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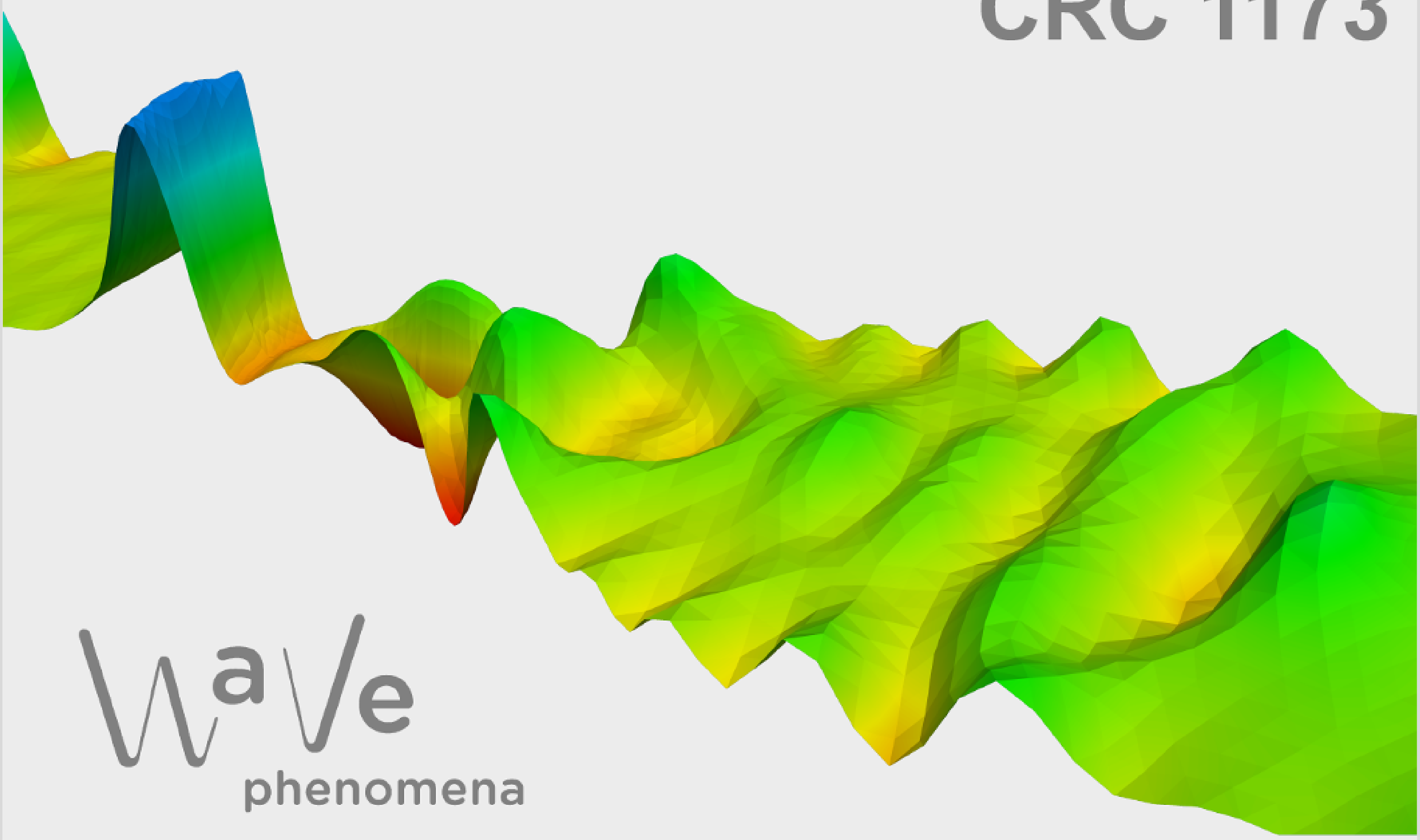

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NORMAL TRACE INEQUALITIES AND DECAY OF SOLUTIONS TO THE NONLINEAR MAXWELL SYSTEM WITH ABSORBING BOUNDARY

RICHARD NUTT AND ROLAND SCHNAUBELT

Abstract. We study the quasilinear Maxwell system with a strictly positive, state dependent boundary conductivity. For small data we show that the solution exists for all times and decays exponentially to 0. As in related literature we assume a nontrapping condition. Our approach relies on a new trace estimate for the linear problem and a observability-type estimate and a detailed regularity analysis.

1. Introduction

The Maxwell system is the foundation of electromagnetic theory. It contains constitutive relations that describe the polarization P and magnetization M of the material in dependence of the electromagnetic fields. In many physical models nonlinear effects occur which lead to nonlinear material laws, see e.g. [2], [4], [13]. In this work we study instantaneous laws, [3], for which the Maxwell system can be written as a quasilinear hyperbolic system. It is well known that such systems can exhibit blow up, see e.g. [8] in the Maxwell case.

We focus on the effect of a strictly positive surface conductivity λ, which may also depend on the electric field E. In the case of linear material laws for P and M, such nonlinear λ had been studied in [16], [10], [12] and for delayed problems in [1]. In the paper [18], co-authored by one of us, for state independent λ and small initial fields it was shown that the solutions of the quasilinear Maxwell system exist globally in time and decay exponentially as $t \to \infty$. In [18] it was assumed that the spatial domain Ω is strictly starshaped. In related problems it is known that this assumption can be removed provided one can show extra regularity of the normal trace of the solutions, see e.g. [14] for the wave equation and [11] for the linear autonomous Maxwell system with the boundary condition of a perfect conductor (which are different from absorbing ones studied here). In the present paper we prove a trace estimate for the solutions of linear nonautonomous Maxwell systems with absorbing boundary conditions and inhomogeneities, see Corollary 4.10 and Proposition 4.12. We use this extra trace regularity to establish an observability-type estimate for such systems in Proposition 3.3. This result had been shown in [18] for strictly starshaped domains only. Based on this estimate, we can extend the analysis of [18] to the case of a nonlinear boundary conductivity $\lambda(x, E)$ and

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show in our main Theorem 3.1 that solutions for small data converge exponentially to 0.

We investigate the Maxwell system

$$\begin{aligned}
&(1.1) && \begin{aligned}\partial_t(\varepsilon(x,E(t,x))E(t,x)) &= \operatorname{curl} H(t,x)\,,\\ \partial_t(\mu(x,H(t,x))H(t,x)) &= -\operatorname{curl} E(t,x)\,,\end{aligned} && t\ge 0\,, x\in\Omega\,,\\
&(1.2) && \begin{aligned}\operatorname{div}(\varepsilon(x,E(t,x))E(t,x)) &= 0\,,\\ \operatorname{div}(\mu(x,H(t,x))H(t,x)) &= 0\,,\end{aligned} && t\ge 0\,, x\in\Omega\,,\\
&(1.3) && H(t,x)\times\nu+\lambda\big(x,E(t,x)\times\nu\big)(E(t,x)\times\nu)\times\nu = 0\,, && t\ge 0\,, x\in\partial\Omega\,,\\
&(1.4) && E(0,x) = E^{(0)}(x)\,, \quad H(0,x) = H^{(0)}(x)\,,
\end{aligned}$$

on a bounded, smooth domain $\Omega\subseteq\mathbb{R}^3$ with connected complement, for the electric and magnetic fields $E(t,x), H(t,x)\in\mathbb{R}^3$ and given initial fields $E^{(0)}, H^{(0)}$. The permittivity $\varepsilon(x,E)$, permeability $\mu(x,E)$ and surface conductivity $\lambda(x,E)$ may depend on position and state and they belong to $\mathbb{R}^{3\times 3}_{\mathrm{sym}}$. So we have nonlinear, inhomogeneous and anisotropic material laws. As stated in Section 2 we assume that the coefficients are C^3, symmetric and uniformly positive definite at least for small $|E|$, respectively $|H|$. In the analysis we often rewrite $\partial_t(\varepsilon E)$ and $\partial_t(\mu H)$ as $\varepsilon^{\mathrm{d}}\partial_t E$ and $\mu^{\mathrm{d}}\partial_t H$ for new coefficients ε^{d} and μ^{d}, which are supposed to have the same properties as ε and μ. This form of the equation facilitates energy estimates.

Local wellposedness of (1.1)–(1.4) was shown in [20] for small data by energy methods. (This smallness restriction is not necessary if $\lambda=\lambda(x)$ is state-independent.) In this approach one has to control the Lipschitz norm of solutions, and thus their H^3-norms in the scale of integer-valued, L^2-based Sobolev spaces. For this reason in [20] the nonlinear problem was solved in H^m for $m\ge 3$. (For full space problems one can reduce the necessary level of regularity below $\frac{5}{2}$ by means of Strichartz estimates in some cases, cf. [19], but so far there are no such results for our boundary conditions.) In this work we stick to H^3. To bound the solutions in this norm, we look at the time-derived Maxwell systems (2.12)–(2.13). Here the coefficients ε^{d}, μ^{d} and the analogue λ^{d} appear, see (2.3). We stress that they are matrix-valued even if the given ε,μ and λ are scalar. Moreover, they depend on time through the inserted solutions, and the system (2.12)–(2.13) contains error terms also at the boundary which will be treated as inhomogeneities.

To obtain H^3-solutions, the initial fields have to satisfy certain compatibility conditions stated in (2.6). We note that these would simplify a lot for scalar-valued (isotropic) coefficients. Applying the divergence to (1.1), we see that the “charges” $\operatorname{div}(\varepsilon E)$ and $\operatorname{div}(\mu H)$ are preserved in time. We assume that the initial charges are 0, see (3.2), and that $\mathbb{R}^3\setminus\Omega$ is connected in order to exclude stationary solutions of the form $(\nabla\varphi,\nabla\psi)$ where φ and ψ are constant on $\partial\Omega$, which would violate the desired decay property.

As in [15] and [20] the decay result follows from three propositions dealing with the time derivates $\partial_t^k(E,H)$ for $k\in\{0,1,2,3\}$, see Section 3. The fields $\partial_t^k(E,H)$ satisfy the boundary condition (1.3) up to lower order terms. This fact is crucial for the analysis. An energy estimate and an observability-type estimate will allow us to control the squared L^2-norm of $\partial_t^k(E,H)$ by a dissipation term plus an error term which is small for small data, but contains space and time derivatives of higher order. Surprisingly the regularity result Proposition 3.4 for the nonlinear problem allows us to absorb the error terms. The theorem then follows by a standard boot strap procedure, see Section 3.

The observability-type estimate Proposition 3.3 is proved by a Morawetz multiplier argument as in [18], which uses ideas from [11] or [17]. For this result we have to assume the lower bound (3.1) on the radial derivatives of $\varepsilon(x,0)$ and $\mu(x,0)$. Heuristically this condition prevents trapping of the solution by back reflections so that they really reach the boundary where damping occurs. A main difficulty is the control of boundary terms. Tangential traces of solutions are bounded by the energy estimate, see also [5] or [20]. To control also the normal trace, in Section 4 we use the so-called collar operator see [11] and also [14] for earlier work on the wave equation. It allows us to trade space into time regularity. We further employ the div-curl estimate from Proposition A.1 and exploit heavily the structure of the (time-differentiated) Maxwell system and the absorbing boundary condition. For Proposition A.1 we have to assume that $\mathbb{R}^3 \setminus \Omega$ is connected. In the trace estimates it is crucial that constants do not depend on time, see Corollary 4.10. We also derive a more concise variant of the estimate in Proposition 4.12, which has time dependent constants though. This result was already shown in [5] using completely different methods. In Sections 5 and 6 we then prove the observability-type estimate Proposition 3.3 and the regularity result Proposition 3.4.

2. Notation, assumptions and auxiliary results

As in [18] we consider a bounded domain $\Omega \subseteq \mathbb{R}^3$ with C^5-boundary $\partial\Omega =: \Gamma$. We introduce the time-space cylinders $\Omega_t := (0,t)\times\Omega$, $\Gamma_t := (0,t)\times\Gamma$, $\Omega_\infty := (-\infty,\infty)\times\Omega$ and $\Gamma_\infty := (-\infty,\infty)\times\Gamma$. The outer unit normal will be denoted as $\nu\colon \Gamma \to \mathbb{R}^3$. Furthermore, we introduce the function spaces

$$\begin{aligned} C^k_\tau(\Gamma) &:= \{f \in C^k(\Gamma) \mid f\cdot\nu = 0\}\,, \\ C^3_\tau(\Gamma\times\mathbb{R}^3, \mathbb{R}^{3\times3}_{\mathrm{sym}}) &:= \{A \in C^3(\Gamma\times\mathbb{R}^3,\mathbb{R}^{3\times3}_{\mathrm{sym}}) \mid A\nu^\perp \subseteq \nu^\perp\}\,, \\ H(\operatorname{div}_\alpha) &:= \left\{ f \in \left(L^2(\Omega)\right)^3 \mid \operatorname{div}(\alpha f) =: \operatorname{div}_\alpha f \in L^2(\Omega)\right\}\,, \\ H(\operatorname{curl}_\beta) &:= \left\{ f \in \left(L^2(\Omega)\right)^3 \mid \operatorname{curl}(\beta f) =: \operatorname{curl}_\beta f \in \left(L^2(\Omega)\right)^3\right\}\,, \end{aligned}$$

for suitable matrix-valued $\alpha, \beta\colon \Omega \to \mathbb{R}^{3\times3}$, and our solution space

$$G^k(\Omega) := G^k\left([0,T_{\max})\right) := \bigcap_{j=1}^{k} C^k\left([0,T_{\max}), H^{k-j}(\Omega)^6\right)$$

equipped with their canonical norms. We also use the analogue of this space for compact time intervals. If $\alpha = \beta = I_3$ is the identity matrix, we will simply write $H(\operatorname{div})$ and $H(\operatorname{curl})$. Often we will drop Ω as well, e.g. writing $G^k := G^k(\Omega)$ and omitting the power in $\left(L^2(\Omega)\right)^3$, etc., to shorten notation. We assume that the permittivity ε, the permeability μ as well as the surface conductivity λ to be of class

$$\varepsilon,\mu \in C^3(\overline{\Omega}\times\mathbb{R}^3, \mathbb{R}^{3\times3}_{\mathrm{sym}}), \quad \lambda \in C^3_\tau(\Gamma\times\mathbb{R}^3,\mathbb{R}^{3\times3}_{\mathrm{sym}}) \tag{2.1}$$

and to be uniformly positive definite for zero fields, i.e., there is a constant η such that

$$\begin{aligned} \varepsilon(x,0) \geq 2\eta I_3, \quad \mu(x,0) &\geq 2\eta I_3 \quad \text{for all } x \in \overline{\Omega} \\ \text{and} \quad \lambda(x,0) &\geq 2\eta I_3 \quad \text{for all } x \in \Gamma\,. \end{aligned} \tag{2.2}$$

Note that the nonlinearities ε and μ are not required to be bounded in $E, H \in \mathbb{R}^3$. Later on, however, the arguments E and H will be bounded and therefore so will be $\varepsilon(\cdot, E)$ as well as $\mu(\cdot, H)$.

In a moment we will also introduce the time derived Maxwell equations for suitably smooth solutions. In order to consider these systems, we introduce the matrices

$$\begin{aligned}
\varepsilon^{\mathrm{d}}_{ij}(x,\xi) &= \varepsilon_{ij}(x,\xi) + \sum_{l=1}^{3} \partial_{\xi_j}\varepsilon_{il}(x,\xi)\xi_l\,, &&\text{for } x\in\overline{\Omega}, \xi\in\mathbb{R}^3\\
\mu^{\mathrm{d}}_{ij}(x,\xi) &= \mu_{ij}(x,\xi) + \sum_{l=1}^{3} \partial_{\xi_j}\mu_{il}(x,\xi)\xi_l\,, &&\text{for } x\in\overline{\Omega}, \xi\in\mathbb{R}^3 \qquad (2.3)\\
\lambda^{\mathrm{d}}_{ij}(x,\xi) &= \lambda_{ij}(x,\xi) + \sum_{l=1}^{3} \partial_{\xi_j}\lambda_{il}(x,\xi)\xi_l\,, &&\text{for } x\in\Gamma, \xi\in\mathbb{R}^3
\end{aligned}$$

for $i,j \in \{1,2,3\}$. To unify notation we set

$$\widehat{\varepsilon}_k = \begin{cases}\varepsilon(\cdot,E), & k=0,\\ \varepsilon^{\mathrm{d}}(\cdot,E), & k\in\{1,2,3\},\end{cases} \qquad \widehat{\mu}_k = \begin{cases}\mu(\cdot,E), & k=0,\\ \mu^{\mathrm{d}}(\cdot,E), & k\in\{1,2,3\},\end{cases}$$
$$\widehat{\lambda}_k = \begin{cases}\lambda(\cdot,E), & k=0,\\ \lambda^{\mathrm{d}}(\cdot,E), & k\in\{1,2,3\}.\end{cases}$$

We also assume that

$$\begin{aligned}
&\partial_{\xi_j}\varepsilon, \partial_{\xi_j}\mu \in C^3(\overline{\Omega}\times\mathbb{R}^3, \mathbb{R}^{3\times3}), \quad \partial_{\xi_j}\lambda \in C^3(\Gamma\times\mathbb{R}^3,\mathbb{R}^{3\times3}),\\
&\varepsilon^{\mathrm{d}} = (\varepsilon^{\mathrm{d}})^{\mathsf{T}}, \quad \mu^{\mathrm{d}} = (\mu^{\mathrm{d}})^{\mathsf{T}}, \quad \lambda^{\mathrm{d}} = (\lambda^{\mathrm{d}})^{\mathsf{T}}\,.
\end{aligned} \qquad (2.4)$$

Note that $\lambda^{\mathrm{d}} = \lambda$ if $\lambda = \lambda(x)$ is linear. We extend λ to a function on $\overline{\Omega}\times\mathbb{R}^3$ satisfying the same conditions as ε and μ. Since the coefficients are continuous we also have uniform positivity in at least a small radius $\delta_0 > 0$, i.e., for small fields

$$\begin{aligned}
\varepsilon(x,\xi), \varepsilon^{\mathrm{d}}(x,\xi) \geq \eta I\,, \quad \mu(x,\xi), \mu^{\mathrm{d}}(x,\xi) &\geq \eta I \quad \text{for all } |\xi|\leq\delta_0, x\in\overline{\Omega}\,,\\
\lambda(x,\xi), \lambda^{\mathrm{d}}(x,\xi) &\geq \eta I \quad \text{for all } |\xi|\leq\delta_0, x\in\Gamma\,.
\end{aligned}$$

An important special case is the Kerr law $\mu = \mu(x)$ and $\varepsilon = \varepsilon(x,\xi) = \varepsilon_{\mathrm{lin}}(x) + \varepsilon_{\mathrm{nl}}(x)\,|\xi|^2$ for scalar coefficients with $\varepsilon_{\mathrm{lin}} \geq 2\eta$ as well. Here one has $\varepsilon^{\mathrm{d}}(x,\xi) = \varepsilon_{\mathrm{lin}}(x) + \varepsilon_{\mathrm{nl}}(x)\xi\xi^{\mathsf{T}}$. Anisotropic examples of polynomial type are indicated in Example 2.1 of [15].

Since the solutions of (1.1)–(1.4) are supposed to satisfy the boundary condition at all times, it has to also hold for the initial values. This leads to so called "compatibility conditions" (of order 3) on $E^{(0)}$ and $H^{(0)}$. Namely for $E^{(0)}, H^{(0)} \in H^3(\Omega)$, a solution $(E,H)\in G^3$ of (1.1) possesses the time derivatives

$$\begin{aligned}
E^{(1)} &:= \big(\varepsilon^{\mathrm{d}}(E^{(0)})\big)^{-1}\operatorname{curl} H^{(0)}\,,\\
H^{(1)} &:= -\big(\mu^{\mathrm{d}}(H^{(0)})\big)^{-1}\operatorname{curl} E^{(0)}\,,\\
E^{(2)} &:= \big(\varepsilon^{\mathrm{d}}(E^{(0)})\big)^{-1}\left[\operatorname{curl} H^{(1)} - \Big(\sum_{\ell=1}^{3}\partial_{\xi_\ell}\varepsilon^{\mathrm{d}}_{i,j}(E^{(0)})E^{(1)}_\ell\Big)_{i,j}E^{(1)}\right]\,, \qquad (2.5)\\
H^{(2)} &:= -\big(\mu^{\mathrm{d}}(H^{(0)})\big)^{-1}\left[\operatorname{curl} E^{(1)} - \Big(\sum_{\ell=1}^{3}\partial_{\xi_\ell}\mu^{\mathrm{d}}_{i,j}(H^{(0)})H^{(1)}_\ell\Big)_{i,j}H^{(1)}\right]\,,
\end{aligned}$$

at time 0, which leads to the compatibility conditions

$$\begin{aligned}
&H^{(0)} \times \nu + \lambda(E^{(0)} \times \nu)(E^{(0)} \times \nu) \times \nu = 0\,,\\
&H^{(1)} \times \nu + \lambda^{\mathrm{d}}(E^{(0)} \times \nu)(E^{(1)} \times \nu) \times \nu = 0\,,\\
&H^{(2)} \times \nu + \lambda^{\mathrm{d}}(E^{(0)} \times \nu)(E^{(2)} \times \nu) \times \nu \\
&\qquad = -\bigg(\Big(\sum_{\ell=1}^{3} \partial_{\xi_\ell} \lambda^{\mathrm{d}}_{i,j}(E^{(0)} \times \nu)(E^{(1)} \times \nu)_\ell\Big)_{i,j} E^{(1)} \times \nu\bigg) \times \nu\,.
\end{aligned} \tag{2.6}$$

Let c_0 be the norm of the Sobolev embedding $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$. Set $\tilde{\delta} := \min\{1, \frac{\delta_0}{c_0}\}$. Under the assumptions in Theorem 6.4 of [20] above, it was shown that for $\delta \in (0, \tilde{\delta}]$ and sufficiently small initial values satisfying (2.6), i.e.,

$$\Big\|(E^{(0)}, H^{(0)})\Big\|_{H^3(\Omega)^6} \le r(\delta) \tag{2.7}$$

for a constant $r(\delta) > 0$ depending on δ, there exists a unique classical solution $(E, H) \in G^3([0, T_{\max}))$ such that $T_{\max} > 1$ and

$$\max_{0\le j\le 3} \Big(\Big\|\partial_t^j E(t)\Big\|^2_{H^{k-j}(\Omega)} + \Big\|\partial_t^j H(t)\Big\|^2_{H^{k-j}(\Omega)} \Big) \le \delta^2 \quad \text{for } 0 \le t \le T\,. \tag{2.8}$$

We set

$$T_* := \sup\{\hat{T} \in [0, T_{\max}] \mid (2.8) \text{ holds for } t \in [0, \hat{T}]\} \ge 1\,. \tag{2.9}$$

The blow-up condition in Theorem 6.4 of [20] furthermore shows that, unless $T_* = \infty$,

$$\max_{0\le j\le 3} \Big(\Big\|\partial_t^j E(T_*)\Big\|^2_{H^{k-j}(\Omega)} + \Big\|\partial_t^j H(T_*)\Big\|^2_{H^{k-j}(\Omega)} \Big) = \delta^2\,. \tag{2.10}$$

Below we will assume $0 \le t < T_*$ if we work with the solution (E, H) to (1.1) – (1.4) with data satisfying (2.7).

Furthermore we define the commutator terms

$$\begin{aligned}
&f_0 = f_1 = 0\,, \quad f_2 = (\partial_t \varepsilon^{\mathrm{d}}(\cdot, E))\partial_t E\,, \quad f_3 = (\partial_t^2 \varepsilon^{\mathrm{d}}(\cdot, E))\partial_t E + 2(\partial_t \varepsilon^{\mathrm{d}}(\cdot, E))\partial_t^2 E\,,\\
&g_0 = g_1 = 0\,, \quad g_2 = (\partial_t \mu^{\mathrm{d}}(\cdot, H))\partial_t H\,, \quad g_3 = (\partial_t^2 \mu^{\mathrm{d}}(\cdot, H))\partial_t H + 2(\partial_t \mu^{\mathrm{d}}(\cdot, H))\partial_t^2 H\,,\\
&h_0 = h_1 = 0\,, \quad h_2 = (\partial_t \lambda^{\mathrm{d}}(\cdot, E \times \nu))\partial_t E \times \nu\,,\\
&h_3 = (\partial_t^2 \lambda^{\mathrm{d}}(\cdot, E \times \nu))\partial_t E \times \nu + 2(\partial_t \lambda^{\mathrm{d}}(\cdot, E \times \nu))\partial_t^2 E \times \nu\,.
\end{aligned}$$

Let $k \in \{0, 1, 2, 3\}$ With the definitions above the time derived solutions satisfy

$$\begin{aligned}
\partial_t^k(\varepsilon E) &= \widehat{\varepsilon}_k \partial_t^k E + f_k\,, \qquad \partial_t^k(\mu H) = \widehat{\mu}_k \partial_t^k H + g_k \qquad &&\text{on } \Omega\,,\\
\partial_t^k(\lambda E \times \nu) &= \widehat{\lambda}_k \partial_t^k E \times \nu + h_k &&\text{on } \Gamma\,.
\end{aligned} \tag{2.11}$$

We thus arrive at the inhomogeneous system

$$\left.\begin{aligned}
\partial_t(\widehat{\varepsilon}_k \partial_t^k E) &= \operatorname{curl} \partial_t^k H - \partial_t f_k\,,\\
\partial_t(\widehat{\mu}_k \partial_t^k H) &= -\operatorname{curl} \partial_t^k E - \partial_t g_k\,,
\end{aligned}\right. \qquad t \ge 0, x \in \Omega, \tag{2.12}$$

$$\partial_t^k H \times \nu + \widehat{\lambda}_k(\partial_t^k E \times \nu) \times \nu = -h_k \times \nu\,. \tag{2.13}$$

Notice that (1.1), (1.2) and (2.11) imply

$$\operatorname{div}(\widehat{\varepsilon}_k \partial_t^k E) = -\operatorname{div}(f_k)\,, \quad \operatorname{div}(\widehat{\mu}_k \partial_t^k H) = -\operatorname{div}(g_k)\,. \tag{2.14}$$

Finally, in order to discuss the observability and energy estimates, we introduce the energy dissipation and error terms

$$e_k(t) = \tfrac{1}{2} \max_{0\le j\le k} \left(\left\| \widehat{\varepsilon}_j^{1/2} \partial_t^j E(t) \right\|_{L^2(\Omega)}^2 + \left\| \widehat{\mu}_j^{1/2} \partial_t^j H(t) \right\|_{L^2(\Omega)}^2 \right), \qquad e := e_3\,,$$
$$d_k(t) = \max_{0\le j\le k} \left\| \lambda^{1/2} \operatorname{tr}_t \partial_t^j E(t) \right\|_{L^2(\Gamma)}^2, \qquad d := d_3\,,$$
$$z_k(t) = \max_{0\le j\le k} \left(\left\| \partial_t^j E(t) \right\|_{H^{k-j}(\Omega)}^2 + \left\| \partial_t^j H(t) \right\|_{H^{k-j}(\Omega)}^2 \right), \qquad z := z_3\,,$$

for the time-derived fields. Here

$$\operatorname{tr}_t \colon H(\mathrm{curl}) \to \left(H^{-1/2}(\Gamma)\right)^3;\ u \mapsto u \times \nu$$

denotes the tangential trace. Similarly we define the normal trace

$$\operatorname{tr}_n \colon H(\mathrm{div}) \to H^{-1/2}(\Gamma);\ u \mapsto u \cdot \nu\,.$$

These linear maps are bounded, see Theorem 2.2 and 2.3 of [6]. We also use the rotated tangential trace

$$\operatorname{tr}_\tau \colon H(\mathrm{curl}) \to \left(H^{-1/2}(\Gamma)\right)^3;\ u \mapsto \nu \times (u \times \nu) = \operatorname{tr} u - (\operatorname{tr}_n u)\nu\,.$$

One can show the following estimates for the commutator terms

$$\begin{aligned} \max_{2\le k\le 3, 0\le j\le 1} \left\| \partial_t^j f_k \right\|_{H^{4-j-k}(\Omega)} + \left\| \partial_t^j g_k \right\|_{H^{4-j-k}(\Omega)} &\lesssim z(t)\,, \\ \max_{2\le k\le 3, 0\le j\le 1} \left\| \partial_t^j h_k \right\|_{H^{3+1/2-j-k}(\Gamma)} &\lesssim z(t)\,, \end{aligned} \tag{2.15}$$

of equation (2.22) in [18]. We write "$\lesssim$" if the inequality holds up to a constant not depending on $t \in [0, T_*), T_*, \delta \in (0, \tilde{\delta}], r \in (0, r(\delta)]$ and $E^{(0)}, H^{(0)}$ fulfilling (2.6), (2.7) and (3.2) below. Moreover, such constants are denoted by c, c_j, C or C_j. We stress that z is quadratic in the fields and bounded by $\delta z(t)^{1/2}$ due to (2.8).

Sketch of proof of (2.15). We carry out the proof for f_k. For g_k the argument can be repeated and the same is true for h_k after extending h_k to $\overline{\Omega}$ using the trace theorem. For $k \in \{0, 1\}$ the commutator term f_k vanishes and there is nothing to show. Notice that $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ and therefore E and $\partial_t E$ are bounded. Since ε^{d} is assumed to belong to $C^3(\Omega \times \mathbb{R}^3)$, we can estimate the time derivative of

$$\begin{aligned} f_3 = {} & \Big[\sum_{i,j=1}^3 \partial_{\xi_i,\xi_j} \varepsilon^{\mathrm{d}}(\cdot, E) \partial_t E_i \partial_t E_j \Big] \partial_t E + \Big[\sum_{i=1}^3 \partial_{\xi_i} \varepsilon^{\mathrm{d}}(\cdot, E) \partial_t^2 E_i \Big] \partial_t E \\ & + 2 \Big[\sum_{i=1}^3 \partial_{\xi_i} \varepsilon^{\mathrm{d}}(\cdot, E) \partial_t E_i \Big] \partial_t^2 E =: f_{3,1} + f_{3,2} + 2 f_{3,3} \end{aligned}$$

for instance by

$$\begin{aligned} \|\partial_t f_{3,1}\|_{L^2(\Omega)} &= \left\| \partial_t \Big[\Big(\sum_{i,j=1}^3 \partial_{\xi_i,\xi_j} \varepsilon^{\mathrm{d}}(\cdot, E) \partial_t E_i \partial_t E_j \Big) \partial_t E \Big] \right\|_{L^2(\Omega)} \\ &\le \left\| \Big(\sum_{i,j,k=1}^3 \partial_{\xi_i,\xi_j,\xi_k} \varepsilon^{\mathrm{d}}(\cdot, E) \partial_t E_i \partial_t E_j \partial_t E_k \Big) \partial_t E \right\|_{L^2(\Omega)} \end{aligned}$$

$$
\begin{aligned}
&+ \left\| 2\Big(\sum_{i,j=1}^{3} \partial_{\xi_i,\xi_j} \varepsilon^{\mathrm{d}}(\cdot, E) \partial_t^2 E_i \partial_t E_j \Big) \partial_t E \right\|_{L^2(\Omega)} \\
&+ \left\| \Big(\sum_{i,j=1}^{3} \partial_{\xi_i,\xi_j} \varepsilon^{\mathrm{d}}(\cdot, E) \partial_t E_i \partial_t E_j \Big) \partial_t^2 E \right\|_{L^2(\Omega)} \\
\lesssim &\sum_{i,j=1}^{3} \|\partial_t E_i\|_{L^\infty(\Omega)} \|\partial_t E_j\|_{L^2(\Omega)} + \left\|\partial_t^2 E_i\right\|_{L^2(\Omega)} \|\partial_t E_j\|_{L^\infty(\Omega)} \\
&+ \|\partial_t E_i\|_{L^\infty(\Omega)} \left\|\partial_t^2 E_j\right\|_{L^2(\Omega)} \\
\lesssim &\sum_{i,j=1}^{3} \|\partial_t E_i\|_{H^1(\Omega)} \|\partial_t E_j\|_{L^2(\Omega)} + \left\|\partial_t^2 E_i\right\|_{L^2(\Omega)} \|\partial_t E_j\|_{H^1(\Omega)} \\
&+ \|\partial_t E_i\|_{H^1(\Omega)} \left\|\partial_t^2 E_j\right\|_{L^2(\Omega)} \\
\lesssim &\, z(t)
\end{aligned}
$$

Analogous estimates for $\|\partial_t f_{3,2}\|_{L^2(\Omega)}$ and $\|\partial_t f_{3,3}\|_{L^2(\Omega)}$ can be shown. In a similar way one obtains bounds on $\|f_2\|_{H^2(\Omega)}$, $\|\partial_t f_2\|_{H^1(\Omega)}$ and $\|f_3\|_{H^1(\Omega)}$. $\square$

We also define the function $m(x) = x - x_0$ on Ω for a fixed point $x_0 \in \Omega$. It is used in the multiplier argument from [18].

3. Main result

We state our main decay theorem and the core ingredients for its proof. Based on them, we show the theorem at the end of the section. Note that compared to Theorem 2.2 in [18] we no longer require the domain to be strictly starshaped and also permit semilinear boundary damping. In [18] the strict starshapedness of the domain was used to estimate trace terms, which we will treat here in a more delicate manner, utilizing methods from microlocal analysis as previously introduced in [11]. On that note we should mention that Propositions 3.1 and 4.1 in [18] do not require strict starshapedness contrary to the statements there.

We require the the following *non-trapping condition* on ε and μ, which emanates from the Morawetz multiplier technique used in [18]:

$$
\begin{aligned}
\varepsilon(x,0) + \big(m(x) \cdot \nabla_x\big)\varepsilon(x,0) &\geq \overline{\eta}\varepsilon(x,0)\,, \\
\mu(x,0) + \big(m(x) \cdot \nabla_x\big)\mu(x,0) &\geq \overline{\eta}\mu(x,0)
\end{aligned}
\tag{3.1}
$$

for a constant $\overline{\eta} > 0$ and $x \in \overline{\Omega}$. It says that ε and μ do not decay too rapidly in radial directions for small fields. Heuristically this should reduce back reflections preventing the fields to reach the boundary. Similar conditions were used in [11], [9] and [17], for instance.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^5-boundary and connected complement. Furthermore assume that the permittivity ε and permeability μ satisfy* (2.1)*,* (2.2)*,* (2.4) *and* (3.1)*. We assume that the initial values $E^{(0)}, H^{(0)} \in \big(H^3(\Omega)\big)^3$ satisfy the compatibility conditions* (2.6) *as well as the initial "charge" conditions*

$$
\operatorname{div}\big(\varepsilon(E^{(0)})E^{(0)}\big) = \operatorname{div}\big(\mu(H^{(0)})H^{(0)}\big) = 0 \text{ on } \Omega\,. \tag{3.2}
$$

Then there exist constants $M, \omega, r > 0$ such that for $\left\|E^{(0)}\right\|_{H^3(\Omega)}^2 + \left\|H^{(0)}\right\|_{H^3(\Omega)}^2 \leq r^2$ the solutions $(E,H) \in G^3$ exist for all times $t \geq 0$, are unique, and decay exponentially, i.e.,

$$\max_{0\leq j\leq 3} \left\| \left(\partial_t^j E(t), \partial_t^j H(t)\right)\right\|_{H^{3-j}(\Omega)}^2 \leq M e^{-\omega t} \left\| \left(E^{(0)}, H^{(0)}\right)\right\|_{H^3(\Omega)}^2 .$$

Theorem 3.1 is a consequence of the following three propositions on solutions to (2.12) and (2.13): an energy inequality, an observability-type estimate and a regularity result. All contain small error terms of highest order. It is crucial that constants do not depend on time.

Proposition 3.2. *Assume the hypotheses of Theorem 3.1, except for the connectedness assumptions on $\mathbb{R}^3 \setminus \Omega$, (3.1) and (3.2). Then we have*

$$e(t) + \int_s^t d(\tau)\,\mathrm{d}\tau \leq e(s) + c_1 \int_s^t z^{3/2}(\tau)\,\mathrm{d}\tau$$

for $0 \leq s \leq t < T_*$.

This proposition controls energy and dissipation by the initial energy plus error terms. The result was shown in Proposition 3.1 of [18] for linear $\lambda = \lambda(x)$ and hence for $h_2 = h_3 = 0$. It is straightforward to extend the proof in [18] to the present situation, using (2.15) to estimate the terms with h_2 and h_3. In the next proposition we bound the time integral of the energy by the dissipation and the energy of initial and present time plus error terms.

Proposition 3.3. *Assume that the assumptions of Theorem 3.1 are satisfied. For $0 \leq s \leq t < T_*$ we obtain*

$$\int_s^t e(\tau)\,\mathrm{d}\tau \leq c_2 \int_s^t d(\tau)\,\mathrm{d}\tau + c_3(e(t) + e(s)) + c_4 \int_s^t z^{3/2}(\tau)\,\mathrm{d}\tau .$$

This result is shown in Section 5 and it improves Proposition 3.3 from [18]. We want to explain its role in the autonomous, homogenous and linear case. In this setting the above estimates are true without z and for all $t \geq 0$ and one has equality in Proposition 3.2. (See Lemma 3.2 in [18].) We thus obtain

$$\begin{aligned} T(e(0) - \int_0^T d(\tau)\,\mathrm{d}\tau) = Te(T) &\leq \int_0^T e(\tau)\,\mathrm{d}\tau \\ &\leq c_2 \int_0^T d(\tau)\,\mathrm{d}\tau + c_3(e(T) + e(0)) \\ &\leq c_2 \int_0^T d(\tau)\,\mathrm{d}\tau + 2c_3 e(0) . \end{aligned}$$

Taking $T > 2c_3$, we conclude

$$(T - 2c_3)e(0) \leq (c_2 + T) \int_0^T d(\tau)\,\mathrm{d}\tau . \tag{3.3}$$

The dissipation thus controls the initial value, which is closely related to observation estimates.

Exactly as in Corollary 3.5 of [18], Propositions 3.2 and 3.3 imply the estimate

$$e(t) + \int_s^t e(\tau)\,\mathrm{d}\tau \leq C_1 e(s) + C_2 \int_s^t z^{3/2}(\tau)\,\mathrm{d}\tau \tag{3.4}$$

for $0 \leq s \leq t < T_*$. Without the z-term it is wellknown how to derive exponential stability from here. However, despite $\sqrt{z} \leq \delta$ being small, we cannot absorb the error term with z by the left-hand side. To achieve this, we have to bound the space derivatives of (E, H) by their time derivatives. This can be done exploiting the structure of the Maxwell systems (1.1)–(1.4) resulting in the third ingredient of the proof of Theorem 3.1.

Proposition 3.4. *Under the assumptions of Theorem 3.1, with the exception of* (3.1)*, there exist constants* $c_5, c_6 > 0$ *such that*

$$z(t) \leq c_5 e(t) + c_6 z^2(t)$$

for $0 \leq t < T_*$.

We sketch the proof of this result in Section 6, heavily relying on the proof of Proposition 4.1 in [18]. Based on the propositions above, we can now show our main result.

Proof of Theorem 3.1. The reasoning follows the lines of the proof of Theorem 2.2 of [18]. For convenience we present the arguments briefly. We first combine estimate (3.4) and Proposition 3.4. Fixing a sufficiently small $\overline{\delta} \in (0, \hat{\delta}]$, we derive

$$z(t) + \int_s^t z(\tau)\,\mathrm{d}\tau \leq C z(s) \tag{3.5}$$

for all $0 \leq s \leq t < T_*$ and a constant $C \geq 1$. Suppose that $T_* < \infty$. It follows $z(T_*) = \overline{\delta}^2$ by (2.10). On the other hand, the inequality (3.5) yields $z(t) \leq Cz(s)$. For initial fields with $\big\|\big(E^{(0)}, H^{(0)}\big)\big\| \leq r^2$, one can bound $z(0) \leq c_0 r^2$ using equations (2.5). As a result for a fixed, sufficiently small radius $\overline{r} \in (0, r(\overline{\delta})]$ we infer the contradiction $z(T_*) \leq \frac{1}{2}\overline{\delta}^2$. Therefore the solution exists and (3.5) is valid for all times. For $T = C(2C-1)$ we deduce

$$\begin{aligned} z(T) + \frac{T}{C} z(T) &\leq z(T) + \int_0^T z(\tau)\,\mathrm{d}\tau \leq C z(0)\,, \\ z(T) &\leq \frac{C}{1+\frac{T}{C}} z(0) = \frac{1}{2} z(0)\,. \end{aligned}$$

The assertion follows by iteration. □

Before proving Proposition 3.3, we simplify the notation. Instead of (2.12) and (2.13) we study the system

$$\begin{aligned} \partial_t(\alpha(t,x)u(t,x)) &= \operatorname{curl} v(t,x) + \partial_t \varphi\,, \\ \partial_t(\beta(t,x)v(t,x)) &= -\operatorname{curl} u(t,x) + \partial_t \psi\,, \qquad t \geq 0, x \in \Omega\,, \\ v \times \nu + (\gamma(u \times \nu)) \times \nu &= \omega\,, \\ u(0) = u^{(0)}\,, \quad v(0) &= v^{(0)}\,, \end{aligned} \tag{3.6}$$

where we assume

$$\begin{aligned} &\alpha, \beta \in C^1\big([0,T] \times \overline{\Omega}, \mathbb{R}^{3\times3}_{\mathrm{sym}}\big), \quad \gamma \in C^1_\tau\big([0,T] \times \Gamma, \mathbb{R}^{3\times3}_{\mathrm{sym}}\big), \quad \text{with } \alpha, \beta, \gamma \geq \eta\,, \\ &\varphi, \psi \in G^1 = C^1\big([0,T], L^2(\Omega)^3\big) \cap C\big([0,T], H^1(\Omega)^3\big) \\ &\omega \in C([0,T], H^{1/2}(\Gamma)) \text{ with } \nu \cdot \omega = 0\,, \end{aligned} \tag{3.7}$$

with initial values $u^{(0)}, v^{(0)} \in (L^2(\Omega))^3$ such that

$$\operatorname{div}\left(\alpha(0))u^{(0)}\right) = \operatorname{div}\varphi(0) \text{ and } \operatorname{div}\left(\beta(0)v^{(0)}\right) = \operatorname{div}\psi(0)\,. \tag{3.8}$$

To recover the original system, we simply resubstitute $\alpha = \widehat{\varepsilon}_k, \beta = \widehat{\mu}_k, \gamma = \widehat{\lambda}_k, \varphi = -f_k, \psi = -g_k$ and $\omega = -h_k \times \nu$ for $k \in \{0, 1, 2, 3\}$.

According to Prop 3.1 of [20] there exists a unique weak solution $(u, v) \in C^0([0, T], (L^2(\Omega))^6)$ of (3.6) with tangential trace $(\operatorname{tr}_\tau u, \operatorname{tr}_\tau v) \in L^2(\Omega_T, \mathbb{R}^6)$. Observe that $\operatorname{div}(\alpha u) = \operatorname{div}(\varphi)$ and $\operatorname{div}(\beta v) = \operatorname{div}(\psi)$ belong to $C([0, T], L^2(\Omega))$.

4. Trace inequality

The main goal of this section is to establish new bounds on the normal traces of the electric and magnetic fields and their derivatives, which are needed to show Proposition 3.3.

The pseudodifferential collar operator X. To obtain the desired trace regularity, we introduce the pseudodifferential operator X as defined in Definition 2.1 of [11]. This operator has the advantageous property that it allows us to trade time regularity for spatial regularity on the boundary Γ of our domain, as shown in Lemma 2.2 of [11]. We restate its definition.

Definition 4.1. *By assumption our domain Ω has a C^2 boundary. Hence there exists a tubular neighborhood U of Ω and we can partition $\Omega \cup U = \bigcup_{i=0}^m U_i$ such that*

- *$U_0 \subseteq \Omega$,*
- *for $i \geq 1$ $U_i \subseteq U$ with $U_i \cap \partial\Omega \neq \emptyset$ and*
- *there exists a partition of unity $(\phi_i)_{i=0,\dots,m}$ on Ω such that $\phi_i \in C_0^\infty(U_i)$ and $\sum_i \phi_i(x) = 1, x \in \Omega$.*

Furthermore we choose a neighborhood $V_i \subseteq \mathbb{R}^3$ around the origin and coordinate mappings $\Phi_i \colon V_i \to U_i \in C^2(V_i)$ such that

- *$\Phi_i(V_i \cap \{(x, y, z) \mid z \geq 0\}) = U_i \cap \Omega$*
- *$\Phi_i(V_i \cap \{(x, y, z) \mid z = 0\}) = U_i \cap \partial\Omega$.*

We begin constructing X locally on $(-\infty, \infty) \times U_i$. Let $\varkappa > 0$, let $\tilde{\chi} \in C^\infty(\mathbb{R}^3 \setminus \{0\})$ with $0 \leq \tilde{\chi} \leq 1$ and

$$\tilde{\chi}(\xi_0, \eta_1, \eta_2) = \begin{cases} 1\,, & |\xi_0| \leq \varkappa\,|(\eta_1, \eta_2)|\,/2\,, \\ 0\,, & |\xi_0| \geq \varkappa\,|(\eta_1, \eta_2)|\;, \end{cases}$$

and let $0 \leq \rho \leq 1$ be a cutoff at the origin

$$\rho(\xi_0, \eta_1, \eta_2) = \begin{cases} 1\,, & |(\xi_0, \eta_1, \eta_2)|^2 \geq 2\,, \\ 0\,, & |(\xi_0, \eta_1, \eta_2)|^2 \leq 1\,. \end{cases}$$

We set $\chi(\xi_0, \eta_1, \eta_2, \eta_3) = (\rho\tilde{\chi})\,(\xi_0, \eta_1, \eta_2) \in C^\infty(\mathbb{R}^4)$. For $u \in L^2(\mathbb{R}, H^s(\Omega))$ define $u_i = \phi_i u$ and $v_i(\xi_0, \eta) = u_i(\xi_0, \Phi_i(\eta))$ für $(\xi_0, \eta) \in (-\infty, \infty) \times V_i$. Take $\sigma_i \in C_0^\infty(V_i)$ such that $\sigma_i|_{\operatorname{supp}(\phi_i \circ \Phi_i)} \equiv 1$. The operator X is then given by

$$X(x, D)u = \sum_{i=1}^m [\sigma_i \chi(D_t, D_{y_1}, D_{y_2}) v_i] \circ \Phi_i^{-1}\,.$$

Its symbol is of class $C^1 S^0_{cl}$, cf. [21] and [11].

The restriction of X to the boundary Γ as a pseudodifferential operator on a manifold is also denoted by X.

We recall the central properties of this operator as stated and proven in Lemma 2.2 and 2.3 of [11].

Lemma 4.2. *Let $u \in L^2\big(\mathbb{R}, H^s(\Omega)\big)$. Then there exists a constant $K_1 > 0$ such that*

$$\int_{\mathbb{R}} \|X\partial_t u\|^2_{H^{s-1}(\Omega)} \,\mathrm{d}\tau \le K_1 \left[\int_{\mathbb{R}} \|Xu\|^2_{H^s(\Omega)} \,\mathrm{d}\tau + \int_{\mathbb{R}} \|u\|^2_{H^{s-1}(\Omega)} \,\mathrm{d}\tau\right]$$

for $0 \le s \le 1$.

The constant $K_1 = K_1(\varkappa)$ belongs to $\mathcal{O}(\varkappa^2)$ for $\varkappa \to 0$ and can therefore be chosen arbitrarily small, if one lets $\varkappa$ tend to 0 in the construction of X.

Lemma 4.3. *Let $u \in L^2(\Gamma_\infty)$. Then there exists a constant $K_2 = K_2(\varkappa) > 0$ such that*

$$\|(1-X)u\|^2_{L^2(\Gamma_\infty)} \le K_2 \left[\|u\|^2_{H^{-1}(\Gamma_\infty)} + \|\partial_t u\|^2_{H^{-1}(\Gamma_\infty)}\right].$$

Using the fact that the operator is essentially a Fourier-Multiplier at least locally, we obtain the following regularity result, which follows from Chapter 4 of [21].

Lemma 4.4. *Let $-1 \le s \le 1$. Then $X\colon L^2(\mathbb{R}, H^s(\Omega)) \to L^2(\mathbb{R}, H^s(\Omega))$ and $X\colon L^2(\mathbb{R}, H^s(\Gamma)) \to L^2(\mathbb{R}, H^s(\Gamma))$ are continuous.*

Trace inequality. For a moment we consider the slightly more general inhomogeneities $\xi, \zeta \in L^2([0,T]\times\Omega)$ on the right hand side of (3.6), i.e., the system

$$\begin{aligned}
\partial_t(\alpha(t,x)u(t,x)) &= \operatorname{curl} v(t,x) + \xi\,, && t \ge 0\,, x \in \Omega,\\
\partial_t(\beta(t,x)v(t,x)) &= -\operatorname{curl} u(t,x) + \zeta\,, && t \ge 0\,, x \in \Omega,\\
v \times \nu + (\gamma(u \times \nu))\nu &= \omega\,, && t \ge 0\,, x \in \Gamma,\\
u(0) = u^{(0)}\,, \quad & v(0) = v^{(0)}
\end{aligned} \tag{4.1}$$

with coefficients and data as in (3.7) and (3.8) above. Then Proposition 3.1 of [20] still yields a unique weak solution $(u,v) \in C^0([0,T], (L^2(\Omega))^6)$.

In order to apply the pseudodifferential operator X from Definition 4.1 to the solutions (u,v) defined on $[0,T]$, we have to extend (u,v) by 0 to $L^2((-\infty,\infty), L^2(\Omega))$. We still denote this extension by u respectively v, and proceed analogously with ξ, ζ and ω. By the following div-curl estimate we can bound the image of the pseudodifferential operator in H^1. Here and below we often omit variables. Recall that we also write L^2 instead of $\big(L^2(\Omega)\big)^3$, etc. Initially we assume that (u,v) belong to $G^1 = G^1([0,T])$. This assumption will be removed in Corollary 4.10.

Lemma 4.5. *Let $(u,v) \in G^1$ be solutions to the system (4.1). Then*

$$\begin{aligned}
\int_{\mathbb{R}} \Big(\|Xu\|^2_{H^1(\Omega)} + \|Xv\|^2_{H^1(\Omega)}\Big) \,\mathrm{d}\tau &\lesssim \int_{\mathbb{R}} \Big(\|X \operatorname{div}(\alpha u)\|^2_{L^2} + \|X \operatorname{curl} u\|^2_{L^2}\\
&+ \|X \operatorname{div}(\beta u)\|^2_{L^2} + \|X \operatorname{curl} v\|^2_{L^2} + \|(u,v)\|^2_{L^2} + \|(u,v)\|^2_{L^2(\Gamma)} + \|\omega\|^2_{H^{1/2}(\Gamma)}\Big) \mathrm{d}\tau.
\end{aligned}$$

Proof. By Lemma 4.4 the operator $X\colon L^2(\mathbb{R}, H^s(\Omega)) \to L^2(\mathbb{R}, H^s(\Omega))$ is continuous for $-1 \le s \le 1$. We can apply Lemma A.1 a) to Xu and Xv, obtaining

$$\begin{aligned}
\|Xu\|_{H^1} + \|Xv\|_{H^1} &\lesssim \|\operatorname{div}_\alpha Xu\|_{L^2} + \|\operatorname{curl} Xu\|_{L^2} + \|Xu\|_{L^2} + \|Xv\|_{L^2}\\
&+ \|\operatorname{div}_\beta Xu\|_{L^2} + \|\operatorname{curl} Xv\|_{L^2} + \|Xv \times \nu + (\gamma((Xu)\times\nu))\times\nu\|_{H^{1/2}(\Gamma)}\,.
\end{aligned} \tag{4.2}$$

We now estimate the last term. Since $\Gamma \in C^5$, the cross product with ν can be written as a multiplication with a matrix B whose entries are C^1 and bounded.

The same holds for γ and therefore the entries of $B\gamma B$ are also bounded in C^1. We write

$$Xv \times \nu + (\gamma((Xu) \times \nu)) \times \nu = B(Xv) + B\gamma B(Xu)$$

The commutators satisfy $[X, B], [X, B\gamma B] \colon L^2(\mathbb{R}, H^{-1/2}(\Gamma)) \to L^2(\mathbb{R}, H^{1/2}(\Gamma))$ (cf. Chapter 4 of [21]). We then compute

$$\begin{aligned}
&\int_{\mathbb{R}} \|Xv \times \nu + (\gamma((Xu) \times \nu)) \times \nu\|^2_{H^{1/2}(\Gamma)} \,\mathrm{d}\tau \\
&\quad = \int_{\mathbb{R}} \|X(v \times \nu + \gamma(u \times \nu) \times \nu) + [X, B]v + [X, B\gamma B]u\|^2_{H^{1/2}(\Gamma)} \,\mathrm{d}\tau \\
&\quad \lesssim \int_{\mathbb{R}} \Big(\|\omega\|^2_{H^{1/2}(\Gamma)} + \|v\|^2_{H^{-1/2}(\Gamma)} + \|u\|^2_{H^{-1/2}(\Gamma)} \Big) \,\mathrm{d}\tau\,,
\end{aligned}$$

using the boundary condition $v \times \nu + \gamma(u \times \nu) \times \nu = w$ of system (4.1) and the continuity of X on Γ. Since curl, div_α and div_β have $C^1 S^1_{cl}$-symbols, the commutators with X are bounded on L^2 by Proposition 4.1.C of [21]. The assertion now follows easily. $\square$

Exploiting the properties of X we can simplify the estimate above.

Corollary 4.6. *In the situation of Lemma 4.5 there is a number $\varkappa_0 > 0$ so that for all $\varkappa \in (0, \varkappa_0]$ in Definition 4.1 we have*

$$\begin{aligned}
\int_{\mathbb{R}} \Big(\|Xu\|^2_{H^1(\Omega)} + \|Xv\|^2_{H^1(\Omega)} \Big) \,\mathrm{d}\tau \lesssim \int_{\mathbb{R}} \Big(&\|u\|^2_{H(\mathrm{div}_\alpha)} + \|v\|^2_{H(\mathrm{div}_\beta)} + \|(u, v)\|^2_{L^2(\Gamma)} \\
&+ \|X(\xi, \zeta)\|^2_{L^2(\Omega)} + \|\omega\|^2_{H^{1/2}(\Gamma)} \Big) \,\mathrm{d}\tau\,.
\end{aligned}$$

Proof. Via (4.1) we rewrite the curl-terms in Lemma 4.5 as

$$\|X \operatorname{curl} u\|^2_{L^2} + \|X \operatorname{curl} v\|^2_{L^2} = \|X\partial_t(\alpha u) - X\xi\|^2_{L^2} + \|X\partial_t(\beta v) - X\zeta\|^2_{L^2}\,.$$

By Chapter 4 in [21] the commutator $[X, \alpha] \colon L^2(\mathbb{R}, H^1(\Omega)) \to L^2(\mathbb{R}, L^2(\Omega))$ is bounded. This fact and Lemmas 4.2 and 4.5 lead to

$$\begin{aligned}
\int_{\mathbb{R}} (\|Xu\|^2_{H^1} + \|Xv\|^2_{H^1}) \,\mathrm{d}\tau \lesssim \int_{\mathbb{R}} \Big(&\|\mathrm{div}_\alpha\, u\|^2_{L^2} + K_1(\varkappa)\, \|X(\alpha u)\|^2_{H^1} + \|\mathrm{div}_\beta\, v\|^2_{L^2} \\
&+ K_1(\varkappa)\, \|X(\beta v)\|^2_{H^1} + \|X(\xi, \zeta)\|^2_{L^2} + \|(u, v)\|^2_{L^2} \\
&+ \|(u, v)\|^2_{L^2(\Gamma)} + \|\omega\|^2_{H^{1/2}(\Gamma)} \Big) \,\mathrm{d}\tau \\
\lesssim \int_{\mathbb{R}} \Big(&\|\mathrm{div}_\alpha\, u\|^2_{L^2} + \|\mathrm{div}_\beta\, v\|^2_{L^2} + K_1(\varkappa)\, \|Xu\|^2_{H^1} \\
&+ K_1(\varkappa)\, \|Xv\|^2_{H^1} + \|(u, v)\|^2_{L^2} + \|(u, v)\|^2_{L^2(\Gamma)} \\
&+ \|X(\xi, \zeta)\|^2_{L^2} + \|\omega\|^2_{H^{1/2}(\Gamma)} \Big) \,\mathrm{d}\tau\,, \\
\int_{\mathbb{R}} (\|Xu\|^2_{H^1} + \|Xv\|^2_{H^1}) \,\mathrm{d}\tau \lesssim \int_{\mathbb{R}} \Big(&\|u\|^2_{H(\mathrm{div}_\alpha, \Omega)} + \|v\|^2_{H(\mathrm{div}_\beta, \Omega)} + \|(u, v)\|^2_{L^2(\Gamma)} + \\
&\|X(\xi, \zeta)\|^2_{L^2} + \|\omega\|^2_{H^{1/2}(\Gamma)} \Big) \,\mathrm{d}\tau\,,
\end{aligned}$$

if we choose $\varkappa$ and hence $K_1(\varkappa)$ small enough. $\square$

We will use this inequality in a moment to establish an estimate for the normal trace of αu and βv. For this we will have to split the trace term $\|(u,v)\|^2_{L^2(\Gamma)} \lesssim \int_0^T (\alpha u \cdot u + \beta v \cdot v)\,\mathrm{d}\tau$ appearing on the right-hand side of Corollary 4.6 into a normal and tangential part.

In the following, $u^\nu = (u\cdot\nu)\nu$ denotes the part of u in normal direction, whereas $u^\tau = \nu\times(u\times\nu)$ is the tangential part of u. We can thus decompose u as $u = u^\nu + u^\tau$.

Lemma 4.7. *In the situation of Lemma 4.5 above we have*

$$\int_s^t \int_\Gamma (\alpha u \cdot u + v \cdot \beta v)\,\mathrm{d}\sigma\,\mathrm{d}\tau \lesssim \int_s^t \int_\Gamma \left(|\nu\cdot\alpha u|^2 + |\nu\cdot\beta v|^2 + |u^\tau|^2 + |v^\tau|^2\right)\mathrm{d}\sigma\,\mathrm{d}\tau\,.$$

Proof. We restrict ourselves to the estimate for $\operatorname{tr}(u\cdot\alpha u)$. The estimate for v is shown analogously. Since the matrix operator α is assumed to be bounded and uniformly positive definite we can estimate

$$\begin{aligned}
\int_s^t \int_\Gamma \alpha u\cdot u\,\mathrm{d}\sigma\,\mathrm{d}\tau &= \int_s^t \int_\Gamma (u^\nu\cdot\alpha u^\nu + 2u^\tau \alpha u^\nu + u^\tau\cdot\alpha u^\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
&\lesssim \int_s^t \int_\Gamma \left(\sqrt{\delta}(u\cdot\nu)\frac{1}{\sqrt{\delta}}[\nu\cdot\alpha u^\nu] + |u^\tau|^2\right)\mathrm{d}\sigma\,\mathrm{d}\tau \\
&\lesssim \int_s^t \int_\Gamma \left(\delta\,|u\cdot\nu|^2 + \frac{1}{\delta}\,|\nu\cdot\alpha u^\nu|^2 + |u^\tau|^2\right)\mathrm{d}\sigma\,\mathrm{d}\tau \\
&\lesssim \int_s^t \int_\Gamma \left(\delta\,|u|^2 + \frac{1}{\delta}(|\nu\cdot\alpha u|^2 + |\nu\cdot\alpha u^\tau|^2) + |u^\tau|^2\right)\mathrm{d}\sigma\,\mathrm{d}\tau \\
&\lesssim \int_s^t \int_\Gamma \left(\delta u\cdot\alpha u + \frac{1}{\delta}(|\nu\cdot\alpha u|^2 + |u^\tau|^2) + |u^\tau|^2\right)\mathrm{d}\sigma\,\mathrm{d}\tau\,.
\end{aligned}$$

For $\delta > 0$ sufficiently small we thus obtain the assertion. ☐

The next lemma allows us to control the normal trace of the curl terms. See also Chapter 2.3 in [6] for related results.

Lemma 4.8. *For $f\in H^1(\Omega)$ we can estimate the normal surface curl by*

$$\|\nu\cdot(\operatorname{curl} f)\|_{H^{-1}(\Gamma)} \lesssim \|\nu\times f\|_{L^2(\Gamma)}\,.$$

Proof. Let $\phi\in H^1(\Gamma)$. this function can be extended in $H^{3/2}(\Omega)$ with $\|\phi\|_{H^{3/2}(\Omega)} \lesssim \phi_{H^1(\Gamma)}$. Let $f_n, \phi_m \in C_c^\infty(\overline{\Omega})$ with $f_n \to f$ in $H^1(\Omega)$ and $\phi_m \to \phi$ in $H^{3/2}(\Omega)$. The divergence theorem and integration by parts yield

$$\begin{aligned}
\int_\Gamma \nu\cdot(\operatorname{curl} f_n)\phi_m\,\mathrm{d}\sigma &= (\operatorname{curl} f_n, \nabla\phi_m)_{L^2(\Omega)} + (\nabla\cdot(\operatorname{curl} f_n), \phi_m)_{L^2(\Omega)} \\
&= (\operatorname{curl} f_n, \nabla\phi_m)_{L^2(\Omega)} \\
&= (f_n, \operatorname{curl}\nabla\phi_m)_{L^2(\Omega)} + \int_\Gamma (\nu\times f_n)\cdot\nabla\phi_m\,\mathrm{d}\sigma \\
&\leq 0 + \|\nu\times f_n\|_{L^2(\Gamma)}\,\|\operatorname{tr}(\nabla\phi_m)\|_{L^2(\Gamma)} \\
&\lesssim \|\nu\times f_n\|_{L^2(\Gamma)}\,\|\nabla\phi_m\|_{H^{1/2}(\Omega)}
\end{aligned}$$

$$\lesssim \|\nu\times f_n\|_{L^2(\Gamma)}\,\|\phi_m\|_{H^1(\Gamma)}\,.$$

As $m\to\infty$ we obtain the estimate

$$\|\nu\cdot(\operatorname{curl} f_n)\|_{H^{-1}(\Gamma)} \leq \|\nu\times f_n\|_{L^2(\Gamma)}\,.$$

Letting n tend to infinity then shows the claim. □

The main difficulty in the proof of Proposition 3.3 is that one has to bound trace terms of the electric and magnetic field by the tangential trace of $\partial_t^k E$. The boundary condition directly connects this trace with the tangential trace of the magnetic field. We thus inspect the normal boundary terms in detail. In the next lemma, one can remove the operator X on the right-hand side by continuity. We keep it in the statement in view of Corollary 4.10.

Lemma 4.9. *Let $(u, v) \in G^1$ be a solution to the system (4.1). Then there exists a constant $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0]$ and a constant c_δ depending on δ we get*

$$\begin{aligned}
\int_s^t \Big(\|\nu\cdot\alpha u\|^2_{L^2(\Gamma)} + \|\nu\cdot\beta v\|^2_{L^2(\Gamma)}\Big)\,\mathrm{d}\tau &\lesssim \int_s^t \Big(\delta\,\|u\|^2_{H(\mathrm{div}_\alpha)} + \delta\,\|v\|^2_{H(\mathrm{div}_\beta)} \\
&\quad + c_\delta\,\|\nu\cdot(\alpha u,\beta v)\|^2_{H^{-1}(\Gamma)} + \|\nu\times u\|^2_{L^2(\Gamma)} + \|\omega\|^2_{H^{1/2}(\Gamma)}\,\mathrm{d}\tau \\
&\quad + \int_{\mathbb{R}} \delta\,\|X(\xi,\zeta)\|^2_{L^2(\Omega)}\Big)\,\mathrm{d}\tau + \|(\nu\cdot\xi,\nu\cdot\zeta)\|^2_{H^{-1}(\Gamma_\infty)}
\end{aligned} \tag{4.3}$$

for $0 \le s \le t \le T$.

Proof. We show the estimate only for $s = 0$ and $t = T$ since general times can be treated analogously.

As $\alpha u \in G^1([0,T])$, the normal trace belongs to $C([0,T], L^2(\Gamma))$. Extending αu by 0 to $\mathbb{R}$ and applying Young's inequality, we obtain

$$\begin{aligned}
\int_0^T \|\nu\cdot\alpha u\|^2_{L^2(\Gamma)}\,\mathrm{d}\tau &= \|\nu\cdot\alpha u\|^2_{L^2(\Gamma_\infty)} \\
&\lesssim \|X(\nu\cdot\alpha u)\|^2_{L^2(\Gamma_\infty)} + \|(1-X)(\nu\cdot\alpha u)\|^2_{L^2(\Gamma_\infty)}\,.
\end{aligned} \tag{4.4}$$

We first estimate the second term on the right-hand side. Lemma 4.3 yields

$$\|(1-X)(\nu\cdot\alpha u)\|^2_{L^2(\Gamma_\infty)} \lesssim \|\nu\cdot\alpha u\|^2_{H^{-1}(\Gamma_\infty)} + \|\partial_t(\nu\cdot\alpha u)\|^2_{H^{-1}(\Gamma_\infty)}\,.$$

We have $\nu\cdot\partial_t(\alpha u) = \nu\cdot(\operatorname{curl} v) + \nu\cdot\xi$ by (4.1) and

$$\|\nu\cdot(\operatorname{curl} v)\|_{H^{-1}(\Gamma_\infty)} \lesssim \|\nu\times v\|_{L^2(\Gamma_\infty)} < \infty$$

by Lemma 4.8. Hence the difference $\nu\cdot\xi$ is also contained in $H^{-1}(\Gamma_T)$, and we can continue the above estimate by

$$\begin{aligned}
&\|(1-X)\nu\cdot(\alpha u)\|^2_{L^2(\Gamma_\infty)} \\
&\lesssim \int_0^T \Big(\|\nu\cdot\alpha u\|^2_{H^{-1}(\Gamma)} + \|\nu\times v\|^2_{L^2(\Gamma)}\Big)\,\mathrm{d}\tau + \|\nu\cdot\xi\|^2_{H^{-1}(\Gamma_\infty)}
\end{aligned} \tag{4.5}$$

By interpolation, the first term of equation (4.4) can be bounded by

$$\begin{aligned}
\|X(\nu\cdot\alpha u)\|^2_{L^2(\Gamma_\infty)} &\lesssim \int_{\mathbb{R}} \Big(\tilde\delta\,\|X(\nu\cdot\alpha u)\|^{2/3}_{H^{1/2}(\Gamma)}\,\frac{1}{\tilde\delta}\,\|X(\nu\cdot\alpha u)\|^{1/3}_{H^{-1}(\Gamma)}\Big)^2\,\mathrm{d}\tau\,. \\
&\lesssim \int_{\mathbb{R}} \Big(\tilde\delta^{3/2}\,\|X\nu\cdot\alpha u\|^2_{H^{1/2}(\Gamma)} + \tilde\delta^{-3}\,\|X\nu\cdot\alpha u\|^2_{H^{-1}(\Gamma)}\Big)\,\mathrm{d}\tau
\end{aligned}$$

for an arbitrary $\tilde\delta = \delta^{2/3} > 0$. Since $\Gamma \in C^5$ and $\alpha \in C^1$, the commutators $[X,\nu]$ and $[X,\alpha]$ are $L^2(\mathbb{R}, H^{-1/2})$-$L^2(\mathbb{R}, H^{1/2})$-bounded (cf. Chapter 4 in [21]). It follows that

$$\begin{aligned}
J &:= \int_{\mathbb{R}} (\delta \left\| X\nu \cdot \alpha u \right\|^2_{H^{1/2}(\Gamma)} + c_\delta \left\| X\nu \cdot \alpha u \right\|^2_{H^{-1}(\Gamma)}) \,\mathrm{d}\tau \\
&\lesssim \int_{\mathbb{R}} \Big(\delta(\left\| Xu \right\|^2_{H^{1/2}(\Gamma)} + \left\| u \right\|^2_{H^{-1/2}(\Gamma)}) + c_\delta \left\| \nu \cdot \alpha u \right\|^2_{H^{-1}(\Gamma)} \Big) \,\mathrm{d}\tau \\
&\lesssim \int_{\mathbb{R}} \Big(\delta(\left\| Xu \right\|^2_{H^{1}(\Omega)} + \left\| u \right\|^2_{L^2(\Gamma)}) + c_\delta \left\| \nu \cdot \alpha u \right\|^2_{H^{-1}(\Gamma)} \Big) \,\mathrm{d}\tau \,. \\
&\lesssim \int_{\mathbb{R}} \Big(\delta(\left\| u \right\|^2_{H(\operatorname{div}_\alpha)} + \left\| v \right\|^2_{H(\operatorname{div}_\beta)} + \left\| (u,v) \right\|^2_{L^2(\Gamma)} + \left\| X(\xi,\zeta) \right\|^2_{L^2(\Omega)} \\
&\qquad + \left\| \omega \right\|^2_{H^{1/2}(\Gamma)}) + c_\delta \left\| \nu \cdot \alpha u \right\|^2_{H^{-1}(\Gamma)} \Big) \,\mathrm{d}\tau \,,
\end{aligned}$$

where we also used Corollary 4.6. Lemma 4.7 now allows us to split the trace term into normal and tangential parts

$$\begin{aligned}
J &\lesssim \int_{\mathbb{R}} \Big(\delta\big(\left\| u \right\|^2_{H(\operatorname{div}_\alpha)} + \left\| v \right\|^2_{H(\operatorname{div}_\beta)} + \left\| \nu \cdot (\alpha u, \beta v) \right\|^2_{L^2(\Gamma)} + \left\| \nu \times (u,v) \right\|^2_{L^2(\Gamma)} \\
&\qquad + \left\| X(\xi,\zeta) \right\|^2_{L^2(\Omega)} + \left\| \omega \right\|^2_{H^{1/2}(\Gamma)} \big) + c_\delta \left\| \nu \cdot \alpha u \right\|^2_{H^{-1}(\Gamma)} \Big) \,\mathrm{d}\tau \\
&\lesssim \delta \int_0^T \left\| \nu \cdot (\alpha u, \beta v) \right\|^2_{L^2(\Gamma)} \,\mathrm{d}\tau + \delta \int_{\mathbb{R}} \left\| X(\xi,\zeta) \right\|^2_{L^2(\Omega)} \,\mathrm{d}\tau + \int_0^T \Big(\delta\big(\left\| u \right\|^2_{H(\operatorname{div}_\alpha)} \\
&\quad + \left\| v \right\|^2_{H(\operatorname{div}_\beta)} + \left\| \nu \times (u,v) \right\|^2_{L^2(\Gamma)} + \left\| \omega \right\|^2_{H^{1/2}(\Gamma)} \big) + c_\delta \left\| \nu \cdot \alpha u \right\|^2_{H^{-1}(\Gamma)} \Big) \,\mathrm{d}\tau
\end{aligned}$$

We have the same estimate for $\nu \cdot \beta v$ in (4.4). By choosing $\delta > 0$ sufficiently small, we absorb the normal trace term by the left-hand side of (4.4). We infer

$$\begin{aligned}
\int_0^T \Big(\left\| \nu \cdot \alpha u \right\|^2_{L^2(\Gamma)} + \left\| \nu \cdot \beta v \right\|^2_{L^2(\Gamma)} \Big) \,\mathrm{d}\tau &\lesssim \int_0^T \Big(\delta \left\| u \right\|^2_{H(\operatorname{div}_\alpha)} + \delta \left\| v \right\|^2_{H(\operatorname{div}_\beta)} \\
&\quad + c_\delta \left\| \nu \cdot (\alpha u, \beta v) \right\|^2_{H^{-1}(\Gamma)} + \left\| \nu \times (u,v) \right\|^2_{L^2(\Gamma)} + \delta \left\| \omega \right\|^2_{H^{1/2}(\Gamma)} \Big) \,\mathrm{d}\tau \\
&\quad + \int_{\mathbb{R}} \delta \left\| X(\xi,\zeta) \right\|^2_{L^2(\Omega)} \,\mathrm{d}\tau + \left\| (\nu \cdot \xi, \nu \cdot \zeta) \right\|^2_{H^{-1}(\Gamma_\infty)} \,,
\end{aligned}$$

using also (4.5). Finally we note that the boundary condition in (4.1) yields

$$\int_0^T \left\| \nu \times v \right\|^2_{L^2(\Gamma)} \,\mathrm{d}\tau \lesssim \int_0^T \Big(\left\| \nu \times u \right\|^2_{L^2(\Gamma)} + \left\| \omega \right\|^2_{L^2(\Gamma)} \Big) \,\mathrm{d}\tau \,, \tag{4.6}$$

from which the claim follows. □

We now rewrite the above result for the problem (3.6) where $\xi = \partial_t \varphi$ and $\zeta = \partial_t \psi$. Here we can apply Lemma 4.2 to the inhomogeneities in (4.3).

Corollary 4.10. *Let $\alpha, \beta, \varphi, \psi, \omega, u^{(0)}$ and $v^{(0)}$ satisfy* (3.7) *and* (3.8). *Then the solution $(u, v) \in C([0,T], L^2(\Omega))$ to* (3.6) *possesses a trace in $L^2([0,T], L^2(\Gamma))$ satisfying*

$$\begin{aligned}
&\int_0^T \Big(\|\nu \cdot \alpha u\|_{L^2(\Gamma)}^2 + \|\nu \cdot \beta v\|_{L^2(\Gamma)}^2 \Big) \,\mathrm{d}\tau \\
&\quad \lesssim \int_0^T \Big(\delta \, \|u\|_{H(\mathrm{div}_\alpha)}^2 + \delta \, \|v\|_{H(\mathrm{div}_\beta)}^2 + \|(\varphi, \psi)\|_{H^1(\Omega)}^2 \\
&\qquad + c_\delta \, \|\nu \cdot (\alpha u, \beta v)\|_{H^{-1}(\Gamma)}^2 + \|\nu \times u\|_{L^2(\Gamma)}^2 + \|\omega\|_{H^{1/2}(\Gamma)}^2 \Big) \,\mathrm{d}\tau \\
&\qquad + \|(\nu \cdot \partial_t \varphi, \nu \cdot \partial_t \psi)\|_{H^{-1}(\Gamma_\infty)}^2 \,.
\end{aligned} \tag{4.7}$$

Proof. For $(u, v) \in G^1$ the assertion follows from Lemma 4.9 with $\xi = \partial_t \varphi$ and $\zeta = \partial_t \psi$, where we apply Lemma 4.2 to the term with X on the right-hand side. It remains to remove the assumption $(u, v) \in G^1$. This can be done as in the proof of Lemma 3.4 in [18]. One approximates the initial data in $H(\mathrm{div}_{\alpha(0)}) \times H(\mathrm{div}_{\beta(0)})$ by $(u_n^{(0)}, v_n^{(0)}) \in H^1(\Omega)$ satisfying the compatibility condition $v_n^{(0)} \times \nu + (\gamma(u_n^{(0)} \times \nu))\nu = 0$, (φ, ψ) in G^1 by $(\varphi_n, \psi_n) \in G^2$ and ω in $L^2([0,T], H^{1/2}(\Gamma))$ with $\omega_n \in H^1([0,T] \times \Gamma)$ with $\omega \cdot \nu = 0$ and $\omega_n(0) = 0$. By Theorem 1.3 and Remark 2.1 in [5] there are solutions $(u_n, v_n) \in G^1$. According to Proposition 1.1 in [5] these solutions converge to (u, v) in $C([0,T], L^2(\Omega))$ and $(\mathrm{tr}_t \, u_n, \mathrm{tr}_t \, v_n)$ to $(\mathrm{tr}_t \, u, \mathrm{tr}_t \, v)$ in $L^2([0,T] \times \Omega)$. Moreover $\mathrm{div}(\alpha u_n) = \mathrm{div}(\varphi_n) \to \mathrm{div}(\alpha u)$ and $\mathrm{div}(\beta v_n) = \mathrm{div}(\psi_n) \to \mathrm{div}(\beta v)$ in $C([0,T], L^2(\Omega))$. As a result the traces $\mathrm{tr}_n(\alpha u_n)$ and $\mathrm{tr}_n(\beta v_n)$ converge in $C([0,T], H^{-1/2}(\Gamma))$. The assertion for (u, v) now follows from the claim for (u_n, v_n). $\square$

Remark 4.11. In Corollary 4.10 we can insert $\mathrm{div}(\alpha u) = \varphi$ and $\mathrm{div}(\beta v) = \psi$ and estimate

$$\|(\nu \cdot \partial_t \varphi, \nu \cdot \partial_t \psi)\|_{H^{-1}(\Gamma_\infty)}^2 \lesssim \|(\nu \cdot \varphi, \nu \cdot \psi)\|_{L^2([0,T], L^2(\Gamma)}^2 \lesssim \|(\varphi, \psi)\|_{L^2([0,T], H^1(\Omega))}^2 \,.$$

Moreover, we can replace $c_\delta \, \|\nu \cdot (\alpha u, \beta v)\|_{H^{-1}(\Gamma)}^2$ by

$$c_\delta \big(\, \|u\|_{H(\mathrm{div}_\alpha)}^2 + \|v\|_{H(\mathrm{div}_\beta)}^2 \, \big) \,.$$

The above estimate has time independent constants and can therefore be used below to show Proposition 3.3 and thus Theorem 3.1. Concluding this section, we also prove a variant only having norms of data on the right-hand side but with time depending constants. In this way we give a different proof of Theorem 1.2 in [5]. In view of Remark 4.11 we have to remove the L^2-norm of u and v and the tangential trace term on the right-hand side of (4.3). We control these terms by means of the energy estimate for (3.6). It was shown in Lemma 3.2 of [18] for $\omega = 0$. An obvious modification of this proof gives the equality

$$\begin{aligned}
&\Big\| \alpha(t)^{1/2} u(t) \Big\|_{L^2}^2 + \Big\| \beta(t)^{1/2} v(t) \Big\|_{L^2}^2 + 2 \int_0^t \Big\| \gamma^{1/2} \, \mathrm{tr}_t \, u(\tau) \Big\|_{L^2(\Gamma)}^2 \,\mathrm{d}\tau \\
&= \Big\| \alpha(0)^{1/2} u(0) \Big\|_{L^2}^2 + \Big\| \beta(0)^{1/2} v(0) \Big\|_{L^2}^2 + 2 \int_0^t \int_\Omega (u \cdot \partial_t \varphi + v \cdot \partial_t \psi) \,\mathrm{d}x \,\mathrm{d}\tau \\
&\quad + \int_0^t \int_\Gamma \mathrm{tr}_t \, u \cdot \omega \,\mathrm{d}\sigma \,\mathrm{d}\tau + \int_0^t \int_\Omega (u \cdot (\partial_t \alpha) u + v \cdot (\partial_t \beta) v) \,\mathrm{d}x \,\mathrm{d}\tau \,,
\end{aligned} \tag{4.8}$$

for $0 \le t \le T$. To remove the trace term on the right-hand side we estimate

$$\left| \int_0^t \int_\Gamma \operatorname{tr}_t u \cdot \omega \, \mathrm{d}\sigma \, \mathrm{d}\tau \right| \le \delta \int_0^t \|\operatorname{tr}_t u\|_{L^2(\Gamma)}^2 \, \mathrm{d}\tau + c_\delta \int_0^t \|\omega\|_{L^2(\Gamma)}^2 \, \mathrm{d}\tau \,. \tag{4.9}$$

Since $\gamma \ge \eta$, setting $\delta = \frac{\eta}{2}$ we arrive at

$$\begin{aligned} \|(u(t), v(t))\|_{L^2}^2 + \int_0^t \|\operatorname{tr}_t u(\tau)\|_{L^2(\Gamma)}^2 \, \mathrm{d}\tau &\lesssim \|(u(0), v(0))\|_{L^2}^2 \\ &+ \overline{c} \int_0^t \|(u,v)\|_{L^2}^2 \, \mathrm{d}\tau + \int_0^t \Big(\|(\partial_t \varphi, \partial_t \psi)\|_{L^2}^2 + \|\omega\|_{L^2(\Gamma)}^2 \Big) \, \mathrm{d}\tau \end{aligned} \tag{4.10}$$

with $\overline{c} := c + \max\{\|\partial_t \alpha\|_\infty, \|\partial_t \beta\|_\infty\}$ where $c = 0$ if $\partial_t \varphi = \partial_t \psi = 0$ and $c = 1$ otherwise. Gronwall's inequality now implies

$$\begin{aligned} &\|(u(t), v(t))\|_{L^2}^2 + \int_0^t \|\operatorname{tr}_t u(\tau)\|_{L^2(\Gamma)}^2 \, \mathrm{d}\tau \\ &\qquad \lesssim \bigg(\|(u(0), v(0))\|_{L^2}^2 + \int_0^t \Big(\|(\partial_t \varphi, \partial_t \psi)\|_{L^2}^2 + \|\omega\|_{L^2(\Gamma)}^2 \Big) \, \mathrm{d}\tau \bigg) \mathrm{e}^{\overline{c}t} \end{aligned} \tag{4.11}$$

Using also $\operatorname{div}(\alpha u) = \varphi$ and $\operatorname{div}(\beta v) = \psi$, from Corollary 4.10 and Remark 4.11 we derive a trace estimate for the inhomogeneous, linear Maxwell System (3.6). It can directly be applied to (1.1)–(1.4).

Proposition 4.12. *Assume that conditions* (3.7) *and* (3.8) *hold and let* $(u, v) \in C([0, T], L^2(\Omega))$ *solve* (3.6). *We obtain*

$$\begin{aligned} \int_0^t \|\operatorname{tr}(u,v)\|_{L^2(\Gamma)}^2 \, \mathrm{d}\tau &\lesssim_T \left\| \left(u^{(0)}, v^{(0)}\right) \right\|_{L^2 \Omega}^2 \\ &+ \int_0^t \Big(\|(\partial_t \varphi, \partial_t \psi)\|_{L^2(\Omega)}^2 + \|(\varphi, \psi)\|_{H^1(\Omega)}^2 + \|\omega\|_{L^2(\Gamma)}^2 \Big) \, \mathrm{d}\tau \end{aligned}$$

for $0 \le t \le T$.

As in Theorem 1.2 of [5] the constant depends on T. We will thus only use Corollary 4.10 in the following.

5. Proof of Proposition 3.3

We again consider system (3.6) assuming (3.7), (3.8) and

$$\alpha + (m \cdot \nabla)\alpha \ge \tilde{\eta}\alpha, \quad \beta + (m \cdot \nabla)\beta \ge \tilde{\eta}\beta \tag{5.1}$$

for $\tilde{\eta}$. The proof of Lemma 3.4 in [18] then yields the following result. In [18] it was assumed that Ω is strictly starshaped giving control on the full trace of (u, v) which now appears on the right-hand side of (5.2). Moreover the boundary inhomogeneity ω can simply be included in the calculations of [18].

Lemma 5.1. *Let* (3.7), (3.8) *and* (5.1) *hold. Then the weak solutions* $(u,v) \in C([0,T], L^2(\Omega,\mathbb{R}^6))$ *of system* (3.6) *fulfill*

$$
\begin{aligned}
&\int_s^t \int_\Omega (\alpha u\cdot u + \beta v\cdot v)\,\mathrm{d}(x,\tau)\\
&\quad\lesssim \int_s^t \int_\Gamma |u\times\nu|^2\,\mathrm{d}x\,\mathrm{d}\tau + \|(u(s),v(s))\|_{L^2}^2 + \|(u(t),v(t))\|_{L^2}^2\\
&\qquad + \int_s^t \int_\Omega (|\partial_t\varphi|\,|v| + |\partial_t\psi|\,|u| + |\operatorname{div}\varphi|\,|u| + |\operatorname{div}\psi|\,|v|)\,\mathrm{d}x\,\mathrm{d}\tau\\
&\qquad + \int_s^t \int_\Gamma \operatorname{tr}(\alpha u\cdot u + \beta v\cdot v)\,\mathrm{d}\sigma\,\mathrm{d}\tau + \int_s^t \int_\Gamma |\omega\cdot\operatorname{tr} v|\,\mathrm{d}\sigma\,\mathrm{d}\tau
\end{aligned}
\tag{5.2}
$$

for $0\le s\le t\le T$. *Note that the last term is bounded by* $c\int_s^t\int_\Gamma \operatorname{tr}(\beta v\cdot v)\,\mathrm{d}\sigma\,\mathrm{d}\tau + c\int_s^t \|\omega\|_{L^2(\Gamma)}^2\,\mathrm{d}\tau$.

We still have to estimate the integral $\int_s^t\int_\Gamma \operatorname{tr}(\alpha u\cdot u+\beta v\cdot v)\,\mathrm{d}\sigma\,\mathrm{d}\tau$. By Lemma 4.7 it can be split into

$$
\int_s^t \int_\Gamma \operatorname{tr}(\alpha u\cdot u + \beta v\cdot v)\,\mathrm{d}\sigma\,\mathrm{d}\tau \lesssim \int_s^t \int_\Gamma \Big(|\nu\cdot\alpha u|^2 + |u^\tau|^2 + |\nu\cdot\beta v|^2 + |v^\tau|^2\Big)\,\mathrm{d}\sigma\,\mathrm{d}\tau
$$

The tangential trace of v can be controlled through the boundary condition by

$$
\int_s^t \|v^\tau\|_{L^2(\Gamma)}^2\,\mathrm{d}\tau \lesssim \int_s^t \Big(\|\nu\times u\|_{L^2(\Gamma)}^2 + \|\omega\|_{L^2(\Gamma)}^2\Big)\,\mathrm{d}\tau\,.
$$

Using Corollary 4.10 and Remark 4.11, we derive

$$
\begin{aligned}
\int_s^t \int_\Gamma \Big(|\nu\cdot\alpha u|^2 + |\nu\cdot\beta v|^2\Big)\,\mathrm{d}\sigma\,\mathrm{d}\tau &\lesssim \int_s^t \Big(\hat\delta\,\|u\|_{L^2(\Omega)}^2 + \hat\delta\,\|v\|_{L^2(\Omega)}^2 + \|(\varphi,\psi)\|_{H^1(\Omega)}^2\\
&\quad + \|\nu\times u\|_{L^2(\Gamma)}^2 + \|\omega\|_{H^{1/2}(\Gamma)}^2 + c_{\hat\delta}\,\|\nu\cdot(\alpha u,\beta v)\|_{H^{-1}(\Gamma)}^2\Big)\,\mathrm{d}\tau\,.
\end{aligned}
$$

For sufficiently small $\hat\delta>0$ the norms in L^2 above can be absorbed by the left-hand side of (5.2). We have shown the following estimate.

Lemma 5.2. *Let* (3.7), (3.8) *and* (5.1) *be true. Then the weak solutions* $(u,v)\in C([0,T],L^2(\Omega,\mathbb{R}^6))$ *of system* (3.6) *fulfill*

$$
\begin{aligned}
&\int_s^t \int_\Omega (\alpha u\cdot u + \beta v\cdot v)\,\mathrm{d}x\,\mathrm{d}\tau + \int_s^t \int_\Gamma \operatorname{tr}(\alpha u\cdot u + \beta v\cdot v)\\
&\quad\lesssim \int_s^t \int_\Gamma |u\times\nu|^2\,\mathrm{d}x\,\mathrm{d}\tau + \|(u(s),v(s))\|_{L^2}^2 + \|(u(t),v(t))\|_{L^2}^2\\
&\qquad + \int_s^t \int_\Omega \big(|\partial_t\varphi|\,|v| + |\partial_t\psi|\,|u| + |\operatorname{div}\varphi|\,|u| + |\operatorname{div}\psi|\,|v|\big)\,\mathrm{d}x\,\mathrm{d}\tau\\
&\qquad + \int_s^t \big(\|(\varphi,\psi)\|_{H^1} + \|\omega\|_{H^{1/2}(\Gamma)} + \|\nu\cdot(\alpha u,\beta v)\|_{H^{-1}(\Gamma)}\big)\,\mathrm{d}\tau\,.
\end{aligned}
\tag{5.3}
$$

for $0\le s\le t\le T$.

One could estimate the terms in the second line of the right-hand side by

$$
\delta\int_s^t \int_\Omega \big(|u|^2 + |v|^2\big)\,\mathrm{d}x\,\mathrm{d}\tau + c_\delta\int_s^t \int_\Omega \big(|\partial_t\varphi|^2 + |\partial_t\psi|^2 + |\operatorname{div}\varphi|^2 + |\operatorname{div}\phi|^2\big)\,\mathrm{d}x\,\mathrm{d}\tau\,,
$$

and absorb the first summand for small $\delta > 0$ by the left-hand side. In the following this is not needed, since these summands can be put into small error terms.

In order to obtain Proposition 3.3 it remains to estimate the H^{-1}-norm of the normal traces. To proceed we return to our original problem. Here it does not help to bound them by the norm of u and v in $H(\operatorname{div}_\alpha)$ and $H(\operatorname{div}_\beta)$, since the time integrals of L^2-norms of u and v cannot be absorbed by the left-hand side.

We exploit that the time derivatives of solutions still solve a Maxwell system. We begin with the case for $\left(\partial_t^k E, \partial_t^k H\right)$ with $k \geq 1$ and then use the following result to prove the case $k = 0$. Again we work under the assumptions of Theorem 3.1.

Lemma 5.3. *For the solution $(E, H) \in G^3$ of (1.1)–(1.4) we can estimate the normal trace by*

$$\int_s^t \left\|\nu \cdot (\widehat{\varepsilon}_k \partial_t^k E, \widehat{\mu}_k \partial_t^k H)\right\|_{H^{-1}(\Gamma)}^2 \,\mathrm{d}\tau \lesssim \int_s^t \left(d(\tau) + z(\tau)^2\right) \mathrm{d}\tau$$

for $0 \leq s \leq t < T_*$ and $k \in \{1, 2, 3\}$.

Proof. Let $k \in \{1, 2, 3\}$. Due to (2.11) we can rewrite $\widehat{\varepsilon}_k \partial_t^k E$ as

$$\widehat{\varepsilon}_k \partial_t^k E = \partial_t^k(\varepsilon E) - f_k = \partial_t(\widehat{\varepsilon}_{k-1} \partial_t^{k-1} E + f_{k-1}) - f_k\,.$$

The Maxwell system (2.12) then leads to

$$\widehat{\varepsilon}_k \partial_t^k E = \operatorname{curl} \partial_t^{k-1} H - \partial_t f_{k-1} + \partial_t f_{k-1} - f_k = \operatorname{curl} \partial_t^{k-1} H - f_k$$

Let $\Phi \in H^1(\Gamma)$. These formulas and Lemma 4.8 yield

$$\begin{aligned} \langle \nu \cdot \widehat{\varepsilon}_k \partial_t^k E, \Phi \rangle_{H^{-1/2}(\Gamma)} &= \langle \nu \cdot (\operatorname{curl} \partial_t^{k-1} H), \Phi \rangle_{H^{-1}(\Gamma)} - \langle \nu \cdot f_k, \Phi \rangle_{L^2(\Gamma)} \\ &\lesssim \left\|\nu \times \partial_t^{k-1} H\right\|_{L^2(\Gamma)} \|\Phi\|_{H^1(\Gamma)} + \|f_k\|_{L^2(\Gamma)} \|\Phi\|_{L^2(\Gamma)}\,. \end{aligned}$$

Dividing by $\|\Phi\|_{H^1(\Gamma)}$, we infer

$$\begin{aligned} \left\|\nu \cdot \widehat{\varepsilon}_k \partial_t^k E\right\|_{H^{-1}(\Gamma)} &\lesssim \left\|\nu \times \partial_t^{k-1} H\right\|_{L^2(\Gamma)} + \|f_k\|_{H^1(\Omega)} \\ &\lesssim \left\|\nu \times \partial_t^{k-1} E\right\|_{L^2(\Gamma)} + \|h_{k-1}\|_{L^2(\Gamma)} + \|f_k\|_{H^1(\Omega)}\,, \end{aligned}$$

where we also used the boundary condition (2.13). Analogously we treat $\widehat{\mu}_k \partial_t^k H$. Estimate (2.15) now implies the assertion. □

Lemma 5.4. *For the solution $(E, H) \in G^3$ of (1.1)–(1.4) we estimate*

$$\int_s^t \|\nu \cdot (\widehat{\varepsilon}_0 E, \widehat{\mu}_0 H)\|_{H^{-1}(\Gamma)}^2 \lesssim \int_s^t d(\tau) \,\mathrm{d}\tau + (e_1(t) + e_1(s)) + \int_s^t z^2(\tau) \,\mathrm{d}\tau$$

for $0 \leq s \leq t < T_*$.

Proof. The usual normal trace estimate, (1.2), Lemma A.1 b), (2.11) and (2.12) yield

$$\begin{aligned} \int_s^t \|\nu \cdot (\widehat{\varepsilon}_0 E, \widehat{\mu}_0 H)\|_{H^{-1}(\Gamma)}^2 \,\mathrm{d}\tau &\lesssim \int_s^t \|(\widehat{\varepsilon}_0 E, \widehat{\mu}_0 H)\|_{H(\operatorname{div})}^2 \,\mathrm{d}\tau \\ &= \int_s^t \|(\widehat{\varepsilon}_0 E, \widehat{\mu}_0 H)\|_{L^2(\Omega)}^2 \,\mathrm{d}\tau \\ &\lesssim \int_s^t \|(\operatorname{curl} E, \operatorname{curl} H)\|_{L^2(\Omega)}^2 \,\mathrm{d}\tau \end{aligned}$$

$$= \int_s^t \|(\widehat{\varepsilon}_1 \partial_t E, \widehat{\mu}_1 \partial_t H)\|_{L^2(\Omega)}^2 \,\mathrm{d}\tau$$
$$\lesssim \int_s^t (\partial_t E \cdot \widehat{\varepsilon}_1 \partial_t E + \partial_t H \cdot \widehat{\mu}_1 \partial_t H) \,\mathrm{d}\tau \,.$$

Now using Lemma 5.2 with $u = \partial_t E$, $v = \partial_t H$, $\alpha = \widehat{\varepsilon}_1$, $\beta = \widehat{\mu}_1$ and $\varphi = \psi = \omega = 0$, we obtain

$$\int_s^t \|\nu \cdot (\widehat{\varepsilon}_0 E, \widehat{\mu}_0 H)\|_{H^{-1}(\Gamma)}^2 \,\mathrm{d}\tau \lesssim \int_s^t d(\tau) \,\mathrm{d}\tau + (e_1(t) + e_1(s)) + \int_s^t \|\nu \cdot (\widehat{\varepsilon}_1 \partial_t E, \widehat{\mu}_1 \partial_t H)\|_{H^{-1}(\Gamma)}^2 \,\mathrm{d}\tau \,.$$

By the previous Lemma the last term can be estimated by $\int_s^t \big(d(\tau) + z(\tau)^2\big) \,\mathrm{d}\tau$ and the claim follows. □

Combining the lemmas above and (2.15), we have shown Proposition 3.3.

6. Proof of Proposition 3.4

We recall the crucial regularity result from Section 3.

Proposition 3.4. *Under the conditions of Theorem 3.1 with the exception of* (3.1) *the following estimate holds*

$$z(t) \lesssim e(t) + z(t)^2$$

for all $t \in [0, T_*)$.

Due to nonlinear boundary conditions, one has to account for extra terms compared to the proof of Proposition 4.1 in [18], Section 6, but we can still follow the reasoning given there. Therefore we will only outline the proof and focus on the differences to [18].

Proof. Remember that z is given by

$$z(t) = \max_{0 \le j \le 3} \left(\left\| \partial_t^j E(t) \right\|_{H^{3-j}(\Omega)}^2 + \left\| \partial_t^j H(t) \right\|_{H^{3-j}(\Omega)}^2 \right)$$

and e is defined as

$$e(t) = \tfrac{1}{2} \max_{0 \le j \le 3} \left(\left\| \widehat{\varepsilon}_j^{1/2} \partial_t^j E(t) \right\|_{L^2(\Omega)}^2 + \left\| \widehat{\mu}_j^{1/2} \partial_t^j H(t) \right\|_{L^2(\Omega)}^2 \right) .$$

The L^2-terms of z can thus be trivially bounded by $e(t)$. The other (squared) norms will be estimated by $e(t)$ or $z(t)^2$ or by means of previous steps.

Compared to [18] we only have to modify the arguments that involve boundary conditions of differentiated problems. Here additional error terms appear as λ also depends on E. They enter our reasoning only through the div-curl Lemma A.1 which controls the H^1-norm for fields (u, v) by the estimate (A.1). Here we have to control the $H^{1/2}$-norm of the boundary inhomogeneities on the right-hand side. We follow the steps of the proof given in Section 6 of [18].

1) H^1-estimates for $\partial_t^k E$ and $\partial_t^k H$. The (squared) H^1-norm of $\partial_t^k (E, H)$ for $k \in \{0, 1, 2\}$ is bounded vie the div-curl estimate. Only for $k = 2$ the boundary inhomogeneity h_2 is nonzero, where we have $\|h_2\|_{H^{1/2}(\Gamma)} \lesssim z(t)^2$ by (2.15), as desired.

2) Interior spatio-temporal estimates for E and H. One decomposes $(E,H) = (1-\chi)(E,H)+\chi(E,H)$ for a cutoff function χ, being equal to 1 near the boundary Γ. In the estimates for $(1-\chi)\partial_t^k(E,H)$ the boundary does not play a role so that the reasoning from [18] has not to be modified.

3) Preparation for the boundary collar estimates for E and H. To get H^2 or H^3 bounds on $\partial_t^k\chi(E,H)$, we employ tangential and normal derivatives ∂_τ and ∂_ν (extended to a neighborhood of Γ). We use the div-curl Lemma A.1 for $\partial_\tau\partial_t^k(\chi(E,H))$ with $k\in\{0,1\}$ and for $\partial_\tau^2(\chi(E,H))$. Differentiating (2.13) with $h_0=0$ or $h_1=0$ we obtain

$$
\begin{aligned}
&\partial_\tau\partial_t^k\chi H\times\nu + \Big(\hat\lambda_k(\partial_\tau\partial_t^k\chi E\times\nu)\Big)\times\nu = -\partial_t^k\chi H\times\partial_\tau\nu\\
&\qquad -\Big(\partial_\tau\hat\lambda_k(\partial_t^k\chi E\times\nu)\Big)\times\nu - \Big(\hat\lambda_k(\partial_t^k\chi E\times\partial_\tau\nu)\Big)\times\nu\\
&\qquad -\Big(\hat\lambda_k(\partial_t^k\chi E\times\nu)\Big)\times\partial_\tau\nu\,.
\end{aligned}\tag{6.1}
$$

and

$$
\begin{aligned}
&\partial_\tau^2(\chi H)\times\nu + \big(\lambda\partial_\tau^2(\chi E)\times\nu\big)\times\nu = -2\partial_\tau(\chi H)\times\partial_\tau\nu - (\chi H)\times\partial_\tau^2\nu\\
&\quad -\big((\partial_\tau^2\lambda)(\chi E\times\nu)\big)\times\nu - \big(\lambda(\chi E)\times\partial_\tau^2\nu\big)\times\nu - \big(\lambda(\chi E)\times\nu\big)\times\partial_\tau^2\nu\\
&\quad -2\big((\partial_\tau\lambda)(\partial_\tau(\chi E)\times\nu)\big)\times\nu - 2\big((\partial_\tau\lambda)(\chi E\times\partial_\tau\nu)\big)\times\nu\\
&\quad -2\big((\partial_\tau\lambda)(\chi E\times\nu)\big)\times\partial_\tau\nu - 2\big(\lambda\partial_\tau(\chi E)\times\partial_\tau\nu\big)\times\nu\\
&\quad -2\big(\lambda(\partial_\tau\chi E)\times\nu\big)\times\partial_\tau\nu - 2\big(\lambda\chi E\times\partial_\tau\nu\big)\times\partial_\tau\nu\,.
\end{aligned}
$$

with $\hat\lambda_0=\lambda(\cdot,E)$ and $\hat\lambda_1=\lambda^{\mathrm{d}}(\cdot,E)$. The remaining estimates involving ∂_ν are entirely based on the curl and div equations (2.12) and (2.14) and thus carry over from [18].

4) H^2-estimates for E and H. When estimating (E,H) in H^2 we need the boundary condition (6.1) with $k=0$. Compare to [18], the only new term is

$$\tilde h_k := \Big(\partial_\tau\hat\lambda_k(\partial_t^k E\times\nu)\Big)\times\nu$$

with $k=0$, where we note that

$$\partial_j\hat\lambda_k(x,t) = (\partial_j\hat\lambda_k)(x,E(x,t)) + \sum_{i=1}^3 \partial_{\xi_i}\hat\lambda_k(x,E(x,t))\partial_j E_i(x,t)\,.\tag{6.2}$$

We can thus bound this error term by

$$
\begin{aligned}
\big\|\tilde h_0\big\|_{H^{1/2}(\Gamma)} &\lesssim \|\partial_\tau\lambda(\cdot,E)E\|_{H^1}\\
&\lesssim \|\partial_x\lambda\|_{W^{1,\infty}}\|E\|_{H^1} + \|\partial_\xi\lambda\|_{W^{1,\infty}}\|E\|_{W^{1,\infty}}\|E\|_{H^1}\\
&\lesssim \|E\|_{H^1} + z(t)
\end{aligned}\tag{6.3}
$$

as desired, using Sobolev's embedding and $z\le 1$ by (2.8). (Observe that $\|E\|_{H^1}$ and $\|\partial_t E\|_{H^1}$ were already handled in step *1)*.)

5) H^2-estimates for $\partial_t E$ and $\partial_t H$. To bound $\partial_t(E,H)$ in H^2 we use (6.1) with $k=1$. In (6.3) one only has to replace $\|E\|_{H^1}$ by $\|\partial_t E\|_{H^1}$ in order to show

$$\big\|\tilde h_1\big\|_{H^{1/2}(\Gamma)} \lesssim \|\partial_t E\|_{H^1} + z(t)$$

as before.

6) H^3-estimates for E and H. We finally treat the H^3-norm of (E,H). Compared to steps *4)* and *5)*, in the new boundary only the term $\hat{h}$ with $\partial_\tau^2\lambda$ pose new difficulties. To tackle it we differentiate (6.2) once more in x and employ Sobolev's embedding. It follows

$$\begin{aligned}\|\hat{h}\|_{H^{1/2}(\Gamma)} &\lesssim \|\partial_\tau^2\lambda(\cdot,E)E\|_{H^1} \lesssim \|\partial_x^2\lambda\|_{W^{1,\infty}}\|E\|_{H^1} + \|\partial_{x\xi}\lambda\|_{W^{1,\infty}}\|E\|_{H^2}^2\\ &\quad + \|\partial_\xi^2\lambda\|_{W^{1,\infty}}\|E\|_{H^2}^2\|E\|_{H^3} + \|\partial_\xi\lambda\|_{W^{1,\infty}}\|E\|_{H^3}\|E\|_{H^2}\\ &\lesssim \|E\|_{H^1} + z(t)\,. \end{aligned}$$ □

Appendix A. Div-curl estimate

We show a corrected and partially improved version of Lemma 5.1 of [18]. We note that part a) was shown in Lemma 4.5.5 of [7] for scalar λ. In [18] the L^2-norms of u and v on the right-hand side of (A.1) were omitted erroneously. The estimate (A.1) below suffices for [18]. Actually this version does not require additional geometric properties. In part b) we can remove the L^2-norms, as needed in Lemma 5.4, assuming that $\mathbb{R}^3\setminus\Omega$ is connected. This conditions is needed, without it the operator A in the proof would have a non-trivial kernel of the form $(\nabla\varphi,\nabla\psi)$, where $\operatorname{div}(\alpha\nabla\varphi) = \operatorname{div}(\beta\nabla\psi) = 0$ and φ and ψ are constant on components of $\Gamma = \partial\Omega$.

Lemma A.1. *Let Ω be bounded with $\Gamma \in C^2$. Assume that $(u,v) \in H(\mathrm{curl})$, $\alpha,\beta \in W^{1,\infty}(\Omega,\mathbb{R}^{3\times3}_{\mathrm{sym}})$, $\lambda \in W^{1,\infty}(\Gamma,\mathbb{R}^{3\times3}_{\mathrm{sym}})$ satisfy $\alpha,\beta,\lambda \geq \eta > 0$ and $\lambda\nu^\perp \subseteq \nu^\perp$ with $u \in H(\mathrm{div}_\alpha), v \in H(\mathrm{div}_\beta)$ and $v\times\nu + \lambda(u\times\nu)\times\nu =: h \in H^{1/2}(\Gamma)^3$.*
a) Then u and v belong to $H^1(\Omega)^3$ and

$$\|u\|_{H^1} + \|v\|_{H^1} \lesssim \|u\|_{H(\mathrm{curl})} + \|v\|_{H(\mathrm{curl})} + \|\operatorname{div}(\alpha u)\|_{L^2} + \|\operatorname{div}(\beta v)\|_{L^2} + \|h\|_{H^{1/2}(\Gamma)} =: \varkappa(u,v,h)\,. \tag{A.1}$$

b) Assume in addition that $\mathbb{R}^3\setminus\Omega$ is connected. We then obtain

$$\|u\|_{H^1} + \|v\|_{H^1} \lesssim \|\operatorname{curl} u\|_{L^2} + \|\operatorname{curl} v\|_{L^2} + \|\operatorname{div}(\alpha u)\|_{L^2} + \|\operatorname{div}(\beta v)\|_{L^2} + \|h\|_{H^{1/2}(\Gamma)}\,. \tag{A.2}$$

Proof. a) As in Lemma 4.5.5 of [7] or Proposition 6.1 of [15], we use a finite partition of unity χ_j for $\overline{\Omega}$ such that $\operatorname{supp}\chi_j \subseteq \overline{\Omega}_j \subseteq \overline{\Omega}$ for simply connected open sets Ω_j with a connected smooth boundary. As in Lemma 5.1 of [18], one can prove that $\chi_j u$ and $\chi_j v$ belong to $H^1(\Omega_j)$ and

$$\|\chi_j u\|_{H^1} + \|\chi_j v\|_{H^1} \leq \varkappa(\chi_j u,\chi_j v,\chi_j h) \leq \varkappa(u,v,h)$$

Summing these pieces, we obtain (A.1).
b) Now assume that $\mathbb{R}^3\setminus\Omega$ is connected. We show that the L^2-norms appearing on the right-hand side of (A.1) can be estimated by the div-curl-terms. Assume that

$$\|(u,v)\|_{H^1} \not\lesssim \|\operatorname{curl} u\|_{L^2} + \|\operatorname{curl} v\|_{L^2} + \|\operatorname{div}\alpha u\|_{L^2} + \|\operatorname{div}\beta v\|_{L^2} + \|h\|_{H^{1/2}(\Gamma)}\,.$$

Hence there exists a sequence $(u_k,v_k) \in H^1$ with $\|(u_k,v_k)\|_{H^1} = 1$ such that the right-hand side tends to zero. By employing the Banach-Alaoglu theorem as well as the Rellich-Kondrachov compactness theorem, we can choose a subsequence, again denoted by (u_k,v_k), which converges to a limit $(\overline{u},\overline{v})$ in L^2.

Let $A = \begin{pmatrix} 0 & \mathrm{curl} \\ -\mathrm{curl} & 0\end{pmatrix}$ be an operator on $X = (\ker(\mathrm{div}_\alpha)\cap\ker(\mathrm{div}_\beta), \|\cdot\|_{L^2})$ with

$$D(A) := \{(u,v) \in X \mid \operatorname{curl} u, \operatorname{curl} v \in L^2, v\times\nu + \lambda(u\times\nu)\times\nu = 0\}\,.$$

Note that $(\overline{u},\overline{v})$ belongs to the kernel of A.

We will show below that A is injective. Therefore the sequence (u_k,v_k) converges to $(\overline{u},\overline{v})=0$ in L^2. The estimate (A.1) then shows that $\|(u_k,v_k)\|_{H^1}\to 0$, contradicting $\|(u_k,v_k)\|_{H^1}=1$ and the claim follows.

So take $u,v\in D(A)$ with $0=A\binom{u}{v}$ and consider

$$0=\Big(A\tbinom{u}{v}\,\Big|\,\tbinom{u}{v}\Big)=\int_\Omega(\operatorname{curl}v\cdot u-\operatorname{curl}u\cdot v)\,\mathrm{d}x\,.$$

Integrating by parts and using the boundary condition, we compute

$$0=\int_\Gamma u\times\nu\cdot v^\tau\,\mathrm{d}\sigma=\int_\Gamma u\cdot(v\times\nu)\,\mathrm{d}\sigma=-\int_\Gamma(u\times\nu)\lambda(u\times\nu)\,\mathrm{d}\sigma\,.$$

The positive definiteness of λ implies that $u\times\nu=0$ and therefore $v\times\nu=0$ as well. Thus $(u,v)\in\ker A$ satisfy

$$\begin{aligned}\operatorname{curl}u&=\operatorname{curl}v=0\\ u\times\nu&=v\times\nu=0\,.\end{aligned}$$

Theorem 2.8*ii)* of [6] yields functions $\varphi,\psi\in H_0^1(\Omega)$ such that $\nabla\varphi=u,\nabla\psi=v$. (Here we use that $\mathbb{R}^3\setminus\Omega$ is connected.) Because of $u,v\in X$ we also have $\operatorname{div}\alpha\nabla\varphi=0$ and $\operatorname{div}\beta\nabla\psi=0$, so that $\varphi=0=\psi$ and $u=0=v$. Hence A is injective as desired. $\square$

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