Available online at www.sciencedirect.com

ScienceDirect

Nuclear Physics B 985 (2022) 115988

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

Adler function, Bjorken Sum Rule and Crewther-Broadhurst-Kataev relation with generic fermion representations at order $O(\alpha_s^4)$

K.G. Chetyrkin

Institut für Theoretische Teilchenphysik, Karlsruher Institut für Technologie, Karlsruhe, Germany

Received 11 July 2022; received in revised form 14 September 2022; accepted 4 October 2022
Available online 12 October 2022
Editor: Hong-Jian He

Abstract

We compute the nonsinglet Adler D-function and the coefficient function for Bjorken polarized sum rules S^{Bjp} at order $O(\alpha_s^4)$ in an extended QCD model with arbitrary number of fermion representations. The Crewther-Broadhurst-Kataev (CBK) relation in this order is confirmed.

---

1. Introduction

The Crewther-Broadhurst-Kataev (CBK) relation [1,2] demonstrates a non-trivial connection between two (at first sight seemingly unrelated) important physical quantities, namely the (non-singlet) Adler D-function

$$D(L,a) = 1 + 3\,C_F\,a + \sum_{i=2}^{\infty} d_i(L)\,a^i(\mu^2) \tag{1}$$

and the (non-singlet) coefficient function for the Bjorken polarized sum rules

$$S^{Bjp}(L,a) = 1 - 3\,C_F\,a + \sum_{i=2}^{\infty} c_i(L)\,a^i(\mu^2). \tag{2}$$

E-mail address: konstantin.chetyrkin@partner.kit.edu.

https://doi.org/10.1016/j.nuclphysb.2022.115988

Here $L = \ln \frac{\mu^2}{Q^2}$, μ is the normalization scale in $\overline{\text{MS}}$-scheme [3,4] (which we will assume throughout the paper) and $a = \frac{g_s^2}{16\pi^2} = \frac{\alpha_s}{4\pi}$ (precise definitions for both functions and color factors involved will be given in Sections 2.2, 2.3 and 2.1 correspondingly).

The functions (1) and (2) are very well studied in perturbative QCD. Due to works [5–14] they are known to impressively high order α_s^4. The CBK relation connecting both functions reads[1]:

$$D(a)\, C^{Bjp}(a) = 1 + \beta(a)\, K(a), \quad K(a) = a\, K_1 + a^2\, K_2 + a^3\, K_3 + \ldots \tag{3}$$

Here

$$\beta(a) = \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \ln a(\mu) = \sum_{i \geq 1} \beta_i a^i \tag{4}$$

is the QCD β-function describing the *running* of the coupling constant a with respect to a change of the normalization scale μ and with its first term

$$\beta_1 = -\frac{11}{3} C_A + \frac{4}{3} T_F\, n_f$$

being responsible for asymptotic freedom of QCD. The term proportional the β-function responsible for deviation from the limit of exact conformal invariance, with the deviation starting in order α_s^2, and was suggested [2] on the basis of $\mathcal{O}(\alpha_s^3)$ calculations of $D(a)$ [8,15] and $C^{Bjp}(a)$ [16]. The original relation without this term was first proposed in [1].

The fact that the CBK relation is valid up to maximally known order in α_s is highly non-trivial. Indeed, a simple counting of available color factors shows that fulfillment of (3) sets as many as 6 constraints at the sum $d_4 + c_4$ and all of them are met identically. At lower orders the number of constraints is 2 and 3 for the sums $d_2 + c_2$ and $d_3 + c_3$ correspondingly (see discussions in [2,14] and Section 4).

Some formal arguments in favor of (3) were suggested in [17,18]. Unfortunately, these considerations can not replace a real proof. Such a proof should demonstrate at least how it works in detail and in which renormalization schemes it holds.[2] Finally, it would be highly desirable if the future proof would clarify a way of computing the factor $K(a)$ *directly* that is without previous calculations of $D(a)$ and $C^{Bjp}(a)$.

In the present work we use an extended QCD (eQCD) model with arbitrary number of fermion representations in order to subject the CBK relation to one more non-trivial test. We compute both components $D(a)$ and $C^{Bjp}(a)$ within the extended QCD to order α_s^4 and demonstrate the validity of the resulting CBK relation. Let us stress that the knowledge of both $D(a)$ and $C^{Bjp}(a)$ in QCD with multiple fermion representations provides important ingredients to obtain the so-called β-expansion representation [20–23] for observables.

This representation allows one to apply the extended BLM (eBLM) approach to optimize the PT series [20,23,24]. Note that there exists a method known as the Principle of Maximum Conformality (PMC) [25–28], which suggests a systematic and self-consistent way to solve the scale-setting problem. However, in the PMC approach content of β-expansion in general differ from those in eBLM. Both of these approaches suggest the ways to resume the non-conformal parts of various QCD observables into the scale of the coupling for any optimization task. But the results of optimization are different partly due to the different β-expansions.

---

[1] We omit direct indication on L-dependence in places where it can not lead to misunderstandings.

[2] It has been shown in [19] that the CBK relation ceases to take place in the 't Hooft $\overline{\text{MS}}$-based scheme.

Any detailed comparison of both methods is certainly beyond the scope of this paper. We refer the reader for a thorough discussion of both approaches to [29] as well as to a detailed review [30].

2. Preliminaries

2.1. QCDe Lagrangian and notations for color factors

The Lagrangian of a (massless) QCD-like model extended to include several fermion representations of the gauge group (to be referred as QCDe) is given by (our notations essentially follow those of [31])

$$\begin{aligned}\mathcal{L}_{QCD} = &-\frac{1}{4}G^a_{\mu\nu}G^{a\,\mu\nu} - \frac{1}{2\lambda}\left(\partial_\mu A^{a\,\mu}\right)^2 + \partial_\mu\bar{c}^a\partial^\mu c^a + g_s f^{abc}\,\partial_\mu\bar{c}^a A^{b\,\mu}c^c \\ &+\sum_{r=1}^{N_{\text{rep}}}\sum_{q=1}^{n_{f,r}}\left\{\frac{i}{2}\bar{\psi}_{q,r}\overset{\leftrightarrow}{\not\partial}\,\psi_{q,r} + g_s\bar{\psi}_{q,r}\not{A}^a T^{a,r}\psi_{q,r}\right\},\end{aligned} \tag{5}$$

with

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc}A^b_\mu A^c_\nu, \quad \left[T^{a,r}, T^{b,r}\right] = i f^{abc}T^{c,r}, \tag{6}$$

and f^{abc} being the structure constants of the gauge group. The index r specifies the fermion representation and the index q the fermion flavor, $\psi_{q,r}$ is the corresponding fermion field. The number of fermion flavors in representation r is $n_{f,r}$ for any of the N_{rep} fermion representations.

For every fermion representation r we have two quadratic Casimir operators $C_{F,r}$ and $T_{F,r}$

$$\delta_{ij}C_{F,r} = T^{a,r}_{ik}T^{a,r}_{kj}, \quad T_{F,r}\delta^{ab} = \mathbf{Tr}\left(T^{a,r}T^{b,r}\right) = T^{a,r}_{ij}T^{b,r}_{ji}. \tag{7}$$

The dimension of r will be denoted as $d_{F,r}$. As for gluon (adjoint) representation we use the standard notation C_A and N_A for the corresponding quadratic Casimir operator and dimension of the gluon representation. The standard QCD corresponds to the case of $N_{\text{rep}} = 1$. If $N_{\text{rep}} > 1$ we will consider the first fermion representation as a special one in what follows with

$$C_{F,1} \equiv C_F, \;\; d_{F,1} \equiv d_F, \;\; n_{f,1} \equiv n_f, \;\; T_{F,1} \equiv T_F \;\text{ and }\; T^{a,1} \equiv T^a.$$

Let us stress that all external operators (like the EM current) which appear later are assumed to involve only fermion fields $\psi_{q,1}$ which we will refer also as ψ_q.

In addition to quadratic Casimir operators we need also quartic ones which are expressed in terms of symmetric tensors (see [32] for details)

$$d_R^{a_1a_2a_3a_4} = \frac{1}{n!}\sum_{\text{perm }\pi}\text{Tr}\left\{T^{a_{\pi(1)},R}T^{a_{\pi(2)},R}T^{a_{\pi(3)},R}T^{a_{\pi(4)},R}\right\}, \tag{8}$$

where R can be any fermion representation, $R = F, r$ ($r = 1\ldots N_{\text{rep}}$) or the adjoint representation, $R = A$ where $T^{a,A}_{bc} = -i\,f^{abc}$.

The following quartic Casimir operators appear in our results at order α_s^4:

$$\tilde{d}_{FA} = \frac{d_F^{abcd}d_A^{abcd}}{d_F}, \quad \tilde{d}_{FF,r} = \frac{d_F^{abcd}d_{F,r}^{abcd}}{d_F}, \tag{9}$$

with $d_F^{abcd} \equiv d_{F,1}^{abcd}$ and $\tilde{d}_{FF} \equiv \frac{d_F^{abcd}d_{F,1}^{abcd}}{d_F}$.

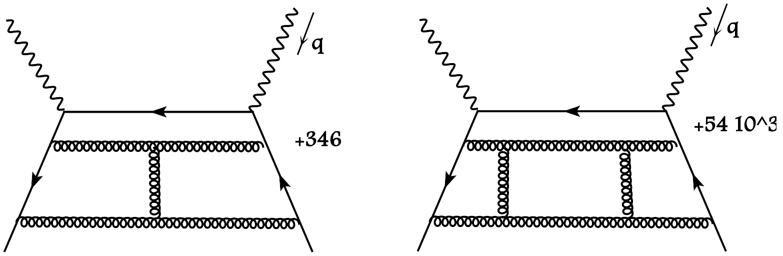

Fig. 1. Examples of diagrams contributing to the coefficient function C_{NS}^{Bjp} at three and four loops.

2.2. *Adler function in QCDe*

We start with the (non-singlet) polarization function $\Pi(L,a)$ of the vector current $j_\alpha = \overline{\psi}_q \gamma_\alpha \psi_q$ and defined as

$$(-g_{\alpha\beta}q^2 + q_\alpha q_\beta)\, \Pi^{\mathrm{NS}}(L,a) = i \int \mathrm{d}^4 x\, e^{iq\cdot x} \langle 0|\mathrm{T} j_\alpha(x)\, j_\beta(0)|0\rangle^{\mathrm{NS}}, \tag{10}$$

where $Q^2 = -q^2$, $L = \ln \frac{\mu^2}{Q^2}$. It is understood that the rhs of (10) includes only non-singlet diagrams, that is those with both external currents belonging to one and the same quark loop. The Adler function is defined as (the normalization factor below is conventionally fixed by the requirement that in Born approximation the Adler function starts from one) [33]

$$d_F\, n_f\, D(L,a_s) = -12\pi^2\, Q^2 \frac{\mathrm{d}}{\mathrm{d}Q^2} \Pi^{\mathrm{NS}}(L,a). \tag{11}$$

It is worthwhile to note that the Adler function unlike the polarization operator (10) is scale invariant due to the derivative in Q^2 which kills a quadratic UV divergence of (10) around integration region $x \approx 0$.

2.3. *Bjorken function in QCDe*

The most convenient for us definition of the coefficient function S^{Bjp} comes from the following Operator Product Expansion (OPE):

$$\int T[j_\alpha(x)\, j_\beta(0)]\, e^{iqx}\, dx|_{q^2\to -\infty} \approx \frac{q^\sigma}{q^2} C_{\mathrm{NS}}^{Bjp}(L,a)\, \epsilon_{\alpha\beta\rho\sigma}\, A_\rho(0) \tag{12}$$
$$+ \ldots \text{ (singlet and other terms)},$$

where $A_\rho = \overline{\psi}_q\, \gamma_\rho \gamma_5\, \psi_q$ is the axial vector current. The function C_{NS}^{Bjp} is by definition contributed by non-singlet diagrams only (see Fig. 1). In what follows we will not write index NS explicitly. Singlet contributions to OPE (12) were discussed in [34]. There is a technical subtlety in definition of the γ_5 matrix appearing in the definition of the axial current. Following works [16,14] we will use so-called Larin's approach [35]. It means that renormalized axial current is defined as

$$A_\rho \equiv z_A^{\mathrm{NS}} \frac{1}{6} \epsilon_{\alpha\beta\sigma\rho} [\,\overline{\psi}_q \gamma_{[\alpha\beta\sigma]} \psi_q\,]^{\mathrm{MS}}, \quad \gamma_{[\alpha\beta\sigma]} \equiv \frac{1}{2}(\gamma_\alpha\gamma_\beta\gamma_\sigma - \gamma_\sigma\gamma_\beta\gamma_\alpha), \tag{13}$$

Table 1
Proliferation of the n_f-dependent color factors in the QCDe-model.

QCD	QCDe
α_s^2	
$n_f T_f$	$\mathbf{nT}$
α_s^3	
$C_F^2 n_f T_f$	$C_F^2 \mathbf{nT},\ C_f\, \mathbf{nTC1}$
$C_F n_f^2 T_f^2$	$C_F (\mathbf{nT})^2$
$C_F C_A\, n_f T_f$	$C_F C_A\, \mathbf{nT}$
α_s^4	
$C_F^3 n_f T_f$	$C_F^3 \mathbf{nT},\ C_f^2\, \mathbf{nTC1},\ C_f\, \mathbf{nTC2}$
$C_F^2 n_f^2 T_f^2$	$C_F (\mathbf{nT})^2, C_f\, \mathbf{nTC1}$
$C_F^2 C_A\, n_f T_f$	$C_F^2 C_A\, \mathbf{nT}, C_F C_A\, \mathbf{nTC1}$
$C_F n_f^3 T_f^3$	$C_F (\mathbf{nT})^3$
$C_F C_A n_f^2 T_f^2$	$C_F C_A\, (\mathbf{nT})^2$
$C_F C_A^2 n_f T_f$	$C_F C_A^2 (\mathbf{nT})$
$n_f \tilde{d}_{FF}$	$\sum_r n_{f,r}\, \tilde{d}_{FF,r}$

where $[\,\overline{\psi}_q \gamma_{[\alpha\beta\sigma]} \psi_q\,]^{\mathrm{MS}}$ stands for the $\overline{\mathrm{MS}}$-renormalized current. The (finite) factor z_A^{NS} is chosen in such a way to effectively restore the anticommutativity of γ_5 (see corresponding discussion in [35,36]).

2.4. *Color factors*

For future reference let us describe color factors which appear in all three components of the CBK relation. First, we note an obvious fact that one and the same collection of color factors may appear in d_n, c_n. Second, due to the prefactor $\beta(a)$ in (3), the same set of color factors describes coefficient K_{n+1}. This is true for both QCD and QCDe cases [14]. Another important fact is that transition from QCD to QCDe does touch only n_f-dependent color factors.[3] The corresponding modifications are shown in Table 1. Here we use the following notations:

$$\mathbf{nT} \equiv \sum_i n_{f,i} T_{F,i}, \quad \mathbf{nTC1} \equiv \sum_i n_{f,i} T_{F,i} C_{F,i}, \quad \mathbf{nTC2} \equiv \sum_i n_{f,i} T_{F,i} C_{F,i}^2. \tag{14}$$

Note that if a color structure in the left column of Table 1 does not proliferate then the corresponding contributions should be identical in QCD and QCDe results.

In order to transform a QCDe result to the corresponding one in the standard QCD one should make the following replacements:

$$\begin{gathered} \mathbf{nT} \to n_f T_F, \quad \mathbf{nTC1} \to n_f T_F C_F, \\ \mathbf{nTC2} \to n_f T_F C_F^2, \ \sum_r n_{f,r} \tilde{d}_{FF,r} \to n_f \tilde{d}_{FF}. \end{gathered} \tag{15}$$

---

[3] This means that contributions proportional to n_f-independent color factors are identical in both cases.

An inspection of Table 1 clearly shows that in the case of the QCDe-model the number of extra constraints imposed by the CBK relation on the combinations $d_3 + c_3$ and $d_4 + c_4$ is increased from 3 and 6 to 4 and 9 correspondingly.

3. Calculation and results

3.1. Results for D and S^{Bjp}

We have computed the functions D and S^{Bjp} to order $\mathcal{O}(\alpha_s^4)$ using essentially the same methods as in [14] (for a short review see [37]). All momentum diagrams have been generated with QGRAF [38] and reduced to master integrals (well known from [39,40]) with the help of the $1/D$ expansion [41,42].

For calculation of color factors we have employed a generalization of the FORM [43] package COLOR [32] developed by M. Zoller [31]. Below we present our results for the Adler fuction and the coefficient function S^{Bjp} as defined by (1), (12). Note that we set $\mu^2 = Q^2$; the full dependence on μ can be easily restored by expressing $a(Q^2)$ by $a(\mu^2)$ with the help of the standard RG evolution equation for a (the β-function for QCDe is known at four loops from [31]).

$$d_1 = 3C_F, \tag{16}$$

$$d_2 = -\frac{3}{2}C_F^2 + C_F C_A\left(\frac{123}{2} - 44\zeta_3\right) - 2C_F(\mathbf{nT})(11 - 8\zeta_3), \tag{17}$$

$$\begin{aligned}
d_3 = {} & -\frac{69}{2}C_F^3 + \\
& C_F^2\Big[C_A(-127 - 572\zeta_3 + 880\zeta_5) + (\mathbf{nT})(72 + 208\zeta_3 - 320\zeta_5)\Big] + \\
& C_F C_A^2\left(\frac{90445}{54} - \frac{10948}{9}\zeta_3 - \frac{440}{3}\zeta_5\right) + \\
& C_F C_A(\mathbf{nT})\left(-\frac{31040}{27} + \frac{7168}{9}\zeta_3 + \frac{160}{3}\zeta_5\right) + \\
& C_F(\mathbf{nT})^2\left(\frac{4832}{27} - \frac{1216}{9}\zeta_3\right) + C_F(\mathbf{nTC1})(-101 + 96\zeta_3),
\end{aligned} \tag{18}$$

$$\begin{aligned}
d_4 = {} & C_F^4\left(\frac{4157}{8} + 96\zeta_3\right) + \\
& C_F^3\Big[C_A(-2024 - 278\zeta_3 + 18040\zeta_5 - 18480\zeta_7) \\
& \qquad -\mathbf{nT}(-298 + 56\zeta_3 + 6560\zeta_5 - 6720\zeta_7)\Big] + \\
& C_F^2\Big[C_A^2\left(-\frac{592141}{72} - \frac{87850}{3}\zeta_3 + \frac{104080}{3}\zeta_5 + 9240\zeta_7\right) + \\
& \quad C_A(\mathbf{nT})\left(\frac{67925}{9} + \frac{61912}{3}\zeta_3 - \frac{83680}{3}\zeta_5 - 3360\zeta_7\right) + \\
& (\mathbf{nT})^2\left(-\frac{13466}{9} - \frac{10240}{3}\zeta_3 + \frac{16000}{3}\zeta_5\right) + \mathbf{nTC1}(251 + 576\zeta_3 - 960\zeta_5)\Big] +
\end{aligned}$$

$$
\begin{aligned}
& C_F\Bigg[C_A^3\left(\frac{52207039}{972}-\frac{912446}{27}\zeta_3-\frac{155990}{9}\zeta_5+4840\zeta_3^2-1540\zeta_7\right)\\
&\quad C_A^2(\mathbf{nT})\left(-\frac{4379861}{81}+\frac{275488}{9}\zeta_3+\frac{150440}{9}\zeta_5-1408\zeta_3^2+560\zeta_7\right)+\\
&\quad C_A(\mathbf{nT})^2\left(\frac{1363372}{81}-\frac{83624}{9}\zeta_3-\frac{43520}{9}\zeta_5-128\zeta_3^2\right)+\\
&\quad C_A(\mathbf{nTC1})\left(-\frac{375193}{54}+7792\zeta_3+400\zeta_5-2112\zeta_3^2\right)+\\
&\quad (\mathbf{nT})^3\left(-\frac{392384}{243}+\frac{25984}{27}\zeta_3+\frac{1280}{3}\zeta_5\right)+\\
&\quad (\mathbf{nT})(\mathbf{nTC1})\left(\frac{63250}{27}-2784\zeta_3+768\zeta_3^2\right)+\\
&\qquad \mathbf{nTC2}\left(\frac{355}{3}+272\zeta_3-480\zeta_5\right)\Bigg]-\\
& 16\left[\sum_r n_{f,r}\,\tilde{d}_{\mathrm{FF},r}\cdot(13+16\zeta_3-40\zeta_5)+\tilde{d}_{\mathrm{FA}}\cdot(-3+4\zeta_3+20\zeta_5)\right]. \qquad (19)
\end{aligned}
$$

The results for c_k of the Bjorken SR in QCDe,

$$c_1=-3C_F, \qquad (20)$$

$$c_2=\frac{21}{2}C_F^2-23C_AC_F+8C_F(\mathbf{nT}), \qquad (21)$$

$$
\begin{aligned}
c_3={}&-\frac{3}{2}C_F^3+C_F^2\left[C_A\left(\frac{1241}{9}-\frac{176}{3}\zeta_3\right)-\mathbf{nT}\left(\frac{664}{9}-\frac{64}{3}\zeta_3\right)\right]+\\
&C_FC_A^2\left(-\frac{10874}{27}+\frac{440}{3}\zeta_5\right)+\\
&C_FC_A(\mathbf{nT})\left(\frac{7070}{27}+48\zeta_3-\frac{160}{3}\zeta_5\right)-\\
&C_F(\mathbf{nT})^2\frac{920}{27}+C_F(\mathbf{nTC1})(59-48\zeta_3), \qquad (22)
\end{aligned}
$$

$$
\begin{aligned}
c_4={}&-C_F^4\left(\frac{4823}{8}+96\zeta_3\right)+ \qquad (23)\\
&C_F^3\left[-C_A\left(\frac{3707}{18}+\frac{7768}{3}\zeta_3-\frac{16720}{3}\zeta_5\right)+\right.\\
&\qquad\left.\mathbf{nT}\left(\frac{5912}{9}+\frac{3296}{3}\zeta_3-\frac{6080}{3}\zeta_5\right)\right]+\\
&C_F^2\Bigg[C_A^2\left(\frac{1071641}{216}+\frac{25456}{9}\zeta_3-\frac{22000}{9}\zeta_5-6160\zeta_7\right)-\\
&\quad C_A(\mathbf{nT})\left(\frac{106081}{27}+\frac{9104}{9}\zeta_3-\frac{8000}{9}\zeta_5-2240\zeta_7\right)+\\
&\quad (\mathbf{nT})^2\left(\frac{16114}{27}-\frac{512}{3}\zeta_3\right)-\mathbf{nTC1}\left(\frac{1399}{3}-400\zeta_3\right)\Bigg]+
\end{aligned}
$$

$$
\begin{aligned}
&C_F\Bigg[C_A^3\left(-\frac{8004277}{972}+\frac{4276}{9}\zeta_3+\frac{25090}{9}\zeta_5-\frac{968}{3}\zeta_3^2+1540\zeta_7\right)+\\
&\quad C_A^2(\mathbf{nT})\left(\frac{1238827}{162}+236\zeta_3-\frac{14840}{9}\zeta_5+\frac{704}{3}\zeta_3^2-560\zeta_7\right)-\\
&\quad C_A(\mathbf{nT})^2\left(\frac{165283}{81}+\frac{688}{9}\zeta_3-\frac{320}{3}\zeta_5+\frac{128}{3}\zeta_3^2\right)+\\
&\quad C_A(\mathbf{nTC1})\left(\frac{124759}{54}-1280\zeta_3-400\zeta_5\right)+\\
&\quad \frac{38720}{243}(\mathbf{nT})^3-(\mathbf{nT})(\mathbf{nTC1})\left(\frac{19294}{27}-480\zeta_3\right)-\\
&\quad \mathbf{nTC2}\left(\frac{292}{3}+296\zeta_3-480\zeta_5\right)\Bigg]+\\
&16\left[\sum_r n_{f,r}\,\tilde{d}_{\mathrm{FF},r}\cdot(13+16\zeta_3-40\zeta_5)+\tilde{d}_{\mathrm{FA}}\cdot(-3+4\zeta_3+20\zeta_5)\right].
\end{aligned}
$$

4. CBK relation in QCDe

Using the color structures of d_2 and d_3 as templates we find that the CBK relation (3) is indeed fulfilled identically with the following values for the coefficients K_i:

$$K_1 = C_F\left(-\frac{21}{2}+12\,\zeta_3\right), \tag{24}$$

$$
\begin{aligned}
K_2 = {} & C_F^2\left(\frac{397}{6}+136\,\zeta_3-240\,\zeta_5\right)+C_FC_A\left(-\frac{629}{2}+\frac{884}{3}\,\zeta_3\right)\\
&+(C_F\mathbf{nT})\left(\frac{326}{3}-\frac{304}{3}\,\zeta_3\right),
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
K_3 = {} & C_F^3\left(\frac{2471}{12}+488\,\zeta_3-5720\,\zeta_5+5040\,\zeta_7\right)\\
&+C_F^2C_A\left(\frac{99757}{36}+\frac{16570}{3}\,\zeta_3-\frac{24880}{3}\,\zeta_5-840\,\zeta_7\right)\\
&+C_FC_A^2\left(-\frac{406043}{36}+\frac{72028}{9}\,\zeta_3-1232\,\zeta_3^2+\frac{11900}{3}\,\zeta_5\right)\\
&+C_FC_A(\mathbf{nT})\left(\frac{67520}{9}-\frac{40336}{9}\,\zeta_3-\frac{8000}{3}\,\zeta_5-128\,\zeta_3^2\right)\\
&+C_F(\mathbf{nT})^2\left(-\frac{9824}{9}+\frac{6496}{9}\,\zeta_3+320\,\zeta_5\right)\\
&+C_F^2(\mathbf{nT})\left(-\frac{11573}{9}-2288\,\zeta_3+4000\,\zeta_5\right)\\
&+C_F(\mathbf{nTC1})\left(\frac{1713}{2}-1380\,\zeta_3+576\,\zeta_3^2\right).
\end{aligned}
\tag{26}
$$

As expected from Table 1 and relations (15) and (15) coefficient K_2 in QCD is essentially identical to the one in QCD (that is after identification **nT** with $n_f T_f$). Coefficient K_3 in QCDe is different from the case of QCD only by 2 last terms. All constraints imposed by the CBK relation are fulfilled.

5. Conclusion

We have computed the nonsinglet Adler D-function and the coefficient function for Bjorken polarized sum rules S^{Bjp} at order $O(\alpha^4)$ in the extended QCD model. The CBK relation is confirmed.

These results have been extensively used for construction and analyzing explicit expressions for the elements of the $\{\beta\}$-expansion for the nonsinglet Adler D-function and Bjorken polarized sum rules S^{Bjp} in the N^{4}LO and higher orders in [44].

They may be also useful for renormalization group analysis of the D and S^{Bjp} functions in large-N_c and large-N_f limits [45,46].

For readers's convenience all our results are collected in an ancillary file.

CRediT authorship contribution statement

Chetyrkin: Methodology, Software, Computation, Writing, Reviewing and Editing

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

Author is grateful to P. A. Baikov and S. V. Mikhailov for essential help, useful comments and good advice. I would like to express my special thanks to M. Zoller for his kind permission to use his extension of the package COLOR.

The work was supported in part by DFG grant CH 1479/2-1.

References

[1] R.J. Crewther, Nonperturbative evaluation of the anomalies in low-energy theorems, Phys. Rev. Lett. 28 (1972) 1421, https://doi.org/10.1103/PhysRevLett.28.1421.

[2] D.J. Broadhurst, A.L. Kataev, Connections between deep inelastic and annihilation processes at next to next-to-leading order and beyond, Phys. Lett. B 315 (1993) 179–187, https://doi.org/10.1016/0370-2693(93)90177-J, arXiv:hep-ph/9308274.

[3] G. 't Hooft, Dimensional regularization and the renormalization group, Nucl. Phys. B 61 (1973) 455–468, https://doi.org/10.1016/0550-3213(73)90376-3.

[4] W.A. Bardeen, A. Buras, D. Duke, T. Muta, Deep inelastic scattering beyond the leading order in asymptotically free gauge theories, Phys. Rev. D 18 (1978) 3998, https://doi.org/10.1103/PhysRevD.18.3998.

[5] K.G. Chetyrkin, A.L. Kataev, F.V. Tkachov, Higher order corrections to Sigma-t (e+ e- —> hadrons) in quantum chromodynamics, Phys. Lett. B 85 (1979) 277–279, https://doi.org/10.1016/0370-2693(79)90596-3.

[6] M. Dine, J.R. Sapirstein, Higher order QCD corrections in e+ e- annihilation, Phys. Rev. Lett. 43 (1979) 668, https://doi.org/10.1103/PhysRevLett.43.668.

[7] W. Celmaster, R.J. Gonsalves, An analytic calculation of higher order quantum chromodynamic corrections in e+ e, Phys. Rev. Lett. 44 (1980) 560, https://doi.org/10.1103/PhysRevLett.44.560.

[8] S.G. Gorishnii, A.L. Kataev, S.A. Larin, The $O(\alpha_s^3)$-corrections to $\sigma_{tot}(e^+e^- \to hadrons)$ and $\Gamma(\tau^- \to \nu_\tau + hadrons)$ in QCD, Phys. Lett. B 259 (1991) 144–150, https://doi.org/10.1016/0370-2693(91)90149-K.

[9] K.G. Chetyrkin, Corrections of order alpha-s**3 to R(had) in pQCD with light gluinos, Phys. Lett. B 391 (1997) 402–412, https://doi.org/10.1016/S0370-2693(96)01478-5, arXiv:hep-ph/9608480.

[10] P.A. Baikov, K.G. Chetyrkin, J.H. Kuhn, Order alpha**4(s) QCD corrections to Z and tau decays, Phys. Rev. Lett. 101 (2008) 012002, https://doi.org/10.1103/PhysRevLett.101.012002, arXiv:0801.1821.

[11] F. Herzog, B. Ruijl, T. Ueda, J.A.M. Vermaseren, A. Vogt, On Higgs decays to hadrons and the R-ratio at N^{4}LO, J. High Energy Phys. 08 (2017) 113, https://doi.org/10.1007/JHEP08(2017)113, arXiv:1707.01044.

[12] S.G. Gorishnii, S.A. Larin, F.V. Tkachov, K.G. Chetyrkin, Higher QCD corrections to the Bjorken Sum Rule, Phys. Lett. B 137 (1984) 230, https://doi.org/10.1016/0370-2693(84)90235-1.

[13] S.A. Larin, F.V. Tkachov, J.A.M. Vermaseren, The alpha(s**3) correction to the Bjorken sum rule, Phys. Rev. Lett. 66 (1991) 862–863, https://doi.org/10.1103/PhysRevLett.66.862.

[14] P.A. Baikov, K.G. Chetyrkin, J.H. Kuhn Adler, Function, Bjorken Sum Rule, and the Crewther relation to order α_s^4 in a general gauge theory, Phys. Rev. Lett. 104 (2010) 132004, https://doi.org/10.1103/PhysRevLett.104.132004, arXiv:1001.3606.

[15] L.R. Surguladze, M.A. Samuel, Total hadronic cross-section in e+ e- annihilation at the four loop level of perturbative QCD, Phys. Rev. Lett. 66 (1991) 560–563, https://doi.org/10.1103/PhysRevLett.66.560, Phys. Rev. Lett. 66 (1991) 2416, Erratum.

[16] S.A. Larin, J.A.M. Vermaseren, The alpha-s**3 corrections to the Bjorken sum rule for polarized electroproduction and to the Gross-Llewellyn Smith sum rule, Phys. Lett. B 259 (1991) 345–352, https://doi.org/10.1016/0370-2693(91)90839-I.

[17] R.J. Crewther, Relating inclusive e+ e- annihilation to electroproduction sum rules in quantum chromodynamics, Phys. Lett. B 397 (1997) 137–142, https://doi.org/10.1016/S0370-2693(97)00157-3, arXiv:hep-ph/9701321.

[18] V.M. Braun, G.P. Korchemsky, D. Müller, The uses of conformal symmetry in QCD, Prog. Part. Nucl. Phys. 51 (2003) 311–398, https://doi.org/10.1016/S0146-6410(03)90004-4, arXiv:hep-ph/0306057.

[19] A.V. Garkusha, A.L. Kataev, The absence of QCD β-function factorization property of the generalized Crewther relation in the ‘t Hooft $\bar{M}S$-based scheme, Phys. Lett. B 705 (2011) 400–404, https://doi.org/10.1016/j.physletb.2011.10.036, arXiv:1108.5909.

[20] S.V. Mikhailov, Generalization of BLM procedure and its scales in any order of pQCD: a practical approach, J. High Energy Phys. 06 (2007) 009, https://doi.org/10.1088/1126-6708/2007/06/009, arXiv:hep-ph/0411397.

[21] S.V. Mikhailov, On a realization of $\{\beta\}$-expansion in QCD, J. High Energy Phys. 04 (2017) 169, https://doi.org/10.1007/JHEP04(2017)169, arXiv:1610.01305.

[22] A.L. Kataev, S.V. Mikhailov, New perturbation theory representation of the conformal symmetry breaking effects in gauge quantum field theory models, Theor. Math. Phys. 170 (2012) 139–150, https://doi.org/10.1007/s11232-012-0016-7, arXiv:1011.5248.

[23] A.L. Kataev, S.V. Mikhailov, Generalization of the Brodsky-Lepage-Mackenzie optimization within the β-expansion and the principle of maximal conformality, Phys. Rev. D 91 (1) (2015) 014007, https://doi.org/10.1103/PhysRevD.91.014007, arXiv:1408.0122.

[24] D. Kotlorz, S.V. Mikhailov, Optimized determination of the polarized Bjorken sum rule in pQCD, Phys. Rev. D 100 (5) (2019) 056007, https://doi.org/10.1103/PhysRevD.100.056007, arXiv:1810.02973.

[25] S.J. Brodsky, X.-G. Wu, Scale setting using the extended renormalization group and the principle of maximum conformality: the QCD coupling constant at four loops, Phys. Rev. D 85 (2012) 034038, https://doi.org/10.1103/PhysRevD.85.034038, arXiv:1111.6175, Phys. Rev. D 86 (2012) 079903, Erratum.

[26] S.J. Brodsky, L. Di Giustino, Setting the renormalization scale in QCD: the principle of maximum conformality, Phys. Rev. D 86 (2012) 085026, https://doi.org/10.1103/PhysRevD.86.085026, arXiv:1107.0338.

[27] S.J. Brodsky, M. Mojaza, X.-G. Wu, Systematic scale-setting to all orders: the principle of maximum conformality and commensurate scale relations, Phys. Rev. D 89 (2014) 014027, https://doi.org/10.1103/PhysRevD.89.014027, arXiv:1304.4631.

[28] J.-M. Shen, X.-G. Wu, Y. Ma, S.J. Brodsky, The generalized scheme-independent Crewther relation in QCD, Phys. Lett. B 770 (2017) 494–499, https://doi.org/10.1016/j.physletb.2017.05.022, arXiv:1611.07249.

[29] H.-H. Ma, X.-G. Wu, Y. Ma, S.J. Brodsky, M. Mojaza, Setting the renormalization scale in perturbative QCD: comparisons of the principle of maximum conformality with the sequential extended Brodsky-Lepage-Mackenzie approach, Phys. Rev. D 91 (9) (2015) 094028, https://doi.org/10.1103/PhysRevD.91.094028, arXiv:1504.01260.

[30] X.-G. Wu, J.-M. Shen, B.-L. Du, X.-D. Huang, S.-Q. Wang, S.J. Brodsky, The QCD renormalization group equation and the elimination of fixed-order scheme-and-scale ambiguities using the principle of maximum conformality, Prog. Part. Nucl. Phys. 108 (2019) 103706, https://doi.org/10.1016/j.ppnp.2019.05.003, arXiv:1903.12177.

[31] M.F. Zoller, Four-loop QCD β-function with different fermion representations of the gauge group, J. High Energy Phys. 10 (2016) 118, https://doi.org/10.1007/JHEP10(2016)118, arXiv:1608.08982.

[32] T. Van Ritbergen, A. Schellekens, J. Vermaseren, Group theory factors for feynman diagrams, Int. J. Mod. Phys. A 14 (1) (1999) 41–96.

[33] S.L. Adler, Some simple vacuum polarization phenomenology: $e^+e^- \to$ hadrons: the μ-mesic atom x-ray discrepancy and $g_\mu - 2$, Phys. Rev. D 10 (1974) 3714, https://doi.org/10.1103/PhysRevD.10.3714.

[34] S.A. Larin, The singlet contribution to the Bjorken sum rule for polarized deep inelastic scattering, Phys. Lett. B 723 (2013) 348–350, https://doi.org/10.1016/j.physletb.2013.05.026, arXiv:1303.4021.

[35] S. Larin, The renormalization of the axial anomaly in dimensional regularization, Phys. Lett. B 303 (1993) 113–118, https://doi.org/10.1016/0370-2693(93)90053-K, arXiv:hep-ph/9302240.

[36] K.G. Chetyrkin, J.H. Kühn, Neutral current in the heavy top quark limit and the renormalization of the singlet axial current, Z. Phys. C 60 (1993) 497–502, https://doi.org/10.1007/BF01560047.

[37] P.A. Baikov, K.G. Chetyrkin, J.H. Kühn, Massless propagators, $R(s)$ and multiloop QCD, Nucl. Part. Phys. Proc. 261–262 (2015) 3–18, https://doi.org/10.1016/j.nuclphysbps.2015.03.002, arXiv:1501.06739.

[38] P. Nogueira, Automatic Feynman graph generation, J. Comput. Phys. 105 (1993) 279–289, https://doi.org/10.1006/jcph.1993.1074.

[39] P.A. Baikov, K.G. Chetyrkin, Four loop massless propagators: an algebraic evaluation of all master integrals, Nucl. Phys. B 837 (2010) 186–220, https://doi.org/10.1016/j.nuclphysb.2010.05.004, arXiv:1004.1153.

[40] R.N. Lee, A.V. Smirnov, V.A. Smirnov, Master integrals for four-loop massless propagators up to transcendentality weight twelve, Nucl. Phys. B 856 (2012) 95–110, https://doi.org/10.1016/j.nuclphysb.2011.11.005, arXiv:1108.0732.

[41] P.A. Baikov, A practical criterion of irreducibility of multi-loop Feynman integrals, Phys. Lett. B 634 (2006) 325–329, https://doi.org/10.1016/j.physletb.2006.01.052, arXiv:hep-ph/0507053.

[42] P.A. Baikov, Explicit solutions of the three loop vacuum integral recurrence relations, Phys. Lett. B 385 (1996) 404–410, https://doi.org/10.1016/0370-2693(96)00835-0, arXiv:hep-ph/9603267.

[43] J.A.M. Vermaseren, New features of FORM, arXiv:math-ph/0010025.

[44] P.A. Baikov, S.V. Mikhailov, The $\{\beta\}$-expansion for Adler function, Bjorken Sum Rule, and the Crewther-Broadhurst-Kataev relation at order $O(\alpha_s^4)$, arXiv:2206.14063.

[45] T.A. Ryttov, R. Shrock, Scheme-independent calculations of properties at a conformal infrared fixed point in gauge theories with multiple fermion representations, Phys. Rev. D 98 (9) (2018) 096003, https://doi.org/10.1103/PhysRevD.98.096003, arXiv:1809.02242.

[46] S. Girmohanta, T.A. Ryttov, R. Shrock, Large-N_c and large-N_F limits of SU(N_c) gauge theories with fermions in different representations, Phys. Rev. D 99 (11) (2019) 116022, https://doi.org/10.1103/PhysRevD.99.116022, arXiv:1903.09672.