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CRC Preprint 2022/70, December 2022

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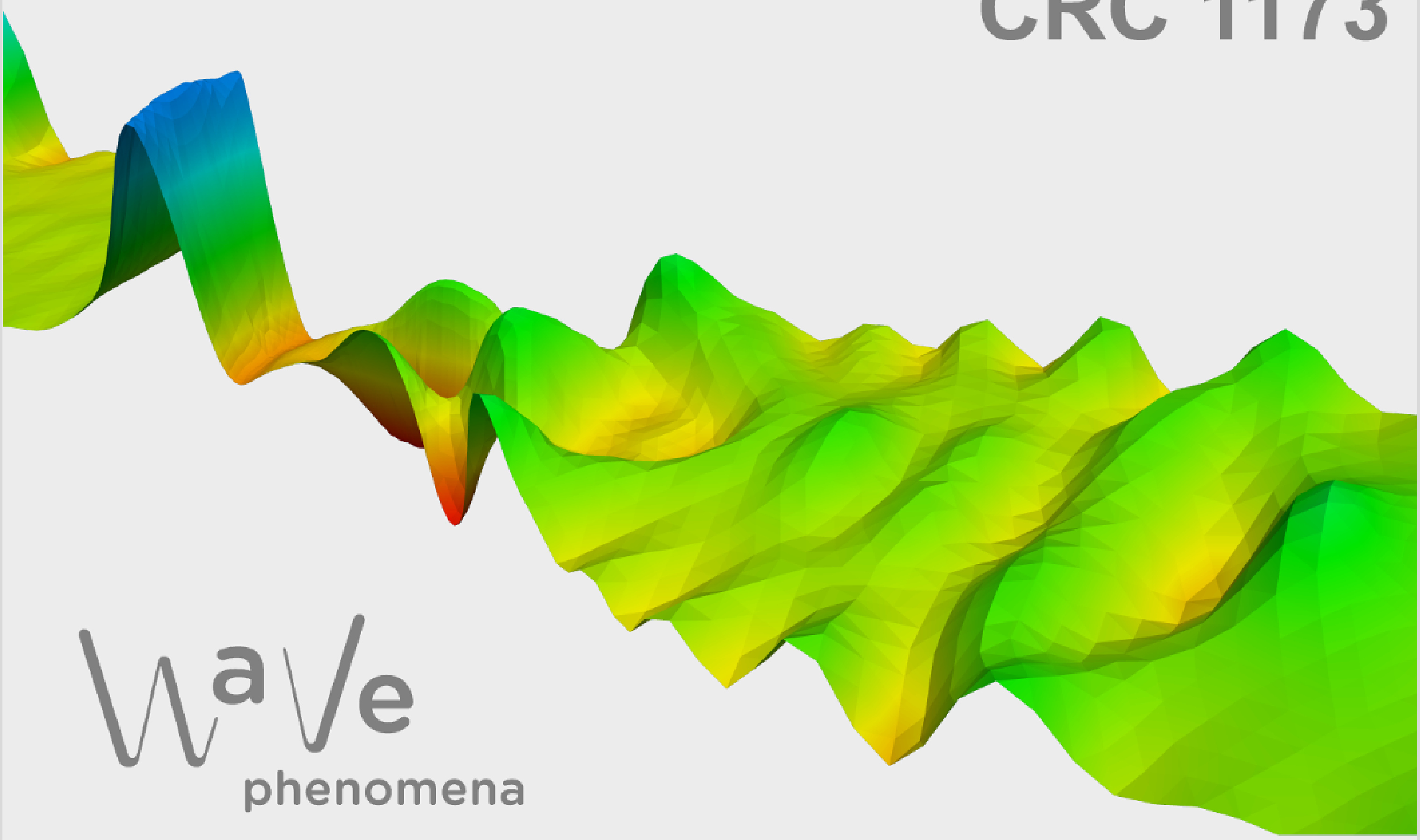
CRC 1173



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ISSN 2365-662X

ON ECHO CHAINS IN THE LINEARIZED BOUSSINESQ EQUATIONS AROUND TRAVELING WAVES

CHRISTIAN ZILLINGER

ABSTRACT. We consider the 2D Boussinesq equations with viscous but without thermal dissipation and observe that in any neighborhood of Couette flow and hydrostatic balance (with respect to local norms) there are time-dependent traveling wave solutions of the form $\omega = -1 + f(t)\cos(x - ty)$, $\theta = \alpha y + g(t)\sin(x - ty)$. As our main result we show that the linearized equations around these waves for $\alpha = 0$ exhibit echo chains and norm inflation despite viscous dissipation of the velocity. Furthermore, we construct initial data in a critical Gevrey 3 class, for which temperature and vorticity diverge to infinity in Sobolev regularity as $t \to \infty$ but for which the velocity still converges.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

In this article we study the long-time asymptotic stability of the 2D Boussinesq equations without thermal dissipation and with isotropic viscous dissipation:

$$\begin{aligned} \partial_t v + v \cdot \nabla v + \nabla p &= \nu \Delta v + \theta e_2, \\ \partial_t \theta + v \cdot \nabla \theta &= 0, \\ \nabla \cdot v &= 0, \\ (t, x, y) &\in (0, \infty) \times \mathbb{T} \times \mathbb{R}. \end{aligned} \tag{1}$$

Here v denotes the velocity of a fluid, p denotes the pressure and θ denotes the temperature of the fluid.

The Boussinesq equations are a coupled system of the Navier-Stokes equations and a diffusion equation. They describe the evolution of a heat conducting viscous fluid, where the term θe_2 causes hot fluid to rise above cold fluid and thus models buoyancy. In particular, if a layer of hot fluid lies beneath a layer of cold fluid, the system may exhibit a so-called Rayleigh-Bénard instability [DWZZ18], which can be suppressed by sufficiently strong shear flow or dissipation [Zil20].

2010 *Mathematics Subject Classification.* 35Q35,35Q79,76D05,35B40.
Key words and phrases. Boussinesq equations, partial dissipation, resonances, blow-up.

The study of the stability and asymptotic behavior of the Boussinesq equations, in particular with anisotropic or partial dissipation, is an area of very active research. We in particular mention the recent works [EW15, Wid18, DWZZ18, YL18, WXZ19, DWZ20, WSP20, DWXZ20, MSHZ20, TWZZ20, LWX$^+$21, Zil20, Zil21b] and the classical wellposedness results [CKN99, Cha06].

Following the seminal works of Villani and Mouhot [MV11] on Landau damping in plasma physics and of Bedrossian and Masmoudi [BM14] on inviscid damping for the Euler equations, questions of the effects of mixing have attracted strong interest. In the Euler setting one observes that for small, smooth perturbations of an affine shear flow, $v = (y, 0)$, the perturbation of the velocity field asymptotically converges. This asymptotic stability of the velocity field is known as *inviscid damping* and related to the Orr mechanism [Orr07]. Subsequently it was shown that this mechanism is very robust and linear inviscid damping holds for rather general classes of monotone flows at rather low Sobolev regularity [WZZ18, WZZ19, Zil16, Zil19, Zil17, CZZ19, BCZV19, IJ19, Jia20]. However, for the *nonlinear equations* very high, Gevrey 2 regularity requirements are imposed to establish stability and inviscid damping [IJ19, IJ20]. Already in [BM15] it was sketched in terms of a toy model that iterated nonlinear resonances might lead to norm inflation with an exponential dependence on the frequency, which gave strong evidence of the necessity of Gevrey regularity.

In [DM18] it was shown that there indeed exists data exhibiting chains of resonances (called echo chains) and associated norm inflation behavior (see also [Bed20] for similar results for the Vlasov-Poisson equations). This mechanism has been further studied in detail in [DZ19] for the Euler equations linearized around traveling waves. As the main results of [DZ19] it is shown that the norm inflation does *not* necessarily imply that inviscid damping fails. On the contrary, there exists data in a critical Gevrey class such that the solution not only exhibits norm inflation, but even blow-up: the *vorticity* diverges to infinity in Sobolev regularity as time tends to infinity. Yet, for the same data the *velocity* still is damped in L^2 to another shear flow as time tends to infinity. Hence, damping of the *velocity*, which is the physical effect of inviscid damping, persists *despite* blow-up of the *vorticity*. Similar results also holds for the Vlasov-Poisson equations [Zil21a].

In this article we study whether such resonances are also be present in the Boussinesq equations without thermal dissipation and whether the Gevrey regularity requirements of [MSHZ20] are necessary. In these equations there is competition of viscous dissipation, destabilization by buoyancy and resonance effects. In particular, it is a priori not clear whether these equations can sustain resonances and, if so, what the implications for norm inflation and asymptotic stability are. As the main results of this article, we show that the Boussinesq equations linearized around traveling waves indeed exhibit norm inflation and blow-up of the temperature in Sobolev regularity as time tends to infinity. Yet, damping of the velocity field (in lower Sobolev norms) persists despite this blow-up.

The key mechanism of this article's results is given by a (nonlinear) resonance mechanism, which exploits the system structure of the Boussinesq equations. Similar resonance mechanisms also underlie instability results in the Euler equations [DZ19], where they are known as fluid echoes, and the Vlasov-Poisson equations where they are called plasma echoes [Bed20]. In both the Euler setting [YOD05] and the plasma setting [MWGO68] these echoes have been experimentally observed.

We observe that a combination of a shear flow and hydrostatic balance

$$v = \begin{pmatrix} y \\ 0 \end{pmatrix}, \theta = \alpha y, \tag{2}$$

is a stationary solution of the Boussinesq equations (1)

$$\begin{aligned} \partial_t v + v \cdot \nabla v + \nabla p &= \nu \Delta v + \theta e_2, \\ \partial_t \theta + v \cdot \nabla \theta &= 0, \\ \nabla \cdot v &= 0. \end{aligned}$$

for any $\alpha \geq 0$, where in this article, for simplicity, we restrict to the case $\alpha = 0$. In this setting the echo mechanism then works as follows:

- At the initial time one introduces a perturbation of the temperature of the form $\epsilon e^{ikx+i\eta y}$. According to the linearized dynamics this solution will be mixed and will weakly converge to zero as time tends to infinity.
- At a later time $\tau < \frac{\eta}{k}$ we introduce another perturbation of the temperature at a different frequency in x of the form ϵe^{ilx}. According to the linearized equation also this perturbation will weakly converge to zero as time tends to infinity.
- In the linearized equations around the *stationary state* (2) both perturbations do not interact. However, in the nonlinear evolution (and in the linearization around a traveling wave) one observes a large time- and frequency-localized correction. Both perturbations result in a nonlinear *echo*:
  - By the buoyancy term the first perturbation at frequency k generates a perturbation of the vorticity $\omega = \nabla \times v$.
  - This vorticity perturbation leads to a frequency-localized resonance in the velocity at the resonant time $\frac{\eta}{k}$.
  - By the nonlinearity $v \cdot \nabla \theta$ this velocity resonance then interacts with the second perturbation of the temperature at mode l, exciting the temperature at frequency $k + l$ in x.

We stress the presence of viscous dissipation in the velocity equations. Thus, in contrast to the Euler equations or Vlasov-Poisson equations, where the density directly generates the velocity, the nonlinear echo effect here relies on the system structure of the Boussinesq equations. The resonance mechanism starts in θ, then excites v and in turn excites θ.

Building on this heuristic of a single echo interaction, in this article we show that such perturbations of the temperature can be identified with *traveling wave* solutions. Moreover, they can result in not just one echo but rather a chain of echoes. That is, let there be a traveling wave solution with $l = -1$ and introduce another perturbation at frequency k in x and η in y with $\eta > k > 1$. Then by the above sketch the interaction of the second perturbation with the underlying wave will result in an echo at frequency $k - 1$ at around the resonant time $\frac{\eta}{k}$. In turn, this echo correction at frequency $k - 1$ will interact with the underlying wave to generate an echo at frequency $k - 2$ at around the later time $\frac{\eta}{k-1}$. Iterating this procedure, we thus generate an *echo chain*:

$$k \to k-1 \to k-2 \dots \to 1,$$

where the size of the steps in frequency corresponds to the frequency of the underlying wave. As we show in Theorem 1.1 of Section 1.2 the associated norm inflation along this chain can be of size $\exp(C\sqrt[3]{\eta})$, which corresponds to a Gevrey 3 regularity class. This agrees with recent nonlinear stability results of [MSHZ20].

In the following subsections we provide an outline of the main results of this article.

1.1. **Traveling Waves.** As a main result of Section 2 we show that the tuple

$$\begin{aligned} v &= \begin{pmatrix} y \\ 0 \end{pmatrix} + \frac{f(t)}{1+t^2}\sin(x-ty)\begin{pmatrix} t \\ 1 \end{pmatrix}, \\ \omega &= \nabla v = -1 + f(t)\cos(x-ty), \\ \theta &= \alpha y + g(t)\sin(x-ty), \end{aligned} \tag{3}$$

yields a solution of the nonlinear Boussinesq equations with $\nu \geq 0$ for any $f(t), g(t)$, which solve an associated ODE (see Proposition 2.1). In particular, choosing $f(0), g(0)$ small, we can view these *traveling wave* solutions as initially arbitrarily small perturbations of the stationary solutions (2). Therefore, we suggest that in order to understand the nonlinear perturbation problem around (2), one should first study the linearized problem around the waves (3). We remark that such waves are also solutions of the inviscid problem. However, in that case generically $f(t)$ does not remain bounded as $t \to \infty$ (see Lemma 2.2).

The linearized problem around such a wave in vorticity formulation then reads

$$\begin{aligned} \partial_t\omega + y\partial_x\omega + \frac{f(t)}{1+t^2}\sin(x-ty)(\partial_y - t\partial_x)\omega + v\cdot\nabla f(t)\cos(x-ty) &= \nu\Delta\omega + \partial_x\theta, \\ \partial_t\theta + y\partial_x\theta + \frac{f(t)}{1+t^2}\sin(x-ty)(\partial_y - t\partial_x)\theta + v\cdot\nabla g(t)\cos(x-ty) + v_2\alpha &= 0. \end{aligned} \tag{4}$$

As a simplification, throughout this article we consider the following setup:

- We consider the case $\alpha = 0$, which implies that $g(t) = g(0)$ is independent of time. Furthermore, we assume that $g = 2c\nu$ for a small constant $0 < c < 0.001$, which by Lemma 2.2 further implies that
$$f(t) \leq \frac{4c}{1+t^2}.$$
- We remove the shear term $\frac{f(t)}{1+t^2}\sin(x-ty)(\partial_y - t\partial_x)$ and we fix the x average of θ and ω to be zero by a small forcing. As we discuss in Section 2 we expect the shear term to not change the dynamics qualitatively since $\frac{f(t)}{1+t^2}$ is small, rapidly decaying and integrable in time. Similarly, a change of x average would correspond to a change of the underlying shear flow $U(y,t)$ which we expect to be controlled in terms of a change of coordinates $z = U(y,t)$. However, the associated change of variables would introduce small further than nearest neighbor interactions and variable coefficients in the differential operators, thus making the analysis technically much more involved. We hence neglect these effects in the present article.

The equations studied in this article thus read (see Definition 2.4)

$$\begin{aligned}
&\partial_t\omega + y\partial_x\omega = -(v\cdot\nabla f\cos(x-ty))_{\neq} + \nu\Delta\omega + \partial_x\theta,\\
&\partial_t\theta + y\partial_x\theta + (v\cdot\nabla g\cos(x-ty))_{\neq} = 0,\\
&(t,x,y)\in(0,\infty)\times\mathbb{T}\times\mathbb{R},
\end{aligned} \tag{5}$$

where $()_{\neq}$ denotes the projection removing the x-average and $g = c\nu$, $f(t) \leq \frac{c}{1+t^2}$ (see Lemmas 2.2 and 2.3). We then show that this system exhibits chains of resonances.

We remark that the structure of the equations (5) is very similar to the one of the linearized Euler equations around a traveling wave, $\omega = -1 + c\cos(x-ty)$ (see [DZ19]), with the following main differences:

- Let $G := \nu\partial_x\omega + \partial_x^2\Delta^{-1}\theta$. Then the replacement of the Biot Savart law $v_1 = \partial_y\phi = \partial_y\Delta^{-1}\omega$ in the Euler equations is given by
$$\partial_y\phi = \nu^{-1}\partial_x\partial_y\Delta^{-2}\theta + \nu^{-1}\partial_y\partial_x^{-1}\Delta^{-1}G.$$
In particular, this mapping is of order -2 with respect to θ instead of -1 and corresponds to a real-valued instead of an imaginary Fourier multiplier. For this reason we here choose the underlying wave to be given by a cosine instead of a sine.
- Here $\frac{g}{2\nu}$ serves as a parameter of the strength of the interaction. In this article we will focus on the setting where this parameter is small, i.e. $g = 2c\nu$ for a small constant. We point out that this coupling parameter is a large challenge if one were to consider the inviscid limit $\nu \downarrow 0$ with g fixed (or slowly decaying in ν) instead.
- We observe that in terms of frequency η with respect to y, in the Euler setting we have a decay of multipliers with a rate η^{-1} as $\eta\to\infty$, since the Biot-Savart law is of order -1. In contrast in the present setting we have decay with a rate η^{-3}. This implies that the decoupling of neighboring modes becomes much stronger for large η.
- As we discuss in Section 2 the resonances rely on the coupling between the temperature θ and the vorticity ω by G. More precisely, the underlying wave in θ leads to a growth of the velocity at a critical time. While the perturbation of the vorticity is then subsequently damped, this velocity induces a growth of a different mode of the temperature perturbation, which then excites the velocity again at a later time. We stress that this system structure of the resonance mechanism strongly differs from the one in the Euler equations [DZ19]. In particular, here the vorticity and velocity experience strong, mixing-enhanced dissipation. The resonance mechanism hence has to exploit the absence of thermal dissipation while making use of resonances in the velocity.

We in particular stress the η^{-3} decay. In stark contrast to the Euler equations considered in [DZ19], here the regime where η is arbitrarily large actually turns out to be *better behaved* due to stronger decay of coefficients and as result stronger separation of frequencies.

Indeed, in Section 3.3 we study the resonance mechanism for those frequencies η with

$$|\eta| \geq c^{-1}, \tag{6}$$

where $g = c\nu$. This restriction is justified in Section 3.1, where we show that otherwise the evolution is asymptotically stable and does not exhibit large norm inflation.

1.2. **Main Results.** As the main result of our article we show that the linearized equations (5) around a wave indeed exhibit the sketched echo chain mechanism. Moreover, in addition to norm inflation there is a critical Gevrey 3 class of initial data for which the *temperature* and *vorticity* diverge to infinity in Sobolev regularity as time tends to infinity, but the *velocity* still converges. Thus damping of the *velocity* may persist despite blow-up and viscous dissipation is *not* sufficient to suppress resonances in the form of echo chains.

Theorem 1.1 (Stability, norm inflation and blow-up). *Consider the equation* (5) *and suppose that* $\frac{g}{2\nu} =: c$ *satisfies* $c < 0.001$. *Further define* $G = \nu\partial_x\omega + \partial_x^2\Delta^{-1}\theta$ *(see Lemma 2.5 for a formulation of* (5) *in terms of* G*).*

- *There exists* $C > 0$ *such that if the (Fourier transform of the) initial data satisfy*
$$\int \exp(C\sqrt[3]{|\eta|})(1+k^2)^N(|\mathcal{F}\theta_0|^2 + \eta|\mathcal{F}G_0|^2) < \infty,$$
then for all times $t > 0$ *it holds that*
$$\int \exp(\frac{C}{2}\sqrt[3]{|\eta - kt|})(1+k^2)^N(|\mathcal{F}\theta(t)|^2 + \eta|\mathcal{F}G(t)|^2) < \infty.$$
The evolution preserves Gevrey 3 regularity *up to a loss of constant.*
- *For* $c\eta > \nu^{-3/2}$ *there exists initial data* $\theta_0 \in L^2$ *localized at frequency* η *and* $G_0 = 0$, *such that for all* $t > 2\eta$ *the solution satisfies*
$$\|\theta(t)\|_{L^2} \geq \exp(\sqrt[3]{c\eta}).$$
There exists frequency localized initial data which exhibits norm inflation. *However, after attaining this norm inflation the solution is stable for all future times.*
- *Moreover, for every* $s_0 \in \mathbb{R}$ *there exists* $0 < C' < C$ *and initial data with*
$$\int \exp(C'\sqrt[3]{\eta})(1+k^2)^N(|\mathcal{F}\theta_0|^2 + |\mathcal{F}\omega_0|^2) < \infty,$$
such that $\theta(t)$ *converges in* H^s *for* $s < s_0$ *and diverges to infinity in* H^s *for* $s > s_0$, *as* $t \to \infty$. *The Gevrey* 3 *regularity class is hence a critical space for stability and damping may persist despite* blow-up.

Let us comment on these results:

- The resonance mechanism here relies on the coupling between temperature and vorticity by means of G (as we discuss in Section 2 a similar unknown has previously been introduced in [MSHZ20]). In particular, while the equations for ω exhibit very strong, mixing-enhanced dissipation, the norm of the temperature does not asymptotically decay and hence norm inflation persists.
- We remark that in the Euler equations the critical Gevrey class is given by Gevrey 2. The Boussinesq equations thus rely on a different resonance mechanism where it is not the interplay of vorticity and velocity resulting in resonances, but of the temperature and the velocity.

- In [MSHZ20] stability of the nonlinear Boussinesq equations without thermal dissipation was established for Gevrey 3 regular data. There a toy model suggests a norm inflation for frequency localized data, thus giving evidence of the necessity of Gevrey regularity. This article shows that this norm inflation indeed happens for (5) and that chains of resonances are a feature of the linear equations around traveling waves. Furthermore, there is not only norm inflation but blow-up, yet for a critical class of data damping of the velocity still persists. This hence raises the question whether then in the nonlinear problem there exists a critical class and whether there, as in the linear dynamics around waves, damping may persist despite instability.
- In [BBCZD21] it is shown that the inviscid, nonlinear Boussinesq equations with stably stratified temperature exhibit damping of the velocity for small Gevrey regular initial data, but algebraic instability of the vorticity and of the gradient of the temperature.

As we discuss in Section 2 this growth is driven by a chain of resonances which happens on a time interval $(c^{-1/3}\eta^{2/3}, 2\eta)$. Here we used that the coefficient functions in (5) in coordinates $(x-ty, y)$ do not depend on y anymore. Thus, in these coordinates the evolution equations decouple with respect to η and we may hence treat η as a given parameter. Therefore in order to establish the Gevrey regularity bounds of Theorem 1.1 we may consider η to be arbitrary but fixed and show that for $\omega(t,x)$, $\theta(t,x)$ the norms with respect to x grow at most by a factor $\exp(\sqrt[3]{c\eta})$ as time tends to infinity.

1.3. **Outline of the Article.** The remainder of the article is structured as follows:

- In Section 2 we construct traveling wave solutions of the Boussinesq equations. Due to its system structure here waves propagate both in temperature and vorticity and the magnitude of both waves is coupled. We further show that for the viscous problem with $\alpha = 0$, the wave in the vorticity decays as time tends to infinity while the wave in the temperature keeps it shape. In contrast, for the inviscid problem one encounters algebraic instability of the vorticity wave in agreement with the Miles-Howard criterion [How61].
- In Section 3 we establish stability of the evolution in Gevrey 3 regularity. In particular, we discuss the growth in different time regimes in different subsections. For instance in Section 3.1 we show that no large norm inflation may happen until a time of size about $c^{-1/3}\eta^{2/3}$ or after time 2η. Hence, all resonances have to happen inside this time interval, where we distinguish between intermediate time intervals where $t \approx \frac{\eta}{k}$ and $\frac{c\eta}{k^3}$ is not yet large, treated in Section 3.2 and the main resonance mechanism discussed in Section 3.3. The proof of the stability result of Theorem 1.1 is then given in Section 3.4.
- In Section 4 we construct data exhibiting norm inflation and blow-up. Here we study the evolution of frequency-localized initial data throughout the various time regimes. In Subsection 4.1 we use a contraction mapping approach in strongly weighted spaces to show that the solution remains localized until time about $c^{-1/3}\eta^{2/3}$. We then control the solution for intermediate times in Subsection 4.2 by means of a bootstrap approach. Subsection 4.3 then forms the core of our norm inflation argument where we show that this data achieves the norm inflation estimated in Section 3.3.

Subsequently we show that this norm inflation of the temperature persists for all future times.

Given these global in time solutions exhibiting norm inflation, we further construct data in critical Gevrey 3 class which exhibit blow-up in Sobolev regularity.

1.4. **Notation.** In this article we consider the linearized Boussinesq equations around traveling waves

$$\begin{aligned} \omega &= -1 + f(t)\cos(k(x-ty)), \\ \theta &= g\sin(k(x-ty)), \end{aligned}$$

where

$$\begin{aligned} g &= 2c\nu, \\ f(t) &\leq \frac{4c}{1+t^2}, \end{aligned}$$

and $0 < c < 0.001$ is a small constant.

Since these waves are stationary in coordinates $(x-ty, y)$ throughout this article we work in these coordinates and note that (after some simplification) the linear system (10) around these waves reads

$$\begin{aligned} \partial_t \omega &= (\frac{f(t)}{\nu}\cos(x)\partial_y(\partial_x^2 + (\partial_y - t\partial_x)^2)^{-1}\omega)_{\neq} + \nu(\partial_x^2 + (\partial_y - t\partial_x)^2)\omega + \partial_x\theta, \\ \partial_t\theta &= (\frac{g}{\nu}\cos(x)\partial_y(\partial_x^2 + (\partial_y - t\partial_x)^2)^{-1}\omega)_{\neq}, \\ (t,x,y) &\in (0,\infty)\times\mathbb{T}\times\mathbb{R}, \end{aligned}$$

where $()_{\neq}$ denotes the projection removing the x-average.

Here it turns out to be advantageous to consider the *good unknown* (see Lemma 2.5)

$$G = \nu\partial_x\omega + \partial_x^2(\partial_x^2 + (\partial_y - t\partial_x)^2)^{-1}\theta$$

in place of the vorticity ω (see Section 2 for a discussion), which leads to the system (11)

$$\begin{aligned} &\partial_t\theta = (\frac{g}{\nu}\cos(x)\partial_y\Delta_t^{-1}(\partial_x^{-1}G + \partial_x\Delta_t^{-1}\theta))_{\neq} \\ &\partial_t G = \nu\Delta_t G + (f\partial_x\cos(x)\partial_y\Delta_t^{-1}(\partial_x^{-1}G + \partial_x\Delta_t^{-1}\theta))_{\neq} + 2\partial_x^3(\partial_y - t\partial_x)\Delta_t^{-2}\theta + \partial_x^2\Delta_t^{-1}(\partial_t\theta), \end{aligned}$$

where for brevity, we use the short notation

$$\Delta_t := \partial_x^2 + (\partial_y - t\partial_x)^2.$$

We observe that none of the coefficient functions in this evolution equation depend on y explicitly. Therefore the equations decouple with respect to the Fourier variable

$$\eta \in \mathbb{R}$$

with respect to y. We thus tend to consider η as arbitrary but fixed and study θ and G as functions of t and $x \in \mathbb{T}$ only.

Furthermore, we observe that also with respect to x the only explicit coefficient functions are given by $\cos(x)$, which corresponds to a shift by ± 1 in Fourier space.

We may this equivalently consider this evolution equation as a nearest neighbor system for the Fourier modes

$$\theta_k, G_k,$$

for $k \in \mathbb{Z} \setminus \{0\}$. The corresponding ODE system and its integral formulation are given in Section 3.3.

Some of our estimates consider the regime where η is very large or use that c is very small. Here, specifically our notation of very large or very small is given by a factor 1000 and we use

$$a \ll b$$

if

$$1000|a| < |b|.$$

Similarly, we write

$$a \gtrsim b$$

if there exists a universal constant $C > \frac{1}{1000}$ such that

$$|a| \geq C|b|.$$

2. Traveling Wave Solutions and a Good Unknown

While the linearized Euler, Navier-Stokes or Boussinesq equations around an affine velocity $v = (y, 0)$ have been well-studied, the nonlinear problems have proved much more challenging with very active research in recent years. Similarly to the Vlasov-Poisson equations of plasma physics a main challenge in this nonlinear analysis is given by nonlinear resonances called *echoes*, which have also been experimentally observed [YOD05, MWGO68]. We briefly recall the experiment for the Euler setting:

- At an initial time one introduces a first perturbation of the vorticity which looks like a mode e^{ikx}. It is then mixed by the evolution and the perturbation of the velocity field is damped.
- At a later time one introduces a second perturbation of the vorticity which looks like a mode e^{ilx}. According to the linearized equations around the stationary state both perturbations do not interact (the linearized equations decouple with respect to frequency in x) and both perturbations are expected to be damped.
- Yet, at a predictable later time one observes a peak of the velocity field at a mode $e^{i(k+l)x}$. Both perturbations have interacted by means of the nonlinearity to induce a correction at frequency $k + l$ in x (and some frequency in y), which later unmixes and yields a peak. The perturbations result in an *echo* at a later time.

This effect, or more accurately chains of echoes where one echo causes another echo at a later time and so on, underlies the Gevrey regularity requirement of the nonlinear analysis [Bed20, Zil21a, DZ19, DM18].

Since the stationary states $\omega = -1, \theta = \alpha y$ of the Boussinesq equation are independent of x, the linearized problem around them does not include this chain of resonances. Hence these linearized equations exhibit radically different stability properties than the nonlinear problem. We thus aim to find a different nearby solutions of the Boussinesq equations which are x-dependent and whose linearized equations include this resonance mechanism. Such solutions are given by *traveling waves*:

Proposition 2.1 (Traveling waves). *Let $\alpha \in \mathbb{R}, k \in \mathbb{Z} \setminus \{0\}$ and $\nu \geq 0$ be given. Then the tuple*

$$\begin{aligned} v &= \begin{pmatrix} y \\ 0 \end{pmatrix} + \frac{f(t)}{k^2 + k^2 t^2} \sin(k(x - ty)) \begin{pmatrix} kt \\ 1 \end{pmatrix}, \\ \omega &= -1 + f(t) \cos(k(x - ty)), \\ \theta &= \alpha y + g(t) \sin(k(x - ty)), \end{aligned}$$

solves the Boussinesq equations (1) if and only if f and g solve the ODE system

$$\begin{aligned} \partial_t f &= -\nu(k^2 + k^2 t^2) f - kg, \\ \partial_t g &= \frac{\alpha}{k^2 + k^2 t^2} f. \end{aligned} \tag{7}$$

We call such solutions traveling waves*.*

We stress that these traveling wave solutions exist both in the inviscid and viscous problem. However, due to the dissipation, the asymptotic behavior as $t \to \infty$ is very different in both cases, as we discuss in Lemma 2.2.

Proof of Proposition 2.1. We make the ansatz that

$$\begin{aligned} \omega &= -1 + f(t) \cos(k(x - ty)), \\ \theta &= \alpha y + g(t) \sin(k(x - ty)), \end{aligned}$$

with $f(t)$ and $g(t)$ to be determined.

Then since $\cos(k(x - ty))$ is an eigenfunction of the Laplacian, the stream function $\phi := (-\Delta)^{-1}\omega$ is given by

$$\phi = -\frac{y^2}{2} + \frac{1}{k^2 + k^2 t^2} \cos(k(x - ty))$$

and $v = \nabla^\perp \phi$ is of the claimed form. In particular, we note that

$$\begin{aligned} (\partial_t + y\partial_x) f(t) \cos(k(x - ty)) &= \dot{f}(t) \cos(k(x - ty)), \\ (\partial_t + y\partial_x) g(t) \sin(k(x - ty)) &= \dot{g}(t) \sin(k(x - ty)) \end{aligned}$$

and, since $\sin(x) = \pm\sqrt{1 - \cos^2(x)}$ can locally be expressed as a function of $\sin(x)$ it also holds that

$$\begin{aligned} (\nabla^\perp \frac{1}{k^2 + k^2 t^2} \cos(k(x - ty))) \cdot \nabla f(t) \cos(k(x - ty)) &= 0, \\ (\nabla^\perp \frac{1}{k^2 + k^2 t^2} \cos(k(x - ty))) \cdot \nabla g(t) \sin(k(x - ty)) &= 0. \end{aligned}$$

Therefore, plugging in this ansatz the Boussinesq equations read

$$\begin{aligned} \dot{f}(t) \cos(k(x - ty)) &= -\nu(k^2 - k^2 t^2) f(t) \cos(k(x - ty)) - g(t) k \cos(k(x - ty)), \\ \dot{g}(t) \sin(k(x - ty)) &= \frac{\alpha}{k^2 + k^2 t^2} f(t) \sin(k(x - ty)). \end{aligned}$$

Since these equations are supposed to hold for all x, we may assume that $\sin(k(x - ty)) \neq 0 \neq \cos(k(x - ty))$ and hence obtain the claimed system of ordinary differential equations. $\square$

In this article we will focus on the case $k = 1$ and thus traveling waves at the lowest non-trivial frequency. Moreover, we restrict to considering the case $\alpha = 0$ for which $g(t)$ is constant in time and thus allows for an explicit characterization of the evolution.

Lemma 2.2 (Bounds on f and g)**.** *Let $\alpha = 0, \nu > 0$, $k = 1$ and let $f(t), g(t)$ be the solution of (7) with initial data f_0, g_0. Then for all $t > 0$ it holds that*

$$\begin{aligned} |f(t) - \exp(-\nu(t + t^3/3))f_0| &\leq \frac{4}{\nu(1+t^2)}|g_0|, \\ g(t) &= g_0. \end{aligned}$$

Let instead $\nu = 0$ and $\alpha \in \mathbb{R}$ and define

$$\gamma = \Re\sqrt{\frac{1}{4} - \alpha},$$

then $f(t), g(t)$ can be explicitly computed in terms of hypergeometric functions and there exist solutions for which $|f(t)| \sim t^{\frac{1}{2}+\gamma}$ and $|g(t)| \sim t^{-1/2+\gamma}$.

In particular, we note that if $f_0 = 0$ and $g = c\nu > 0$ it follows that

$$\begin{aligned} \frac{g}{\nu} &= c, \\ f(t) &\leq \frac{3c}{1+t^2}, \end{aligned} \tag{8}$$

for all times $t > 0$.

In the inviscid case, we instead observe that for $\alpha \geq \frac{1}{4}$, γ vanishes and we hence observe growth and decay with rates $t^{\pm\frac{1}{2}}$. We remark that here another common notational convention is to normalize α instead of gravity, so that

$$\begin{aligned} \partial_t f(t) &= -\beta^2 g, \\ \partial_t g(t) &= \frac{1}{1+t^2} f(t), \end{aligned}$$

where β^2 is the Richardson number. In this convention γ vanishes if $\beta^2 \geq \frac{1}{4}$, which agrees with the Miles-Howard criterion [How61]. Since our focus in this article is on the viscous problem, we do not pursue this further.

Proof. We observe that $\partial_t g = 0$ and thus $g(t) = g_0$ for all times. Furthermore, since f solves

$$\partial_t f = -\nu(1+t^2)f - g_0$$

it follows that

$$f(t) = \exp(-\nu(t + t^3/3))f_0 + \int_0^t \exp(-\nu(t - s + t^3/3 - s^3/3))ds g_0. \tag{9}$$

It thus only remains to bound the integral

$$\begin{aligned} \int_0^t \exp(-\nu(t - s + t^3/3 - s^3/3))ds &= \int_0^{t/2} \exp(-\nu(t - s + t^3/3 - s^3/3))ds \\ &\quad + \int_{t/2}^t \exp(-\nu(t - s + t^3/3 - s^3/3))ds. \end{aligned}$$

On the interval $(0, t/2)$ we may control

$$t - s \geq t/2 \geq s,$$
$$t^3/3 - s^3/3 \geq t^3/3 - t^3/24 \geq t^3/4 \geq \frac{t^2}{2} s.$$

Thus this integral can be bounded by

$$\int_0^{t/2} \exp(-\nu s - \nu \frac{t^2}{2} s) ds = \frac{1}{\nu(1+\frac{t^2}{2})}(1 - \exp(-\nu t/2 - \nu t^3/8))$$
$$\leq \frac{1}{\nu(1+\frac{t^2}{2})}.$$

On the interval $(t/2, t)$ we may integrate by parts to obtain

$$\begin{aligned}
&\int_{t/2}^t \frac{1}{\nu(1+s^2)} \partial_s \exp(-\nu(t - s + t^3/3 - s^3/3)) ds \\
&= \frac{1}{\nu(1+s^2)} \exp(-\nu(t - s + t^3/3 - s^3/3))|_{s=t/2}^t \\
&\quad - \int_{t/2}^t \exp(-\nu(t - s + t^3/3 - s^3/3)) \partial_s \frac{1}{\nu(1+s^2)} ds \\
&\leq \frac{1}{\nu(1+\frac{t^2}{2})},
\end{aligned}$$

where we estimated $0 \leq \exp(-\nu(t - s + t^3/3 - s^3/3)) \leq 1$.

Combining the estimates for both intervals we deduce that

$$|f(t) - \exp(-\nu(t + t^3/3)) f_0| \leq \frac{2}{\nu(1+\frac{t^2}{2})}|g_0| \leq \frac{4}{\nu(1+t^2)}|g_0|,$$

which concludes the proof of this case.

We next turn to the case $\nu = 0$, $\alpha \in \mathbb{R}$, for which the equations read

$$\partial_t f = -g,$$
$$\partial_t g = \frac{\alpha}{1+t^2} f.$$

It then follows that

$$\partial_t^2 f = -\frac{\alpha}{1+t^2} f.$$

Thus $f(t)$ is explicitly given in terms of hypergeometric functions of the second kind:

$$\begin{aligned}
f(t) &= c_1 F_2^1(-\frac{1}{4} - \frac{1}{4}\sqrt{1-4\alpha}, -\frac{1}{4} + \frac{1}{4}\sqrt{1-4\alpha}, 1/2, -t^2) \\
&\quad + c_2 F_2^1(\frac{1}{4} - \frac{1}{4}\sqrt{1-4\alpha}, \frac{1}{4} + \frac{1}{4}\sqrt{1-4\alpha}, 1/2, -t^2)
\end{aligned}$$

and

$$g = -\partial_t f.$$

In particular, for t large these hypergeometric functions behave as

$$t^{\frac{1}{2} - \frac{1}{2}\sqrt{1-4\alpha}}, \ t^{\frac{1}{2} + \frac{1}{2}\sqrt{1-4\alpha}},$$

which concludes the proof. $\square$

In the following we discuss the linearized equations around a given traveling wave.

Lemma 2.3. *Consider a traveling wave solution*

$$\begin{aligned}\omega_\star &= -1 + f(t)\cos(k(x-ty)),\\ \theta_\star &= \alpha y + g(t)\sin(k(x-ty)),\end{aligned}$$

as in Proposition 2.1. Then the linearized equations for the perturbations ω, θ around this wave are given by the system:

$$\begin{aligned}\partial_t\omega + y\partial_x\omega + \frac{f(t)}{1+t^2}\sin(x-ty)(\partial_y - t\partial_x)\omega + v\cdot\nabla f(t)\cos(x-ty) &= \nu\Delta\omega + \partial_x\theta,\\ \partial_t\theta + y\partial_x\theta + \frac{f(t)}{1+t^2}\sin(x-ty)(\partial_y - t\partial_x)\theta + v\cdot\nabla g(t)\cos(x-ty) + v_2\alpha &= 0.\end{aligned}$$

Proof of Lemma 2.3. In the linearization we omit the quadratic nonlinearities $v\cdot\nabla\omega$ and $v\cdot\nabla\theta$. □

With these preparations, we can sketch the echo mechanism for the linearized equations around a traveling wave:

(1) At the initial time we introduce a perturbation $e^{ik_0x}e^{i\eta y}$ to the temperature θ. This perturbation roughly evolves by transport for a length of time.
(2) At around the critical time $\frac{\eta}{k_0}$ the buoyancy term $\partial_x\theta$ in the evolution equation for ω will also cause the mode $e^{ik_0x}e^{i\eta y}$ of ω to grow.
(3) The mode $e^{ik_0x}e^{i\eta y}$ of the vorticity leads to a peak of the velocity at that mode at the resonant time. This perturbation *interacts with the underlying wave* in θ at frequencies ± 1 to yield a contribution to the temperature at frequencies $e^{i(k_0\pm 1)x}e^{i\eta y}$.
(4) We repeat the cycle with $k_0 - 1$.

We remark that the *underlying wave* here determines the *step size* of the chain, $k_0 \rightsquigarrow k_0 - 1 \rightsquigarrow k_0 - 2 \rightsquigarrow \cdots \rightsquigarrow 1$ and hence choosing the wave with the lowest frequency, $k = 1$, yields the longest chains. At each resonant time our linear perturbation picks up some energy from the underlying wave and moves to a lower frequency. It stops once it has reached the lowest frequency 1.

As noted following Proposition 2.1 throughout this article we thus make two simplifications:

- We only study the case $\alpha = 0$, so g is constant. We further assume that $g = c\nu$ for a small constant $c > 0$ and $f_0 = 0$. In particular, as noted in (8) this implies that $f(t) \leq \frac{c}{1+t^2}$ is also small and decreasing in time.
- We omit the advection term $\frac{f(t)}{1+t^2}\sin(x-ty)(\partial_y - t\partial_x)$, which is a shear by $(0, \frac{1}{1+t^2}\sin(x))$ in coordinates $(x-ty, y)$ and fix the x average of both θ and ω as zero. Since $\frac{1}{1+t^2}$ is quickly decaying and integrable in time, we expect that it is possible to remove this simplification. Similarly, we expect that changes to the x-average can be controlled by using the dissipation and mixing effects. However, the resulting change of variables and the associated modification of integro-differential operators makes this problem technically very challenging and is hence omitted in this article. This has the additional benefit that after a Fourier transform in x the evolution for frequencies $k > 0$ and $k < 0$ decouple.

For later reference we formulate this problem in coordinates $(x-ty, y)$ moving with the shear as a definition.

Definition 2.4. *Let $0<g<\nu$ be a given constant and formally set $f \equiv 0$. Then the* wave perturbation equations *in coordinates $(x-ty,y)$ moving with the shear are given by*

(10)
$$\begin{aligned}
&\partial_t \omega = (\frac{f(t)}{\nu}\cos(x)\partial_y(\partial_x^2+(\partial_y-t\partial_x)^2)^{-1}\omega)_{\neq} + \nu(\partial_x^2+(\partial_y-t\partial_x)^2)\omega + \partial_x\theta,\\
&\partial_t\theta = (\frac{g}{\nu}\cos(x)\partial_y(\partial_x^2+(\partial_y-t\partial_x)^2)^{-1}\omega)_{\neq},\\
&(t,x,y)\in(0,\infty)\times\mathbb{T}\times\mathbb{R},
\end{aligned}$$

where $()_{\neq}$ projects out the x-average. We also introduce the notation $\Delta_t = \partial_x^2 + (\partial_y - t\partial_x)^2$.

Similarly to [MSHZ20] it turns out to be advantageous to equivalently reformulate this system in terms of another unknown.

Lemma 2.5 (Good unknown)**.** *Let ω,θ be given functions and $\nu>0$ be given. We further define the* good unknown

$$G = \nu\partial_x\omega + \partial_x^2\Delta_t^{-1}\theta.$$

Then (ω,θ) is a solution of (10) if and only if (G,θ) solve

$$\begin{aligned}
\partial_t\theta &= (\frac{g}{\nu}\cos(x)\partial_y\Delta_t^{-1}(\partial_x^{-1}G+\partial_x\Delta_t^{-1}\theta))_{\neq}\\
\partial_t G &= \nu\Delta_t G + (f\partial_x\cos(x)\partial_y\Delta_t^{-1}(\partial_x^{-1}G+\partial_x\Delta_t^{-1}\theta))_{\neq}\\
&\quad + 2\partial_x^2(\partial_y-t\partial_x)\Delta_t^{-2}\theta + \partial_x\Delta_t^{-1}(\partial_t\theta).
\end{aligned} \tag{11}$$

Here we use the short notation $\Delta_t = \partial_x^2+(\partial_y-t\partial_x)^2$.

Proof. Direct calculation. We further use that by assumption the x-averages of ω vanishes and we hence lose no information by considering the x derivative only. □

We remark that in [MSHZ20] instead the unknown $K=\Delta_t\partial_x^{-1}G$ is considered. The present choice is made for two reasons:

- Since $\partial_x^2\Delta_t^{-1}$ is an order 0 multiplier, G and θ can be treated similarly and we for instance do not have to introduce energies such as $\|\nabla\omega\|_{L^2}^2+\|\theta\|_{L^2}^2$ with different numbers of derivatives.
- The evolution equation of G follows more immediately from the equation for ω and $\partial_x^2\Delta_t^{-1}\partial_t\theta$ exhibits further cancellations. More precisely, an operator such as $\partial_x^2\Delta_t^{-1}(\cos(x)\Delta_t^{-1})$ exhibits good bounds since $\cos(x)$ induces a Fourier shift and if a frequency k in x resonant, then $k+1, k-1$ are non-resonant (see Section 3.3 for a definition of resonance and corresponding estimates).

In order to introduce ideas, we first consider a further simplified model problem.

Definition 2.6. *In a* model problem *we formally set $G\equiv 0$, which yields the following system:*

$$\begin{aligned}
&G=0,\\
&\partial_t\theta = (\frac{g}{\nu}\cos(x)\partial_y\Delta_t^{-1}\partial_x\Delta_t^{-1}\theta)_{\neq},
\end{aligned} \tag{12}$$

and its equivalent Fourier characterization (in coordinates $(x+ty, y)$):

$$\begin{aligned}\partial_t \tilde{\theta}(t,k,\eta) + i\eta \frac{g(t)}{2}\tilde{\phi}(t,k+1,\eta) + i\eta\frac{g(t)}{2}\tilde{\phi}(t,k-1,\eta) = 0,\\ \nu(k^2+(\eta-kt)^2)^2\tilde{\phi}(t,k,\eta) + ik\tilde{\theta}(t,k,\eta) = 0.\end{aligned}$$

Inserting the second equation into the first, we obtain the following nearest neighbor ode system for θ:

$$\begin{aligned}&\partial_t\tilde{\theta}(k) + c_{k-1}\tilde{\theta}(k-1) + c_{k+1}\tilde{\theta}(k+1) = 0,\\ &c_l = \frac{g(t)}{2}\frac{\eta l}{\nu(l^2+(\eta-lt)^2)^2}\\ &\quad = \frac{g(t)}{2\nu}\frac{\eta}{l^3}\frac{1}{(1+(\frac{\eta}{l}-t)^2)^2}\end{aligned}$$

Next, for a heuristic argument assume that $\frac{\eta}{k^2}$ is very large, $t\approx\frac{\eta}{k}$ and that $c:=\frac{g}{2\nu}$ is small. Then $c_k(t) = c\frac{\eta}{k^3}\frac{1}{(1+(\frac{\eta}{k}-t)^2)^2}$ will be resonant and possibly very large. In contrast, if $l\neq k$, then $(\frac{\eta}{l}-t)^2 \geq \frac{1}{4}\max(\frac{\eta}{k^2},\frac{\eta}{l^2})$ and thus c_l can be controlled in terms of $\frac{c}{l}\max(\frac{\eta}{k^2},\frac{\eta}{l^2})^{-3}$ and will be very small. Thus, it seems reasonable to expect that the growth mechanism is largely determined by

$$\partial_t\tilde{\theta}(k-1) \approx c\frac{\eta}{k^3}\frac{1}{(1+(\frac{\eta}{k}-t)^2)^2}\tilde{\theta}(k).$$

Since

$$\int_{\mathbb{R}}\frac{1}{(1+(\frac{\eta}{k}-t)^2)^2} = \frac{\pi}{2}$$

and $c\frac{\eta}{k^3}$ is large, this causes $\tilde{\theta}(k-1)$ to grow by a large multiple of $\tilde{\theta}(k)$. Iterating this procedure with decreasing k, thus suggests that

$$\theta(1, t\approx 1) \approx \frac{(c\eta)^k}{(k!)^3}\tilde{\theta}(k, t\approx\frac{\eta}{k}).$$

This product is maximal for $k\approx\sqrt[3]{c\eta}$

$$\frac{(c\eta)^k}{(k!)^3} \approx \frac{1}{(c\eta)^{2/3}}\exp(\sqrt[3]{c\eta}).$$

Thus, these heuristics suggest that the model problem exhibits norm inflation at this exponential rate and hence Gevrey 3 regularity is critical.

A main challenge in this article is to show that one indeed can reduce to the case $t\approx\frac{\eta}{k}$ with $\frac{\eta}{k^2}$ large and that the full evolution matches the growth behavior of (12), though resonances in G and the system structure make this problem more challenging.

For later reference, we introduce the following notational conventions:

Definition 2.7. *Let $\eta>0$ be given. Then for any $k>0$ we define*

$$t_k = \frac{1}{2}\left(\frac{\eta}{k+1}+\frac{\eta}{k}\right)$$

and $t_0 := 2\eta$. Then the k-th resonant time interval I_k is given by $I_k = (t_k, t_{k-1})$.

In particular, we observe that for $t \in I_k$ it holds that for any $l \neq k$

$$\begin{aligned}|t - \frac{\eta}{l}| &\geq \min(|t - \frac{\eta}{k-1}|, |t - \frac{\eta}{k+1}|) \\ &\geq \frac{1}{4} \max\left(\frac{|\eta|}{k^2}, \frac{|\eta|}{l^2}\right).\end{aligned}$$

Thus, for any $l \neq k$ and $t \in I_k$ we may estimate

$$c_l \leq \frac{g(t)}{2k\nu}(\frac{\eta}{k^2})^{-3} \ll 1.$$

Furthermore, we note that the interval I_k has length about $\frac{\eta}{k^2}$ and thus also $\int_{I_k} c_l$ is small.

In order to make our stability analysis in the following sections precise, we need to specify a space X with respect to which stability is measured.

Definition 2.8 (The space X)**.** *Consider a weight function $\lambda(l) > 0$ on ℓ^2 with*

$$\sup_{l \in \mathbb{Z}} \frac{\lambda(l \pm 1)}{\lambda(l)} < 2.$$

Then the Hilbert space X is given by all sequences $(u_l)_l$ such that $(\lambda_l u_l)_l \in \ell^2$ with the associated inner product.

The main examples we are interested in are

- $\lambda(l) \equiv 1$, which yields $X = \ell^2$,
- $\lambda(l) = 1 + 2^{-N}|l|^N$, which corresponds to H^N in physical space, and
- $\lambda(l) = 2^{|l|}$, which implies that u is analytic in physical space.

Throughout the remainder of the article we will consider X to be an arbitrary but fixed such space. Our plan for the remainder of this article is the following:

- In Section 3.1 we show that if we pick initial data $\theta, G \in X$ which is localized at frequency η, then the solution remains stable until a time much larger than $\eta^{2/3}$ (and for all time if η is small) and is stable again after the time 2η. Thus, any norm inflation has to happen between these two times. Furthermore, while the evolution of (θ, G) is not invertible due to dissipation, for small G we show that the evolution of θ is a small perturbation of the identity if the frequency η is large.
- In Section 3.3 we study the resonance mechanism on the remaining time interval. Here we show that resonances could result in bounded norm inflation. More precisely, we establish upper bounds on possible growth, which are valid for all initial data. Lower bounds are studied in Section 4
- In Section 3.4 we combine the bounds establish for the various time regimes to prove the Gevrey 3 regularity result of Theorem 1.1.
- In Section 4 we show that these upper bounds are optimal (up to changes of constants in the exponents) by constructing data exhibiting norm inflation. Moreover, we construct data in a critical Gevrey regularity class that not only exhibits norm inflation but blow-up as time tends to infinity. We stress here that in the Euler equations or Vlasov-Poisson equations [DZ19, Zil21a] the lack of dissipation allowed for an inversion of the time direction and thus to more easily construct initial data producing desired final data. In contrast, the viscous dissipation of the Boussinesq equations prevents any

invertibility in time. It thus is a very challenging problem to ensure the existence of data achieving norm inflation.

3. Stability and Gevrey 3 regularity

In the heuristic model of Section 2 we showed that most growth is expected to occur when

$$t \approx \frac{\eta}{k}$$

and k is such that $c\frac{\eta}{k^3}$ is large and that this growth is expected to be bounded by

$$\exp(C\sqrt[3]{c\eta})$$

for some constant C.

In this section we prove a corresponding stability estimate which shows that this factor indeed provides an upper bound. As a complementary result, in Section 4 we show that there exist solutions attaining such growth (possibly with smaller constant C). We then use the norm inflation solutions as building blocks to construct solutions exhibiting blow-up.

In our analysis we first show that if η is much smaller than c^{-1}, then the evolution is globally stable for all times. If η is not this small we consider four time regimes (see also Figure 1):

- Define $k_0 \approx \sqrt[3]{c\eta\pi}$ (rounded down) and let $k_1 = 20k_0$. Then for all $k \geq k_1$ it holds that $\frac{c\eta}{k^3}$ is much smaller than 1. We hence consider the interval of times

  $$0 \leq t \leq t_{k_1}$$

  with $t_{k_1} = \frac{1}{2}(\frac{\eta}{k_1+1} + \frac{\eta}{k_1}) < \frac{\eta}{k_1}$ (see Definition 2.7) as the *small time* regime. Here we show that the evolution is stable and, in a suitable sense, close to the identity.
- We observe that for $k \geq k_0$, $\frac{c\eta}{k^3}$ is bounded by 1 but not necessarily small. We hence call

  $$(t_{k_1}, t_{k_0})$$

  the *intermediate* time regime. We here derive rough upper bounds showing that the norm at most grows by $\exp(40\sqrt[3]{c\eta})$.
- We next consider the interval

  $$(t_{k_0}, 2\eta),$$

  which is composed of intervals I_k with

  $$\frac{c\eta\pi}{k^3} \geq 1$$

  and potentially very large. Here we encounter the main resonance mechanism discussed in the heuristic model of Section 2 and establish an upper growth bound by $\exp(10\sqrt[3]{c\eta})$. We call this interval the *resonant* regime. In Section 4 we further introduce an additional time t_{k_3} with $k_3 = \frac{k_0}{1000}$ up to which we also establish lower bounds on norm inflation.
- Finally, we show that on the interval

  $$(2\eta, \infty)$$

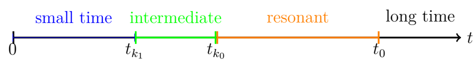

FIGURE 1. The time regimes are determined by the size of $\frac{c\eta}{k^3}$ for $t \in I_k$. If k is very large, i.e. for small times, this factor is very small. If instead $k \leq k_0$, then this factor is large, leading to possibly large resonances.

the evolution is stable and that the evolution of θ is a bounded perturbation of the identity. Thus, there can be no further norm inflation in this *long time* regime.

We stress that the asymptotic stability in the long time regime highlights that norm inflation by a finite time does not imply instability as time tends to infinity. Hence in Section 4 we need to combine infinitely many solutions exhibiting norm inflation to obtain non-trivial asymptotic behavior like blow-up.

3.1. **Small Time, Large Time and Small Frequencies.** In this subsection we consider the regimes where t or η are small or when $t > 2\eta$ is large. In all these regimes it turns out that the evolution is well-controlled and, in a sense, a small perturbation of the identity for θ and a dissipative equation for G. Our main results are summarized in the following theorem.

Theorem 3.1 (Stable regimes)**.** *Let $\eta \in \mathbb{R}$ be given and for simplicity of notation assume $\eta > 0$. Let further $0 \leq \frac{g}{\nu} = c < 0.001$.*

Then we have the following stable time intervals depending on η:

(1) *If $c\eta \ll 1$, then for all times $t > 0$ it holds that*
$$40^2\|\theta(t)\|_X + \|G(t)\|_X \leq 4(40^2\|\theta(0)\|_X + \|G(0)\|_X).$$
(2) *Suppose that $c\eta \geq 0.001$, then for all times $t > 2\eta =: t_0$ it holds that*
$$\|\theta(t)\|_X + \|G(t)\|_X \leq 4(\|\theta(t_0)\|_X + \|G(t_0)\|_X).$$
Moreover,
$$\|\theta(t) - \theta(t_0)\|_X \leq c\eta^{-1}2(\|\theta(t_0)\|_X + \|G(t_0)\|_X).$$
The evolution of θ is a bounded perturbation of the identity.

(3) *Consider the early time regime given by $0 < t < \frac{\eta}{\sqrt{8000c\eta\pi}}$. Then on this time interval it holds that*
$$40^2\|\theta(t)\|_X + \|G(t)\|_X \leq 2(40^2\|\theta(0)\|_X + \|G(0)\|_X).$$

The factor 40 here is used to control a potentially large interaction term between θ and G. Since this term is small in the large time regime, there we may consider more symmetric energies.

We remark that this discussion omits the two regimes $t \in I_k$, where in the intermediate time regime
$$\frac{c\eta}{k^3}\pi \in (\frac{1}{8000}, 1)$$
or in the resonant time regime
$$\frac{c\eta}{k^3}\pi \geq 1.$$

Establishing bounds on the growth in these regimes requires significantly more effort and is the main challenge of Sections 3.3 and 3.2.

Proof of Theorem 3.1. We recall from Lemma 2.5 that the evolution equations with respect to θ and G are given by (11):

$$\begin{aligned}
&\partial_t G = \nu\Delta_t G + \nu f \partial_x \cos(x)\partial_y \phi + 2\partial_x^3(\partial_y - t\partial_x)\Delta_t^{-2}\theta - \partial_x^2\Delta_t^{-1}(g\cos(x)\partial_y\phi),\\
&\partial_t\theta = g\cos(x)\partial_y\phi,
\end{aligned}$$

where

$$\begin{aligned}
G &= \nu\partial_x\omega - \partial_x^2\Delta_t^{-1}\theta,\\
\phi &= \nu^{-1}\Delta_t^{-1}\partial_x^{-1}G + \nu^{-1}\partial_x\Delta_t^{-2}\theta.
\end{aligned}$$

The model case: In order to introduce ideas, let us first consider the model problem of Definition 2.6 where we formally set $G \equiv 0$. Then we define the Fourier multiplier

$$A(t,l,\eta) = \exp\left(C(\arctan(\frac{\eta}{l-1} - t) + \arctan(\frac{\eta}{l} - t) + \arctan(\frac{\eta}{l+1} - t))\right),$$

where we omit terms in which we would divide by 0 and $C = 0.01$ is a small constant. In particular, we note that $A(t,l,\eta) \approx 1$ uniformly in t, l, η and that A is decreasing in time. We then aim to show that the energy

$$\|A\theta\|_X^2$$

is non-increasing in time and thus serves as a Lyapunov functional. We hence compute the time derivative of this energy and observe that on the one hand

$$\langle A\theta, \dot{A}\theta\rangle \leq 0$$

is non-positive provides decay in terms of the Fourier multipliers

$$-C\frac{1}{1+(\frac{\eta}{l-1}-t)^2} - C\frac{1}{1+(\frac{\eta}{l}-t)^2} - C\frac{1}{1+(\frac{\eta}{l+1}-t)^2}.$$

On the other hand the contribution

$$\langle A\theta, A\partial_t\theta\rangle$$

is possibly positive, but can be bounded from above in terms of the Fourier multipliers

$$c\frac{(l-1)\eta}{((l-1)^2+(\eta-(l-1)t)^2)^2} + c\frac{l\eta}{(l^2+(\eta-lt)^2)^2} + c\frac{(l+1)\eta}{((l+1)^2+(\eta-(l+1)t)^2)^2},$$

by using Young's inequality and the fact that there is only interaction between neighboring modes. It hence suffices to show that

$$c\frac{l\eta}{(l^2+(\eta-lt)^2)^2} = \frac{c\eta}{l^3}\frac{1}{(1+(\frac{\eta}{l})^2)^2} \leq \frac{C}{2}\frac{1}{1+(\frac{\eta}{l}-t)^2}, \tag{13}$$

In any of the regimes considered we can then use different arguments to show that (13) holds:

(1) If $c\eta \ll 1$ is small, $\frac{c\eta}{l^3}$ is small for all l and thus

$$c\frac{l\eta}{(l^2+(\eta-lt)^2)^2} = c\frac{\eta}{l^3}\frac{1}{(1+(\frac{\eta}{l}-t)^2)^2} \leq c\eta\frac{1}{1+(\frac{\eta}{l}-t)^2} \leq \frac{C}{2}\frac{1}{1+(\frac{\eta}{l}-t)^2}$$

satisfies (13) for all l.

(2) Similarly, if $t > 2\eta$, then we can control by
$$c\frac{\eta}{l^3}\frac{1}{1+\eta^2}\frac{1}{1+(\frac{\eta}{l}-t)^2} \leq c\frac{1}{1+(\frac{\eta}{l}-t)^2} \leq \frac{C}{2}\frac{1}{1+(\frac{\eta}{l}-t)^2},$$
and hence (13) holds for all l.

(3) Finally, let $c\eta \gtrsim 1$ and suppose that $t \approx \frac{\eta}{k} \lesssim c^{-1/3}\eta^{2/3}$ is not too large. Then it follows that
$$c\frac{\eta}{k^3} \approx c\eta^{-2}t^3 \ll 1,$$
which is the reason for our choice of upper bound on t.

We then distinguish two cases. If $l > \frac{k}{2}$, then
$$c\frac{\eta}{l^3} \leq 8c\frac{\eta}{k^3} \ll 1,$$
and thus the estimate (13) holds for such l. If instead $l \leq \frac{k}{2}$, then $\frac{\eta}{l} \geq 2\frac{\eta}{k} \geq 2t$ and hence
$$c\frac{\eta}{l^3}\frac{1}{1+(\frac{\eta}{l}-t)^2} \leq c\frac{\eta}{l^3}\frac{1}{1+\frac{1}{4}(\frac{\eta}{l})^2} \leq 4c \ll 1.$$
Thus, the estimate (13) also holds for such l, which concludes the proof for the model case.

The general case: Building on the insights developed in the model case, we turn to the full problem:
$$\begin{aligned}
\partial_t G &= \nu\Delta_t G + \nu f\partial_x \cos(x)\partial_y\phi + \partial_x^2\Delta_t^{-1}(g\cos(x)\partial_y\phi) + 2(\partial_y - t\partial_x)\partial_x^3\Delta_t^{-2}\theta,\\
\partial_t\theta &= g\cos(x)\partial_y\phi,\\
\phi &= \nu^{-1}\Delta_t^{-1}\partial_x^{-1}G + \nu^{-1}\partial_x\Delta_t^{-2}\theta.
\end{aligned}$$
Here, main additional challenges are given by the slower decay of Δ_t^{-1} as compared to Δ_t^{-2} and the potentially large size of the multiplier corresponding to $(\partial_y - t\partial_x)\partial_x^3\Delta_t^{-2}$.

The large time case $t > 2\eta$: We first discuss the simplest case where $t > 2\eta$ is large.

Here we observe the following multiplier estimates:
$$\begin{aligned}
g\nu^{-1}\partial_y\Delta_t^{-2} &\mapsto \frac{c\eta}{k^3}\frac{1}{(1+(\frac{\eta}{k}-t)^2)^2} \leq c\frac{1}{(1+(t/2)^2)^{3/2}},\\
g\nu^{-1}\partial_y\Delta_t^{-1}\partial_x^{-1} &\mapsto \frac{c\eta}{k^3}\frac{1}{1+(\frac{\eta}{k}-t)^2} \leq c\eta\frac{1}{(1+(t/2)^2)},\\
2(\partial_y - t\partial_x)\partial_x^3\Delta_t^{-2} &\mapsto 2\frac{\frac{\eta}{k}-t}{(1+(\frac{\eta}{k}-t)^2)^2} \leq \frac{2}{(1+(t/2)^2)^{3/2}},\\
\partial_x^2\Delta_t^{-1} &\mapsto \frac{1}{1+(\frac{\eta}{k}-t)^2} \leq \frac{1}{1+(t/2)^2},\\
\nu f &\leq \frac{c}{1+t^2},
\end{aligned} \tag{14}$$
where we used that $t - \frac{\eta}{k} \geq t - \eta \geq t/2$.

We thus conclude that
$$\partial_t(\|G\|_X^2 + \|\theta\|_X^2) \leq -\nu\|\nabla_t G\|_X^2 + \max(c\eta\frac{1}{(1+(t/2)^2)}, \frac{2}{(1+(t/2)^2)^{3/2}})(\|G\|_X^2 + \|\theta\|_X^2).$$

Since

$$\int_{2\eta}^{\infty} c\eta \frac{1}{(1+(t/2)^2)} + \frac{2}{(1+(t/2)^2)^{3/2}} dt \leq c + (1+|\eta|)^{-1}$$

it hence follows that

$$\|G(t)\|_X^2 + \|\theta(t)\|_X \leq \exp(c + (1+|\eta|)^{-1})(\|G(2\eta)\|_X^2 + \|\theta(2\eta)\|_X).$$

The solution is stable. Moreover, we may insert these bounds into the integral equations to conclude that

$$\begin{aligned}\|\theta(t) - \theta(2\eta)\|_X &\leq \int_{2\eta}^{\infty} \|\partial_t \theta\|_X \leq c\|G\|_{L^\infty X} + (1+|\eta|)^{-1}\|\theta\|_{L^\infty X} \\ &\leq (c + (1+|\eta|)^{-1}) \exp(c + (1+|\eta|)^{-1})(\|G(2\eta)\|_X^2 + \|\theta(2\eta)\|_X).\end{aligned}$$

In view of the dissipation term, for $G(t)$ we instead estimate

$$\|G(t)\|_X^2 - \|G(2\eta)\|_X^2 \leq (1+|\eta|)^{-1} \exp(c + (1+|\eta|)^{-1})(\|G(2\eta)\|_X^2 + \|\theta(2\eta)\|_X).$$

Small frequencies: We next turn to considering the regime where

$$c\eta \leq 0.001$$

is very small. Again considering the multipliers shown in (14) we observe that, for instance, we may estimate

$$\begin{aligned} g\nu^{-1}\partial_y \Delta_t^{-2} &\mapsto \frac{c\eta}{k^3} \frac{1}{(1+(\frac{\eta}{k}-t)^2)^2} \leq 0.001 \frac{1}{(1+(\frac{\eta}{k}-t)^2)^2}, \\ g\nu^{-1}\partial_y \Delta_t^{-1}\partial_x^{-1} &\mapsto \frac{c\eta}{k^3} \frac{1}{1+(\frac{\eta}{k}-t)^2} \leq 0.001 \frac{1}{1+(\frac{\eta}{k}-t)^2}, \end{aligned} \tag{15}$$

for all $k \in \mathbb{Z} \setminus \{0\}$. The bound on the right-hand-side here is integrable in time with small norm for any fixed k. However, if one were to first take the supremum with respect to k and then integrate the resulting norm would be large, since for any t there exists some k such that $t \approx \frac{\eta}{k}$.

We thus consider a *frequency-dependent* multiplier A of a similar form as in the model problem in order to construct a Lyapunov functional:

$$A(t,k) = \exp\left(0.1 \sum_{l \in \{k-1,k,k+1\}} (\arctan(\frac{\eta}{l} - t) - \arctan(\frac{\eta}{l} - t_k))\right).$$

We note that $0.1|\arctan(\frac{\eta}{l} - t) - \arctan(\frac{\eta}{l} - t_k)| \leq 0.1\pi$ and hence A is comparable to 1 at all times. Moreover, $A(t)$ is decreasing and

$$\dot{A} = A\mathcal{F}^{-1} \frac{-0.1}{1+(\frac{\eta}{k}-t)^2} \mathcal{F}$$

is sufficiently negative to absorb multipliers such as in (15).

We observe that one multiplier

$$2(\partial_y - t\partial_x)\partial_x^3 \Delta_t^{-2} \mapsto 2\frac{\frac{\eta}{k} - t}{(1+(\frac{\eta}{k}-t)^2)^2} \leq 2\frac{1}{1+(\frac{\eta}{k}-t)^2} \tag{16}$$

does not necessarily include a small prefactor. For this reason we make the following ansatz for our energy

$$E(t) := 40^2\|A(t)\theta\|_X^2 + \|A(t)G\|_X^2,$$

which weighs θ more highly. Here, with slight abuse of notation, we identify A and the corresponding Fourier multiplier and hence write $A\theta$ instead of $\mathcal{F}^{-1}(A\mathcal{F}\theta)$.

We then claim that $E(t)$ is non-increasing and thus provides a Lyapunov functional. We compute the time-derivative of $E(t)$ as

$$\begin{aligned}\partial_t E(t) &= 2 \cdot 40^2 \langle A\theta, \dot{A}\theta\rangle + 2\langle AG, \dot{A}G\rangle \\ &\quad + 2 \cdot 40^2 \langle A\theta, A\partial_t \theta\rangle + 2\langle AG, A\partial_t G\rangle,\end{aligned}$$

where $\langle , \rangle$ denotes the inner product in X.

By the estimate (15) it then follows that

$$\begin{aligned}40^2\langle A\theta, A\partial_t\theta\rangle \leq 0.001 &\left\| \sum_{l \in \{k-1,k,k+1\}} \frac{1}{\sqrt{1+(\frac{\eta}{l}-t)^2}} 40 A\theta \right\|_X^2 \\ &+ 40 \cdot 0.001 \left\| \sum_{l \in \{k-1,k,k+1\}} \frac{1}{\sqrt{1+(\frac{\eta}{l}-t)^2}} 40 A\theta \right\|_X \\ &\cdot \left\| \sum_{l \in \{k-1,k,k+1\}} \frac{1}{\sqrt{1+(\frac{\eta}{l}-t)^2}} AG \right\|_X,\end{aligned}$$

where we used a sum to account for the Fourier shift due to $\cos(x)$.

Using Young's inequality in the last factor we thus conclude that any possible growth due to $40^2\langle A\theta, A\partial_t\theta\rangle$ can be absorbed by the decay terms due to $\dot{A}$.

Similarly, for the time derivative of G we compute:

$$\begin{aligned}\langle AG, A\partial_t G\rangle &= -\nu\langle A\nabla_t G, A\nabla_t G\rangle \\ &\quad + \langle AG, A\partial_x^3(\partial_y - t\partial_x)\Delta_t^{-2}A\theta\rangle \\ &\quad + \langle AG, A\nu f \cos(x)\partial_y\phi\rangle \\ &\quad + \langle AG, A\partial_x^2\Delta_t^{-1}\partial_t\theta\rangle.\end{aligned}$$

The dissipation term is non-positive and thus potentially helpful. However, as ν is allowed to be arbitrarily small, we do not make use of it in the following. For the next term we make use of the estimate (16) and express

$$\langle AG, A\partial_x^3(\partial_y - t\partial_x)\Delta_t^{-2}A\theta\rangle = \langle AG, A\frac{1}{40}\partial_x^3(\partial_y - t\partial_x)\Delta_t^{-2}40A\theta\rangle$$

to obtain a small factor $\frac{1}{40}$.

Finally, for the contributions by $\nu f \partial_y \phi$ and by

$$\partial_x^2\Delta_t^{-1}\partial_t\theta = \partial_x^2\Delta_t^{-1} g\cos(x)\partial_y\phi$$

we simply control the multiplier corresponding to $\partial_x^2\Delta_t^{-1}$ by 1 and $\nu f \leq \frac{c}{1+t^2} \leq c$. The estimate of this term then follows as for $\langle A\theta, A\partial_t\theta\rangle$.

In conclusion, we have thus shown that

$$\partial_t E \leq 40^2\langle A\theta, \dot{A}\theta\rangle + \langle AG, \dot{A}G\rangle \leq 0$$

and thus $E(t)$ is non-increasing, which is our desired stability estimate.

The small time regime: As a last regime we consider the case when $c\eta$ is allowed to be large but t is bounded by $\frac{\eta}{\sqrt[3]{8000c\eta}}$. Here we again aim to construct a Lyapunov functional by combining the estimates of the previous two regimes, but will require small changes to the multiplier A.

Consider for instance the multiplier

$$g\nu^{-1}\partial_y\Delta_t^{-2} \mapsto \frac{c\eta}{k^3}\frac{1}{(1+(\frac{\eta}{k}-t)^2)^2},$$

and let $k_1 \approx \sqrt[3]{8000c\eta}$ (rounded up). Then for all $k \geq 0.5k_1$ it holds that

$$\frac{c\eta}{k^3} \leq \frac{2^3}{8000} \leq 0.001,$$

and for these k we may hence argue as in the small frequency regime. Conversely, if $k < 0.5k_1$, then

$$\frac{\eta}{k} > 2t \Rightarrow |\frac{\eta}{k}-t| \geq \frac{1}{2}|\frac{\eta}{k}| \tag{17}$$

and hence

$$\frac{c\eta}{k^3}\frac{1}{(1+(\frac{\eta}{k}-t)^2)^2} \leq c\frac{1}{1+(\frac{\eta}{k}-t)^2}.$$

Similarly, for the multiplier

$$g\nu^{-1}\partial_y\Delta_t^{-1}\partial_x^{-1} \mapsto \frac{c\eta}{k^3}\frac{1}{1+(\frac{\eta}{k}-t)^2}$$

we may estimate as in the small frequency regime if $k \geq 0.5k_1$. If $k < 0.5k_1$ we cannot spare powers of t but note that by (17):

$$\begin{aligned}
&\int_{t\in\mathbb{R}:|t|<\frac{1}{2}|\frac{\eta}{k}|}\frac{c\eta}{k^3}\frac{1}{1+(\frac{\eta}{k}-t)^2}\\
&\leq \frac{c\eta}{k^3}\int_{\tau\in R:|\tau|\geq\frac{1}{2}|\frac{\eta}{k}|}\frac{1}{1+\tau^2}\\
&\leq \frac{c\eta}{k^3}\frac{1}{1+\frac{1}{2}|\frac{\eta}{k}|} \leq \frac{c}{k}.
\end{aligned}$$

We thus consider a slightly different multiplier

$$A(t) = \exp(0.01\arctan(\frac{\eta}{k}-t) - \arctan(\frac{\eta}{k}-t_k) + 1_{k<k_0}\int_{t_k}^{t}10\frac{c\eta}{k^3}\frac{1}{1+(\frac{\eta}{k}-s)^2}ds).$$

Then by the same argument as in the previous regime it follows that

$$E(t) := 40^2\|A(t)\theta\|_X^2 + \|A(t)G\|_X^2$$

is non-increasing. Since $A(t,k)$ is comparable to 1, E is thus a Lyapunov functional. □

This theorem shows that any norm inflation has to happen in the remaining time regimes, where we distinguish between the intermediate time regime

$$t \in I_k : \frac{c\eta}{k^3} \in (\frac{1}{8000}, \frac{1}{\pi}),$$

which is considered in the following Section 3.2 and the resonant time regime

$$t \in I_k : \frac{c\eta}{k^3} \geq \pi^{-1},$$

which is considered in Section 3.3.

In both cases we determine the growth on individual intervals I_k and then show that the total growth obtained by iterating over all possible values of k is bounded by $\exp(C\sqrt[3]{c\eta})$ for a suitable constant C, which is consistent with Gevrey 3 regularity.

3.2. **The Intermediate Regime.** In this section we consider the time intervals I_k where

$$c\frac{\eta}{k^3} \in (\frac{1}{8000}, \frac{1}{\pi}).$$

We recall that the heuristic model of Section 2 suggested a growth of the neighbors of the resonant frequency by a factor $c\frac{\eta}{k^3}\pi$. In the present regime this factor is still smaller than 1 but not very small anymore.

As a main result of this subsection we show that on each interval I_k the norm in X grows at most by a factor C independent of η. As we show in Section 4 this bound is probably far from optimal, but sufficient for our stability estimates. Indeed, we observe that if $t \approx \frac{\eta}{k}$ then in this regime t is proportional to $\eta^{2/3}$ and hence might be very large, but k is contained in the interval

$$(\sqrt[3]{\frac{1}{\pi}c\eta}, \sqrt[3]{8000c\eta}).$$

In particular, there are at most $20\sqrt[3]{c\eta}$ such k.

Hence, a growth bound by a factor C for each k implies a total growth bound by

$$C^{20\sqrt[3]{c\eta}} = \exp(20\ln(C)\sqrt[3]{c\eta}),$$

which is consistent with Gevrey 3 regularity.

Our results are summarized in the following theorem.

Theorem 3.2. *Let $\eta \gg 1$, $0 < c < 0.001$ be given. Let further $k \in \mathbb{N}$ with*

$$\sqrt[3]{\pi c\eta} \leq k \leq \sqrt[3]{8000c\eta\pi}.$$

Then on the time interval $I_k = (t_k, t_{k-1})$ and for $t = t_{k-1}$ it holds that

$$\|\theta(t)\|_X + \|G(t)\|_X \leq e^{3\pi}(\|\theta(t_k)\|_X + \|G(t_k)\|_X). \tag{18}$$

In particular, if $k_0 \approx \sqrt[3]{\pi c\eta}$, $k_1 \approx \sqrt[3]{8000c\eta\pi}$ (rounded to an integer), then it holds that

$$\|\theta(t_{k_0})\|_X + \|G(t_{k_0})\|_X \leq e^{20\cdot 3\pi\sqrt[3]{c\eta}}(\|\theta(t_{k_1})\|_X + \|G(t_{k_1})\|_X).$$

Proof of Theorem 3.2. Suppose for the moment the estimate (18) holds for all $k_0 \leq k \leq k_1$. Then it follows that

$$\|\theta(t_{k_0})\|_X \leq e^2\|\theta(t_{k_0+1})\|_X \leq e^2e^2\|\theta(t_{k_0+2})\|_X \leq e^{2|k_0-k_1|}\|\theta(t_{k_1}).$$

Thus the claimed bound follows by noting that $|k_0 - k_1| \leq \sqrt[3]{8000c\eta} = 20\sqrt[3]{c\eta}$.

It remains to prove the estimate (18), for which we again want to use a multiplier argument. More precisely, we note that on the interval I_k the coefficients in front of the mode k are the largest. For instance,

$$\frac{c\eta}{k^3}\frac{1}{(1+(\frac{\eta}{k}-t)^2)^2} \geq |\frac{c\eta}{l^3}\frac{1}{(1+(\frac{\eta}{l}-t)^2})^2|$$

for all $l \in \mathbb{Z} \setminus \{0\}$.

We thus consider the *frequency-independent* multiplier

$$A(t) = \exp(-3\int_{t_k}^{t} \frac{1}{1+(\frac{\eta}{k}-\tau)^2} d\tau),$$

Unlike in the regimes considered in Section 3.1 we here do not need to control commutators due to the evaluation of A at neighboring modes. Hence, here we do not require a small exponent 0.01 but can choose 3 as a comparably large exponent. We remark that a similar idea has been used in the analysis of the nonlinear equations in [MSHZ20]. We now claim that

$$E(t) := A(t)(\|\theta(t)\|_X^2 + \|G(t)\|_X^2)$$

is non-increasing for $t \in I_k$. Since $A(t_k) = 1$, $A(t)$ is decreasing in time and $A(t) \geq \exp(-3\pi)$, this further implies

$$\|\theta(t)\|_X^2 + \|G(t)\|_X^2 = A(t)^{-1}E(t) \leq e^{3\pi}E(t_k) = e^{3\pi}(\|\theta(t_k)\|_X^2 + \|G(t_k)\|_X^2),$$

and thus establishes the desired bound.

It remains to prove that $E(t)$ indeed is non-increasing. We recall that

$$\partial_t\theta = c\cos(x)(\partial_y\partial_x^{-1}\Delta_t^{-1}G + \partial_y\partial_x\Delta_t^{-2}\theta)$$

and that at frequency l

$$\begin{aligned} c\partial_y\Delta_t^{-2} &\mapsto \frac{c\eta}{l^3}\frac{1}{(1+(\frac{\eta}{l}-t)^2)^2}, \\ c\partial_y\Delta_t^{-1}\partial_x^{-1} &\mapsto \frac{c\eta}{l^3}\frac{1}{1+(\frac{\eta}{l}-t)^2}. \end{aligned}$$

Since $t \in I_k$ these multipliers are largest (in absolute value) if $l = k$ and are hence both bounded by

$$\frac{c\eta}{k^3}\frac{1}{1+(\frac{\eta}{k}-t)^2}$$

and can therefore be absorbed into the decay of $A(t)$.

Similarly, we recall that

$$\begin{aligned} &\partial_t G = \nu\Delta_t G + \nu f\partial_x\cos(x)\partial_y\phi + \partial_x^2\Delta_t^{-1}(g\cos(x)\partial_y\phi) + 2(\partial_y - t\partial_x)\partial_x^3\Delta_t^{-2}\theta, \\ &\nu\partial_y\theta = \partial_y\partial_x^{-1}\Delta_t^{-1}G + \partial_y\partial_x\Delta_t^{-2}\theta. \end{aligned}$$

Then the dissipation term yields a non-positive contribution by

$$-\nu A(t)\|\nabla_t G\|_X^2 \leq 0.$$

For the contributions by

$$\nu f\partial_x\cos(x)\partial_y\phi + \partial_x^2\Delta_t^{-1}(g\cos(x)\partial_y\phi),$$

we may very roughly estimate $f\nu \leq c$, $\frac{1}{1+(\frac{\eta}{l}-t)^2} \leq 1$ and argue as for the estimate of $\partial_t\theta$. It thus only remains to discuss

$$2(\partial_y - t\partial_x)\partial_x^3\Delta_t^{-2} \mapsto \frac{2(\frac{\eta}{l}-t)}{(1+(\frac{\eta}{l}-t)^2)},$$

which in absolute value can be bounded by

$$2\frac{1}{1+(\frac{\eta}{k}-t)^2}$$

and hence can also be absorbed by the decay of $A(t)$.

In conclusion, we have shown that $E(t)$ indeed is non-increasing, which completes the proof. $\square$

We next turn to studying the main resonance mechanism for times $t \in I_k$ with

$$\frac{c\eta}{k^3}\pi \geq 1.$$

Here we derive a slightly suboptimal upper bound of possible growth by a factor

$$1 + \frac{c\eta}{k^3}\pi \leq 2\frac{c\eta}{k^3}\pi,$$

which then implies a total growth bound by

$$\prod_{k=1}^{k_0} 2\frac{c\eta}{k^3}\pi \leq \exp(\sqrt[3]{2c\eta}).$$

In Section 4 we show that for special initial data this growth is attained up to loss of factor in the exponent. We remark that in the Euler equations or Vlasov-Poisson equations [DZ19, Zil21a] one can explicitly construct initial data generating desired data at a later time t_{k_0} by inverting the time direction. As the Boussinesq equations include viscous dissipation this is not possible in the present setting and we instead have to invest considerable effort to characterize which data can be generated at time t_{k_0} starting from suitable initial data.

3.3. **The Resonance Mechanism.** In this section we study the norm inflation mechanism on the time interval I_k, when

$$1 \leq \pi\frac{c\eta}{k^3} \leq \pi c\eta.$$

Since this factor might be very large, we cannot allow for rough, Gronwall-type bound of growth by

$$\exp(\pi\frac{c\eta}{k^3}).$$

Instead, we show that, as sketched for the model problem of Section 2, only the neighbors of the mode k grow by a factor at most

$$2\pi\frac{c\eta}{k^3},$$

and all other modes only change mildly.

The main estimates of this section are summarized in the following theorem.

Theorem 3.3 (Resonance Mechanism)**.** *Let $0 < c < 0.01$ be as in Theorem 1.1 and let $\eta \gtrsim c^{-1}$ be given and $k \in \mathbb{N}$ be such that $c\frac{\eta}{k^3} \gtrsim 1$. Then for any data $\theta^\star, G^\star \in X$ prescribed at the time $t_k = \frac{1}{2}(\frac{\eta}{k+1} + \frac{\eta}{k-1})$ the corresponding solution of the wave perturbation equation (11) satisfies*

$$\|\theta(t) - \theta^\star\|_X \leq 2c\frac{\eta}{k^3}\pi(\|\theta^\star\|_X + \|G^\star\|_X)$$

and

$$\|G(t)\|_X \leq 2c\frac{\eta}{k^3}\pi(\|\theta^\star\|_X + \|G^\star\|_X).$$

for all $t_k \leq t \leq t_{k-1}$.

Moreover, the following mode-wise bounds hold:

$$|\theta_{k\pm1}(t_{k-1}) - \theta_{k\pm1}(t_k) - c\frac{\eta}{k^3}\pi\theta_k(t_k)| \leq \frac{1}{k}c\frac{\eta}{k^3}\pi(\|\theta^\star\|_X + k\|G^\star\|_X),$$
$$|\theta_k(t_{k-1}) - \theta_k(t_k)| \leq c(\frac{\eta}{k^2})^{-1}\pi(\|\theta^\star\|_X + k\|G^\star\|_X)$$

and for all $l \notin \{k-1, k, k+1\}$ it holds that

$$|\theta_l(t_{k-1}) - \theta_l(t_k)| \leq c(\frac{\eta}{k^2})^{-1}\pi(\|\theta^\star\|_X + \|G^\star\|_X).$$

We emphasize that the present theorem only provides an upper bound on growth. In Section 4 we will show that there indeed exists data saturating this growth (up to a factor). In particular, we construct global in time solutions exhibiting norm inflation due to *echo chains*. Using these solutions as building blocks we then construct a critical class of initial data exhibiting blow-up in Sobolev regularity.

The proof of Theorem 3.3 concludes in Subsection 3.3.1 and builds on multiple steps which are formulated as propositions and lemmas. Unlike the Lyapunov energy approach of Sections 3.1 and 3.2 we here iteratively construct solutions in weighted ℓ^∞ spaces, which we will then use to deduce analogous estimates in the space X.

Before stating these estimates we introduce an integral formulation of the equations and collect estimates on time integrals of the coefficient functions. The interplay of these estimates then determines admissible weights.

The equations (11) in Fourier variables read

$$\begin{aligned}\partial_t\theta_l = {}&\frac{g}{2\nu}\frac{\eta}{(l+1)^3}\frac{1}{(1+(\frac{\eta}{l+1}-t)^2)^2}\theta_{l+1} + \frac{g}{2\nu}\frac{\eta}{(l-1)^3}\frac{1}{(1+(\frac{\eta}{l-1}-t)^2)^2}\theta_{l-1}\\ &+\frac{g}{2\nu}\frac{\eta}{(l+1)^3}\frac{1}{1+(\frac{\eta}{l+1}-t)^2}G_{l+1}\\ &+\frac{g}{2\nu}\frac{\eta}{(l-1)^3}\frac{1}{1+(\frac{\eta}{l-1}-t)^2}G_{l-1}\\ &=: c_l^+\theta_{l+1} + c_l^-\theta_{l-1} + d_l^+G_{l+1} + d_l^-G_{l-1}.\end{aligned} \tag{19}$$

and

$$\begin{aligned}\partial_t G_l = {}&-\nu(l^2+(\eta-lt)^2)G_l + f\frac{\nu}{g}il(d_l^+G_{l+1} + d_l^{-1}G_{l-1})\\ &+ f\frac{\nu}{g}il(c_l^+\theta_{l+1} + c_l^{-1}\theta_{l-1})\\ &+ 2\frac{(\frac{\eta}{l}-t)}{(1+(\frac{\eta}{l}-t)^2)^2}\theta_l\\ &+ \frac{1}{1+(\frac{\eta}{l}-t)^2}\left(c_l^+\theta_{l+1} + c_l^-\theta_{l-1}\right)\\ &+ \frac{1}{1+(\frac{\eta}{l}-t)^2}\left(d_l^+G_{l+1} + d_l^-G_{l-1}\right),\end{aligned} \tag{20}$$

where we use G_l and θ_l to denote the Fourier coefficients. We recall that, as a simplification, throughout this article we assume that the x-averages θ_0, G_0 identically vanish. Hence, these equations should be interpreted as being valid for $l \in \mathbb{Z} \setminus \{0\}$ and all terms involving modes G_0, θ_0 are trivial.

These differential equations are equivalent to the following integral equations:

(21)

$$\theta_l(T)-\theta_l(t_k)=\int_{t_k}^T c_l^+\theta_{l+1}+c_l^-\theta_{l-1}+d_l^+G_{l+1}+d_l^-G_{l-1}dt,$$

$$\begin{aligned}
G_l(T)-&\exp\left(-\nu\int_{t_k}^T l^2+(\eta-lt)^2dt\right)G_l(t_k)\\
&=\int_{t_k}^T\exp(-\nu\int_t^T l^2+(\eta-l\tau)^2d\tau)2\frac{(\frac{\eta}{l}-t)}{(1+(\frac{\eta}{l}-t)^2)^2}\theta_l dt\\
&\quad+\int_{t_k}^T\exp(-\nu\int_t^T l^2+(\eta-l\tau)^2d\tau)fil\frac{\nu}{g}(d_l^+G_{l+1}+d_l^-G_{l-1})dt\\
&\quad+\int_{t_k}^T\exp(-\nu\int_t^T l^2+(\eta-l\tau)^2d\tau)fil\frac{\nu}{g}(c_l^+\theta_{l+1}+c_l^-\theta_{l-1})dt\\
&\quad+\int_{t_k}^T\exp(-\nu\int_t^T l^2+(\eta-l\tau)^2d\tau)\frac{1}{1+(\frac{\eta}{l}-t)^2}\left(c_l^+\theta_{l+1}+c_l^-\theta_{l-1}\right)dt\\
&\quad+\int_{t_k}^T\exp(-\nu\int_t^T l^2+(\eta-l\tau)^2d\tau)\frac{1}{1+(\frac{\eta}{l}-t)^2}\left(d_l^+G_{l+1}+d_l^-G_{l-1}\right)dt.
\end{aligned}$$

In the following we will show by means of a bootstrap argument that the modes G_l,θ_l when adjusted with a suitable weight function remain bounded uniformly in time.

For easier reference and to motivate our choice of weight function the L^1_t estimates on the coefficients required for the control of θ are collected in the following lemma.

Lemma 3.4. *Let k,η,c be as in Theorem 3.3 and let $t\in I_k$. Let further $c_l^\pm,d_l^\pm$ be defined by equation (19):*

$$\begin{aligned}
c_l^+&=c\frac{\eta}{(l+1)^3}\frac{1}{(1+(\frac{\eta}{l+1}-t)^2)^2},\\
c_l^-&=c\frac{\eta}{(l-1)^3}\frac{1}{(1+(\frac{\eta}{l-1}-t)^2)^2},\\
d_l^+&=c\frac{\eta}{(l+1)^3}\frac{1}{1+(\frac{\eta}{l+1}-t)^2},\\
d_l^-&=c\frac{\eta}{(l-1)^3}\frac{1}{1+(\frac{\eta}{l-1}-t)^2}.
\end{aligned}$$

Then it holds that

$$\begin{aligned}
\int c_{k\pm1}^\mp&=c\frac{\eta}{k^3}\int\frac{1}{(1+(\frac{\eta}{k}-t)^2)^2},\\
\int d_{k\pm1}^\mp&=c\frac{\eta}{k^3}\int\frac{1}{1+(\frac{\eta}{k}-t)^2},
\end{aligned}\tag{22}$$

and

$$\begin{aligned}
\int_{\mathbb{R}}\frac{1}{(1+(\frac{\eta}{k}-t)^2)^2}&=\frac{\pi}{2},\\
\int_{\mathbb{R}}\frac{1}{1+(\frac{\eta}{k}-t)^2}&=\pi.
\end{aligned}$$

Since $c\frac{\eta}{k^3}$ is possibly very large, we call these the resonant *cases.*

For the remaining non-resonant *cases, the following bounds hold:*

$$
\begin{aligned}
&\int_{I_k} c_l^{\pm} \leq \frac{4c}{k}(\frac{\eta}{k^2})^{-2}, \\
&\int_{I_k} d_l^{\pm} \leq \frac{4c}{k}.
\end{aligned} \tag{23}
$$

We stress that unlike in Section 3.1 here $c\frac{\eta}{k^3}$ can be very large. Hence, the integrals in (22) can be very large. On the other hand the integrals in (23) are quite small and the integral over $c_l^{\pm}$ becomes even smaller the larger $\frac{\eta}{k^2}$ is. As we discuss after the following lemma, these large resonant coefficients and small non-resonant coefficients determine the structure of our choice of weight function.

The estimates required to control the evolution of G are also collected in a lemma.

Lemma 3.5. *Let k, η, c be as in Theorem 3.3 and let $t \in I_k$ and consider the integrals stated in (21).*

Then for the resonant case $l = k$ *it holds that*

$$
\int_{t_k}^{T} \exp(-\nu \int_t^T k^2 + (\eta - k\tau)^2 d\tau) 2 \frac{(\frac{\eta}{k} - t)}{(1 + (\frac{\eta}{k} - t)^2)^2} \leq 2. \tag{24}
$$

For the non-resonant *cases $l \neq k$ we instead estimate*

$$
\int_{t_k}^{T} \exp(-\nu \int_t^T l^2 + (\eta - l\tau)^2 d\tau) \left| 2 \frac{(\frac{\eta}{l} - t)}{(1 + (\frac{\eta}{l} - t)^2)^2} \right| dt \leq 2(\frac{\eta}{k^2})^{-2}. \tag{25}
$$

For the coefficient functions involving f we control:

$$
\begin{aligned}
&\int_{t_k}^{t} f \frac{\nu}{g} i l c_l^{\pm} \leq \begin{cases} c(\frac{\eta}{k^2})^{-1} & , \text{ if } l = k \mp 1, \\ c(\frac{\eta}{k^2})^{-4} & , \text{ else.} \end{cases}, \\
&\int_{t_k}^{t} f \frac{\nu}{g} i l d_l^{\pm} \leq \begin{cases} c(\frac{\eta}{k^2})^{-1} & , \text{ if } l = k \mp 1, \\ c(\frac{\eta}{k^2})^{-2} & , \text{ else.} \end{cases}.
\end{aligned} \tag{26}
$$

We further estimate

(27)

$$
\int_{t_k}^{T} \exp(-\nu \int_t^T l^2 + (\eta - l\tau)^2 d\tau) \frac{1}{1 + (\frac{\eta}{l} - t)^2} c_l^{\pm} \leq \begin{cases} 32\frac{c}{k}(\frac{\eta}{k^2})^{-4}, & \text{if } l \neq k \neq l \pm 1, \\ \frac{c}{k}(\frac{\eta}{k^2})^{-3} 16\pi, & \text{if } l = k, \\ \frac{c}{k}(\frac{\eta}{k^2})^{-1}\frac{\pi}{2}, & \text{if } l = k \pm 1. \end{cases}
$$

Finally, we control

(28)

$$
\int_{t_k}^{T} \exp(-\nu \int_t^T l^2 + (\eta - l\tau)^2 d\tau) \frac{1}{1 + (\frac{\eta}{l} - t)^2} d_l^{\pm} \leq \begin{cases} \frac{c}{k}(\frac{\eta}{k^2})^{-2}, & \text{if } l \neq k \neq l \pm 1, \\ \frac{c}{k}(\frac{\eta}{k^2})^{-1}\pi, & \text{else.} \end{cases}
$$

We stress that the contributions by resonant frequencies as stated in (24) while not large in an absolute sense, are not small. In contrast all other coefficients provide a gain of negative powers of $(\frac{\eta}{k^2})$. We postpone the proof of the coefficient estimates formulated in Lemmas 3.5 and 3.4 to Subsection 3.3.2.

The estimates of Lemmas 3.4 and 3.5 suggest that if initially

$$
\theta_l(t_k) = \delta_{lk}, \; G_l(t_k) = 0,
$$

then for $t_k < t < t_{k-1}$ one should expect the following *heuristic* bounds:

$$\begin{aligned}
&\theta_k(T) \approx 1, && G_k(T) \leq 1, \\
&\theta_{k-1}(T) \approx c\frac{\eta}{k^3}, && G_{k-1}(T) \leq \frac{c}{k}(\frac{\eta}{k^2})^{-1}, \\
&\theta_{k-1-j}(T) \leq c\frac{\eta}{k^3}(c(\frac{\eta}{k^2})^{-2})^j, && G_{k-1-j}(T) \leq \frac{c}{k}(\frac{\eta}{k^2})^{-1}(c(\frac{\eta}{k^2})^{-2})^j,
\end{aligned} \tag{29}$$

for $j \in \mathbb{N}$ and analogously for θ_l, G_l with $l > k$.

The following Proposition 3.6 proves that this heuristic is indeed valid and establishes error bounds on the heuristic approximations. The cases of initial data concentrated on a different mode

$$\theta_l(t_k) = \delta_{k_0 l}, G_l(t_k) = 0,$$

or

$$\theta_l(t_k) = 0, G_l(t_k) = \delta_{k_0 l},$$

are considered in Proposition 3.7. In Subsection 3.3.1 we then show how to pass from weighted ℓ^∞ estimates to bounds in X and thus establish Theorem 3.3.

Proposition 3.6. *Let η, c, k be as in Theorem 3.3. Consider the evolution equation (21) with*

$$\theta_l(t_k) = \delta_{lk}, G_l(t_k) = 0$$

for all $l \in \mathbb{Z}$. Let further $c_l^\pm$, $d_l^\pm$ be defined as in Lemma 3.4.

Then the following estimates hold for all times $t_k \leq T \leq t_{k-1}$:

(B1) For the resonant mode θ_k it holds that

$$|\theta_k(T) - 1| \leq \frac{10c}{k}(\frac{\eta}{k^2})^{-1}.$$

(B2) For $l \in \{k-1, k+1\}$ it holds that

$$\left|\theta_l(T) - \int_{t_k}^T c_l^\mp dt - \int_{t_k}^T d_l^\mp G_k(t) dt\right| \leq \frac{0.5}{k} c\frac{\eta}{k^3}.$$

(B3) For all $l \notin \{k-1, k, k+1\}$ it holds that

$$|\theta_l(T)| \leq c\frac{\eta}{k^3}(c(\frac{\eta}{k^2})^{-2})^{|l-k|+1}.$$

(B4) For the mode G_k it holds that

$$|G_k(T) - \int_{t_k}^t \exp(-\nu \int_{t_k}^t k^2 + (\eta - ks)^2 ds) 2\frac{(\frac{\eta}{k} - t)}{(1 + (\frac{\eta}{k} - t)^2)^2} dt| \leq \frac{2}{k}.$$

(B5) For all $l \neq k$ it holds that

$$|G_l(T)| \leq \frac{\eta}{k^3}(c(\frac{\eta}{k^2})^{-2})^{|l-k|}.$$

If we instead consider $\theta_l(t_k) = 0$, $G_l(t_k) = \delta_{lk}$, then (B3) and (B5) still hold and (B1), (B2), (B4) are replaced by

(B1') It holds that

$$|\theta_k(T)| \leq \frac{c}{k}.$$

(B2') It holds that

$$\left|\theta_l(T) - \int_{t_k}^{T} d_l^{\mp} G_k(t)dt\right| \leq \frac{0.5}{k} c \frac{\eta}{k^3}.$$

(B4') For the mode G_k it holds that

$$|G_k(T) - \exp(-\nu \int_{t_k}^{t} k^2 + (\eta - ks)^2 ds)| \leq \frac{2}{k}.$$

These estimates quantify our heuristic estimates (29). In particular, in addition to an upper bound, (B2) also provides a lower bound. This norm inflation mechanism then will form the core of our echo chain construction of Section 3.4.

Proof of Proposition 3.6. In our proof we use (B1)–(B5) as bootstrap estimates. More precisely, our strategy is the following:

- We first show that these estimates hold at least on a small time interval $(t_k, T_\star)$.
- Subsequently, we prove that on that time interval the estimates self-improve. That is, variants of (B1)–(B5) hold where the right-hand-side is improved by a factor 0.9.
- Choosing $T_\star$ maximal with the property that (B1)–(B5) are satisfied, it follows that equality in these estimates is not attained. However, by local in time arguments this implies that if $T_\star$ were smaller than t_{k-1}, the estimates would remain valid at least for a small additional time. Since this contradicts the maximality of $T_\star$ it follows that $T_\star = t_{k-1}$, which concludes the proof.

Establishing the initial bootstrap: We note that the right-hand-side estimates in (B1)–(B5) include a power law $(c(\frac{\eta}{k^2})^{-2})^{|l-k|}$, which is not bounded below and thus at first sight might seem problematic for local in time continuity results. We thus instead consider the equivalent unknowns

$$(c(\frac{\eta}{k^2})^{-2})^{|l-k|}\theta_l, (c(\frac{\eta}{k^2})^{-2})^{|l-k|}G_l, \tag{30}$$

where we note that at time t_k, these unknowns equal δ_{lk} and 0, respectively.

We further observe that the equations (21) only include nearest neighbor interactions and that the quotients

$$\frac{(c(\frac{\eta}{k^2})^{-2})^{|l\pm1-k|}}{(c(\frac{\eta}{k^2})^{-2})^{|l-k|}}$$

are bounded above and below. Thus expressing the integral equations (21) with respect to the unknowns (30) we observe that all integrands are bounded uniformly in time (but may depend on c and η). Thus, choosing $T_\star$ such that $|T_\star - t_k|$ is sufficiently small, we obtain a contraction mapping in ℓ^∞ and at least for a small time it holds that

$$\|(c(\frac{\eta}{k^2})^{-2})^{|l-k|}\theta_l - \delta_{lk}\|_{\ell^\infty} + \|(c(\frac{\eta}{k^2})^{-2})^{|l-k|}G_l\|_{\ell^\infty}$$

is sufficiently small that (B1)–(B5) are satisfied.

Having established our initial bootstrap estimates, we now let $T_\star \leq t_{k-1}$ be the maximal time such that these estimates hold. We then show that all bootstrap estimates improve by a factor and thus $T_\star$ can only have been maximal if $T_\star = t_{k-1}$.

Improving (B1): Let $t_k \leq T \leq T_\star$ and consider the integral equation (21) for $\theta_k(T)$. Then by our choice of initial data it holds that

$$\theta_k(T) - 1 = \int_{t_k}^T c_k^+ \theta_{k+1} + c_k^- \theta_{k-1} + d_k^+ G_{k+1} + d_k^- G_{k-1} dt.$$

Since $c_k^\pm$ and $d_k^\pm$ are non-resonant, by the estimates (23) of Lemma 3.4 it follows that

$$|\theta_k(T) - 1| \leq \frac{4c}{k} (\frac{\eta}{k^2})^{-2} (\|\theta_{k+1}\|_{L^\infty} + \|\theta_{k-1}\|_{L^\infty}) + \frac{4c}{k} (\|G_{k+1}\|_{L^\infty} + \|G_{k-1}\|_{L^\infty}).$$

By the bootstrap estimate (B5) it follows that

$$\frac{4c}{k} (\|G_{k+1}\|_{L^\infty} + \|G_{k-1}\|_{L^\infty}) \leq \frac{8c}{k^2} (\frac{\eta}{k^2})^{-1},$$

and by the bootstrap estimate (B2):

$$\begin{aligned}
&\frac{4c}{k} (\frac{\eta}{k^2})^{-2} (\|\theta_{k+1}\|_{L^\infty} + \|\theta_{k-1}\|_{L^\infty}) \\
&\leq \frac{4c}{k} (\frac{\eta}{k^2})^{-2} c \frac{\eta}{k^3} \pi \leq \frac{4c^2 \pi}{k^2} (\frac{\eta}{k^2})^{-1}.
\end{aligned}$$

Thus combining both estimates we obtain

$$|\theta_k(T) - 1| \leq \frac{8c}{k} (\frac{\eta}{k^2})^{-1} + \frac{4c^2 \pi}{k^2} (\frac{\eta}{k^2})^{-1} < \frac{10c}{k} (\frac{\eta}{k^2})^{-1}.$$

Improving (B2): We next consider θ_l for $l \in \{k-1, k+1\}$, for which by the integral equation (21) it holds that

$$\begin{aligned}
\theta_{k\pm1}(T) = &\int_{t_k}^T c_{k\pm1}^\mp (\theta_k - 1 + 1) + \int_{t_k}^T d_{k\pm1}^\mp G_k \\
&+ \int_{t_k}^T c_{k\pm1}^\pm \theta_{k\pm2} + \int_{t_k}^T d_{k\pm1}^\pm G_{k\pm2}.
\end{aligned}$$

We then subtract the contributions by $\theta_k(t_k) = 1$ from both sides and control:

$$\begin{aligned}
\int_{t_k}^T c_{k\pm1}^\mp (\theta_k - 1) &\underset{(22),(B1)}{\leq} c \frac{\eta}{k^3} \pi \|\theta_k - 1\|_{L^\infty} \leq \frac{10c^2}{k^2} \pi, \\
\int_{t_k}^T d_{k\pm1}^\mp (G_k - \hat{G}_k) &\underset{(22),(B4)}{\leq} c \frac{\eta}{k^3} \pi \|G_k - \hat{G}_k\| \leq \frac{1}{k} c \frac{\eta}{k^3} \pi \\
\int_{t_k}^T c_{k\pm1}^\pm \theta_{k\pm2} &\underset{(23),(B3)}{\leq} \frac{4c}{k} (\frac{\eta}{k^2})^{-2} c \frac{\eta}{k^3} (c (\frac{\eta}{k^2})^{-2}) \\
\int_{t_k}^T d_{k\pm1}^\pm G_{k\pm2} &\underset{(23),(B5)}{\leq} c \frac{1}{k} (\frac{\eta}{k^2}) (c (\frac{\eta}{k^2})^{-2})^2,
\end{aligned}$$

where

$$\hat{G}_k = \int_{t_k}^t \exp(-\nu \int_{t_k}^t k^2 + (\eta - ks)^2 ds) 2 \frac{(\frac{\eta}{k} - t)}{(1 + (\frac{\eta}{k} - t)^2)^2} dt$$

is the contribution to G_k by $\theta_k(t_k) = 1$.

Here the contributions by $\theta_{k\pm2}, G_{k\pm2}$ are much smaller than $\theta_{k\pm1}$ by the bootstrap assumption and moreover the coefficient functions are small. The size of $\theta_{k\pm1}$ is thus mostly determined by the modes θ_k and G_k, which form the core of the resonance mechanism.

The claimed estimate thus follows.

Improving (B3): For $l \notin \{k-1, k, k+1\}$ the integral equations (21) read

$$\theta_l(T) = \int_{t_k}^{T} c_l^+ \theta_{l+1} + c_l^- \theta_{l-1} + d_l^+ G_{l+1} + d_l^- G_{l-1} dt.$$

By our choice of l all coefficient functions $c_l^\pm, d_l^\pm$ are controlled by the estimate (23) of Lemma 3.4.

Combining these estimates, we deduce that

$$\begin{aligned}|\theta_l(T)| \leq 4c\frac{1}{k}(\frac{\eta}{k^2})^{-2}(\|\theta_{l+1}\|_{L^\infty} + \|\theta_{l-1}\|_{L^\infty}) \\ + \frac{c}{k}(\|G_{l+1}\|_{L^\infty} + \|G_{l-1}\|_{L^\infty})\end{aligned}$$

As in the estimate of (B2) we here observe that by the bootstrap assumptions the nodes $l \pm 1$ closer to k are potentially much larger and thus determine the achievable upper bound. In particular, we gain a factor $\frac{4}{k}c(\frac{\eta}{k^2})^{-2}$ with respect to the neighbors in θ and a factor $\frac{c}{k}$ with respect to the neighboring modes of G, which by the bootstrap assumptions themselves satisfy bounds with an additional power $(\frac{\eta}{k^2})^{-2}$.

Hence, using the bootstrap estimates (B2), (B3) and (B5) we deduce that

$$\begin{aligned}|\theta_l(T)| &\leq \frac{1}{k}c(\frac{\eta}{k^2})^{-2+\min(|l-1-k|,|l+1-k|)+1}\frac{0.5}{k}c\frac{\eta}{k^3} \\ &\quad + c\frac{1}{k}(\frac{\eta}{k^2})(c(\frac{\eta}{k^2})^{-2})^{\min(|l+1-k|,|l-1-k|} \\ &< \frac{0.5}{k}c\frac{\eta}{k^3}(c(\frac{\eta}{k^2})^{-2})^{|l-k|+1}.\end{aligned}$$

Improving (B4): We next turn to studying the evolution of G_k:

$$\begin{aligned}G_k(T) = &= \int_{t_k}^{T} \exp(-\nu\int_t^T k^2 + (\eta - k\tau)^2 d\tau)\frac{2}{k}\frac{(\frac{\eta}{k} - t)}{(1 + (\frac{\eta}{k} - t)^2)^2}(\theta_k - 1 + 1)dt \\ &\quad + \int_{t_k}^{T} \exp(-\nu\int_t^T k^2 + (\eta - k\tau)^2 d\tau) f\frac{\nu}{g} ik(d_k^+ G_{k+1} + d_k^- G_{k-1})dt \\ &\quad + \int_{t_k}^{T} \exp(-\nu\int_t^T k^2 + (\eta - k\tau)^2 d\tau) f\frac{\nu}{g} ik(c_k^+ \theta_{k+1} + c_k^- \theta_{k-1})dt \\ &\quad + \int_{t_k}^{T} \exp(-\nu\int_t^T k^2 + (\eta - k\tau)^2 d\tau)\frac{1}{1 + (\frac{\eta}{k} - t)^2}\left(c_k^+ \theta_{k+1} + c_k^- \theta_{k-1}\right) dt \\ &\quad + \int_{t_k}^{T} \exp(-\nu\int_t^T k^2 + (\eta - k\tau)^2 d\tau)\frac{1}{1 + (\frac{\eta}{k} - t)^2}\left(d_k^+ G_{k+1} + d_k^- G_{k-1}\right) dt.\end{aligned}$$

We then subtract the contribution by $\theta_k(t_k)=1$ from both sides and estimate

$$
\begin{aligned}
\int_{t_k}^T \exp(-\nu\int_t^T k^2+(\eta-k\tau)^2 d\tau)\frac{2}{k}\frac{(\frac{\eta}{k}-t)}{(1+(\frac{\eta}{k}-t)^2)^2}(\theta_k-1) &\underset{(24),(B1)}{\leq} \frac{2}{k}\frac{0.5}{k}=\frac{1}{k^2},\\
\int_{t_k}^T \exp(-\nu\int_t^T k^2+(\eta-k\tau)^2 d\tau) f\frac{\nu}{g} ik(d_k^+ G_{k+1}+d_k^- G_{k-1})dt &\underset{(26),(B5)}{\leq} c(\frac{\eta}{k})^{-2}\frac{1}{k}(\frac{\eta}{k^2})^{-1},\\
\int_{t_k}^T \exp(-\nu\int_t^T k^2+(\eta-k\tau)^2 d\tau) f\frac{\nu}{g} ik(c_k^+ \theta_{k+1}+c_k^- \theta_{k-1}) &\underset{(26),(B2)}{\leq} c(\frac{\eta}{k})^{-4}c\frac{\eta}{k^3},\\
\int_{t_k}^T \exp(-\nu\int_t^T k^2+(\eta-k\tau)^2 d\tau)\frac{1}{1+(\frac{\eta}{k}-t)^2}\left(c_k^+ \theta_{k+1}+c_k^- \theta_{k-1}\right) &\underset{(27),(B2)}{\leq} \frac{c}{k}(\frac{\eta}{k^2})^{-3}16\pi c\frac{\eta}{k^3},\\
\int_{t_k}^T \exp(-\nu\int_t^T k^2+(\eta-k\tau)^2 d\tau)\frac{1}{1+(\frac{\eta}{k}-t)^2}\left(d_k^+ G_{k+1}+d_k^- G_{k-1}\right) &\underset{(28),(B5)}{\leq} \frac{c}{k}(\frac{\eta}{k^2})^{-2}\pi\frac{c}{k}(\frac{\eta}{k^2})^{-1}.
\end{aligned}
$$

We note that here the largest contribution is given by the first line and that hence estimate (B4) improves.

Improving (B5): Finally, we consider $G_l(T)$ for $l\neq k$, which satisfies:

$$
\begin{aligned}
G_l(T) = &\int_{t_k}^T \exp(-\nu\int_t^T l^2+(\eta-l\tau)^2 d\tau)\frac{2}{l}\frac{(\frac{\eta}{l}-t)}{(1+(\frac{\eta}{l}-t)^2)^2}\theta_l dt\\
&+\int_{t_k}^T \exp(-\nu\int_t^T l^2+(\eta-l\tau)^2 d\tau) f\frac{\nu}{g} il(d_l^+ G_{l+1}+d_l^- G_{l-1})dt\\
&+\int_{t_k}^T \exp(-\nu\int_t^T l^2+(\eta-l\tau)^2 d\tau) f\frac{\nu}{g} il(c_l^+ \theta_{l+1}+c_l^- \theta_{l-1})dt\\
&+\int_{t_k}^T \exp(-\nu\int_t^T l^2+(\eta-l\tau)^2 d\tau)\frac{1}{1+(\frac{\eta}{l}-t)^2}\left(c_l^+ \theta_{l+1}+c_l^- \theta_{l-1}\right)dt\\
&+\int_{t_k}^T \exp(-\nu\int_t^T l^2+(\eta-l\tau)^2 d\tau)\frac{1}{1+(\frac{\eta}{l}-t)^2}\left(d_l^+ G_{l+1}+d_l^- G_{l-1}\right)dt.
\end{aligned}
$$

We remark that if, for instance, $c\leq\nu$ then all coefficient functions on the right-hand-side can easily be dominated by the exponential decay and using that $(\eta-lt)^2$ is bounded below, since $l\neq k$. A key effort of this proof hence lies in establishing estimates that are valid when $\nu<c$ as well.

Since $l\neq k$ is non-resonant, we may control

$$
\begin{aligned}
&\int_{t_k}^T \exp(-\nu\int_t^T l^2+(\eta-l\tau)^2 d\tau)\frac{2}{l}\frac{(\frac{\eta}{l}-t)}{(1+(\frac{\eta}{l}-t)^2)^2}\theta_l\\
&\underset{(25),(B2),(B3)}{\leq} \frac{2}{k}(\frac{\eta}{k^2})^{-2}c\frac{\eta}{k^3}(c(\frac{\eta}{k^2})^{-2})^{|l-k|+1}.
\end{aligned}
$$

For the remaining estimates, we first study the case $l \notin \{k-1,k,k+1\}$, for which

$$\begin{aligned}
&\int_{t_k}^T \exp(-\nu \int_t^T l^2+(\eta-l\tau)^2 d\tau) f\frac{\nu}{g} il(d_l^+ G_{l+1}+d_l^- G_{l-1}) \\
&\underset{(26),(B5)}{\leq} c(\frac{\eta}{k^2})^{-2}(c(\frac{\eta}{k^2})^{-2})^{|l-k|}, \\
&\int_{t_k}^T \exp(-\nu \int_t^T l^2+(\eta-l\tau)^2 d\tau) f\frac{\nu}{g} il(c_l^+ \theta_{l+1}+c_l^- \theta_{l-1}) \\
&\underset{(26),(B3)}{\leq} c(\frac{\eta}{k^2})^{-4} c\frac{\eta}{k^3}(c(\frac{\eta}{k^2})^{-2})^{|l-k|} \\
&\int_{t_k}^T \exp(-\nu \int_t^T l^2+(\eta-l\tau)^2 d\tau) \frac{1}{1+(\frac{\eta}{l}-t)^2}\left(c_l^+ \theta_{l+1}+c_l^- \theta_{l-1}\right) \\
&\underset{(27),(B3)}{\leq} 32\frac{c}{k}(\frac{\eta}{k^2})^{-4} c\frac{\eta}{k^3}(c(\frac{\eta}{k^2})^{-2})^{|l-k|-1} \\
&\int_{t_k}^T \exp(-\nu \int_t^T l^2+(\eta-l\tau)^2 d\tau) \frac{1}{1+(\frac{\eta}{l}-t)^2}\left(d_l^+ G_{l+1}+d_l^- G_{l-1}\right) \\
&\underset{(28),(B5)}{\leq} \frac{c}{k}(\frac{\eta}{k^2})^{-1}\frac{1}{k}(c(\frac{\eta}{k^2})^{-2})^{|l-k|-1}.
\end{aligned}$$

In particular, all estimates indeed yield an improvement by a factor $c(\frac{\eta}{k^2})^{-2}$ compared to its neighbors and thus this bootstrap estimate is improved.

Finally, we discuss the case $l \in \{k-1,k+1\}$. Here the above estimates also apply to the contributions due to $\theta_{k\pm1}$, $\theta_{k\pm2}$ and $G_{k\pm2}$. For the contributions by θ_k and G_k we instead establish the following estimates:

$$\begin{aligned}
\int_{t_k}^T \exp(-\nu \int_t^T l^2+(\eta-l\tau)^2 d\tau) f\frac{\nu}{g} il d_l^{\mp} G_k &\underset{(26),(B4)}{\leq} c(\frac{\eta}{k^2})^{-1}\frac{2}{k}, \\
\int_{t_k}^T \exp(-\nu \int_t^T l^2+(\eta-l\tau)^2 d\tau) f\frac{\nu}{g} il c_l^{\mp} \theta_k &\underset{(26),(B1)}{\leq} \frac{c}{k}(\frac{\eta}{k^2})^{-1} 2, \\
\int_{t_k}^T \exp(-\nu \int_t^T l^2+(\eta-l\tau)^2 d\tau) \frac{1}{1+(\frac{\eta}{l}-t)^2} c_l^{\mp}\theta_k &\underset{(27),(B1)}{\leq} \frac{c}{k^2}(\frac{\eta}{k^2})^{-1}\pi, \\
\int_{t_k}^T \exp(-\nu \int_t^T l^2+(\eta-l\tau)^2 d\tau) \frac{1}{1+(\frac{\eta}{l}-t)^2} d_l^{\mp} G_k &\underset{(28),(B4)}{\leq} \frac{c}{k}(\frac{\eta}{k^2})^{-1}\frac{2\pi}{k}.
\end{aligned}$$

It thus follows that

$$|G_{k\pm1}(T)| \leq 10\frac{c}{k}(\frac{\eta}{k^2})^{-1},$$

which is exactly the desired estimate (B5) in that case.

Improving (B1'), (B2'), (B4'): These estimates follow by the same argument as in the unmodified cases. We omit the details for brevity. □

We next turn to the non-resonant cases, where the initial data is localized on a mode $k_0 \neq k$. Similarly to the resonant case we here show that the effect on neighboring modes decreases in terms of powers of $c(\frac{\eta}{k^2})^{-2}$ except for the interact of the mode k with its neighbors, which increases by a factor $c\frac{\eta}{k^3}$. In particular, we observe that since the evolution equations (21) only explicitly include nearest neighbor interactions (and one interaction $\theta_l \mapsto G_l$) the estimates derived in Proposition 3.6 only relied on the relative growth or decrease of these weights. Hence, the proofs in the non-resonant case are largely identical to the resonant case except for a different choice of initial data and multiplication by a suitable factor.

The estimates for the non-resonant case are summarized in the following proposition.

Proposition 3.7. *Let $k_0 \neq k$ and suppose that at time t_k, it holds that*

$$\theta_l = \delta_{lk_0},\ G_l \equiv 0.$$

We then define a weight function $\gamma : \mathbb{Z} \to \mathbb{R}_+$ to satisfy the following properties:

(1) $\gamma(k_0) = 1$.

(2) $\gamma(k+1) = \gamma(k-1) = c\frac{\eta}{k^3}\gamma(k)$.

(3) $\gamma(l+1) = c(\frac{\eta}{k^2})^{-2}\gamma(l)$ *if* $l > k_0$ *and* $\gamma(l-1) = c(\frac{\eta}{k^2})^{-2}\gamma(l)$ *if* $l < k_0$ *unless this would violate the second property.*

Then the following bootstrap estimates hold:

(C1) For all l it holds that

$$|\theta_l(T) - \delta_{lk_0}| \leq \gamma(l).$$

(C2) For all l it holds that

$$|G_l(T)| \leq c(\frac{\eta}{k^2})^{-2}\gamma(l).$$

If instead $G_l(t_k) = \delta_{lk_0}$ for some $k_0 \neq k$, then the following bootstrap estimates hold:

(D1) For all l it holds that

$$|\theta_l(T)| \leq \frac{0.5}{k}\gamma(l).$$

(D2) For all l it holds that

$$|G_l(T) - \delta_{lk_0}\exp(-\nu\int_{t_k}^{T} k_0^2 + (\eta - k_0 s)^2 ds)| \leq \gamma(l).$$

Proof of Proposition 3.7. In this proof we follow the same bootstrap strategy as in Proposition 3.6, where the initial bootstrap estimates again follow by a local contraction argument.

It thus remains to be shown that the bootstrap estimates (C1), (C2) and (D1), (D2) self-improve. Here we note that in our proof the estimates (B1)–(B5) we only used the relative size of the bounds on neighboring modes compared to the size of the desired bound on the current mode. Therefore, in the current estimate we only need to control

$$\frac{\gamma(l)}{\gamma(l \pm 1)},$$

which by construction satisfies the same estimates as in Proposition 3.6. In particular, most bounds follow by the exact same argument. In the interest of brevity we hence only comment on possible differences in the proof.

Improving (C1), (C2): Since $\theta_k(t_k) = 0$, $G_k(t_k) = 0$ the estimates for $\theta_{k\pm 1}$ simplify and follow by the same argument as for (B1) to (B5). As the only difference we note that for $\theta_{k_0}(T)$ we, of course, consider

$$\theta_{k_0}(T) - \theta_{k_0}(t_k) = \theta_{k_0}(T) - 1.$$

Improving (D1),(D2): We again note that in our bounds (B1)–(B5) we only required control on the relative size of the desired estimates. For instance, in order to control $\|\theta_k - 1\|_{L^\infty}$ in (B1) we only used that $\|G_{k\pm 1}\|_{L^\infty}$ and $\|\theta_{k\pm1}\|_{L^\infty}$ were controlled in terms of $c\frac{\eta}{k^3}$ as compared to the bound by 1 imposed on θ_k by the bootstrap assumption.

The estimates hence follow completely analogously, with the only difference that for G_{k_0} the integral equations (21) include a contribution by the initial data. □

We have thus shown that Propositions 3.6 and 3.7 follow as consequences of the coefficient estimates collected in Lemmas 3.4 and 3.5. In turn, in Section 3.3.1 we will use these propositions to establish Theorem 3.3. The coefficient estimates are then proven in Subsection 3.3.2.

3.3.1. *Proof of Theorem 3.3.* In Propositions 3.6 and 3.7 we have established that θ and G satisfy suitable estimates in weighted ℓ^∞ spaces. In the following we discuss how to pass from these ℓ^∞ estimates to estimates on X and thus prove Theorem 3.3.

Proof of Theorem 3.3. Let $\theta(t_k), G(t_k) \in X$ be given initial data. Then by linearity we may express the solution $\theta(T), G(T)$ as the sum over the solutions with initial data localized on a single mode. If we denote the weight of Proposition 3.7 for a mode k_0 by γ_{k_0}, it thus follows from Propositions 3.6, 3.7 that for any $l \notin \{k-1, k, k+1\}$:

$$\begin{aligned}|\theta_l(T) - \theta_l(t_k)| &\underset{(B3),(C1),(D1)}{\leq} |\theta_k(t_k)| c\frac{\eta}{k^3}(c(\frac{\eta}{k^2})^{-2})^{|l-k|+1} \\ &\quad + \sum_{k_0 \neq k} |\theta_{k_0}(t_k)|\gamma_{k_0}(l) + |G_{k_0}(t_k)|\gamma_{k_0}(l),\end{aligned}$$

and

$$\begin{aligned}&|G_l(T) - G_l(t_k)\exp(-\nu\int_{t_k}^T l^2 + (\eta - ls)^2 ds)| \\ &\underset{(B4),(C2),(D2)}{\leq} \frac{\eta}{k^3}(c(\frac{\eta}{k^2})^{-2})^{|l-k|}|\theta_k(t_k)| + \frac{2}{k}|G_k(t_k)| \\ &\quad + \sum_{k_0 \neq k} \gamma_{k_0}(l)|\theta_k(t_k)| + |G_{k_0}(t_k)|\gamma_{k_0}(l).\end{aligned}$$

We recall that $\gamma_{k_0}(l)$ rapidly decays in $|k_0 - l|$ and thus interpret the right-hand-side as discrete convolutions or rather integral kernels applied to the (absolute values of the) initial data.

Hence, in the following we intend to pass from a point-wise bound

$$|\theta_l(T) - \theta_l(t_k)| \leq \sum_{k_0} K(k_0, l)(|\theta_{k_0}| + |G_{k_0}|)$$

to a bound on weighted ℓ^2 spaces, X.

For this purpose we note that by Schur's test if the kernel $K(\cdot,\cdot)$ satisfies

$$\sup_l \sum_{k_0} |K(k_0, l)| \leq C_1 < \infty,$$

$$\sup_{k_0} \sum_l |K(k_0, l)| \leq C_2 < \infty,$$

then the associated integral operator maps ℓ^2 to ℓ^2 with operator norm bounded by $\sqrt{C_1 C_2}$.

Applied to our case we observe that

$$K(k_0,l) \leq c\frac{\eta}{k^3}(c(\frac{\eta}{k^2})^{-2})^{\max(0,|l-k_0|-1)},$$

and therefore by the geometric series

$$C_1 = C_2 \leq c\frac{\eta}{k^3}\frac{1}{1-c(\frac{\eta}{k^2})^{-2}} \leq 1.1c\frac{\eta}{k^3}.$$

It thus follows that

$$\|(\theta(T)-\theta(t_k))_{l\not\in\{k-1,k,k+1\}}\|_{\ell^2} \leq 2c\frac{\eta}{k^3}(\|\theta(t_k)\|_{\ell^2} + \|G(t_k)\|_{\ell^2}),$$

and analogous estimates hold for G. Moreover, by the Definition 2.8 of the space X its weight function is bounded by $2^{|l|}$ and we may thus apply the same argument with $2^{|k_0-l|}|K(k_0,l)|$ to also deduce bounds on X.

It thus remains to discuss the modes $l \in \{k-1,k,k+1\}$. Here we observe that by the bootstrap estimates and the triangle inequality

$$|\theta_k(T)-\theta_k(t_k)| \leq \frac{10c}{k}(\frac{\eta}{k^2})^{-1}|\theta_k(t_k)| + \sum_{k_0\neq k}\gamma_{k_0}(l)(|\theta_{k_0}(t_k)| + |G_{k_0}(t_k)|).$$

Similarly, the modes $k-1, k+1$ by (B2) and the other bootstrap estimates satisfy

$$\begin{aligned}&|\theta_{k\pm1}(T)-\theta_{k\pm1}(t_k) - \int_{t_k}^T c_{k+1}^-\theta_k(t_k) - \int_{t_k}^T d_{k+1}^-\hat{G_K}| \\ &\leq \frac{0.5}{k}c\frac{\eta}{k^3}|\theta_k(t_k)| + \sum_{k_0\neq k}\gamma(l)(|\theta_{k_0}(l)| + |G_{k_0}(l)|),\end{aligned}$$

where

$$\begin{aligned}\hat{G_k}(t) &= G_k(t_k)\exp(-\nu\int_{t_k}^t k^2 + (\eta-ks)^2 ds) \\ &+ \theta_k(t_k)\int_{t_k}^t \exp(-\nu\int_{t_k}^t k^2+(\eta-ks)^2ds)2\frac{(\frac{\eta}{k}-\tau)}{(1+(\frac{\eta}{k}-\tau)^2)^2}d\tau\end{aligned}$$

accounts for the explicit influence of $G_k(t_k)$ and $\theta_k(t_k)$.

We further recall that by Lemma 3.4

$$\begin{aligned}\int_{t_k}^{t_{k-1}} c_{k+1}^- dt &\approx c\frac{\eta}{k^3}\pi, \\ \int_{t_k}^{t_{k-1}} d_{k+1}^- dt &\approx c\frac{\eta}{k^2}\frac{\pi}{2}.\end{aligned}$$

and note that

$$\int_{t_k}^{t_{k-1}}\exp(-\nu\int_{t_k}^t k^2+(\eta-ks)^2ds)\frac{2}{k}\frac{(\frac{\eta}{k}-t)}{(1+(\frac{\eta}{k}-t)^2)^2}dt \leq 2.$$

Hence, θ_{k-1} satisfies the desired upper bound

$$\begin{aligned}&|\theta_{k\pm1}(T)-\theta_{k\pm1}(t_k)\int_{t_k}^T c_{k+1}^-\theta_k(t_k) - \int_{t_k}^T d_{k+1}^-\hat{G_K}| \\ &\leq \max(c\frac{\eta}{k^3},\frac{1}{k})(\|\theta(t_k)\|_X + \|G(t_k)\|_X),\end{aligned}$$

where we may insert $T = t_{k-1}$. $\square$

In the following subsection we provide the proof of the coefficient estimates of Lemma 3.4 and 3.5.

3.3.2. *Proof of Coefficient Estimates.* In Section 3.3 we have stated estimates on the coefficients in the evolution equations (21) and used them to establish bounds on the resonance mechanism. In the following we prove these estimates.

Proof of Lemma 3.4. We note that for $l \in \{k-1, k+1\}$ it holds that

$$\begin{aligned} c_{k\pm1}^{\mp} &= c\frac{\eta}{k^3}\frac{1}{(1+(\frac{\eta}{k}-t)^2)^2}, \\ d_{k\pm1}^{\mp} &= c\frac{\eta}{k^3}\frac{1}{1+(\frac{\eta}{k}-t)^2}, \end{aligned}$$

which we integrate in time to obtain (22). In particular, we stress that if one integrates over all of I_k and $\frac{\eta}{k^2}$ is large, the integral is comparable to integral over all of $\mathbb{R}$ and

$$\begin{aligned} \int_{\mathbb{R}} \frac{1}{(1+(\frac{\eta}{k}-t)^2)^2} dt &= \frac{\pi}{2}, \\ \int_{\mathbb{R}} \frac{1}{1+(\frac{\eta}{k}-t)^2} dt &= \pi, \end{aligned}$$

by explicit calculation. Therefore the resonant contributions (22), for this choice of times, are comparable to $\frac{g}{\nu}\frac{\eta}{k^3}$ and thus potentially very large.

We next turn to estimating all other coefficients. We recall that:

$$\begin{aligned} c_l^{\pm} &= c\frac{\eta}{(l\pm1)^3}\frac{1}{(1+(\frac{\eta}{l\pm1}-t)^2)^2}, \\ d_l^{\pm} &= c\frac{\eta}{(l\pm1)^3}\frac{1}{1+(\frac{\eta}{l\pm1}-t)^2}. \end{aligned}$$

In this non-resonant-case considered in (23), it holds that $l \pm 1 \neq k$ and thus

$$(\frac{\eta}{l\pm1}-t)^2 \geq \frac{1}{4}\max(t, \frac{\eta}{(l\pm1)^2}, \frac{\eta}{k^2})^2.$$

It thus follows that

$$c_l^{\pm} \leq 16c\frac{\eta}{(l\pm1)^3}\max(t, \frac{\eta}{(l\pm1)^2}, \frac{\eta}{k^2})^{-4} \leq 16\frac{c}{k}(\frac{\eta}{k^2})^{-3},$$

and

$$d_l^{\pm} \leq 4c\frac{\eta}{(l\pm1)^3}\max(t, \frac{\eta}{(l\pm1)^2}, \frac{\eta}{k^2})^{-2} \leq 4\frac{c}{k}(\frac{\eta}{k^2})^{-1}.$$

Using the fact that $|I_k| = \frac{1}{2}(\frac{\eta}{k(k-1)} + \frac{\eta}{k(k+1)}) \leq 2\frac{\eta}{k^2}$, the estimates (23) thus follow. $\square$

We next turn to the coefficient estimates required to control the evolution of G.

Proof of Lemma 3.5. Estimating (24): We first consider the integral

$$\int_{t_k}^{T} \exp(-\nu \int_t^T l^2 + (\eta - l\tau)^2 d\tau) \left| 2\frac{(\frac{\eta}{l} - t)}{(1 + (\frac{\eta}{l} - t)^2)^2} \right|,$$

for $k = l$, which is related to the forcing exerted by the mode θ_k on G_k. Since we are searching for estimates uniform in ν, we bound the exponential by 1 and observe that

$$\int_{\mathbb{R}} \left| \frac{(\frac{\eta}{l} - t)}{(1 + (\frac{\eta}{l} - t)^2)^2} \right| = 1,$$

which yields the desired upper bound. However, we remark that if the absolute value signs are not introduced then the fraction is anti-symmetric with respect to $t = \frac{\eta}{k}$.

Estimating (25): We next consider the same integral for $l \neq k$. In this case it holds that

$$(\frac{\eta}{l} - t)^2 \geq \frac{1}{4} \max(t, \frac{\eta}{k^2}, \frac{\eta}{l^2})^2$$

and thus

$$\begin{aligned} 2&\frac{(\frac{\eta}{k} - t)}{(1 + (\frac{\eta}{k} - t)^2)^2} \\ &\leq 2(\frac{\eta}{k^2})^{-3}. \end{aligned}$$

The claimed estimate then follows by again noting that the length of I_k is controlled in terms of $\frac{\eta}{k^2}$.

Estimating (26): Since $f \leq \frac{g}{\nu} \frac{1}{1+t^2}$, we need to control

$$\int_{t_k}^{T} \frac{c}{1+t^2} \frac{\eta}{l^3} \frac{1}{(1 + (\frac{\eta}{l} - t)^2)^2} dt,$$

$$\int_{t_k}^{T} \frac{c}{1+t^2} \frac{\eta}{l^3} \frac{1}{1 + (\frac{\eta}{l} - t)^2} dt.$$

Consider the first integral. We recall that $t \approx \frac{\eta}{k}$ and argue as in Section 3.1. That is, for $l > \frac{k}{2}$, $l \neq k$ we may bound

$$\frac{c}{1+t^2} \frac{\eta}{l^3} \frac{1}{(1 + (\frac{\eta}{l} - t)^2)^2} \leq c(\frac{\eta}{k})^{-2} \frac{\eta}{k^3} (\frac{\eta}{k})^{-4} \leq c(\frac{\eta}{k})^{-5},$$

which is then integrated over a time interval of length bounded by $\frac{\eta}{k^2}$ and we thus gain

$$c(\frac{\eta}{k})^{-4}.$$

If $l = k$, this estimate is slightly worse as

$$c(\frac{\eta}{k})^{-2} \frac{\eta}{k^3} \int_{t_k}^{T} \frac{1}{1 + (\frac{\eta}{l} - t)^2} dt \leq c(\frac{\eta}{k})^{-1}.$$

Finally, if $l < \frac{k}{2}$, we bound

$$\frac{c}{1+t^2} \frac{\eta}{l^3} \frac{1}{1 + (\frac{\eta}{l} - t)^2} \leq c\frac{\eta}{l^3} \frac{1}{1+t^2} \frac{1}{(1 + \frac{1}{4}\max(\frac{\eta}{l}, t)^2)^2} \leq \frac{c}{k} (\frac{\eta}{k})^{-5}$$

and thus bound the integral by

$$\frac{c}{k}(\frac{\eta}{k})^{-4}.$$

For the second integral we argue similarly. If $l=k$, then we control the integral by

$$\frac{c}{1+(\frac{\eta}{k})^2}\frac{\eta}{k^3}\pi \leq \frac{c}{k}(\frac{\eta}{k})^{-1}.$$

If $l \neq k$, then

$$\frac{\eta}{l^3}\frac{1}{1+(\frac{\eta}{l}-t)^2} \leq \frac{1}{k}(\frac{\eta}{k})^{-1}$$

is uniformly integrable and we control by

$$\frac{c}{1+t^2} \leq c(\frac{\eta}{k})^{-2}.$$

Estimating 27 We next consider the integral

$$\int_{t_k}^T \exp(-\nu\int_t^T l^2+(\eta-l\tau)^2 d\tau)\frac{1}{1+(\frac{\eta}{l}-t)^2}c_l^{\pm},$$

where we again bound the exponential by 1, since we allow for ν to be very small. Then for

$$\frac{1}{1+(\frac{\eta}{l}-t)^2}c\frac{\eta}{(l\pm 1)^3}\frac{1}{(1+(\frac{\eta}{l}-t)^2)^2} \tag{31}$$

we distinguish three cases.

If $l \neq k$ and $l\pm 1 \neq k$, that is all frequencies are non-resonant, then we may control

$$(\frac{\eta}{l}-t)^2 \geq \frac{1}{4}\max(\frac{\eta}{l^2},\frac{\eta}{k^2})^2$$

and thus (31) is bounded by

$$\frac{\eta}{(l\pm 1)^3}16c\max(\frac{\eta}{l^2},\frac{\eta}{k^2})^{-6} \leq 16\frac{c}{k}(\frac{\eta}{k^2})^{-5}.$$

Since the length of interval I_k is bounded by $2\frac{\eta}{k^2}$, the claimed bound hence follows.

If $l=k$, then $l\pm 1\neq k$ and we may control (31) by a constant times

$$\begin{aligned}&c\frac{\eta}{(k\pm 1)^3}(\frac{\eta}{k^2})^{-4}\frac{1}{1+(\frac{\eta}{k}-t)^2}\\ &\leq \frac{c}{k}(\frac{\eta}{k^2})^{-3}\frac{1}{1+(\frac{\eta}{k}-t)^2}\end{aligned}$$

Since the last factor is integrable in time with integral bounded by π, the claimed estimate follows.

Finally, if $l\pm 1=k$ then $l\neq k$ and we may control (31) by a constant times

$$\begin{aligned}&c\frac{\eta}{k^3}(\frac{\eta}{k^2})^{-2}\frac{1}{(1+(\frac{\eta}{l}-t)^2)^2}\\ &\leq \frac{c}{k}(\frac{\eta}{k^2})^{-1}\frac{1}{(1+(\frac{\eta}{l}-t)^2)^2}.\end{aligned}$$

We again observe that the last factor is integrable in time with integral bounded by $\frac{\pi}{2}$. This concludes the estimate of (27).

Estimating 28 It remains to estimate

$$\int_{t_k}^{T} \exp(-\nu \int_t^T l^2 + (\eta - l\tau)^2 d\tau) \frac{1}{1+(\frac{\eta}{l}-t)^2} c \frac{\eta}{(l\pm 1)^2} \frac{1}{1+(\frac{\eta}{l}-t)^2},$$

where we argue similarly as in the case of estimate (27) and bound the exponential by 1. We thus have to estimate

$$\frac{1}{1+(\frac{\eta}{l}-t)^2} c \frac{\eta}{(l\pm 1)^2} \frac{1}{1+(\frac{\eta}{l}-t)^2}, \tag{32}$$

where distinguish two cases.

If $l \neq k \neq k-1$ all frequencies are non-resonant and we can again bound $(\frac{\eta}{l}-t)^2$ and $(\frac{\eta}{l}-t)^2$ from below. We may thus bound (32) by a constant times

$$c\frac{\eta}{(l\pm 1)^3} \max(\frac{\eta}{l^2}, \frac{\eta}{k^2})^{-4} \leq \frac{c}{k}(\frac{\eta}{k^2})^{-3},$$

and the desired bound again follows by estimating the length of I_k from above.

Suppose that $l = k$ and hence $l \pm 1 \neq k$ (the case $l \pm 1 = k$ is analogous). Then we may instead control (32) by a constant times

$$\begin{aligned} &c\frac{\eta}{(k\pm 1)^3}(\frac{\eta}{k^2})^{-2}\frac{1}{1+(\frac{\eta}{k}-t)^2} \\ &\leq 4\frac{c}{k}(\frac{\eta}{k^2})^{-1}\frac{1}{1+(\frac{\eta}{k}-t)^2}. \end{aligned}$$

The desired bound then follows by noting that the last factor is integrable in time with integral bounded by π. $\square$

3.4. **Proof of Stability in Theorem 1.1.** As a final result of this section we combine the estimates of Theorems 3.1, 3.2 and 3.3 to establish global in time stability.

Proof of Theorem 1.1. As sketched in Section 1 the strategy of our construction here is the following:

- We start with given initial data at time 0 and control it up to a time t_k of size $C\sqrt{\eta}$ by Theorem 3.1.
- On the intermediate range of times (for which $c\frac{\eta}{k^3}\pi \leq 1$) we control the evolution by Theorem 3.2.
- On the time interval $(\frac{\eta}{\sqrt[3]{c\eta\pi}}, 2\eta)$ we encounter a sequences of resonances, each possibly leading to norm inflation by a factor. This corresponds to an echo chain
$$k \mapsto k-1 \mapsto \cdots \mapsto 1.$$
There we use Theorem 3.3 to control the growth due to each echo.
- Finally, after the time $t_0 = 2\eta$ the evolution is stable by Theorem 3.1 and we have thus established global in time control of solutions. We thus have constructed global in time solutions exhibiting echo chains.

We recall that the coefficient functions of the linearized problem (5) do not depend on y explicitly and that the problem decouples after a Fourier transform in y. In this proof we will hence consider $\eta \in \mathbb{R}$ as a given parameter, where the estimate for general data follows by integration with respect to η.

Let thus c and η be given and let $\theta(0), G(0) \in X$. In Theorem 3.1 we have shown that the evolution is stable globally in time if $c\eta \ll 1$, we hence in the following without loss of generality restrict to the case $c\eta \gtrsim 1$. For simplicity of notation let

$$\begin{aligned} k_1 &\approx \sqrt{8000 c\eta\pi}, \\ k_0 &\approx \sqrt{c\eta\pi}, \end{aligned}$$

where we round down to an integer.

Then by Theorem 3.1 it holds that for all $0 < t \leq t_{k_1}$

$$40^2\|\theta(t)\|_X^2 + \|G(t)\|_X^2 \leq 2(40^2\|\theta(0)\|_X^2 + \|G(0)\|_X^2).$$

In particular, it follows that

$$\|\theta(t_{k_1})\|_X^2 + \|G(t_{k_1})\|_X^2 \leq 2 \cdot 40^2(\|\theta(0)\|_X^2 + \|G(0)\|_X^2).$$

Next, we use Theorem 3.2 to show that for $t_{k_1} \leq t \leq t_{k_0}$ it holds that

$$\begin{aligned} \|\theta(t)\|_X + \|G(t)\|_X &\leq e^{20\cdot 3\pi \sqrt[3]{c\eta}}(\|\theta(t_{k_1})\|_X^2 + \|G(t_{k_1})\|_X^2) \\ &\leq 2 \cdot 40^2 e^{20\cdot 3\pi \sqrt[3]{c\eta}}(\|\theta(0)\|_X^2 + \|G(0)\|_X^2). \end{aligned}$$

Next, on the time interval $(t_{k_0}, t_0 = 2\eta)$, we have shown in Theorem 3.3 that our solution grows at most by a factor

$$\prod_{k=1}^{k_0} 2\frac{c\eta\pi}{k^3} = \frac{(2c\eta\pi)^{k_0}}{(k_0!)^3} \leq \frac{C}{(c\eta)^{3/2}} e^{3\sqrt[3]{2c\eta\pi}},$$

where we used Stirling's approximation to approximate

$$k_0! \sim \sqrt{2\pi k_0} k_0^{k_0} e^{-k_0}$$

in the last step. Therefore $k_0^{3k_0}$ and $(2c\eta\pi)^{k_0}$ then cancel by our choice of k_0 and a bound in terms of e^{3k_0} remains.

In particular, it follows that

$$\begin{aligned} \|\theta(t_0)\|_X + \|G(t_0)\|_X &\leq (c\eta)^{-3/2} e^{\sqrt[3]{2c\eta\pi}}(\|\theta(t_{k_0})\|_X + \|G(t_{k_0})\|_X) \\ &\leq 2(c\eta)^{-3/2} 40^2 e^{200\sqrt[3]{c\eta}}(\|\theta(0)\|_X^2 + \|G(0)\|_X^2). \end{aligned}$$

Finally, by Theorem 3.1, for all times $t \geq t_0 = 2\eta$ it holds that

$$\begin{aligned} \|\theta(t)\|_X + \|G(t)\|_X &\leq 1.1(\|\theta(t_0)\|_X + \|G(t_0)\|_X) \\ &\leq \frac{1.1}{(c\eta)^{3/2}} 2 \cdot 40^2 e^{200\sqrt[3]{c\eta}}(\|\theta(0)\|_X^2 + \|G(0)\|_X^2), \end{aligned}$$

which concludes the proof.

The evolution preserves Gevrey 3 regularity with a possible loss of constant. ☐

Having established this stability result in Gevrey 3 regularity in the following we show that the estimate is optimal (up to the choice of constant). More precisely, we construct initial data which achieves growth at least by

$$e^{\sqrt[3]{c\eta}}.$$

4. Echo Chains and Blow-up

As a complementary result to the stability estimates for initial data in a Gevrey 3 class with large constant, we show that there exists data in a critical Gevrey class that not only achieves norm inflation but blow-up in Sobolev regularity as time tends to infinity.

Theorem 4.1. *Let $c, \nu > 0$ be as in Theorem 1.1 and suppose that $\sqrt[3]{c\eta} \gg \nu^{-1/2}$. Let further*

$$k_2 \approx \frac{1}{10}\sqrt[3]{c\eta\frac{\pi}{2}}$$

(rounded down). Then the solution of (11) with initial data

$$\theta_l(0) = \delta_{lk_2},\ G(0) = 0$$

satisfies

$$\exp(\sqrt[3]{c\eta})\|\theta(0)\|_X \leq \|\theta(t)\|_X \leq \|\theta(t)\|_X + \|G(t)\|_X \leq \exp(50\sqrt[3]{c\eta})\|\theta(0)\|_X$$

for all $t > 2\eta$ and $\theta(t)$ converges in X as $t \to \infty$. There thus exist global in time, asymptotically stable solutions achieving norm inflation.

Moreover, when considering the y-dependent formulation (10), for each $\sigma \in \mathbb{R}$ there exists initial data $\theta(0) \in \mathcal{G}_3 X$, $G(0) = 0$, such that $\theta(t)$ converges in $H^s X$ for all $s < \sigma$, but diverges to infinity in $H^s X$ for all $s > \sigma$.

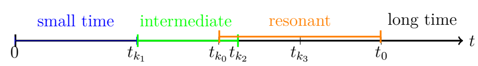

Figure 2. The time regimes considered in Section 4. Compared to Section 3 we here allow an overlap of the intermediate and resonant regime. Furthermore, we divide the resonant regime into a part where $k \geq k_3$ is still large and one where $k < k_3$ is possibly small.

We remark that the asymptotic stability of solutions has already been establish in Section 3.1. The effort of this section lies in the construction of the global in time solutions exhibiting norm inflation and showing that lower bounds persist for all times. Indeed, given such solutions we can construct solutions exhibiting blow-up as follows:

Proof of the blow-up result of Theorem 4.1. Let $c, \nu > 0$ be given and suppose that for all η as in Theorem 4.1 there exist initial data $\theta[\eta](0)$ with $\|\theta[\eta](0)\|_X = 1$ such that the associated evolution is asymptotically stable and such that

$$\theta^\infty[\eta] := \theta[\eta](t)$$

satisfies $\|\theta^\infty[\eta]\|_X =: \psi(\eta) \geq \exp(\sqrt[3]{c\eta})$.

Then for given $\sigma \in \mathbb{R}$ there exists a density $\rho \in H^\sigma$ with $\rho \notin H^s$ for any $s > \sigma$ and such that the support of its Fourier transform is contained in the set $\{\eta : \sqrt[3]{c|\eta|} \gg \nu^{-1/2}\}$. For instance, such data can be explicitly constructed in Fourier space in terms of $|\eta|^\alpha |\log(\eta)|^\beta$ for suitable α, β.

We then consider the initial $\theta(0)$ with Fourier transform given by

$$\mathcal{F}(\theta)(0) := \frac{1}{\psi(\eta)}\mathcal{F}(\rho)(\eta)\theta[\eta](0).$$

Since $\|\theta[\eta](0)\|_X = 1$ and $\frac{1}{\psi(\eta)} \leq e^{-\sqrt[3]{c\eta}}$, clearly $\theta(0) \in \mathcal{G}_3 X$.

Moreover, by the asymptotic stability of the frequency-localized initial data it holds that

$$\mathcal{F}(\theta)(t) \to \mathcal{F}(\rho)(\eta)\frac{\theta^\infty[\eta]}{\psi(\eta)}.$$

pointwise in η. By definition of $\psi(\eta)$ the last factor is normalized in X and hence this pointwise (in frequency) limit is an element of $H^\sigma X$. In particular, by compactness of the embedding $H^\sigma \subset H^s$ for $s < \sigma$ we obtained the claimed convergence in $H^s X$ for $s < \sigma$. Since $\rho \notin H^s$ for $s > \sigma$ we also obtain divergence in H^s, $s > \sigma$. This concludes the proof of the blow-up construction. □

Our main aim in the remainder of this section is thus to construct global in time solutions for given η which achieve the desired norm inflation. As discussed in the heuristic model of Section 2 the main growth is expected to happen in the resonant time regime $(t_{k_0}, t_0 = 2\eta)$, where

$$k_0 = \sqrt[3]{c\eta\frac{\pi}{2}}.$$

For technical reasons we do not consider the extremal case of a full chain starting at frequency k_0, but instead begin at frequency $k_2 = \frac{k_0}{4}$ and only establish lower bounds on the norm inflation until the time $k_3 = \frac{k_0}{1000}$.

The corresponding time regimes and behavior of the solution are described in more detail in the following proposition, which thus states the main steps of the proof of Theorem 4.1.

Proposition 4.2. *Let c, η, ν be given and assume that $\sqrt[3]{c\eta} \gg \max(\nu^{-1}, 1)$ (i.e. choose η large enough). Furthermore define the following threshold values:*

$$\begin{aligned} k_0 &= \sqrt[3]{c\eta\pi}, \\ k_1 &= 4k_0, \\ k_2 &= \frac{k_0}{10}, \\ k_3 &= \frac{k_0}{1000}. \end{aligned}$$

Then the solution with initial data

$$\theta_l(0) = \delta_{lk_2}.$$

has the following properties:

- *At time t_{k_1} it holds that*

$$|\theta_{k_2} - 1| \leq c^2(\frac{\eta}{k_1^2})^{-3},$$

$$|\theta_l| \leq c(\frac{\eta}{k^2})^{-2}\prod_{j=l}^{k_2} c(\frac{\eta}{j^2})^{-1}, \text{ if } l \leq k_1,$$

$$|\theta_l| \leq 2^{-|l-k_1|}c(\frac{\eta}{k_1^2})^{-2}\prod_{j=k_1}^{k_2} c(\frac{\eta}{j^2})^{-1}.$$

Thus at time t_{k_1} the mode k_2 is by far the largest (the factor $c(\frac{\eta}{k_1^2})^{-2} \ll 1$) and we have very rapid decay of all other modes.

- *At time t_{k_2} it holds that*

$$
\begin{aligned}
|\theta_{k_2} - 1| &\leq c(\frac{\eta}{k_1^2})^{-1} e^2, \\
|\theta_l| &\leq c(\frac{\eta}{k_1^2})^{-1} e^2, \text{ if } l \geq k_2, \\
|\theta_l| &\leq c(\frac{\eta}{k_1^2})^{-2} e^2 \prod_{j=l}^{k_2} c(\frac{\eta}{j^2})^{-1}, \text{ if } l < k_2.
\end{aligned}
$$

  Thus at time time still θ_{k_2} is the largest mode. Furthermore, while modes $l > k_2$ do not exhibit decay in $|l - k_2|$ anymore, this is still the case for $l < k_2$.

- *At the time t_{k_3} it holds that*

$$
\begin{aligned}
|\theta_{k_3}|| &\geq e^{k_0}, \\
|\theta_{k_3-2}| &\geq e^{k_0}, \\
|\theta_l| &\leq \frac{1}{1000} |\theta_{k_3}| \text{ for all } l \not\in \{k_3 - 1, k_3 + 1\}.
\end{aligned}
$$

  At time t_{k_3} the modes θ_{k_3} and θ_{k_3-2} are by far the largest modes and have achieved significant norm inflation.

- *For all times $t \geq t_0 = 2\eta$ it holds that*

$$
|\theta_{k_3-2}(t)| \geq e^{k_0}.
$$

  While other modes might have grown even more, this growth persists. In particular $\|\theta(t)\|_{\ell^2} \geq e^{k_0} \|\theta(0)\|_{\ell^2}$.

Since each time regime requires rather different techniques, we discuss the regimes in different subsections. The corresponding estimates on G are then included in the respective Lemmas 4.3, 4.4, 4.6 and Proposition 4.5.

4.1. **The Small Time Regime and Contraction Mappings.** In this section we consider the evolution of θ and G in the small time regime

$$
(0, t_{k_1}) \approx (0, \frac{1}{10} \frac{\eta}{\sqrt[3]{c\eta\pi/2}}).
$$

We note that this choice of time interval implies that

$$
\frac{c}{\eta} k^3 \leq 10^{-3} = 0.001
$$

for all $k \geq k_1$ and that for k smaller than this $\frac{\eta}{k}$ is not part of this interval. We will show that this implies that the associated integral equation for the modes θ_l, $\frac{1}{10} G_l$ is a contraction mapping in $L^\infty \ell^\infty$ on this interval. The resulting ℓ^∞ bound is then subsequently improved to the weighted decay estimate of Proposition 4.2.

Lemma 4.3. *Let $\eta, c\nu, k_1, k_2$ be as in Proposition 4.2 and consider the solution of (11) with initial data*

$$
\theta_l(0) = \delta_{lk_2}, \ G(0) = 0.
$$

Then at the time t_{k_1} it holds that

$$
\begin{aligned}
|\theta_{k_2} - 1| &\leq c^2 (\frac{\eta}{k_1^2})^{-2}, \\
|\theta_l| &\leq c(\frac{\eta}{k^2})^{-2} \prod_{j=l}^{k_2} c(\frac{\eta}{j^2})^{-1}, \ \text{if } l \leq k_1, \\
|\theta_l| &\leq 2^{-|l-k_1|} c(\frac{\eta}{k_1^2})^{-2} \prod_{j=k_1}^{k_2} c(\frac{\eta}{j^2})^{-1}, \ \text{if } l \neq k_2, l \geq k_1, \\
|G_l| &\leq 10c(\frac{\eta}{k^2})^{-2} \prod_{j=l}^{k_2} c(\frac{\eta}{j^2})^{-1}, \ \text{if } l \leq k_1, l \neq k_2, \\
|G_l| &\leq 10 \cdot 2^{-|l-k_1|} c(\frac{\eta}{k_1^2})^{-2} \prod_{j=k_1}^{k_2} c(\frac{\eta}{j^2})^{-1}, \ \text{if } l \geq k_1, \\
|G_{k_2}| &\leq c^2 (\frac{\eta}{k_1^2})^{-2}.
\end{aligned}
$$

We thus observe that at time t_{k_1} our data exhibits a sharp concentration on the mode θ_{k_2} (see Figure 3 for an illustration). Moreover, the exponential decay in terms of $|k - k_2|$ is stronger than possible growth by $\frac{c\eta}{k^3}$ due to the resonance mechanism.

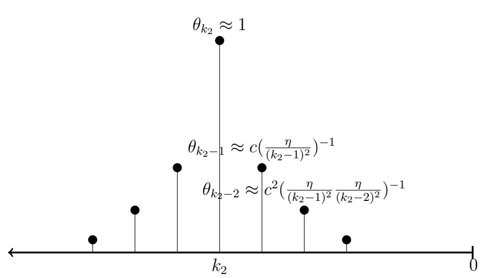

FIGURE 3. Distribution of θ_l at time t_{k_1}. We observe a peak at the frequency k_2 and rapid decay away from this frequency.

Proof of Lemma 4.3. Given our choice of initial data we observe that θ and G satisfy the integral equations (21):

$$\begin{aligned}
\theta_l(T)-\delta_{lk_2} &= \int_0^T c_l^+\theta_{l+1}+c_l^-\theta_{l-1}+d_l^+G_{l+1}+d_l^-G_{l-1}dt,\\
G_l(T) &= \int_0^T \exp(-\nu\int_t^T l^2+(\eta-l\tau)^2 d\tau)2\frac{(\frac{\eta}{l}-t)}{(1+(\frac{\eta}{l}-t)^2)^2}\theta_l dt\\
&+\int_0^T \exp(-\nu\int_t^T l^2+(\eta-l\tau)^2 d\tau) fil\frac{\nu}{g}(d_l^+G_{l+1}+d_l^-G_{l-1})dt\\
&+\int_0^T \exp(-\nu\int_t^T l^2+(\eta-l\tau)^2 d\tau) fil\frac{\nu}{g}(c_l^+\theta_{l+1}+c_l^-\theta_{l-1})dt\\
&+\int_0^T \exp(-\nu\int_t^T l^2+(\eta-l\tau)^2 d\tau)\frac{1}{1+(\frac{\eta}{l}-t)^2}\left(c_l^+\theta_{l+1}+c_l^-\theta_{l-1}\right)dt\\
&+\int_0^T \exp(-\nu\int_t^T l^2+(\eta-l\tau)^2 d\tau)\frac{1}{1+(\frac{\eta}{l}-t)^2}\left(d_l^+G_{l+1}+d_l^-G_{l-1}\right)dt.
\end{aligned}$$

We then observe that

$$\begin{aligned}
\int_0^T c_l^\pm &\leq 0.01,\\
\int_0^T d_l^\pm &\leq 0.01,
\end{aligned}$$

since $\frac{c\eta}{k^3}\leq 4^{-3}$ for $k\geq k_1$ and the frequencies are not yet resonant for $k\leq k_1$. Similarly,

$$\begin{aligned}
\int_0^T \exp(-\nu\int_t^T l^2+(\eta-l\tau)^2 d\tau) fil\frac{\nu}{g}d_l^\pm &\leq 0.01,\\
\int_0^T \exp(-\nu\int_t^T l^2+(\eta-l\tau)^2 d\tau) fil\frac{\nu}{g}c_l^\pm &\leq 0.01,\\
\int_0^T \exp(-\nu\int_t^T l^2+(\eta-l\tau)^2 d\tau)\frac{1}{1+(\frac{\eta}{l}-t)^2}c_l^\pm &\leq 0.01,\\
\int_0^T \exp(-\nu\int_t^T l^2+(\eta-l\tau)^2 d\tau)\frac{1}{1+(\frac{\eta}{l}-t)^2}d_l^\pm &\leq 0.01.
\end{aligned}$$

The only possibly large contribution is hence given by

$$\int_0^T \exp(-\nu\int_t^T l^2+(\eta-l\tau)^2 d\tau)2|\frac{(\frac{\eta}{l}-t)}{(1+(\frac{\eta}{l}-t)^2)^2}|dt\leq 2.$$

Similarly to the results of Section 3.1 we thus consider the equations as equations for θ and $\frac{G}{10}$ instead, so that all coefficient functions are bounded by 0.1.

We then define

$$\hat{\theta}_l := \theta_l-\delta_{lk_2}$$

and view these equations as a fixed point iteration for

$$\begin{pmatrix}\hat{\theta}\\ \frac{1}{10}G\end{pmatrix} = B[k_2]+L\begin{pmatrix}\hat{\theta}\\ \frac{1}{10}G\end{pmatrix},$$

on the space
$$L^\infty \ell^\infty.$$
By the above choice L is a contraction with norm less than $1/2$ and hence we can control
$$\|\theta(t)\|_{\ell^\infty} + \frac{1}{10}\|G(t)\|_{\ell^\infty} \leq 2\|B[k_2]\|_{L^\infty \ell^\infty}.$$
We note that here the components of $B[k_2]$ are given by
$$\int_0^T \frac{c\eta}{k_2^3}\frac{1}{(1+(\frac{\eta}{k_2}-t)^2)^2}dt \leq \frac{c}{k_2}(\frac{\eta}{k_2^2})^{-2}$$
and
$$\int_0^T \exp(-\nu\int_t^T k_2^2 + (\eta - k_2\tau)^2 d\tau) 2\frac{(\frac{\eta}{k_2}-t)}{(1+(\frac{\eta}{k_2}-t)^2)^2} \leq \min((\frac{\eta}{k_2})^{-2}, \frac{1}{\nu k_2^2}(\frac{\eta}{k_2})^{-3}),$$
$$\int_0^T \frac{\exp(-\nu\int_t^T (k_2\pm 1)^2 + (\eta-(k_2\pm 1)\tau)^2 d\tau)}{(1+(\frac{\eta}{k}-t)^2)^2} f(t)\frac{c\eta}{k_2^3} \leq c(\frac{\eta}{k_2})^{-2},$$
$$\int_0^T \frac{\exp(-\nu\int_t^T (k_2\pm 1)^2 + (\eta-(k_2\pm 1)\tau)^2 d\tau)}{1+(\frac{\eta}{k}-t)^2} f(t)\frac{c\eta}{k_2^3} \leq c(\frac{\eta}{k_2})^{-2},$$
$$\int_0^T \frac{\exp(-\nu\int_t^T (k_2\pm 1)^2 + (\eta-(k_2\pm 1)\tau)^2 d\tau)}{(1+(\frac{\eta}{k_2\pm 1}-t)^2)(1+(\frac{\eta}{k}-t)^2)^2} \frac{c\eta}{k_2^3} \leq c(\frac{\eta}{k_2})^{-2},$$
$$\int_0^T \frac{\exp(-\nu\int_t^T (k_2\pm 1)^2 + (\eta-(k_2\pm 1)\tau)^2 d\tau)}{(1+(\frac{\eta}{k_2\pm 1}-t)^2)(1+(\frac{\eta}{k}-t)^2)} \frac{c\eta}{k_2^3} \leq c(\frac{\eta}{k_2})^{-2},$$
respectively. Therefore, it holds that
$$\|\theta(t)\|_{\ell^\infty} + \frac{1}{10}\|G(t)\|_{\ell^\infty} \leq C(\frac{\eta}{k_2})^{-2}$$
for a universal constant C.

We next use these rough upper bounds to establish the claimed improved bounds and decay. For this purpose we observe that
$$\int_0^{t_{k_1}} c_l^\pm \leq \begin{cases} 0.01 & \text{if } l\pm 1 > k_1, \\ c(\frac{\eta}{(l\pm 1)^2})^{-2} & \text{if } (l\pm 1) \leq k_1, \end{cases}$$
since the latter frequencies have not yet been resonant. In particular, it follows that
$$|\theta_{k_2}(t) - 1| \leq c(\frac{\eta}{l^2})^{-2}(\|\hat{\theta}\|_{L^\infty\ell^\infty} + \|G\|_{L^\infty\ell^\infty}),$$
since k_2-1, k_2+1 are non-resonant.

Given this size of θ_{k_2} the claimed decay in $|l-k_2|$ then follows by repeated insertion of the above estimates into the integral equation (using that $\hat{\theta}(0) = 0$, $G(0)=0$). More precisely, we observe that
$$\begin{aligned}|G_{k_2}(T)| &\leq \min((\frac{\eta}{k_2})^{-2}, \frac{1}{\nu k_2^2}(\frac{\eta}{k_2})^{-3})\|\theta_{k_2}\|_{L^\infty\ell^\infty} \\ &\quad + c(\frac{\eta}{k_2})^{-2}(\|\theta_{k_2+1}\|_{L^\infty\ell^\infty} + \|\theta_{k_2-1}\|_{L^\infty\ell^\infty} + \|G_{k_2+1}\|_{L^\infty\ell^\infty} + \|G_{k_2-1}\|_{L^\infty\ell^\infty}).\end{aligned}$$
Thus, inserting our bounds by 1 and $\|B[k_2]\|_{L^\infty\ell^\infty}$ we observe the desired improvement for $|G_{k_2}(T)|$.

For bounds on modes at frequencies further away from k_2, we require multiple iterations of this argument. For instance, we observe that after the first insertion

$$\begin{aligned}\|\theta_{k_2-10}\|_{L^\infty\ell^\infty} &\leq c(\frac{\eta}{l^2})^{-2}(\|\theta_{k_2-9}\|_{L^\infty\ell^\infty}+\|\theta_{k_2-11}\|_{L^\infty\ell^\infty})\\ &+c(\|G_{k_2-9}\|_{L^\infty\ell^\infty}+\|G_{k_2-11}\|_{L^\infty\ell^\infty})\end{aligned}$$

does not enjoy a better bound than θ_{k_2-2}. However, these improved estimates then hold for all frequencies $\leq k_2-2$. Thus, inserting the improved estimates once more we establish the desired bound for k_2-3 and the same (then suboptimal) bound for all modes smaller than k_2-3. Thus, repeating the argument N times we obtain the desired bounds for modes with $|k-k_2|\leq N$ and thus the full result by letting $N\to\infty$.

□

4.2. **The Intermediate Time Regime and Upper Bounds.**

We next study the intermediate time regime

$$(t_{k_1}, t_{k_2}).$$

Since in this regime $\frac{c\eta\pi}{k^3}$ is not necessarily small anymore a fixed point iteration is not possible anymore. Moreover, resonances can result in growth of certain modes. However, since by Lemma 4.3 we know that at least at time t_{k_1} the corresponding modes possibly becoming resonant are small we can control the growth on each interval I_k in this time regime by induction.

Lemma 4.4. *Let c,η,k_1,k_2 be as in Proposition 4.2 and for $k_2\leq k\leq k_1$ define the constant C_k by*

$$\begin{aligned}C_{k_1}&=1,\\ C_{k-1}&=(1+(\frac{\eta}{k^2})^{-1})C_k.\end{aligned}$$

Then for all such k it holds that

$$\begin{aligned}
&|\theta_{k_2}(t_k)|\leq C_k,\\
&|\theta_l(t_k)|\leq C_k\prod_{j=l}^{k_2}(c(\frac{\eta}{j^2})^{-1}) \text{ for } j\leq k,\\
&|\theta_l(t_k)|\leq C_k\prod_{j=k-1}^{k_2}(c(\frac{\eta}{j^2})^{-1}) \text{ for } j\geq k,\\
&|G_l(t_k)|\leq 10C_kc(\frac{\eta}{k_1^2})^{-2}\prod_{j=l}^{k_2}c(\frac{\eta}{j^2})^{-1}, \text{ if } l\leq k, l\neq k_2,\\
&|G_l(t_k)|\leq 10C_k\cdot 2^{-|l-k_1|}c(\frac{\eta}{k_1^2})^{-2}\prod_{j=k-1}^{k_2}c(\frac{\eta}{j^2})^{-1}, \text{ if } l\geq k,\\
&|G_{k_2}(t_k)|\leq C_kc^2(\frac{\eta}{k_1^2})^{-2}.
\end{aligned}\tag{33}$$

We note that the upper bounds (33) here have a "dent", where the bounds for θ_{k+2} and θ_k are the same and much larger than the one for θ_{k+1} (see Figure 4). The reason for this is that the resonance during the time interval I_{k+1} may cause the modes $k+1\pm1=k+2,k$ to grow while θ_k remains relatively unchanged. Thus the upper bounds for $k+2, k$ are much larger than for k, resulting in the "dent". On the time interval I_k the mode θ_k then is resonant and may cause the modes θ_{k+1} and θ_{k-1} to grow by a large factor (shown in red in Figure 4), thus resulting in a new "dent".

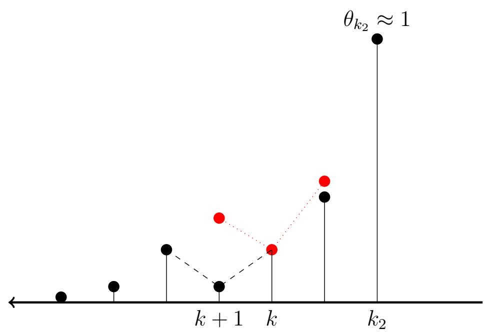

FIGURE 4. The resonance mechanism causes neighboring modes to grow. In this figure we show the upper bounds of Lemma 4.4 at the time t_k as black dots. In particular, we observe a "dent" at the frequency $k+1$. During the time interval I_k then the mode θ_k becomes resonant leading to the change of upper bounds colored in red. The "dent" moves.

We further remark that (33) holds for $k = k_1$ by Lemma 4.3. Moreover, since $1 \leq k \leq k_1$ it holds that

$$\frac{\eta}{k^2} \geq \frac{\eta}{k_1^2} \geq \frac{1}{10} c^{-1} k_1 \geq 100 k_1.$$

In particular, since there are less than k_1 many such k we observe that

$$C_k \leq (1 + \frac{1}{100}\frac{1}{k_1})^{k_1} \leq e^{\frac{1}{100}}$$

and thus C_k remains uniformly bounded and comparable to 1. The estimates (33) hence formalize the statement that modes other than $k-1, k+1$ only change slightly, while the modes $k-1, k+1$ might potentially grow. However, since the upper bound on the θ_k was much smaller than the one on the mode θ_{k-1} it suffices to that all modes $l \geq k-1$ then at time t_{k-1} satisfy the same upper bound as the mode $k-1$.

Additionally, using the integral equation we deduce that

$$|\theta_{k_2}(t_{k_2}) - \theta_{k_2}(t_{k_1})| \leq 4e^{\frac{1}{100}} c(\frac{\eta}{k_2^2})^{-1},$$

where we used the bound on $\int c_{k_2}^{\pm}$ by the definition of k_2 and the above induction bound. Thus, the claimed control at time t_{k_2} stated in Proposition 4.2 indeed follows from this lemma.

Proof of Lemma 4.4. As remarked Lemma 4.3 ensures that (33) holds for $k = k_1$ and we then aim to proceed by induction, where we again use a bootstrap argument.

More precisely, let k be given and suppose that (33) holds for this k. Since $C_{k+1} > C_k$ and the products also only become larger when replacing k by $k-1$, it follows that there exists a time interval $(t_k, T_\star)$ such that (33) with $k-1$ holds on this interval with slight modification:

$$
\begin{aligned}
|\theta_{k_2}| &\leq C_{k-}, \\
|\theta_l| &\leq C_{k-1} \prod_{j=l}^{k_2} (c(\frac{\eta}{j^2})^{-1}) \text{ for } j \leq k-1, \\
|\theta_l| &\leq C_{k-1} \prod_{j=k-2}^{k_2} (c(\frac{\eta}{j^2})^{-1}) \text{ for } j \geq k-1, \\
|G_l| &\leq 10 C_{k-1} c(\frac{\eta}{k_1^2})^{-2} \prod_{j=l}^{k_2} c(\frac{\eta}{j^2})^{-1}, \text{ if } l < k-1, l \neq k_2, \\
|G_l| &\leq 10 C_{k-1} \cdot 2^{-|l-k_1|} c(\frac{\eta}{k_1^2})^{-2} \prod_{j=k-2}^{k_2} c(\frac{\eta}{j^2})^{-1}, \text{ if } l > k-1, l \neq k \\
|G_k| &\leq 10 C_{k-1} \prod_{j=k-2}^{k_2} (c(\frac{\eta}{j^2})^{-1}), \\
|G_{k_2}| &\leq C_{k-1} c^2 (\frac{\eta}{k_1^2})^{-2}.
\end{aligned}
$$

for all $t \in (t_k, T_\star)$. We emphasize here that G_k is only required to satisfy an upper bound comparable to the one of θ_k. The reason for this is that the contribution by

$$
\int_{t_k}^{T} \exp(-\nu \int_t^T k^2 + (\eta - k\tau)^2 d\tau) 2 \frac{(\frac{\eta}{k} - t)}{(1 + (\frac{\eta}{k} - t)^2)^2}
$$

is potentially large and thus for times $T \approx \frac{\eta}{k}$ we do not necessarily expect the unmodified estimate of (33) to hold. However, we claim that the above modified bootstrap assumptions allow us to recover the unmodified estimates at the final time. Indeed, we observe that for $t_{k-1} - 3 \leq \tau \leq t_{k-1}$ it holds that

$$
(\eta - k\tau)^2 \geq 0.5 (\frac{\eta}{k})^2 = 0.5 \eta \frac{\eta}{k^2}.
$$

Hence, using the fact that $0.5\eta\nu \gg 1$, it follows that for $t_{k-1} - 3 \leq t \leq T \leq t_{k-1}$

$$
\exp(-\nu \int_t^T k^2 + (\eta - k\tau)^2 d\tau) \leq \exp(-\frac{\eta}{k^2}(T - t)).
$$

We may then use the integral equations to express

$$
\begin{aligned}
G_k(t_{k-1}) = \exp(-\nu \int_{t_{k-1}-3}^{t_{k_1}} k^2 + (\eta - k\tau)^2 d\tau) G_k(t_{k-1} - 3) \\
+ \int_{t_{k-1}-3}^{t_{k-1}} \exp(-\nu \int_t^T k^2 + (\eta - k\tau)^2 d\tau) \mathrm{rhs}(t) dt,
\end{aligned}
$$

where we abbreviated the terms of the evolution equation as a right-hand-side(t). We then observe that

$$
\exp(-\nu \int_{t_{k-1}-3}^{t_{k_1}} k^2 + (\eta - k\tau)^2 d\tau) \leq \exp(-3 \frac{\eta}{k^2})
$$

yields exponential decay and that all coefficients included in the right-hand-side are non-resonant on the interval $(t_{k-1}-3, t_{k-1})$. Thus $G_k(t_{k-1})$ satisfies the desired improved bounds.

It remains to be shown that the maximal time $T_\star \leq t_{k-1}$ for which the (modified) bootstrap assumptions are satisfied is given by $T_\star = t_{k-1}$. Indeed, suppose not then

$$\begin{aligned}\theta_{k_2}(t) - \theta_{k_2}(t_k) &= \int_{t_k}^{t} c^+_{k_2}\theta_{k_2+1} + c^{-1}_{k_2}\theta^{-1}_{k_2} \\ &\leq c(\frac{\eta}{k_1^2})^{-2} C_{k-1} c(\frac{\eta}{k_1^2})^{-1} \\ &\leq c^2(\frac{\eta}{k_1^2})^{-2} C_{k-1}\end{aligned}$$

Thus, using the induction assumption

$$|\theta_{k_2}(t)| \leq C_k + c^2(\frac{\eta}{k_1^2})^{-2} C_{k-1} < C_{k-1}.$$

Similarly, if $l \not\in \{k-1, k+1\}$ the non-resonant integrals are bounded by $c(\frac{\eta}{l^2})^{-2}$, while the larger of the neighbors is larger by a factor at most $c^{-1}\frac{\eta}{l^2}$ compared to the bound on the mode l. Thus, in total we lose at most a factor $(1+2(\frac{\eta}{k^2})^{-1})$, as claimed. The estimates for G in the non-resonant cases are analogous.

It remains to discuss the effect of resonant modes. Here we observe that

$$\int_{t_k}^{t_{k-1}} \frac{c\eta}{k_3} \frac{1}{(1+(\frac{\eta}{k}-t)^2)^2} dt \leq \frac{c\eta}{k^3}\pi$$

might cause the modes θ_{k-1} and θ_{k+1} to grow by this factor times the bound on the mode θ_k. However, the bound on θ_{k-1} formulated in the bootstrap is already larger than $c^{-1}(\frac{\eta}{k^2})$ times this bound. Hence, this growth is consistent with (33). Similarly the growth of θ_{k+1} is controlled since the "dent" moved in the induction step and now θ_{k+1} needs only to satisfy the same upper bound as θ_{k-1}.

We also observe that G_k may grow due to

$$\int_{t_k}^{T} \exp(-\nu \int_{t}^{T} k^2 + (\eta - kt)^2) \frac{2(\frac{\eta}{k}-t)}{(1+(\frac{\eta}{k}-t)^2)^2} \theta_k dt.$$

Since $k \geq k_2$ this contribution can be estimated from above by $\frac{1}{\nu k_2^3} \ll 1$ in the present setting. However, for later reference we remark that for our upper bounds it suffices to note that

$$\int_{t_k}^{T} |\frac{2(\frac{\eta}{k}-t)}{(1+(\frac{\eta}{k}-t)^2)^2}| \leq 2.$$

$\square$

4.3. **Norm Inflation in the Resonant Regime.** The core of our norm inflation mechanism is given by the resonant growth in the time interval

$$(t_{k_2}, t_0),$$

which is formulate in the following proposition.

Proposition 4.5. *Let c, η be as in Theorem 4.1 and let $k_0 \in N$ be such that (by rounding down)*

$$k_0 \approx \sqrt[3]{c\eta\pi}.$$

Let further $1 \leq k \leq \frac{k_0}{4}$ be such that $\nu k^2 \geq 4$. Then if at time t_k it holds that

$$\begin{aligned} |\theta_k(t_k)| &\geq 0.5 \max(\|\theta(t_k)\|_{\ell^\infty}, \|G(t_k)\|_{\ell^\infty}), \\ |\theta_k(t_k)| &\geq 4|G_k(t_k)|, \end{aligned} \tag{34}$$

then at time t_{k-1} it holds that

$$|\theta_{k\pm1}(t_{k-1})| \geq \frac{c\eta\pi}{3k^3}|\theta_k(t_k)| \tag{35}$$

and (34) holds with k replaced by $k-1$.

In particular it holds that

$$\begin{aligned} |\theta_{k_3}(t_{k_3})| &\geq e^{k_0}|\theta_{k_2}(t_{k_2})|, \\ |\theta_{k_3-2}(t_{k_3})| &\geq e^{k_0}|\theta_{k_2}(t_{k_2})|, \end{aligned}$$

We remark that here Lemma 4.4 ensures that (34) holds for $k = k_2$. The condition $\nu k^2 \geq 4$ here is the first and only time we explicitly use the dissipation and is used to establish lower bounds (but not required for upper bounds). More precisely, we recall that

$$\nu\partial_y\phi = \partial_y\partial_x\Delta_t^{-2}\theta + \partial_y\partial_x^{-1}\Delta_t^{-1}G.$$

Hence, in general control of the velocity requires control of both θ and G. In particular, if G is not smaller than θ but of comparable size and with an opposite sign it could be that even at resonant times $\partial_y\theta$ is very small even if θ is large. Thus, in order to avoid such cancellations we use dissipation to ensure that G is much smaller than θ. Here we stress that for $\nu \geq 4$ this condition is trivial, but for ν small restricts us to considering k large. We thus introduced a time threshold t_{k_3} with $k_3 = \frac{1}{1000}k_0$ until which this lower bound is satisfied. As we show in Section 4.4 after this time all upper bounds can be established in the same way and lower bounds remain true at least for frequencies larger than $k_3 + 1$.

Before beginning the proof, we briefly discuss the lower bound. Iteratively applying (35) we observe that

$$|\theta_{k_3+1\pm1}(t_{k_3})| \geq |\theta_{k_2}(t_{k_2})| \prod_{j=k_2}^{k_3} \frac{c\eta\pi}{3j^3}.$$

Hence, it suffices to bound the latter product from below. Indeed, we obtain a lower bound by

$$\prod_{k=\frac{k_0}{1000}}^{k_0/2} \frac{c\eta\pi}{2k^3} = \frac{(c\eta\pi/2)^{k_0/2}}{(\frac{k_0}{2}!)^3} \frac{(\frac{k_0}{1000}!)^3}{(c\eta\pi/2)^{k_0/1000}}.$$

We may then use Stirling's approximation to compute the first factor as approximately

$$2^{3\frac{k_0}{2}} e^{3\frac{k_0}{2}} (\sqrt{2\pi\frac{k_0}{2}})^{-3}$$

and the second factor as

$$1000^{-3\frac{k_0}{1000}} e^{-3\frac{k_0}{1000}} (\sqrt{2\pi \frac{k_0}{1000}})^3.$$

Using the fact that $n^{1/n} \to 1$ as $n \to \infty$ (and is about 1.007 for $n = 1000$) the first factors are easily dominated by the growth of $2^{3\frac{k_0}{2}}$. Thus, we obtain a lower bound by

$$e^{\frac{3}{2}k_0 - \frac{3}{1000}k_0} \geq e^{k_0},$$

as claimed.

Proof of Proposition 4.5. The heuristic idea of our proof is that at time t_{k-1}

$$\begin{aligned}\theta_{k-1}(t_{k-1}) &\approx \theta_{k-1}(t_k) + \theta_k(t_k) \int_{t_k}^{t_{k-1}} \frac{c\eta}{k^3} \frac{1}{(1 + (\frac{\eta}{k} - t)^2)^2} \\ &\approx \theta_{k-1}(t_k) + \theta_k(t_k) \frac{c\eta\pi}{2k^3}.\end{aligned}$$

Since by our choice of k it holds that

$$\frac{c\eta\pi}{2k^3} \geq 4^3 = 64$$

this suggests that $\theta_k(t_k)\frac{c\eta\pi}{2k^3}$ dominates all other contributions, which implies the lower bound (35). Moreover, while all other modes except $\theta_{k\pm1}$ may also grow by some factor, this factor is much smaller than $\frac{c\eta\pi}{2k^3}$. Hence, at time t_{k-1} the mode θ_{k-1} will be one of the largest modes and hence satisfy (34) with k replaced by $k - 1$.

It remains to make this heuristic rigorous, for which we employ a bootstrap approach similar to the one of Section 3.3. We remark that, since we do not require ℓ^2-based estimates, we here do not need to decompose into single mode data. For simplicity of notation we may without loss of generality assume that

$$\theta_k(t_k).$$

We then make the bootstrap assumptions that for $t_k \leq T \leq T_\star$ it holds that

- (E1) $|\theta_k(T) - 1| \leq 0.1$.
- (E2) $|\theta_l(T)| \leq 4$ for all $l \notin \{k-1, k+1\}$.
- (E3) $|\theta_{k\pm1}(T) - \theta_{k\pm1}(t_k) - \int_{t_k}^T \frac{c\eta}{k^3} \frac{1}{(1+(\frac{\eta}{k}-t)^2)^2} \theta_k(t) dt - \int_{t_k}^T \frac{c\eta}{k^3} \frac{1}{1+(\frac{\eta}{k}-t)^2} G_k(t) dt|$ ≤ 0.1.
- (E4) $|G_k(T) - \exp(-\nu \int_{t_k}^T k^2 + (\eta - ks)^2 ds) G_k(t_k)$ $- \int_{t_k}^T \exp(\nu \int_{t_k}^T k^2 + (\eta - ks)^2 ds) \frac{2(\frac{\eta}{k}-t)}{(1+(\frac{\eta}{k}-t)^2)^2} \theta_k(t) dt| \leq 0.1$.
- (E5) $|G_l(T)| \leq 4$.

By local well-posedness these estimates are satisfied at least for a short time. Similarly to the proof of Proposition 3.6 we in the following show that these estimates self-improve and hence remain valid until time t_{k-1}.

Improving (E1): We recall that by the integral equations (21)

$$\theta_k(T) - 1 = \int_{t_k}^T c_k^+ \theta_{k+1} + c_k^- \theta_{k-1} + d_k^+ G_{k+1} + d_k^- G_{k-1}.$$

Furthermore, by the estimates collected in Lemma 3.4 it holds that

$$\begin{aligned}\int_{t_k}^{t} c_k^{\pm} &\leq \frac{4c}{k}(\frac{\eta}{k^2})^{-2},\\ \int_{t_k}^{t} d_k^{\pm} &\leq \frac{4c}{k}.\end{aligned}\tag{36}$$

Using the bootstrap estimates (E3)–(E5) it then follows that

$$|\theta_k(T)-1| \leq 20c < 0.1,$$

and thus (E1) is improved.

Improving (E2): We recall that by the integral equations (21) it holds that

$$\theta_l(T)-\theta_l(t_k) = \int_{t_k}^{T} c_l^{+}\theta_{l+1} + c_l^{-}\theta_{l-1} + d_l^{+}G_{l+1} + d_l^{-}G_{l-1}dt.$$

Since $l \not\in \{k-1, k+1\}$ all coefficient functions are non-resonant and hence satisfy (36). Using the bootstrap assumptions (E3)–(E5) it then follows that the right-hand-side is much smaller than 2, while $|\theta_l(t_k)| \leq 2|\theta_k(t_k)| = 2$ by (34). Hence, this bootstrap bound improves.

Improving (E3): By the integral equations it holds that (21)

$$\theta_{k\pm 1}(T)-\theta_{k\pm 1}(t_k) = \int_{t_k}^{T} c_{k\pm 1}^{\mp}\theta_k + c_{k\pm 1}^{\pm}\theta_{k\pm 2} + d_{k\pm 1}^{\mp}G_k + d_{k\pm 2}^{\pm}G_{k\pm 2}.$$

We thus only need to estimate

$$\int_{t_k}^{T} c_{k\pm 1}^{\pm}\theta_{k\pm 2} + d_{k\pm 2}^{\pm}G_{k\pm 2}$$

from above. Here we again use the estimates (36) and control $\theta_{k\pm 2}, G_{k\pm 2}$ by the bootstrap assumptions (E2), (E5).

We remark that

$$\begin{aligned}\int_{t_k}^{T} c_{k\pm 1}^{\mp} &\leq \frac{c\eta\pi}{2k^3},\\ \int_{t_k}^{T} d_{k\pm 1}^{\mp} &\leq \frac{c\eta\pi}{k^3}\end{aligned}$$

and that $\theta_k(t), G_k(t) \leq 2$ by the bootstrap assumptions (E1),(E4). Hence, this estimate also provides an upper bound on the size of $\theta_{k\pm 1}(T)$.

Improving (E5): We recall that by the integral equations (21) it holds that

$$
\begin{aligned}
G_l(T) - \exp&\left(-\nu\int_{t_k}^T l^2+(\eta-lt)^2dt\right)G_l(t_k)\\
&=\int_{t_k}^T \exp(-\nu\int_t^T l^2+(\eta-l\tau)^2d\tau)2\frac{(\frac{\eta}{l}-t)}{(1+(\frac{\eta}{l}-t)^2)^2}\theta_l dt\\
&\quad+\int_{t_k}^T \exp(-\nu\int_t^T l^2+(\eta-l\tau)^2d\tau)f\frac{\nu}{g}il(d_l^+G_{l+1}+d_l^-G_{l-1})dt\\
&\quad+\int_{t_k}^T \exp(-\nu\int_t^T l^2+(\eta-l\tau)^2d\tau)f\frac{\nu}{g}il(c_l^+\theta_{l+1}+c_l^-\theta_{l-1})dt\\
&\quad+\int_{t_k}^T \exp(-\nu\int_t^T l^2+(\eta-l\tau)^2d\tau)\frac{1}{1+(\frac{\eta}{l}-t)^2}\left(c_l^+\theta_{l+1}+c_l^-\theta_{l-1}\right)dt\\
&\quad+\int_{t_k}^T \exp(-\nu\int_t^T l^2+(\eta-l\tau)^2d\tau)\frac{1}{1+(\frac{\eta}{l}-t)^2}\left(d_l^+G_{l+1}+d_l^-G_{l-1}\right)dt.
\end{aligned}
$$

Since $l\neq k$ the estimate (25) of Lemma 3.5 holds:

$$
\int_{t_k}^T \exp(-\nu\int_t^T l^2+(\eta-l\tau)^2d\tau)\left|2\frac{(\frac{\eta}{l}-t)}{(1+(\frac{\eta}{l}-t)^2)^2}\right|dt \leq 2(\frac{\eta}{k^2})^{-2}.
$$

Similarly, by the estimates (26), (27) and (28) all other integrals can be controlled by

$$
c(\frac{\eta}{k^2})^{-1}
$$

times the L^∞ norms of $G_{l\pm1},\theta_{l\pm1}$, which are controlled by the bootstrap assumptions (E1)–(E5). Therefore, in conclusion

$$
\begin{aligned}
|G_l(T)-\exp&\left(-\nu\int_{t_k}^T l^2+(\eta-lt)^2dt\right)G_l(t_k)|\leq 0.1\\
&\Rightarrow |G_l(T)|\leq 2+0.1<4.
\end{aligned}
$$

Improving (E4): Similarly to the improvement of (E5) we observe that

$$
\begin{aligned}
G_k(T) - \exp&\left(-\nu\int_{t_k}^T k^2+(\eta-kt)^2dt\right)G_k(t_k)\\
&-\int_{t_k}^T \exp(-\nu\int_t^T k^2+(\eta-k\tau)^2d\tau)2\frac{(\frac{\eta}{k}-t)}{(1+(\frac{\eta}{k}-t)^2)^2}\theta_k dt\\
&=\int_{t_k}^T \exp(-\nu\int_t^T k^2+(\eta-k\tau)^2d\tau)f\frac{\nu}{g}ik(d_k^+G_{k+1}+d_k^-G_{k-1})dt\\
&\quad+\int_{t_k}^T \exp(-\nu\int_t^T k^2+(\eta-k\tau)^2d\tau)f\frac{\nu}{g}ik(c_k^+\theta_{k+1}+c_k^-\theta_{k-1})dt\\
&\quad+\int_{t_k}^T \exp(-\nu\int_t^T k^2+(\eta-k\tau)^2d\tau)\frac{1}{1+(\frac{\eta}{k}-t)^2}\left(c_k^+\theta_{k+1}+c_k^-\theta_{k-1}\right)dt\\
&\quad+\int_{t_k}^T \exp(-\nu\int_t^T k^2+(\eta-k\tau)^2d\tau)\frac{1}{1+(\frac{\eta}{k}-t)^2}\left(d_k^+G_{k+1}+d_k^-G_{k-1}\right)dt.
\end{aligned}
$$

The contributions on the right-hand-side are then controlled by (26), (27) and (28) of Lemma 3.5 and the bootstrap assumptions and thus (E4) improves.

Thus, all bootstrap estimates improve and hence remain valid at least until time t_{k-1}.

Establishing the lower bound (35) and (34): As the last step of our proof we show that the bootstrap estimates (E1)–(E5) at time t_{k-1} imply the desired lower bound and that the solution satisfies (34) at that time.

Here we first observe that by (E1), (E2), (E4), (E5) all modes except $\theta_{k\pm 1}$ are bounded above by 4.

We next study (E3):

$$\begin{aligned}|\theta_{k\pm1}(T) - \theta_{k\pm1}(t_k) - \int_{t_k}^{T} \frac{c\eta}{k^3} \frac{1}{(1+(\frac{\eta}{k}-t)^2)^2}\theta_k(t)dt \\ - \int_{t_k}^{T} \frac{c\eta}{k^3} \frac{1}{1+(\frac{\eta}{k}-t)^2} G_k(t)dt| \leq 0.1\end{aligned}$$

Here it holds that

$$\int_{t_k}^{t_{k-1}} \frac{c\eta}{k^3} \frac{1}{(1+(\frac{\eta}{k}-t)^2)^2} \approx \frac{c\eta\pi}{2k^3} \geq 32, \tag{37}$$

within a factor 1.2, since θ_k is controlled by (E1) and $|t_k - \frac{\eta}{k}|, |t_{k-1} - \frac{\eta}{k}|$ are sufficiently large to approximate by the integral over all of $\mathbb{R}$.

It thus only remains to show that the contribution by

$$\int_{t_k}^{t_{k-1}} \frac{c\eta}{k^3} \frac{1}{1+(\frac{\eta}{k}-t)^2} G_k(t)dt$$

does not cancel this growth. That is, this integral should not be close to

$$\int_{t_k}^{t_{k-1}} \frac{c\eta}{k^3} \frac{1}{(1+(\frac{\eta}{k}-t)^2)^2} \theta_k(t)dt.$$

We here recall that by (E4)

$$\begin{aligned}&|G_k(T) - \exp(\nu \int_{t_k}^{T} k^2 + (\eta - ks)^2 ds) G_k(t_k) \\ &\quad - \int_{t_k}^{T} \exp(-\nu \int_{t_k}^{T} k^2 + (\eta-ks)^2 ds) \frac{2(\frac{\eta}{k}-t)}{(1+(\frac{\eta}{k}-t)^2)^2}\theta_k(t)dt| \leq 0.1.\end{aligned}$$

Hence $G_k(T)$ is determined by $G_k(t_k)$ and $\theta_k(t_k)$ up to a negligible error (compared to (37)).

Let us first discuss

$$\int_{t_k}^{t_{k-1}} \frac{c\eta}{k^3} \frac{1}{1+(\frac{\eta}{k}-t)^2} \exp(\nu \int_{t_k}^{T} k^2 + (\eta-ks)^2 ds) G_k(t_k) dt.$$

Here, by assumption (34), we know that

$$\exp(-\nu \int_{t_k}^{T} k^2 + (\eta-ks)^2 ds)|G_k(t_k)| \leq |G_k(t_k)| \leq \frac{1}{4}.$$

and hence

$$\int_{t_k}^{t_{k-1}} \frac{c\eta}{k^3} \frac{1}{1+(\frac{\eta}{k}-t)^2} \exp(-\nu \int_{t_k}^{t} k^2 + (\eta-ks)^2 ds) G_k(t_k) dt \leq \frac{1}{2} \frac{c\eta\pi}{k^3 2}$$

is smaller than (37) regardless of ν. Moreover, if k and ν are such that νk^2 is large, then we may instead control

$$\begin{aligned}\int_{t_k}^{t_{k-1}} \frac{c\eta}{k^3}\frac{1}{1+(\frac{\eta}{k}-t)^2} \exp(-\nu \int_{t_k}^{t} k^2+(\eta-ks)^2 ds) G_k(t_k) \\ \leq \frac{c\eta}{k^3}\frac{1}{\nu k^2}|G_k(t_k)|,\end{aligned}$$

which is smaller than (37) provided $\frac{1}{\nu k^2}$ is much smaller than π.

It hence remains to discuss

$$\int_{t_k}^{T} \exp(-\nu \int_{t_k}^{T} k^2+(\eta-ks)^2 ds)\frac{2(\frac{\eta}{k}-t)}{(1+(\frac{\eta}{k}-t)^2)^2}\theta_k(t)dt.$$

Here we observe that

$$\int_{t_k}^{T} \frac{2(\frac{\eta}{k}-t)}{(1+(\frac{\eta}{k}-t)^2)^2}dt = \frac{1}{1+(\frac{\eta}{k}-T)^2} - \frac{1}{1+(\frac{\eta}{k}-t_k)^2}$$

and that

$$G_k(t) = -\frac{1}{1+(\frac{\eta}{k}-t)^2}\theta_k(t)$$

would imply that the contribution by $G_k(t)$ exactly cancels (37). Thus, we need to make use of the decay due to the dissipation to rule out such cancellation. Indeed, bounding

$$\int_{t_k}^{T} k^2+(\eta-ks)^2 ds \geq k^2(T-t_k),$$

we may control

$$\int_{t_k}^{T} \exp(-\nu \int_{t_k}^{T} k^2+(\eta-ks)^2 ds)|\frac{2(\frac{\eta}{k}-t)}{(1+(\frac{\eta}{k}-t)^2)^2}| \leq \frac{2}{\nu k^2},$$

which is much smaller than 1 by assumption on ν and k.

Thus, in summary

$$|\theta_{k\pm1}(t_{k-1})| \approx \frac{c\eta\pi}{2k^3}|\theta_k(t_k)| \geq 32$$

as claimed in (35) and all other modes are bounded above by 4 and thus also (34) holds. □

4.4. **Persistence of Lower Bounds.** In this subsection we consider the evolution on the time interval

$$(t_{k_3}, \infty),$$

where we recall that the evolution after time $t_0 = 2\eta$ is asymptotically stable by Theorem 3.1. As remarked in Sections 4.2 and 4.3 here we do not anymore derive lower bounds on the norm inflation since a priori there could be cancellation. Instead we derive upper bounds on the possible growth, which then allow us to also prove lower bounds on $\theta_{k_3+2}(t)$ for all times $t \geq t_{k_3}$. Thus norm inflation persists for all times.

Lemma 4.6. *Suppose that at the time t_{k_3} it holds that*

$$|\theta_{k_3+2}| \geq 0.5\|\theta\|_{\ell^\infty} + 10\|G\|_{\ell^\infty}.$$

and for $k \leq k_3$ define

$$C_{k_3} = 2|\theta_{k_3+2}(t_{k_3})|,$$
$$C_{k-1} = (1 + (\frac{\eta}{k^2})^{-1})C_k.$$

Then for all $k \leq k_3$ it holds that

$$\begin{aligned} |\theta_l(t_k)| &\leq C_k, \text{ for } l \geq k_3+2, \\ |\theta_l(t_k)| &\leq C_k \prod_{j=k}^{k_1} \frac{c\eta\pi}{j^3}, \text{ for } k_3+1 > l > k+1, \\ |\theta_l(t_k) &\leq C_k \prod_{j=k+1}^{k_1} \frac{c\eta\pi}{j^3}, \text{ for } l \leq k, \\ |G_l(t_k)| &\leq C_k(\frac{\eta}{k^2})^{-2} \prod_{j=k}^{k_1} \frac{c\eta\pi}{j^3}, \text{ for } k_3+1 > l > k+1, \\ |G_l(t_k)| &\leq 2C_k \prod_{j=k+1}^{k_1} \frac{c\eta\pi}{j^3}, \text{ for } l \leq k. \end{aligned} \tag{38}$$

Moreover, the bounds for t_0 persist for all times $t > t_0$ up to a possible growth by a constant $(1+c)$ and for all $t \geq t_{k_2}$ it holds that

$$|\theta_{k_3+2}(t)| \geq e^{-2}|\theta_{k_3+2}(t_{k_3})|.$$

As in Section 4.2 we here note that the upper bounds (38) have a "dent", where θ_{k+2} and θ_k satisfy the same upper bounds and θ_{k-1} is potentially much smaller (see Figure 5). Here we further observe that the resonance mechanism during a time interval I_k may only cause the modes $k-1, k+1$ to exhibit large change, while modes larger than $k+1$ remain mostly unchanged. In particular, the mode k_3+2 only mildly changes after time t_{k_3} and hence the lower bound persists. We further

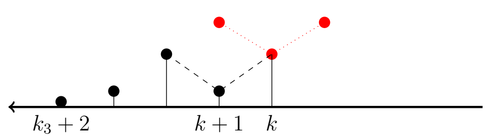

FIGURE 5. Growth bounds after time t_{k_3}. Following a similar strategy as in Section 4.2 we obtain upper bounds for the resonances during the time intervals I_k. Here the mode θ_k highlighted in red may cause the modes θ_{k+1} and θ_{k-1} to grow, thus resulting in a new dent.

remark that the assumptions at the time t_{k_3} are ensured by Proposition 4.5 of the preceding section. This lemma hence proves that the lower bound on θ_{k_3+2} persists until time t_0 up to a loss of constant. Thus norm inflation in ℓ^∞ (and hence also in ℓ^2) has been achieved at that time. Moreover, this lower bound then persists also for all future times by the same argument as in Section 3.1.

Proof of Lemma 4.6. After normalization we may assume that $C_{k_3} = 1$ and deduce that deduce that $C_k \leq e^{\frac{1}{1000}}$ for all k. Hence, supposing that the claimed estimates hold we observe that for all $t \geq t_{k_2}$ it holds that

$$|\theta_{k_3+1}| \leq e^{\frac{1}{1000}} \frac{c\eta\pi}{k_3^3},$$
$$|\theta_{k_3+3}| \leq e^{\frac{1}{1000}}.$$

Since $k_3 + 2$ is non-resonant on the interval (t_{k_3}, ∞) this then implies that

$$|\theta_{k_3+2}(t) - \theta_{k_3+2}(t_{k_3})| \ll 1$$

and thus $\theta_{k_3+2}(t) \approx \theta_{k_3+2}(t_{k_3})$, as desired.

The proof of the induction step for (38) follows by the same argument as in the proofs of Proposition 4.5 and of Lemma 4.4. In particular, we again establish slightly rougher upper bounds for the mode G_k on $(t_k, t_{k-1} - 3)$ and then recover the desired bounds by using the fast exponential decay. We omit the details for brevity. We further remark that for $t \geq t_0$ all integrals are non-resonant (see also Section 3.1). Thus we may use the same proof on the interval (t_0, ∞).

This concludes the proof of Proposition 4.2 and thus of Theorem 4.1. □

In this article we have shown that, despite viscous dissipation, the Boussinesq equations linearized around *traveling waves* exhibit norm inflation and blow-up. Yet, for certain critical data we can prescribe the blow-up in a fine enough way that damping of the velocity field persists. These results hence show that damping of the *velocity* field is a more robust effect than asymptotic stability of the *vorticity* and *temperature*. Moreover, while classically considered a nonlinear effect, we show that *echoes* and their corresponding resonances are a feature of the linearized problem around traveling waves.

Acknowledgments. Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173.

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