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# Numerical methods for scattering problems from multi-layers with different periodicities

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**Abstract**

In this paper, we consider a numerical method to solve scattering problems with multi-periodic layers with different periodicities. The main tool applied in this paper is the Bloch transform. With this method, the problem is written into an equivalent coupled family of quasi-periodic problems. As the Bloch transform is only defined for one fixed period, the inhomogeneous layer with another period is simply treated as a non-periodic one. First, we approximate the refractive index by a periodic one where its period is an integer multiple of the fixed period, and it is decomposed by finite number of quasi-periodic functions. Then the coupled system is reduced into a simplified formulation. A convergent finite element method is proposed for the numerical solution, and the numerical method has been applied to several numerical experiments. At the end of this paper, relative errors of the numerical solutions will be shown to illustrate the convergence of the numerical algorithm.

**KEYWORDS**

multi-periodic layers, numerical methods, scattering problems

## 1 | INTRODUCTION

The easier case, for example, when either the periodicities are the same, or the quotient of the periodicities is rational, the problem is naturally reduced into a problem with one periodic layer, which is

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easily treated in the classic for quasi-periodic scattering problems [1, 2]. However, if the quotient of the periodicities is either irrational or extremely large/small, the problem becomes much more complicated. For the first case, the original problem is impossible to be reduced into any bounded domain naturally, thus it is a scattering problem with unbounded inhomogeneous medium; while for the second case, although the problem could be reduced into one periodic cell, the cell will be very large. For both cases, numerical simulations of these problems are very challenging.

Scattering problems with unbounded structures been investigated by many mathematicians in decades. Based on the integral equation method, the well-posedness of these scattering problems has been established [3–6], and numerical methods have been proposed for rough surface scattering problems [7–9]. The variational method, on the other hand, has also been applied to theoretical analysis of scattering from unbounded obstacles [10–13]. An important extension of the variational method is to consider the well-posedness in weighted Sobolev spaces [14], and more generalized cases (e.g., incident plane waves) are included. Similar results in weighted Sobolev spaces been shown for more generalized boundary conditions in [15].

As far as the author knows, the first paper that adopted this method is [16] for scattering problems with locally perturbed periodic. Inspired by this paper, the method has been extended to scattering problems with non-periodic incident fields with (locally perturbed) periodic surfaces [17–19]. Based on the theoretical results, Bloch-transform based numerical methods have been proposed [19–21]. The Bloch transform was also applied to other cases, that is, scattering problems in locally perturbed periodic waveguides [22]. For all these works listed above, the perturbations of periodic are assumed to be compactly supported. In this case, the Bloch transformed problem has a simplified variational form. However, for more general cases, that is, when the perturbations are non-compactly supported, the problems become much more complicated and difficult to be dealt with. Further study on the Bloch transform is then required for the globally perturbed problems.

In this paper, the Bloch transformed scattering problems from different periodic layers in  $\mathbb{R}^2$  will be investigated. The original problem is approximated by a new one with a periodic layer, and the weak formulation for the Bloch transformed new problem is established, and the equivalence, well-posedness and regularity results are proved following [1]. Based on the weak formulation, the numerical method will be introduced. The key step is the approximation of periodic inhomogeneous media by a finite series of quasi-periodic functions with another different period. The inhomogeneous media is first approximated by a periodic one with a relatively larger period, and the compactly supported function is then approximated by a finite Fourier series. With the method inspired by the decomposition (52) in [23], the Fourier series is written into the sum of finite number of quasi-periodic functions.

The rest of the paper is organized as follows. In Section 2, we will describe the mathematical model of the scattering problems and show the well-posedness of the problem. In Section 3, we approximate the original scattering problem by replacing the inhomogeneous layer with a periodic one. Then we apply the Bloch transform to the new problem in Section 4. In Sections 5 and 6, we formulate the discretization of the transformed problem. Finally, we show some numerical examples in the last section.

## 2 | SCATTERING PROBLEMS: MATHEMATICAL MODEL

In this section, we describe the mathematical model for scattering problems with periodic layers with different periods in two dimensional spaces (see Figure 1). Let the straight line  $\Gamma_h := \mathbb{R} \times \{h\}$  for any  $h \in \mathbb{R}$ , and assume that  $\Gamma_{H_0}$  where  $H_0 > 0$  is a sound-soft surface. Define the domains by

$$D := \mathbb{R} \times (H_0, \infty); \quad D_H = \mathbb{R} \times (H_0, H)$$

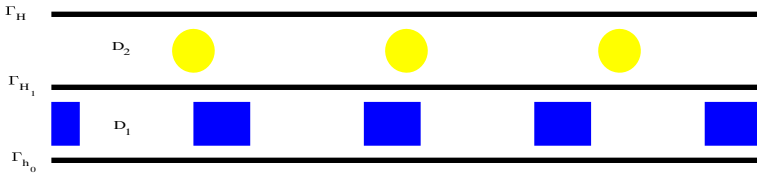


FIGURE 1 Inhomogeneous layers with different periodicities

where  $H > H_0$ . Assume that the infinite layer is embedded in  $D_H$  for some fixed positive number  $H$ , and it is divided into two layers by a straight line  $\Gamma_{H_1}$ , for some  $H_1 \in (H_0, H)$ . Let  $D_1 = \mathbb{R} \times (H_0, H_1)$  and  $D_2 = \mathbb{R} \times (H_1, H)$ . Let

$$n(x_1, x_2) = \begin{cases} n_1(x_1, x_2), & \text{when } x \in D_1; \\ n_2(x_1, x_2), & \text{when } x \in D_2; \\ 0, & \text{when } x_2 \geq H, \end{cases}$$

where  $n_1$  and  $n_2$  both periodic functions in  $x_1$ -direction. The period of  $n_1$  is  $\Lambda_1 > 0$  and that of  $n_2$  is  $\Lambda > 0$ . We simply assume that  $\Lambda_1 \neq \Lambda$  without further conditions.

*Remark 1*  $n$  is simply assumed to be in the space  $L^\infty(D)$ . However, to guarantee the convergence of the numerical method, we may assume that the refractive index has a higher regularity later.

Consider a scattering problem with an inhomogeneous medium, which is by the Helmholtz equation with a Dirichlet boundary condition on  $\Gamma_{H_0}$ :

$$\Delta u^s + k^2(1+n)u^s = -k^2nu^i \quad \text{in } D, \quad u^s = -u^i \text{ on } \Gamma_{H_0}, \tag{1}$$

where the incident field  $u^i \in H^1(D)$  satisfies  $\Delta u^i + k^2u^i = 0$  in  $D$ . To guarantee that the scattered field  $u^s$  is upward propagating, it is required that  $u^s$  satisfies the following boundary condition on  $\Gamma_H$

$$\frac{\partial u^s}{\partial x_2}(x_1, H) = T^+ [u^s|_{\Gamma_H}], \tag{2}$$

where  $T^+$  is the Dirichlet-to-Neumann map from  $H^{1/2}(\Gamma_H)$  to  $H^{-1/2}(\Gamma_H)$  (see [10]) defined by

$$T^+ \varphi = \frac{i}{2\pi} \int_{\mathbb{R}} \sqrt{k^2 - |\xi|^2} e^{ix_1 \cdot \xi} \widehat{\varphi}(\xi) d\xi, \quad \text{where } \varphi = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix_1 \cdot \xi} \widehat{\varphi}(\xi) d\xi. \tag{3}$$

Let  $u = u^s + u^i \mathcal{X}(x_2)$  where  $\mathcal{X}(x_2)$  is a smooth cutoff function that equals to 1 when  $x_2$  is close to  $H_0$  and vanishes when  $x_2 > H - \delta$  for some  $\delta > 0$ . Then  $u$  satisfies the following equations:

$$\Delta u + k^2(1+n)u = g \quad \text{in } D, \quad u = 0 \text{ on } \Gamma_{H_0}, \tag{4}$$

where  $g = k^2(\mathcal{X}(x_2) - 1)nu^i + 2\frac{\partial u^i}{\partial x_2}\mathcal{X}'(x_2) + u^i\mathcal{X}''(x_2)$  is the source term supported in  $D_H$ , with the boundary condition (2) on  $\Gamma_H$ . The scattering problem is now formulated into the following variational problem, that is, given any  $g \in L^2(D_H)$ , a solution  $u \in \widetilde{H}^1(H)$  such that

$$\int_{D_H} [\nabla u \cdot \nabla \bar{v} - k^2(1+n)u\bar{v}] dx - \int_{\Gamma_H} T^+ [u|_{\Gamma_H}] \bar{v} ds = - \int_{D_H} g\bar{v} dx, \tag{5}$$

for all  $v \in \widetilde{H}^1(D_H)$  with compact support in  $\overline{D_H}$ . Note that the tilde in  $\widetilde{H}^1(D_H)$  shows that the functions in this space belong to  $H^1(D_H)$  and satisfy the homogeneous Dirichlet boundary condition on  $\Gamma_{H_0}$ .

Similar notations are adopted for other spaces,  $\tilde{H}_r^1(D_H)$  and  $H_0^r(W_{\Lambda^*}; \tilde{H}_\alpha^s(D_H^\Lambda))$ , in the following parts.

Following [14], we consider the solution of the scattering problem in weighted Sobolev spaces. Define the weighted Sobolev space in  $D_H$  for any fixed  $r \in \mathbb{R}$  by:

$$H_r^s(D_H) := \left\{ \varphi \in \mathcal{D}'(D_H) : (1 + |x|^2)^{r/2} \varphi(x) \in H^s(D_H) \right\}.$$

The definitions for  $H_r^{1/2}(\Gamma_H)$  and  $H_r^{-1/2}(\Gamma_H)$  are similar. From [14] again, the operator  $T^+$  is bounded from  $H_r^{1/2}(\Gamma_H)$  to  $H_r^{-1/2}(\Gamma_H)$  for any  $|r| < 1$ , thus the left-hand-side of (5) is a bounded sesquilinear form defined in  $H_r^1(D_H) \times H_{-r}^1(D_H)$ . For any  $g \in L_r^2(D_H)$ , we are looking for a solution  $u \in \tilde{H}_r^1(D_H)$  such that (5) holds for any  $v \in \tilde{H}_{-r}^1(D_H)$ . From Riesz's lemma, there is a bounded linear operator depending on  $n$ , that is,  $\mathcal{B}_r(n) : H_r^1(D_H) \rightarrow (H_r^1(D_H))^*$ , such that

$$\int_{D_H} [\nabla u \cdot \nabla \bar{v} - k^2(1+n)u\bar{v}] dx - \int_{\Gamma_H} T^+ [u|_{\Gamma_H}] \bar{v} ds = (\mathcal{B}_r(n)u, v)_{(H_{-r}^1(D_H))^* \times H_{-r}^1(D_H)}.$$

Especially, when  $n = 0$  in  $D$ , the problem is reduced to the scattering problem from the sound soft surface  $\Gamma_{H_0}$  with homogeneous media in  $D$ . The well-posedness for this problem in the space  $\tilde{H}_r^1(D_H)$  has been proved in [14], thus the operator  $\mathcal{B}_r(0)$  is invertible. Then the operator

$$\mathcal{B}_r(n) := \mathcal{B}_r(0) + [\mathcal{B}_r(n) - \mathcal{B}_r(0)]$$

is a perturbation of the isomorphism  $\mathcal{B}_r(0)$ . The perturbation  $\mathcal{B}_r(n) - \mathcal{B}_r(0)$  satisfies

$$([\mathcal{B}_r(n) - \mathcal{B}_r(0)]u, v)_{(H_{-r}^1(D_H))^* \times H_{-r}^1(D_H)} = -k^2 \int_{D_H} nu\bar{v} dx.$$

**Lemma 2** *The operator  $\mathcal{K}_r(n) := \mathcal{B}_r(n) - \mathcal{B}_r(0)$  is bounded from  $H_r^1(D_H)$  to  $(H_{-r}^1(D_H))^*$ , and the norm is bounded by*

$$\|\mathcal{K}_r(n)\| \leq k^2 \|n\|_\infty,$$

where  $\|\cdot\|$  is the operator norm and  $\|\cdot\|_\infty$  is the  $L^\infty(D_H)$  norm.

The proof is trivial thus is omitted.

As  $\mathcal{B}_r(0)$  is invertible and  $\mathcal{K}_r(n)$  is bounded by  $k^2 \|n\|_\infty$ , when  $k^2 \|n\|_\infty$  is small enough,  $\mathcal{B}_r(n) = \mathcal{B}_r(0) + \mathcal{K}_r(n)$  is invertible. We conclude the well-posedness result for (5) in the following theorem.

**Theorem 3** *Suppose  $k^2 \|n\|_\infty$  is small enough, that is,  $k^2 \|n\|_\infty \leq \|\mathcal{B}_r(0)^{-1}\|^{-1}$ . Given any function  $g \in L_r^2(D_H)$  for some fixed  $|r| < 1$ , the variational problem (5) is uniquely solvable in the space  $\tilde{H}_r^1(D_H)$ . Moreover, there is a constant that depends on  $k$  and  $n$  such that*

$$\|u\|_{H_r^1(D_H)} \leq C \|g\|_{L_r^2(D_H)}. \quad (6)$$

**Remark 4** The condition in Theorem 3 is not optimal. In fact, a number of research papers are devoted to the well-posedness of the scattering problems from rough layers, for details we refer to [11, 13, 24, 25] for Helmholtz equations and [26, 27] for Maxwell's equations. However, in this paper, as we are only interested in the numerical solutions for this kind of, we simply assume that  $k^2 \|n\|_\infty$  is small enough to guarantee that the problem (5) is uniquely solvable with the stability result addressed in Theorem 3.

*Remark 5* The well-posedness and stability results in Theorem 3 could be extended to other boundary conditions on  $\Gamma_{H_0}$ , for example, impedance boundary conditions (see [28]) or transmission, if the well-posedness in weighted Sobolev spaces could be proved when  $n = 0$ . As  $g$  in (4) is defined by  $u^i$ , we only need to consider the original problem

$$\Delta u^s + k^2 u^s = 0 \quad \text{in } D, \quad \frac{\partial u^s}{\partial x_2}(x_1, H) = T^+ [u^s|_{\Gamma_H}] \quad \text{on } \Gamma_H.$$

First consider the impedance boundary condition. Then

$$\frac{\partial u^s}{\partial x_2} + i\lambda u^s = -\frac{\partial u^i}{\partial x_2} - i\lambda u^i \quad \text{on } \Gamma_{H_0}$$

where  $\lambda > 0$ . Given any  $u^i \in H_r^1(D_H)$ , the solution  $u^s$  as

$$u^s(x_1, x_2) = \frac{1}{2\pi} \int \frac{\sqrt{k^2 - \xi^2} - \lambda}{\sqrt{k^2 - \xi^2} + \lambda} e^{i\xi x_1 + i\sqrt{k^2 - \xi^2}(x_2 - H_0)} \widehat{u^i}(\xi, H_0) d\xi,$$

where  $\widehat{u^i}(\xi, H_0)$  is the Fourier transform of  $u^i(\cdot, H_0)$ . Note that the decaying rate of a function depends on the regularity of the Fourier transform, it is easily checked that if  $\widehat{u^i}(\xi, H_0)$  has square root singularities when  $\xi = \pm k$ ,  $u^s \in H_r^1(D_H)$ .

We can use the same method to find the analytic formulation of the solution with respect to transmission conditions, and the decaying rate of  $u^s$  can be obtained in the same way.

### 3 | APPROXIMATION OF SOLUTIONS WITH UNBOUNDED REFRACTIVE INDEX

#### 3.1 | Volume integral equation

In this section, we introduce the Lippmann–Schwinger equation to estimate the decay of the solutions. First, define the volume potential on the unbounded domain  $D_H$  as

$$(V\phi)(x) := \int_{D_H} G(x, y)\phi(y) dy, \quad x \in D_H,$$

where  $G(x, y)$  is the half-space Green’s function defined by

$$G(x, y) = \frac{i}{4} \left[ H_0^{(1)}(k|x - y|) - H_0^{(1)}(k|x - y'|) \right]$$

for  $y = (y_1, y_2)^T$  and  $y' = (y_1, 2H_0 - y_2)^T$ .  $H_0^{(1)}$  is the Hankel function of the first kind. For simplicity, we define the space

$$C_r^\ell(D) := \{ \varphi \in C^\ell(D) : \nabla^j \varphi(x) = O(|x|^{-r}), |x| \rightarrow \infty \text{ for all } j = 0, 1, \dots, \ell \},$$

where the index  $r$  indicates the decay/increase rate of the functions, when  $|x| \rightarrow \infty$ . First we recall the decay property of the Green’s function and its derivatives, for details we refer to [3].

**Lemma 6** *The Green’s function and its first- and second-order derivatives decay as*

$$|G(x, y)|, |\nabla G(x, y)|, |\nabla^2 G(x, y)| \sim |x_1 - y_1|^{-3/2}.$$

*Proof.* The decay of the Green’s function and its first order derivatives comes from [3]. For the second order derivatives, the results can be proved in the similar way, thus is omitted. ■

We also extend the result of theorem 8.2, [29] to unbounded domains.

**Theorem 7** Given  $\varphi \in L^2(D_H)$ ,  $v = V\phi \in H_{loc}^2(D_H)$ .

*Proof.* For any bounded domain  $S \subset D_H$ , let  $S \subset (-L, L) \times (H_0, H) \subset D$  where  $L$  is large such that  $\sup_{x \in S, y \in D_H \setminus (-L, L) \times (H_0, H)} \geq \delta > 0$ . Then

$$v = \int_{(-L, L) \times (H_0, H)} G(x, y)\varphi(y) dy + \int_{D_H \setminus (-L, L) \times (H_0, H)} G(x, y)\varphi(y) dy := v_1 + v_2.$$

First, from theorem 8.2, [29], there is a constant  $C > 0$  such that

$$\|v_1\|_{H^2(S)} \leq C\|\varphi\|_{L^2((-L, L) \times (H_0, H))}.$$

For the second term, as  $G(x, y)$  is analytic for  $x \in S$ , we only need to consider the decay of  $G(x, y)$  and its derivatives. From Lemma 6, the Green's functions and its first- and second-order derivatives decay at the rate of (or even faster than)  $|x_1 - y_1|^{-3/2}$ , the integral is well defined. Thus  $v, \nabla v$  and  $\nabla^2 v$  are all bounded in  $S$ . Thus  $v \in H_{loc}^2(D_H)$ . The proof is finished. ■

The mapping property of  $V$  is concluded in the following theorem, for details we refer to [30].

**Theorem 8** (Corollary 8, [30])  $V$  is a bounded operator in  $C_{3/2}(D_H)$ .

The result is also extended to first- and second order derivatives of  $V$ , from Lemma 6:

**Corollary 9** If  $\varphi \in C_{3/2}(D_H)$ ,  $V\phi, \nabla V\phi, \nabla^2 V\phi$  belong to the space  $C_{3/2}(D_H)$ .

Then we show that the scattering problem (1) and (2) is equivalent to the following Lippmann–Schwinger equation:

$$u(x) + k^2 \int_{D_H} G(x, y)n(y)u(y) dy = \int_{D_H} G(x, y)g(y) dy, \quad x \in D. \quad (7)$$

**Theorem 10** If  $u \in H_{loc}^2(D) \cap C_{3/2}(D)$  is a solution of (1) and (2), then  $u$  is a solution of (7).

For proof we refer to Theorem 3.3 in [3]. To guarantee (7) is uniquely solvable, we make the following assumption.

**Assumption 11** Suppose  $\|n\|_\infty$  is so small such that (7) is uniquely solvable in  $H_{loc}^2(D) \cap C_{3/2}(D)$ . We also assume that  $k^2\|n\|_\infty < \|\mathcal{B}_r(0)^{-1}\|^{-1}$ , such that (5) is uniquely solvable in  $H_r^1(D_H)$ .

As  $u$  is the solution to the Lippmann–Schwinger equation (7), we can estimate its decay from the right hand side. Extend the result of Corollary 4.5, [31], when  $u$  solves the equation with the right hand side decays as  $O(|x_1|^{-3/2})$ ,  $u$  decays as  $O(|x_1|^{-3/2})$  as well. Then we arrive at the final result in this section.

**Theorem 12** Let Assumption 11 holds,  $n \in L^\infty(D)$  and  $g \in H_r^1(D_H) \cap C_{3/2}(D_H)$ . The variational problem (5) and The Lippmann-Schwinger equation (7) are equivalent. Moreover, let  $u$  be the unique solution, then  $u \in H_{loc}^2(D_H)$  is the solution of (1) and (2). Moreover,  $u, \nabla u, \nabla^2 u \in C_{3/2}(D_H)$ .

### 3.2 | Approximation

To solve a problem defined in an unbounded domain, to approximate it by one defined in a bounded. However, for this case, to approximate it by a periodic one. Thus we fix one periodic layer, and modify another layer based on the period of the layer.

Let  $N > 0$  be a sufficiently large integer, and the smooth cutoff function  $\mathcal{X}(t)$  satisfies

$$\mathcal{X}(t) = \begin{cases} 1, & |t| \leq N\Lambda/4; \\ 0, & |t| \geq N\Lambda/2; \\ \text{smooth,} & \text{otherwise.} \end{cases}$$

We define a new function by

$$n_1^N(x_1, x_2) := n_1(x_1, x_2) \mathcal{X}(x_1), \quad -\frac{N\Lambda}{2} \leq x_1 \leq \frac{N\Lambda}{2}.$$

We extend  $n_1^N$  into an  $N\Lambda$ -periodic function in  $x_1$ -direction, and it is still denoted by  $n_1^N$ . Let

$$n_N(x) = \begin{cases} n_1^N(x), & \text{when } x \in D_1; \\ n_2(x), & \text{when } x \in D_2; \\ 0, & \text{when } x_2 \geq H. \end{cases}$$

As  $n_2$  is  $\Lambda$ -periodic and  $n_1^N$  is  $N\Lambda$ -periodic, the function  $n_N$  is  $N\Lambda$ -periodic as well. Define  $D_H^N := \{x \in D_1 : |x_1| \leq N\Lambda/4\} \cup D_2$ , then  $n = n_N$  when  $x \in D_H^N$ . When  $x \in D_H \setminus D_H^N$ ,

$$\|n_N\|_\infty \leq \|n\|_\infty; \quad \|n - n_N\|_\infty \leq 2\|n\|_\infty.$$

We consider the new variational problem, with  $n$  replaced by  $n_N$  in (5). Give any  $g \in L_r^2(D_H)$ , we are looking for a solution  $u_N \in \tilde{H}_r^1(D_H)$  such that

$$\int_{D_H} [\nabla u_N \cdot \nabla \bar{v} - k^2 (1 + n_N) u_N \bar{v}] dx - \int_{\Gamma_H} T^+ [u_N|_{\Gamma_H}] \bar{v} ds = - \int_{D_H} g \bar{v} dx \tag{8}$$

holds for any  $v \in \tilde{H}_r^1(D_H)$ . From the definition of  $\mathcal{B}_r(n)$ , the left hand side is equivalent to  $(\mathcal{B}_r(n_N) u_N, v)_{(H_{-r}^1(D_H))^* \times H_{-r}^1(D_H)}$ . We obtain the invertibility of  $\mathcal{B}_r(n_N)$  in the following theorem.

**Theorem 13** *Suppose Assumption 11 is satisfied. For any  $g \in L_r^2(D_H)$ , there is a unique solution  $u_N \in \tilde{H}_r^1(D_H)$  such that (8) is satisfied. Moreover,*

$$\|u_N\|_{\tilde{H}_r^1(D_H)} \leq C \|g\|_{L_r^2(D_H)} \tag{9}$$

*holds uniformly for  $N \in \mathbb{N}$ , where  $C$  is the same as that in (6).*

Let  $\delta u_N := u - u_N$ , then it satisfies

$$\Delta \delta u_N + k^2 (1 + n_N) \delta u_N = k^2 (n_N - n) u \text{ in } D, \quad \delta u_N = 0 \text{ on } \Gamma_{H_0}$$

with the boundary condition (2) on  $\Gamma_H$ . As  $n - n_N \in L^\infty(D)$  and  $u \in H_r^1(D_H)$ ,  $k^2 (n - n_N) u \in H_r^0(D_H)$ . From Theorem 13, the problem is uniquely solvable and  $\delta u_N \in \tilde{H}_r^1(D_H)$ . To study the decay property of  $\delta u_N$  with respect to  $N$ , first we need the following lemma.

**Lemma 14** *If  $\text{supp}(\varphi) \subset (\mathbb{R} \setminus [-L, L]) \times (H_0, H)$  for a sufficient large  $L > 0$ , and  $\varphi \in C_{3/2}(D_H)$ , then the function  $\|V\phi\|_{L^\infty(D_H)} \leq CL^{-3/2}$ .*

*Proof.* From Theorem 8,  $V\phi \in C_{3/2}(D_H)$ . Thus for  $|x_1| \geq L/2$ ,

$$|(V\phi)(x)| \leq C(L/2)^{-3/2} = CL^{-3/2}.$$

We only consider the case that  $|x_1| \leq L/2$ . For  $|y_1| \geq L$ ,  $|x_1 - y_1| \geq |y_1|/2$ . Then

$$\begin{aligned} |(V\phi)(x)| &= \left| \int_{D_H} G(x, y)\varphi(y) \right| \leq C \int_{\mathbb{R} \setminus [-L, L] \times (H_0, H)} |x_1 - y_1|^{-3/2} |y_1|^{-3/2} dy \\ &\leq C \int_{H_0}^H \int_{\mathbb{R} \setminus [-L, L]} |y_1|^{-3} dy_1 dy_2 \leq CL^{-2} \leq CL^{-3/2}. \end{aligned}$$

The proof is finished.  $\blacksquare$

Now we apply this lemma to the following estimation of  $\delta u_N$ .

**Theorem 15** *When Assumption 11 holds, then the problems (5) and (8) are uniquely solvable solutions, respectively. There is a constant  $C > 0$  such that  $|\delta u_N(x)| \leq CN^{-3/2}$  for any  $x \in D_H$ , where  $C$  does not depend on  $N$ .*

*Proof.* the solution to the following Lippmann-Schwinger equation:

$$\delta u_N(x) + k^2 \int_{D_H} G(x, y)\tilde{n}(y)\delta u_N(y) dy = k^2 \int_{D_H} G(x, y)(\tilde{n} - n)(y)u(y) dy.$$

From Assumption 11, the above equation is uniquely solvable. From Lemma 14, as  $u \in C_{3/2}(D_H)$ , the right hand side is bounded by  $CN^{-3/2}$ . From the well-posedness of the problem, the proof is finished.  $\blacksquare$

**Corollary 16** *For any bounded domain  $S \subset D_H$ ,  $\|\delta u_N\|_{L^2(S)} \leq CN^{-3/2}$ .*

#### 4 | THE BLOCH TRANSFORM OF THE APPROXIMATED PROBLEM

In this section, we apply the Bloch transform (for its definition see Appendix) to analyze the approximated problem (8). The periodic cell for  $x_1$ , which is called the Wigner–Seitz-cell, and the dual cell, called Brillouin zone, are defined by

$$W_\Lambda := \left(-\frac{\Lambda}{2}, \frac{\Lambda}{2}\right], \quad W_{\Lambda^*} := \left(-\frac{\Lambda^*}{2}, \frac{\Lambda^*}{2}\right) = \left(-\frac{\pi}{\Lambda}, \frac{\pi}{\Lambda}\right),$$

where  $\Lambda^* := 2\pi/\Lambda$ . Let  $\Gamma_H^\Lambda$  and  $D_H^\Lambda$  be restrictions of  $\Gamma_H$  and  $D_H$  in one periodic cell  $W_\Lambda \times \mathbb{R}$ , that is,

$$\Gamma_H^\Lambda = \Gamma_H \cap [W_\Lambda \times \mathbb{R}], \quad D_H^\Lambda = D_H \cap [W_\Lambda \times \mathbb{R}].$$

The definitions are similar for other domains restricted in one periodic cell  $W_\Lambda \times \mathbb{R}$ . We also need the domain which corresponds to the period  $N\Lambda$ . Thus define

$$W_{N\Lambda} := \left(-\frac{N\Lambda}{2}, \frac{N\Lambda}{2}\right], \quad W_{\Lambda^*/N} := \left(-\frac{\Lambda^*}{2N}, \frac{\Lambda^*}{2N}\right) = \left(-\frac{\pi}{N\Lambda}, \frac{\pi}{N\Lambda}\right)$$

and

$$\Gamma_H^{N\Lambda} := \Gamma_H \cap [W_{N\Lambda} \times \mathbb{R}], \quad D_H^{N\Lambda} = D_H \cap [W_{N\Lambda} \times \mathbb{R}].$$

Let  $\mathcal{J}$  be the Floquet–Bloch transform with the period  $\Lambda$  and  $\mathcal{J}_N$  be the one with the period  $N\Lambda$ . First, we consider the transform  $\mathcal{J}_N$  with  $N\Lambda$ .



4.1 | Bloch transform with  $N\Lambda$

From the definition, the Floquet–Bloch transform  $\mathcal{J}_N$  is defined by

$$(\mathcal{J}_N \varphi)(\gamma, x) = \sum_{j \in \mathbb{Z}} \varphi \left( x + \begin{pmatrix} N\Lambda j \\ 0 \end{pmatrix} \right) e^{-i\gamma N\Lambda j}, \quad \gamma \in W_{\Lambda^*/N}, x \in D_H^{N\Lambda}.$$

Now we apply the Floquet–Bloch transform  $\mathcal{J}_N$  to the problem (8). Let  $w_N(\gamma, x) := (\mathcal{J}_N u_N)(\gamma, x)$ , then it satisfies the variational problem

$$\int_{W_{\Lambda^*/N}} \left( \int_{D_H^{N\Lambda}} [\nabla w_N(\gamma x) \cdot \nabla \overline{\varphi_N}(\gamma, x) - k^2 (1 + n_N) w_N(\gamma, x) \overline{\varphi_N}(\gamma, x)] dx - \int_{\Gamma_H^{N\Lambda}} T_{\gamma, N}^+ w(\gamma x) \overline{\varphi_N}(\gamma, x) ds \right) d\gamma = \int_{W_{\Lambda^*/N}} \int_{D_H^{N\Lambda}} [\mathcal{J}_N g](\gamma, x) \overline{\varphi_N}(\gamma, x) dx d\gamma, \quad (10)$$

where  $T_{\gamma, N}^+$  is the  $\gamma$ -quasi-periodic DtN map with period  $N\Lambda$  defined by

$$T_{\gamma, N}^+ \psi = i \sum_{j \in \mathbb{Z}} \sqrt{k^2 - |\Lambda^* j/N - \gamma|^2} \widehat{\psi}(j) e^{i(\Lambda^* j/N - \gamma)x_1}, \quad \psi = \sum_{j \in \mathbb{Z}} \widehat{\psi}(j) e^{i(\Lambda^* j/N - \gamma)x_1}.$$

When  $g \in H_r^0(D_H)$  for some  $r \in (1/2, 1)$ ,  $u \in \widetilde{H}_\alpha^1(D_H)$  thus  $w_N \in H_0^r(W_{\Lambda^*/N}; \widetilde{H}_\alpha^1(D_H^{N\Lambda}))$ . From Sobolev embedding that  $H^r(W_{\Lambda^*}) \subset C(W_{\Lambda^*})$ ,  $w_N$  is a continuous function with respect to  $\gamma$ . Thus for any fixed  $\gamma \in W_{\Lambda^*/N}$ , it satisfies

$$\int_{D_H^{N\Lambda}} [\nabla w_N(\gamma x) \cdot \nabla \overline{\varphi_N}(\gamma, x) - k^2 (1 + n_N) w_N(\gamma, x) \overline{\varphi_N}(\gamma, x)] dx - \int_{\Gamma_H^{N\Lambda}} T_{\gamma, N}^+ w_N(\gamma, x) \overline{\varphi_N}(\gamma, x) ds = \int_{D_H^{N\Lambda}} (\mathcal{J}_N g)(\gamma, x) \overline{\varphi_N}(\gamma, x) dx. \quad (11)$$

Note that when  $n_N = 0$ , (11) is always uniquely solvable for any  $\gamma \in W_{\Lambda^*/N}$ . We make the following assumption to guarantee the for non-vanishing  $n_N$ .

**Assumption 17** We assume that  $\|n\|_\infty$  is sufficient small such that (11) is uniquely solvable for any  $\gamma \in W_{\Lambda^*/N}$ .

With Assumption 17, the variational problem (11) is uniquely solvable. From inverse Floquet–Bloch transform, the solution  $u_N$  is obtained by

$$u_N(x) = \frac{N\Lambda}{2\pi} \int_{W_{\Lambda^*/N}} w_N(\gamma, x) d\gamma, \quad x_1 \in D_H^{N\Lambda}.$$

use  $w_N(0, x)$  to approximate  $w_N(\gamma, x)$  with  $\gamma \in W_{\Lambda^*/N}$ . Thus the approximation  $\widetilde{u}_N$  is obtained for  $x \in D_H^{N\Lambda}$ :

$$\widetilde{u}_N(x) = \frac{N\Lambda}{2\pi} \int_{W_{\Lambda^*/N}} w_N(0, x) d\gamma = w_N(0, x) = \sum_{j \in \mathbb{Z}} u_N \left( x + \begin{pmatrix} N\Lambda j \\ 0 \end{pmatrix} \right).$$

We estimate the difference between  $u_N$  and  $\widetilde{u}_N$  by Theorem 12. When Assumption 11 is satisfied,  $u \in C_{3/2}(D_H)$ , that is, there is a constant  $C > 0$  such that  $|u(x)| \leq C|x|^{-3/2}$ . Thus for any fixed  $x \in D_H^{N\Lambda}$ ,

$$|(u_N - \tilde{u}_N)(x)| \leq \sum_{j \in \mathbb{Z}, j \neq 0} \left| u_N \left( x + \begin{pmatrix} N\Lambda j \\ 0 \end{pmatrix} \right) \right| \leq C \sum_{j \in \mathbb{Z}, j \neq 0} |x_1 + N\Lambda j|^{-3/2}.$$

With the estimation above, we have the following result.

**Theorem 18** For any bounded domain  $S \subset D_H$ , when  $N$  is large enough,  $\|u_N - \tilde{u}_N\|_{L^2(S)} \leq CN^{-3/2}$ .

*Proof.* For  $x \in S$ , there is constant  $C(S) > 0$  such that  $|x_1 + N\Lambda j| < C(S)|N\Lambda j|$  for any  $j \in \setminus \{0\}$ . From the fact that  $\zeta(1.5) = \sum_{j=1}^{\infty} j^{-3/2}$  where  $\zeta$  is the Riemann–Zeta function,

$$|(u_N - \tilde{u}_N)(x)| \leq C \sum_{j \in \mathbb{Z}, j \neq 0} |x_1 + N\Lambda j|^{-3/2} \leq CC(S) \sum_{j \in \mathbb{Z}, j \neq 0} |N\Lambda j|^{-3/2} = 2CC(S)\zeta(3/2)\Lambda^{-3/2}N^{-3/2}.$$

Thus  $\|u_N - \tilde{u}_N\|_{L^2(S)} \leq CN^{-3/2}$ . The proof is finished. ■

From now on, we use  $w_N(0, x)$  to approximate  $u$  in a fixed bounded domain  $S$ . For simplicity, let  $w_0(x) := w_N(0, x)$ . From Corollary 16 and Theorem 18, when  $N$  is sufficient large,

$$\|u - w_0\|_{L^2(S)} \leq CN^{-3/2}. \tag{12}$$

$w_0$  is the solution of the following variational formulation:

$$\int_{D_H^{N\Lambda}} [\nabla w_0 \cdot \nabla \bar{\varphi} - k^2(1 + n_N)w_0\bar{\varphi}] dx - \int_{\Gamma_H^{N\Lambda}} T_{0,N}^+ w_0 \bar{\varphi} ds = \int_{D_H^{N\Lambda}} G\bar{\varphi} dx, \tag{13}$$

for any  $\varphi \in \tilde{H}_0^1(D_H^{N\Lambda})$ , where  $G(x) = (\mathcal{J}_N g)(0, x)$ .

### 4.2 | Decomposition of periodic solution

Inspired by [23], any  $N\Lambda$ -periodic function is decomposed as  $N$  quasi-periodic functions with period  $\Lambda$ . Let  $\varphi$  be such a function, then

$$\varphi(x_1, x_2) = \sum_{\ell \in \mathbb{Z}_N} \varphi_\ell(x_1, x_2) \exp\left(i \frac{2\pi}{N\Lambda} \ell x_1\right), \tag{14}$$

where  $\varphi_\ell$  is a  $\Lambda$ -periodic function in  $x_1$ -direction for any  $\ell \in \mathbb{Z}_N$ , and  $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$ . The formulation for the  $\varphi_\ell$  is easily obtained from the Fourier series. Let

$$\varphi(x_1, x_2) = \sum_{j \in \mathbb{Z}} \hat{\varphi}_j(x_2) \exp\left(i \frac{2\pi}{N\Lambda} j x_1\right) = \sum_{j \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}_N} \hat{\varphi}_{Nj+\ell}(x_2) \exp\left(i \frac{2\pi}{N\Lambda} (Nj + \ell)x_1\right).$$

For any  $\ell \in \mathbb{Z}_N$ , set

$$\varphi_\ell(x_1, x_2) := \sum_{j \in \mathbb{Z}} \hat{\varphi}_{Nj+\ell}(x_2) \exp\left(i \frac{2\pi}{\Lambda} j x_1\right), \tag{15}$$

then the equation (14) holds.

Let  $\tilde{n}_N$  be the  $\Lambda$ -periodic function defined by

$$\tilde{n}_N = \begin{cases} 1 + n_2, & x \in D_2; \\ 1, & \text{otherwise.} \end{cases}$$

Extend  $n_1^N$  by 0 to the half space  $D$ , it is still  $N\Lambda$ -periodic in  $x_1$ -direction, then

$$1 + n_N = \tilde{n}_N + n_1^N.$$

Thus  $n_1^N$  is a  $N\Lambda$ -periodic function. From (15), it is decomposed as  $N$  quasi-periodic functions with period  $\Lambda$ , that is,

$$n_1^N(x_1, x_2) = \sum_{\ell \in \mathbb{Z}_N} \hat{n}_N^\ell(x) \exp\left(i \frac{2\pi}{N\Lambda} \ell x_1\right) := \sum_{\ell \in \mathbb{Z}_N} n_N^\ell(x), \tag{16}$$

where for any  $\ell \in \mathbb{Z}_N$ ,  $\hat{n}_N^\ell$  is a  $\Lambda$ -periodic function in  $x_1$ -direction and  $n_N^\ell$  is a  $2\pi\ell/(N\Lambda)$ -quasi-periodic function. Here we always assume that  $n_1^N$  is smooth enough such that the series in (15) converges uniformly w.r.t.  $x \in D_H^\Lambda$ .

As  $w_0$  is also  $N\Lambda$ -periodic in  $x_1$ -direction, it has the following decomposition:

$$w_0(x) = \sum_{\ell \in \mathbb{Z}_N} \hat{w}_N^\ell(x) \exp\left(i \frac{2\pi}{N\Lambda} \ell x_1\right) := \sum_{\ell \in \mathbb{Z}_N} w_N^\ell(x),$$

where  $\hat{w}_N$  are  $\Lambda$ -periodic functions and  $w_N^\ell$  are  $\frac{2\pi}{N\Lambda} \ell$ -quasi-periodic functions. Moreover,

$$T_{0,N}^+ w_0 = \sum_{\ell \in \mathbb{Z}_N} T_{(2\pi\ell)/(N\Lambda)}^+ w_N^\ell,$$

where  $T_\alpha^+$  with period  $\Lambda$  defined as:

$$T_\alpha^+ \psi = i \sum_{j \in \mathbb{Z}} \sqrt{k^2 - |\Lambda^* j - \alpha|^2} \hat{\psi}(j) e^{i(\Lambda^* j - \alpha) \cdot x_1}, \quad \psi = \sum_{j \in \mathbb{Z}} \hat{\psi}(j) e^{i(\Lambda^* j - \alpha) \cdot x_1}.$$

We write  $T_\ell^+$  instead of  $T_{(2\pi\ell)/(N\Lambda)}^+$  for simplicity.

In the following, we consider the decomposition of the function  $n_1^N(x)w_0(x)$ , which is also  $N\Lambda$ -periodic.

$$\begin{aligned} n_1^N(x)w_0(x) &= \left[ \sum_{j \in \mathbb{Z}_N} \hat{n}_N^j(x) \exp\left(i \frac{2\pi}{N\Lambda} j x_1\right) \right] \left[ \sum_{\ell \in \mathbb{Z}_N} \hat{w}_N^\ell(x) \exp\left(i \frac{2\pi}{N\Lambda} \ell x_1\right) \right] \\ &= \sum_{\ell \in \mathbb{Z}_N} \left[ \sum_{j \in \mathbb{Z}_N} \hat{n}_N^j(x) \hat{w}_N^{\ell-j}(x) \right] \exp\left(i \frac{2\pi}{N\Lambda} \ell x_1\right) \\ &= \sum_{\ell \in \mathbb{Z}_N} \left[ \sum_{j \in \mathbb{Z}_N} n_N^j(x) w_N^{\ell-j}(x) \exp\left(-i \frac{2\pi}{N\Lambda} \ell x_1\right) \right] \exp\left(i \frac{2\pi}{N\Lambda} \ell x_1\right). \end{aligned}$$

Moreover, let  $G$  be decomposed as:

$$G(x) = \sum_{\ell \in \mathbb{Z}_N} G_N^\ell(x),$$

where  $G_N^\ell$  are  $\frac{2\pi}{N\Lambda} \ell$ -quasi-periodic functions.

Put the of  $w_0$ ,  $G$  and  $n_1^N w_0$  into the variational (13), we finally get the equation for  $w_N^\ell$ :

$$\begin{aligned} &\int_{D_H^{N\Lambda}} [\nabla w_N^\ell \cdot \nabla \bar{\varphi} - k^2 \tilde{n}_N w_N^\ell \bar{\varphi}] dx - \int_{\Gamma_H^{N\Lambda}} T_\ell^+ w_N^\ell \bar{\varphi} ds \\ &- k^2 \int_{D_H^\Lambda} \left[ \sum_{j \in \mathbb{Z}_N} n_N^j(x) w_N^{\ell-j}(x) \exp\left(-i \frac{2\pi}{N\Lambda} \ell x_1\right) \right] \bar{\varphi} dx = \int_{D_H^{N\Lambda}} G_N^\ell \bar{\varphi} dx, \tag{17} \end{aligned}$$

for any  $\frac{2\pi}{N\Lambda}\ell$ -quasi-periodic function  $\varphi \in \tilde{H}^1_{(2\pi\ell)/(N\Lambda)}(D_H^\Lambda)$ . This is a coupled system of  $W_N := \{w_N^\ell : \ell \in \mathbb{Z}_N\} \subset \oplus_{\ell \in \mathbb{Z}_N} \tilde{H}^1_{(2\pi\ell)/(N\Lambda)}(D_H^\Lambda)$ . Define  $a_N^\ell(\cdot, \cdot)$ ,  $b_N^\ell(\cdot, \cdot)$  and  $F_N^\ell(\cdot)$  by

$$a_N^\ell(w, \varphi) := \int_{D_H^\Lambda} [\nabla w \cdot \nabla \bar{\varphi} - k^2 \tilde{n}_N w \bar{\varphi}] dx - \int_{\Gamma_H^\Lambda} T_\ell^+(w) \bar{\varphi} ds, \quad (18)$$

$$b_N^{\ell,j}(w, \varphi) = \int_{D_H^\Lambda} n_N^j(x) w(x) \exp\left(-i \frac{2\pi}{N\Lambda} \ell x_1\right) \bar{\varphi} dx. \quad (19)$$

$$F_N^\ell(\varphi) = \int_{D_H^\Lambda} G_N^\ell \bar{\varphi} dx. \quad (20)$$

Thus we obtain the variational problem (11):

$$a_N^\ell(w_N^\ell, \varphi) + \sum_{j \in \mathbb{Z}_N} b_N^{\ell,j}(w_N^{\ell-j}, \varphi) = F_N^\ell(\varphi) \quad (21)$$

for any  $\varphi \in H^1_{(2\pi\ell)/(N\Lambda)}(D_H^\Lambda)$ .

*Remark 19* In this paper, the period of the Floquet–Bloch transform is chosen as the period of  $n_2$ . In fact, we can also choose the period of  $n_1$  and all of the arguments are similar.

We can even choose the parameter which is neither the period of  $n_1$  nor that of  $n_2$ , then the problem is treated generally as the scattering problem with a rough layer. In this case, it is possible that the computational complexity is increased in numerical implementation.

## 5 | THE FINITE ELEMENT METHOD

In this section, as was shown in the last section, the field  $w(\alpha, \cdot)$  has been approximated by the  $N\Lambda$ -periodic function  $w_0$ , which is decomposed into  $N$  quasi-periodic functions. Thus we only need to continue with the discretization with respect to  $x$ .

Define the uniformly distributed grid points

$$\alpha_N^\ell = \frac{2\pi\ell}{N\Lambda} \in W_{-1pt\Lambda^*}, \ell = 1, \dots, N.$$

Note that the integer  $N$  is the same as previous sections. It is more convenient to consider functions that are periodic in  $x_1$ , that is,  $\hat{w}_N^\ell$ . Let  $\hat{\varphi}_N^\ell$  be the periodic test function. Replace  $w_N^\ell$  and  $\varphi$  by  $\tilde{w}_N^\ell$  and  $\tilde{z}_N^\ell$  in (17), then

$$\hat{a}_N^\ell(\hat{w}_N^\ell, \hat{\varphi}_N^\ell) - k^2 \sum_{j=1}^N \hat{b}_N^{\ell-j,j}(\hat{w}_N^{\ell-j}, \hat{\varphi}_N^j) = \hat{g}_N^\ell(\hat{\varphi}_N^\ell), \quad (22)$$

where

$$\hat{a}_N^\ell(\varphi, \psi) = \int_{D_H^\Lambda} [(\nabla + i\alpha_N^\ell \mathbf{e}_1) \varphi \cdot (\nabla - i\alpha_N^\ell \mathbf{e}_1) \bar{\psi} - k^2 \tilde{n}_N \varphi \bar{\psi}] dx - \int_{\Gamma_H^\Lambda} \tilde{T}_\ell^+(\varphi) \bar{\psi} ds,$$

$$\hat{b}_N^{\ell,j}(\varphi, \psi) = \int_{D_H^\Lambda} n_N^j(x) \exp\left(i \frac{2\pi}{N\Lambda} \ell x_1\right) \varphi(x) \bar{\psi}(x) dx,$$

$$\hat{g}_N^\ell(\psi) = - \int_{D_H^\Lambda} G_N^\ell(x) e^{i\alpha_N^\ell x_1} \bar{\psi}(x) dx.$$

$\tilde{T}_\ell^+$  is the periodic Dirichlet-to-Neumann map defined by

$$\tilde{T}_\ell^+ \psi = i \sum_{j \in \mathbb{Z}} \sqrt{k^2 - \left| \Lambda^* j - \frac{2\pi}{N\Lambda} \ell \right|^2} \hat{\psi}_j e^{i\Lambda^* j x_1}, \quad \psi = \sum_{j \in \mathbb{Z}} \hat{\psi}_j e^{i\Lambda^* j x_1}. \tag{23}$$

Now the problem is written as the problem defined in  $(\tilde{H}_0^1(D_H^\Lambda))^N$ . The next step is Assume that  $\mathcal{M}_h$  is a family of regular and quasi-uniform meshes [32] for the periodic cell  $D_H^\Lambda$ , where  $0 < h \leq h_0$  and  $h_0$  is a small enough positive number. To obtain periodic basic functions, it is required that the nodal points on the left and right boundaries have the same heights. By omitting the nodal points on the left boundary, let  $\{\varphi_M^{(\ell)}\}_{\ell=1}^M$  be the family of piecewise linear and globally continuous nodal functions. For any  $\varphi_M^{(\ell)}$ , it equals to one at the  $\ell$ -th point (except for the lower boundary) and zero at other nodal points. Then  $V_h := \text{span}\{\varphi_M^{(\ell)}\}_{\ell=1}^M$  is a subspace of  $\tilde{H}_0^1(D_H^\Lambda)$ . Let  $\hat{w}_N^\ell$  be approximated by the piecewise linear function

$$W_{N,h}^\ell(x) = \sum_{j=1}^M w_{N,h}^{(\ell,j)} \varphi_M^{(j)}(x). \tag{24}$$

Let the test function be  $\varphi_M^{(\ell')}(x)$ , then (22) has the discretized form

$$\sum_{\ell=1}^M \delta_{j,j'} A_{\ell,\ell'}^j w_{N,h}^{(\ell,j)} - k^2 \sum_{m=1}^N \sum_{\ell=1}^M B_{\ell,\ell'}^m w_{N,h}^{(\ell,j'-m)} = g_{\ell'}^{j'}, \tag{25}$$

where  $\delta(j, j') = 1$  if and only if  $j = j'$ , otherwise it equals to 0. The coefficients are defined as follows:

$$\begin{aligned} A_{\ell,\ell'}^j &= \tilde{a}_N^{(j)}(\varphi_M^{(\ell)}, \varphi_M^{(\ell')}); \\ B_{\ell,\ell'}^m &= \int_{D_H^\Lambda} n_1^N(m)(x) \varphi_M^{(\ell)}(x) \overline{\varphi_M^{(\ell')}(x)} dx; \\ g_{\ell'}^{j'} &= \tilde{g}_T^{(j')}(\varphi_M^{(\ell')}, \end{aligned}$$

Define the matrices and vectors as follows:

$$A_j = \begin{pmatrix} A_{1,1}^j & A_{1,2}^j & \cdots & A_{1,M}^j \\ A_{2,1}^j & A_{2,2}^j & \cdots & A_{2,M}^j \\ \vdots & \vdots & \ddots & \vdots \\ A_{M,1}^j & A_{M,2}^j & \cdots & A_{M,M}^j \end{pmatrix}; B_m = \begin{pmatrix} B_{1,1}^m & B_{1,2}^m & \cdots & B_{1,M}^m \\ B_{2,1}^m & B_{2,2}^m & \cdots & B_{2,M}^m \\ \vdots & \vdots & \ddots & \vdots \\ B_{M,1}^m & B_{M,2}^m & \cdots & B_{M,M}^m \end{pmatrix}; W_j = \begin{pmatrix} w_{N,h}^{1,j} \\ w_{N,h}^{2,j} \\ \vdots \\ w_{N,h}^{M,j} \end{pmatrix}; G_j = \begin{pmatrix} g_1^j \\ g_2^j \\ \vdots \\ g_M^j \end{pmatrix}.$$

Thus the discretization equation (25) has the form of

$$[A_M^N - k^2 B_M^N] \begin{pmatrix} W_1 \\ W_2 \\ \vdots \\ W_N \end{pmatrix} = \begin{pmatrix} G_1 \\ G_2 \\ \vdots \\ G_N \end{pmatrix} \tag{26}$$

where

$$\mathbf{A}_M^N = \begin{pmatrix} \mathbf{A}_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \mathbf{A}_2 & 0 & & & \vdots \\ \vdots & 0 & \mathbf{A}_3 & 0 & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & 0 & \mathbf{A}_{N-1} & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & \mathbf{A}_N \end{pmatrix};$$

$$\mathbf{B}_M^N = \begin{pmatrix} \mathbf{B}_N & \mathbf{B}_{N-1} & \mathbf{B}_{N-2} & \cdots & \cdots & \cdots & \mathbf{B}_2 & \mathbf{B}_1 \\ \mathbf{B}_1 & \mathbf{B}_N & \mathbf{B}_{N-1} & \mathbf{B}_{N-2} & & & & \mathbf{B}_2 \\ \mathbf{B}_2 & \mathbf{B}_1 & \mathbf{B}_N & \mathbf{B}_{N-1} & \mathbf{B}_{N-2} & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \mathbf{B}_2 & \mathbf{B}_1 & \mathbf{B}_N & \mathbf{B}_{N-1} & \mathbf{B}_{N-2} \\ \mathbf{B}_{N-2} & & & & \mathbf{B}_2 & \mathbf{B}_1 & \mathbf{B}_N & \mathbf{B}_{N-1} \\ \mathbf{B}_{N-1} & \mathbf{B}_{N-2} & \cdots & \cdots & \cdots & \mathbf{B}_2 & \mathbf{B}_1 & \mathbf{B}_N \end{pmatrix}.$$

By solving (22), finally we obtain the approximated solution

$$u_{N,h} = \sum_{\ell \in \mathbb{Z}_N} W_{N,h}^\ell(x) \exp\left(i \frac{2\pi}{N\Lambda} \ell x_1\right) = \sum_{\ell \in \mathbb{Z}_N} \sum_{j=1}^M w_{N,h}^{(\ell,j)}(x) \exp\left(i \frac{2\pi}{N\Lambda} \ell x_1\right) \varphi_M^{(j)}(x).$$

Note that, we only need to consider the error estimation for the finite element method of the latter one.

### 6 | NUMERICAL EXAMPLES

In this section, we present eight examples to illustrate the convergence result of the numerical algorithm. We choose two different groups of refractive indexes  $(n_1, n_2)$ , both of which are embedded in the half domain above the line  $\mathbb{R} \times \{1\}$ :

Group 1.

$$n_1^{(1)} = 0.1 \sin\left(\frac{\sqrt{2}x_1}{2}\right) \mathcal{X}_{0.3}(|x_2 - 1.5|);$$

$$n_2^{(1)} = 0.25 \sin(x_1) \mathcal{X}_{0.3}(|x_2 - 2.5|).$$

Group 2.

$$n_1^{(2)} = -0.25 \mathcal{X}_4(|x_1|) \mathcal{X}_{0.3}(|x_2 - 1.5|) \text{ and is extended } 15 - \text{ periodically in } x_1 - \text{ direction;}$$

$$n_2^{(2)} = 0.25 \mathcal{X}_{0.3}\left(\sqrt{x_1^2 + (x_2 - 2.5)^2}\right) \text{ and is extended } 2\pi - \text{ periodically in } x_1 - \text{ direction.}$$

Note that  $\mathcal{X}_a$  is a  $C^1$ -continuous function with a bounded second order derivative defined in  $[0, \infty)$  as:

$$\mathcal{X}_a(t) = \begin{cases} 1, & \text{when } t \leq a/2; \\ -\frac{4(a-x)^2(a-4x)}{a^3}, & \text{when } a/2 < t < a; \\ 0, & \text{when } t \geq a. \end{cases}$$

Both  $n_2^{(1)}$  and  $n_2^{(2)}$  are  $2\pi$ -periodic functions in  $x_1$ -direction supported in the strip  $\mathbb{R} \times (2, 3)$ ; while both the function  $n_1^{(1)}$  and  $n_1^{(2)}$  are supported in  $\mathbb{R} \times (1, 2)$ . Moreover,  $n_1^{(1)}$  is  $2\sqrt{2}\pi$ -periodic while  $n_1^{(2)}$  is 15-periodic.

In numerical examples, the following parameters are fixed:

$$\Lambda = 2\pi, \Lambda^* = 1, H = 3, H_1 = 2, H_0 = 1.$$

In the numerical implementation, we use very large number of points to approximate the Fourier series of  $n_1^N(x_1, x_2)$ , that is,  $1000N$  points in  $x_1$ -direction and 1000 points in  $x_2$ -direction, where  $N$  is the number of uniform sub-intervals introduced in Section 3. Then we use the  $-8N$ -th to  $8N$ -th coefficients to construct the decomposition (16), and use the ‘‘pchip’’ interpolation in MATLAB to obtain values on mesh points. As the Fourier coefficients at the rate of  $O(N^{-2})$ , we assume that the error brought by this approximation is sufficient small such that it is ignored.

### 6.1 | Numerical examples with exact solutions

Recall the half space Green’s function

$$G(x, y) = \frac{i}{4} \left[ H_0^{(1)}(k|x - y|) - H_0^{(1)}(k|x - y'|) \right],$$

where  $y = (y_1, y_2)^T$  and  $y' = (y_1, -y_2)^T$ . Furthermore, we assume that  $0 < y_2 < H_0$ , thus the point source is located in  $\mathbb{R} \times (0, 1)$  (see Figure 2). It is easy to check that,  $G(x, y) \in H_r^1(D_H)$  for any  $|r| < 1$ .

For a fixed point  $y$ ,  $G(\cdot, y)$  solves the following equations:

$$\Delta u + k^2(1 + n)u = g \quad \text{in } D_H;$$

$$u = G(\cdot, y) \quad \text{on } \Gamma_{H_0};$$

$$\frac{\partial u}{\partial x_2} - T^+ u = 0 \quad \text{on } \Gamma_H;$$

where

$$g(x) = k^2 n(x)G(x, y) \quad \text{in } D_H.$$

From the property of  $G(x, y)$  and  $n, g \in H_r^0(D_H)$ .

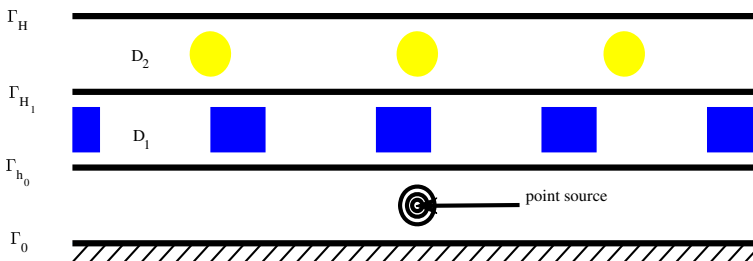


FIGURE 2 Locations of the periodic layers and point sources

**Remark 20** There is a difference between the numerical examples in this subsection (and also the next subsection) and the original problem (5), as the homogeneous boundary conditions on  $\Gamma_{H_0}$  or  $\Gamma_H$  are. For numerical implementation, we the algorithm with introduced in [19]. For error, we can modify the problem to get an equivalent one in the form of (5). Let  $\tilde{u} := u - u_0$ , where  $u_0 = G(\cdot, y)$  on  $\Gamma_{H_0}$ , and is extended to a smooth function with a support in  $\mathbb{R} \times [H_0, H']$  for some  $H_0 < H' < H$ . Then  $u_0$  satisfies (5) with  $g = (\Delta + k^2(1+n))u_0$ . The regularity of the right hand side is decided by both  $g$  and  $u_0$ , and we can carry out the error estimation in the same way as (5). For numerical examples in the next subsection, similar technique can be employed as well.

We choose one fixed Green's function in this subsection located at the point

$$P = (0.5, 0.4).$$

The numerical scheme is carried out for the mesh size  $h$  is chosen as 0.64, 0.32, 0.16, 0.08 for  $k = 1$  and 0.16, 0.08, 0.04, 0.02 for  $k = 6$ , and the parameter  $N$  is taken as 10, 20, 40, 80. Then the following four examples are considered for different  $h$  and  $N$ , and the relative  $L^2$ -errors on  $\Gamma_H$ , defined by

$$err = \frac{\|u_{N,h} - u\|_{L^2(\Gamma_H^\Delta)}}{\|u\|_{L^2(\Gamma_H^\Delta)}}$$

are listed in Tables 1–4, where the exact solution is  $u = G(\cdot, y)$ .

**Example 1** The wave number  $k = 1$ , the refractive indexes are defined by  $n_1^{(1)}$  and  $n_2^{(1)}$ , the relative errors are listed in Table 1.

**Example 2** the wave number  $k = 6$ , the refractive indexes are defined by  $n_1^{(1)}$  and  $n_2^{(1)}$ , the relative errors are listed in Table 2.

**Example 3** The wave number  $k = 1$ , the refractive indexes are defined by  $n_1^{(2)}$  and  $n_2^{(2)}$ , the relative errors are listed in Table 3.

**Example 4** the wave number  $k = 6$ , the refractive indexes are defined by  $n_1^{(2)}$  and  $n_2^{(2)}$ , the relative errors are listed in Table 4.

TABLE 1 Relative  $L^2$ -errors for Example 1

	$h = 0.64$	$h = 0.32$	$h = 0.16$	$h = 0.08$
$N = 10$	6.3E-02	4.5E-02	4.6E-02	4.7E-02
$N = 20$	5.1E-02	2.1E-02	1.6E	1.7E-02
$N = 40$	4.8E-02	1.5E-02	6.6E-03	5.9E-03
$N = 80$	4.8E-02	1.4E-02	3.9E-03	2.2E-02

TABLE 2 Relative  $L^2$ -errors for Example 2

	$h = 0.16$	$h = 0.08$	$h = 0.04$	$h = 0.02$
$N = 10$	3.8E-01	1.4 E -01	1.1 E -01	1.1 E -01
$N = 20$	3.8-01	1.1 E -01	4.9 E -02	4.2 E -02
$N = 40$	3.8-01	1.0 E -01	3.1 E -02	1.7 E -02
$N = 80$	3.8-01	1.0 E -01	2.8 E -02	8.9 E -03



TABLE 3 Relative  $L^2$ -errors for Example 3

	$h = 0.64$	$h = 0.32$	$h = 0.16$	$h = 0.08$
$N = 10$	$6.5E - 02$	$4.6E - 02$	$4.6E - 02$	$4.7E - 02$
$N = 20$	$5.2E - 02$	$2.2E - 02$	$1.7E - 02$	$1.7E - 02$
$N = 40$	$4.9E - 02$	$1.6E - 02$	$6.7E - 03$	$5.9E - 03$
$N = 80$	$4.8E - 02$	$1.5E - 02$	$4.0E - 03$	$2.2E - 03$

TABLE 4 Relative  $L^2$ -errors for Example 4

	$h = 0.16$	$h = 0.08$	$h = 0.04$	$h = 0.02$
$N = 10$	$4.0E - 01$	$1.2E - 01$	$9.1E - 02$	$1.1E - 01$
$N = 20$	$4.0E - 01$	$1.1E - 01$	$4.2E - 02$	$3.9E - 02$
$N = 40$	$4.0E - 01$	$1.1E - 01$	$3.1E - 02$	$1.6E - 02$
$N = 80$	$4.0E - 01$	$1.1E - 01$	$2.9E - 02$	$8.9E - 03$

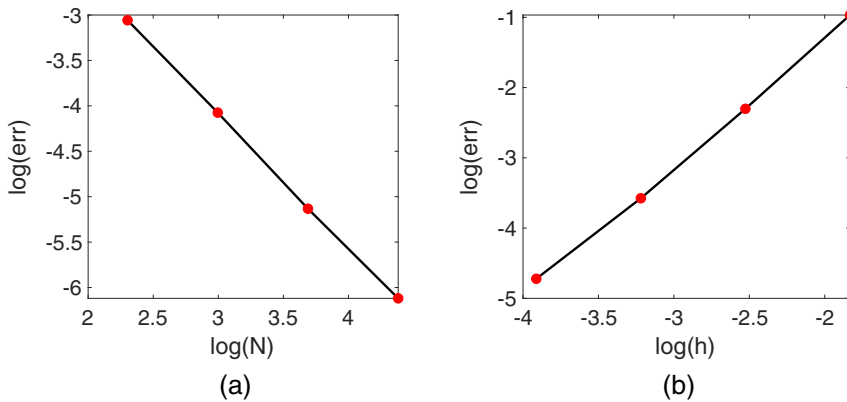


FIGURE 3 (a): The relative  $L^2$ -errors for Example 1 with  $h = 0.02$  plotted in logarithmic scale over  $N$ . (b): The relative  $L^2$ -errors for Example 2 with  $N = 80$  plotted in logarithmic scale over  $h$

From the relative errors listed in Tables 1–4, the errors decrease when  $N$  and  $h$ . When the wave number is relatively small and  $h$  is small enough (i.e.,  $k = 1$  and  $h = 0.02, 0.04$ ), the error brought by  $N$  is dominant, thus the error brought by  $h$ , see the last two columns in Tables 1 and 3. From Figure 3a, the convergence rate with respect to  $N$  is about  $O(N^{-1.5})$ , which is the same as expected. On the other, when the wave number is large and  $N$  is large enough (i.e.,  $k = 6$  and  $N = 80$ ), the dominant part of the relative error is brought by  $h$ , and the error brought by  $N$ , see the last lines in Tables 2 and 4. From Figure 3b, the convergence rate with respect to  $h$  ( $h^{1.8}$ ), which is almost as high as expected.

### 6.2 | Numerical example with non-exact solutions

In this subsection, the incident field  $G(x, y)$  is located above  $D_H$ , that is,  $y = (\pi, 4)^T$ . Thus  $u$  satisfies the following equations

$$\Delta u + k^2(1 + n)u = 0 \quad \text{in } D_H;$$

$$u = 0 \quad \text{on } \Gamma_{H_0};$$

$$\frac{\partial u}{\partial x_2} - T^+ u = f \quad \text{on } \Gamma_H;$$

where

$$f = \frac{\partial G(\cdot, y)}{\partial x_2} - T^+ G(\cdot, y) \quad \text{on } \Gamma_H.$$

From the property of  $G(\cdot, y)$ ,  $f \in H_r^{-1/2}(\Gamma_H)$  for any  $|r| < 1$ .

As no exact solution is known for the refractive indexes  $(n_1^{(1)}, n_2^{(1)})$  and  $(n_1^{(2)}, n_2^{(2)})$ , we only use finer meshes to compute “exact” solutions. We set the parameters for the finer meshes to be  $h = 0.01$  and  $N = 160$ , and let the solution with respect to these meshes be the “exact” solution  $u$ . We set Example 5–8 as follows:

**Example 5** The wave number  $k = 1$ , the refractive indexes are defined by  $n_1^{(1)}$  and  $n_2^{(1)}$ . The relative errors with  $h = 0.16, 0.08, 0.04, 0.02$  and  $N = 10, 20, 40, 80$  are listed in Table 5.

**Example 6** the wave number  $k = 6$ , the refractive indexes are defined by  $n_1^{(1)}$  and  $n_2^{(1)}$ . The relative errors with  $0.08, 0.04, 0.02$  and  $N = 10, 20, 40, 80$  are listed in Table 6.

**Example 7** The wave number  $k = 1$ , the refractive indexes are defined by  $n_1^{(2)}$  and  $n_2^{(2)}$ . The relative errors with  $h = 0.16, 0.08, 0.04, 0.02$  and  $N = 10, 20, 40, 80$  are listed in Table 7.

**Example 8** the wave number  $k = 6$ , the refractive indexes are defined by  $n_1^{(2)}$  and  $n_2^{(2)}$ . The relative errors with  $0.08, 0.04, 0.02$  and  $N = 10, 20, 40, 80$  are listed in Table 8.

From Tables 5–8, we can conclude similar convergence results as in the last subsection. However, due to limited memory of our computers, we are not able to use finer meshes to produce better “exact

TABLE 5 Relative  $L^2$ -errors for Example 5

	$h = 0.16$	$h = 0.08$	$h = 0.04$	$h = 0.02$
$N = 10$	9.9E – 02	9.3E – 02	8.9E – 02	8.9E – 02
$N = 20$	6.0E – 02	4.2E – 02	3.2E – 02	3.1E – 02
$N = 40$	5.4E – 02	3.0E – 02	1.4E – 02	1.1E – 02
$N = 80$	5.3E – 02	2.8E – 02	1.1E – 02	4.6E – 03

TABLE 6 Relative  $L^2$ -errors for Example 6

	$h = 0.08$	$h = 0.04$	$h = 0.02$
$N = 10$	6.7E – 01	2.4E – 01	2.5E – 01
$N = 20$	6.1E – 01	1.1E – 01	9.4E – 02
$N = 40$	6.0E – 01	9.1E – 02	4.0E – 02
$N = 80$	6.0E – 01	9.1E – 02	3.0E – 02

TABLE 7 Relative  $L^2$ -errors for Example 7

	$h = 0.16$	$h = 0.08$	$h = 0.04$	$h = 0.02$
$N = 10$	1.0E – 01	9.7E – 02	9.3E – 02	9.3E – 02
$N = 20$	6.2E – 02	4.3E – 02	3.3E – 02	3.2E – 02
$N = 40$	5.5E – 02	3.1E – 02	1.5E – 02	1.1E – 02
$N = 80$	5.5E – 02	2.9E – 02	1.1E – 02	4.7E – 03

TABLE 8 Relative  $L^2$ -errors for Example 8

	$h = 0.08$	$h = 0.04$	$h = 0.02$
$N = 10$	8.3E - 01	6.3E - 01	7.5E - 01
$N = 20$	6.0E - 01	2.4E - 01	2.8E - 01
$N = 40$	5.7E - 01	1.2E - 01	9.5E - 02
$N = 80$	5.7E - 01	1.1E - 01	4.1E - 02

solutions”, which results in worse relative errors compare with those in Tables 1–4. However, although the numerical results are not as good as Example 1–4, they still shows that our algorithm converges as  $h \rightarrow 0$  and  $N \rightarrow \infty$ .

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#### DATA AVAILABILITY STATEMENT

Data sharing not applicable - no new data generated.

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## APPENDIX: THE FLOQUET-BLOCH TRANSFORM

The main tool used in this paper is the Floquet–Bloch transform. In the [Appendix](#), we recall the definition and some basic properties of the Bloch transform in periodic domains in  $\mathbb{R}^2$  (for details see [1]).

Suppose  $\Omega \subset \mathbb{R}^2$  is  $\Lambda$ -periodic in  $x_1$ -direction, that is, for any  $x = (x_1, x_2)^T \in \Omega$ , the translated point  $(x_1 + \Lambda j, x_2) \in \Omega$ ,  $\forall j \in \mathbb{Z}$ . Define one periodic cell by  $\Omega^\Lambda := \Omega \cap [W_\Lambda \times \mathbb{R}]$  where  $W_\Lambda = (-\Lambda/2, \Lambda/2]$ . For any  $\varphi \in C_0^\infty(\Omega)$ , define the (partial) Bloch transform in  $\Omega$ , that is,  $\mathcal{J}_\Omega$ , of  $\varphi$  as

$$(\mathcal{J}\varphi)(\alpha, x) = C_\Lambda \sum_{j \in \mathbb{Z}} \varphi \left( x + \begin{pmatrix} \Lambda j \\ 0 \end{pmatrix} \right) e^{-i\alpha \cdot \Lambda j}, \quad \alpha \in \mathbb{R}, x \in \Omega^\Lambda,$$

where  $C_\Lambda$  is a constant defined by  $C_\Lambda := \left[ \frac{\Lambda}{2\pi} \right]^{1/2}$ .

*Remark 21* The periodic domain  $\Omega$  is not required to be bounded in  $x_2$ -direction.

The space  $H_0^r(W_{\Lambda^*}; \widetilde{H}_\alpha^s(D_H^\Lambda))$  ( $r, s \in \mathbb{R}$ ) is defined as follows.

**Definition 22** For any  $\ell \in \mathbb{N}, s \in \mathbb{R}$ , we define the following Hilbert space by

$$H^\ell(W_{\Lambda^*}; H^s(D_H^\Lambda)) := \left\{ \psi \in \mathcal{D}'(W_{-1\rho t\Lambda^*} \times D_H^\Lambda) : \left( \sum_{m=0}^{\ell} \int_{W_{-1\rho t\Lambda^*}} \|\partial_\alpha^m \psi(\alpha, \cdot)\|_{H^s(D_H^\Lambda)}^2 d\alpha \right)^{1/2} < \infty \right\}.$$

From interpolation and duality arguments, we can extend the definition of the space  $H_0^r(W_{\Lambda^*}; H^s(D_H^\Lambda))$  for any  $r \in \mathbb{R}$ .

A function  $\psi$  is  $\alpha$ -quasi-periodic if  $\psi(x_1 + \Lambda, x_2) = \exp(i\Lambda\alpha)\psi(x_1, x_2)$ . Let  $H_\alpha^s(D_H^\Lambda)$  be the subspace of  $H^s(D_H^\Lambda)$  of functions such that they can be extended into  $\alpha$ -quasi-periodic functions in  $H_{loc}^s(D_H)$ , then we can also define  $H^\ell(W_{\Lambda^*}; H_\alpha^s(D_H^\Lambda)) \subset H^\ell(W_{\Lambda^*}; H^s(D_H^\Lambda))$  in the same way.

The following properties for the  $d$ -dimensional (partial) Bloch transform  $\mathcal{J}$  is also proved in [1].

**Theorem 23** The Bloch transform  $\mathcal{J}$  extends to an isomorphism between  $H_r^s(\Omega)$  and  $H_0^r(W_{\Lambda^*}; H_\alpha^s(\Omega^\Lambda))$  for any  $s, r \in \mathbb{R}$ . Its inverse has the form of

$$(\mathcal{J}^{-1}\psi) \left( x + \begin{pmatrix} \Lambda j \\ 0 \end{pmatrix} \right) = C_\Lambda \int_{W_{\Lambda^*}} \psi(\alpha, x) e^{i\alpha \cdot \Lambda j} d\alpha, \quad x_1 \in \Omega^\Lambda, j \in \mathbb{Z},$$

and the adjoint operator  $\mathcal{J}_\Omega^*$  with respect to the scalar product in  $L^2(W_{\Lambda^*}; L^2(\Omega^\Lambda))$  equals to the inverse  $\mathcal{J}^{-1}$ . Moreover, when  $r = s = 0$ , the Bloch transform  $\mathcal{J}$  is an isometric isomorphism.

Another important property of the Bloch transform is that it commutes with partial derivatives, see [1]. If  $u \in H^n(\Omega)$  for some  $n \in \mathbb{N}$ , then for any  $\gamma = (\gamma_1, \gamma_2) \in \mathbb{N}^2$  with  $|\gamma| = |\gamma_1| + |\gamma_2| \leq n$ ,

$$\partial_x^\gamma (\mathcal{J}u)(\alpha, x) = \mathcal{J} [\partial^\gamma u](\alpha, x).$$