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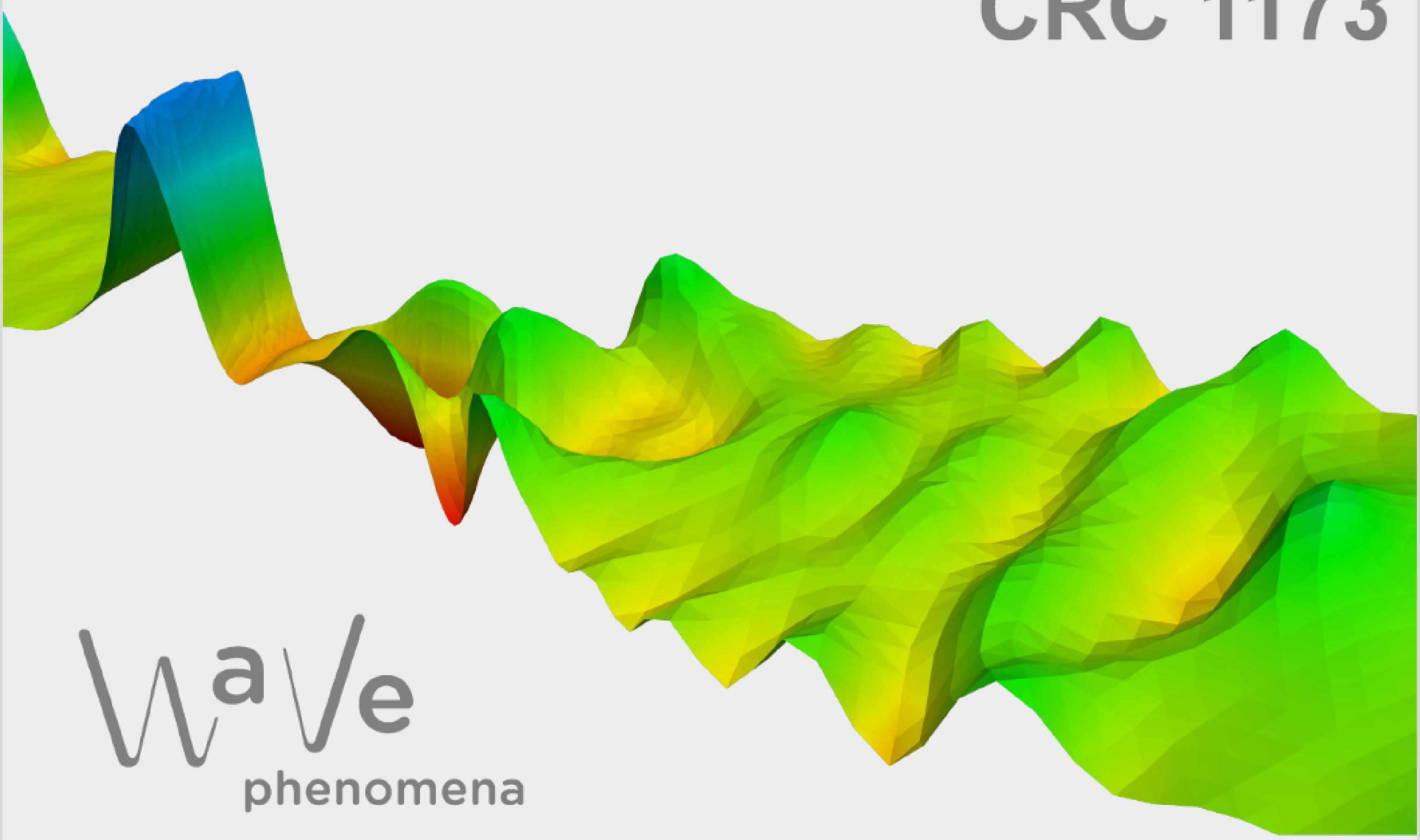

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# Monotonicity-based shape reconstruction for an inverse scattering problem in a waveguide

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**Abstract**

We consider an inverse medium scattering problem for the Helmholtz equation in a closed cylindrical waveguide with penetrable compactly supported scattering objects. We develop novel monotonicity relations for the eigenvalues of an associated modified near field operator, and we use them to establish linearized monotonicity tests that characterize the support of the scatterers in terms of near field observations of the corresponding scattered waves. The proofs of these shape characterizations rely on the existence of localized wave functions, which are solutions to the scattering problem in the waveguide that have arbitrarily large norm in some prescribed region, while at the same time having arbitrarily small norm in some other prescribed region. As a byproduct we obtain a uniqueness result for the inverse medium scattering problem in the waveguide. Numerical examples are presented to document the potentials and limitations of this approach.

Mathematics subject classifications (MSC2010): 35R30, (65N21)
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Short title: Monotonicity in inverse scattering for waveguides

## 1 Introduction

Inverse scattering problems in closed cylindrical waveguides inherit several interesting features that are not present in free space inverse scattering problems. For instance one has to distinguish between propagating and evanescent modes, the latter being virtually undetectable far away from the scatterer for all practical purposes. Moreover, due to the waveguide geometry the available near field scattering data are usually of very limited aperture, which typically increases the instability in reconstruction algorithms. Nevertheless, inverse scattering problems in waveguides are of practical relevance and have thus received increasing attention in recent years. For instance, sampling-type reconstruction methods, which are closely related to the approach considered in this work, have been discussed in [5, 6, 7, 8, 38] (see also [36, 40] for inverse scattering problems modeled by Maxwell's equations). A sampling method for a multi-frequency inverse scattering problem has recently been proposed in [37], and a time-domain sampling method has been established in [39]. Furthermore, optimization schemes have, e.g., been considered in [45, 46].

In this work we extend the results on monotonicity-based shape reconstruction and localized wave functions for the inverse medium scattering problem in unbounded free space from [19] to an inverse scattering problem in a closed straight cylindrical waveguide with Neumann boundary

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conditions featuring all the obstructions mentioned before. Our goal is to detect and recover the support of one or more penetrable scattering objects from a knowledge of near field scattering data using a monotonicity-based reconstruction scheme. Although we focus on a simple model problem, we expect that everything presented here can be generalized to other types of obstacles, other kinds of boundary conditions, and to more complex geometries.

Monotonicity-based shape reconstruction has been proposed in [43] for an inverse problem in electrical impedance tomography. The starting point for this method has been the observation that if $\sigma_1$ and $\sigma_2$ are positive functions representing electric conductivities in some bounded domain $\Omega \subseteq \mathbb{R}^d$ such that $\sigma_1 \leq \sigma_2$, then the associated Neumann-to-Dirichlet operators $\Lambda_{\sigma_1}$ and $\Lambda_{\sigma_2}$ on $\partial\Omega$ satisfy $\Lambda_{\sigma_1} - \Lambda_{\sigma_2} \geq 0$ in the sense that the self-adjoint compact linear operator $\Lambda_{\sigma_1} - \Lambda_{\sigma_2}$ is positive semidefinite, i.e., with respect to the Loewner order. A rigorous theoretical justification of the method has been established in [28]. This analysis combines monotonicity estimates for Neumann-to-Dirichlet maps (see also [30, 31] for earlier contributions in this direction) with the existence of localized potentials for the Laplace equation that has been shown in [18]. Localized potentials are solutions to the Laplace equation in $\Omega$ that have arbitrarily large norm in some prescribed region $B \subseteq \Omega$, while at the same time having arbitrarily small norm in a different prescribed region $E \subseteq \Omega$. A regularization strategy and numerical realizations for monotonicity-based shape reconstruction in electrical impedance tomography have been considered in [15, 16, 17]. The case of impenetrable conductivity inclusions has been discussed in [10]. Recently the method from [28] has been extended to an inverse boundary value problem for the Helmholtz equation in [26, 27] and to an inverse scattering problem with compactly supported penetrable scattering objects in unbounded free space in [19]. These results have been generalized to time-harmonic Maxwell's equations in [1, 25]. Inverse scattering problems with impenetrable obstacles and an inverse crack detection problem have been considered in [2, 13], and the connection to the factorization method [32, 33] has been further clarified in [14]. Monotonicity-based shape reconstruction techniques for eddy current problems and magnetic induction tomography have been proposed in [41, 42, 44], fractional order Schrödinger equations have been discussed in [22, 23], and nonlinear materials have been studied in [9, 12, 21, 24].

In contrast to [19], where the monotonicity relation has been shown for a modified far field operator, we deal with near field observations in the waveguide setting. Accordingly, our analysis of the monotonicity relation as well as the proof of the existence of localized wave functions require a near field variant of the scattering operator appears that is not unitary as in the far field setting, but we show invertibility. Describing the radiation condition in the waveguide by means of modal expansions of the associated Dirichlet-to-Neumann operators, the corresponding terms in the monotonicity relations can be estimated more directly than in [19], which allows to carry over estimates on the dimension of the finite dimensional subspaces that have to be excluded in the monotonicity relations from [27]. However, the improved dimension bounds from [26] do not seem to be applicable straightforwardly. Comparing these theoretical dimension bounds with the number of propagating modes of the waveguide we find that the dimension of the finite dimensional subspaces that have to be excluded might grow much faster than the number of propagating modes when increasing the wave number, in particular if the refractive index of the scatterer is large, or if the scatterer is not just contained in a very narrow section of the waveguide. On the other hand, in our numerical results we observe that the method works reasonably well even if we work with propagating modes only.

This paper is organized as follows. In Section 2 the governing equations for the scattering problem in the Neumann waveguide are presented. In Section 3 we show the monotonicity relation for the near field operator in terms of the Loewner order up to a finite-dimensional subspace. We also discuss the dimension of this subspace and compare it to the dimension of the subspace of propagating modes of the waveguide. In Section 4 we extend the existence result for localized and simultaneously localized wave functions from [19, 20] to the waveguide setting. In this section we also give a uniqueness result for the inverse scattering problem in the waveguide that is proved using the monotonicity relation and the existence of localized wave functions established. Section 5 contains the theoretical justification of linearized monotonicity tests for shape reconstruction for both sign-definite and sign-indefinite scattering configurations. Some numerical results to illustrate our findings are provided in Section 6, and we conclude with some final remarks.

## 2 Scattering by an inhomogeneous obstacle

We are concerned with acoustic wave propagation in a closed straight cylindrical waveguide. The interior of the waveguide will be denoted as $\Omega := \mathbb{R} \times \Sigma$, where $\Sigma \subseteq \mathbb{R}^{d-1}$, $d = 2, 3$, is the cross section. We assume that $\Sigma = (0, h)$ with $h > 0$ when $d = 2$, while $\Sigma$ is a bounded connected Lipschitz domain when $d = 3$. For $x \in \Omega$, we use the notation $x =: (x_1, x_\Sigma)$ with $x_1 \in \mathbb{R}$ and $x_\Sigma \in \Sigma$. Often, we will only consider a finite section of the waveguide. Fixing some $R > 0$, let $\Omega_R := (-R, R) \times \Sigma$. We also use the notation $C_R^\pm := \{\pm R\} \times \Sigma$, and we write $C_R := C_R^+ \cup C_R^-$ for the boundary section $\partial\Omega_R \setminus \partial\Omega$. We will frequently identify $L^2(C_R)$ with $L^2(C_R^+) \times L^2(C_R^-)$.

The propagation of time harmonic acoustic waves in the homogeneous waveguide is governed by the Helmholtz equation with Neumann boundary conditions

$$\Delta u + k^2 u = 0 \quad \text{in } \Omega\,, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega\,, \tag{2.1}$$

where $k > 0$ is the wavenumber. Throughout, we understand Helmholtz equations such as (2.1) to hold in the weak sense with solutions in some Sobolev space, e.g., $H^1_{\mathrm{loc}}(\Omega)$ such that boundary conditions have to be understood in the trace sense. We assume that an incident field $u^i$, satisfying (2.1) in $\Omega \backslash C_R$, is scattered by an inhomogeneous object within the waveguide. This scatterer is described by the refractive index $n^2 = 1 + q$ with a contrast function $q \in L^\infty_{R,+}(\Omega)$, where $L^\infty_{R,+}(\Omega)$ denotes the space of essentially bounded real-valued functions on $\Omega$ that are larger than $-1$ almost everywhere in $\Omega$ and vanish identically outside $\Omega_R$. The total field $u_q$ is then a superposition of the incident and the scattered field due to the inhomogeneity,

$$u_q = u^i + u_q^s\,,$$

such that $u_q$ is a weak solution to the Helmholtz equation with inhomogeneous coefficient,

$$\Delta u_q + k^2(1+q) u_q = 0 \quad \text{in } \Omega \setminus C_R\,, \qquad \frac{\partial u_q}{\partial \nu} = 0 \quad \text{on } \partial\Omega\,.$$

Moreover, $u_q^s$ is assumed to solve

$$\Delta u_q^s + k^2(1+q) u_q^s = -k^2 q u^i \quad \text{in } \Omega\,, \qquad \frac{\partial u_q^s}{\partial \nu} = 0 \quad \text{on } \partial\Omega\,, \tag{2.2}$$

and to be outgoing.

In a waveguide, the notion of outgoing fields is defined via a representation obtained by separation of variables. It is well known that there exists a complete orthonormal system of Neumann eigenfunctions $(\theta_m)_{m\in\mathbb{N}_0} \subseteq L^2(\Sigma)$ of $-\Delta$ in $\Sigma$, and that the corresponding eigenvalues $(k_m^2)_{m\in\mathbb{N}_0}$ form a non-negative, non-decreasing sequence accumulating at $\infty$. A simple application of Green's first identity in $\Sigma$ shows that the sequence $(\theta_m)_{m\in\mathbb{N}_0}$ is also orthogonal in $H^1(\Sigma)$ with $\|\theta_m\|^2_{H^1(\Sigma)} = 1 + k_m^2$. Thus expanding $\varphi = \sum_{m=0}^{\infty} \langle \varphi, \theta_m \rangle_{L^2(\Sigma)} \theta_m$, and using interpolation techniques [35, p. 329], we find that the norms

$$\|\varphi\|_{H^s(\Sigma)}^2 \;=\; \sum_{m=0}^{\infty}(1+k_m^2)^s|\langle\varphi,\theta_m\rangle_{L^2(\Sigma)}|^2\,, \qquad 0\leq s\leq 1\,, \tag{2.3}$$

on $H^s(\Sigma)$ are equivalent to the standard norms. By duality, this extends to $-1 \leq s < 0$. From now on we assume that $k \in (k_N, k_{N+1})$ for some $N \in \mathbb{N}_0$, and for any $m \in \mathbb{N}_0$ we set $\beta_m := \sqrt{k^2 - k_m^2}$. Throughout this work, the square root is such that for any $z = |z|\mathrm{e}^{\mathrm{i}\arg(z)} \in \mathbb{C}$ with $\arg(z) \in [-\pi/2, 3\pi/2)$ we have $\sqrt{z} = \sqrt{|z|}\mathrm{e}^{\mathrm{i}\arg(z)/2}$. Then, the functions

$$u_m^{\pm}(x) \;:=\; \theta_m(x_\Sigma)\mathrm{e}^{\pm\mathrm{i}\beta_m x_1}\,, \qquad x\in\Omega\,,\; m\in\mathbb{N}_0\,,$$

are solutions to (2.1), called the modes of the waveguide. For $m = 0, \dots, N$, the mode $u_m^{\pm}$ propagates along the waveguide from $x_1 = \mp\infty$ to $\pm\infty$, while for $m > N$, $u_m^{\pm}$ is exponentially decaying as $x_1 \to \pm\infty$ and exponentially growing as $x_1 \to \mp\infty$. The radiation condition is that outside the finite section $\Omega_R$, the scattered field $u_q^s$ satisfies

$$u_q^s(x) \;=\; \sum_{m=0}^{\infty}\alpha_m^{\pm}u_m^{\pm}(x)\,, \qquad x\in\Omega\,,\; \pm x_1\geq R\,, \tag{2.4}$$

for some $(\alpha_m^{\pm})_{\mathbb{N}_0} \subseteq \mathbb{C}$.

It is often advantageous to formulate this radiation condition via Dirichlet-to-Neumann maps. Given a function $\varphi^{\pm}$ on $C_R^{\pm}$, we define

$$\left(\Lambda_{\pm}\varphi^{\pm}\right)(\pm R,\,\cdot\,) \;:=\; \sum_{m=0}^{\infty}\mathrm{i}\beta_m\langle\varphi^{\pm}(\pm R,\,\cdot\,),\theta_m\rangle_{L^2(\Sigma)}\theta_m\,.$$

Identifying $H^s(C_R^{\pm})$ with $H^s(\Sigma)$ for $s = \pm 1/2$ and using the norms (2.3), we obtain that $\Lambda_{\pm} : H^{1/2}(C_R^{\pm}) \to H^{-1/2}(C_R^{\pm})$ is bounded. If we denote the unit outward normals on $C_R^{\pm}$ with respect to $\Omega_R$ by $\nu$, the radiation condition (2.4) is equivalent to the boundary conditions

$$\left.\frac{\partial u_q^s}{\partial\nu}\right|_{C_R^+} \;=\; \Lambda_+ u_q^s \qquad\text{and}\qquad \left.\frac{\partial u_q^s}{\partial\nu}\right|_{C_R^-} \;=\; \Lambda_- u_q^s\,. \tag{2.5}$$

*Remark* 2.1. The dual operators $\Lambda_{\pm}^* : H^{1/2}(C_R^{\pm}) \to H^{-1/2}(C_R^{\pm})$ satisfy

$$\left(\Lambda_{\pm}^*\varphi^{\pm}\right)(\pm R,\,\cdot\,) \;=\; -\sum_{m=0}^{\infty}\mathrm{i}\overline{\beta_m}\langle\varphi^{\pm}(\pm R,\cdot),\theta_m\rangle_{L^2(\Sigma)}\theta_m\,. \tag{2.6}$$

Accordingly, $\Lambda_{\pm} - \Lambda_{\pm}^* : H^{1/2}(C_R^{\pm}) \to H^{-1/2}(C_R^{\pm})$,

$$(\Lambda_{\pm} - \Lambda_{\pm}^*)\varphi \;:=\; \sum_{m=0}^{N}2\mathrm{i}\beta_m\langle\varphi^{\pm}(\pm R,\,\cdot\,),\theta_m\rangle_{L^2(\Sigma)}\theta_m\,, \tag{2.7}$$

has finite dimensional range. Therefore, $\Lambda_{\pm} - \Lambda_{\pm}^*$ is also bounded and compact as an operator from $H^{1/2}(C_R^{\pm})$ to $L^2(C_R^{\pm})$. ◊

To simplify the notation, we introduce $\Lambda : H^{1/2}(C_R) \to H^{-1/2}(C_R)$,

$$\Lambda(\varphi) \;:=\; \left(\Lambda_+\varphi|_{C_R^+}, \Lambda_-\varphi|_{C_R^-}\right)\,,$$

where we identify $H^{\pm 1/2}(C_R)$ with $H^{\pm 1/2}(C_R^+) \times H^{\pm 1/2}(C_R^-)$. Accordingly, we will write $\langle \cdot, \cdot \rangle_{C_R}$ for the associated anti-dual bracket (with complex conjugation on the second argument). Therewith, we can state the weak formulation of the scattering problem in the waveguide, which is to find $u_q \in H^1(\Omega_R)$ such that

$$\int_{\Omega_R} \left( \nabla u_q \cdot \overline{\nabla v} - k^2(1+q) u_q \overline{v} \right) \,\mathrm{d}x - \langle \Lambda u_q, v \rangle_{C_R} \\ = \left\langle \frac{\partial u^i}{\partial \nu}\Big|_{C_R}^- - \Lambda u^i, v \right\rangle_{C_R} \qquad \text{for all } v \in H^1(\Omega_R)\,. \quad (2.8)$$

As usual, an additional superscript $+$ or $-$ at a Neumann trace indicates that the trace is taken from the outside or the inside of $\Omega_R$, respectively.

Throughout this paper, we will assume that (2.8) admits a unique solution. This, of course, is not the case for all positive $k$. To obtain solvability results, one may proceed as done in [3, 4] for a related waveguide problem by proving that the operator associated with the bilinear form in (2.8) admits a Garding inequality and establishing analytic dependence on $k$ of the Dirichlet-to-Neumann map $\Lambda$ except along branch cuts of $\sqrt{k^2 - k_m^2}$, $m \in \mathbb{N}_0$. Analytic Fredhom theory then implies that the scattering problem is uniquely solvable for all $k$ except for a sequence $(\hat{k}_j)_{j \in \mathbb{N}_0}$ with $\infty$ as its only accumulation point. We will always assume that $k \neq \hat{k}_j$ for all $j \in \mathbb{N}_0$.

An important tool to represent solutions to waveguide problems are volume and layer potentials. Hence we introduce the Green's function of the waveguide, defined as

$$G(x,y) := -\sum_{m=0}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\beta_m |x_1 - y_1|}}{2\mathrm{i}\,\beta_m} \theta_m(x_\Sigma)\theta_m(y_\Sigma)\,, \qquad x, y \in \Omega_R\,,\ x \neq y\,, \qquad (2.9)$$

see, e.g., [7]. In particular we have that

$$G(x,y) = \Phi(x,y) + \psi(x,y)\,, \qquad x, y \in \Omega_R\,,\ x \neq y\,,$$

where $\psi$ is an analytic function in $\Omega_R \times \Omega_R$ and $\Phi$ denotes the fundamental solution to the Helmholtz equation in free space, i.e.,

$$\Phi(x,y) := \begin{cases} \dfrac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|)\,, & x, y \in \mathbb{R}^2\,,\ x \neq y\,, \\ \dfrac{\mathrm{e}^{\mathrm{i}k|x-y|}}{4\pi|x-y|}\,, & x, y \in \mathbb{R}^3\,,\ x \neq y\,. \end{cases}$$

We will consider incident fields of the form

$$u_g^i(x) := \int_{C_R} \overline{G(x,y)} g(y) \,\mathrm{d}s(y)\,, \qquad x \in \Omega_R\,, \qquad (2.10)$$

where $g \in L^2(C_R)$. Then $u_g^i$ solves (2.1) in $\Omega_R \setminus C_R$ and thus it is a valid incident field in (2.8). Denoting the associated solution of (2.8) by $u_{q,g}$ and the scattered field by $u_{q,g}^s = u_{q,g} - u_g^i$, we define the near field operator $N_q : L^2(C_R) \to H^{1/2}(C_R)$ by

$$N_q g := u_q^s\big|_{C_R}\,. \qquad (2.11)$$

In the next lemma we establish some integral identities for $N_q$.

**Lemma 2.2.** *(a) Let $q \in L^\infty_{R,+}(\Omega)$ and $g \in L^2(C_R)$. For any $v \in H^1(\Omega_R)$,*

$$\int_{\Omega_R} \left(\nabla u^s_{q,g} \cdot \overline{\nabla v} - k^2(1+q)u^s_{q,g}\overline{v}\right) \,\mathrm{d}x - \left\langle \frac{\partial u^s_{q,g}}{\partial \nu}\bigg|_{C_R}, v \right\rangle_{C_R} = k^2 \int_{\Omega_R} q u^i_g \overline{v} \,\mathrm{d}x \,. \tag{2.12}$$

*(b) Let $q \in L^\infty_{R,+}(\Omega)$ and $g, h \in L^2(C_R)$. Then,*

$$\int_{C_R} h\, \overline{N_q g} \,\mathrm{d}s = k^2 \int_{\Omega_R} q u^i_h \overline{u_{q,g}} \,\mathrm{d}x \,. \tag{2.13}$$

*(c) Let $q_1, q_2 \in L^\infty_{R,+}(\Omega)$ and $g, h \in L^2(C_R)$. Then, for any $j, l \in \{1, 2\}$,*

$$\int_{C_R} h\, \overline{\left(N^*_{q_j}(\Lambda - \Lambda^*)N_{q_l}\right) g} \,\mathrm{d}s = \overline{\left\langle \frac{\partial u^s_{q_l,g}}{\partial \nu}\bigg|_{C_R}, u^s_{q_j,h} \right\rangle_{C_R}} - \left\langle \frac{\partial u^s_{q_j,h}}{\partial \nu}\bigg|_{C_R}, u^s_{q_l,g} \right\rangle_{C_R} . \tag{2.14}$$

*(d) Let $q \in L^\infty_{R,+}(\Omega)$. Then,*

$$N_q - N^*_q = N^*_q(\Lambda - \Lambda^*)N_q \,. \tag{2.15}$$

*Proof.* (a) The scattered field $u^s_{q,g}$ satisfies (2.2), and (2.12) is the weak form of this equation.

(b) We obtain from (2.2) that $u^s_{q,g}$ satisfies the Lippmann-Schwinger equation

$$u^s_{q,g}(x) = k^2 \int_{\Omega_R} q(y)u_{q,g}(y)G(x,y) \,\mathrm{d}y \,, \qquad x \in C_R \,,$$

where $G$ denotes the Green's function introduced in (2.9) (see, e.g., [11, Thm. 8.3], where this result is shown for the inhomogeneous medium scattering problem in unbounded free space), and thus

$$\int_{C_R} h\overline{N_q g} \,\mathrm{d}s = \int_{C_R} h\overline{u^s_{q,g}} \,\mathrm{d}s = k^2 \int_{\Omega_R} q(y)\overline{u_{q,g}(y)} \int_{C_R} h(x)\overline{G(x,y)} \,\mathrm{d}s(x) \,\mathrm{d}y \,.$$

Together with (2.10) this gives (2.13).

(c)

$$\begin{aligned}\int_{C_R} h\overline{\left(N^*_{q_j}(\Lambda - \Lambda^*)N_{q_l}\right) g} \,\mathrm{d}s &= \int_{C_R} N_{q_j}h\, \overline{(\Lambda N_{q_l})g} \,\mathrm{d}s - \int_{C_R} (\Lambda N_{q_j})h\, \overline{N_{q_l}g} \,\mathrm{d}s \\ &= \overline{\left\langle \frac{\partial u^s_{q_l,g}}{\partial \nu}\bigg|_{C_R}, u^s_{q_j,h} \right\rangle_{C_R}} - \left\langle \frac{\partial u^s_{q_j,h}}{\partial \nu}\bigg|_{C_R}, u^s_{q_l,g} \right\rangle_{C_R} .\end{aligned} \tag{2.16}$$

(d) If $q_j = q_l = q$, then (2.16), (2.2), and (2.13) give

$$\begin{aligned}\int_{C_R} h\,\overline{\left(N^*_q(\Lambda - \Lambda^*)N_q\right) g} \,\mathrm{d}s &= \overline{\left\langle \frac{\partial u^s_{q,g}}{\partial \nu}\bigg|_{C_R}, u^s_{q,h} \right\rangle_{C_R}} - \left\langle \frac{\partial u^s_{q,h}}{\partial \nu}\bigg|_{C_R}, u^s_{q,g} \right\rangle_{C_R} \\ = \int_{\Omega_R} \left(u^s_{q,h}\overline{\Delta u^s_{q,g}} - \overline{u^s_{q,g}}\Delta u^s_{q,h}\right) \mathrm{d}x &= -k^2 \int_{\Omega_R} q\left(u^s_{q,h}\overline{u^i_g} - \overline{u^s_{q,g}}u^i_h\right) \mathrm{d}x \\ = -k^2 \int_{\Omega_R} q\left(u_{q,h}\overline{u^i_g} - \overline{u_{q,g}}u^i_h\right) \mathrm{d}x &= -\int_{C_R} N_q h\,\overline{g} \,\mathrm{d}s + \int_{C_R} h\,\overline{N_q g} \,\mathrm{d}s \,.\end{aligned}$$

Since $g, h \in L^2(C_R)$ is arbitrary, we have shown (d).

□

We also define the bounded linear operator $\mathcal{S}_q : L^2(C_R) \to L^2(C_R)$ by

$$\mathcal{S}_q := I + (\Lambda - \Lambda^*)N_q \,. \tag{2.17}$$

In the analysis of the monotonicity properties of the near field operator $N_q$ in Section 3 below, the operator $\mathcal{S}_q$ takes the role of the scattering operator in the corresponding analysis for the inverse medium scattering from [19]. Recalling (2.7) we note that $\mathcal{S}_q$ changes elements of the subspace of propagating modes only, while it coincides with the identity on the subspace spanned by the evanescent modes.

**Lemma 2.3.** *The operator $\mathcal{S}_q$ has a bounded inverse.*

*Proof.* In Remark 2.1 we have seen that $\Lambda - \Lambda^*$ is compact from $H^{1/2}(C_R)$ to $L^2(C_R)$, thus $\mathcal{S}_q$ is a Fredholm operator with index zero. Accordingly, it suffices to establish injectivity of $\mathcal{S}_q$ in order to prove that $\mathcal{S}_q$ has a bounded inverse.

Suppose $g \in L^2(C_R)$ with $\mathcal{S}_q g = 0$. Then (2.17) shows that $g = -(\Lambda - \Lambda^*)N_q g$, and denoting by $W := \operatorname{span}\{\theta_0, \dots, \theta_N\} \subseteq L^2(C_R)$ the subspace of propagating modes, the identity (2.7) implies that $g \in W$. Furthermore, again by (2.17) and (2.7), $\mathcal{S}_q$ maps $W$ to $W$, and we denote its restriction to $W$ by $\mathcal{S}_q|_W$. Accordingly, let $\widetilde{\mathcal{S}_q} : W \to W$ be defined by

$$\widetilde{\mathcal{S}_q} := (\Lambda - \Lambda^*)^{-1/2}\mathcal{S}_q|_W \,. \tag{2.18}$$

Here, $(\Lambda - \Lambda^*)^{-1/2} : W \to W$ is given by

$$(\Lambda_\pm - \Lambda_\pm^*)^{-1/2}h := \sum_{m=1}^{N} \frac{1}{\sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{\beta_m}} \langle \varphi, \theta_m \rangle_{L^2(C_R^\pm)} \theta_m \,, \qquad h \in W \,.$$

Therewith we find that

$$\begin{aligned} \widetilde{\mathcal{S}_q}^*\widetilde{\mathcal{S}_q} &= \Big( (\Lambda^* - \Lambda)^{-1/2} + N_q^*(\Lambda^* - \Lambda)^{1/2} \Big) \Big( (\Lambda - \Lambda^*)^{-1/2} + (\Lambda - \Lambda^*)^{1/2} N_q \Big) \\ &= \mathrm{i}(\Lambda - \Lambda^*)^{-1} + \mathrm{i}\left( N_q - N_q^* - N_q^*(\Lambda - \Lambda^*)N_q \right) \,. \end{aligned}$$

Applying (2.15) shows that

$$\widetilde{\mathcal{S}_q}^*\widetilde{\mathcal{S}_q} = \mathrm{i}(\Lambda - \Lambda^*)^{-1} \,,$$

which is injective, and thus we have shown that $\widetilde{\mathcal{S}_q}$ is injective. Therefore, (2.18) implies that $\mathcal{S}_q|_W$ is injective, and we obtain that $g = 0$. Accordingly, $\mathcal{S}_q$ is injective and thus it has a bounded inverse. □

*Remark* 2.4. From (2.15) we find that

$$N_q = N_q^* \left( I + (\Lambda - \Lambda^*)N_q \right) = N_q^* \mathcal{S}_q \,.$$

Substituting this into (2.17) gives

$$\mathcal{S}_q = I + (\Lambda - \Lambda^*)N_q^*\mathcal{S}_q = \mathcal{S}_q^{-1}\mathcal{S}_q + (\Lambda - \Lambda^*)N_q^*\mathcal{S}_q \,.$$

Accordingly,

$$\mathcal{S}_q^{-1} = I - (\Lambda - \Lambda^*)N_q^* = \left( I + N_q(\Lambda - \Lambda^*) \right)^* \,. \tag{2.19}$$

This will be used in the proof of Theorem 4.4 below. ♢

## 3 A monotonicity relation for the measurement operator

We discuss a monotonicity relation for the near field operator with respect to the refractive index of the scatterer. This relation will be formulated in terms of the following extension of Loewner order from [27]. Let $A, B : X \to X$ be compact self-adjoint operators on a Hilbert space $X$, and let $r \in \mathbb{N}$. We say that

$$A \leq_r B \qquad \text{for some } r \in \mathbb{N}_0\,,$$

if $B - A$ has at most $r$ negative eigenvalues. Moreover, we write $A \leq_{\text{fin}} B$ if $A \leq_r B$ holds for some $r \in \mathbb{N}$. The following characterization of this partial ordering has been established using the min-max principle in [27, Cor 3.3].

**Lemma 3.1.** *Let $A, B : X \to X$ be self-adjoint compact linear operators on a Hilbert space $X$ and $r \in \mathbb{N}$. Then $A \leq_r B$ if and only if there is a finite dimensional space $V \subseteq X$ with $\dim(V) \leq r$ such that*

$$\langle v, (B - A)v\rangle_X \,\geq\, 0 \qquad \textit{for all } v \in V^{\perp}\,.$$

We denote by $I : H^1(\Omega_R) \to H^1(\Omega_R)$ the identity operator, and by $J : H^1(\Omega_R) \to L^2(\Omega_R)$ the compact imbedding. Accordingly, we define for any $q \in L^{\infty}_{R,+}(\Omega)$ the compact self-adjoint operators $K : H^1(\Omega_R) \to H^1(\Omega_R)$ and $K_q : H^1(\Omega_R) \to H^1(\Omega_R)$ by

$$Kv \,:=\, J^*Jv \qquad \text{and} \qquad K_q v \,:=\, J^*((1+q)Jv)\,, \qquad v \in H^1(\Omega_R)\,.$$

Then, for any $v \in H^1(\Omega_R)$,

$$\big\langle (I - K - k^2 K_q)v, v\big\rangle_{H^1(\Omega_R)} \,=\, \int_{\Omega_R} \big(|\nabla v|^2 - k^2(1+q)|v|^2\big)\, \mathrm{d}x\,. \tag{3.1}$$

The following definition from [27] is used to describe the dimension of the subspace of $H^1(\Omega_R)$ where this sesquilinear form is positive semidefinite.

**Definition 3.2.** Let $q \in L^{\infty}_{R,+}(\Omega)$, consider the eigenvalues of $K + k^2 K_q$ that are larger than 1, and let $V(q) \subseteq H^1(\Omega_R)$ be the sum of the associated eigenspaces. We define $d(q) := \dim(V(q))$.

It follows immediately from the spectral theorem for compact self-adjoint operators that $d(q)$ is finite, and that

$$\int_{\Omega_R} \big(|\nabla v|^2 - k^2(1+q)|v|^2\big)\, \mathrm{d}x \,\geq\, 0 \qquad \text{for all } v \in V(q)^{\perp}\,.$$

Now we establish a monotonicity relation between the index of refraction and the near field operator.

**Theorem 3.3.** *For any $q_1, q_2 \in L^{\infty}_{R,+}(\Omega)$, there is a subspace $V \subseteq L^2(C_R)$ with $\dim(V) \leq d(q_2)$ such that*

$$\mathrm{Re}\left(\int_{C_R} g\,\overline{\mathcal{S}^*_{q_1}(N_{q_2} - N_{q_1})g}\, \mathrm{d}s\right) \,\geq\, k^2 \int_{\Omega_R} (q_2 - q_1)|u_{q_1,g}|^2\, \mathrm{d}x \qquad \textit{for all } g \in V^{\perp}\,. \tag{3.2}$$

*In particular, $q_1 \leq q_2$ implies that $\mathrm{Re}(\mathcal{S}^*_{q_1} N_{q_1}) \leq_{d(q_2)} \mathrm{Re}(\mathcal{S}^*_{q_1} N_{q_2})$.*

*Remark* 3.4. Using (2.17), we find for any $q_1$, $q_2 \in L^{\infty}_{R,+}(\Omega)$ that

$$\mathrm{Re}\,\big(\mathcal{S}^*_{q_1}(N_{q_2} - N_{q_1}) - \mathcal{S}^*_{q_2}(N_{q_2} - N_{q_1})\big) \,=\, -\,\mathrm{Re}\,\big((N^*_{q_2} - N^*_{q_1})(\Lambda^* - \Lambda)(N_{q_2} - N_{q_1})\big) \,=\, 0\,.$$

Therefore, (3.2) remains true if we replace by $\mathcal{S}^*_{q_1}$ by $\mathcal{S}^*_{q_2}$ in this formula. $\Diamond$

Exchanging the roles of $q_1$ and $q_2$, except for $\mathcal{S}_{q_1}$ (see Remark 3.4), we obtain the following corollary.

**Corollary 3.5.** *For any $q_1, q_2 \in L^\infty_{R,+}(\Omega)$, there is a subspace $V \subseteq L^2(C_R)$ with $\dim(V) \leq d(q_1)$ such that*

$$\operatorname{Re}\left(\int_{C_R} g\,\overline{\mathcal{S}^*_{q_1}(N_{q_2} - N_{q_1})g}\;\mathrm{d}s\right) \leq k^2 \int_{\Omega_R} (q_2 - q_1)|u_{q_2,g}|^2\;\mathrm{d}x \qquad \textit{for all } g \in V^\perp\,. \tag{3.3}$$

*Remark* 3.6. Choosing $q_1 = 0$ and $q_2 = q$ in Theorem 3.3, the monotonicity relation (3.2) shows that $q \geq 0$ implies $\operatorname{Re}(N_q) \geq_{d(q)} 0$. Similarly, $q_1 = 0$ and $q_2 = q$ in (3.3) shows that $q \leq 0$ implies $\operatorname{Re}(N_q) \leq_{d(q)} 0$. ◊

The proof of Theorem 3.3 is obtained from Lemmas 3.7, 3.8, 3.9, and 3.10 which we present next.

**Lemma 3.7.** *Let $q_1, q_2 \in L^\infty_{R,+}(\Omega)$ and $g \in L^2(C_R)$. Then,*

$$\begin{aligned}\int_{C_R} g\,\overline{N_{q_2}g}\;\mathrm{d}s - \int_{C_R} \overline{g}\,N_{q_1}g\;\mathrm{d}s - \int_{C_R} g\,\overline{\left(N^*_{q_1}(\Lambda - \Lambda^*)N_{q_2}\right)g}\;\mathrm{d}s + k^2\int_{\Omega_R}(q_1 - q_2)|u_{q_1,g}|^2\;\mathrm{d}x \\ = \int_{\Omega_R}\left(|\nabla(u^s_{q_2,g} - u^s_{q_1,g})|^2 - k^2(1+q_2)|u^s_{q_2,g} - u^s_{q_1,g}|^2\right)\mathrm{d}x \\ - \left\langle \frac{\partial(u^s_{q_2,g} - u^s_{q_1,g})}{\partial\nu}\bigg|_{C_R}, u^s_{q_2,g} - u^s_{q_1,g}\right\rangle_{C_R}.\end{aligned} \tag{3.4}$$

*Proof.* The right hand side of (3.4) can be expanded as

$$\begin{aligned}&\int_{\Omega_R}\left(|\nabla(u^s_{q_2,g} - u^s_{q_1,g})|^2 - k^2(1+q_2)|u^s_{q_2,g} - u^s_{q_1,g}|^2\right)\mathrm{d}x \\ &\quad - \left\langle \frac{\partial(u^s_{q_2,g} - u^s_{q_1,g})}{\partial\nu}\bigg|_{C_R}, u^s_{q_2,g} - u^s_{q_1,g}\right\rangle_{C_R} \\ &= \int_{\Omega_R}\left(|\nabla u^s_{q_2,g}|^2 - k^2(1+q_2)|u^s_{q_2,g}|^2\right)\mathrm{d}x + \int_{\Omega_R}\left(|\nabla u^s_{q_1,g}|^2 - k^2(1+q_2)|u^s_{q_1,g}|^2\right)\mathrm{d}x \\ &\quad - 2\operatorname{Re}\left(\int_{\Omega_R}\left(\nabla u^s_{q_2,g}\cdot\overline{\nabla u^s_{q_1,g}} - k^2(1+q_2)u^s_{q_2,g}\overline{u^s_{q_1,g}}\right)\mathrm{d}x\right) \\ &\quad - \left\langle \frac{\partial(u^s_{q_2,g} - u^s_{q_1,g})}{\partial\nu}\bigg|_{C_R}, u^s_{q_2,g} - u^s_{q_1,g}\right\rangle_{C_R}.\end{aligned} \tag{3.5}$$

Using (2.12) we find that

$$\begin{aligned}&\int_{\Omega_R}\left(|\nabla(u^s_{q_2,g} - u^s_{q_1,g})|^2 - k^2(1+q_2)|u^s_{q_2,g} - u^s_{q_1,g}|^2\right)\mathrm{d}x \\ &\qquad - \left\langle \frac{\partial(u^s_{q_2,g} - u^s_{q_1,g})}{\partial\nu}\bigg|_{C_R}, u^s_{q_2,g} - u^s_{q_1,g}\right\rangle_{C_R} \\ &\quad = \left\langle \frac{\partial u^s_{q_2,g}}{\partial\nu}\bigg|_{C_R}, u^s_{q_2,g}\right\rangle_{C_R} + k^2\int_{\Omega_R} q_2 u^i_g\overline{u^s_{q_2,g}}\;\mathrm{d}x + \left\langle \frac{\partial u^s_{q_1,g}}{\partial\nu}\bigg|_{C_R}, u^s_{q_1,g}\right\rangle_{C_R} \\ &\qquad + k^2\int_{\Omega_R} q_1 u^i_g\overline{u^s_{q_1,g}}\;\mathrm{d}x + k^2\int_{\Omega_R}(q_1 - q_2)|u^s_{q_1,g}|^2\;\mathrm{d}x\end{aligned}$$

$$- 2\,\mathrm{Re}\left(\left\langle \frac{\partial u^s_{q_2,g}}{\partial\nu}\Big|_{C_R}, u^s_{q_1,g}\right\rangle_{C_R} + k^2 \int_{\Omega_R} q_2 u^i_g \overline{u^s_{q_1,g}}\,\mathrm{d}x\right)$$
$$- \left\langle \frac{\partial (u^s_{q_2,g} - u^s_{q_1,g})}{\partial\nu}\Big|_{C_R}, u^s_{q_2,g} - u^s_{q_1,g}\right\rangle_{C_R}$$
$$= k^2 \int_{\Omega_R} q_2 u^i_g \overline{u_{q_2,g}}\,\mathrm{d}x - k^2 \int_{\Omega_R} q_1 \overline{u^i_g} u_{q_1,g}\,\mathrm{d}x + k^2 \int_{\Omega_R} (q_1 - q_2)|u_{q_1,g}|^2\,\mathrm{d}x$$
$$+ \left\langle \frac{\partial u^s_{q_1,g}}{\partial\nu}\Big|_{C_R}, u^s_{q_2,g}\right\rangle_{C_R} - \overline{\left\langle \frac{\partial u^s_{q_2,g}}{\partial\nu}\Big|_{C_R}, u^s_{q_1,g}\right\rangle_{C_R}}\,.$$

Applying (2.13) and (2.14) gives (3.4). □

The real part of the left hand side of (3.2) can be simplified using the following identity.

**Lemma 3.8.** *Let $q_1, q_2 \in L^\infty_{R,+}(\Omega)$ and $g \in L^2(C_R)$.*

$$\mathrm{Re}\left(\int_{C_R} g\,\overline{N_{q_2} g}\,\mathrm{d}s - \int_{C_R} \overline{g}\, N_{q_1} g\,\mathrm{d}s - \int_{C_R} g\,\overline{\left(N^*_{q_1}(\Lambda - \Lambda^*)N_{q_2}\right) g}\,\mathrm{d}s\right)$$
$$= \mathrm{Re}\left(\int_{C_R} g\,\overline{\left(\mathcal{S}^*_{q_1}(N_{q_2} - N_{q_1})\right) g}\,\mathrm{d}s\right)\,.$$

*Proof.* Observing that $\mathcal{S}^*_{q_1} = I + N^*_{q_1}(\Lambda^* - \Lambda)$ and $\mathrm{Re}(N^*_{q_1}(\Lambda - \Lambda^*)N_{q_1}) = 0$, we find that

$$\mathrm{Re}\left(\mathcal{S}^*_{q_1}(N_{q_2} - N_{q_1})\right) = \mathrm{Re}\left(N_{q_2} - N_{q_1} - N^*_{q_1}(\Lambda - \Lambda^*)N_{q_2}\right)\,. \tag{3.6}$$

Accordingly, the first three terms on the left hand side of (3.4) satisfy

$$\mathrm{Re}\left(\int_{C_R} g\,\overline{N_{q_2} g}\,\mathrm{d}s - \int_{C_R} \overline{g}\, N_{q_1} g\,\mathrm{d}s - \int_{C_R} g\,\overline{\left(N^*_{q_1}(\Lambda - \Lambda^*)N_{q_2}\right) g}\,\mathrm{d}s\right)$$
$$= \mathrm{Re}\left(\int_{C_R} g\,\overline{\left(N_{q_2} - N_{q_1} - N^*_{q_1}(\Lambda - \Lambda^*)N_{q_2}\right) g}\,\mathrm{d}s\right)$$
$$= \mathrm{Re}\left(\int_{C_R} g\,\overline{\left(\mathcal{S}^*_{q_1}(N_{q_2} - N_{q_1})\right) g}\,\mathrm{d}s\right)\,.$$

□

Next we discuss the right hand side of (3.4).

**Lemma 3.9.** *Let $q_1, q_2 \in L^\infty_{R,+}(\Omega)$ and $g \in L^2(C_R)$. Then,*

$$- \mathrm{Re}\left(\left\langle \frac{\partial (u^s_{q_2,g} - u^s_{q_1,g})}{\partial\nu}\Big|_{C_R}, u^s_{q_2,g} - u^s_{q_1,g}\right\rangle_{C_R}\right) \geq 0\,. \tag{3.7}$$

*Proof.* Using the radiation condition (2.5) and the orthogonality of the Neumann eigenfunctions $(\theta_m)_{m\in\mathbb{N}_0}$ of $-\Delta$ in $\Sigma$, we find that

$$- \mathrm{Re}\left(\left\langle \frac{\partial (u^s_{q_2,g} - u^s_{q_1,g})}{\partial\nu}\Big|_{C_R}, u^s_{q_2,g} - u^s_{q_1,g}\right\rangle_{C_R}\right)$$

$$= -\operatorname{Re}\left(\sum_{m=0}^{\infty} \mathrm{i}\beta_m \left(\left|\left\langle (u^s_{q_2,g} - u^s_{q_1,g})(-R,\,\cdot\,), \theta_m\right\rangle_{L^2(\Sigma)}\right|^2 + \left|\left\langle (u^s_{q_2,g} - u^s_{q_1,g})(R,\,\cdot\,), \theta_m\right\rangle_{L^2(\Sigma)}\right|^2\right)\right)$$

$$= \sum_{m=0}^{\infty} \operatorname{Im}(\beta_m) \left(\left|\left\langle (u^s_{q_2,g} - u^s_{q_1,g})(-R,\,\cdot\,), \theta_m\right\rangle_{L^2(\Sigma)}\right|^2 + \left|\left\langle (u^s_{q_2,g} - u^s_{q_1,g})(R,\,\cdot\,), \theta_m\right\rangle_{L^2(\Sigma)}\right|^2\right).$$

If $0 \leq m \leq N$, i.e., for the propagating modes, we have that $\operatorname{Im}(\beta_m) = 0$, and if $m > N$, i.e., for the evanescent modes, $\operatorname{Im}(\beta_m) > 0$. This gives (3.7). □

As a consequence of the proof, we note that the propagating part of the left hand side of (3.7) vanishes identically.

**Lemma 3.10.** *Let $q_1, q_2 \in L^2(\Omega_R)$. Then there is a subspace $V \subseteq L^2(C_R)$ with $\dim(V) \leq d(q_2)$ such that, for all $g \in V^\perp$,*

$$\int_{\Omega_R} \left(|\nabla(u^s_{q_2,g} - u^s_{q_1,g})|^2 - k^2(1+q_2)|u^s_{q_2,g} - u^s_{q_1,g}|^2\right) \mathrm{d}x \\ - \operatorname{Re}\left(\left\langle \frac{\partial(u^s_{q_2,g} - u^s_{q_1,g})}{\partial\nu}\Big|_{C_R}, u^s_{q_2,g} - u^s_{q_1,g}\right\rangle_{C_R}\right) \geq 0\,.$$

*Proof.* For $j = 1,2$ let $A_{q_j} : L^2(C_R) \to H^1(\Omega_R)$ be the bounded linear operator that maps $g \in L^2(C_R)$ to the restriction of the scattered field $u^s_{q_j,g}$ to $\Omega_R$. Combining (3.1) and (3.7) we find that, for any $g \in L^2(C_R)$,

$$\int_{\Omega_R} \left(|\nabla(u^s_{q_2,g} - u^s_{q_1,g})|^2 - k^2(1+q_2)|u^s_{q_2,g} - u^s_{q_1,g}|^2\right) \mathrm{d}x \\ - \operatorname{Re}\left(\left\langle \frac{\partial(u^s_{q_2,g} - u^s_{q_1,g})}{\partial\nu}\Big|_{C_R}, u^s_{q_2,g} - u^s_{q_1,g}\right\rangle_{C_R}\right) \\ \geq \left\langle (I - K - k^2K_{q_2})(A_{q_2} - A_{q_1})g, (A_{q_2} - A_{q_1})g\right\rangle_{H^1(\Omega_R)}\,.$$

Let $V(q_2)$ be the sum of eigenspaces of the compact and self-adjoint operator $K + k^2K_{q_2}$ associated with eigenvalues greater than 1. Then $\dim(V(q_2)) = d(q_2)$ is finite, and

$$\left\langle (I - K - k^2K_{q_2})w, w\right\rangle_{H^1(\Omega_R)} \geq 0 \qquad \text{for all } w \in V(q_2)^\perp\,.$$

Since, for any $g \in L^2(C_R)$,

$$(A_{q_2} - A_{q_1})g \in V(q_2)^\perp \qquad \text{if and only if} \qquad g \in ((A_{q_2} - A_{q_1})^*V(q_2))^\perp\,,$$

and $\dim((A_{q_2} - A_{q_1})^*V(q_2)) \leq d(q_2)$, choosing $V := (A_{q_2} - A_{q_1})^*V(q_2)$ ends the proof. □

*Proof of Theorem 3.3.* Taking the real part of (3.4) and substituting (3.6), we find that

$$\operatorname{Re}\left(\left\langle \mathcal{S}^*_{q_1}(N_{q_2} - N_{q_1})\,g, g\right\rangle_{C_R}\right) + k^2\int_{\Omega_R} (q_1 - q_2)|u_{q_1,g}|^2\,\mathrm{d}x \\ = \int_{\Omega_R} \left(|\nabla(u^s_{q_2,g} - u^s_{q_1,g})|^2 - k^2(1+q_2)|u^s_{q_2,g} - u^s_{q_1,g}|^2\right) \mathrm{d}x \\ - \operatorname{Re}\left(\left\langle \frac{\partial(u^s_{q_2,g} - u^s_{q_1,g})}{\partial\nu}\Big|_{C_R}, u^s_{q_2,g} - u^s_{q_1,g}\right\rangle_{C_R}\right).$$

Applying Lemma 3.10 shows that there is a subspace $V \subseteq L^2(C_R)$ with $\dim(V) \leq d(q_2)$ such that (3.2) holds. □

At the end of this section we now discuss an upper bound for the dimension $d(q)$ of the subspaces $V \subseteq L^2(C_R)$ that have to be excluded in (3.2) and (3.3). To this end, we quote two results from [27]. The first lemma relates the dimension $d(q)$ (see Definition 3.2) to the number of negative Neumann eigenvalues of $-\Delta - k^2(1+q)$ in $\Omega_R$.

**Lemma 3.11** ([27, Lem. 3.10]). *Let $q \in L^\infty_{R,+}(\Omega)$.*

*(a) There exists a complete orthonormal system of Neumann eigenfunctions $(v_m)_{m\in\mathbb{N}_0} \subseteq L^2(\Omega_R)$ of $-\Delta - k^2(1+q)$ in $\Omega_R$, i.e., each $v_m \in H^1(\Omega_R)$ solves*

$$-\Delta v_m - k^2(1+q)v_m \;=\; \lambda_m v_m \quad \text{in } \Omega_R\,, \qquad \frac{\partial v_m}{\partial \nu} \;=\; 0 \quad \text{on } \partial\Omega_R\,,$$

*for some $\lambda_m \in \mathbb{R}$. The Neumann eigenvalues $(\lambda_m)_{m\in\mathbb{N}_0}$ form a non-decreasing sequence accumulating at $\infty$.*

*(b) $d(q)$ is the number of negative Neumann eigenvalues of $-\Delta - k^2(1+q)$ in $\Omega_R$.*

The next lemma is an immediate consequence of Definition 3.2.

**Lemma 3.12** ([27, Lem. 3.9]). *Let $q_1, q_2 \in L^\infty_{R,+}(\Omega)$. If $q_1 \leq q_2$ a.e. in $\Omega_R$, then $d(q_1) \leq d(q_2)$.*

*Proof.* Let $v \in V(q_1)$. Then,

$$\begin{aligned}\big\langle (K + k^2 K_{q_2})v, v\big\rangle_{H^1(\Omega_R)} \;&=\; \int_{\Omega_R} (1+k^2 q_2)|v|^2 \,\mathrm{d}x \;\geq\; \int_{\Omega_R} (1+k^2 q_1)|v|^2 \,\mathrm{d}x\\ &=\; \big\langle (K + k^2 K_{q_1})v, v\big\rangle_{H^1(\Omega_R)} \;>\; 1\,.\end{aligned}$$

Accordingly, Lemma 3.2(b) in [27] implies that $d(q_2) \geq \dim(V(q_1)) = d(q_1)$. □

Combining these two lemmas we obtain the following upper bound for $d(q)$ (see also [27, Cor. 3.11]).

**Corollary 3.13.** *Let $q \in L^\infty_{R,+}(\Omega)$ with $q \leq q_{\max}$ a.e. in $\Omega_R$ for some $q_{\max} \in \mathbb{R}$. Then $d(q) \leq d(q_{\max}\mathbf{1}_{\Omega_R})$, and $d(q_{\max}\mathbf{1}_{\Omega_R})$ is the number of Neumann eigenvalues of $-\Delta$ in $\Omega_R$ that are smaller than $k^2(1+q_{\max})$.*

Next we will explore the relation between the upper bound $d(q_{\max}\mathbf{1}_{\Omega_R})$ and the number of propagating modes of the wave guide.

**Example 3.14.** We consider the two-dimensional case and assume that $\Omega_R = (-R,R)\times(0,1)$, $D \subseteq \Omega_R$, and

$$q(x) \;=\; \begin{cases} a\,, & x \in D\,,\\ 0\,, & x \in \Omega\setminus\overline{D}\,,\end{cases}$$

is piecewise constant for some $a \in (-1,0)\cup(0,\infty)$.

The cross section of the waveguide is $\Sigma = (0,1)$. The Neumann eigenfunctions of $-\Delta$ in $\Sigma$ are given by

$$\theta_m(x_2) \;=\; c_m \cos(m\pi x_2)\,, \qquad m \in \mathbb{N}_0\,,$$

with $c_0 = 1$ and $c_m = \sqrt{2}$ for $m \geq 1$. The associated eigenvalues are $k_m^2 = m^2\pi^2$, and accordingly the number of propagating modes is $N + 1 = \lceil k/\pi \rceil$.

For $l, m \in \mathbb{N}_0$ we define $v_{l,m} \in L^2(\Omega_R)$ by

$$v_{l,m}(x_1, x_2) := b_l c_m \cos\left(\frac{l\pi}{R} x_1\right) \cos(m\pi x_2)\,, \qquad (x_1, x_2) \in \Omega_R\,,$$

with $b_0 = 1/\sqrt{2R}$ and $b_l = 1/\sqrt{R}$ for $l \geq 1$. Then,

$$-\Delta v_{l,m} = \pi^2 \left(\left(\frac{l}{R}\right)^2 + m^2\right) v_{l,m} \quad \text{in } \Omega_R\,, \qquad \frac{\partial v_{l,m}}{\partial \nu} = 0 \quad \text{on } \partial\Omega\,,$$

i.e., the functions $v_{l,m}$ are Neumann eigenfunctions of $-\Delta$ in $\Omega_R$ and we denote the associated eigenvalues by $\lambda_{l,m}^2 := \pi^2((l/R)^2 + m^2)$.

Since $q \leq a$ in $\Omega$, Corollary 3.13 says that the number $d(q)$ from Definition 3.2 is bounded by the number of Neumann eigenvalues $\lambda_{l,m}$ that are smaller than $k^2(1+a)$. This is equivalent to

$$\begin{aligned} d(q) &\leq \#\left\{(l,m) \in \mathbb{N}_0 \times \mathbb{N}_0 \;\middle|\; \frac{l^2}{R^2} + m^2 < \left(\frac{k}{\pi}\right)^2 (1+a)\right\} \\ &\leq \#\left\{(l,m) \in \mathbb{N}_0 \times \mathbb{N}_0 \;\middle|\; \frac{l^2}{R^2} + m^2 < (N+1)^2(1+a)\right\}. \end{aligned} \tag{3.8}$$

We define $\rho_a := (N+1)\sqrt{1+a}$. The constraint on the right hand side of the second line of (3.8) describes a quadrant of an ellipse with semi-axes of length $R\rho_a$ and $\rho_a$. Accordingly, an upper bound for $d(q)$ is given by

$$d(q) \leq \frac{\pi}{4}(R\rho_a + \sqrt{2})(\rho_a + \sqrt{2}) = \frac{\pi}{4}\left(R\rho_a^2 + \sqrt{2}(R+1)\rho_a + 2\right),$$

which grows quadratically in the number of propagating modes $N+1$ as the wave number $k$ increases unless $R(1+a) \lesssim 1/(1+N) = 1/\lceil k/\pi \rceil$. $\diamondsuit$

# 4 Localized wave functions

In order to exploit the monotonicity relations from Theorem 3.3 and Corollary 3.5 in a shape reconstruction algorithm for the support of the contrast function, we require localized wave functions. Given two open bounded subsets $E, M \subseteq \Omega_R$ such that $E \not\subseteq M$ and $\Omega_R \setminus \overline{M}$ is connected, a localized wave function has arbitrarily large norm on the set $E$ while at the same time having arbitrarily small norm on $M$.

Following [28] we say that a relatively open subset $O \subseteq \overline{\Omega_R}$ is connected to $C_R^\pm$ if $O$ is connected and $C_R^\pm \cap O \neq \emptyset$.

**Theorem 4.1.** *Let $q \in L^\infty_{R,+}(\Omega)$ and $E, M \subseteq \Omega_R$ be relatively open domains such that $\overline{\Omega_R} \setminus \overline{M}$ is connected to $C_R^+$ or $C_R^-$. If $E \not\subseteq M$, then for any finite dimensional subspace $V \subseteq L^2(C_R)$ there exists a sequence $(g_m)_{m\in\mathbb{N}_0}$ in $V^\perp$ such that*

$$\int_E |u_{q,g_m}|^2 \,\mathrm{d}x \to \infty \qquad \textit{and} \qquad \int_M |u_{q,g_m}|^2 \,\mathrm{d}x \to 0 \qquad \textit{as } m \to \infty\,,$$

*where $u_{q,g_m} \in H^1(\Omega_R)$ denotes the solution of (2.8) with the incident field $u^i_{g_m}$ as in (2.10) with density $g = g_m$.*

We will prove this theorem using two lemmas which are concerned with properties of the operator

$$L_{q,O} : L^2(C_R) \to L^2(O), \qquad g \mapsto u_{q,g}|_O,$$

where we assume that $O \subseteq \Omega_R$ is relatively open. The compact imbedding of $L^2(C_R)$ in $H^{-1/2}(C_R)$ already tells us that $L_{q,O}$ is a compact linear operator. We will proceed to characterize the adjoint operator $L_{q,O}^*$. To this end, we note that $\overline{u_g^i}$ is simply the single layer potential on $C_R$ with density $\overline{g}$. Hence this function is outgoing outside of $\Omega_R$ and thus

$$\left.\frac{\partial \overline{u_g^i}}{\partial \nu}\right|_{C_R}^{+} = \Lambda \overline{u_g^i}.$$

By (2.6), we have $\Lambda^* \varphi = \overline{\Lambda \overline{\varphi}}$, so we conclude

$$\left.\frac{\partial u_g^i}{\partial \nu}\right|_{C_R}^{+} = \Lambda^* u_g^i.$$

**Lemma 4.2.** *Let $O \subseteq \Omega_R$ be relatively open. The adjoint operator of $L_{q,O}$ is*

$$L_{q,O}^* : L^2(O) \to L^2(C_R), \qquad f \mapsto \mathcal{S}_q^* \left( w_{q,f}|_{C_R} \right),$$

*where $\mathcal{S}_q$ is given by (2.17) and $w_{q,f} \in H^1_{\mathrm{loc}}(\Omega)$ is the unique outgoing function satisfying*

$$\Delta w_{q,f} + k^2(1+q) w_{q,f} = -f \quad \text{in } \Omega, \qquad \frac{\partial w_{q,f}}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{4.1}$$

*Proof.* We write down the weak formulation of the boundary value problem satisfied by $w_{q,f}$ with $u_{q,g}$ as the test function and conclude using (2.8) that, for any $g \in L^2(C_R)$ and $f \in L^2(O)$,

$$\begin{aligned}
\int_O L_{q,O} g \, \overline{f} \, \mathrm{d}x &= \int_{\Omega_R} u_{q,g} \, \overline{f} \, \mathrm{d}x \\
&= \int_{\Omega_R} \left( \nabla u_{q,g} \cdot \nabla \overline{w_{q,f}} - k^2(1-q) u_{q,g} \overline{w_{q,f}} \right) \mathrm{d}x - \overline{\left\langle \left.\frac{\partial w_{q,f}}{\partial \nu}\right|_{C_R}, u_{q,g} \right\rangle_{C_R}} \\
&= \left\langle \left.\frac{\partial u_{q,g}}{\partial \nu}\right|_{C_R}^{-}, w_{q,f} \right\rangle_{C_R} - \overline{\left\langle \left.\frac{\partial w_{q,f}}{\partial \nu}\right|_{C_R}, u_{q,g} \right\rangle_{C_R}}.
\end{aligned}$$

Using the radiation condition (2.5) and the jump relation for the normal derivative of the single layer potential, we have

$$\left.\frac{\partial u_{q,g}}{\partial \nu}\right|_{C_R}^{-} = \left.\frac{\partial u_g^i}{\partial \nu}\right|_{C_R}^{-} + \left.\frac{\partial u_{q,g}^s}{\partial \nu}\right|_{C_R}^{-} = g + \left.\frac{\partial u_g^i}{\partial \nu}\right|_{C_R}^{+} + \Lambda u_{q,g}^s = g + \Lambda^* u_g^i + \Lambda u_{q,g}^s.$$

Noting that $w_{q,f}$ is outgoing, we obtain

$$\begin{aligned}
\int_O L_{q,O} g \, \overline{f} \, \mathrm{d}x &= \langle g + \Lambda^* u_g^i + \Lambda u_{q,g}^s, w_{q,f} \rangle_{C_R} - \overline{\langle \Lambda w_{q,f}, u_{q,g} \rangle_{C_R}} \\
&= \int_{C_R} \left( g + (\Lambda - \Lambda^*) u_{q,g}^s \right) \overline{w_{q,f}} \, \mathrm{d}s.
\end{aligned}$$

The assertion now follows from the definitions (2.11) and (2.17) of $N_q$ and $\mathcal{S}_q$, respectively. □

**Lemma 4.3.** *Let $q \in L^\infty_{R,+}(\Omega)$ and $E$, $M \subseteq \Omega_R$ be non-empty relatively open domains such that $\overline{\Omega_R} \setminus \overline{(E \cup M)}$ is connected to $C_R^+$ or $C_R^-$ and $\overline{E} \cap \overline{M} = \emptyset$. Then $\mathcal{R}(L^*_{q,E}) \cap \mathcal{R}(L^*_{q,M}) = \{0\}$ and $\mathcal{R}(L^*_{q,E})$, $\mathcal{R}(L^*_{q,M})$ are dense in $L^2(C_R)$.*

*Proof.* First we prove the injectivity of $L_{q,M}$, and note that the same proof applies to $L_{q,E}$. Suppose $L_{q,M} g = 0$ for some $g \in L^2(C_R)$. It follows that $u_{q,g}|_M = 0$ and from unique continuation [27, Thm. 2.4], we find that $u_{q,g}$ vanishes throughout $\Omega$. As $u_{q,g}$ is the solution of (2.8), it satisfies the Lippmann-Schwinger equation

$$u_{q,g}(x) \;=\; u^i_g(x) + k^2 \int_\Omega q(y) G(x,y) u_{q,g}(y)\, \mathrm{d}y\,, \qquad x \in \;\Omega\,,$$

where $G$ denotes the Green's function introduced in (2.9) (see, e.g., [11, Thm. 8.3], where this result is shown for the inhomogeneous medium scattering problem in unbounded free space). This implies that $u^i_g = 0$ in $\Omega$ and hence the jumps of the normal derivative of $u^i_g$ vanish across $C_R^\pm$. We conclude $g = 0$ from jump relations of the single layer potential. Thus $L_{q,M}$ is an injection, which implies $\mathcal{R}(L^*_{q,M})$ is dense in $L^2(C_R)$.

Next, we prove $\mathcal{R}(L^*_{q,E}) \cap \mathcal{R}(L^*_{q,M}) = \{0\}$. Let $h \in \mathcal{R}(L^*_{q,E}) \cap \mathcal{R}(L^*_{q,M})$. Then there exist $f_E \in L^2(E)$ and $f_M \in L^2(M)$ such that $h = L^*_{q,E} f_E = L^*_{q,M} f_M$. Let $w_{q,f_E}$ and $w_{q,f_M}$ denote the corresponding outgoing solutions of (4.1). Then,

$$\mathcal{S}^*_q \left( w_{q,f_E}\big|_{C_R} \right) \;=\; \mathcal{S}^*_q \left( w_{q,f_M}\big|_{C_R} \right) \;=\; h\,.$$

From Lemma 2.3, we obtain $w_{q,f_E} = w_{q,f_M}$ on $C_R$. As these functions are outgoing, their Cauchy data on $C_R$ coincide. Using the variant of Holmgrem's theorem formulated as stated in part (b) of [27, Thm. 2.4], we obtain $w_{q,f_E} = w_{q,f_M}$ in $\Omega \setminus \overline{(E \cup M)}$. Define

$$w_q \;=\; \begin{cases} w_{q,f_E} \;=\; w_{q,f_M}\,, & x \in \Omega \setminus \overline{(E \cup M)}\,,\\ w_{q,f_E}\,, & x \in M\,,\\ w_{q,f_M}\,, & x \in E\,. \end{cases}$$

Then $w$ is the outgoing solution of

$$\Delta w_q + k^2(1+q) w_q \;=\; 0 \quad \text{in } \Omega\,, \qquad \frac{\partial w_q}{\partial \nu} \;=\; 0 \quad \text{on } \partial\Omega\,.$$

and thus $w_q = 0$ in $\Omega$. Therefore,

$$h \;=\; \mathcal{S}^*_q \left( w_q\big|_{C_R} \right) \;=\; 0\,.$$

□

We can now carry out the proof of Theorem 4.1 which is obtained from straightforward modifications of the proof of Theorem 4.1 in [19].

*Proof of Theorem 4.1:* We note first that we may assume that $\overline{E} \cap \overline{M} = \emptyset$ and that $\overline{\Omega_R} \setminus \overline{(E \cup M)}$ is connected to $C_R^+$ or $C_R^-$. If this is not the case, replace $E$ by $E^\dagger$ such that $E^\dagger \subseteq E$ and $\overline{E^\dagger} \cap \overline{M^\dagger} = \emptyset$ for some open $M^\dagger \supseteq \overline{M}$.

Let $V \subseteq L^2(C_R)$ denote a finite dimensional subspace and $P_V : L^2(C_R) \to V$ the orthogonal projection. Assume that $\mathcal{R}(L^*_{q,E}) \subseteq \mathcal{R}(L^*_{q,M}) + V$. As $\mathcal{R}(L^*_{q,E}) \cap \mathcal{R}(L^*_{q,M}) = \{0\}$, this implies

$\dim(\mathcal{R}(L_{q,E}^*)) \le \dim(V) < \infty$ by [27, Lmm. 4.7]. However, this contradicts that $\mathcal{R}(L_{q,E}^*)$ is dense in $L^2(C_R)$. Hence

$$\mathcal{R}(L_{q,E}^*) \not\subseteq \mathcal{R}(L_{q,M}^*) + V \;=\; \mathcal{R}(L_{q,M}^*) + \mathcal{R}(P_V^*) \;=\; \mathcal{R}\left(\begin{bmatrix} L_{q,M} \\ P_V \end{bmatrix}^*\right).$$

Lemma 4.6 in [27] now implies that there exists no constant $C > 0$ such that, for all $g \in L^2(C_R)$,

$$\|L_{q,E}g\|_{L^2(E)}^2 \;\le\; C^2 \left(\|L_{q,M}g\|_{L^2(M)}^2 + \|P_V g\|_{L^2(C_R)}^2\right).$$

Hence there exists a sequence $(\widetilde{g}_m)_{m\in\mathbb{N}_0}$ in $L^2(C_R)$ with

$$\|L_{q,E}\widetilde{g}_m\|_{L^2(E)} \to \infty\,, \qquad \|L_{q,M}\widetilde{g}_m\|_{L^2(M)} + \|P_V\widetilde{g}_m\|_{L^2(C_R)} \to 0 \qquad \text{as } m \to \infty\,.$$

We now set $g_m = (I - P_V)\,\widetilde{g}_m \in V^\perp$ and obtain

$$\begin{aligned}
\|u_{q,g_m}\|_{L^2(E)} &= \|L_{q,E}g_m\|_{L^2(E)} \\
&\ge \|L_{q,E}\widetilde{g}_m\|_{L^2(E)} - \|L_{q,E}\|\,\|P_V\widetilde{g}_m\|_{L^2(C_R)} \to \infty \qquad && \text{as } m \to \infty\,, \\
\|u_{q,g_m}\|_{L^2(M)} &= \|L_{q,M}g_m\|_{L^2(M)} \\
&\le \|L_{q,M}\widetilde{g}_m\|_{L^2(M)} + \|L_{q,M}\|\,\|P_V\widetilde{g}_m\|_{L^2(C_R)} \to 0 \qquad && \text{as } m \to \infty\,.
\end{aligned}$$

□

As a further consequence of Lemma 4.2, one obtains that the $L^2$-norms of total fields for the same incident field but for two different contrast functions may be estimated against each other on the support of the difference.

**Theorem 4.4.** *Suppose that $q_1, q_2 \in L^\infty_{R,+}(\Omega)$, and let $M \subseteq \Omega_R$ be relatively open. If $q_1 = q_2$ almost everywhere in $\Omega_R \setminus \overline{M}$, then there exist constants $c$, $C > 0$ such that*

$$c\int_M |u_{q_1,g}|^2 \,\mathrm{d}x \;\le\; \int_M |u_{q_2,g}|^2 \,\mathrm{d}x \;\le\; C\int_M |u_{q_1,g}|^2 \,\mathrm{d}x \qquad \textit{for all } g \in L^2(C_R)\,.$$

*Proof.* Let $w_j$ denote the outgoing solution of (4.1) for $q = q_j$ $j = 1, 2$. Then

$$L_{q_1,M}^* f \;=\; \mathcal{S}_{q_1}^*(w_1|_{C_R})\,, \qquad L_{q_2,M}^* f \;=\; \mathcal{S}_{q_2}^*(w_2|_{C_R})\,. \tag{4.2}$$

We can rewrite the Helmholtz equations as

$$\begin{aligned}
\Delta w_1 + k^2(1+q_2)w_1 \;&=\; -(f + k^2\,(q_1 - q_2)\,w_1) \qquad && \text{in } \Omega\,, \\
\Delta w_1 + k^2(1+q_1)w_2 \;&=\; -(f + k^2(q_2 - q_1)w_2) \qquad && \text{in } \Omega\,.
\end{aligned}$$

As $\operatorname{supp}(q_1 - q_2) \subseteq \overline{M}$, it follows that

$$\mathcal{S}_{q_2}^*(w_1|_{C_R}) \;=\; L_{q_2,M}^*(f + k^2\,(q_1 - q_2)\,w_1)\,, \tag{4.3a}$$
$$\mathcal{S}_{q_1}^*(w_2|_{C_R}) \;=\; L_{q_1,M}^*(f + k^2\,(q_2 - q_1)\,w_2)\,. \tag{4.3b}$$

Combining (4.2) and (4.3), we obtain

$$\begin{aligned}
\mathcal{S}_{q_1}^{-*}L_{q_1,M}^* f \;&=\; (w_1|_{C_R}) \;=\; \mathcal{S}_{q_2}^{-*}L_{q_2,M}^*(f + k^2(q_1 - q_2)(w_1|_{C_R}))\,, \\
\mathcal{S}_{q_2}^{-*}L_{q_2,M}^* f \;&=\; (w_2|_{C_R}) \;=\; \mathcal{S}_{q_1}^{-*}L_{q_1,M}^*(f + k^2\,(q_2 - q_1)\,(w_2|_{C_R}))\,.
\end{aligned}$$

From this we conclude $\mathcal{R}(\mathcal{S}_{q_1}^{-*}L_{q_1,M}^*) = \mathcal{R}(\mathcal{S}_{q_2}^{-*}L_{q_2,M}^*)$. It remains to show that $\mathcal{R}(\mathcal{S}_{q_j}^{-*}L_{q_j,M}^*) = \mathcal{R}(L_{q_j,M}^*)$ for $j=1,2$. Then the assertion follows from Lemma 4.6 in [27].

Using (2.19) we find that for any $f \in L^2(\Omega_R)$ and $j=1,2$,

$$\mathcal{S}_{q_j}^{-*}L_{q_j,M}^* f \;=\; L_{q_j,M}^* f + N_{q_j}(\Lambda-\Lambda^*)L_{q_j,M}^* f\,. \tag{4.4}$$

The definition of the near field operator $N_{q_j}$ shows that $N_{q_j}(\Lambda-\Lambda^*)L_{q_j,M}^* f = u_{q_j,p_{j,f}}^s|_{C_R}$ with $p_{j,f} := (\Lambda-\Lambda^*)L_{q_j,M}^* f$. Since

$$\Delta u_{q_j,p_{j,f}}^s + k^2(1+q_j)u_{q_j,p_{j,f}}^s \;=\; -k^2 q_j u_{p_{j,f}}^i \quad \text{in } \Omega\,, \qquad \frac{\partial u_{q_j,p_{j,f}}^s}{\partial \nu} = 0 \quad \text{on } \partial\Omega\,,$$

we find using Lemma 4.2 that

$$u_{q_j,p_{j,f}}^s|_{C_R} \;=\; \mathcal{S}_{q_j}^{-*}L_{q_j,M}^*(k^2 q_j u_{p_{j,f}}^i)\,.$$

Substituting this into (4.4) and rearranging terms shows that, for any $f \in L^2(\Omega_R)$,

$$L_{q_j,M}^* f \;=\; \mathcal{S}_{q_j}^{-*}L_{q_j,M}^* f - \mathcal{S}_{q_j}^{-*}L_{q_j,M}^*(k^2 q_j u_{p_{j,f}}^i)\,,$$

i.e., $\mathcal{R}(L_{q_j,M}^*) \subseteq \mathcal{R}(\mathcal{S}_{q_j}^{-*}L_{q_j,M}^*)$.

Similarly, using (4.2) and (2.19) we have that, for any $f \in L^2(\Omega_R)$ and $j=1,2$,

$$\begin{aligned} \mathcal{S}_{q_j}^{-*}L_{q_j,M}^* f \;&=\; \mathcal{S}_{q_j}^{-*}\mathcal{S}_{q_j}^*(w_j|_{C_R}) \;=\; \mathcal{S}_{q_j}^*\mathcal{S}_{q_j}^{-*}(w_j|_{C_R}) \\ &=\; \mathcal{S}_{q_j}^*(w_j|_{C_R}) + \mathcal{S}_{q_j}^* N_{q_j}(\Lambda-\Lambda^*)(w_j|_{C_R})\,. \end{aligned} \tag{4.5}$$

Writing $p_{j,f} := (\Lambda-\Lambda^*)(w_j|_{C_R})$, we obtain as before that

$$N_{q_j}(\Lambda-\Lambda^*)(w_j|_{C_R}) \;=\; u_{p_{j,f}}^s \;=\; \mathcal{S}_{q_j}^{-*}L_{q_j,M}^*(k^2 q_j u_{p_{j,f}}^i)\,.$$

Substituting this into (4.5) and applying (4.2) we find that, for any $f \in L^2(\Omega_R)$,

$$\mathcal{S}_{q_j}^{-*}L_{q_j,M}^* f \;=\; L_{q_j,M}^* f + L_{q_j,M}^*(k^2 q_j u_{p_{j,f}}^i)\,,$$

i.e., $\mathcal{R}(\mathcal{S}_{q_j}^{-*}L_{q_j,M}^*) \subseteq \mathcal{R}(L_{q_j,M}^*)$. □

Combining the monotonicity relation in Theorem 3.3 and the localized wave functions from Theorem 4.1, we can prove the following uniqueness result for the inverse medium scattering problem in the waveguide which is a variant of the local uniqueness results in [27, Thm. 5.1] and [29, Thm. 1.1].

**Theorem 4.5.** *Let $q_1, q_2 \in L^\infty_{R,+}(\Omega)$. Furthermore, let $O \subseteq \Omega_R$ be relatively open such that $\overline{O}$ is connected to $C_R^+$ or $C_R^-$ and $q_1 \leq q_2$ a.e. in $O$. If there is a non-empty open subset $B \subseteq O$ such that*

$$q_1 \leq q_2 - c \qquad \text{a.e. in } B \text{ for some } c > 0\,,$$

*then*

$$\mathrm{Re}(\mathcal{S}_{q_1}^* N_{q_1}) \not\geq_{\mathrm{fin}} \mathrm{Re}(\mathcal{S}_{q_1}^* N_{q_2})\,.$$

*This means that the operator $\mathrm{Re}(\mathcal{S}_{q_1}^*(N_{q_2} - N_{q_1}))$ has infinitely many positive eigenvalues, and it implies that $N_{q_1} \neq N_{q_2}$.*

*Proof.* Assume on the contrary that $\mathrm{Re}(\mathcal{S}_{q_1}^*(N_{q_2}-N_{q_1})) \leq_{\text{fin}} 0$ and let $V_1$ denote the finite-dimensional space spanned by all eigenfunctions corresponding to positive eigenvalues of this operator. Let $V_2$ denote the space in Theorem 3.3 and set $V = V_1 + V_2$. Then, by Theorem 3.3, for all $g \in V^\perp$,

$$\begin{aligned} 0 &\geq \mathrm{Re}\left(\int_{C_R} g\,\overline{\mathcal{S}_{q_1}^*(N_{q_2}-N_{q_1})g}\,\mathrm{d}s\right) \geq k^2 \int_{\Omega_R} (q_2-q_1)|u_{q_1,g}|^2\,\mathrm{d}x \\ &= k^2 \int_{O\cap\Omega_R} (q_2-q_1)|u_{q_1,g}|^2\,\mathrm{d}x + k^2 \int_{\Omega_R\setminus\overline{O}} (q_2-q_1)|u_{q_1,g}|^2\,\mathrm{d}x \\ &\geq ck^2 \int_E |u_{q_1,g}|^2\,\mathrm{d}x - Ck^2 \int_{\Omega_R\setminus\overline{O}} |u_{q_1,g}|^2\,\mathrm{d}x\,, \end{aligned} \tag{4.6}$$

where we have set $C := \|q_1\|_{L^\infty(\Omega)} + \|q_2\|_{L^\infty(\Omega)}$. Let $M := \Omega_R\setminus\overline{O}$. Note that since $\overline{O}$ is connected to $C_R^+$ or $C_R^-$, it holds that $\overline{\Omega_R}\setminus\overline{M}$ is connected to $C_R^+$ or $C_R^-$ and Theorem 4.1 may be applied. However, this contradicts (4.6), as the theorem guarantees the existence of a sequence $(g_m)_{m\in\mathbb{N}_0}$ in $V^\perp$ such that

$$\int_E |u_{q_1,g_m}|^2\,\mathrm{d}x \to \infty\,, \qquad \int_M |u_{q_1,g_m}|^2\,\mathrm{d}x \to 0 \qquad \text{as } m\to\infty\,.$$

It follows that $\mathrm{Re}(\mathcal{S}_{q_1}^*(N_{q_2}-N_{q_1})) \not\leq_{\text{fin}} 0$. □

Next we consider a refined version of Theorem 4.1, where we establish the existence of simultaneously localized wave functions. These have arbitrarily large norm on some prescribed region $E \subseteq \Omega_R$ while at the same time having arbitrarily small norm in a different region $M \subseteq \Omega_R$. In contrast to Theorem 4.1 we not only control the total field but also the incident field.

**Theorem 4.6.** *Suppose that $q \in L^\infty_{R,+}(\Omega)$, and let $E, M \subseteq \Omega_R$ be relatively open and Lipschitz bounded such that $\mathrm{supp}(q) \subseteq \overline{E}\cup\overline{M}$, $\overline{\Omega_R}\setminus(\overline{E}\cup\overline{M})$ is connected to $C_R^+$ or $C_R^-$, and $E\cap M = \emptyset$. Assume that there is a connected subset $\Gamma \subseteq \partial E\setminus\overline{M}$ that is relatively open such that $\overline{\Gamma}$ is $C^{1,1}$-smooth.*

*Then for any finite dimensional subspace $V \subseteq L^2(C_R)$ there exists a sequence $(g_m)_{m\in\mathbb{N}} \subseteq V^\perp$ such that*

$$\int_E |u_{q,g_m}|^2\,\mathrm{d}x \to \infty \qquad \textit{and} \qquad \int_M \left(|u_{q,g_m}|^2 + |u_{g_m}^i|^2\right)\mathrm{d}x \to 0 \qquad \textit{as } m\to\infty\,,$$

*where $u_{q,g_m} \in H^1(\Omega_R)$ denotes the solution of (2.8) with the incident field $u_{g_m}^i$ as in (2.10) with density $g = g_m$.*

The proof of this theorem is the same as the proof of [20, Thm. 2.1] with similar modifications as required in the proof of Theorem 4.1, when compared to [19]. Therefore it is omitted.

## 5 Shape reconstruction

In this section, it is our goal to develop an algorithm to recover the support of $q$ using the monotonicity relation for the near field operator that we developed in Section 3. In this algorithm, we will relate the near field operator $N_q$ associated to the unknown scatterer to the Born approximation of near field operators associated to certain probing domains. For any open set $B \subseteq \Omega_R$, the incident field (2.10) naturally defines an operator $H_B : L^2(C_R) \to L^2(B)$ by

$$(H_B g)(x) := u_g^i(x) = \int_{C_R} \overline{G(x,y)} g(y)\,\mathrm{d}s(y)\,, \qquad x\in B\,.$$

The scattered field in the Born approximation is obtained by replacing $u_{q,g}$ by $u_g^i$ in the boundary value problem

$$\Delta u_{q,g}^s + k^2 u_{q,g}^s \;=\; -k^2 q u_{q,g} \quad \text{in } \Omega\,, \qquad \frac{\partial u_{q,g}^s}{\partial \nu} \;=\; 0 \quad \text{on } \partial\Omega\,.$$

We consider the case in which $q = \mathbf{1}_B$ which gives rise to the operator $T_B : L^2(C_R) \to L^2(C_R)$,

$$(T_B g)(x) \;:=\; k^2 \int_B G(x,y) u_g^i(y) \,\mathrm{d}y \;=\; k^2 \int_B G(x,y) \int_{C_R} \overline{G(y,z)} g(z) \,\mathrm{d}s(z) \,\mathrm{d}y\,, \qquad x \in C_R\,. \tag{5.1}$$

Combining both equations, we obtain the representation $T_B = k^2 H_B^* H_B$.

We will now use the operator $T_B$ to formulate criteria with which to determine the support of $q$. The proofs of these results essentially require no new arguments and are hence rather similar to similar proofs in [19] for free space scattering. However, in contrast to [19] we establish upper bounds on the dimensions of the finite dimensional subspaces that have to be excluded in these criteria (see Theorems 5.1–5.3 below) similar to [27]. To begin with, we consider the special case when the contrast function $q$ is either strictly positive or strictly negative a.e. on its support. The general case will be treated in Theorem 5.3 below.

**Theorem 5.1.** *Let $B, D \subseteq \Omega_R$ be relatively open such that $\overline{\Omega_R} \setminus \overline{D}$ is connected to $C_R^+$ or $C_R^-$, and let $q \in L^\infty_{R,+}(\Omega)$ with $\mathrm{supp}(q) = \overline{D}$. Further suppose that $0 \le q_{\min} \le q \le q_{\max} < \infty$ a.e. in $D$ for some constants $q_{\min}, q_{\max} \in \mathbb{R}$.*

*(a) If $B \subseteq D$, then $\alpha T_B \le_{d(q)} \mathrm{Re}(N_q)$ for all $\alpha \le q_{\min}$.*

*(b) If $B \not\subseteq D$, then for all $\alpha > 0$, $\alpha T_B \not\le_{\mathrm{fin}} \mathrm{Re}(N_q)$, and hence the operator $\mathrm{Re}(N_q) - \alpha T_B$ has infinitely many negative eigenvalues.*

*Proof.* To show part (a), we apply Theorem 3.3 with $q_1 = 0$ and $q_2 = q$. Hence there exists a subspace $V \subseteq L^2(C_R)$ with $\dim(V) \le d(q)$ such that

$$\mathrm{Re}\left(\int_{C_R} g\,\overline{N_q g} \,\mathrm{d}s\right) \;\ge\; k^2 \int_D q |u_g^i|^2 \,\mathrm{d}x \qquad \text{for all } g \in V^\perp\,.$$

On the other hand, from $B \subseteq D$ and $\alpha \le q_{\min}$, for any $g \in L^2(C_R)$, we obtain

$$\alpha \int_{C_R} g\,\overline{T_B g} \,\mathrm{d}s \;=\; \alpha k^2 \|H_B g\|^2_{L^2(B)} \;=\; k^2 \int_B \alpha |u_g^i|^2 \,\mathrm{d}x \;\le\; k^2 \int_B q |u_g^i|^2 \,\mathrm{d}x\,.$$

For part (b), assume that for some $B \not\subseteq D$ and $\alpha > 0$ there holds $\alpha T_B \le_{\mathrm{fin}} \mathrm{Re}(N_q)$, i.e., there exists a finite dimensional subspace $V_1 \subseteq L^2(C_R)$ such that

$$\alpha k^2 \int_D |u_g^i|^2 \,\mathrm{d}x \;\le\; \mathrm{Re}\biggl(\int_{C_R} g\,\overline{N_q g} \,\mathrm{d}s\biggr) \qquad \text{for all } g \in V_1^\perp\,.$$

We apply Corollary 3.5 with $q_1 = 0$ and $q_2 = q$ to see that there exists a finite dimensional subspace $V_2 \subseteq L^2(C_R)$ such that

$$\mathrm{Re}\left(\int_{C_R} g\,\overline{N_q g} \,\mathrm{d}s\right) \;\le\; k^2 \int_D q |u_{q,g}|^2 \,\mathrm{d}x \qquad \text{for all } g \in V_2^\perp\,.$$

Combining both inequalities, we obtain that there exists a finite dimensional subspace $V := V_1 + V_2 \subseteq L^2(C_R)$ such that

$$\alpha k^2 \int_B |u_g^i|^2 \,\mathrm{d}x \;\le\; k^2 \int_D q\,|u_{q,g}|^2 \,\mathrm{d}x \;\le\; k^2 q_{\max} \int_D |u_{q,g}|^2 \,\mathrm{d}x \qquad \text{for all } g \in V^\perp\,.$$

We next apply Theorem 4.4 with $q_1 = 0$ and $q_2 = q$ to see there exists a constant $C > 0$ such that

$$\alpha k^2 \int_B |u_g^i|^2 \,\mathrm{d}x \,\leq\, Ck^2 q_{\max} \int_D |u_g^i|^2 \,\mathrm{d}x \qquad \text{for all } g \in V^\perp .$$

However, this contradicts Theorem 4.1 with $q = 0$, $E = B$, and $M = D$. □

An analogous theorem holds if the contrast function is negative on its support.

**Theorem 5.2.** *Let $B, D \subseteq \Omega_R$ be relatively open such that $\overline{\Omega_R} \setminus \overline{D}$ is connected to $C_R^+$ or $C_R^-$, and let $q \in L^\infty_{R,+}(\Omega)$ with $\mathrm{supp}(q) = \overline{D}$. Further suppose that $-1 < q_{\min} \leq q \leq q_{\max} \leq 0$ a.e. in $D$ for some constants $q_{\min}, q_{\max} \in \mathbb{R}$.*

*(a) If $B \subseteq D$, then $\alpha T_B \geq_{d(q)} \mathrm{Re}(N_q)$ for all $\alpha \geq C q_{\max}$ with the constant $C > 0$ from Theorem 4.4.*

*(b) If $B \not\subseteq D$, then for all $\alpha < 0$, $\alpha T_B \not\geq_{\mathrm{fin}} \mathrm{Re}(N_q)$, and hence the operator $\mathrm{Re}(N_q) - \alpha T_B$ has infinitely many positive eigenvalues.*

*Proof.* Let $B \subseteq D$. We use Corollary 3.5 and Theorem 4.4 with $q_1 = 0$ and $q_2 = q$ to show that there exists a constant $C > 0$ and a subspace $V \subseteq L^2(C_R)$ with $\dim(V) \leq d(q)$ such that

$$\mathrm{Re}\left(\int_{C_R} g\, \overline{N_q g} \,\mathrm{d}s\right) \,\leq\, k^2 \int_D q\, |u_{q,g}|^2 \,\mathrm{d}x \,\leq\, k^2 q_{\max} \int_D |u_{q,g}|^2 \,\mathrm{d}x \,\leq\, Ck^2 q_{\max} \int_D |u_g^i|^2 \,\mathrm{d}x$$

for all $g \in V^\perp$. We immediately obtain the assertion of (a) for $\alpha \geq C q_{\max}$.

For the proof of part (b), we let $B \not\subseteq D$, $\alpha < 0$ and assume, contrary to the assertion, that $\alpha T_B \geq_{\mathrm{fin}} \mathrm{Re}(N_q)$, i.e., there exists a finite dimensional subspace $V_1 \subseteq L^2(C_R)$ such that

$$\alpha k^2 \int_D |u_g^i|^2 \,\mathrm{d}x \,\geq\, \mathrm{Re}\left(\int_{S^2} g\, \overline{N_q g} \,\mathrm{d}s\right) \qquad \text{for all } g \in V_1^\perp .$$

From Theorem 3.3 for $q_1 = 0$ and $q_2 = q$, we obtain that there exists a finite dimensional subspace $V_2 \subseteq L^2(C_R)$ such that

$$\mathrm{Re}\left(\int_{C_R} g\, \overline{N_q g} \,\mathrm{d}s\right) \,\geq\, k^2 \int_D q |u_g^i|^2 \,\mathrm{d}x \qquad \text{for all } g \in V_2^\perp .$$

Combining both inequalities yields the existence of a finite dimensional subspace $V := V_1 + V_2 \subseteq L^2(C_R)$ such that

$$\alpha k^2 \int_B |u_g^i|^2 \,\mathrm{d}x \,\geq\, k^2 \int_D q |u_g^i|^2 \,\mathrm{d}x \,\geq\, k^2 q_{\min} \int_D |u_g^i|^2 \,\mathrm{d}x \qquad \text{for all } g \in V^\perp .$$

Noting that $\alpha < 0$, we again have a contradiction to Theorem 4.1 with $q = 0$, $E = B$, and $M = D$. Thus our assumption was wrong, which finishes the proof of (b). □

Finally, we consider the general case when the contrast function $q$ is neither strictly positive nor strictly negative a.e. on its support. In contrast to the criteria developed in Theorems 5.1 and 5.2, which determine whether a probing domain $B$ is contained in the scattering object $D$ or not, the criterion in Theorem 5.3 characterizes whether a probing domain $B$ contains the scatterer $D$ or not.

**Theorem 5.3.** *Let $D \subseteq \Omega_R$ be relatively open and bounded such that $\overline{\Omega_R} \setminus \overline{D}$ is connected to $C_R^+$ or $C_R^-$ and $\partial D$ is piecewise $C^{1,1}$-smooth. Let $q \in L^\infty_{R,+}(\Omega)$ with $\mathrm{supp}(q) = \overline{D}$, and suppose that $-\infty < q_{\min} \leq q \leq q_{\max} < \infty$ a.e. on $D$ for some constants $q_{\min}, q_{\max} \in \mathbb{R}$. Moreover, we assume that for any $x \in \partial D$, and for any neighborhood $U \subseteq \overline{D}$ of $x$ in $\overline{D}$, there exists a relatively open subset $O \subseteq \overline{\Omega_R}$ that is connected to $C_R^+$ or $C_R^-$ with $\emptyset \neq E := O \cap D \subseteq U$ such that*

$$q|_E \geq q_{\min,E} > 0 \qquad \text{or} \qquad q|_E \leq q_{\max,E} < 0 \tag{5.2}$$

*for some constants $q_{\min,E}, q_{\max,E} \in \mathbb{R}$.*

*Let $B \subseteq \Omega_R$ be relatively open such that $\overline{\Omega_R} \setminus \overline{B}$ is connected to $C_R^+$ or $C_R^-$, and let $T_B$ denote the corresponding probing operator from (5.1).*

*(a) If $D \subseteq B$, then*

$$\alpha T_B \leq_{d(q)} \mathrm{Re}(N_q) \leq_{d(0)} \beta T_B \quad \text{for all } \alpha \leq \min\{0, q_{\min}\}\,,\ \beta \geq \max\{0, C q_{\max}\}$$

*with the constant $C > 0$ from Theorem 4.4.*

*(b) If $D \not\subseteq B$, then*

$$\alpha T_B \not\leq_{\mathrm{fin}} \mathrm{Re}(N_q) \quad \text{for any } \alpha \in \mathbb{R} \quad \text{or} \quad \mathrm{Re}(N_q) \not\leq_{\mathrm{fin}} \beta T_B \quad \text{for any } \beta \in \mathbb{R}\,.$$

*Proof.* To show part (a) we assume that $D \subseteq B$, and we apply Corollary 3.5 and Theorem 4.4 with $q_1 = 0$ and $q_2 = q$. Accordingly, there is a constant $C > 0$ and a subspace $V_1 \subseteq L^2(C_R)$ with $\dim(V_1) \leq d(0)$ such that, for all $g \in V_1^\perp$ and any $\beta \geq \max\{0, C q_{\max}\}$,

$$\begin{aligned}\mathrm{Re}\left(\int_{C_R} g\,\overline{N_q g}\,\mathrm{d}s\right) &\leq k^2 \int_D q |u_{q,g}|^2\,\mathrm{d}x \leq k^2 q_{\max} \int_D |u_{q,g}|^2\,\mathrm{d}x \\ &\leq k^2 C q_{\max} \int_D |u_g^i|^2\,\mathrm{d}x \leq k^2 \beta \int_B |u_g^i|^2\,\mathrm{d}x\,.\end{aligned}$$

On the other hand, Theorem 3.3 with $q_1 = 0$ and $q_2 = q$ shows that there exists a subspace $V_2 \subseteq L^2(C_R)$ with $\dim(V_2) \leq d(q)$ such that, for all $g \in V_2^\perp$ and any $\alpha \leq \min\{0, q_{\min}\}$,

$$\mathrm{Re}\left(\int_{C_R} g\,\overline{N_q g}\,\mathrm{d}s\right) \geq k^2 \int_D q |u_g^i|^2\,\mathrm{d}x \geq k^2 q_{\min} \int_D |u_g^i|^2\,\mathrm{d}x \geq k^2 \alpha \int_B |u_g^i|^2\,\mathrm{d}x\,.$$

For part (b) we observe that $D \not\subseteq B$ implies that $U := D \setminus B$ is not empty. Accordingly, we choose a point $x \in \overline{U} \cap \partial D$ and an open neighborhood $O \subseteq \overline{\Omega_R}$ of $x$ with $O \cap D \subseteq U$ and $O \cap B = \emptyset$, such that (5.2) is satisfied with $E := O \cap D$. Without loss of generality we suppose that $O$ and $\overline{\Omega_R} \setminus \overline{O}$ are connected to $C_R^+$ or $C_R^-$.

Suppose that $q|_E \geq q_{\min,E} > 0$ and $\mathrm{Re}(N_q) \leq_{\mathrm{fin}} \beta T_B$ for some $\beta \in \mathbb{R}$. Then we apply Theorem 3.3 with $q_1 = 0$ and $q_2 = q$ to see that there exists a finite dimensional subspace $V_3 \subseteq L^2(C_R)$ such that, for any $g \in V_3^\perp$,

$$\begin{aligned}0 &\geq \int_{C_R} g\left(\overline{\mathrm{Re}(N_q) g - \beta T_B g}\right)\mathrm{d}s \geq k^2 \int_{\Omega_R} (q - \beta\chi_B)\,|u_g^i|^2\,\mathrm{d}x \\ &= k^2 \int_{\Omega_R \setminus \overline{O}} (q - \beta\chi_B)\,|u_g^i|^2\,\mathrm{d}x + k^2 \int_{\Omega_R \cap O} (q - \beta\chi_B)\,|u_g^i|^2\,\mathrm{d}x \\ &\geq -k^2\left(\|q\|_{L^\infty(\Omega)} + |\beta|\right) \int_{\Omega_R \setminus \overline{O}} |u_g^i|^2\,\mathrm{d}x + k^2 q_{\min,E} \int_E |u_g^i|^2\,\mathrm{d}x\,.\end{aligned}$$

This contradicts Theorem 4.1 with $M = \overline{\Omega_R} \setminus \overline{O}$ and $q = 0$, which guarantees the existence of a sequence $(g_m)_{m\in\mathbb{N}} \subseteq V_3^\perp$ with

$$\int_E |u^i_{g_m}|^2 \, \mathrm{d}x \to \infty \qquad \text{and} \qquad \int_{\Omega_R\setminus\overline{O}} |u^i_{g_m}|^2 \, \mathrm{d}x \to 0 \qquad \text{as } m \to \infty\,.$$

Hence, $\mathrm{Re}(N_q) \not\leq_{\mathrm{fin}} \beta T_B$ for all $\beta \in \mathbb{R}$.

If $q|_E \leq q_{\max,E} < 0$ and $\alpha T_B \leq_{\mathrm{fin}} \mathrm{Re}(N_q)$ for some $\alpha \in \mathbb{R}$, then the Corollary 3.5 with $q_1 = 0$ and $q_2 = q$ shows that there is a finite dimensional subspace $V_4 \subseteq L^2(C_R)$ such that, for any $g \in V_4^\perp$,

$$\begin{aligned}
0 \;&\leq\; \int_{C_R} g\left(\overline{\mathrm{Re}(N_q)g - \alpha T_B g}\right) \mathrm{d}s \;\leq\; k^2 \int_{\Omega_R} \left(q|u_{q,g}|^2 - \alpha\chi_B|u^i_g|^2\right) \mathrm{d}x\\
&= k^2 \int_{\Omega_R\setminus\overline{O}} \left(q|u_{q,g}|^2 - \alpha\chi_B|u^i_g|^2\right) \mathrm{d}x + k^2 \int_{\Omega_R\cap O} \left(q|u_{q,g}|^2 - \alpha\chi_B|u^i_g|^2\right) \mathrm{d}x\\
&\leq k^2 q_{\max} \int_{\Omega_R\setminus\overline{O}} |u_{q,g}|^2 \, \mathrm{d}x + k^2|\alpha| \int_{\Omega_R\setminus\overline{O}} |u^i_g|^2 \, \mathrm{d}x + k^2 q_{\max,E} \int_E |u_{q,g}|^2 \, \mathrm{d}x\,.
\end{aligned}$$

We define $M := \Omega_R \setminus \overline{O}$. Since $\partial D$ is piecewise $C^{1,1}$ smooth, there is a connected subset $\Gamma \subseteq \partial E \setminus \overline{M}$ that is relatively open and $C^{1,1}$ smooth. Applying Theorem 4.6 we find that there exists a sequence $(g_m)_{m\in\mathbb{N}} \subseteq V^\perp$ such that

$$\int_E |u_{q,g_m}|^2 \, \mathrm{d}x \to \infty \qquad \text{and} \qquad \int_{\Omega_R\setminus\overline{O}} |u_{q,g_m}|^2 + |u^i_{g_m}|^2 \, \mathrm{d}x \to 0 \qquad \text{as } m \to \infty\,.$$

Since $q_{\max,E} < 0$, this gives a contradiction. Accordingly, $\alpha T_B \not\leq_{\mathrm{fin}} \mathrm{Re}(N_q)$ for all $\alpha \in \mathbb{R}$. □

## 6 Numerical examples

To demonstrate the feasibility of the shape reconstruction algorithm, we present some examples for two-dimensional scattering problems with sign-definite contrast functions. We consider obstacles with constant refractive index which differs from the background medium. In this case, the scattering problem may be formulated as a transmission problem. The solution can be obtained by solving a system of two second kind weakly singular boundary integral equations using the approach of [34]. We solve these integral equations by a Nyström method.

To discretize the operators $N_q$ from (2.11) and $T_B$ from (5.1), a suitable subspace of $L^2(C_R)$ needs to be chosen. We define a complete orthonormal system on $L^2(C_R)$ induced by the Neumann eigenfunctions of $-\Delta$ on $\Sigma$,

$$g^{(1)}_m(x) := \begin{pmatrix}\theta_m(x_\Sigma)\\ 0\end{pmatrix}, \quad g^{(2)}_m(x) := \begin{pmatrix}0\\ \theta_m(x_\Sigma)\end{pmatrix} \in L^2(C_R) = L^2(C_R^-) \times L^2(C_R^+)\,, \qquad m \in \mathbb{N}_0\,.$$

As we consider a two-dimensional problem in $\Omega = \mathbb{R} \times (0,h)$ in this section, the Neumann eigenvalues are $k_m^2 = m^2\pi^2/h^2$, $m \in \mathbb{N}_0$, with the corresponding eigenfunctions $\theta_0(y_2) = \sqrt{1/h}$ and $\theta_m(y_2) = \sqrt{2/h}\cos(k_m y_2)$, $m \in \mathbb{N}$, for $y_2 \in \Sigma$. Fixing $M \in \mathbb{N}$, we obtain the $2M+2$ dimensional subspace

$$X_M = \mathrm{span}\left\{ g^{(1)}_m,\; g^{(2)}_m \;\middle|\; m = 0,1,\dots,M \right\} \subseteq L^2(C_R)\,.$$

The incident fields corresponding to these basis functions are easily worked out using the modal representation (2.9) of the Green's function

$$u^i_{g^{(1)}_m}(x) = \sum_{\ell=0}^\infty \frac{\mathrm{e}^{-\mathrm{i}\overline{\beta_\ell}(R+x_1)}}{2\mathrm{i}\overline{\beta_\ell}}\, \theta_\ell(x_2)\, \langle\theta_m, \theta_\ell\rangle_{L^2(0,h)} = \frac{\mathrm{e}^{-\mathrm{i}\overline{\beta_m}(R+x_1)}}{2\mathrm{i}\overline{\beta_m}}\, \theta_m(x_2)\,, \qquad x \in \Omega_R\,,$$

and likewise

$$u^i_{g_m^{(2)}}(x) \;=\; \frac{\mathrm{e}^{-\mathrm{i}\overline{\beta_m}(R-x_1)}}{2\mathrm{i}\overline{\beta_m}}\,\theta_m(x_2)\,, \qquad x\in\Omega_R\,.$$

We now obtain a discretization $\boldsymbol{N}_q := (N^{\nu,\mu}_{q,\ell,m})_{\substack{\nu,\ell\\ \mu,m}} \in \mathbb{C}^{2M+2\times 2M+2}$ of the near field operator $N_q$ by computing the orthogonal projections of the scattered field on the basis functions of $X_M$,

$$N^{\nu,\mu}_{q,\ell,m} \;:=\; \langle u^s_{g_m^{(\mu)}}, g_\ell^{(\nu)}\rangle_{L^2(C_R)}\,, \qquad \nu,\mu=1,2\,,\ \ell,m=0,\dots,M\,.$$

In our implementation, these scalar products are computed by the composite trapezoidal rule on $(0,h)$ which is highly accurate as the integrands extend to even $2h$-periodic smooth functions. In the examples below, we use a rule with 81 quadrature points.

In what follows, we implement the criteria from Theorems 5.1 and 5.2. As the test domain, we chose a square $B=\xi+(-a,a)^2\subseteq\Omega_R$ with center point $\xi\in\Omega_R$ and lateral length $2a>0$. The Born scattering operator $T_B$ from (5.1) applied to one of the basis functions $g_m^{(\mu)}$ of $X_M$ with $\mu=1,2$ and $m\in\mathbb{N}_0$ satisfies

$$\left(T_B g_m^{(\mu)}\right)(x) \;=\; -\frac{k^2}{2\mathrm{i}}\sum_{\ell=0}^{\infty}\frac{\theta_\ell(x_2)}{\beta_\ell}\int_B \mathrm{e}^{\mathrm{i}\beta_\ell|x_1-y_1|}\,\theta_\ell(y_2)\,u^i_{g_m^{(\mu)}}(y)\;\mathrm{d}y\,, \qquad x\in C_R\,.$$

Introducing the coefficients

$$\gamma_{\ell,m} \;:=\; \int_{\xi_2-a}^{\xi_2+a}\theta_\ell(y_2)\,\theta_m(y_2)\;\mathrm{d}y_2\,, \qquad \ell,m\in\mathbb{N}_0\,,$$

and inserting the expressions for the incident fields, we obtain

$$\left(T_B g_m^{(\mu)}\right)(x) \;=\; \frac{k^2}{4}\sum_{\ell=0}^{\infty}\frac{\gamma_{\ell,m}\theta_\ell(x_2)}{\beta_\ell\overline{\beta_m}}\int_{\xi_1-a}^{\xi_1+a}\mathrm{e}^{\mathrm{i}\beta_\ell|x_1-y_1|}\,\mathrm{e}^{-\mathrm{i}\overline{\beta_m}(R-(-1)^\mu y_1)}\;\mathrm{d}y_1\,, \qquad x\in C_R\,.$$

The discretization $\boldsymbol{T}_B := (T^{\nu,\mu}_{\ell,m})_{\substack{\nu,\ell\\ \mu,m}} \in \mathbb{C}^{2M+2\times 2M+2}$ of $T_B$ is again obtained by orthogonal projection on $X_M$, which gives

$$T^{\nu,\mu}_{\ell,m} \;:=\; \langle T_B g_m^{(\mu)}, g_\ell^{(\nu)}\rangle_{L^2(C_R)} \;=\; \frac{k^2}{4}\frac{\gamma_{\ell,m}}{\beta_\ell\overline{\beta_m}}\int_{\xi_1-a}^{\xi_1+a}\mathrm{e}^{\mathrm{i}\beta_\ell(R-(-1)^\nu y_1)}\mathrm{e}^{-\mathrm{i}\overline{\beta_m}(R-(-1)^\mu y_1)}\;\mathrm{d}y_1\,.$$

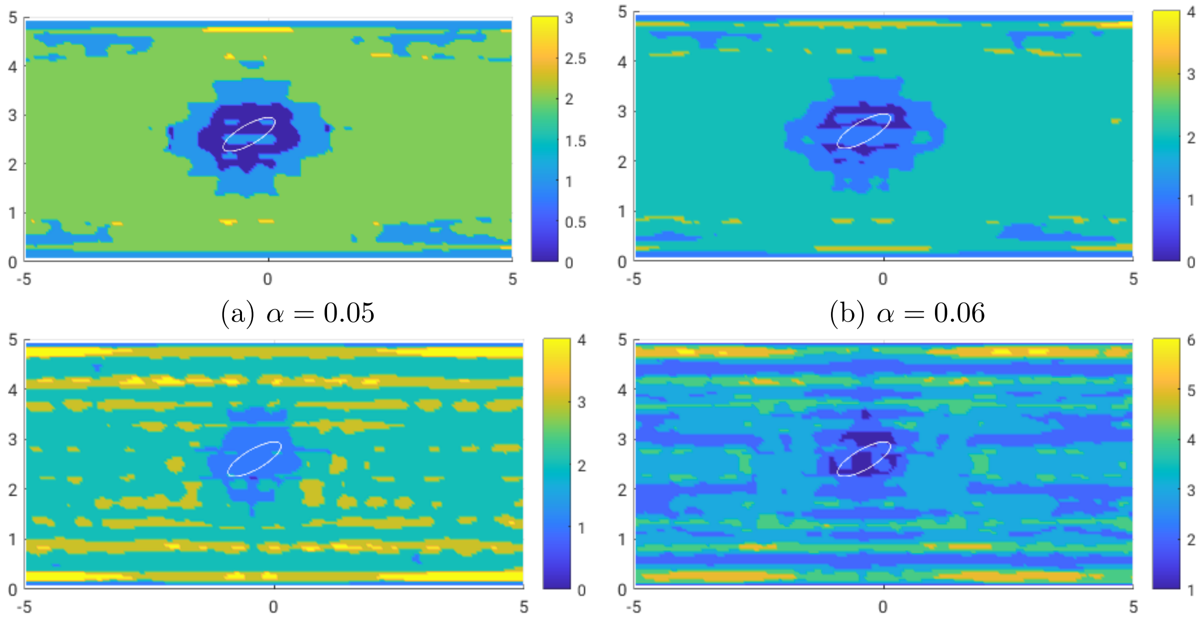
Both this remaining integral and the coefficients $\gamma_{\ell,m}$ can easily be computed analytically.

In the case $q>0$, for a given parameter $\alpha$, lateral length $a$, cut off parameter $\delta$ and grid of center points $\xi$, we compute the indicator function $I_\alpha:\Omega_R\to\mathbb{N}$,

$$I_\alpha(\xi) \;:=\; \#\{\lambda<-\delta\,|\,\lambda \text{ is eigenvalue of } \operatorname{Re}(\boldsymbol{N}_q)-\alpha\boldsymbol{T}_B\}\,.$$

Theorem 5.1 suggests that $I_\alpha$ is larger for test domains $B=\xi+(-a,a)^2$ that do not intersect the support $\operatorname{supp}(q)$ of the scattering object than on test domains $B$ contained in $\operatorname{supp}(q)$. Appropriate value of $\delta$ was estimated from a plot of magnitude of the eigenvalues of the matrix $\boldsymbol{N}_q$. In all examples below, we have chosen $\delta=2\cdot10^{-6}$.

In a first example, we consider a waveguide of height $h=5$ and the wavenumber $k=6.0$. This corresponds to $N=\lceil hk/\pi\rceil=10$ propagating modes in the waveguide in either direction. We take one evanescent mode in either direction into account and accordingly, choose the 22 incident fields generated by the Neuman eigenfunctions $\theta_m$, $m=0,\dots,10$, on $C_R^+$ and $C_R^-$, respectively, for $R=6$. Including these two evanescent modes in our computation slightly



(a) $\alpha = 0.05$ (b) $\alpha = 0.06$ (c) $\alpha = 0.1$ (d) $\alpha = 0.2$

Figure 6.1: Indicator function for an ellipse in a waveguide at wavenumber $k = 6.0$ and contrast $q = 2.0$ for lateral pixel size $2a = 0.02$ and various values of $\alpha$. The boundary of the true obstacle is shown in white.

improved the reconstructions. Contributions from further evanescent modes are so small that they offer no further improvement. The obstacle in this example is an ellipse located near the center of the waveguide with constant contrast $q = 2$.

We have chosen equidistant grid points $\xi \in \Omega_R$ with a step width $\tau = 0.05$ and a lateral length $2a = 0.02$ for the test squares. Plots of the corresponding indicator function $I_\alpha$ for four different values of $\alpha$ are displayed in Figure 6.1. As expected, the value of the indicator function is lower for points inside or close to the obstacle than for points away from the obstacle.

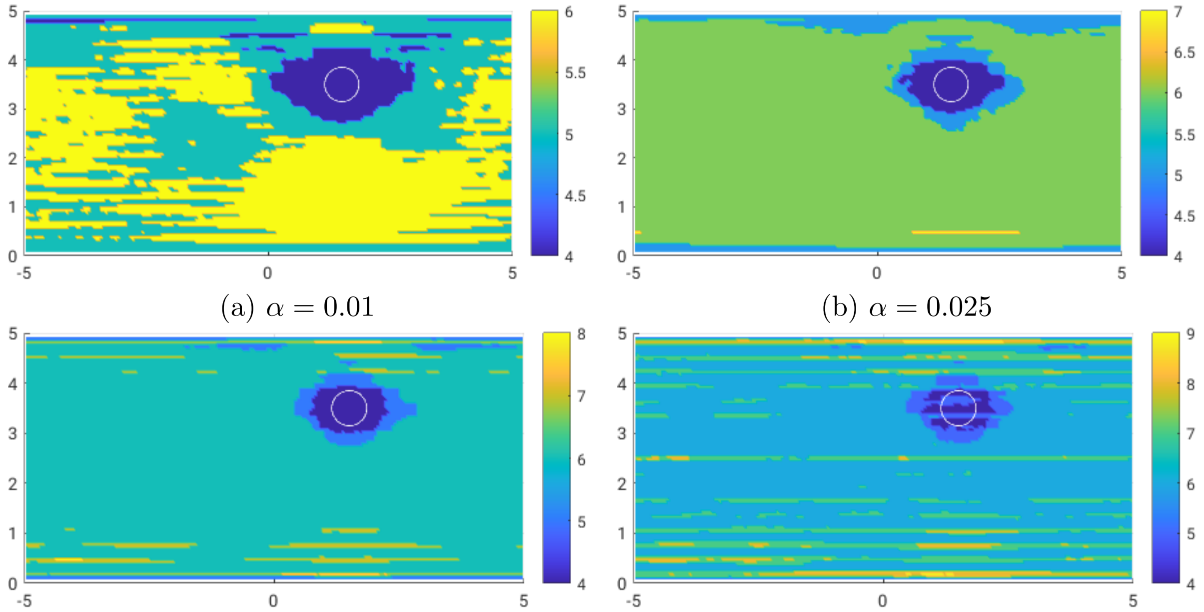
In the example shown in Figure 6.2, we consider a similar scattering problem, but at the wave number $k = 11.0$. The obstacle is a circle with $q = 2$ and it is placed towards one of the waveguide edges. In this example, there are 18 eigenfunctions corresponding to modes propagating in either direction. These are used in the reconstruction, i.e., we choose 36 incident fields, and do not include any evanescent modes. Again we plot the indicator function for four different values of $\alpha$, but with the same mesh and lateral test domain length as previously.

In the final example, we return to the wavenumber $k = 6.0$ but now consider an ellipse shaped obstacle with a negative contrast $q = -0.5$. The same 20 propagating modes as before, but no evanescent modes, are used to discretize the near field operator and the test operator. In this example, due to the negative contrast in the refractive index, the indicator function needs to be modified to

$$I_\alpha(\xi) := \#\{\lambda > \delta \mid \lambda \text{ is eigenvalue of } \operatorname{Re}(\boldsymbol{N}_q) - \alpha \boldsymbol{T}_B\}$$

(see Theorem 5.1). We again choose $\delta = 2 \cdot 10^{-6}$. The results for various values of $\alpha$ are displayed in Figure 6.3.

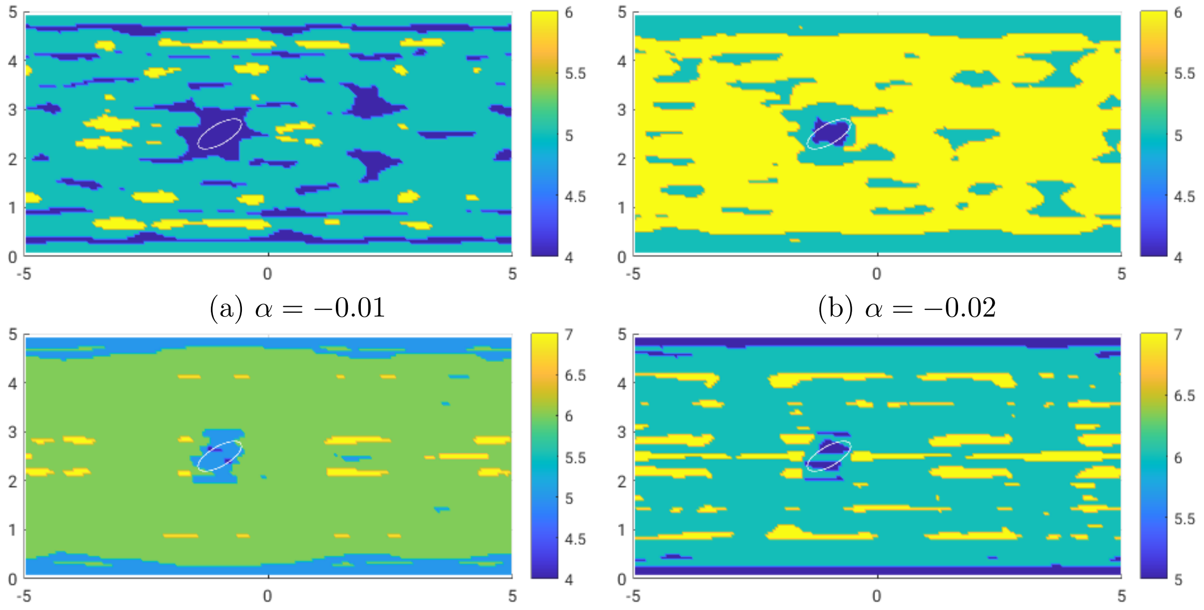
In all examples, with an appropriately chosen value of $\alpha$, the indicator function clearly has lowest values for points located inside or close to the obstacle. Low values of the indicator function also occur near the waveguide edges. Some of the examples also show that a badly chosen value of $\alpha$ leads to a the appearance of patches with low values of the indicator throughout



(a) $\alpha = 0.01$ (b) $\alpha = 0.025$

(c) $\alpha = 0.0375$ (d) $\alpha = 0.05$

Figure 6.2: Indicator function for an ellipse in a waveguide at wavenumber $k = 11.0$ and contrast $q = 2.0$ for lateral pixel size $2a = 0.02$ various values of $\alpha$. The boundary of the true obstacle is shown in white.



(a) $\alpha = -0.01$ (b) $\alpha = -0.02$

(c) $\alpha = -0.03$ (d) $\alpha = -0.04$

Figure 6.3: Indicator function for an ellipse in a waveguide at wavenumber $k = 6.0$ and contrast $q = -0.5$ for lateral pixel size $2a = 0.02$ various values of $\alpha$. The boundary of the true obstacle is shown in white.

the waveguide. A clear reconstruction of the shape of the obstacle appears to be beyond what can be achieved from the available data. The magnitude of the cutoff value $\delta$ had to chosen significantly larger than for the numerical results presented in [19], mainly due to the accuracy with which the waveguide Green's function $G$ is evaluated in the boundary integral method to generate the scattering data. In particular, at the higher wavenumber, where more propagating modes and thus a larger data set are available, the reconstructions are significantly better.

## Conclusions

We have shown that the rigorous monotonicity-based shape characterizations for inverse boundary value and inverse medium scattering problems from [19, 27, 29, 43] can be transferred to inverse medium scattering problems in closed cylindrical waveguides. In particular the treatment of the near field operator in the proofs of the monotonicity relation and of the existence of localized wave functions required some additional nontrivial estimates and the introduction of a near field equivalent of the scattering operator in the waveguide. Having established the monotonicity relation and the existence of localized wave functions the final proofs of the shape characterizations turned out to be rather close to the corresponding proofs in [19, 27]. In our numerical examples we have seen that the method works reasonably well using only the propagating part of the scattered fields in the waveguide, which is not covered by our theoretical results.

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