\Gamma} \psi_h \, ds + k_{\Gamma} \int_{\Gamma_h} z_h \psi_h \, ds, \end{aligned} \tag{8b}$$

$$m_h(\mathcal{D}_h(\mathbf{v}_h), \boldsymbol{\varphi}_h) = \sum_{\Gamma_h} c_{\Omega} (\theta(v_h) - \eta(z_h)) \varphi_h + (dz_h + \rho v_h) \psi_h \Delta \mathbf{s}, \tag{8c}$$

$$m_h(\mathbf{f}_h, \boldsymbol{\varphi}_h) = \int_{\Omega_h} I_{h,\Omega} f_{\Omega} \varphi_h \, d\mathbf{x} + \mu \int_{\Gamma_h} \frac{1}{\mu} I_{h,\Gamma} f_{\Gamma} \psi_h \, ds. \tag{8d}$$

Then, the spatial discretization of (1) is given by: seek  $\mathbf{u}_h : [0, T] \rightarrow V_h$  s.t.

$$\begin{aligned} m_h(\mathbf{u}''_h, \mathbf{v}'_h) + m_h(\mathcal{D}_h(\mathbf{u}'_h), \mathbf{v}'_h) + a_h(\mathbf{u}_h, \mathbf{v}'_h) &= m_h(\mathbf{f}_h, \mathbf{v}'_h), \quad \text{for } t \geq 0, \mathbf{v}'_h \in V_h, \\ \mathbf{u}_h(0) = \mathbf{u}_h^0, \quad \mathbf{u}'_h(0) &= \mathbf{v}_h^0. \end{aligned}$$

**Remark 2.4** The use of the quadrature formulas instead of the interpolation in the definition of the discretized nonlinearity  $\mathcal{D}_h$  is required to prove that  $\mathcal{D}_h$  is quasi-monotone, cf. Lemma 5.3.

To prove an error bound of the discretization we pose the following assumptions on the exact solution and the data:

**Assumption 2.5** a) Let  $T > 0$ . For the inhomogeneities and the nonlinearities in (1) we assume the additional regularity

$$f_{\Omega} \in L^{\infty}([0, T]; H^{\max\{2,p\}}(\Omega)), \quad f_{\Gamma} \in L^{\infty}([0, T]; H^{\max\{2,p\}}(\Gamma)), \tag{9a}$$

$$\theta, \eta \in C^{\max\{2,p\}}(\mathbb{R}; \mathbb{R}). \tag{9b}$$

Furthermore, we assume that the strong solution  $u, \delta$  of (1) satisfies on  $[0, T]$

$$\begin{aligned} u &\in L^\infty([0, T]; H^{p+1}(\Omega)), & u' &\in L^\infty([0, T]; H^{p+2}(\Omega) \cap W^{p+1,\infty}(\Omega)), \\ u'' &\in L^\infty([0, T]; H^{\max\{2,p\}}(\Omega)), \\ \delta &\in L^\infty([0, T]; H^{p+1}(\Gamma)), & \delta' &\in L^\infty([0, T]; H^{p+1}(\Gamma) \cap W^{p,\infty}(\Gamma)), \\ \delta'' &\in L^\infty([0, T]; H^{\max\{2,p\}}(\Gamma)). \end{aligned}$$

b) Let the discrete initial values satisfy

$$\left\| \mathbf{u}_h^0 - I_h \mathbf{u}^0 \right\|_{\mathbb{H}^1} + \left\| \mathbf{v}_h^0 - I_h \mathbf{v}^0 \right\|_{\mathbb{H}^0} \leq C_{iv} h^p$$

with a constant  $C_{iv}$  independent of  $h$ .

As main theorem, we state the following error bound for the finite element discretization of the wave equation with nonlinear acoustic boundary conditions.

**Theorem 2.6** *Let Assumption 2.2 be satisfied and  $\mathbf{u} = [u, \delta]^\top$  be the solution of (1) on  $[0, T]$ . Further, let Assumption 2.5 be satisfied and let  $\mathbf{u}_h = [u_h, \delta_h]^\top$  be the spatial approximation of  $\mathbf{u}$ , obtained with the bulk-surface finite element method of order  $p$ . Then, there exists some  $h_0 > 0$  s.t. the error bound*

$$\left\| \mathbf{u}_h^\ell - \mathbf{u} \right\|_{\mathbb{H}^1} + \left\| (\mathbf{u}'_h)^\ell(t) - \mathbf{u}'(t) \right\|_{\mathbb{H}^0} \leq C e^{(\frac{1}{2}+c')t} (1+t) h^p$$

holds true for all  $h < h_0$ , where  $c' = \frac{1}{\mu} \left( \frac{L_\eta^2 c_\Omega}{4\theta_0} - d \right)$  and  $C$  is a constant independent of  $h$ .

In the next two sections we will now present a general theory for the error analysis of non-conforming space discretizations which we then use to proof Theorem 2.6 in Sect. 5.

### 3 Abstract space discretizations of first-order evolution equations with monotone operators

In this section we present the unified error analysis for abstract space discretizations of first-order evolution equations with maximal monotone operators. This generalizes the results from [16] and [18] for linear and semilinear equations, respectively. The results of this section are part of the dissertation [20].

We first present the continuous equation and the corresponding abstract space discretization, before we prove an error bound.

### 3.1 Analytical setting

Let  $X$  be a Hilbert space with scalar product  $(\cdot, \cdot)_X$  in which we consider the evolution equation

$$x'(t) + \mathcal{S}(x(t)) = g(t), \quad t \geq 0, \tag{10a}$$

$$x(0) = x^0 \in D(\mathcal{S}). \tag{10b}$$

In the following, we omit the  $t$  arguments in evolution equations. We pose the following classical assumptions to ensure that (10) is wellposed.

**Assumption 3.1** a) The nonlinear operator  $\mathcal{S} : D(\mathcal{S}) \rightarrow X$  is quasi-monotone and maximal, i.e., there is a  $c_{\text{qm}} > 0$  s.t.

$$(\mathcal{S}(y) - \mathcal{S}(z), y - z)_X \geq -c_{\text{qm}} \|y - z\|_X^2 \quad \text{for all } y, z \in D(\mathcal{S}),$$

and there exists some  $\lambda > c_{\text{qm}}$  s.t.  $\text{range}(\lambda + \mathcal{S}) = X$ .

b) The inhomogeneity satisfies  $g \in W_{loc}^{1,1}([0, \infty); X)$ .

The following wellposedness result can, e.g., be found in [25, Corollary IV.4.1].

**Theorem 3.2** *Let Assumption 3.1 hold true. Then, the evolution equation (10) is globally wellposed, i.e., (10) has a unique strong solution  $x \in C([0, \infty); X)$  which satisfies  $x(t) \in D(\mathcal{S})$  for all  $t \in [0, \infty)$ ,  $x(0) = x^0$ , and (10a) is satisfied for almost all  $t \in [0, \infty)$ .*

We further state the following stability result which is essential for the latter error analysis.

**Theorem 3.3** *Let Assumption 3.1 be satisfied and for  $T > 0$  and  $i = 1, 2$  let  $x_i$  be the strong solutions of*

$$\begin{aligned} x_i' + \mathcal{S}(x_i) &= g_i, \quad t \in [0, T], \\ x_i(0) &= x_i^0 \in D(\mathcal{S}) \end{aligned}$$

with  $g_i \in W^{1,1}([0, T]; X)$ . Then for all  $t \in [0, T]$

$$\|x_1(t) - x_2(t)\|_X \leq e^{c_{\text{qm}}t} \left( \|x_1^0 - x_2^0\|_X + \int_0^t e^{-c_{\text{qm}}s} \|g_1(s) - g_2(s)\|_X \, ds \right).$$

**Proof** The result can be derived with energy estimates similar to [25, Theorem IV.4.1A]. □

### 3.2 Abstract space discretization

We now present an abstract space discretization of the evolution equation (10). Let  $(X_h)_h$  be a family of finite dimensional vector spaces with scalar products  $(\cdot, \cdot)_{X_h}$ ,



where  $h$  is a discretization parameter, e.g., the maximal mesh width of a finite element discretization. For all  $X_h \in (X_h)_h$  we seek an approximations  $x_h \in X_h$  to the solution  $x$  of (10). Therefore, let  $\mathcal{S}_h$  and  $g_h$  be approximations of  $\mathcal{S}$  and  $g$ , respectively, which satisfy the following assumptions similar to Assumption 3.1.

**Assumption 3.4**

- a) The nonlinear operator  $\mathcal{S}_h : X_h \rightarrow X_h$  is quasi-monotone, i.e., there is a  $\widehat{c}_{qm} > 0$  independent of  $h$  s.t.

$$(\mathcal{S}_h(y_h) - \mathcal{S}_h(z_h), y_h - z_h)_{X_h} \geq -\widehat{c}_{qm} \|y_h - z_h\|_X^2 \quad \text{for all } y_h, z_h \in X_h. \tag{11}$$

- b) The inhomogeneity satisfies  $g_h \in W_{loc}^{1,1}([0; \infty); X_h)$ .

The discretized evolution equation is then given by

$$x'_h + \mathcal{S}_h(x_h) = g_h, \quad t \geq 0, \tag{12a}$$

$$x_h(0) = x_h^0. \tag{12b}$$

Since these assumptions are similar to the continuous case, we obtain by Theorem 3.2 that (12) is globally wellposed.

In the following we introduce a framework for the error analysis of the abstract space discretization that is similar to the linear case presented in [16]. To cover non-conforming space discretizations where  $X_h \not\subseteq X$ , as they appear in Sect. 2, we make the following assumptions to relate the discrete and the continuous problem.

**Assumption 3.5**

- a) There exists a *lift operator*  $\mathcal{L}_h \in \mathcal{L}(X_h, X)$  which satisfies

$$\|\mathcal{L}_h y_h\|_X \leq \widehat{C}_X \|y_h\|_{X_h} \quad \text{for all } y_h \in X_h \tag{13}$$

for some constant  $\widehat{C}_X > 0$  independent of  $h$ . The adjoint of the lift operator  $\mathcal{L}_h^* \in \mathcal{L}(X, X_h)$  is defined via

$$(\mathcal{L}_h^* y, y_h)_{X_h} = (y, \mathcal{L}_h y_h)_X, \quad \text{for all } y \in X, y_h \in X_h.$$

- b) Let  $Z \hookrightarrow X$  be a densely embedded subspace of  $X$  on which a *reference operator*  $J_h \in \mathcal{L}(Z; X_h)$  is defined which satisfies

$$\|J_h\|_{X_h \leftarrow Z} \leq \widehat{C}_{J_h}$$

for some constant  $\widehat{C}_{J_h} > 0$  independent of  $h$ .

As the term  $\mathcal{L}_h J_h - I$  appears in the error bound in Theorem 3.7, the reference operator should be chosen such that the approximation  $\mathcal{L}_h J_h z \approx z$  ( $z \in Z$ ) is of the order of convergence which one wants to proof for the spatial discretization. It could, e.g., be an interpolation or a projection operator. One possible example for a lift operator is the lift defined in (6). We will consider this in Sect. 5.

The space discretization error bound is given in terms of the following terms:

**Definition 3.6** (Remainder and error terms)

a) The remainder of the nonlinear monotone operator is given by

$$R_h : D(\mathcal{S}) \cap Z \rightarrow X_h, \quad R_h(z) := \mathcal{L}_h^* \mathcal{S}(z) - \mathcal{S}_h(J_h z). \tag{14}$$

b) We define the error term

$$E_h(t) = \|x_h^0 - J_h X^0\|_{X_h} + t \|e^{-\widehat{c}_{qm} \cdot} (\mathcal{L}_h^* - J_h)x'\|_{L^\infty([0,t]; X_h)} + t \|e^{-\widehat{c}_{qm} \cdot} R_h(x)\|_{L^\infty([0,t]; X_h)} + t \|e^{-\widehat{c}_{qm} \cdot} \mathcal{L}_h^* g - g_h\|_{L^\infty([0,t]; X_h)}. \tag{15}$$

We now can state and prove an error bound of the abstract space discretization, cf. [20, Thm. 2.10].

**Theorem 3.7** *Let Assumptions 3.1, 3.4, and 3.5 be satisfied and  $x$  be the strong solution of (10) on  $[0, T]$  with  $x, x' \in L^\infty([0, T]; Z)$ . Furthermore, let  $x_h$  be the solution of (12) on  $[0, T]$ . Then, for all  $t \in [0, T]$  the lifted discrete solution satisfies the error bound*

$$\|\mathcal{L}_h x_h(t) - x(t)\|_X \leq \widehat{C}_X e^{\widehat{c}_{qm} t} E_h(t) + \|(I - \mathcal{L}_h J_h)x(t)\|_X. \tag{16}$$

**Proof** We split the error via  $\mathcal{L}_h x_h(t) - x(t) = \mathcal{L}_h e_h + (\mathcal{L}_h J_h - I)x(t)$ , where

$$e_h(t) = x_h(t) - J_h x(t) \in X_h$$

is the discrete error. The full error can thus be bounded by

$$\|\mathcal{L}_h x_h(t) - x(t)\|_X \leq \widehat{C}_X \|e_h\|_{X_h} + \|(\mathcal{L}_h J_h - I)x(t)\|_X \tag{17}$$

and we further investigate the discrete error. By applying the adjoint lift to (10a) we obtain

$$\mathcal{L}_h^* x' + \mathcal{L}_h^* \mathcal{S}(x) = \mathcal{L}_h^* g.$$

Adding  $J_h x', \mathcal{S}_h(J_h x)$ , and  $g_h$  on both sides yields

$$J_h x' + \mathcal{S}_h(J_h x) = g_h + \Delta_h \tag{18}$$

where

$$\Delta_h = (J_h - \mathcal{L}_h^*) x' + \mathcal{S}_h(J_h x) - \mathcal{L}_h^* \mathcal{S}(x) + \mathcal{L}_h^* g - g_h. \tag{19}$$

Under Assumption 3.4, the stability estimate from Theorem 3.3 holds also true in the discrete case with  $\widehat{c}_{qm}$  instead of  $c_{qm}$ . Hence, we obtain by Theorem 3.3 applied to

(12) and (18) the following bound for the discrete error

$$\begin{aligned} \|e_h(t)\|_{X_h} &\leq e^{\widehat{c}_{qm}t} \left( \|x_h^0 - J_h x^0\|_{X_h} + \int_0^t e^{-\widehat{c}_{qm}s} \|\Delta_h(s)\| \, ds \right) \\ &\leq e^{\widehat{c}_{qm}t} \left( \|x_h^0 - J_h x^0\| + t \|e^{-\widehat{c}_{qm}\cdot} \Delta_h\|_{L^\infty([0,T]; X_h)} \right) \\ &\leq e^{\widehat{c}_{qm}t} E_h(t), \end{aligned} \tag{20}$$

where we used (19) and (14). Together with (17), we finally obtain (16). □

In the following section we will use this result to derive error bounds for second-order nonlinear wave-type equations.

### 4 Abstract space discretizations of second-order evolution equations with nonlinear damping

In this section we apply the theory of Sect. 3 to second-order evolution equations. As in the previous section, we first introduce the continuous problem and then present and analyze the abstract space discretization. This is a generalization of the linear unified error analysis introduced in [16] and also an extension of the framework considered in the dissertation [20] which does not cover the acoustic boundary conditions with nonlinear coupling from Sect. 2, cf. Remark 4.2 and Sect. 5.

#### 4.1 Analytical setting

Let  $V, H$  be Hilbert spaces and let  $V$  be densely embedded in  $H$ . We consider the following variational equation, which is typical for a weak formulation of a second-order partial differential equation. Seek  $u \in C^2([0, T]; H) \cap C^1([0, T]; V)$  with

$$\begin{aligned} m(u'', \varphi) + \langle \mathcal{D}(u'), \varphi \rangle_{V^* \times V} + a(u, \varphi) &= m(f, \varphi), \quad \text{for } t \geq 0 \text{ and all } \varphi \in V, \\ u(0) = u^0, \quad u'(0) = v^0, \end{aligned} \tag{21}$$

To ensure the wellposedness of (21) we pose the following assumptions.

#### Assumption 4.1

- a) The bilinear form  $m : H \times H \rightarrow \mathbb{R}$  is a scalar product on  $H$  with induced complete norm  $\|\cdot\|_m$ . In the following, we equip  $H$  with  $m$ .
- b) The bilinear form  $a : V \times V \rightarrow \mathbb{R}$  is symmetric and there exists a constant  $c_G \geq 0$  s.t.

$$\tilde{a} := a + c_G m$$

is a scalar product on  $V$  with induced complete norm  $\|\cdot\|_{\tilde{a}}$ . From now on, we equip  $V$  with  $\tilde{a}$ .

- c) The nonlinearity  $\mathcal{D} \in C(V; V^*)$  satisfies  $\mathcal{D}(0) = 0$  and is quasi-monotone, i.e., there is a constant  $\beta_{\text{qm}} \geq 0$  s.t.

$$\langle \mathcal{D}(v) - \mathcal{D}(w), v - w \rangle_{V^* \times V} \geq -\beta_{\text{qm}} \|v - w\|_m \quad \text{for all } v, w \in V.$$

- d) The inhomogeneity satisfies  $f \in W_{loc}^{1,1}([0, \infty); H)$ .

We denote by  $C_{H,V}$  the embedding constant of  $V$  into  $H$ , i.e.,

$$\|v\| \leq C_{H,V} \|v\|_{\tilde{a}} \quad \text{for all } v \in V. \tag{22}$$

*Formulation as evolution equation* We identify  $H$  with its dual space  $H^*$  to obtain the Gelfand triple

$$V \hookrightarrow H \cong H^* \hookrightarrow V^* \tag{23}$$

with dense embeddings. We thus have for all  $v \in V, w \in H$

$$m(v, w) = \langle w, v \rangle_{V^* \times V}.$$

To reformulate (21) as an evolution equation, we define the operator  $\mathcal{A} \in \mathcal{L}(V, V^*)$  associated to  $a$  via

$$\langle \mathcal{A}v, w \rangle_{V^* \times V} := a(v, w) \quad \text{for all } v, w \in V. \tag{24}$$

Then, we can rewrite (21) equivalently as an evolution equation in  $V^*$ : Seek  $u \in C^2([0, T]; H) \cap C^1([0, T]; V)$  satisfying

$$\begin{aligned} u'' + \mathcal{D}(u') + \mathcal{A}u &= f, & t \geq 0, \\ u(0) = u^0, \quad u'(0) &= v^0. \end{aligned} \tag{25}$$

Note that (25) implicitly contains the condition

$$\mathcal{D}(u') + \mathcal{A}u \in H$$

due to  $u'', f \in H$ .

**Remark 4.2** In [20], the stricter assumption  $\mathcal{D} \in C(V; H)$  was posed. However, this does not cover the acoustic boundary conditions with nonlinear coupling (1c) as we will see in Sect. 5, cf. Remark 5.2.

*First-order formulation* We rewrite (25) into a first-order formulation in the framework of Sect. 3.1. For this let  $u' = v$  and we define

$$x = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \mathfrak{S}(x) = \begin{bmatrix} -v \\ \mathcal{A}u + \mathcal{D}(v) \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ f \end{bmatrix}, \quad x^0 = \begin{bmatrix} u^0 \\ v^0 \end{bmatrix} \tag{26}$$

with

$$X = V \times H, \quad D(\mathcal{S}) = \left\{ [u, v]^T \in V \times V \mid \mathcal{A}u + \mathcal{D}(v) \in H \right\}. \quad (27)$$

Then, (25) is equivalent to the first-order evolution equation (10).

In the following we show that the assumptions of Sect. 3.1 are satisfied. The subsequent lemma is a slight extension of [20, Lemma 2.14].

**Lemma 4.3** *The nonlinear operator  $\mathcal{S}$  is maximal and quasi-monotone with constant*

$$c_{\text{qm}} = \frac{1}{2}c_G C_{H,V} + \beta_{\text{qm}}$$

and  $D(\mathcal{S})$  is dense in  $X$ .

**Proof** We start by proving the quasi-monotonicity. For  $x_1 = [u_1, v_1]^T, x_2 = [u_2, v_2]^T \in D(\mathcal{S})$  we calculate by using Assumption 4.1, (23), and the definitions of  $\mathcal{S}$  and  $\mathcal{A}$

$$\begin{aligned} & (\mathcal{S}(x_1) - \mathcal{S}(x_2), x_1 - x_2)_X \\ &= -\tilde{a}(v_1 - v_2, u_1 - u_2) + m(\mathcal{A}u_1 + \mathcal{D}(v_1) - \mathcal{A}u_2 - \mathcal{D}(v_2), v_1 - v_2) \\ &= -\tilde{a}(v_1 - v_2, u_1 - u_2) + a(u_1 - u_2, v_1 - v_2) + (\mathcal{D}(v_1) - \mathcal{D}(v_2), v_1 - v_2)_{V^* \times V} \\ &\geq -c_G m(v_1 - v_2, u_1 - u_2) - \beta_{\text{qm}} \|v_1 - v_2\|_m^2 \\ &\geq -c_G \|v_1 - v_2\|_m \|u_1 - u_2\|_m - \beta_{\text{qm}} \|v_1 - v_2\|_m^2 \\ &\geq -c_G C_{H,V} \|u_1 - u_2\|_{\tilde{a}} \|v_1 - v_2\|_m - \beta_{\text{qm}} \|v_1 - v_2\|_m^2 \\ &\geq -\frac{1}{2}c_G C_{H,V} \left( \|u_1 - u_2\|_{\tilde{a}}^2 + \|v_1 - v_2\|_m^2 \right) - \beta_{\text{qm}} \|v_1 - v_2\|_m^2 \\ &\geq -\left( \frac{1}{2}c_G C_{H,V} + \beta_{\text{qm}} \right) \|x_1 - x_2\|_X^2. \end{aligned}$$

In the next step we prove the maximality and proceed similar as in the proof of [27, Theorem 4.1]. We have to show that there exists a  $\lambda > 0$  such that for every  $h = [h_1, h_2]^T \in X = V \times H$  there exists a solution  $x = [v, w]^T \in D(\mathcal{S})$  of the stationary problem  $(\lambda + \mathcal{S})x = h$  or equivalently

$$\lambda v - w = h_1, \quad (28a)$$

$$\lambda w + \mathcal{A}v + \mathcal{D}(w) = h_2. \quad (28b)$$

By solving (28a) for  $v$  and plugging it into (28b) we obtain

$$\lambda w + \frac{1}{\lambda} \mathcal{A}w + \mathcal{D}(w) = h_2 - \frac{1}{\lambda} \mathcal{A}h_1 := \tilde{h} \in V^*. \quad (29)$$

We thus investigate the operator  $T = \lambda + \frac{1}{\lambda}\mathcal{A} + \mathcal{D} \in C(V; V^*)$  which can be decomposed via  $T = T_1 + T_2$  with

$$T_1 = \frac{1}{\lambda} \left( \frac{\lambda^2}{2} + \mathcal{A} \right), \quad T_2 = \frac{\lambda}{2} + \mathcal{D}.$$

For

$$\lambda > \max\{c_{qm}, \sqrt{2c_G}, 2\beta_{qm}\}$$

we then have that  $T$  is monotone as the sum of monotone operators. Further, we have for all  $v \in V$

$$\begin{aligned} \langle T(v), v \rangle_{V^* \times V} &= \langle T_1(v), v \rangle_{V^* \times V} + \langle T_2(v), v \rangle_{V^* \times V} \\ &\geq \frac{1}{\lambda} \|v\|_{\tilde{a}}^2 + \langle T_2(v) - T_2(0), v - 0 \rangle_{V^* \times V} \\ &\geq \frac{1}{\lambda} \|v\|_{\tilde{a}}^2, \end{aligned}$$

where we used that  $T_1$  is coercive due to the choice of  $\lambda$ , and  $T_2$  is monotone with  $T_2(0) = 0$ . Thus,  $T$  is coercive, i.e.

$$\frac{\langle T(v), v \rangle_{V^* \times V}}{\|v\|_{\tilde{a}}} \rightarrow \infty \quad \text{for } \|v\|_{\tilde{a}} \rightarrow \infty.$$

We apply [3, Corollary 2.3] stating that continuous, monotone, and coercive operators from a reflexive Banach space to its dual space are surjective. This yields the existence of a solution  $v \in V$  of (29) and thus also of a solution  $x = [v, w]^T \in V \times V$  of (28). We further obtain by (28b)  $x \in D(\mathcal{S})$  since

$$\mathcal{A}v + \mathcal{D}(w) = h_2 - \lambda w \in H.$$

The density of  $D(\mathcal{S})$  in  $X$  follows from the maximality and the quasi-monotonicity of  $\mathcal{S}$  and  $\mathcal{S}(0) = 0$ , cf [25, Prop. I.4.2]. □

**Corollary 4.4** *Assumption 4.1 implies that the first-order formulation of (25) satisfies Assumption 3.1.*

**Proof** By Lemma 4.3 we have that Assumption 3.1 a) is satisfied. Assumption 3.1 b) is directly implied by Assumption 4.1 d). □

By Theorem 3.2 we then directly obtain the wellposedness of (21).

**Corollary 4.5** *Let Assumption 4.1 hold true and let  $[u^0, v^0]^T \in D(\mathcal{S})$ , i.e.,  $u^0, v^0 \in V$  with  $\mathcal{A}u^0 + \mathcal{D}(v^0) \in H$ . Then, (21) is globally wellposed, i.e., there exists a unique strong solution  $[u, v]^T \in C([0, \infty); V \times H)$ .*

### 4.2 Space discretization

We consider a family  $(V_h)_h$  of finite dimensional vector spaces related to a discretization parameter  $h$  and the following discretized version of (21) in  $V_h \in (V_h)_h$ : seek  $u_h \in C^2([0, T]; V_h)$  with

$$m_h(u_h'', \varphi_h) + m_h(\mathcal{D}_h(u_h'), \varphi_h) + a_h(u_h, \varphi_h) = m_h(f_h, \varphi_h), \quad \text{for all } \varphi_h \in V_h, t \geq 0, \\ u_h(0) = u_h^0, \quad u_h'(0) = v_h^0. \tag{30}$$

Here,  $m_h, a_h, \mathcal{D}_h,$  and  $f_h$  are approximations of the corresponding continuous counterparts.

We pose the following assumptions similar to Assumption 4.1.

**Assumption 4.6** All constants in the following statements are independent of  $h$ .

- a) The bilinear form  $m_h$  is a scalar product on  $V_h$ . We denote  $V_h$  equipped with this scalar product  $m_h$  by  $H_h$  and the induced norm by  $\|\cdot\|_{m_h}$ .
- b) The bilinear form  $a_h: V_h \times V_h \rightarrow \mathbb{R}$  is symmetric and there exists a constant  $\widehat{c}_G \geq 0$  s.t.

$$\widetilde{a}_h := a_h + \widehat{c}_G m_h$$

is a scalar product on  $V_h$  with induced norm  $\|\cdot\|_{\widetilde{a}_h}$ . In the following, we equip  $V_h$  with  $\widetilde{a}_h$ .

- c) The nonlinearity  $\mathcal{D}_h \in C(V_h; H_h)$  satisfies  $\mathcal{D}_h(0) = 0$  and is continuous and quasi-monotone with constant  $\widehat{\beta}_{qm}$ .
- d) The inhomogeneity satisfies  $f_h \in W_{loc}^{1,1}([0, \infty); H_h)$ .
- e) There exists a constant  $\widehat{C}_{H,V} > 0$  s.t.

$$\|v_h\|_{m_h} \leq \widehat{C}_{H,V} \|v_h\|_{\widetilde{a}_h} \quad \text{for all } v_h \in V_h. \tag{31}$$

The operator  $\mathcal{A}_h \in \mathcal{L}(V_h; V_h)$  related to  $a_h$  is defined via

$$m_h(\mathcal{A}_h v_h, w_h) := a_h(v_h, w_h) \quad \text{for all } v_h, w_h \in V_h.$$

We then can reformulate (30) as an evolution equation in  $V_h$ :

$$u_h'' + \mathcal{D}_h(u_h') + \mathcal{A}_h u_h = f_h, \quad t \geq 0, \\ u_h(0) = u_h^0, \quad u_h'(0) = v_h^0. \tag{32}$$

Analogously to the continuous equation we rewrite (32) in a first-order formulation and therefore define  $X_h = V_h \times H_h$ . With

$$x_h = \begin{bmatrix} u_h \\ v_h \end{bmatrix}, \quad \mathcal{S}_h(x_h) = \begin{bmatrix} -v_h \\ \mathcal{A}_h u_h + \mathcal{D}_h(v_h) \end{bmatrix}, \quad g_h = \begin{bmatrix} 0 \\ f_h \end{bmatrix}, \quad x_h^0 = \begin{bmatrix} u_h^0 \\ v_h^0 \end{bmatrix}, \tag{33}$$

(32) is then of the form (12).

**Corollary 4.7** *Assumption 4.6 implies that the first-order formulation of (32) satisfies Assumption 3.4. Furthermore, (11) holds true with  $\widehat{c}_{\text{qm}} = \frac{1}{2}\widehat{c}_G\widehat{C}_{H,V} + \widehat{\beta}_{\text{qm}}$ .*

**Proof** Since the setting in the discrete case from Assumption 4.6 is similar to the continuous one from Assumption 4.1 with constants independent of  $h$ , the proof of Lemma 4.3 transfers directly to the discrete case.  $\square$

Similar to the first-order case, we require the existence of suitable operators to relate continuous and discrete functions of the abstract non-conforming space discretization.

**Assumption 4.8**

a) There exists a lift operator  $\mathcal{L}_h^V \in \mathcal{L}(V_h; V)$  satisfying

$$\|\mathcal{L}_h^V v_h\|_m \leq \widehat{C}_H \|v_h\|_{m_h}, \quad \|\mathcal{L}_h^V v_h\|_{\tilde{a}} \leq \widehat{C}_V \|v_h\|_{\tilde{a}_h}, \tag{34}$$

for all  $v_h \in V_h$  with constants  $\widehat{C}_H, \widehat{C}_V > 0$  independent of  $h$ .

b) There exists an interpolation operator  $I_h \in \mathcal{L}(Z^V; V_h)$ , defined on a dense subspace  $Z^V$  of  $V$ , which satisfies

$$\|I_h\|_{H_h \leftarrow Z^V} \leq \widehat{C}_{I_h} \tag{35}$$

with a constant  $\widehat{C}_{I_h} > 0$  independent of  $h$ .

To apply the results of Sect. 3.2, we now define the first-order reference and lift operator.

**Definition 4.9**

a) The *adjoint lift operators*  $\mathcal{L}_h^{V*}: V \rightarrow V_h$  and  $\mathcal{L}_h^{H*}: H \rightarrow H_h$  w.r.t. the scalar products of  $V$  and  $H$  are defined via

$$\begin{aligned} m_h(\mathcal{L}_h^{H*} v, w_h) &:= m(v, \mathcal{L}_h^V w_h) && \text{for all } v \in H, w_h \in H_h, \\ \tilde{a}_h(\mathcal{L}_h^{V*} v, w_h) &:= \tilde{a}(v, \mathcal{L}_h^V w_h) && \text{for all } v \in V, w_h \in V_h. \end{aligned} \tag{36}$$

b) We define the *first-order lift operator*  $\mathcal{L}_h: X_h \rightarrow X$  by

$$\mathcal{L}_h \begin{bmatrix} v_h \\ w_h \end{bmatrix} := \begin{bmatrix} \mathcal{L}_h^V v_h \\ \mathcal{L}_h^V w_h \end{bmatrix}.$$

c) We define the *first-order reference operator*  $J_h: Z \rightarrow X_h$  by

$$J_h \begin{bmatrix} v \\ w \end{bmatrix} := \begin{bmatrix} \mathcal{L}_h^{V*} v \\ I_h w \end{bmatrix} \tag{37}$$

on  $Z = V \times Z^V \xrightarrow{d} X$ .



**Lemma 4.10** *The first-order lift and reference operators from Definition 4.9 satisfy Assumption 3.5 with  $\widehat{C}_X = \max\{\widehat{C}_V, \widehat{C}_H\}$  and  $\widehat{C}_{J_h} = \max\{\widehat{C}_V, \widehat{C}_{I_h}\}$ .*

**Proof** This is a direct consequence of Assumption 4.8. □

In the following we now bound the first-order remainder term which is for  $z = [v, w]^T \in D(\mathcal{S}) \cap Z$  given by

$$R_h(z) = \mathcal{L}_h^* \mathcal{S}(z) - \mathcal{S}_h J_h(z) = \left[ \begin{array}{c} -(\mathcal{L}_h^{V*} - I_h)w \\ \mathcal{L}_h^{H*}(Av + \mathcal{D}(w)) - (A_h \mathcal{L}_h^{V*} v + \mathcal{D}_h(I_h w)) \end{array} \right]. \tag{38}$$

To do so, we use the following error terms in the scalar products, which are for  $v_h, w_h \in V_h$  defined via

$$\begin{aligned} \Delta m(v_h, w_h) &:= m(\mathcal{L}_h^V v_h, \mathcal{L}_h^V w_h) - m_h(v_h, w_h), \\ \Delta \tilde{a}(v_h, w_h) &:= \tilde{a}(\mathcal{L}_h^V v_h, \mathcal{L}_h^V w_h) - \tilde{a}_h(v_h, w_h). \end{aligned} \tag{39}$$

We obtain the following bound for the remainder term, cf.[20, Lem. 2.23]

**Lemma 4.11** *Let Assumption 4.1 and 4.6 be satisfied. Then, for  $z = [v, w]^T \in D(\mathcal{S}) \cap Z$ , the remainder of the monotone operator can be bounded by*

$$\begin{aligned} \|R_h(z)\|_{X_h} \leq C &\left( \max_{\|\varphi_h\|_{\tilde{a}_h}=1} |\Delta \tilde{a}(I_h w, \varphi_h)| + \max_{\|\varphi_h\|_{\tilde{a}_h}=1} |\Delta \tilde{a}(I_h v, \varphi_h)| \right. \\ &+ \max_{\|\psi_h\|_{m_h}=1} |\Delta m(I_h v, \psi_h)| + \|(\mathbf{I} - \mathcal{L}_h^V I_h)v\|_{\tilde{a}} + \|(\mathbf{I} - \mathcal{L}_h^V I_h)w\|_{\tilde{a}} \\ &\left. + \max_{\|\psi_h\|_{m_h}=1} \left| \langle \mathcal{D}(w), \mathcal{L}_h^V \psi_h \rangle_{V^* \times V} - m_h(\mathcal{D}_h(I_h w), \psi_h) \right| \right), \end{aligned} \tag{40}$$

*i.e., against errors in the scalar products, interpolation errors, and the discretization error of the nonlinear operator.*

**Proof** The proof works similar to the proof of [16, Lemma 4.7] and relies on the identity

$$\|R_h(z)\|_{X_h} = \max_{\|y_h\|_{X_h}=1} (R_h(z), y_h)_{X_h},$$

where  $(\cdot, \cdot)_{X_h}$  is the scalar product on  $X_h$ . Thus, let  $y_h = [\varphi_h, \psi_h]^T \in X_h$  with  $\|y_h\|_{X_h} = 1$ . By (38) we obtain

$$\begin{aligned} (R_h(z), y_h)_{X_h} &= -\tilde{a}_h((\mathcal{L}_h^{V*} - I_h)w, \varphi_h) + m_h(\mathcal{L}_h^{H*}(Av + \mathcal{D}(w)) - (A_h \mathcal{L}_h^{V*} v + \mathcal{D}_h(I_h w)), \psi_h) \\ &= -\left(\tilde{a}(w, \mathcal{L}_h^V \varphi_h) - \tilde{a}_h(I_h w, \varphi_h)\right) + \left(a(v, \mathcal{L}_h^V \psi_h) - a_h(\mathcal{L}_h^{V*} v, \psi_h)\right) \\ &\quad + \langle \mathcal{D}(w), \mathcal{L}_h^V \psi_h \rangle_{V^* \times V} - m_h(\mathcal{D}_h(I_h w), \psi_h), \end{aligned}$$

(41)

and we bound the first two summands separately. To bound the first one, we use (39), (34), and  $\|\varphi_h\|_{\tilde{a}_h} \leq 1$  to obtain

$$\begin{aligned} \left| \tilde{a}(w, \mathcal{L}_h^V \varphi_h) - \tilde{a}_h(I_h w, \varphi_h) \right| &\leq \left| \tilde{a}(w, \mathcal{L}_h^V \varphi_h) - \tilde{a}(\mathcal{L}_h^V I_h w, \mathcal{L}_h^V \varphi_h) \right| + \left| \Delta \tilde{a}(I_h w, \varphi_h) \right| \\ &\leq \left\| (\mathbf{I} - \mathcal{L}_h^V I_h) w \right\|_{\tilde{a}} \|\mathcal{L}_h^V \varphi_h\|_{\tilde{a}} + \left| \Delta \tilde{a}(I_h w, \varphi_h) \right| \\ &\leq \widehat{C}_V \left\| (\mathbf{I} - \mathcal{L}_h^V I_h) w \right\|_{\tilde{a}} + \max_{\|\varphi_h\|_{\tilde{a}_h}=1} \left| \Delta \tilde{a}(I_h w, \varphi_h) \right|. \end{aligned} \tag{42}$$

By using the definitions of  $\tilde{a}$ ,  $\tilde{a}_h$ ,  $\|\psi_h\|_{m_h} \leq 1$  and (39), (22), (34), (31), we bound the second summand in (41) via

$$\begin{aligned} &a(v, \mathcal{L}_h^V \psi_h) - a_h(\mathcal{L}_h^{V*} v, \psi_h) \\ &= \tilde{a}(v, \mathcal{L}_h^V \psi_h) - \tilde{a}_h(\mathcal{L}_h^{V*} v, \psi_h) - \left( c_G m(v, \mathcal{L}_h^V \psi_h) - \widehat{c}_G m_h(\mathcal{L}_h^{V*} v, \psi_h) \right) \\ &\leq \max\{c_G, \widehat{c}_G\} \left| m(v, \mathcal{L}_h^V \psi_h) - m_h(\mathcal{L}_h^{V*} v, \psi_h) \right| \\ &\leq \max\{c_G, \widehat{c}_G\} \left( \left| m((\mathbf{I} - \mathcal{L}_h^V I_h)v, \mathcal{L}_h^V \psi_h) \right| + \left| \Delta m(I_h v, \psi_h) \right| \right. \\ &\quad \left. + m_h((I_h - \mathcal{L}_h^{V*})v, \psi_h) \right) \\ &\leq \max\{c_G, \widehat{c}_G\} \left( \widehat{C}_H C_{H,V} \left\| (\mathbf{I} - \mathcal{L}_h^V I_h)v \right\|_{\tilde{a}} + \max_{\|\psi_h\|_{m_h}=1} \left| \Delta m(I_h v, \psi_h) \right| \right. \\ &\quad \left. + \widehat{C}_{H,V} \left\| (I_h - \mathcal{L}_h^{V*})v \right\|_{\tilde{a}_h} \right). \end{aligned}$$

Similar to (42), we further estimate

$$\begin{aligned} \left\| (I_h - \mathcal{L}_h^{V*})v \right\|_{\tilde{a}_h} &= \max_{\|\varphi_h\|_{\tilde{a}_h}=1} \tilde{a}_h((I_h - \mathcal{L}_h^{V*})v, \varphi_h) \\ &= \max_{\|\varphi_h\|_{\tilde{a}_h}=1} \tilde{a}_h(I_h v, \varphi_h) - \tilde{a}(v, \mathcal{L}_h^V \varphi_h) \\ &\leq \widehat{C}_V \left\| (\mathbf{I} - \mathcal{L}_h^V I_h)v \right\|_{\tilde{a}} + \max_{\|\varphi_h\|_{\tilde{a}_h}=1} \left| \Delta \tilde{a}(I_h v, \varphi_h) \right|. \end{aligned}$$

We finally obtain the assertion by collecting all terms. □

We are now in the position to prove the following error bound which is a generalization of [20, Thm. 2.24], cf. Remark 4.2. It is applicable to all equations and space discretizations fitting in the abstract framework of this section. In the dissertation [20], it was used to prove novel convergence rates for the wave equation with nonlinear kinetic boundary conditions and in this paper, we use it to prove Theorem 2.6.

**Theorem 4.12** *Let Assumptions 4.1, 4.6, and 4.8 be satisfied and  $u$  be the strong solution of (25) on  $[0, T]$  with  $u, u', u'' \in L^\infty([0, T]; Z^V)$ . Further, let  $u_h$  be the semidiscrete solution of (32) on  $[0, T]$ . Then, for all  $t \in [0, T]$ , the lifted semidiscrete solution satisfies the error bound*

$$\|\mathcal{L}_h^V u_h(t) - u(t)\|_{\tilde{a}} + \|\mathcal{L}_h^V u'_h(t) - u'(t)\|_m \leq C e^{\widehat{c}_{qm}t} (1+t) \sum_{i=1}^4 E_{h,i} \tag{43}$$

with a constant  $C$  that is independent of  $h$  and  $t$ . The other constants are given by

$$\widehat{c}_{qm} = \frac{1}{2} \widehat{c}_G \widehat{C}_{H,V} + \widehat{\beta}_{qm},$$

and the abstract space discretization errors

$$\begin{aligned} E_{h,1} &= \|u_h^0 - \mathcal{L}_h^{V*} u^0\|_{\tilde{a}_h} + \|v_h^0 - I_h v^0\|_{m_h} + \|e^{-\widehat{c}_{qm}\cdot} (\mathcal{L}_h^{H*} f - f_h)\|_{L^\infty([0,T];H)}, \\ E_{h,2} &= \|e^{-\widehat{c}_{qm}\cdot} \max_{\|\psi_h\|_{m_h}=1} |\langle \mathcal{D}(u'), \mathcal{L}_h^V \psi_h \rangle_{V^* \times V} - m_h(\mathcal{D}_h(I_h u'), \psi_h)|\|_{L^\infty(0,T)}, \\ E_{h,3} &= \|e^{-\widehat{c}_{qm}\cdot} (\mathbf{I} - \mathcal{L}_h^V I_h)u\|_{L^\infty([0,T];V)} + \|e^{-\widehat{c}_{qm}\cdot} (\mathbf{I} - \mathcal{L}_h^V I_h)u'\|_{L^\infty([0,T];V)} \\ &\quad + \|e^{-\widehat{c}_{qm}\cdot} (\mathbf{I} - \mathcal{L}_h^V I_h)u''\|_{L^\infty([0,T];H)}, \\ E_{h,4} &= \|e^{-\widehat{c}_{qm}\cdot} \max_{\|\varphi_h\|_{\tilde{a}_h}=1} \Delta \tilde{a}(I_h u, \varphi_h)\|_{L^\infty(0,T)} \tag{44} \\ &\quad + \|e^{-\widehat{c}_{qm}\cdot} \max_{\|\psi_h\|_{m_h}=1} \Delta m(I_h u, \psi_h)\|_{L^\infty(0,T)} \\ &\quad + \|e^{-\widehat{c}_{qm}\cdot} \max_{\|\varphi_h\|_{\tilde{a}_h}=1} \Delta \tilde{a}(I_h u', \varphi_h)\|_{L^\infty(0,T)} \\ &\quad + \|e^{-\widehat{c}_{qm}\cdot} \max_{\|\psi_h\|_{m_h}=1} \Delta m(I_h u'', \psi_h)\|_{L^\infty(0,T)}. \end{aligned}$$

**Proof** By Corollaries 4.4, 4.7, and Lemma 4.10, we have that the first-order formulations of (25) and (32) satisfy all assumptions of Theorem 3.7.

By applying Theorem 3.7 and employing the error bound (16), we obtain

$$\begin{aligned} &\|\mathcal{L}_h^V u_h(t) - u(t)\|_{\tilde{a}} + \|\mathcal{L}_h^V u'_h(t) - u'(t)\|_m^2 \\ &\leq 2 \left( \|\mathcal{L}_h^V u_h(t) - u(t)\|_{\tilde{a}}^2 + \|\mathcal{L}_h^V u'_h(t) - u'(t)\|_m^2 \right)^{\frac{1}{2}} \\ &= 2 \|\mathcal{L}_h x_h(t) - x(t)\|_X \\ &\leq 2 \widehat{C}_X e^{(\widehat{L}_{T,M_h} + \widehat{c}_{qm})t} E_h(t) + 2 \|(\mathbf{I} - \mathcal{L}_h J_h)x(t)\|_X \end{aligned}$$

with

$$E_h(t) = \|x_h^0 - J_h x^0\|_{X_h(+)} t \|e^{-\widehat{c}_{qm}\cdot} (\mathcal{L}_h^* - J_h)x'\|_{L^\infty([0,T];X_h)}$$

$$+ t \|e^{-\widehat{c}_{qm} \cdot} R_h(x)\|_{L^\infty([0, T]; X_h)} + t \|e^{-\widehat{c}_{qm} \cdot} (\mathcal{L}_h^* g - g_h)\|_{L^\infty([0, T]; X_h)}.$$

In the remaining proof, we bound the different terms against  $E_{h,i}$ ,  $i = 1, \dots, 4$ . For the remainder term we apply the bound (40) and obtain for all  $t \in [0, T]$

$$\|R_h(x(t))\|_{X_h} \leq C(E_{h,2} + E_{h,3} + E_{h,4}),$$

By the definitions of  $J_h$  and  $\mathcal{L}_h^*$  we further have for the discretization errors of the initial values and the inhomogeneity

$$\|x_h^0 - J_h x^0\|_{x_h(+)} \|\mathcal{L}_h^* g - g_h\|_{L^\infty([0, T]; X_h)} \leq C E_{h,1}.$$

The reference error can be decomposed for all  $t \in [0, T]$  via

$$\begin{aligned} \|(I - \mathcal{L}_h J_h)x(t)\|_X &\leq \|(I - \mathcal{L}_h^V \mathcal{L}_h^{V*})u(t)\|_{\tilde{a}} + \|(I - \mathcal{L}_h^V I_h)u'(t)\|_m \|(I - \mathcal{L}_h^V I_h)u'(t)\| \\ &\leq \|(I - \mathcal{L}_h^V \mathcal{L}_h^{V*})u(t)\|_{\tilde{a}} + e^{\widehat{c}_{qm} t} E_{h,3}, \end{aligned}$$

where we have similar to (42)

$$\begin{aligned} \|(I - \mathcal{L}_h^V \mathcal{L}_h^{V*})u\|_{\tilde{a}} &\leq \|(I - \mathcal{L}_h^V I_h)u\|_{\tilde{a}} + \|\mathcal{L}_h^V (I_h - \mathcal{L}_h^{V*})u\|_{\tilde{a}} \\ &\leq C E_{h,3} + \widehat{C}_V \max_{\|\varphi_h\|_{\tilde{a}_h}=1} \left( \tilde{a}_h(I_h u, \varphi_h) - \tilde{a}(u, \mathcal{L}_h^V \varphi_h) \right) \\ &\leq C E_{h,3} + \widehat{C}_V^2 \|(I - \mathcal{L}_h^V I_h)u\|_{\tilde{a}} + \widehat{C}_V \max_{\|\varphi_h\|_{\tilde{a}_h}=1} |\Delta \tilde{a}(I_h u, \varphi_h)| \\ &\leq C e^{\widehat{c}_{qm} t} (E_{h,3} + E_{h,4}). \end{aligned}$$

In the same way, we finally bound

$$\begin{aligned} \|(\mathcal{L}_h^* - J_h)x'\|_{x_h} &\leq \|(\mathcal{L}_h^{H*} - I_h)u''\|_{m_h} \\ &\leq \widehat{C}_H \|(I - \mathcal{L}_h^V I_h)u''\|_m \|(I - \mathcal{L}_h^V I_h)u''\| \\ &\quad + \max_{\|\psi_h\|_{m_h}=1} |\Delta m(I_h u'', \psi_h)| \\ &\leq C e^{\widehat{c}_{qm} t} (E_{h,3} + E_{h,4}). \end{aligned}$$

□

Having this abstract theory at hand, we can now return to the wave equation with nonlinear acoustic boundary conditions from Sect. 2 and give the proof of Theorem 2.6 in the next section.

### 5 Numerical analysis of wave equations with nonlinear acoustic boundary conditions

In this section we will use the unified error analysis for second-order equations from Sect. 4 to prove the error bound from Theorem 2.6. We start by verifying that all assumptions are satisfied.

**Lemma 5.1** *Let Assumption 2.2 be satisfied. Then, with the definitions in (5), Assumption 4.1 is satisfied with  $\beta_{qm} = \frac{1}{\mu} \left( \frac{L_\eta^2 c_\Omega}{4\theta_0} - d \right)$ ,  $c_G = 1$ , and  $C_{H,V} = 1$ .*

**Proof** We clearly have that  $m$  is a scalar product on  $H$  and that  $\tilde{a} := a + m$  is a scalar product on  $V$ . Further, Assumption 2.2 d) directly implies Assumption 4.1 d).

Thus it remains to prove Assumption 4.1 c). By Assumption 2.2 a), b), c) and (5a), (5c) we obtain

$$\begin{aligned} & \langle \mathcal{D}(\mathbf{v}_1) - \mathcal{D}(\mathbf{v}_2), \mathbf{v}_1 - \mathbf{v}_2 \rangle_{V^* \times V} \\ &= \int_\Gamma c_\Omega (\theta(v_1) - \theta(v_2) - (\eta(z_1) - \eta(z_2))) (v_1 - v_2) \\ &\quad + (d(z_1 - z_2) + \rho(v_1 - v_2)) (z_1 - z_2) \, ds \\ &\geq \int_\Gamma c_\Omega \theta_0 |v_1 - v_2|^2 + d |z_1 - z_2|^2 \\ &\quad + c_\Omega \left( \frac{\rho}{c_\Omega} (z_1 - z_2) - (\eta(z_1) - \eta(z_2)) \right) (v_1 - v_2) \, ds \\ &\geq \int_\Gamma c_\Omega \theta_0 |v_1 - v_2|^2 + d |z_1 - z_2|^2 - c_\Omega L_\eta |(z_1 - z_2)(v_1 - v_2)| \, ds \\ &\geq \int_\Gamma c_\Omega \theta_0 |v_1 - v_2|^2 + d |z_1 - z_2|^2 - c_\Omega \theta_0 |v_1 - v_2|^2 - \frac{L_\eta^2 c_\Omega}{4\theta_0} |z_1 - z_2|^2 \, ds \\ &\geq \left( d - \frac{L_\eta^2 c_\Omega}{4\theta_0} \right) \|z_1 - z_2\|_{L^2(\Gamma)}^2 \\ &\geq -\beta_{qm} \|\mathbf{v}_1 - \mathbf{v}_2\|_m^2. \end{aligned}$$

This proves the the quasi-monotonicity of  $\mathcal{D}$ .

In the next step we show  $\mathcal{D} \in C(V; V^*)$ . We emphasize that the trace inequality

$$v \mapsto v|_\Gamma \in C(H^1(\Omega); L^q(\Gamma)) \tag{45}$$

holds true for  $q = \zeta + 1$  with  $\zeta$  from the growth condition (3), cf.[24, Thms. 2.4.2 and 2.4.6]. For  $\mathbf{v}_1 = [v_1, z_1]^\top$ ,  $\mathbf{v}_2 = [v_2, z_2]^\top$ ,  $\boldsymbol{\varphi} = [\varphi, \psi]^\top \in V$  with  $\|\boldsymbol{\varphi}\|_{\tilde{a}} = 1$  this yields together with the Hölder and the Minkowski inequalities and the global

Lipschitz continuity of  $\eta$

$$\begin{aligned}
 & \left| \langle \mathcal{D}(\mathbf{v}_2) - \mathcal{D}(\mathbf{v}_2), \boldsymbol{\varphi} \rangle_{V^* \times V} \right| \\
 &= \left| \int_{\Gamma} c_{\Omega} (\theta(v_1) - \theta(v_2) - (\eta(z_1) - \eta(z_2))) \boldsymbol{\varphi} + (d(z_1 - z_2) + \rho(v_1 - v_2)) \psi \, ds \right| \\
 &\leq c_{\Omega} \left( \|\theta(v_1) - \theta(v_2)\|_{L^{\frac{q}{q-1}}(\Gamma)} + \|\eta(z_1) - \eta(z_2)\|_{L^{\frac{q}{q-1}}(\Gamma)} \right) \|\boldsymbol{\varphi}\|_{L^q(\Gamma)} \\
 &\quad + \left( d \|z_1 - z_2\|_{L^2(\Gamma)} + \rho \|v_1 - v_2\|_{L^2(\Gamma)} \right) \|\psi\|_{L^2(\Gamma)} \\
 &\leq c_{\Omega} \left( \|\theta(v_1) - \theta(v_2)\|_{L^{\frac{q}{q-1}}(\Gamma)} + C \|z_1 - z_2\|_{L^2(\Gamma)} \right) \max\{1, \frac{1}{c_{\Omega}}\} \|\boldsymbol{\varphi}\|_{\bar{a}} \\
 &\quad + \left( d \|z_1 - z_2\|_{L^2(\Gamma)} + \rho C \|v_1 - v_2\|_{H^1(\Omega)} \right) \frac{1}{\mu} \|\boldsymbol{\varphi}\|_{\bar{a}} \\
 &\leq \max\{c_{\Omega}, 1\} \left( \|\theta(v_1) - \theta(v_2)\|_{L^{\frac{q}{q-1}}(\Gamma)} + C \|z_1 - z_2\|_{L^2(\Gamma)} \right) \\
 &\quad + \frac{1}{\mu} \left( d \|z_1 - z_2\|_{L^2(\Gamma)} + \rho C \|v_1 - v_2\|_{H^1(\Omega)} \right).
 \end{aligned} \tag{46}$$

We hence obtain

$$\begin{aligned}
 & \|\mathcal{D}(\mathbf{v}_2) - \mathcal{D}(\mathbf{v}_2)\|_{V^*} \\
 &= \sup_{\|\boldsymbol{\varphi}\|_{\bar{a}}=1} \left| \langle \mathcal{D}(\mathbf{v}_2) - \mathcal{D}(\mathbf{v}_2), \boldsymbol{\varphi} \rangle_{V^* \times V} \right| \\
 &\leq C \left( \|\theta(v_1) - \theta(v_2)\|_{L^{\frac{q}{q-1}}(\Gamma)} + \|z_1 - z_2\|_{L^2(\Gamma)} + \|v_1 - v_2\|_{H^1(\Omega)} \right).
 \end{aligned}$$

By the trace inequality (45), the growth condition (3), the relation  $\zeta = q - 1$ , and [12, Theorem 2] we further have  $v \mapsto \theta(v) \in C(H^1(\Omega); L^{\frac{q}{q-1}}(\Gamma))$ . This yields

$$\|\mathcal{D}(\mathbf{v}_2) - \mathcal{D}(\mathbf{v}_2)\|_{V^*} \rightarrow 0 \quad \text{for } \|\mathbf{v}_1 - \mathbf{v}_2\|_{\bar{a}} \rightarrow 0$$

which proves  $\mathcal{D} \in C(V, V^*)$ . □

**Remark 5.2** It is not possible to prove the stronger condition  $\mathcal{D} \in C(V, H)$ . This is due to the fact that the calculation (46) strongly relies on  $\boldsymbol{\varphi} \in H^1(\Omega)$  and is not possible for a test function  $\boldsymbol{\varphi} \in L^2(\Omega)$ .

Lemma 5.1 ensures that the weak formulation (4) of (1) fits in the setting of Sect. 4.1 and, hence, is locally wellposed by Corollary 4.4.

We now prove, that the bulk-surface finite element space discretization from Sect. 2.2 fits into the abstract setting of Sect. 4.2.

**Lemma 5.3** *Let Assumption 2.2 hold true. Then, the bulk-surface finite element space discretization of (1) satisfies Assumption 4.6 with  $\widehat{\beta}_{\text{qm}} = \frac{1}{\mu} \left( \frac{L_{\eta}^2 c_{\Omega}}{4\theta_0} - d \right)$ ,  $\widehat{c}_G = 1$ , and  $\widehat{C}_{H,V} = 1$ .*

**Proof** Since  $a_h$  and  $m_h$  are defined as in continuous case, Assumption 4.6 a) and b) are satisfied. Assumption 4.6 d) follows from Assumption 2.2 d) and the continuity of the interpolation operator.

It remains to prove Assumption 4.6 c). To show the quasi-monotonicity, we proceed analogously to the proof in the continuous case from Lemma 5.1 and obtain by (8c)

$$\begin{aligned}
 m_h(\mathcal{D}_h(\mathbf{v}_h^1) - \mathcal{D}_h(\mathbf{v}_h^2), \mathbf{v}_h^1 - \mathbf{v}_h^2) &= \sum_{\Gamma_h} c_\Omega \left( \theta(v_h^1) - \theta(v_h^2) - (\eta(z_h^1) - \eta(z_h^2)) \right) (v_h^1 - v_h^2) \\
 &\quad + \left( d(z_h^1 - z_h^2) + \rho(v_h^1 - v_h^2) \right) (z_h^1 - z_h^2) \Delta \mathbf{s} \\
 &\geq \left( d - \frac{L_\eta^2 c_\Omega}{4\theta_0} \right) \sum_{\Gamma_h} |z_h^1 - z_h^2|^2 \Delta \mathbf{s} \\
 &= \left( d - \frac{L_\eta^2 c_\Omega}{4\theta_0} \right) \|z_h^1 - z_h^2\|_{L^2(\Gamma_h)}^2 \\
 &\geq -\widehat{\beta}_{\text{qm}} \|\mathbf{v}_h^1 - \mathbf{v}_h^2\|_{m_h},
 \end{aligned}$$

where we used that the quadrature formula has positive weights and satisfies (7).

Finally,  $\mathcal{D}_h$  is continuous, since  $V_h$  is a finite dimensional space and, thus, convergence in  $V_h$  implies uniform pointwise convergence and especially convergence in all quadrature nodes. □

To prove an error bound for the semidiscretization, we apply the theory of Sect. 4.2 and therefore have to specify the operators from Assumption 4.8.

**Definition 5.4**

a) The lift operator  $\mathcal{L}_h^V \in \mathcal{L}(V_h; V)$  is defined via

$$\mathcal{L}_h^V [v_h, z_h]^T := \left[ v_h^\ell, z_h^\ell \right]^T \quad \text{for all } [v_h, z_h]^T \in V_h$$

with  $v_h^\ell$  from (6).

b) We set  $Z^V := H^2(\Omega) \times H^2(\Gamma) \subset C(\overline{\Omega}) \times C(\Gamma)$ .

c) We define the interpolation operator via  $I_h [v_h, z_h]^T := [I_{h,\Omega} v_h, I_{h,\Gamma} z_h]^T$ .

Our error analysis relies on the following properties of the lift and the interpolation operators.

First of all, there exist element-wise norm equivalences related to the lift, which were shown in [8, Lemmas 5.3 and 7.3].

**Lemma 5.5** *There exist  $C_{\Omega, \Omega_h} > c_{\Omega, \Omega_h} > 0$ ,  $C_{\Gamma, \Gamma_h} > c_{\Gamma, \Gamma_h} > 0$  independent of  $h < h_0$  sufficiently small s.t. for all  $v_h \in V_{h,p}^\Omega$ ,  $\vartheta_h \in V_{h,p}^\Gamma$ ,  $k = 0, 1, \dots, p + 1$ , and  $K_\Omega \in \mathcal{T}_h$ ,  $K_\Gamma \in \mathcal{T}_h^\Gamma$  and for all  $h < h_0$  sufficiently small we have*

$$\begin{aligned}
 c_{\Omega, \Omega_h} \|v_h\|_{H^k(K_\Omega)} &\leq \|v_h^\ell\|_{H^k(K_\Omega^\ell)} \leq C_{\Omega, \Omega_h} \|v_h\|_{H^k(K_\Omega)}, \\
 c_{\Gamma, \Gamma_h} \|\vartheta_h\|_{H^k(K_\Gamma)} &\leq \|\vartheta_h^\ell\|_{H^k(K_\Gamma^\ell)} \leq C_{\Gamma, \Gamma_h} \|\vartheta_h\|_{H^k(K_\Gamma)},
 \end{aligned} \tag{47}$$

where  $K_\Omega^\ell = G_h(K_\Omega)$ ,  $K_\Gamma^\ell = G_h(K_\Gamma)$ . By construction, the lift additionally preserves the  $L^\infty$  norm, i.e.,

$$\begin{aligned} \left\| v_h^\ell \right\|_{L^\infty(K_\Omega^\ell)} &= \|v_h\|_{L^\infty(K_\Omega)}, \\ \left\| \vartheta_h^\ell \right\|_{L^\infty(K_\Gamma^\ell)} &= \|\vartheta_h\|_{L^\infty(K_\Gamma)}. \end{aligned}$$

Further, we have the following bounds of the geometric errors stemming from the domain approximation (cf.[7, proof of Lemma 6.2]).

**Lemma 5.6** For  $u_h, \varphi_h \in V_{h,p}^{\Omega}$  and  $\vartheta_h, \psi_h \in V_{h,p}^\Gamma$ , the following bounds hold true for all  $h < h_0$  sufficiently small:

$$\left| \int_\Omega u_h^\ell \varphi_h^\ell \, dx - \int_{\Omega_h} u_h \varphi_h \, dx \right| \leq Ch^p \|u_h\|_{L^2(\Omega_h)} \|\varphi_h\|_{L^2(\Omega_h)}, \tag{48a}$$

$$\left| \int_\Omega \nabla u_h^\ell \nabla \varphi_h^\ell \, dx - \int_{\Omega_h} \nabla u_h \nabla \varphi_h \, dx \right| \leq Ch^p \|\nabla u_h\|_{L^2(\Omega_h)} \|\nabla \varphi_h\|_{L^2(\Omega_h)}, \tag{48b}$$

$$\left| \int_\Gamma \vartheta_h^\ell \psi_h^\ell \, ds - \int_{\Gamma_h} \vartheta_h \psi_h \, ds \right| \leq Ch^{p+1} \|\vartheta_h\|_{L^2(\Gamma_h)} \|\psi_h\|_{L^2(\Gamma_h)}, \tag{48c}$$

$$\left| \int_\Gamma \nabla_\Gamma \vartheta_h^\ell \nabla_\Gamma \psi_h^\ell \, ds - \int_{\Gamma_h} \nabla_{\Gamma_h} \vartheta_h \nabla_{\Gamma_h} \psi_h \, ds \right| \leq Ch^{p+1} \|\nabla_{\Gamma_h} \vartheta_h\|_{L^2(\Gamma_h)} \|\nabla_{\Gamma_h} \psi_h\|_{L^2(\Gamma_h)}. \tag{48d}$$

The nodal interpolation satisfy the following error bounds, which follow from [8, Theorem 4.28, Theorem 5.9] for the bulk and [8, Theorem 6.24, Theorem 7.10] for the surface interpolation, respectively.

**Lemma 5.7** Let  $1 \leq k \leq p$  and  $h < h_0$  sufficiently small.

a) Globally, the interpolation operators satisfy for all  $v \in H^{k+1}(\Omega)$ , and  $\vartheta \in H^{k+1}(\Gamma)$  the error bounds

$$\left\| v - (I_{h,\Omega} v)^\ell \right\|_{L^2(\Omega)} + h \left\| v - (I_{h,\Omega} v)^\ell \right\|_{H^1(\Omega)} \leq Ch^{k+1} \|v\|_{H^{k+1}(\Omega)}, \tag{49a}$$

$$\left\| \vartheta - (I_{h,\Gamma} \vartheta)^\ell \right\|_{L^2(\Gamma)} + h \left\| \vartheta - (I_{h,\Gamma} \vartheta)^\ell \right\|_{H^1(\Gamma)} \leq Ch^{k+1} \|\vartheta\|_{H^{k+1}(\Gamma)}, \tag{49b}$$

with a constant  $C$  independent of  $h$ .

b) Locally, on each element  $K_\Omega \in \mathcal{T}_h^\Omega$ ,  $K_\Gamma \in \mathcal{T}_h^\Gamma$ , the interpolation operators satisfy for all  $0 \leq r \leq k$  and all  $v \in H^{k+1}(K_\Omega^\ell)$ ,  $\vartheta \in H^{k+1}(K_\Gamma^\ell)$ , the error bounds

$$\left\| v - (I_{h,\Omega} v)^\ell \right\|_{H^r(K_\Omega^\ell)} \leq Ch^{k+1-r} \|v\|_{H^{k+1}(K_\Omega^\ell)}, \tag{50a}$$

$$\left\| \vartheta - (I_{h,\Gamma} \vartheta)^\ell \right\|_{H^r(K_\Gamma^\ell)} \leq Ch^{k+1-r} \|\vartheta\|_{H^{k+1}(K_\Gamma^\ell)}, \tag{50b}$$

with a constant  $C$  independent of  $h$ .



c) *Locally, on each element  $K_\Omega \in \mathcal{T}_h^\Omega$ ,  $K_\Gamma \in \mathcal{T}_h^\Gamma$ , and for every  $v_h \in H^{k+1}(K_\Omega)$ ,  $\vartheta_h \in H^{k+1}(K_\Gamma)$ , the  $L^\infty$  error bounds*

$$\|v_h - I_{h,\Omega} v_h^\ell\|_{L^\infty(K_\Omega)} \leq Ch^{k+1} \|v_h\|_{W^{k+1,\infty}(K_\Omega)}, \tag{51a}$$

$$\|\vartheta_h - I_{h,\Gamma} \vartheta_h^\ell\|_{L^\infty(K_\Gamma)} \leq Ch^{k+1} \|\vartheta_h\|_{W^{k+1,\infty}(K_\Gamma)} \tag{51b}$$

hold true with a constant  $C$  independent of  $h$ .

The following lemma is a direct consequence of Lemmas 5.6 and 5.7.

**Lemma 5.8** *The operators defined in Definition 5.4 satisfy Assumption 4.8 with*

$$\widehat{C}_V = \max\{C_{\Omega,\Omega_h}, C_{\Gamma,\Gamma_h}\},$$

where  $C_{\Omega,\Omega_h}$  and  $C_{\Gamma,\Gamma_h}$  are given in (47).

We are now in the position to prove the error bound of the space discretization. *Proof of Theorem 2.6.* We apply Theorem 4.12. By Lemmas 5.1, 5.3, and 5.8 we have that all assumptions are satisfied and we have to bound the space discretization error terms  $E_{h,i}$  in (44). In the following we always assume that  $h < h_0$  is sufficiently small that all the bounds we apply are valid.

The terms  $E_{h,1}$ ,  $E_{h,3}$  and  $E_{h,4}$  also appeared in the linear case and were bounded under Assumption 2.5 in [16, Proof of Thm 5.3] by order  $h^p$ .

It thus remains to bound  $E_{h,2}$ . For  $t \in [0, T]$  and  $\mathbf{v} = \mathbf{u}'(t) = [v, z]^\top \in V$  we calculate

$$\begin{aligned} & \max_{\|\varphi_h\|_{m_h}=1} \left| \langle \mathcal{D}(\mathbf{v}), \mathcal{L}_h^V \varphi_h \rangle_{V^* \times V} - m_h(\mathcal{D}_h(I_h \mathbf{v}), \varphi_h) \right| \\ & \leq \max_{\|[\varphi_h, \psi_h]^\top\|_{m_h}=1} \left| c_\Omega \left( \int_\Gamma \theta(v) \varphi_h^\ell \, ds - \sum_{\Gamma_h} \theta(I_{h,\Omega} v) \varphi_h \Delta s \right) \right| \\ & \quad + \left| c_\Omega \left( \int_\Gamma \eta(z) \varphi_h^\ell \, ds - \sum_{\Gamma_h} \eta(I_{h,\Gamma} z) \varphi_h \Delta s \right) \right| \\ & \quad + \left| \rho \left( \int_\Gamma v \psi_h^\ell \, ds - \sum_{\Gamma_h} I_{h,\Omega} v \psi_h \Delta s \right) \right| + \left| d \left( \int_\Gamma z \psi_h^\ell \, ds - \sum_{\Gamma_h} I_{h,\Gamma} z \psi_h \Delta s \right) \right|. \end{aligned} \tag{52}$$

We bound the different terms separately. Let  $\|\varphi_h\|_{m_h} = \|[\varphi_h, \psi_h]^T\|_{m_h} = 1$ . We then have

$$\begin{aligned} \left| \int_{\Gamma} \theta(v) \varphi_h^\ell \, ds - \sum_{\Gamma_h} \theta(I_{h,\Omega} v) \varphi_h \, \Delta s \right| &\leq \left| \int_{\Gamma} \theta(v) \varphi_h^\ell \, ds - \int_{\Gamma} (I_{h,\Gamma} \theta(v))^\ell \varphi_h^\ell \, ds \right| \\ &\quad + \left| \int_{\Gamma} (I_{h,\Gamma} \theta(v))^\ell \varphi_h^\ell \, ds - \int_{\Gamma_h} (I_{h,\Gamma} \theta(v)) \varphi_h \, ds \right| \\ &\quad + \left| \int_{\Gamma_h} (I_{h,\Gamma} \theta(v)) \varphi_h \, ds - \sum_{\Gamma_h} \theta(I_{h,\Omega} v) \varphi_h \, \Delta s \right|. \end{aligned} \tag{53}$$

For the first summand on the right-hand side of (53) we have by the continuity of the lift operator and the interpolation error (49b)

$$\begin{aligned} \left| \int_{\Gamma} \theta(v) \varphi_h^\ell \, ds - \int_{\Gamma} (I_{h,\Gamma} \theta(v))^\ell \varphi_h^\ell \, ds \right| &\leq \left\| \theta(v) - (I_{h,\Gamma} \theta(v))^\ell \right\|_{L^2(\Gamma)} \left\| \varphi_h^\ell \right\|_{L^2(\Gamma)} \\ &\leq Ch^p \|\theta(v)\|_{H^{\max\{2,p\}}(\Gamma)}. \end{aligned}$$

The second summand can be bounded using the geometric error estimate (48c) by

$$\begin{aligned} \left| \int_{\Gamma} (I_{h,\Gamma} \theta(v))^\ell \varphi_h^\ell \, ds - \int_{\Gamma_h} (I_{h,\Gamma} \theta(v)) \varphi_h \, ds \right| &\leq Ch^{p+1} \|I_{h,\Gamma} \theta(v)\|_{L^2(\Gamma_h)} \|\varphi_h\|_{L^2(\Gamma_h)} \\ &\leq Ch^p \|\theta(v)\|_{H^2(\Gamma)}. \end{aligned}$$

To bound the third summand on the right-hand side of (53) we use that for the nodal interpolation we have  $I_{h,\Gamma} \theta(v) = I_{h,\Gamma} \theta((I_{h,\Gamma} v)^\ell) \in V_{h,p}^\Gamma$  and that the order of quadrature formula is greater than  $2p$  to obtain

$$\begin{aligned} &\left| \int_{\Gamma_h} (I_{h,\Gamma} \theta(v)) \varphi_h \, ds - \sum_{\Gamma_h} \theta(I_{h,\Gamma} v) \varphi_h \, \Delta s \right| \\ &= \left| \sum_{\Gamma_h} (I_{h,\Gamma} \theta((I_{h,\Gamma} v)^\ell)) \varphi_h \, \Delta s - \sum_{\Gamma_h} \theta(I_{h,\Gamma} v) \varphi_h \, \Delta s \right| \\ &\leq \left( \sum_{\Gamma_h} (I_{h,\Gamma} \theta((I_{h,\Gamma} v)^\ell) - \theta(I_{h,\Gamma} v))^2 \, \Delta s \right)^{\frac{1}{2}} \left( \sum_{\Gamma_h} \varphi_h^2 \, \Delta s \right)^{\frac{1}{2}} \\ &\leq \sigma(\Gamma_h) \left\| I_{h,\Gamma} \theta((I_{h,\Gamma} v)^\ell) - \theta(I_{h,\Gamma} v) \right\|_{L^\infty(\Gamma_h)} \|\varphi_h\|_{L^2(\Gamma_h)} \\ &\leq Ch^p \sum_{F \in \mathcal{F}_h^\Gamma} \|\theta(I_{h,\Gamma} v)\|_{W^{p,\infty}(F)}, \end{aligned}$$

where we denote by  $\sigma(\Gamma_h)$  the measure of  $\Gamma_h$  and we used the  $L^\infty$  interpolation error bound (51b). The terms  $\|\theta(I_{h,\Gamma}v)\|_{W^{p,\infty}(F)}$  are bounded due since we have  $v \in W^{p+1,\infty}(\Omega)$ ,  $v \mapsto v|_\Gamma \in C(W^{p+1,\infty}(\Omega); W^{p,\infty}(\Gamma))$ ,  $I_{h,\Gamma} \in C(W^{p,\infty}(\Gamma); W^{p,\infty}(\Gamma_h))$ , and  $\theta \in C^p$ , cf.(9b). In total we obtain in (53)

$$\left| \int_\Gamma \theta(v)\varphi_h^\ell \, ds - \sum_{\Gamma_h} \theta(I_{h,\Omega}v)\varphi_h \, \Delta s \right| \leq Ch^p.$$

Similarly we obtain

$$\left| \int_\Gamma \eta(z)\varphi_h^\ell \, ds - \sum_{\Gamma_h} \eta(I_{h,\Gamma}z)\varphi_h \, \Delta s \right| \leq Ch^p$$

but since  $z$  is defined on  $\Gamma$  we only require the regularity  $z \in W^{p,\infty}(\Gamma)$  in contrast to  $v \in W^{p+1,\infty}(\Omega)$ .

To bound the third term in (52), we make use of the classical inverse estimate  $\|v_h\|_{H^1(\Omega_h)} \leq Ch^{-1} \|v_h\|_{L^2(\Omega_h)}$  (cf.[6, Lem. 4.5.3]), (49b), and the trace inequality  $\|v\|_{H^{p+1}(\Gamma)} \leq C \|v\|_{H^{p+2}(\Omega)}$  to obtain

$$\begin{aligned} & \left| \left( \int_\Gamma v\psi_h^\ell \, ds - \sum_{\Gamma_h} I_{h,\Omega}v\psi_h \, \Delta s \right) \right| \\ &= \left| \left( \int_\Gamma v\psi_h^\ell \, ds - \int_{\Gamma_h} I_{h,\Omega}v\psi_h \, ds \right) \right| \\ &\leq \left| \left( \int_\Gamma v\psi_h^\ell \, ds - \int_\Gamma (I_{h,\Omega}v)^\ell \psi_h^\ell \, ds \right) \right| + \left| \left( \int_\Gamma (I_{h,\Omega}v)^\ell \psi_h^\ell \, ds - \int_{\Gamma_h} I_{h,\Omega}v\psi_h \, ds \right) \right| \\ &\leq \|v - (I_{h,\Omega}v)^\ell\|_{L^2(\Gamma)} \|\psi_h^\ell\|_{L^2(\Gamma)} + Ch^{p+1} \|I_{h,\Omega}v\|_{L^2(\Gamma_h)} \|\psi_h\|_{L^2(\Gamma_h)} \\ &\leq Ch^{p+1} \|v\|_{H^{p+1}(\Gamma)} \|\psi_h\|_{H^1(\Omega_h)} + Ch^{p+1} \|I_{h,\Omega}v\|_{H^1(\Omega_h)} \|\psi_h\|_{H^1(\Omega_h)} \\ &\leq Ch^{p+1} \|v\|_{H^{p+2}(\Omega)} h^{-1} \|\psi_h\|_{L^2(\Omega_h)} + Ch^{p+1} \|v\|_{H^2(\Omega_h)} h^{-1} \|\psi_h\|_{L^2(\Omega_h)} \\ &\leq Ch^p. \end{aligned}$$

Similarly, we have

$$\left| \int_\Gamma z\psi_h^\ell \, ds - \sum_{\Gamma_h} I_{h,\Gamma}z\psi_h \, \Delta s \right| \leq Ch^p,$$

but we do not need the inverse estimate in this case and hence the smoothness  $z \in H^{p+1}(\Gamma)$  is sufficient. In total we obtain  $E_{h,2} \leq Ch^p$  and

Theorem 4.12 gives then the desired result. □

## 6 Numerical experiment

In this section we illustrate Theorem 2.6 with a numerical experiment.

Let  $\Omega = B_1(0) \subset \mathbb{R}^2$  be the unit disc and in (1) we set

$$\begin{aligned}
 c_\Omega &= c_\Gamma = \rho = \mu = 1, & k_\Omega &= k_\Gamma = d = 0, \\
 \theta(\xi) &= \xi + \xi^3, & \eta(\xi) &= 4\pi \sin(\xi) + 64\pi^3 \sin^3(\xi), \\
 f_\Omega(\mathbf{x}) &= \cos(2\pi t) (-16\pi^2 r^3 + 24\pi^2 r^2 - 144r^2 + 96r), \\
 f_\Gamma(\mathbf{x}) &= 2\pi + 4\pi \sin(2\pi t), \\
 u^0(\mathbf{x}) &= 4r^3 - 6r^2, & v^0(\mathbf{x}) &= 0, & \delta^0(\mathbf{x}) &= 0, & \vartheta^0 &= 0
 \end{aligned}$$

where  $r = r(\mathbf{x}) = \mathbf{x}_1^2 + \mathbf{x}_2^2$ . Then, Assumption 2.2 is satisfied and the exact solution of (1) is given by

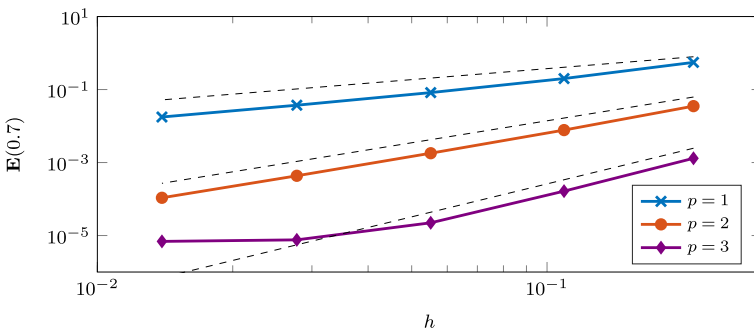
$$u(t, \mathbf{x}) = \cos(2\pi t)(4r^3 - 6r^2), \quad \delta(t, \mathbf{x}) = \pi t^2.$$

We implemented the experiments in the C++ finite element library deal.ii, cf.[1, 2]. The code which was used for the numerical is available at <https://doi.org/10.5445/IR/1000150159>. For the time integration we use the implicit midpoint rule with time step size  $\approx 2 \cdot 10^{-4}$  and solve the arising nonlinear systems with the simplified Newton method. For the spatial discretization we use the bulk-surface finite element methods of orders  $p = 1, 2, 3$ .

We consider the error

$$\begin{aligned}
 \mathbf{E}(t) := & \left\| u_h(t) - u(t) \Big|_{\Omega_h} \right\|_{H^1(\Omega_h)} + \left\| u'_h(t) - u'(t) \Big|_{\Omega_h} \right\|_{L^2(\Omega_h)} \\
 & + \left\| \delta_h(t) - I_{h,\Gamma} \delta(t) \right\|_{H^1(\Gamma_h)} + \left\| \delta'_h(t) - I_{h,\Gamma} \delta'(t) \right\|_{L^2(\Gamma_h)}
 \end{aligned} \tag{54}$$

instead of the error from Theorem 2.6 since the computation of the lift is quite laborious. We evaluated the integrals with a quadrature rule of degree  $2p$ , so that the



**Fig. 1** Error from (54) at  $t = 0.7$  for the test example. The dashed plots are straight lines of slope 1, 2 and 3, respectively

quadrature error is negligible. The restriction of  $u$  to  $\Omega_h$  is possible for this example since we have  $\Omega_h \subset \Omega$ .

In Fig. 1 the error  $\mathbf{E}(t)$  is plotted against the maximal mesh width  $h$ . We chose  $t = 0.7$  since it keeps distance to the roots of  $\sin(2\pi t)$ . We observe that the error converges with order  $p$  as predicted by Theorem 2.6 which indicates that our proven convergence rates are optimal. Note that for order  $p = 3$  and small  $h$ , the plot approaches the plateau of the time discretization error.

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