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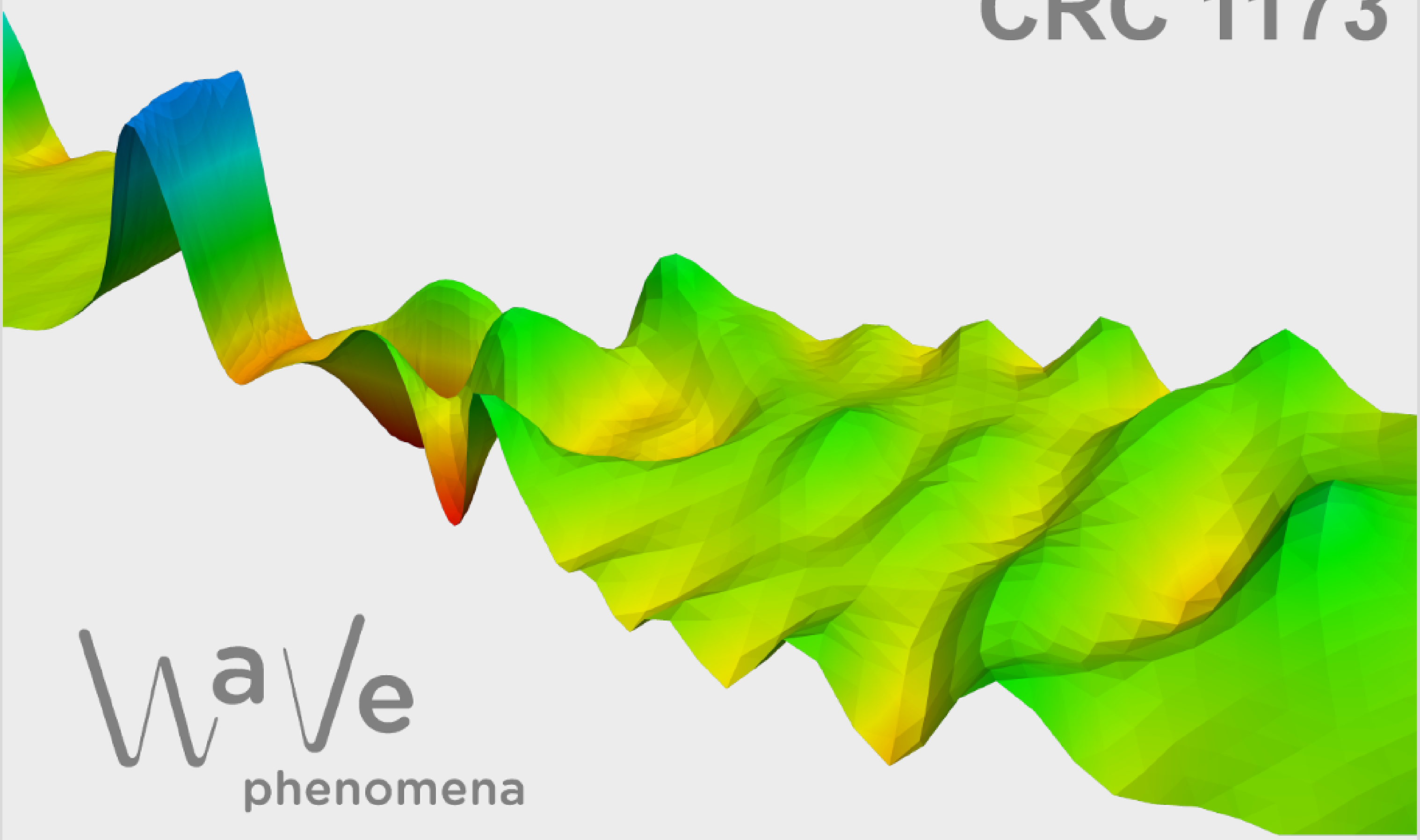

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On Potential-Based Shape Derivatives of the Electromagnetic Transmission Problem

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Abstract

Domain derivatives are an important tool to characterize and compute shape derivatives. If some quantity of interest depends on the shape of an object such as the obstacle in a scattering problem, shape derivatives are used to describe the effect of variations of the shape on that quantity. We here consider the scattering of time-harmonic electromagnetic waves by a penetrable obstacle. As an alternative to the formulation using the Maxwell system, the problem may be posed as a coupled system of Helmholtz equations with complicated transmission conditions. We prove equivalence of the two formulations and then proceed to characterize the domain derivatives of the scattered fields in the potential formulation. Our main result is the equivalence of the characterizations of such derivatives in the Maxwell and in the potential based problem formulation.

Mathematics subject classifications (MSC2010): 35Q61, 78A25
Keywords: Maxwell's equations, domain derivative for penetrable scatterer, potential formulation

1 Introduction

Shape derivatives are an indispensable tool in a range of inverse and optimal design problems. An objective functional such as the misfit between measured far field data and the field produced by a given scatterer in an inverse scattering problem, depends on the shape of the given object. Optimization of the objective functional through gradient based methods naturally leads to the question of how derivatives of this functional with respect to the shape of the object can be computed. Computationally, it is useful that this derivative can be characterized by the domain derivative, a solution of the original PDE, but with different boundary conditions.

In the present paper, we are concerned with shape derivatives for the electromagnetic scattering problem for a bounded penetrable object. We consider a time-harmonic electromagnetic incident field $(\boldsymbol{E}^i, \boldsymbol{H}^i)$ in a homogeneous isotropic background medium characterised by a constant positive permittivity ε_0 and permeability μ_0. Suppose that in a bounded, open set $D \subseteq \mathbb{R}^3$, the permittivity ε and permeability μ differ from the background values. The total field $(\boldsymbol{E}, \boldsymbol{H})$ then is a solution to the system of Maxwell's equations

$$\mathbf{curl}\, \boldsymbol{E} - \mathrm{i}\omega\, \mu \boldsymbol{H} = 0\,, \qquad \mathbf{curl}\, \boldsymbol{H} + \mathrm{i}\omega\, \varepsilon \boldsymbol{E} = 0 \tag{1}$$

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in $\mathbb{R}^3$, while the incident field satisfies this equation for the material parameters ε_0 and μ_0 throughout $\mathbb{R}^3$. Moreover, the scattered field $(\boldsymbol{E}^s, \boldsymbol{H}^s) = (\boldsymbol{E}, \boldsymbol{H}) - (\boldsymbol{E}^i, \boldsymbol{H}^i)$ is assumed to satisfy the Silver-Müller radiation condition

$$\lim_{|\boldsymbol{x}|\to\infty} \left(\sqrt{\mu_0}\, \boldsymbol{H}^s(x) \times \boldsymbol{x} - |\boldsymbol{x}|\, \sqrt{\varepsilon_0}\, \boldsymbol{E}^s(\boldsymbol{x})\right) = 0\,, \tag{2}$$

uniformly in $\hat{\boldsymbol{x}} = \boldsymbol{x}/|\boldsymbol{x}|$. The objective functional typically is a functional of this scattered field and hence inherits the dependence of the scattered field on the shape and physical properties of the scatterer D.

In this paper, we will be mainly concerned with the well-known approach of representing the electromagnetic field by potentials: one may choose a vector field $\boldsymbol{A}$ and a scalar function φ such that

$$\boldsymbol{E} = \mathrm{i}\omega\, \boldsymbol{A} - \mathbf{grad}\, \varphi\,, \qquad \boldsymbol{H} = \frac{1}{\mu}\, \mathbf{curl}\, \boldsymbol{A}\,. \tag{3}$$

Analogous representations for the incident and scattered fields hold. Such representations are not unique. Under the assumption that $\operatorname{div} \boldsymbol{A} = 0$ in an appropriate sense, Helmholtz decompositions will provide them. A different approach, more useful for computational purposes, is to impose the *Lorentz gauge condition*

$$\operatorname{div} \boldsymbol{A} - \mathrm{i}\omega\varepsilon\mu\, \varphi = 0 \tag{4}$$

instead. This approach has been pursued in [6], where the authors show that for piecewise homogeneous media, the approach leads to a well-posed system of integral equations defined on the boundary of the scatterer that can be used to compute the potentials $\boldsymbol{A}^s$ and φ^s of the scattered fields.

In our present work, we pursue how to use these potential representations for computing domain derivatives. The new results are two-fold: Firstly, we give formulations of the method described in [6] in mathematical terms, using the language of weak formulations of boundary value problems, and mathematically rigorously establish equivalence to the original electromagnetic scattering problem. Moreover, and more importantly, in the case when the material properties are piecewise constant, we characterize the domain derivatives of the potentials $\boldsymbol{A}^s$, φ^s and establish their relation to the domain derivative of $\boldsymbol{E}^s$.

Let us motivate why this approach has computational advantages over other established ways to compute domain derivatives in relevant situations. One example of an application in which such scattering problems feature prominently is the optimal design of an object with respect to its electromagnetic chirality [3, 10]. An electromagnetic field can canonically be split into components of pure circular polarizations or helicities. Electromagnetic chirality is a physical concept describing how different a scatterer interacts with fields of the two helicities. The goal is to design a scatterer that is (nearly) invisible to incident fields of one helicity while it strongly interacts with fields of the opposite helicity. Using a simplified model [5], such designs have been realized using shape derivatives [2]. Particular examples have been found at or near optical frequencies and consist of thin elongated structures that are made of noble metals [11]. However, these objects are too thin to realistically be fabricable in the near future, and thus thicker objects need to be considered for which the simplified model is no longer valid. The goal thus is to solve electromagnetic scattering problems and compute domain derivatives for piecewise homogeneous media in a regime where the real part of the electromagnetic permittivity inside the scatterer is negative. Although the problem in principle can be formulated as a conventional uniquely solvable system of integral equations and corresponding domain derivatives can be computed [14], the material properties lead to slow convergence of iterative solvers when solving the corresponding discretized system. This renders the approach unusable in practice.

As discussed in [6] and also below in section 3, the formulation of the scattering problem and characterization of the domain derivatives via potentials can be formulated as a system of integral equations with operators arising in conjunction with the Helmholtz equation rather than the Maxwell system. A solution strategy is provided that requires only the inversion of discretizations of boundary operators that are applied to one scalar unknown, greatly reducing computational complexity. The application of a direct solver becomes viable, avoiding convergence issues with iterative solvers. Moreover, in the numerical computation of a domain derivative the solver typically has to be invoked many times for different right hand sides, making a direct solver even more attractive.

There are possible alternatives that have been derived recently to robustly solve boundary integral equations for electromagnetic scattering for a wide range of material properties. The Debye poential based approach of [7–9] has a very convincing theoretical foundation, however it requires non-standard integral operators that are difficult to implement for complex geometries. Other integral formulations for the time-harmonic Maxwell system with robust dependence on the material properties have been reported on [12, 13, 16, 17], but require substantial implementational efforts to evaluate.

In what follows, we will discuss the mathematical formulation of the electromagnetic scattering problem and its formulation using potentials satisfying (3), (4) in section 2. In particular, we will provide a mathematical formulation of the transmission problem for $\boldsymbol{A}$, φ given in [6] and establish that a solution of this problem in turn gives rise to an electromagnetic field $(\boldsymbol{E}, \boldsymbol{H})$ that solves the original problem. Thus we establish well-posedness of the transmission problem for the potentials. Limiting ourselves to the case of piecewise constant media in section 3, we discuss the solution of the problem via an integral equation formulation. Finally, in section 4, we consider the domain derivatives of the potentials and characterize them as the solutions of a transmission problem with the same operators as in the problem for $\boldsymbol{A}$ and φ. As the central result of this paper, in Theorem 4.5, we prove that the domain derivative of the electromagnetic scattered field can in turn be obtained from the domain derivatives of the potentials. Some results which are not essentially new, have been collected in the appendix. This in particular concerns the characterizations of the domain derivatives which are analogous to previously established results [15, 18] except for the the more complicated transmission conditions.

2 The electromagnetic transmission problem

Throughout this paper we will work with a weak formulation of the problem (1), (2) posed in appropriate Sobolev spaces. Given some bounded Lipschitz domain Ω, we use the usual Hilbert spaces

$$H^1(\Omega) = \{u \in L^2(\Omega) : \mathbf{grad}\, u \in L^2(\Omega)\}\,,$$
$$H^1(\Omega, \mathbb{C}^3) = \{\boldsymbol{U} = (\boldsymbol{U}_1, \boldsymbol{U}_2, \boldsymbol{U}_3)^\top : U_j \in H^1(\Omega)\}\,.$$

For vector valued functions $\boldsymbol{U}$ and a differential operator D, so that $D\boldsymbol{U}$ takes values in $\mathbb{C}^\ell$, we also require the Hilbert spaces

$$H(D, \Omega) = \{\boldsymbol{U} \in L^2(\Omega, \mathbb{C}^3) : D\boldsymbol{U} \in L^2(\Omega, \mathbb{C}^\ell)\}\,.$$

Typcial choices for D are **curl**, div or $\mathbf{curl}^2$. As is usual, a subscript loc will be used to indicate corresponding spaces of functions having the required regularity on every open set compactly contained in Ω.

We also require a number of trace spaces on $\partial\Omega$. The spaces $H^s(\partial\Omega)$, $s \in (-1, 1)$, and the corresponding H^{-s}-H^s-duality $\langle \cdot, \cdot \rangle_{\partial\Omega}$ and their analogues for vector fields, are standard. Let $\gamma_D : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ denote the usual Dirichlet trace operator. For a vector field $\boldsymbol{U} \in H(\mathrm{div}, \Omega)$, the normal trace will be denoted by $\boldsymbol{U}_{\boldsymbol{\nu}} = \boldsymbol{\nu} \cdot \boldsymbol{U} \in H^{-1/2}(\partial\Omega)$. For $u \in H^1(\Omega)$ with $\Delta u \in L^2(\Omega)$, the normal derivative on $\partial\Omega$ will be denoted by $\frac{\partial u}{\partial \boldsymbol{\nu}} \in H^{-1/2}(\partial\Omega)$.

In addition, there are a number of spaces of (in the weak sense) tangential fields. Denote by $\boldsymbol{\nu}$ the outward drawn unit normal vector to $\partial\Omega$ and define for a sufficiently smooth vector field

$$\gamma_t \boldsymbol{U} = \boldsymbol{U} \times \boldsymbol{\nu}\,,$$

and hereby the spaces,

$$L^2_t(\partial\Omega) = \{\boldsymbol{U} \in L^2(\partial\Omega, \mathbb{C}^3) : \boldsymbol{U}\cdot\boldsymbol{\nu} = 0\}$$
$$H^{1/2}_t(\partial\Omega) = \{\gamma_t \boldsymbol{U} : \boldsymbol{U} \in H^1(\Omega)\}\,, \qquad H^{-1/2}_t(\partial\Omega) = (H^{1/2}_t(\partial\Omega))^*$$
$$H^{-1/2}(\mathrm{Div}, \partial\Omega) = \{\boldsymbol{U} \in H^{-1/2}_t(\partial\Omega) : \mathrm{Div}\,\boldsymbol{U} \in H^{-1/2}(\partial\Omega)\}\,.$$

The symmetric bilinear form

$$\langle \boldsymbol{U}, \boldsymbol{V}\rangle_{t,\partial\Omega} = \int_{\partial\Omega} \boldsymbol{U}\cdot(\boldsymbol{\nu}\times\boldsymbol{V})\,\mathrm{d}s\,, \qquad \boldsymbol{U},\boldsymbol{V} \in L^2_t(\partial\Omega)\,,$$

extends to a duality between $H^{-1/2}(\mathrm{Div},\partial\Omega)$ and itself, and the tangential trace extends to a bounded operator $\gamma_t : H(\mathbf{curl},\Omega) \to H^{-1/2}(\mathrm{Div},\partial\Omega)$. Precise definitions of such relations and of the surface differential operators used here may be found in [4]. Note that for surface differential operators such as Div above, it will always be clear from the context with respect to which surfaces these operators are applied. Hence we do not include this surface as an index to the operator.

All these traces may differ when taken from opposite sides of a boundary. As usual, we will indicate that the trace is taken from the direction the normal vector is pointing to by a superscript +, and the opposite side by a superscript −. For normal components, this is indicated by $\boldsymbol{U}_{\boldsymbol{\nu}}|^{\pm} = \boldsymbol{\nu}\cdot\boldsymbol{U}|^{\pm}$ and for normal derivatives by $\frac{\partial u}{\partial \nu}|^{\pm}$. Jumps and mean values of traces taken from opposite sides of a boundary are denoted with the common bracket notation, e.g. in the case of a Dirichlet trace,

$$[\gamma_D u]_{\partial\Omega} = \gamma_D^+ u - \gamma_D^- u\,, \qquad \{\gamma_D u\}_{\partial\Omega} = \frac{1}{2}\left(\gamma_D^+ u + \gamma_D^- u\right).$$

For vector valued functions, additional notation for derivatives is required. We will denote the Jacobian matrix of any function $\boldsymbol{U} : \mathbb{R}^3 \to \mathbb{C}^3$ by $J_{\boldsymbol{U}}$. We will use this notation in particular when dealing with transformations of the domain and domain derivatives in section 4, but we also note the useful identity

$$\sum_{j=1}^{3} \int_\Omega \mathbf{grad}\,\boldsymbol{U}_j \cdot \mathbf{grad}\,\boldsymbol{V}_j \,\mathrm{d}\boldsymbol{x} = \int_\Omega \mathrm{tr}\left(J_{\boldsymbol{U}}\, J_{\boldsymbol{V}}^\top\right) \mathrm{d}\boldsymbol{x}\,.$$

Also, the vector of normal derivatives of the components of $\boldsymbol{U}$ can then be written as $J_{\boldsymbol{U}}\boldsymbol{\nu}$.

To conclude this section on notations, we note that we sometimes will make use of distributions. The action of a distribution f on a test function $\varphi \in C_0^\infty(\mathbb{R}^3)$ is denoted by (f, φ).

To ensure uniqueness of solution to the electromagnetic scattering problem, we need to prescribe regularity assumptions on the material parameters ε and μ. We will assume for now that both are piecewise continuously differentiable functions, that $\mathrm{Im}(\varepsilon) \geq 0$ and that there are constants c_1, $c_2 > 0$ such that $\mu \geq c_1$ on $\mathbb{R}^3$ and either $\mathrm{Re}(\varepsilon) \geq c_2$ on $\mathbb{R}^3$ or $\mathrm{Im}(\varepsilon) \geq c_2$ on some open set. Finally, we assume that outside some bounded Lipschitz domain D contained in the ball B_R of radius R centered at the origin, both functions are equal to positive constants ε_0 and μ_0, respectively. Later on in this paper, stronger assumptions will be imposed.

The transmission problem will be formulated in weak form in B_R. The incident field $\boldsymbol{E}^i, \boldsymbol{H}^i \in H_{\mathrm{loc}}(\mathbf{curl}, \mathbb{R}^3)$ is assumed to be a solution of (1) for ε, μ replaced by ε_0, μ_0 in all of $\mathbb{R}^3$, and gives rise to the scattered field $(\boldsymbol{E}^s, \boldsymbol{H}^s) = (\boldsymbol{E}, \boldsymbol{H}) - (\boldsymbol{E}^i, \boldsymbol{H}^i)$. To ensure uniqueness of solution, the scattered field is assumed to satisfy the Silver-Müller radiation condition (2). The *Calderon operator* is the bounded operator given by

$$C : H^{-1/2}(\mathrm{Div}, \partial B_R) \to H^{-1/2}(\mathrm{Div}, \partial B_R)\,, \qquad \gamma_t \boldsymbol{E}^s \mapsto \gamma_t \boldsymbol{H}^s\,.$$

The variational formulation of the scattering problem, based on the weak form of the **curl curl** equation for the electric field, is to find $\boldsymbol{E} \in H(\mathbf{curl}, B_R)$ such that

$$\int_{B_R} \left(\frac{1}{\mu}\, \mathbf{curl}\, \boldsymbol{E} \cdot \mathbf{curl}\, \boldsymbol{V} - \omega^2 \varepsilon\, \boldsymbol{E} \cdot \boldsymbol{V} \right) \mathrm{d}\boldsymbol{x} - \mathrm{i}\omega \, \langle C \gamma_t \boldsymbol{E}, \gamma_t \boldsymbol{V} \rangle_{t,\partial B_R}$$
$$= \mathrm{i}\omega \, \langle \boldsymbol{H}^i - C\gamma_t \boldsymbol{E}^i, \gamma_t \boldsymbol{V} \rangle_{t,\partial B_R} \qquad \text{for all } \boldsymbol{V} \in H(\mathbf{curl}, B_R)\,. \quad (5)$$

Unique solvability of this problem can be proved as in section 4.6 of [22], also using the unique continuation result of [24]. For the case of $\mathrm{Re}(\varepsilon) < 0$ on D, see also [19].

As described in the introduction, we consider potentials $\boldsymbol{A}$, φ such that (3), (4) hold. This leads to potentials that satisfy a Helmholtz equation with compactly supported distributional source term,

$$\Delta u + k^2\, u = f\,, \qquad (6)$$

and $k = \omega\,\sqrt{\varepsilon\mu}$. This is the approach used in [6] and we will discuss it in more detail below.

The appropriate radiation condition for outgoing solutions to the Helmholtz equation is the well-known Sommerfeld radiation condition. Analogously to the Calderon map, we introduce the DtN-operator on ∂B_R which is the bounded operator given by

$$\Lambda : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)\,, \qquad \gamma_D u^s \mapsto \frac{\partial u^s}{\partial \boldsymbol{\nu}}\,,$$

where u^s denotes a solution to (6) in $\mathbb{R}^3 \setminus \overline{B_R}$ that satisfies the Sommerfeld radiation condition. The variational formulation for the total field $u \in H^1(B_R)$ that solves (6) in the homogeneous case $f = 0$ due to an incident field $u^i \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ then is

$$\int_{B_R} \left(\mathbf{grad}\, u \cdot \mathbf{grad}\, v - k^2\, u\, v \right) \mathrm{d}x - \langle \Lambda \gamma_D u, \gamma_D v \rangle_{\partial B_R} = \left\langle \frac{\partial u^i}{\partial \boldsymbol{\nu}} - \Lambda \gamma_D u^i, v \right\rangle_{\partial B_R} \qquad (7)$$

for all $v \in H^1(B_R)$. Below, we will also encounter a variant of (7) with the scalar product of the gradients multiplied by ε, corresponding to a jump in the Neumann data of u along discontinuities of ε.

Throughout the paper, we will assume that the incident field $\boldsymbol{E}^i$, $\boldsymbol{H}^i$ also has a potential representation of the form (3),

$$\boldsymbol{E}^i = \mathrm{i}\omega\, \boldsymbol{A}^i - \mathbf{grad}\, \varphi^i\,, \qquad \boldsymbol{H}^i = \frac{1}{\mu_0}\, \mathbf{curl}\, \boldsymbol{A}^i$$

with potentials $\varphi^i \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$, $\boldsymbol{A}^i \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ that satisfy (6) and (4) with ε, μ replaced by ε_0 and μ_0, respectively, and vanishing source terms.

Theorem 2.1 *Let $\boldsymbol{E} \in H(\mathbf{curl}, B_R)$ denote a solution to (5) and set $\boldsymbol{H} = 1/(\mathrm{i}\omega\mu)\, \mathbf{curl}\, \boldsymbol{E}$. Then there exists a scalar potential $\varphi \in H^1(B_R)$ and a vector potential $\boldsymbol{A} \in H^1(B_R, \mathbb{C}^3)$ such that (3) and (4) hold in B_R. Moreover φ and $\boldsymbol{A}$ are weak solutions to inhomogeneous Helmholtz equations,*

$$\int_{B_R} \left(\varepsilon\, \mathbf{grad}\, \varphi \cdot \mathbf{grad}\, v - \omega^2 \varepsilon^2 \mu\, \varphi\, v \right) \mathrm{d}\boldsymbol{x} - \varepsilon_0 \, \langle \Lambda \gamma_D \varphi, \gamma_D v \rangle_{\partial B_R}$$
$$= -\mathrm{i}\omega \left(\mathbf{grad}\,(\varepsilon) \cdot \boldsymbol{A}\,,\, v \right) + \varepsilon_0 \left\langle \frac{\partial \varphi^i}{\partial \boldsymbol{\nu}} - \Lambda \gamma_D \varphi^i, v \right\rangle_{\partial B_R} \qquad (8)$$

for all $v \in C_0^\infty(\mathbb{R}^3)$ *and*

$$\int\limits_{B_R} \left(\operatorname{tr}\left(J_{\boldsymbol{A}} J_{\boldsymbol{V}}^\top\right) - \omega^2 \varepsilon\mu\, \boldsymbol{A}\cdot\boldsymbol{V} \right) \mathrm{d}\boldsymbol{x} - \langle \Lambda\gamma_D \boldsymbol{A}, \gamma_D \boldsymbol{V}\rangle_{\partial B_R} = -\mathrm{i}\omega \left(\mathbf{grad}(\varepsilon\mu)\,\varphi\,,\,\boldsymbol{V}\right)$$
$$+ \int_{B_R} (\mu-\mu_0)\,\boldsymbol{H}\cdot\mathbf{curl}\,\boldsymbol{V}\,\mathrm{d}\boldsymbol{x} + \mathrm{i}\omega \int_{B_R} (\mu-\mu_0)\,\varepsilon\boldsymbol{E}\cdot\boldsymbol{V}\,\mathrm{d}\boldsymbol{x} + \left\langle J_{\boldsymbol{A}^i}\boldsymbol{\nu} - \Lambda\gamma_D\boldsymbol{A}^i, \gamma_D\boldsymbol{V}\right\rangle_{\partial B_R} \quad (9)$$

for all $\boldsymbol{V} \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^3)$*, respectively. Both* $\varphi - \varphi^i$*,* $\boldsymbol{A} - \boldsymbol{A}^i$ *may be smoothly extended to radiating solutions to the Helmholtz equation in* $\mathbb{R}^3 \setminus \overline{B_R}$*.*

Remark 2.2 *As* $\varepsilon = \varepsilon_0$*,* $\mu = \mu_0$ *outside of* $\overline{D}$*, the distributions on the right hand sides of (8) and (9) have compact support in* B_R*. For piecewise constant material parameters as we will consider later (see Theorem 2.4 below), they reduce to terms with support on the interfaces between materials and can be formulated as interface conditions for the potentials.*

Proof (of Theorem 2.1): We start by following the arguments in [1, section 3.5] to obtain a standard Helmholtz decomposition of $\boldsymbol{E}$. Denote by $\varphi^{(1)} \in H_0^1(B_R)$ the uniquely determined solution of the Dirichlet problem

$$\int_{B_R} \mathbf{grad}\,\varphi^{(1)}\cdot\mathbf{grad}\,v\,\mathrm{d}\boldsymbol{x} = -\int_{B_R} \boldsymbol{E}\cdot\mathbf{grad}\,v\,\mathrm{d}\boldsymbol{x} \qquad \text{for all } v \in H_0^1(B_R)\,. \quad (10)$$

Set $\boldsymbol{A}^{(1)} = (\mathrm{i}\omega)^{-1}\left(\boldsymbol{E} + \mathbf{grad}\,\varphi^{(1)}\right)$. Then $\operatorname{div}\boldsymbol{A}^{(1)} = 0$ in B_R and hence $\boldsymbol{A}^{(1)} \in H(\mathbf{curl}, B_R) \cap H(\operatorname{div}, B_R)$. Note also that $\varphi^{(1)} = 0$ outside of the support of $\operatorname{div} E$, so in particular in the neighborhood of ∂B_R where $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$. Hence $\mathrm{i}\omega\,\boldsymbol{A}^{(1)} = \boldsymbol{E}$ in this neighborhood. We conclude that $\mathrm{i}\omega\,\boldsymbol{\nu}\cdot\boldsymbol{A}^{(1)} = \boldsymbol{\nu}\cdot\boldsymbol{E}$ is smooth on ∂B_R and thus $\boldsymbol{A}^{(1)} \in H^1(B_R, \mathbb{C}^3)$ by Corollary 2.15 in [1]. It also follows that all Cartesian components of $\boldsymbol{A}^{(1)} - (\mathrm{i}\omega)^{-1}\boldsymbol{E}^i$ can be smoothly extended as radiating solutions to the Helmholtz equation to $\mathbb{R}^3 \setminus \overline{B_R}$.

We now modify the decomposition to satisfy the gauge condition. Continue $\varphi^{(1)}$ by 0 to a function in $H^1(\mathbb{R}^3)$ with compact support. Then the equation

$$\Delta\varphi^{(2)} + \omega^2\varepsilon\mu\,\varphi^{(2)} = -\omega^2\varepsilon\mu\,\varphi^{(1)} \qquad \text{in } \mathbb{R}^3$$

has a unique solution in $\varphi^{(2)} \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$ that satisfies the Sommerfeld radiation condition. Set $\varphi = \varphi^{(1)} + \varphi^{(2)} + \varphi^i$ and $\boldsymbol{A} = \boldsymbol{A}^{(1)} + (\mathrm{i}\omega)^{-1}\,\mathbf{grad}(\varphi^{(2)} + \varphi^i)$, giving (3). Also,

$$\operatorname{div}\boldsymbol{A} = \operatorname{div}\boldsymbol{A}^{(1)} + (\mathrm{i}\omega)^{-1}\,\Delta(\varphi^{(2)} + \varphi^{(i)}) = \mathrm{i}\omega\varepsilon\mu\,\varphi\,,$$

so that (4) holds. Obviously, $\varphi - \varphi^i$ is a radiating solution to the Helmholtz equation in $\mathbb{R}^3 \setminus \overline{B_R}$. Noting

$$\boldsymbol{A} - \boldsymbol{A}^i = \boldsymbol{A} - \frac{1}{\mathrm{i}\omega}\left(\boldsymbol{E}^i + \mathbf{grad}\,\varphi^i\right) = \boldsymbol{A}^{(1)} - \frac{1}{\mathrm{i}\omega}\boldsymbol{E}^i + \frac{1}{\mathrm{i}\omega}\,\mathbf{grad}\,\varphi^{(2)}\,,$$

it also holds that all Cartesian components of $\boldsymbol{A} - \boldsymbol{A}^i$ can be smoothly extended to radiating solutions of the Helmholtz equation in $\mathbb{R}^3 \setminus \overline{B_R}$.

From $\boldsymbol{E} \in H(\mathbf{curl}, B_R)$ satisfying (5) we conclude $\varepsilon\boldsymbol{E} \in H(\operatorname{div}, B_R)$ and, in particular, we have $\operatorname{div}(\varepsilon\boldsymbol{E}) = 0 \in L^2(B_R)$. From the potential representation, the gauge condition and the divergence theorem we now obtain for $v \in C_0^\infty(\mathbb{R}^3)$ that

$$\begin{aligned}
&\int\limits_{B_R} \left(\varepsilon\, \mathbf{grad}\,\varphi \cdot \mathbf{grad}\,v - \omega^2\varepsilon^2\mu\,\varphi\,v\right)\mathrm{d}\boldsymbol{x} - \varepsilon_0 \left\langle \frac{\partial\varphi}{\partial\boldsymbol{\nu}}, \gamma_D v \right\rangle_{\partial B_R} \\
&\quad = \int\limits_{B_R} \left((\mathrm{i}\omega\varepsilon\,\boldsymbol{A} - \varepsilon\boldsymbol{E})\cdot\mathbf{grad}\,v + \mathrm{i}\omega\varepsilon\ \mathrm{div}\,\boldsymbol{A}\,v\right)\mathrm{d}\boldsymbol{x} - \varepsilon_0\langle\boldsymbol{\nu}\cdot(\mathrm{i}\omega\,\boldsymbol{A} - \boldsymbol{E}), \gamma_D v\rangle_{\partial B_R} \\
&\quad = -\mathrm{i}\omega\left(\mathbf{grad}\,(\varepsilon)\cdot\boldsymbol{A}\,,\,v\right)\,. \qquad (11)
\end{aligned}$$

Taking into account that $\varphi - \varphi^i$ is radiating gives (8).

As the last step of the proof, we show the variational equation for $\boldsymbol{A}$. For $\boldsymbol{V} \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^3)$ there holds

$$\int\limits_{B_R} \mathrm{tr}\left(J_{\boldsymbol{A}} J_{\boldsymbol{V}}^\top\right)\mathrm{d}\boldsymbol{x} - \langle J_{\boldsymbol{A}}\boldsymbol{\nu}, \gamma_D\boldsymbol{V}\rangle_{\partial B_R} = -\left(\Delta\boldsymbol{A}, \boldsymbol{V}\right). \qquad (12)$$

Now, from standard identities from vector calculus, (3) and (4), we obtain

$$\begin{aligned}
\Delta\boldsymbol{A} &= \mathbf{grad}\,\mathrm{div}\,\boldsymbol{A} - \mathbf{curl}^2\,\boldsymbol{A} \\
&= \mathrm{i}\omega\ \mathbf{grad}\,(\varepsilon\mu\varphi) - \mathbf{curl}\left(\frac{\mu-\mu_0}{\mu}\ \mathbf{curl}\,\boldsymbol{A}\right) - \mathbf{curl}\left(\frac{\mu_0}{\mu}\ \mathbf{curl}\,\boldsymbol{A}\right) \\
&= \mathrm{i}\omega\ \mathbf{grad}(\varepsilon\mu)\,\varphi + \mathrm{i}\omega\varepsilon\mu\ \mathbf{grad}\,\varphi - \mathbf{curl}((\mu-\mu_0)\,\boldsymbol{H}) - \mu_0\ \mathbf{curl}\,\boldsymbol{H} \qquad (13)\\
&= \mathrm{i}\omega\ \mathbf{grad}(\varepsilon\mu)\,\varphi - \mathbf{curl}((\mu-\mu_0)\,\boldsymbol{H}) - \omega^2\varepsilon\mu\boldsymbol{A} - \mathrm{i}\omega\varepsilon\mu\boldsymbol{E} - \mu_0\,(-\mathrm{i}\omega\varepsilon\boldsymbol{E})\,.
\end{aligned}$$

Inserting into (12) and observing that $\boldsymbol{A} - \boldsymbol{A}^i$ is radiating completes the proof. ∎

Indeed, we may also obtain the original variational equation (5) for $\boldsymbol{E}$ from the equations for the potentials $\boldsymbol{A}$ and φ. This shows equivalence of the original electromagnetic scattering problem (5) to the variational problem (8), (9).

Theorem 2.3 *Assume that* $\varphi \in H^1(B_R)$, $\boldsymbol{A} \in H^1(B_R, \mathbb{C}^3)$ *satisfy* (8), (9) *with* $\boldsymbol{E} = \mathrm{i}\omega\,\boldsymbol{A} - \mathbf{grad}\,\varphi$, $\boldsymbol{H} = (1/\mu)\ \mathbf{curl}\,\boldsymbol{A}$. *Then* φ, $\boldsymbol{A}$ *satisfy the Lorentz gauge condition* (4) *and* $\boldsymbol{E}$ *is a solution of* (5).

Proof: Starting from the definition of $\boldsymbol{E}$, we observe $\boldsymbol{E} \in H(\mathbf{curl}, B_R)$ with $\mathbf{curl}\,\boldsymbol{E} = \mathrm{i}\omega\,\mathbf{curl}\,\boldsymbol{A}$. Additionally, we note that from (8) and (9) φ^s and $\boldsymbol{A}_j^s$ can smoothly be extended to radiating solutions of the Helmholtz equation in $\mathbb{R}^3 \setminus B_R$.

First, we show that the Lorentz gauge condition is satisfied. Let us consider $w = \mathrm{div}\,\boldsymbol{A} - \mathrm{i}\omega\varepsilon\mu\,\varphi \in L^2(B_R)$. We choose a test function $v \in C_0^\infty(B_R)$ and set $\boldsymbol{V} = \mathbf{grad}\,v$. Then

$$\int_{B_R} \mathrm{div}\,\boldsymbol{A}\,\Delta v\,\mathrm{d}\boldsymbol{x} = -\left(\mathbf{grad}\,\mathrm{div}\,\boldsymbol{A}\,,\,\boldsymbol{V}\right) = -\left(\Delta\boldsymbol{A} + \mathbf{curl}^2\,\boldsymbol{A}\,,\,\boldsymbol{V}\right) = -\left(\Delta\boldsymbol{A}\,,\boldsymbol{V}\right),$$

as $\mathbf{curl}\,\boldsymbol{V} = \mathbf{curl}\,\mathbf{grad}\,v = 0$. Hence, from (12), we obtain

$$\begin{aligned}
\int_{B_R} w\left(\Delta v + \omega^2\varepsilon\mu_0 v\right)\mathrm{d}\boldsymbol{x} &= \int_{B_R} \mathrm{tr}\left(J_{\boldsymbol{A}} J_{\boldsymbol{V}}^\top\right)\mathrm{d}\boldsymbol{x} - \mathrm{i}\omega\int_{B_R} \varepsilon\mu\,\varphi\,\Delta v\,\mathrm{d}\boldsymbol{x} \\
&\quad + \omega^2\mu_0\int_{B_R} \varepsilon\ \mathrm{div}(\boldsymbol{A})\,v\,\mathrm{d}\boldsymbol{x} - \mathrm{i}\omega^3\mu_0\int_{B_R} \varepsilon^2\mu\,\varphi\,v\,\mathrm{d}\boldsymbol{x}\,.
\end{aligned}$$

Substituting equation (9) and observing $\mathbf{curl}\, \boldsymbol{V} = 0$ leads to

$$\begin{aligned}
&\int_{B_R} w\left(\Delta v + \omega^2 \varepsilon\mu_0 v\right) \mathrm{d}x = \omega^2 \int_{B_R} \varepsilon\mu\, \boldsymbol{A}\cdot\boldsymbol{V}\,\mathrm{d}x - \mathrm{i}\omega\left(\mathbf{grad}(\varepsilon\mu)\,\varphi\,,\,\boldsymbol{V}\right)\\
&\quad + \mathrm{i}\omega \int_{B_R} (\mu-\mu_0)\,\varepsilon\boldsymbol{E}\cdot\boldsymbol{V}\,\mathrm{d}\boldsymbol{x} - \mathrm{i}\omega\int_{B_R} \varepsilon\mu\,\varphi\,\Delta v\,\mathrm{d}\boldsymbol{x}\\
&\quad + \omega^2\mu_0 \int_{B_R} \varepsilon\ \mathrm{div}(\boldsymbol{A})\, v\,\mathrm{d}\boldsymbol{x} - \mathrm{i}\omega^3\mu_0 \int_{B_R} \varepsilon^2\mu\,\varphi\, v\,\mathrm{d}\boldsymbol{x}\\
&= \omega^2 \int_{B_R} \varepsilon\,(\mu-\mu_0)\,\boldsymbol{A}\cdot\boldsymbol{V}\,\mathrm{d}x + \mathrm{i}\omega\int_{B_R} \varepsilon\mu\ \mathbf{grad}\,\varphi\cdot\boldsymbol{V}\,\mathrm{d}\boldsymbol{x}\\
&\quad + \mathrm{i}\omega \int_{B_R} (\mu-\mu_0)\,\varepsilon\boldsymbol{E}\cdot\boldsymbol{V}\,\mathrm{d}\boldsymbol{x} - \mathrm{i}\omega^3\mu_0 \int_{B_R} \varepsilon^2\mu\,\varphi\, v\,\mathrm{d}\boldsymbol{x} - \omega^2\mu_0\left(\mathbf{grad}(\varepsilon)\cdot\boldsymbol{A}, v\right).
\end{aligned}$$

Next we use equation (8), which yields

$$\int_{B_R} w\left(\Delta v + \omega^2\varepsilon\mu_0 v\right)\,\mathrm{d}x = \mathrm{i}\omega\int_{B_R} \varepsilon\,(\mu-\mu_0)\left[-\mathrm{i}\omega\,\boldsymbol{A} + \mathbf{grad}\,\varphi + \boldsymbol{E}\right]\cdot\boldsymbol{V}\,\mathrm{d}x = 0\,.$$

by the definition of $\boldsymbol{E}$. Theorem 4.38 (b) in [19] implies $w \in H^2(U)$ for any open set U such that $\overline{U} \subseteq B_R$ and $\Delta w = -\omega^2\varepsilon\mu_0 w$ in U. Additionally, since $\boldsymbol{A}^s = \boldsymbol{A} - \boldsymbol{A}^i$ and $\varphi^s = \varphi - \varphi^i$ can be extended to radiating solutions of the corresponding Helmholtz equations, we conclude $\boldsymbol{A}$ and φ to be smooth in a neighborhood of ∂B_R. In particular, we have $w \in H^2(B_R)$ and the divergence theorem leads to

$$\int_{B_R} \mathbf{grad}\left(\mathrm{div}\,\boldsymbol{A} - \mathrm{i}\omega\varepsilon\mu\varphi\right)\cdot\mathbf{grad}\,v - \omega^2\varepsilon\mu_0\left(\mathrm{div}\,\boldsymbol{A} - \mathrm{i}\omega\varepsilon\mu\varphi\right) v\,\mathrm{d}x = \left\langle \frac{\partial}{\partial\boldsymbol{\nu}}\left(\mathrm{div}\,\boldsymbol{A} - \mathrm{i}\omega\mu_0\varepsilon_0\varphi\right), v\right\rangle_{\partial B_R}$$

Thus, with the gauge condition for $\boldsymbol{A}^i$ and φ^i, the function $w = \mathrm{div}\,\boldsymbol{A} - \mathrm{i}\omega\varepsilon\mu\varphi$ satisfies (7) with $k = \omega\sqrt{\varepsilon\mu_0}$ and vanishing incident field. By uniqueness of solution for this problem, we conclude that $w = 0$.

It remains to show that $\boldsymbol{E}$ is a weak solution to the Maxwell system. As the Lorentz gauge condition holds, we may combine (12) and (13) with (9) to obtain

$$0 = \mathrm{i}\omega\mu_0 \int_{B_R} \varepsilon\boldsymbol{E}\cdot\boldsymbol{V}\,\mathrm{d}\boldsymbol{x} + \mu_0\left(\mathbf{curl}\,\boldsymbol{H}\,,\,\boldsymbol{V}\right) \qquad \text{for all } \boldsymbol{V} \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^3)\,.$$

This equation is easily seen to be equivalent to (5) without the terms involving terms on ∂B_R. ■

An important case for applications, which we will focus on in the remainder of the paper, is a piecewise homogeneous medium. For simplicity, we will just consider the case when D is a Lipschitz domain, $\mathbb{R}^3 \setminus \overline{D}$ is connected and for some constants ε_1, μ_1,

$$\varepsilon(\boldsymbol{x}) = \begin{cases} \varepsilon_1\,, & \boldsymbol{x} \in D\,,\\ \varepsilon_0\,, & \boldsymbol{x} \in \mathbb{R}^3 \setminus \overline{D}\,, \end{cases} \qquad \mu(\boldsymbol{x}) = \begin{cases} \mu_1\,, & \boldsymbol{x} \in D\,,\\ \mu_0\,, & \boldsymbol{x} \in \mathbb{R}^3 \setminus \overline{D}\,. \end{cases} \tag{14}$$

We note that the case of several disconnected scatterers, each with different material constants, may be treated in a similar fashion.

Theorem 2.4 *Let (14) be satisfied. Then φ and $\boldsymbol{A}$ from Theorem 2.1 satisfy the variational problem*

$$a\left(\begin{pmatrix}\boldsymbol{A}\\ \varphi\end{pmatrix}, \begin{pmatrix}\boldsymbol{V}\\ v\end{pmatrix}\right) = \ell\left(\begin{pmatrix}\boldsymbol{V}\\ v\end{pmatrix}\right) \quad \textit{for all } \begin{pmatrix}\boldsymbol{V}\\ v\end{pmatrix} \in H^1(B_R, \mathbb{C}^3) \times H^1(B_R)\,, \tag{15}$$

where

$$a\left(\begin{pmatrix}\boldsymbol{A}\\ \varphi\end{pmatrix},\begin{pmatrix}\boldsymbol{V}\\ v\end{pmatrix}\right)=\int\limits_{B_R}\left(\varepsilon\,\mathbf{grad}\,\varphi\cdot\mathbf{grad}\,v-\omega^2\varepsilon^2\mu\,\varphi\,v\right)\mathrm{d}\boldsymbol{x}$$
$$+\int\limits_{B_R}\left(\frac{1}{\mu}\,\mathrm{tr}\left(J_{\boldsymbol{A}}J_{\boldsymbol{V}}^{\top}\right)-\omega^2\varepsilon\,\boldsymbol{A}\cdot\boldsymbol{V}\right)\mathrm{d}\boldsymbol{x}-\varepsilon_0\,\langle\Lambda\gamma_D\varphi,\gamma_D v\rangle_{\partial B_R}-\frac{1}{\mu_0}\,\langle\Lambda\gamma_D\boldsymbol{A},\gamma_D\boldsymbol{V}\rangle_{\partial B_R}$$
$$+\,\mathrm{i}\omega\,\langle[\varepsilon]_{\partial D}\,\boldsymbol{\nu}\cdot\boldsymbol{A},\gamma_D v\rangle_{\partial D}+\frac{\mathrm{i}\omega}{\mu_0}\,\langle[\varepsilon\mu]_{\partial D}\,\gamma_D\varphi,\boldsymbol{\nu}\cdot\boldsymbol{V}\rangle_{\partial D}$$
$$+\left\langle\left[\frac{1}{\mu}\right]_{\partial D}J_{\boldsymbol{A}}\boldsymbol{\nu}|^-\,,\,\gamma_D\boldsymbol{V}\right\rangle_{\partial D}-\left\langle\left[\frac{1}{\mu}\right]_{\partial D}\gamma_t^-\,\mathbf{curl}\,\boldsymbol{A}\,,\,\gamma_t\boldsymbol{V}\right\rangle_{t,\partial D}\,, \tag{16}$$

$$\ell\left(\begin{pmatrix}\boldsymbol{V}\\ v\end{pmatrix}\right)=\varepsilon_0\left\langle\frac{\partial\varphi^i}{\partial\boldsymbol{\nu}}-\Lambda\gamma_D\varphi^i,\gamma_D v\right\rangle_{\partial B_R}+\frac{1}{\mu_0}\,\langle J_{\boldsymbol{A}^i}\boldsymbol{\nu}-\Lambda\gamma_D\boldsymbol{A}^i,\gamma_D\boldsymbol{V}\rangle_{\partial B_R}\,. \tag{17}$$

Proof: We rewrite the distribution on the right hand side of (8) as

$$(\mathbf{grad}(\varepsilon)\cdot\boldsymbol{A}\,,\,v)=-\int_{\mathbb{R}^3}(\varepsilon\,v\,\,\mathrm{div}\,\boldsymbol{A}+\varepsilon\boldsymbol{A}\cdot\mathbf{grad}\,v)\,\mathrm{d}\boldsymbol{x}$$

for all $v\in C_0^\infty(\mathbb{R}^3)$. Applying the divergence theorem in D and in $\mathbb{R}^3\setminus\overline{D}$ gives

$$(\mathbf{grad}(\varepsilon)\cdot\boldsymbol{A}\,,\,v)=\langle[\varepsilon]_{\partial D}\,\boldsymbol{A}_{\boldsymbol{\nu}}\,,\,\gamma_D v\rangle_{\partial D}\,.$$

Next, we multiply (9) by $1/\mu_0$ and obtain

$$\frac{1}{\mu_0}\int\limits_{B_R}\left(J_{\boldsymbol{A}}J_{\boldsymbol{V}}^{\top}-\omega^2\varepsilon\mu\,\boldsymbol{A}\cdot\boldsymbol{V}\right)\mathrm{d}\boldsymbol{x}-\frac{1}{\mu_0}\,\langle\Lambda\gamma_D\boldsymbol{A},\gamma_D\boldsymbol{V}\rangle_{\partial B_R}-\frac{1}{\mu_0}\,\langle J_{\boldsymbol{A}^i}\boldsymbol{\nu}-\Lambda\gamma_D\boldsymbol{A}^i,\gamma_D\boldsymbol{V}\rangle_{\partial B_R}$$
$$=-\frac{\mathrm{i}\omega}{\mu_0}\,(\mathbf{grad}(\varepsilon\mu)\,\varphi\,,\,\boldsymbol{V})+\left(\frac{1}{\mu_0}-\frac{1}{\mu_1}\right)\int_D(\mu_1\,\boldsymbol{H}\cdot\mathbf{curl}\,\boldsymbol{V}+\mathrm{i}\omega\mu_1\varepsilon_1\,\boldsymbol{E}\cdot\boldsymbol{V})\,\mathrm{d}\boldsymbol{x}$$

The first term on the right hand side is treated as above:

$$(\mathbf{grad}(\varepsilon\mu)\,\varphi\,,\,\boldsymbol{V})=\langle[\varepsilon\mu]_{\partial D}\gamma_D\varphi\,,\,\boldsymbol{V}_{\boldsymbol{\nu}}\rangle_{\partial D}\,.$$

Using the potentials, divergence theorem and gauge condition, we rewrite the second integral as

$$\int_D(\mu_1\,\boldsymbol{H}\cdot\mathbf{curl}\,\boldsymbol{V}+\mathrm{i}\omega\mu_1\varepsilon_1\,\boldsymbol{E}\cdot\boldsymbol{V})\,\mathrm{d}\boldsymbol{x}$$
$$=\int_D\left(\mathbf{curl}\,\boldsymbol{A}\cdot\mathbf{curl}\,\boldsymbol{V}-\omega^2\varepsilon_1\mu_1\,\boldsymbol{A}\cdot\boldsymbol{V}-\mathrm{i}\omega\varepsilon_1\mu_1\,\mathbf{grad}\,\varphi\cdot\boldsymbol{V}\right)\mathrm{d}\boldsymbol{x}$$
$$=\int_D\left(\mathbf{curl}\,\boldsymbol{A}\cdot\mathbf{curl}\,\boldsymbol{V}+\mathrm{div}\,\boldsymbol{A}\,\,\mathrm{div}\,\boldsymbol{V}-\omega^2\varepsilon_1\mu_1\,\boldsymbol{A}\cdot\boldsymbol{V}\right)\mathrm{d}\boldsymbol{x}-\langle\mathrm{i}\omega\varepsilon_1\mu_1\gamma_D^-\varphi\,,\,\boldsymbol{\nu}\cdot\boldsymbol{V}\rangle_{\partial D}\,.$$

The theorem now follows by adding (8) and (9), using the last three equations and Lemma A.1. ■

3 A system of integral equations

In this section, we will assume that (14) holds and additionally, that $\mu = \mu_0 = \mu_1$. In principle, a similar analysis can be carried out for the magnetic case, however the non-magnetic case is more important for practical applications while its analysis is simpler.

For the materials under consideration, the problem from Theorem 2.4 may be formulated as follows: both φ and $\boldsymbol{A}$ satisfy the Helmholtz equation in D with wavenumber $k_1 = \sqrt{\mu_0\varepsilon_1}\,\omega$, and in $\mathbb{R}^3 \setminus \overline{D}$ with wavenumber $k_0 = \sqrt{\mu_0\varepsilon_0}\,\omega$, respectively. The potentials of the scattered fields $\varphi - \varphi^i$, $\boldsymbol{A} - \boldsymbol{A}^i$ satisfy the Sommerfeld radiation condition and the following interface conditions hold on ∂D:

$$[\gamma_D \varphi]_{\partial D} = 0\,, \qquad [\gamma_D \boldsymbol{A}]_{\partial D} = 0\,, \tag{18}$$

$$\left[\varepsilon\left(\mathrm{i}\omega\, \boldsymbol{A}_{\boldsymbol{\nu}} - \frac{\partial\varphi}{\partial\boldsymbol{\nu}}\right)\right]_{\partial D} = 0\,, \tag{19}$$

$$[\mathrm{i}\omega\mu_0\varepsilon\,\gamma_D\varphi\,\boldsymbol{\nu} - J_{\boldsymbol{A}}\boldsymbol{\nu}]_{\partial D} = 0\,. \tag{20}$$

Here, (18) is obtained from H^1 regularity across ∂D. The conditions (19) and (20) are obtained from (15) – (17) and applications of the divergence theorem.

The fundamental solutions of the Helmholtz equation for the materials under consideration are

$$\Phi^{(j)}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi}\,\frac{\mathrm{e}^{\mathrm{i}k_j|\boldsymbol{x}-\boldsymbol{y}|}}{|\boldsymbol{x}-\boldsymbol{y}|}\,, \qquad \boldsymbol{x} \neq \boldsymbol{y}\,, \qquad j = 0,\,1\,.$$

The corresponding single layer potentials on ∂D are

$$\mathrm{SL}^{(j)}\,\zeta(\boldsymbol{x}) = \int_{\partial D} \Phi^{(j)}(\boldsymbol{x},\boldsymbol{y})\,\zeta(\boldsymbol{x})\,\mathrm{d}s(\boldsymbol{y})\,, \qquad \boldsymbol{x} \in \mathbb{R}^3 \setminus \partial D\,,$$

for smooth enough ζ such that the integral exists. However, by density arguments, the definition of the single layer potentials extends to more general densities [21], and will use these both with scalar densities in $H^{-1/2}(\partial D)$ and vectorial densities in $H^{-1/2}(\partial D, \mathbb{C}^3)$. They represent solutions to the corresponding Helmholtz equations in D and $\mathbb{R}^3 \setminus \overline{D}$, respectively, and satisfy the Sommerfeld radiation condition. The corresponding boundary operators are

$$S^{(j)}\zeta = \gamma_D\,\mathrm{SL}^{(j)}\,\zeta\,, \qquad \tilde{D}^{(j)}\zeta = \frac{1}{2}\left\{\frac{\partial\,\mathrm{SL}^{(j)}\,\zeta}{\partial\boldsymbol{\nu}}\right\}_{\partial D}$$

on ∂D, and from the jump relations we have

$$\left.\frac{\partial\,\mathrm{SL}^{(j)}\,\zeta}{\partial\boldsymbol{\nu}}\right|^{\pm} = \mp\frac{\zeta}{2} + \tilde{D}^{(j)}\zeta \qquad \text{on } \partial D\,.$$

From (18) – (20), we immediately obtain the following equivalent formulation of the boundary value problem for the potentials as a system of integral equations.

Theorem 3.1 *Let $\phi^{(j)} \in H^{-1/2}(\partial D)$, $\boldsymbol{\psi}^{(j)} \in H^{-1/2}(\partial D, \mathbb{C}^3)$, $j = 0,\,1$ and set*

$$\varphi(\boldsymbol{x}) = \begin{cases} \mathrm{SL}^{(0)}\,\phi^{(0)}(\boldsymbol{x}) + \varphi^i(\boldsymbol{x})\,, & \boldsymbol{x} \in \mathbb{R}^3 \setminus \overline{D}\,,\\ \mathrm{SL}^{(1)}\,\phi^{(1)}(\boldsymbol{x})\,, & \boldsymbol{x} \in D\,,\end{cases}$$

$$\boldsymbol{A}(\boldsymbol{x}) = \begin{cases} \mathrm{SL}^{(0)}\,\boldsymbol{\psi}^{(0)}(\boldsymbol{x}) + \boldsymbol{A}^i(x)\,, & \boldsymbol{x} \in \mathbb{R}^3 \setminus \overline{D}\,,\\ \mathrm{SL}^{(1)}\,\boldsymbol{\psi}^{(1)}(\boldsymbol{x})\,, & \boldsymbol{x} \in D\,.\end{cases}$$

Then φ, $\boldsymbol{A}$ are solutions to the problem (15) if and only if $\phi^{(j)}$, $\boldsymbol{\psi}^{(j)}$ are solutions to the system of integral equations

$$
\begin{aligned}
S^{(0)}\phi^{(0)} - S^{(1)}\phi^{(1)} &= f_1\,, \\
S^{(0)}\boldsymbol{\psi}^{(0)} - S^{(1)}\boldsymbol{\psi}^{(1)} &= \boldsymbol{f}_2\,, \\
\frac{1}{2}\left(\varepsilon_0\phi^{(0)} + \varepsilon_1\phi^{(1)}\right) - \varepsilon_0\tilde{D}^{(0)}\phi^{(0)} + \varepsilon_1\tilde{D}^{(1)}\phi^{(1)} + \mathrm{i}\omega\,\boldsymbol{\nu}\cdot\left(\varepsilon_0 S^{(0)}\boldsymbol{\psi}^{(0)} - \varepsilon_1 S^{(1)}\boldsymbol{\psi}^{(1)}\right) &= f_3, \\
\frac{1}{2}\left(\boldsymbol{\psi}^{(0)} + \boldsymbol{\psi}^{(1)}\right) - \tilde{D}^{(0)}\boldsymbol{\psi}^{(0)} + \tilde{D}^{(1)}\boldsymbol{\psi}^{(1)} + \mathrm{i}\omega\mu_0\,\boldsymbol{\nu}\left(\varepsilon_0 S^{(0)}\phi^{(0)} - \varepsilon_1 S^{(1)}\phi^{(1)}\right) &= \boldsymbol{f}_4
\end{aligned}
\tag{21}
$$

on ∂D, where

$$
f_1 = -\gamma_D\varphi^i\,, \qquad \boldsymbol{f}_2 = -\gamma_D\boldsymbol{A}^i\,,
$$
$$
f_3 = \varepsilon_0\left(\frac{\partial\varphi^i}{\partial\boldsymbol{\nu}} - \mathrm{i}\omega\,\boldsymbol{A}^i_{\boldsymbol{\nu}}\right), \qquad \boldsymbol{f}_4 = J_{\boldsymbol{A}^i}\boldsymbol{\nu} - \mathrm{i}\omega\mu_0\varepsilon_0\,\varphi^i\,\boldsymbol{\nu}\,.
$$

Theorem 3.2 *Suppose that k_j^2 is not a Dirichlet eigenvalue of $-\Delta$ in D, $j = 0, 1$. Then the system of integral equations (21) has at most one solution.*

Proof: Let $\phi^{(0)}$, $\phi^{(1)}$, $\boldsymbol{\psi}^{(0)}$, $\boldsymbol{\psi}^{(1)}$ denote a solution to (21) for $\varphi^i = 0$, $\boldsymbol{A}^i = 0$. Let φ and $\boldsymbol{A}$ be defined as in Theorem 3.1. Then these functions are solutions to the transmission problem for vanishing incident fields and hence by uniqueness of this problem, vanish themselves. Thus we have

$$
S^{(j)}\phi^{(j)} = 0\,, \qquad S^{(j)}\boldsymbol{\psi}^{(j)} = 0\,, \qquad j = 0, 1\,.
$$

By the assumptions, $S^{(j)} : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$ is an isomorphism [21, Theorem 7.6], hence the assertion follows. ∎

Under the conditions of Theorem 3.2, it is also possible to obtain explicit expressions for the solution of (21). This requires to introduce some auxiliary operators and establish their bijectivity. As $S^{(0)}$ and $S^{(1)}$ are isomorphisms, we can define

$$
K^{(j)} = \left(-\frac{1}{2} + \tilde{D}^{(j)}\right)(S^{(j)})^{-1}\,, \quad j = 0, 1, \qquad L = K^{(0)} - K^{(1)}\,.
$$

Lemma 3.3 *Under the assumptions of Theorem 3.2, the operator $L : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ is invertible.*

Proof: Given $\zeta \in H^{1/2}(\partial D)$, consider the problems

$$
\begin{aligned}
\Delta w + k_0^2 w = 0 &\quad \text{in } \mathbb{R}^3 \setminus \overline{D}\,, \\
\gamma_D^+ w = \zeta &\quad \text{on } \partial D\,, \\
w \text{ satisfies the SRC.}
\end{aligned}
\qquad\qquad
\begin{aligned}
\Delta w + k_1^2 w = 0 &\quad \text{in } D\,, \\
\gamma_D^- w = \zeta &\quad \text{on } \partial D\,,
\end{aligned}
$$

L maps ζ onto the jump in the normal derivative of w across ∂D, $L\zeta = [\partial w/\partial\boldsymbol{\nu}]_{\partial D}$. Note that w is the solution of a transmission problem with vanishing jump of the Dirichlet trace. This is a special case of the problem treated in [25]. Theorem 4.5 in [25] states that for every prescribed jump of the Neumann trace there exists a unique solution w, i.e. L is bijective. ∎

From the second equation in (21), we obtain

$$\left(-\frac{1}{2} + \tilde{D}^{(0)}\right) \boldsymbol{\psi}^{(0)} = K^{(0)} \left(S^{(1)} \boldsymbol{\psi}^{(1)} + \boldsymbol{f}_2\right),$$
$$\left(\frac{1}{2} + \tilde{D}^{(1)}\right) \boldsymbol{\psi}^{(1)} = K^{(1)} \left(S^{(0)} \boldsymbol{\psi}^{(0)} - \boldsymbol{f}_2\right).$$

Subtracting these expressions from the fourth equation in (21), respectively, gives the equations

$$\begin{aligned} -LS^{(1)}\boldsymbol{\psi}^{(1)} + \mathrm{i}\omega\mu_0\,\boldsymbol{\nu}\Big(\varepsilon_0 S^{(0)}\phi^{(0)} - \varepsilon_1 S^{(1)}\phi^{(1)}\Big) &= \boldsymbol{f}_4 + K^{(0)}\boldsymbol{f}_2 \\ -LS^{(0)}\boldsymbol{\psi}^{(0)} + \mathrm{i}\omega\mu_0\,\boldsymbol{\nu}\Big(\varepsilon_0 S^{(0)}\phi^{(0)} - \varepsilon_1 S^{(1)}\phi^{(1)}\Big) &= \boldsymbol{f}_4 + K^{(1)}\boldsymbol{f}_2\,. \end{aligned} \tag{22}$$

Hence

$$\begin{aligned} L\left(\varepsilon_0 S^{(0)}\boldsymbol{\psi}^{(0)} - \varepsilon_1 S^{(1)}\boldsymbol{\psi}^{(1)}\right) = \mathrm{i}\omega\mu_0(\varepsilon_0 - \varepsilon_1)\left(\varepsilon_0\,\boldsymbol{\nu} S^{(0)}\phi^{(0)} - \varepsilon_1\,\boldsymbol{\nu} S^{(1)}\phi^{(1)}\right) \\ - (\varepsilon_0 - \varepsilon_1)\,\boldsymbol{f}_4 - \left(\varepsilon_0 K^{(1)} - \varepsilon_1 K^{(0)}\right)\boldsymbol{f}_2\,. \end{aligned}$$

Applying $\mathrm{i}\omega\boldsymbol{\nu}\cdot L^{-1}$ to this equation, we can insert into the third equation in (21) to obtain

$$\begin{aligned} -\,\varepsilon_0\left(K^{(0)} + \omega^2\mu_0\,(\varepsilon_0 - \varepsilon_1)\boldsymbol{\nu}\cdot L^{-1}\boldsymbol{\nu}\right) S^{(0)}\phi^{(0)} + \varepsilon_1\left(K^{(1)} + \omega^2\mu_0\,(\varepsilon_0 - \varepsilon_1)\boldsymbol{\nu}\cdot L^{-1}\boldsymbol{\nu}\right) S^{(1)}\phi^{(1)} \\ = f_3 + \mathrm{i}\omega\boldsymbol{\nu}\cdot L^{-1}\left[(\varepsilon_0 - \varepsilon_1)\,\boldsymbol{f}_4 + \left(\varepsilon_0 K^{(1)} - \varepsilon_1 K^{(0)}\right)\boldsymbol{f}_2\right] \end{aligned} \tag{23}$$

We now introduce another operator $M : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ by setting

$$M\zeta = \varepsilon_1 K^{(1)}\zeta - \varepsilon_0 K^{(0)}\zeta - \omega^2\mu_0\,(\varepsilon_0 - \varepsilon_1)^2\,\boldsymbol{\nu}\cdot L^{-1}(\boldsymbol{\nu}\zeta)\,, \qquad \zeta \in H^{1/2}(\partial D)\,.$$

Combining the first equation in (21) with (23) finally gives the expressions for $\phi^{(0)}$, $\phi^{(1)}$,

$$\begin{aligned} MS^{(0)}\phi^{(0)} &= f_3 + \varepsilon_1 K^{(1)} f_1 + \mathrm{i}\omega\boldsymbol{\nu}\cdot L^{-1}\big[(\varepsilon_0 - \varepsilon_1)\,(\boldsymbol{f}_4 - \mathrm{i}\omega\mu_0\varepsilon_1\,\boldsymbol{\nu} f_1) + \left(\varepsilon_0 K^{(1)} - \varepsilon_1 K^{(0)}\right)\boldsymbol{f}_2\big] \\ MS^{(1)}\phi^{(1)} &= f_3 + \varepsilon_0 K^{(0)} f_1 + \mathrm{i}\omega\boldsymbol{\nu}\cdot L^{-1}\big[(\varepsilon_0 - \varepsilon_1)\,(\boldsymbol{f}_4 - \mathrm{i}\omega\mu_0\varepsilon_1\,\boldsymbol{\nu} f_1) + \left(\varepsilon_0 K^{(1)} - \varepsilon_1 K^{(0)}\right)\boldsymbol{f}_2\big] \end{aligned} \tag{24}$$

The following Lemma now ensures solvability of the system of integral equations (21).

Lemma 3.4 *Under the assumptions of Theorem 3.2, the operator $M : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ is invertible. Thus, the system of integral equations (21) has exactly one solution for every right hand side.*

Proof: First consider $\tilde{L} = \varepsilon_1 K^{(1)} - \varepsilon_0 K^{(0)}$. This operator can be interpreted as L in the proof of Lemma 3.3, but with the Neumann traces of w multiplied by weight factors ε_0 and ε_1, respectively. Analogous arguments as in that proof show that $\tilde{L}$ is invertible.

For $\zeta \in H^{1/2}(\partial D)$, multiplication with $\boldsymbol{\nu} \in L^\infty(\partial D, \mathbb{R}^3)$ yields $\boldsymbol{\nu}\zeta \in L^2(\partial D, \mathbb{C}^3)$. As L^{-1} is bounded from $H^{-1/2}(\partial D)$ to $H^{1/2}(\partial D)$ and the imbedding of $L^2(\partial D)$ into $H^{-1/2}(\partial D)$ is compact, it follows that $M - \tilde{L}$ is compact, and hence M is Fredholm with index 0.

In the arguments above, we have shown that the system (21) is equivalent to (23) and (24). Suppose that M is not injective. Then, for a homogeneous right hand side, (24) has a non-trivial solution pair $(\phi^{(0)}, \phi^{(1)})$ and using (23) we obtain a corresponding non-trivial solution of (21). This is a contradiction to Theorem 3.2. ∎

Let us note a possible extension of this approach: If one replaces the ansatz for the solution in Theorem 3.1 by a version combining single and double layer potentials in the spirit of Brakhage/Werner,

$$\varphi(\boldsymbol{x}) = \begin{cases} (\mathrm{DL}^{(0)} - \mathrm{i}\,\mathrm{SL}^{(0)})\,\phi^{(0)}(\boldsymbol{x}) + \varphi^{i}(\boldsymbol{x})\,, & \boldsymbol{x} \in \mathbb{R}^3 \setminus \overline{D}\,, \\ (\mathrm{DL}^{(1)} - \mathrm{i}\,\mathrm{SL}^{(1)})\,\phi^{(1)}(\boldsymbol{x})\,, & \boldsymbol{x} \in D\,, \end{cases}$$

$$\boldsymbol{A}(\boldsymbol{x}) = \begin{cases} (\mathrm{DL}^{(0)} - \mathrm{i}\,\mathrm{SL}^{(0)})\,\boldsymbol{\psi}^{(0)}(\boldsymbol{x}) + \boldsymbol{A}^{i}(x)\,, & \boldsymbol{x} \in \mathbb{R}^3 \setminus \overline{D}\,, \\ (\mathrm{DL}^{(1)} - \mathrm{i}\,\mathrm{SL}^{(1)})\,\boldsymbol{\psi}^{(1)}(\boldsymbol{x})\,, & \boldsymbol{x} \in D\,. \end{cases}$$

one may dispense with the conditions on the wave numbers in Theorem 3.2. The corresponding combined operators are always invertible, and the rest of the solution procedure will not change. However, the resulting system will be more complicated to implement in that it contains double-layer and hypersingular boundary operators.

4 Domain derivatives

In this section we will investigate the Fréchet derivative of the scattered field with respect to perturbations of the boundary of the domain D. We will make the same assumptions about the material constants as in section 3. It is then known that the derivative $\boldsymbol{E}'$ of the scattered electric field exists and that it can be characterized by a solution of a boundary value problem which is essentially (5) but with a more complicated transmission condition [18]. We will report these results in more detail below and then prove a similar results for derivatives of $\boldsymbol{A}^s$ and φ^s. Finally, we will investigate the relation between these derivatives and $\boldsymbol{E}'$.

Let us clarify how we describe perturbations of D and of the corresponding solutions of the scattering problems. Consider first a diffeomorphism $\boldsymbol{\eta} : B_R \to B_R$ and the transformation of a scalar function v defined on B_R, $\widetilde{v} = v \circ \boldsymbol{\eta}$. Then we have

$$(\mathbf{grad}\, v) \circ \boldsymbol{\eta} = J_{\boldsymbol{\eta}}^{-\top}\, \mathbf{grad}\, \widetilde{v}\,.$$

This transformation is also applied to vector solutions of Helmholtz equations. A vector field $\boldsymbol{U}$ that solves the Maxwell system, needs to be transformed as $\widehat{\boldsymbol{U}} = J_{\boldsymbol{\eta}}^{\top}\, \boldsymbol{U} \circ \boldsymbol{\eta}$ in order to preserve the property of being an element of $H(\mathbf{curl}, B_R)$. It then follows (see [22, section 3.9]) that

$$(\mathbf{curl}\, \boldsymbol{U}) \circ \boldsymbol{\eta} = \frac{1}{\det J_{\boldsymbol{\eta}}} J_{\boldsymbol{\eta}}\, \mathbf{curl}\, \widehat{\boldsymbol{U}}\,.$$

Introduce $\boldsymbol{h} \in C_0^1(\overline{B_R}, \mathbb{R}^3)$ with $\|\boldsymbol{h}\|_{1,\infty}$ sufficiently small such that the transformation $\boldsymbol{\eta}(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{h}(\boldsymbol{x})$ is a diffeomorphism. Then we obtain the perturbed scatterer

$$D_{\boldsymbol{h}} = \{\boldsymbol{\eta}(\boldsymbol{x}) : \boldsymbol{x} \in D\}$$

with boundary $\partial D_{\boldsymbol{h}} = \{\boldsymbol{\eta}(\boldsymbol{x}) : \boldsymbol{x} \in \partial D\}$.

Below, we will always denote by $\boldsymbol{E}$ the solution of (5) for the unperturbed scatterer D and by $\boldsymbol{H}$ the corresponding magnetic field. We will denote the solution of the variational equation (5) with D replaced by $D_{\boldsymbol{h}}$ as $\boldsymbol{E}_h$. The main idea to derive an expression for the Fréchet derivative of the scattered field as pursued in [18] is to consider the difference of $\boldsymbol{E}$ and $\widehat{\boldsymbol{E}_{\boldsymbol{h}}} = J_{\boldsymbol{\eta}}^{\top} \boldsymbol{E}_{\boldsymbol{h}} \circ \boldsymbol{\eta}$. The following theorem sums up the main results from [18]:

Theorem 4.1 *There exists a vector field $\boldsymbol{W} \in H(\mathbf{curl}, B_R)$, called the material derivative, such that*

$$\lim_{\|\boldsymbol{h}\|_{1,\infty}\to 0} \frac{1}{\|\boldsymbol{h}\|_{1,\infty}} \|\widehat{\boldsymbol{E}_{\boldsymbol{h}}} - \boldsymbol{E} - \boldsymbol{W}\|_{H(\mathbf{curl},B_R)} = 0\,.$$

The map $\boldsymbol{h} \mapsto \boldsymbol{W}$ is linear and bounded from $C^1(\overline{B_R}, \mathbb{R}^3)$ to $H(\mathbf{curl}, B_R)$.
Moreover, if D is of class C^1 (and thus $\boldsymbol{E}|_D \in H^1(D, \mathbb{C}^3)$, $\boldsymbol{E}|_{B_R\setminus\overline{D}} \in H^1(B_R \setminus \overline{D}, \mathbb{C}^3)$), defining the domain derivative $\boldsymbol{E}' = \boldsymbol{W} - J_{\boldsymbol{h}}^\top \boldsymbol{E} - J_{\boldsymbol{E}}\, \boldsymbol{h}$, we have $\boldsymbol{E}'|_D \in H(\mathbf{curl}, D)$, $\boldsymbol{E}'|_{B_R\setminus\overline{D}} \in H(\mathbf{curl}, B_R \setminus \overline{D})$, and $\boldsymbol{E}'$ is the unique radiating weak solution to the transmission problem

$$\mathbf{curl}^2\, \boldsymbol{E}' - \omega^2\varepsilon\mu\, \boldsymbol{E}' = 0 \quad \text{in } D \text{ and in } \mathbb{R}^3 \setminus \overline{D}\,, \tag{25}$$

$$\left[\gamma_t \boldsymbol{E}'\right]_{\partial D} = \left[\mathbf{Grad}\,(h_{\boldsymbol{\nu}}\, E_{\boldsymbol{\nu}}) \times \boldsymbol{\nu}\right]_{\partial D}\,, \tag{26}$$

$$\left[\gamma_t \boldsymbol{H}'\right]_{\partial D} = \left[\mathbf{Grad}\,(h_{\boldsymbol{\nu}}\, H_{\boldsymbol{\nu}}) \times \boldsymbol{\nu}\right]_{\partial D} + \mathrm{i}\omega\, [\varepsilon]_{\partial D}\, h_{\boldsymbol{\nu}}\, (\boldsymbol{\nu} \times \gamma_t \boldsymbol{E})\,. \tag{27}$$

A similar representation can be found for domain derivatives of φ and $\boldsymbol{A}$, which we present next with detailed calculations collected in the appendix. Consider the variational problem (15) with D replaced by $D_{\boldsymbol{h}}$ and corresponding solution $(\boldsymbol{A}_{\boldsymbol{h}}, \varphi_{\boldsymbol{h}})$. Applying the transformation $\boldsymbol{\eta}$, we obtain the variational equation

$$a_{\boldsymbol{h}}\left(\begin{pmatrix}\widetilde{\boldsymbol{A}_{\boldsymbol{h}}}\\ \widetilde{\varphi_{\boldsymbol{h}}}\end{pmatrix}, \begin{pmatrix}\boldsymbol{V}\\ v\end{pmatrix}\right) = \ell\left(\begin{pmatrix}\boldsymbol{V}\\ v\end{pmatrix}\right) \quad \text{for all } \begin{pmatrix}\boldsymbol{V}\\ v\end{pmatrix} \in H^1(B_R, \mathbb{C}^3) \times H^1(B_R)\,, \tag{28}$$

with the variational form $a_{\boldsymbol{h}}$ defined as

$$\begin{aligned}
a_{\boldsymbol{h}}\left(\begin{pmatrix}\widetilde{\boldsymbol{A}_{\boldsymbol{h}}}\\ \widetilde{\varphi_{\boldsymbol{h}}}\end{pmatrix}, \begin{pmatrix}\boldsymbol{V}\\ v\end{pmatrix}\right) &= \int_{B_R} \varepsilon \left(\mathbf{grad}\, \widetilde{\varphi_{\boldsymbol{h}}}^\top J_{\boldsymbol{\eta}}^{-1} J_{\boldsymbol{\eta}}^{-\top} \mathbf{grad}\, v - \omega^2 \mu_0 \varepsilon \widetilde{\varphi_{\boldsymbol{h}}} v\right) \det(J_{\boldsymbol{\eta}})\, \mathrm{d}\boldsymbol{x}\\
&+ \int_{B_R} \left(\frac{1}{\mu_0}\, \mathrm{tr}\left(J_{\widetilde{\boldsymbol{A}_{\boldsymbol{h}}}} J_{\boldsymbol{\eta}}^{-1} J_{\boldsymbol{\eta}}^{-\top} J_{\boldsymbol{V}}^\top\right) - \omega^2 \varepsilon \widetilde{\boldsymbol{A}_{\boldsymbol{h}}} \cdot \boldsymbol{V}\right) \det(J_{\boldsymbol{\eta}})\, \mathrm{d}\boldsymbol{x}\\
&+ \mathrm{i}\omega \left\langle [\varepsilon]_{\partial D}\, \gamma_D \widetilde{\varphi_{\boldsymbol{h}}}\,,\, \boldsymbol{\nu} \cdot J_{\boldsymbol{\eta}}^{-1} \boldsymbol{V}\, \det(J_{\boldsymbol{\eta}})\right\rangle_{\partial D} + \mathrm{i}\omega \left\langle [\varepsilon]_{\partial D} \boldsymbol{\nu} \cdot J_{\boldsymbol{\eta}}^{-1} \widetilde{\boldsymbol{A}_{\boldsymbol{h}}}\,,\, \gamma_D v\, \det(J_{\boldsymbol{\eta}})\right\rangle_{\partial D}\\
&- \varepsilon_0 \left\langle \Lambda \gamma_D \widetilde{\varphi}\,,\, \gamma_D v\right\rangle_{\partial B_R} - \frac{1}{\mu_0} \left\langle \Lambda \gamma_D \widetilde{\boldsymbol{A}}\,,\, \gamma_D \boldsymbol{V}\right\rangle_{\partial B_R}\,.
\end{aligned} \tag{29}$$

Analysing the difference between (15) and (28) leads to the definition of a corresponding material derivative and the following result. As the steps of the proof are analogous to the corresponding result for an acoustic tranmsission problem, we omit the proofs of the this and the next result here but present them in the appendix for completeness.

Theorem 4.2 *Let $\boldsymbol{A}$, φ, $\widetilde{\boldsymbol{A}_{\boldsymbol{h}}}$ and $\varphi_{\boldsymbol{h}}$ as above and let the material derivative $(\boldsymbol{A}_W, \varphi_W)$ be defined as the solution of*

$$a\left(\begin{pmatrix}\boldsymbol{A}_W\\ \varphi_W\end{pmatrix}, \begin{pmatrix}\boldsymbol{V}\\ v\end{pmatrix}\right) = \ell_W\left(\begin{pmatrix}\boldsymbol{V}\\ v\end{pmatrix}\right) \quad \text{for all } \begin{pmatrix}\boldsymbol{V}\\ v\end{pmatrix} \in H^1(B_R, \mathbb{C}^3) \times H^1(B_R)\,, \tag{30}$$

where

$$\begin{aligned}
\ell_W\left(\begin{pmatrix}\boldsymbol{V}\\ v\end{pmatrix}\right) &= \int_{B_R} \varepsilon \left(\mathbf{grad}\, \varphi^\top \left(J_{\boldsymbol{h}} + J_{\boldsymbol{h}}^\top - \mathrm{div}(\boldsymbol{h}) I\right) \mathbf{grad}\, v + \omega^2 \mu_0 \varepsilon\, \mathrm{div}(\boldsymbol{h}) \varphi v\right) \mathrm{d}\boldsymbol{x}\\
&+ \int_{B_R} \left(\frac{1}{\mu_0}\, \mathrm{tr}\left(J_{\boldsymbol{A}} \left(J_{\boldsymbol{h}} + J_{\boldsymbol{h}}^\top - \mathrm{div}(\boldsymbol{h}) I\right) J_{\boldsymbol{V}}^\top\right) + \omega^2 \varepsilon\, \mathrm{div}(\boldsymbol{h}) \boldsymbol{A} \cdot \boldsymbol{V}\right) \mathrm{d}\boldsymbol{x}\\
&- \mathrm{i}\omega \left\langle [\varepsilon]_{\partial D}\, \gamma_D \varphi\,,\, \boldsymbol{\nu}^\top \left(\mathrm{div}(\boldsymbol{h}) I - J_{\boldsymbol{h}}\right) \boldsymbol{V}\right\rangle_{\partial D} - \mathrm{i}\omega \left\langle [\varepsilon]_{\partial D}\, \boldsymbol{\nu}^\top \left(\mathrm{div}(\boldsymbol{h}) I - J_{\boldsymbol{h}}\right) \boldsymbol{A}\,,\, \gamma_D v\right\rangle_{\partial D}\,.
\end{aligned}$$

Then

$$\frac{1}{\|h\|_{1,\infty}}\left\|\begin{pmatrix}\widetilde{\boldsymbol{A}_{\boldsymbol{h}}}\\ \widetilde{\varphi_{\boldsymbol{h}}}\end{pmatrix}-\begin{pmatrix}\boldsymbol{A}\\ \varphi\end{pmatrix}-\begin{pmatrix}\boldsymbol{A}_W\\ \varphi_W\end{pmatrix}\right\|_{H^1}\to 0$$

as $h\to 0$ *in* $C^1(\overline{B_R},\mathbb{R}^3)$.

Proof: See appendix C. ∎

In order to numerically compute the domain derivative, a characterization as in Theorem 4.1 is desirable. As is usual with such result, stronger regularity assumptions are required for the domain. Formally, we consider the Taylor expansion with respect to $\boldsymbol{h}$, which motivates the definition of the *domain derivative* $(\boldsymbol{A}',\varphi')$ by $\boldsymbol{A}'=\boldsymbol{A}_W-J_{\boldsymbol{A}}\boldsymbol{h}$, and $\varphi'=\varphi_W-\mathbf{grad}\,\varphi\cdot\boldsymbol{h}$, respectively.

Theorem 4.3 *Let D be of class $C^{1,1}$. Then there holds $\boldsymbol{A}'\in L^2(B_R,\mathbb{C}^3)$ with $\boldsymbol{A}'|_D\in H^1(D,\mathbb{C}^3)$ and $\boldsymbol{A}'|_{B_R\setminus\overline{D}}\in H^1(B_r\setminus\overline{D},\mathbb{C}^3)$ and $\varphi'\in L^2(B_R)$ with $\varphi'|_D\in H^1(D)$ and $\varphi'|_{B_R\setminus\overline{D}}\in H^1(B_R\setminus\overline{D})$ can be extended to radiating weak solutions of the transmission problem*

$$\left.\begin{aligned}\Delta\boldsymbol{A}'+\omega^2\mu_0\varepsilon\boldsymbol{A}'=0\,,\\ \Delta\varphi'+\omega^2\mu_0\varepsilon\varphi'=0\end{aligned}\right\}\quad \text{in } \mathbb{R}^3\setminus\partial D\,, \tag{31}$$

$$\left[\gamma_D\boldsymbol{A}'\right]_{\partial D}=-\boldsymbol{h}_{\boldsymbol{\nu}}\left[J_{\boldsymbol{A}}\boldsymbol{\nu}\right]_{\partial D}\,, \tag{32}$$

$$\left[\gamma_D\varphi'\right]_{\partial D}=-\boldsymbol{h}_{\boldsymbol{\nu}}\left[\frac{\partial\varphi}{\partial\boldsymbol{\nu}}\right]_{\partial D} \tag{33}$$

$$\begin{aligned}\left[\varepsilon\left(\mathrm{i}\omega\,\boldsymbol{A}'_{\boldsymbol{\nu}}-\frac{\partial\varphi'}{\partial\boldsymbol{\nu}}\right)\right]_{\partial D}=\;&\big[\mathrm{i}\omega\varepsilon\left(\boldsymbol{A}_{\boldsymbol{\tau}}\cdot\mathbf{Grad}(\boldsymbol{h}_{\boldsymbol{\nu}})-\boldsymbol{h}_{\boldsymbol{\nu}}\operatorname{div}(A_{\boldsymbol{\nu}}\boldsymbol{\nu})\right)\\ &-\varepsilon\operatorname{Div}(\boldsymbol{h}_{\boldsymbol{\nu}}\,\mathbf{Grad}\,\varphi)-\varepsilon^2\omega^2\mu_0\boldsymbol{h}_{\boldsymbol{\nu}}\varphi\big]_{\partial D}\,,\end{aligned} \tag{34}$$

$$\left[\mathrm{i}\omega\mu_0\varepsilon\,\gamma_D\varphi'\,\boldsymbol{\nu}-J_{\boldsymbol{A}'}\boldsymbol{\nu}\right]_{\partial D}=\left[\mathrm{i}\omega\mu_0\varepsilon\left(\varphi\,\mathbf{Grad}(\boldsymbol{h}_{\boldsymbol{\nu}})-\boldsymbol{h}_{\boldsymbol{\nu}}\operatorname{div}(\varphi\boldsymbol{\nu})\boldsymbol{\nu}\right)-\omega^2\mu_0\varepsilon\boldsymbol{h}_{\boldsymbol{\nu}}\boldsymbol{A}\right]_{\partial D}. \tag{35}$$

Proof: The proof is given in appendix C. ∎

Remark 4.4 *Note that the above transmission problem exactly corresponds to (15), but with a different right hand side. This becomes more obvious when comparing the interface conditions with (18) – (20).*

Having established characterizations for both the domain derivatives of $\boldsymbol{E}^s$ and of φ^s, $\boldsymbol{A}^s$, it remains to establish how these are connected.

Theorem 4.5 *Let φ', $\boldsymbol{A}'$ denote the domain derivatives of φ^s and $\boldsymbol{A}^s$ and hence the radiating solutions of (31)–(35). Let*

$$\boldsymbol{E}'=\mathrm{i}\omega\boldsymbol{A}'-\mathbf{grad}\,\varphi'\,,\qquad \boldsymbol{H}'=\frac{1}{\mu_0}\,\mathbf{curl}\,\boldsymbol{A}'\,.$$

Then $\boldsymbol{E}'$, $\boldsymbol{H}'$ is a solution of the transmission problem from Theorem 4.1 and

$$\operatorname{div}\boldsymbol{A}'-\mathrm{i}\omega\varepsilon\mu_0\varphi'=0$$

holds.

Proof: We remark that in this proof we omit the Dirichlet trace operator throughout to simplify notation. We will also employ the splitting of vector fields into tangential and normal components as described in Appendix B.

We first prove that $\boldsymbol{E}'$, $\boldsymbol{H}'$ satisfy the transmission condition (27). With the definition of $\boldsymbol{H}'$ and the tangential component of (35) we have

$$
\begin{aligned}
[\boldsymbol{\nu}\times\boldsymbol{H}']_{\partial D} &= \frac{1}{\mu_0}[\boldsymbol{\nu}\times\mathbf{curl}\,\boldsymbol{A}']_{\partial D} = \frac{1}{\mu_0}\left[\left((J_{\boldsymbol{A}'}^\top - J_{\boldsymbol{A}'})\boldsymbol{\nu}\right)_{\boldsymbol{\tau}}\right]_{\partial D}\\
&= \frac{1}{\mu_0}\left[\left(J_{\boldsymbol{A}'}^\top\boldsymbol{\nu}\right)_{\boldsymbol{\tau}}\right]_{\partial D} + \frac{1}{\mu_0}\left[\mathrm{i}\omega\mu_0\varepsilon\,\varphi\,\mathbf{Grad}\,\boldsymbol{h}_{\boldsymbol{\nu}} - \omega^2\mu_0\varepsilon\,\boldsymbol{h}_{\boldsymbol{\nu}}\,\boldsymbol{A}_{\boldsymbol{\tau}}\right]_{\partial D}\\
&= \frac{1}{\mu_0}\left[\left(J_{\boldsymbol{A}'}^\top\boldsymbol{\nu}\right)_{\boldsymbol{\tau}}\right]_{\partial D} + [\mathrm{i}\omega\varepsilon\boldsymbol{h}_{\boldsymbol{\nu}}(-\,\mathbf{Grad}\,\varphi + \mathrm{i}\omega\boldsymbol{A}_{\boldsymbol{\tau}})]_{\partial D} + [\mathrm{i}\omega\varepsilon\,\mathbf{Grad}(\boldsymbol{h}_{\boldsymbol{\nu}}\varphi)]_{\partial D}\\
&= \frac{1}{\mu_0}\left[\left((J_{\boldsymbol{A}'}^\top\boldsymbol{\nu}\right)_{\boldsymbol{\tau}}\right]_{\partial D} + \mathrm{i}\omega[\varepsilon]_{\partial D}\boldsymbol{h}_{\boldsymbol{\nu}}\,\boldsymbol{E}_{\boldsymbol{\tau}} + [\mathrm{i}\omega\varepsilon\ \mathbf{Grad}(\boldsymbol{h}_{\boldsymbol{\nu}}\varphi)]_{\partial D}
\end{aligned}
$$

in the sense of traces in $H^{-\frac{1}{2}}(\partial D)$. Furthermore, from (32) and (20), as well as properties of $J_{\boldsymbol{\nu}}$ from Appendix B,

$$
\begin{aligned}
\left[\left((J_{\boldsymbol{A}'}^\top\boldsymbol{\nu}\right)_{\boldsymbol{\tau}}\right]_{\partial D} &= \left[\boldsymbol{\nu}\times(J_{\boldsymbol{A}'}^\top\boldsymbol{\nu}\times\boldsymbol{\nu})\right]_{\partial D} = \left[\boldsymbol{\nu}\times\left((\mathbf{grad}(\boldsymbol{A}'\cdot\boldsymbol{\nu}) - J_{\boldsymbol{\nu}}^\top\boldsymbol{A}')\times\boldsymbol{\nu}\right)\right]_{\partial D}\\
&= \left[\mathbf{Grad}(\boldsymbol{A}'\cdot\boldsymbol{\nu})\right]_{\partial D} + \left[\boldsymbol{h}_{\boldsymbol{\nu}}\,\boldsymbol{\nu}\times(J_{\boldsymbol{\nu}}^\top(J_{\boldsymbol{A}}\boldsymbol{\nu})\times\boldsymbol{\nu})\right]_{\partial D}\\
&= \left[\mathbf{Grad}(\boldsymbol{A}'\cdot\boldsymbol{\nu})\right]_{\partial D} + \left[\mathrm{i}\omega\mu_0\varepsilon\,\boldsymbol{h}_{\boldsymbol{\nu}}\,\varphi\,\boldsymbol{\nu}\times(J_{\boldsymbol{\nu}}^\top\boldsymbol{\nu}\times\boldsymbol{\nu})\right]_{\partial D}\\
&= \left[\mathbf{Grad}(\boldsymbol{A}'\cdot\boldsymbol{\nu})\right]_{\partial D}\,.
\end{aligned}
$$

Using the normal component of (32) and the transmission condition (20), we conclude

$$
\begin{aligned}
\frac{1}{\mu_0}\left[\left((J_{\boldsymbol{A}'}^\top\boldsymbol{\nu}\right)_{\boldsymbol{\tau}}\right]_{\partial D} + [\mathrm{i}\omega\varepsilon\ \mathbf{Grad}(\boldsymbol{h}_{\boldsymbol{\nu}}\varphi)]_{\partial D} &= \frac{1}{\mu_0}\left[\mathbf{Grad}(\boldsymbol{A}'\cdot\boldsymbol{\nu} + \mathrm{i}\omega\varepsilon\mu_0\,\boldsymbol{h}_{\boldsymbol{\nu}}\varphi)\right]_{\partial D}\\
&= \frac{1}{\mu_0}\left[\mathbf{Grad}(-\boldsymbol{h}_{\boldsymbol{\nu}}\boldsymbol{\nu}^\top J_A\boldsymbol{\nu} + \mathrm{i}\omega\varepsilon\mu_0\,\boldsymbol{h}_{\boldsymbol{\nu}}\varphi)\right]_{\partial D} = 0\,.
\end{aligned}
$$

Thus, together with

$$
\begin{aligned}
[\boldsymbol{\nu}\times\mathbf{Grad}(\boldsymbol{h}_{\boldsymbol{\nu}}\boldsymbol{H}_{\boldsymbol{\nu}})]_{\partial D} &= -\frac{1}{\mu_0}[\boldsymbol{\nu}\times\mathbf{Grad}(\boldsymbol{h}_{\boldsymbol{\nu}}\,\mathbf{curl}(\boldsymbol{A})\cdot\boldsymbol{\nu}]_{\partial D}\\
&= -\frac{1}{\mu_0}[\boldsymbol{\nu}\times\mathbf{Grad}(\boldsymbol{h}_{\boldsymbol{\nu}}\,\mathrm{Div}(\boldsymbol{\nu}\times\boldsymbol{A}))]_{\partial D} = 0\,,
\end{aligned}
$$

we conclude the second transmission condition in Theorem 4.1.

In a second step, we show that the Gauge condition holds if $\boldsymbol{A}'$, φ' satisfy (31)–(35). Define

$$
w' = \mathrm{div}\,\boldsymbol{A}' - \mathrm{i}\omega\mu_0\varepsilon\varphi'\,.
$$

From the Helmholtz equation we obtain $\mathbf{grad}\,\mathrm{div}\,\boldsymbol{A}' - \mathbf{curl}^2\,\boldsymbol{A}' = \Delta\boldsymbol{A}' \in H^1(B_R\setminus\partial D)$. Thus, we obtain the Helmholtz equation

$$
\Delta w' = \mathrm{div}(\mathbf{grad}\,\mathrm{div}\,\boldsymbol{A}') - \mathrm{i}\omega^3\varepsilon^2\mu_0^2\varphi' \;=\; -\omega^2\mu_0\varepsilon\left(\mathrm{div}\,\boldsymbol{A}' - \mathrm{i}\omega\varepsilon\mu_0\varphi'\right) \;=\; -\omega^2\mu_0\varepsilon w'
$$

in $L^2(D)$ and in $L^2(B_R\setminus\overline{D})$. Additionally, w' is uniquely extendable satisfying the Sommerfeld radiation condition.

We use (37) to obtain from (32) and the normal component of (34)

$$\left[\operatorname{div} \boldsymbol{A}'\right]_{\partial D} = \left[\mathrm{i}\omega\mu_0\varepsilon\varphi' + \mathrm{i}\omega\mu_0\varepsilon \boldsymbol{h_\nu} \operatorname{div}(\varphi\boldsymbol{\nu}) + \omega^2\mu_0\varepsilon \boldsymbol{h_\nu} \boldsymbol{A_\nu}\right]_{\partial D} - [\operatorname{Div}(\boldsymbol{h_\nu}(J_{\boldsymbol{A}}\boldsymbol{\nu})_\tau)]_{\partial D} - 2\kappa[\boldsymbol{h_\nu}\boldsymbol{\nu}^\top J_{\boldsymbol{A}}\boldsymbol{\nu}]_{\partial D}\,.$$

As $\boldsymbol{A} \in H^1(B_R, \mathbb{C}^3)$, tangential components and their derivatives do not jump across ∂D. Hence $[\operatorname{Div}(\boldsymbol{h_\nu}(J_{\boldsymbol{A}}\boldsymbol{\nu})_{\boldsymbol{\tau}})]_{\partial D} = 0$. From the identity $2\kappa = \operatorname{div} \boldsymbol{\nu}$ and (19), we obtain

$$\left[\operatorname{div} \boldsymbol{A}'\right]_{\partial D} = \left[\mathrm{i}\omega\mu_0\varepsilon\varphi'\right]_{\partial D} + 2\kappa \boldsymbol{h_\nu} \left[\mathrm{i}\omega\mu_0\varepsilon\varphi - \boldsymbol{\nu}^\top J_{\boldsymbol{A}}\boldsymbol{\nu}\right]_{\partial D}\,.$$

Hence, the Dirichlet traces of w' are in $H^{\frac{1}{2}}(\partial D)$, and (20) implies that they do not jump across ∂D. In particular, we conclude $w' \in H^1(B_R)$.

Furthermore, since $\Delta w' \in L^2(D)$ and $\Delta w' \in L^2(B_R \setminus \overline{D})$, exterior and interior Neumann traces of w', and in particular of $\operatorname{div} \boldsymbol{A}'$ exist on ∂D. Thus, in the trace space $H^{-\frac{1}{2}}(\partial D)$ we obtain from the transmission condition (34)

$$\begin{aligned}\left[\boldsymbol{\nu} \cdot \mathbf{grad}\, w'\right]_{\partial D} &= \left[\boldsymbol{\nu} \cdot \mathbf{grad} \operatorname{div} \boldsymbol{A}' + \omega^2\varepsilon\mu_0\, \boldsymbol{\nu} \cdot \boldsymbol{A}' - \omega^2\mu_0\varepsilon\, (\boldsymbol{A_\tau} \cdot \mathbf{Grad}\, \boldsymbol{h_\nu})\right]_{\partial D} \\ &\quad + \left[\omega^2\mu_0\varepsilon\, \boldsymbol{h_\nu} \operatorname{div}(\boldsymbol{A_\nu}\boldsymbol{\nu}) - \mathrm{i}\omega\mu_0\varepsilon \operatorname{Div}(\boldsymbol{h_\nu}\, \mathbf{Grad}\, \varphi) - \mathrm{i}\omega^3\mu_0^2\varepsilon^2 \boldsymbol{h_\nu}\varphi\right]_{\partial D}\,.\end{aligned}$$

It holds

$$\boldsymbol{\nu} \cdot \left(\mathbf{grad} \operatorname{div} \boldsymbol{A}' + \omega^2\varepsilon\mu_0\, \boldsymbol{A}'\right) = \boldsymbol{\nu} \cdot \mathbf{curl}^2\, \boldsymbol{A}' = -\operatorname{Div}(\boldsymbol{\nu} \times \mathbf{curl}\, \boldsymbol{A}') = -\mu_0 \operatorname{Div}(\boldsymbol{\nu} \times \boldsymbol{H}')\,.$$

We now use the second transmission condition in Theorem 4.1, which has already been established above, and observe $\operatorname{Div}(\boldsymbol{\nu} \times \mathbf{Grad}\, f) = 0$ for any scalar function f. Hence

$$\begin{aligned}\left[\boldsymbol{\nu} \cdot \mathbf{grad}\, w'\right]_{\partial D} &= \left[-\operatorname{Div}\left(\mathrm{i}\omega\mu_0\varepsilon \boldsymbol{h_\nu}\left(\mathrm{i}\omega \boldsymbol{A_\tau} - \mathbf{Grad}\, \varphi\right)\right)\right]_{\partial D} - \left[\omega^2\mu_0\varepsilon\, (\boldsymbol{A_\tau} \cdot \mathbf{Grad}\, \boldsymbol{h_\nu})\right]_{\partial D} \\ &\quad + \left[\omega^2\mu_0\varepsilon\, \boldsymbol{h_\nu} \operatorname{div}(\boldsymbol{A_\nu}\boldsymbol{\nu}) - \mathrm{i}\omega\mu_0\varepsilon \operatorname{Div}(\boldsymbol{h_\nu}\, \mathbf{Grad}\, \varphi) - \mathrm{i}\omega^3\mu_0^2\varepsilon^2 \boldsymbol{h_\nu}\varphi\right]_{\partial D} \\ &= \left[\omega^2\mu_0\varepsilon\left(\operatorname{Div}\left(\boldsymbol{h_\nu}\, \boldsymbol{A_\tau}\right) - (\boldsymbol{A_\tau} \cdot \mathbf{Grad}\, \boldsymbol{h_\nu}) + \boldsymbol{h_\nu} \operatorname{div}(\boldsymbol{A_\nu}\boldsymbol{\nu})\right)\right]_{\partial D} - \left[\mathrm{i}\omega^3\mu_0^2\varepsilon^2 \boldsymbol{h_\nu}\varphi\right]_{\partial D}\,.\end{aligned}$$

We further compute

$$\operatorname{Div}\left(\boldsymbol{h_\nu}\, \boldsymbol{A_\tau}\right) - (\boldsymbol{A_\tau} \cdot \mathbf{Grad}\, \boldsymbol{h_\nu}) = \boldsymbol{h_\nu} \operatorname{Div} \boldsymbol{A_\tau} = \boldsymbol{h_\nu}\left(\operatorname{div} \boldsymbol{A_\tau} - \boldsymbol{\nu}^\top J_{\boldsymbol{A_\tau}}\boldsymbol{\nu}\right) = \boldsymbol{h_\nu} \operatorname{div} \boldsymbol{A_\tau}\,,$$

as

$$0 = \mathbf{grad}(\boldsymbol{A_\tau} \cdot \boldsymbol{\nu})\big)^\top \boldsymbol{\nu} = \boldsymbol{\nu}^\top J_{\boldsymbol{A_\tau}}\boldsymbol{\nu} + \boldsymbol{A_\tau}^\top J_{\boldsymbol{\nu}}^\top \boldsymbol{\nu} = \boldsymbol{\nu}^\top J_{\boldsymbol{A_\tau}}\boldsymbol{\nu}\,.$$

We have already established in the proof of Theorem 2.3 that $w = \operatorname{div} \boldsymbol{A} - \mathrm{i}\omega\varepsilon\mu_0\, \varphi = 0$ in $H^2(U)$ for any open subset $U \subseteq B_R$ and thus has vanishing exterior and interior trace on ∂D. We conclude

$$\left[\boldsymbol{\nu} \cdot \mathbf{grad}\, w'\right]_{\partial D} = \omega^2\mu_0\left[\varepsilon \boldsymbol{h_\nu}\left(\operatorname{div}(\boldsymbol{A_\tau} + \boldsymbol{A_\nu}\boldsymbol{\nu}) - \mathrm{i}\omega\mu_0\varepsilon\, \varphi\right)\right]_{\partial D} = 0\,,$$

Thus, we have established that also the Neumann trace of w' does not jump across ∂D. Collecting all results, we see that w' is a radiating solution of a homogeneous transmission problem for the Helmholtz equation, and by uniqueness w' vanishes.

We now show that $\boldsymbol{E}'$, $\boldsymbol{H}'$ are solutions to the Maxwell system. Immediately from the definition of both functions, we have

$$\mathbf{curl}\, \boldsymbol{E}' = \mathrm{i}\omega\, \mathbf{curl}\, \boldsymbol{A}' = \mathrm{i}\omega\mu_0 \boldsymbol{H}'\,.$$

From the Gauge condition established in the second step, we conclude

$$\begin{aligned}\mathbf{curl}\,\boldsymbol{H}' &= \frac{1}{\mu_0}\,\mathbf{curl}^2\,\boldsymbol{A}' = \frac{1}{\mu_0}(\mathbf{grad}\,\mathrm{div} - \Delta)\boldsymbol{A}' = \frac{1}{\mu_0}\left(\mathbf{grad}(\mathrm{i}\omega\mu_0\varepsilon\varphi') - \Delta\boldsymbol{A}'\right) \\ &= \mathrm{i}\omega\varepsilon\,\mathbf{grad}\,\varphi' + \omega^2\varepsilon\boldsymbol{A}' = \mathrm{i}\omega\varepsilon(\mathbf{grad}\,\varphi' - \mathrm{i}\omega\boldsymbol{A}') = -\mathrm{i}\omega\varepsilon\boldsymbol{E}'\,.\end{aligned}$$

It remains to prove the transmission condition (26). From (32) and (33), we find that

$$\begin{aligned}\left[\boldsymbol{\nu}\times\boldsymbol{E}'\right]_{\partial D} &= \left[\boldsymbol{\nu}\times(\mathrm{i}\omega\boldsymbol{A}' - \mathbf{grad}\,\varphi')\right]_{\partial D} = -\mathrm{i}\omega\left[\boldsymbol{h}_{\boldsymbol{\nu}}\,\boldsymbol{\nu}\times J_{\boldsymbol{A}}\boldsymbol{\nu}\right]_{\partial D} - \left[\boldsymbol{\nu}\times\mathbf{Grad}\,\varphi'\right]_{\partial D} \\ &= -\mathrm{i}\omega\,\boldsymbol{h}_{\boldsymbol{\nu}}\left[\boldsymbol{\nu}\times J_{\boldsymbol{A}}\boldsymbol{\nu}\right]_{\partial D} + \left[\boldsymbol{\nu}\times\mathbf{Grad}(\boldsymbol{h}_{\boldsymbol{\nu}}\,\boldsymbol{\nu}\cdot\mathbf{grad}\,\varphi)\right]_{\partial D}\,.\end{aligned}$$

With

$$\begin{aligned}\boldsymbol{\nu}\times\mathbf{grad}(\boldsymbol{A}\cdot\boldsymbol{\nu}) = \boldsymbol{\nu}\times(J_{\boldsymbol{A}}^\top\boldsymbol{\nu} + J_{\boldsymbol{\nu}}^\top\boldsymbol{A}) &= \boldsymbol{\nu}\times(J_{\boldsymbol{A}}^\top - J_{\boldsymbol{A}})\,\boldsymbol{\nu} + \boldsymbol{\nu}\times J_{\boldsymbol{A}}\boldsymbol{\nu} + \boldsymbol{\nu}\times J_{\boldsymbol{\nu}}^\top\boldsymbol{A} \\ &= -\boldsymbol{\nu}\times(\mathbf{curl}\,\boldsymbol{A}\times\boldsymbol{\nu}) + \boldsymbol{\nu}\times J_{\boldsymbol{A}}\boldsymbol{\nu} + \boldsymbol{\nu}\times J_{\boldsymbol{\nu}}^\top\boldsymbol{A}\,,\end{aligned}$$

we have, as $[\boldsymbol{A}]_{\partial D} = 0$ and $[\boldsymbol{\nu}\times H]_{\partial D} = 0$,

$$\begin{aligned}\boldsymbol{h}_{\boldsymbol{\nu}}\left[\boldsymbol{\nu}\times J_{\boldsymbol{A}}\boldsymbol{\nu}\right]_{\partial D} &= \boldsymbol{h}_{\boldsymbol{\nu}}\left[\boldsymbol{\nu}\times\mathbf{grad}(\boldsymbol{A}\cdot\boldsymbol{\nu}) + \boldsymbol{\nu}\times(\mathbf{curl}\,\boldsymbol{A}\times\boldsymbol{\nu}) - \boldsymbol{\nu}\times J_{\boldsymbol{\nu}}^\top\boldsymbol{A}\right]_{\partial D} \\ &= \left[\boldsymbol{\nu}\times\mathbf{grad}(\boldsymbol{h}_{\boldsymbol{\nu}}\,\boldsymbol{A}\cdot\boldsymbol{\nu}) - \boldsymbol{\nu}\times\boldsymbol{A}_{\boldsymbol{\nu}}\,\mathbf{grad}(\boldsymbol{h}_{\boldsymbol{\nu}}) + \mu_0\,\boldsymbol{\nu}\times(\boldsymbol{H}\times\boldsymbol{\nu}) - \boldsymbol{h}_{\boldsymbol{\nu}}\,\boldsymbol{\nu}\times J_{\boldsymbol{\nu}}^\top\boldsymbol{A}\right]_{\partial D} \\ &= \left[\boldsymbol{\nu}\times\mathbf{Grad}(\boldsymbol{h}_{\boldsymbol{\nu}}\,\boldsymbol{A}_{\boldsymbol{\nu}})\right]_{\partial D}\,.\end{aligned}$$

Thus, we conclude the first transmission condition ,

$$\left[\boldsymbol{\nu}\times\boldsymbol{E}'\right]_{\partial D} = -\boldsymbol{\nu}\times\left[\mathbf{Grad}(\boldsymbol{h}_{\boldsymbol{\nu}}\,(\mathrm{i}\omega\boldsymbol{A} - \mathbf{grad}\,\varphi)\cdot\boldsymbol{\nu})\right]_{\partial D} = -\boldsymbol{\nu}\times\left[\mathbf{Grad}(\boldsymbol{h}_{\boldsymbol{\nu}}\,\boldsymbol{E}_{\boldsymbol{\nu}})\right]_{\partial D}\,.$$

■

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A An integral identity

We make use of the following weak form of a well-known identity of differential operators.

Lemma A.1 *Let D a bounded Lipschitz domain and $\boldsymbol{U}$, $\boldsymbol{V} \in H^1(D, \mathbb{C}^3)$ with $\Delta \boldsymbol{U} \in L^2(D, \mathbb{C}^3)$. Then*

$$\int_D \Big(J_{\boldsymbol{U}} J_{\boldsymbol{V}}^\top - \mathbf{curl}\, \boldsymbol{U} \cdot \mathbf{curl}\, \boldsymbol{V} - \operatorname{div} \boldsymbol{U} \, \operatorname{div} \boldsymbol{V} \Big) \, \mathrm{d}\boldsymbol{x}$$

$$= \sum_{j=1}^{3} \langle J_{\boldsymbol{U}} \boldsymbol{\nu} \, , \, \gamma_D \boldsymbol{V} \rangle_{\partial D} - \langle \gamma_t \, \mathbf{curl}\, \boldsymbol{U} \, , \, \gamma_t \boldsymbol{V} \rangle_{t, \partial D} - \langle \gamma_D \operatorname{div} \boldsymbol{U} \, , \, \boldsymbol{\nu} \cdot \boldsymbol{V} \rangle_{\partial D} .$$

Proof: The identity follows for smooth function from the identity $\mathbf{curl}^2 \boldsymbol{U} = \mathbf{grad} \operatorname{div} \boldsymbol{U} - \Delta \boldsymbol{U}$ and applying partial integration formulas to the individual terms. The lemma then follows from a density argument. ∎

B Identities for surfaces

The analysis of domain derivatives relies on a number of identities for surface differential operators and smooth extensions of functions defined on these surfaces. Such results can be found for C^2 smooth surfaces in the literature [23]. However, for the analysis presented in section 4, $C^{1,1}$ regular surfaces appear to be more natural and hence we will describe here, how the necessary results can be obtained in this case.

For a bounded domain $D \subseteq \mathbb{R}^3$ of class $C^{1,1}$, for any point $\boldsymbol{x}$ sufficiently close to ∂D, there exists a uniquely defined projection $p(\boldsymbol{x}) \in \partial D$ such that

$$|\boldsymbol{x} - p(\boldsymbol{x})| = \min_{\boldsymbol{z} \in \partial D} |\boldsymbol{x} - \boldsymbol{z}| = d \, .$$

The existence of such a minimizer follows from compactness of ∂D, and a variational argument establishes that for every minimizer $\boldsymbol{y}$ there holds $\boldsymbol{x} = \boldsymbol{y} \pm d \, \boldsymbol{\nu}(\boldsymbol{y})$. Now, uniqueness of the minimizer in a small enough neighborhood of ∂D follows from Lipschitz continuity of $\boldsymbol{\nu}$.

Lemma B.1 *For every bounded domain of class $C^{1,1}$ the exterior unit normal can be Lipschitz continuously extended to a neighborhood $\mathcal{N}$ of ∂D. This extension is given by*

$$\boldsymbol{\nu}(\boldsymbol{x}) = \pm \, \mathbf{grad} \, |\boldsymbol{x} - p(\boldsymbol{x})| \, , \qquad \boldsymbol{x} \in \mathbb{R}^3 \setminus \partial D \, ,$$

where the $+$ sign is taken for points in the exterior of $\overline{D}$, the $-$ sign in the interior of D. The extension satisfies $\mathbf{curl}\, \boldsymbol{\nu} = 0$ and $J_{\boldsymbol{\nu}} \boldsymbol{\nu} = 0$. Also, $J_{\boldsymbol{\nu}}$ is symmetric.

Proof: With the remarks above, everything in the lemma can be established as in [23, Section 2.5] or [15]. ∎

The mean curvature of ∂D can now be defined almost everywhere in $\mathcal{N}$ as $\kappa = (1/2) \operatorname{div} \boldsymbol{\nu}$. We also decompose a vector field $\boldsymbol{U}$ defined in $\mathcal{N}$ into its normal part $U_{\boldsymbol{\nu}}$ and tangential part $\boldsymbol{U}_{\boldsymbol{\tau}}$,

$$\boldsymbol{U} = U_{\boldsymbol{\nu}} \boldsymbol{\nu} + \boldsymbol{U}_{\boldsymbol{\tau}} \, ,$$

where $\boldsymbol{U}_{\boldsymbol{\nu}} = \boldsymbol{U}\cdot\boldsymbol{\nu}$. We may then establish the following relations for $\boldsymbol{U}$, $\boldsymbol{V} \in H^1(\mathcal{N},\mathbb{C}^3)$ by straightforward calculations.

$$\mathrm{Div}(\boldsymbol{U}_{\boldsymbol{\nu}}\boldsymbol{V}_{\boldsymbol{\tau}}) = \boldsymbol{U}_{\boldsymbol{\nu}}\,\mathrm{Div}(\boldsymbol{V}_{\boldsymbol{\tau}}) + \mathbf{Grad}(\boldsymbol{V}_{\boldsymbol{\nu}})\cdot\boldsymbol{U}_{\boldsymbol{\tau}}\,, \tag{36}$$

$$\mathrm{div}(\boldsymbol{U}) = \mathrm{Div}(\boldsymbol{U}_{\boldsymbol{\tau}}) + 2\kappa\,\boldsymbol{U}_{\boldsymbol{\nu}} + \frac{\partial \boldsymbol{U}_{\boldsymbol{\nu}}}{\partial\boldsymbol{\nu}}\,, \tag{37}$$

$$\boldsymbol{\nu}^\top J_{\boldsymbol{U}}\boldsymbol{V} = \boldsymbol{V}_{\boldsymbol{\tau}}\cdot\mathbf{Grad}(\boldsymbol{U}_{\boldsymbol{\nu}}) + \boldsymbol{V}_{\boldsymbol{\nu}}\frac{\partial \boldsymbol{U}_{\boldsymbol{\nu}}}{\partial\boldsymbol{\nu}} + \boldsymbol{V}_{\boldsymbol{\tau}}^\top J_{\boldsymbol{\nu}}\boldsymbol{U}_{\boldsymbol{\tau}}\,, \tag{38}$$

$$\mathrm{div}(U_{\boldsymbol{\nu}}\boldsymbol{\nu}) = 2\kappa\,\boldsymbol{U}_{\boldsymbol{\nu}} + \frac{\partial \boldsymbol{U}_{\boldsymbol{\nu}}}{\partial\boldsymbol{\nu}}\,. \tag{39}$$

C The domain derivatives of φ^s and $\boldsymbol{A}^s$

In this sections we will provide the proofs of Theorems 4.2 and 4.3. We start with some remarks concerning the transformation $\boldsymbol{\eta}$ from section 4. The outwards directed normal vector $\boldsymbol{\nu}$ at $\boldsymbol{x}\in\partial D$ and the outwards directed normal vector $\boldsymbol{\nu}_{\boldsymbol{h}}$ at $\boldsymbol{\varphi}(\boldsymbol{x})\in\partial D_{\boldsymbol{h}}$ are related by

$$\widetilde{\boldsymbol{\nu}_{\boldsymbol{h}}} = \frac{J_{\boldsymbol{\eta}}^{-\top}\boldsymbol{\nu}}{\left|J_{\boldsymbol{\eta}}^{-\top}\boldsymbol{\nu}\right|}$$

and we have

$$\int_{\partial D_{\boldsymbol{h}}} \mathrm{d}s = \int_{\partial D}\mathrm{Det}(J_{\boldsymbol{\eta}})\,\mathrm{d}s \qquad \text{with} \qquad \mathrm{Det}(J_{\boldsymbol{\eta}}) = \det(J_{\boldsymbol{\eta}})\left|J_{\boldsymbol{\eta}}^{-\top}\boldsymbol{\nu}\right|,$$

see Lemma 3.17 and 3.18 in [15]. We will make frequent use of linearizations: from $J_{\boldsymbol{\eta}} = I + J_{\boldsymbol{h}}$, straightforward calculations show

$$J_{\boldsymbol{\eta}}^{-1} = I - J_{\boldsymbol{h}} + \mathrm{O}\left(\|\boldsymbol{h}\|_{C^1}^2\right) \quad \text{and} \quad \det(J_{\boldsymbol{\eta}}) = 1 + \mathrm{div}(\boldsymbol{h}) + \mathrm{O}\left(\|\boldsymbol{h}\|_{C^1}^2\right) \tag{40}$$

as $\boldsymbol{h}\to 0$ in $C^1(\mathbb{R}^3,\mathbb{R}^3)$ (see e.g. [18]).

Let us also introduce the shorthand notation

$$X = H^1(B_R,\mathbb{C}^3)\times H^1(B_R)$$

for the Hilbert space underlying the variational equations, and its scalar product $\langle\cdot,\cdot\rangle_X$. There are bounded linear operators $\mathcal{A}$, $\mathcal{A}_{\boldsymbol{h}} : X\to X$ and $\mathcal{L}\in X$, such that

$$a\left(x,y\right) = \langle\mathcal{A}x,y\rangle_X\,, \qquad a_{\boldsymbol{h}}\left(x,y\right) = \langle\mathcal{A}_{\boldsymbol{h}}x,y\rangle_X\,, \qquad x,y\in X\,,$$

and

$$\ell\left(y\right) = \langle\mathcal{L},y\rangle_X\,, \qquad y\in X\,.$$

Then, (15) and (28) are equivalent to

$$\mathcal{A}\begin{pmatrix}\boldsymbol{A}\\ \varphi\end{pmatrix} = \mathcal{L}, \qquad \mathcal{A}_{\boldsymbol{h}}\begin{pmatrix}\widetilde{\boldsymbol{A}_{\boldsymbol{h}}}\\ \widetilde{\varphi_{\boldsymbol{h}}}\end{pmatrix} = \mathcal{L}. \tag{41}$$

Lemma C.1 *The solutions of the equations* (41) *satisfy*

$$\left\|\begin{pmatrix}\widetilde{\boldsymbol{A}_{\boldsymbol{h}}}\\ \widetilde{\varphi_{\boldsymbol{h}}}\end{pmatrix} - \begin{pmatrix}\boldsymbol{A}\\ \varphi\end{pmatrix}\right\|_X \le C\,\|\boldsymbol{h}\|_{1,\infty}\,\|\mathcal{L}\|_X\,.$$

Proof: From (16) and (29), we obtain for any $x \in X$ by straightforward estimates that

$$\|(\mathcal{A}-\mathcal{A}_{\boldsymbol{h}})x\|_X^2 = |a\left(x,(\mathcal{A}-\mathcal{A}_{\boldsymbol{h}})\,x\right) - a_{\boldsymbol{h}}\left(x,(\mathcal{A}-\mathcal{A}_{\boldsymbol{h}})\,x\right)| \leq C\left(\left\|I - J_{\boldsymbol{\eta}}^{-1}J_{\boldsymbol{\eta}}^{-\top}\det(J_{\boldsymbol{\eta}})\right\|_\infty \right.$$
$$\left. + \|1-\det(J_{\boldsymbol{\eta}})\|_\infty + \left\|I - J_{\boldsymbol{\eta}}^{-1}\det(J_{\boldsymbol{\eta}})\right\|_\infty\right)\|(\mathcal{A}-\mathcal{A}_{\boldsymbol{h}})\,x\|_X\,\|x\|_X\,.$$

Applying (40), we conclude

$$\|(\mathcal{A}-\mathcal{A}_{\boldsymbol{h}})x\|_X \leq C\,\|\boldsymbol{h}\|_{1,\infty}\,\|x\|_X$$

Since we know $\mathcal{A}$ to have a bounded inverse, we conclude by a perturbation argument [20, Theorem 10.1] that $\left\|\mathcal{A}^{-1}-\mathcal{A}_{\boldsymbol{h}}^{-1}\right\| \leq C\,\|\boldsymbol{h}\|_{1,\infty}$. Using (41) finishes the proof. ∎

We are now in a position to prove the existence of the material derivatives.

Proof of Theorem 4.2: Denote by $(\boldsymbol{A}_W, \varphi_w)$ the solution of (30). We will show that

$$\left\|\mathcal{A}\begin{pmatrix}\widetilde{\boldsymbol{A}_{\boldsymbol{h}}}-\boldsymbol{A}-\boldsymbol{A}_W\\ \widetilde{\varphi_{\boldsymbol{h}}}-\varphi-\varphi_W\end{pmatrix}\right\|_X = \mathrm{o}\left(\|\boldsymbol{h}\|_{1,\infty}\right), \quad \|\boldsymbol{h}\|_{1,\infty}\to 0\,.$$

From linearity and (41) we obtain

$$\mathcal{A}\begin{pmatrix}\widetilde{\boldsymbol{A}_{\boldsymbol{h}}}-\boldsymbol{A}-\boldsymbol{A}_W\\ \widetilde{\varphi_{\boldsymbol{h}}}-\varphi-\varphi_W\end{pmatrix} = \mathcal{A}\begin{pmatrix}\widetilde{\boldsymbol{A}_{\boldsymbol{h}}}\\ \widetilde{\varphi_{\boldsymbol{h}}}\end{pmatrix} - \mathcal{A}_{\boldsymbol{h}}\begin{pmatrix}\widetilde{\boldsymbol{A}_{\boldsymbol{h}}}\\ \widetilde{\varphi_{\boldsymbol{h}}}\end{pmatrix} - \mathcal{A}\begin{pmatrix}\boldsymbol{A}_W\\ \varphi_W\end{pmatrix}.$$

Starting from (15), (28) and (30), and adding and subtracting integrals in a convenient way, leads to

$$\begin{aligned}
&a\left(\begin{pmatrix}\widetilde{\boldsymbol{A}_{\boldsymbol{h}}}-\boldsymbol{A}-\boldsymbol{W}\\ \widetilde{\varphi_{\boldsymbol{h}}}-\varphi-w\end{pmatrix},\begin{pmatrix}\boldsymbol{V}\\ v\end{pmatrix}\right)\\
&= \int_{B_R}\varepsilon\left(\mathbf{grad}\,\widetilde{\varphi_{\boldsymbol{h}}}^\top\left(I-J_{\boldsymbol{\eta}}^{-1}J_{\boldsymbol{\eta}}^{-\top}\det(J_{\boldsymbol{\eta}})-J_{\boldsymbol{h}}-J_{\boldsymbol{h}}^\top+\operatorname{div}(\boldsymbol{h})I\right)\mathbf{grad}\,v\right)\mathrm{d}\boldsymbol{x}\\
&\quad-\omega^2\mu_0\int_{B_R}\varepsilon^2\left(1-\det(J_{\boldsymbol{\eta}})+\operatorname{div}(\boldsymbol{h})\right)\widetilde{\varphi_{\boldsymbol{h}}}\,v\,\mathrm{d}\boldsymbol{x}+\omega^2\mu_0\int_{B_R}\varepsilon^2\operatorname{div}(\boldsymbol{h})\left(\widetilde{\varphi_{\boldsymbol{h}}}-\varphi\right)v\,\mathrm{d}\boldsymbol{x}\\
&\quad+\int_{B_R}\varepsilon\,\mathbf{grad}\left(\widetilde{\varphi_{\boldsymbol{h}}}-\varphi\right)^\top\left(J_{\boldsymbol{h}}+J_{\boldsymbol{h}}^\top-\operatorname{div}(\boldsymbol{h})I\right)\mathbf{grad}\,v\,\mathrm{d}\boldsymbol{x}\\
&\quad+\frac{1}{\mu_0}\sum_{j=1}^3\int_{B_R}\mathbf{grad}\,\widetilde{\boldsymbol{A}_{\boldsymbol{h},j}}^\top\left(I-J_{\boldsymbol{\eta}}^{-1}J_{\boldsymbol{\eta}}^{-\top}\det(J_{\boldsymbol{\eta}})-J_{\boldsymbol{h}}-J_{\boldsymbol{h}}^\top+\operatorname{div}(\boldsymbol{h})I\right)\mathbf{grad}\,\boldsymbol{V}_j\,\mathrm{d}\boldsymbol{x}\\
&\quad-\omega^2\int_{B_R}\varepsilon\left(1-\det(J_{\boldsymbol{\eta}})-1+\operatorname{div}(\boldsymbol{h})\right)\widetilde{\boldsymbol{A}_{\boldsymbol{h}}}\cdot\boldsymbol{V}\,\mathrm{d}\boldsymbol{x}+\omega^2\int_{B_R}\varepsilon\left(\widetilde{\boldsymbol{A}_{\boldsymbol{h}}}-\boldsymbol{A}\right)\cdot\boldsymbol{V}\operatorname{div}(\boldsymbol{h})\,\mathrm{d}\boldsymbol{x}\\
&\quad+\frac{1}{\mu_0}\sum_{j=1}^3\int_{B_R}\mathbf{grad}\left(\widetilde{\boldsymbol{A}_{\boldsymbol{h},j}}-\boldsymbol{A}_j\right)^\top\left(J_{\boldsymbol{h}}+J_{\boldsymbol{h}}^\top-\operatorname{div}(\boldsymbol{h})I\right)\mathbf{grad}\,\boldsymbol{V}_j\,\mathrm{d}\boldsymbol{x}\\
&\quad+\mathrm{i}\omega\left\langle[\varepsilon]_{\partial D}\,\gamma_D\widetilde{\varphi_{\boldsymbol{h}}}\,,\,\boldsymbol{\nu}^\top\left(I-J_{\boldsymbol{\eta}}^{-1}\det(J_{\boldsymbol{\eta}})+\operatorname{div}(\boldsymbol{h})I-J_{\boldsymbol{h}}\right)\boldsymbol{V}\right\rangle_{\partial D}\\
&\quad-\mathrm{i}\omega\left\langle[\varepsilon]_{\partial D}\left(\gamma_D\widetilde{\varphi_{\boldsymbol{h}}}-\gamma_D\varphi\right),\,\boldsymbol{\nu}^\top\left(\operatorname{div}(\boldsymbol{h})I-J_{\boldsymbol{h}}\right)\boldsymbol{V}\right\rangle_{\partial D}\\
&\quad+\mathrm{i}\omega\left\langle[\varepsilon]_{\partial D}\,\boldsymbol{\nu}^\top\left(I-J_{\boldsymbol{\eta}}^{-1}\det(J_{\boldsymbol{\eta}})+\operatorname{div}(h)I-J_{\boldsymbol{h}}\right)\widetilde{\boldsymbol{A}_{\boldsymbol{h}}}\,,\,\gamma_D v\right\rangle_{\partial D}\\
&\quad-\mathrm{i}\omega\left\langle[\varepsilon]_{\partial D}\,\boldsymbol{\nu}^\top\left(\operatorname{div}(\boldsymbol{h})I-J_{\boldsymbol{h}}\right)\left(\widetilde{\boldsymbol{A}_{\boldsymbol{h}}}-\boldsymbol{A}\right),\,\gamma_D v\right\rangle_{\partial D}.
\end{aligned}$$

Substituting the linearizations (40), we obtain for $\boldsymbol{h}$ sufficiently small

$$a\left(\begin{pmatrix}\widetilde{\boldsymbol{A}_{\boldsymbol{h}}}-\boldsymbol{A}-\boldsymbol{A}_W\\ \widetilde{\varphi_{\boldsymbol{h}}}-\varphi-\varphi_w\end{pmatrix},\begin{pmatrix}\boldsymbol{V}\\ v\end{pmatrix}\right)\leqslant C\left(\|\widetilde{\varphi_{\boldsymbol{h}}}\|_{H^1}\|v\|_{H^1}\|\boldsymbol{h}\|_{1,\infty}^2+\|\widetilde{\varphi_{\boldsymbol{h}}}-\varphi\|_{H^1}\|\boldsymbol{h}\|_{1,\infty}\right)$$
$$+C\left(\|\widetilde{\boldsymbol{A}_{\boldsymbol{h}}}\|_{H^1}\|\boldsymbol{V}\|_{H^1}\|\boldsymbol{h}\|_{1,\infty}^2+\|\widetilde{\boldsymbol{A}_{\boldsymbol{h}}}-\boldsymbol{A}\|_{H^1}\|\boldsymbol{V}\|_{H^1}\|\boldsymbol{h}\|_{1,\infty}\right).$$

Now Lemma C.1 implies the assertion. ∎

Proof of Theorem 4.3: For a domain D of class $C^{1,1}$, it follows from standard regularity results for elliptic PDEs [21, Theorem 4.18] that $\boldsymbol{A}|_D\in H^2(D,\mathbb{C}^3)$, $\boldsymbol{A}|_{B_R\setminus\overline{D}}\in H^2(B_R\setminus\overline{D},\mathbb{C}^3)$ and $\varphi|_D\in H^2(D)$, $\varphi|_{B_R\setminus\overline{D}}\in H^2(B_R\setminus\overline{D})$. From the corresponding variational problems it follows that φ, φ_W, and the cartesian components of $\boldsymbol{A}$, $\boldsymbol{A}_W$ are weak solution of the Helmholtz equation in D and radiating weak solutions in $\mathbb{R}^3\setminus\overline{D}$ as asserted. The Dirichlet transmission conditions for $\boldsymbol{A}'$, φ' follow directly from the definition and from the fact that $\boldsymbol{A}$, $\boldsymbol{A}_W\in H^1(B_R,\mathbb{C}^3)$, φ, $\varphi_W\in H^1(B_R)$. Thus it remains to show the two last transmission conditions.

By definition

$$a\left(\begin{pmatrix}\boldsymbol{A}'\\ \varphi'\end{pmatrix},\begin{pmatrix}\boldsymbol{V}\\ v\end{pmatrix}\right)=a\left(\begin{pmatrix}\boldsymbol{A}_W\\ \varphi_W\end{pmatrix},\begin{pmatrix}\boldsymbol{V}\\ v\end{pmatrix}\right)-a\left(\begin{pmatrix}J_{\boldsymbol{A}}\boldsymbol{h}\\ \mathbf{grad}\,\varphi\cdot\boldsymbol{h}\end{pmatrix},\begin{pmatrix}\boldsymbol{V}\\ v\end{pmatrix}\right)$$

Note that $\boldsymbol{h}$ has compact support in B_R. Thus from (30) and the definition of a (see Theorem 2.4), we obtain

$$\begin{aligned}
a\left(\begin{pmatrix}\boldsymbol{A}'\\ \varphi'\end{pmatrix},\begin{pmatrix}\boldsymbol{V}\\ v\end{pmatrix}\right)&=\int_{B_R}\varepsilon\left(\mathbf{grad}\,\varphi^\top\left(J_{\boldsymbol{h}}+J_{\boldsymbol{h}}^\top-\operatorname{div}(\boldsymbol{h})I\right)\mathbf{grad}\,v+\omega^2\mu_0\varepsilon\operatorname{div}(\boldsymbol{h})\varphi v\right)\mathrm{d}\boldsymbol{x}\\
&+\int_{B_R}\left(\frac{1}{\mu_0}\sum_{j=1}^{3}\mathbf{grad}\,\boldsymbol{A}_j^\top\left(J_{\boldsymbol{h}}+J_{\boldsymbol{h}}^\top-\operatorname{div}(\boldsymbol{h})I\right)\mathbf{grad}\,\boldsymbol{V}_j+\omega^2\varepsilon\operatorname{div}(\boldsymbol{h})\boldsymbol{A}\cdot\boldsymbol{V}\right)\mathrm{d}\boldsymbol{x}\\
&-\mathrm{i}\omega\left\langle[\varepsilon]_{\partial D}\,\gamma_D\varphi\,,\,\boldsymbol{\nu}^\top\left(\operatorname{div}(\boldsymbol{h})I-J_{\boldsymbol{h}}\right)\boldsymbol{V}\right\rangle_{\partial D}-\mathrm{i}\omega\left\langle[\varepsilon]_{\partial D}\,\boldsymbol{\nu}^\top\left(\operatorname{div}(\boldsymbol{h})I-J_{\boldsymbol{h}}\right)\,\boldsymbol{A}\,,\,\gamma_D v\right\rangle_{\partial D}\\
&-\int_{B_R}\varepsilon\left(\mathbf{grad}\left(\mathbf{grad}\,\varphi\cdot\boldsymbol{h}\right)\cdot\mathbf{grad}\,v-\omega^2\varepsilon\mu_0\,\mathbf{grad}\,\varphi\cdot\boldsymbol{h}\,v\right)\mathrm{d}\boldsymbol{x}\\
&-\int_{B_R}\left(\frac{1}{\mu_0}\sum_{j=1}^{3}\mathbf{grad}\left(\mathbf{grad}\,\boldsymbol{A}_j\cdot\boldsymbol{h}\right)\cdot\mathbf{grad}\,\boldsymbol{V}_j-\omega^2\varepsilon\,J_{\boldsymbol{A}}\boldsymbol{h}\cdot\boldsymbol{V}\right)\mathrm{d}\boldsymbol{x}\\
&-\mathrm{i}\omega\left\langle[\varepsilon\,\gamma_D\,\mathbf{grad}\,\varphi\cdot\boldsymbol{h}]_{\partial D}\,,\,\boldsymbol{\nu}^\top\boldsymbol{V}\right\rangle_{\partial D}-\mathrm{i}\omega\left\langle\left[\varepsilon\,\boldsymbol{\nu}^\top J_{\boldsymbol{A}}\boldsymbol{h}\right]_{\partial D}\,,\,\gamma_D v\right\rangle_{\partial D}.
\end{aligned}\tag{42}$$

From elementary calculations we have

$$\begin{aligned}
&\operatorname{div}\left((\boldsymbol{h}\cdot\mathbf{grad}\,\varphi)\,\mathbf{grad}\,v+(\boldsymbol{h}\cdot\mathbf{grad}\,v)\,\mathbf{grad}\,\varphi-(\mathbf{grad}\,\varphi\cdot\mathbf{grad}\,v)\boldsymbol{h}\right)\\
&=\mathbf{grad}\,\varphi^\top\left(J_{\boldsymbol{h}}+J_{\boldsymbol{h}}^\top-\operatorname{div}(\boldsymbol{h})I\right)\mathbf{grad}\,v+(\boldsymbol{h}\cdot\mathbf{grad}\,\varphi)\Delta v+(\boldsymbol{h}\cdot\mathbf{grad}\,v)\Delta\varphi.
\end{aligned}$$

and Green's first identity leads to

$$-\int_{B_R}\varepsilon\,\mathbf{grad}\left(\mathbf{grad}\,\varphi\cdot\boldsymbol{h}\right)\cdot\mathbf{grad}\,v\,\mathrm{d}\boldsymbol{x}=\int_{B_R}\varepsilon\left(\boldsymbol{h}\cdot\mathbf{grad}\,\varphi\right)\Delta v\,\mathrm{d}\boldsymbol{x}$$
$$+\left\langle\varepsilon_0\,\gamma_D^+(\boldsymbol{h}\cdot\mathbf{grad}\,\varphi)\,,\,\frac{\partial v}{\partial\boldsymbol{\nu}}\bigg|^+\right\rangle_{\partial D}-\left\langle\varepsilon_1\,\gamma_D^-(\boldsymbol{h}\cdot\mathbf{grad}\,\varphi)\,,\,\frac{\partial v}{\partial\boldsymbol{\nu}}\bigg|^-\right\rangle_{\partial D}.$$

Combining these equations with the Helmholtz equation for φ, we obtain

$$
\begin{aligned}
&\int_{B_R} \varepsilon \Big(\mathbf{grad}\,\varphi^\top \left(J_{\boldsymbol{h}} + J_{\boldsymbol{h}}^\top - \operatorname{div}(\boldsymbol{h}) I\right) \mathbf{grad}\, v + \omega^2 \mu_0 \varepsilon \operatorname{div}(\boldsymbol{h}) \varphi v\Big)\, \mathrm{d}\boldsymbol{x} \\
&\quad - \int_{B_R} \varepsilon \left(\mathbf{grad}\,(\mathbf{grad}\,\varphi \cdot \boldsymbol{h}) \cdot \mathbf{grad}\, v - \omega^2 \mu_0 \varepsilon\, \mathbf{grad}\,\varphi \cdot \boldsymbol{h} v\right) \mathrm{d}\boldsymbol{x} \\
&= \int_{B_R} \varepsilon \operatorname{div}\Big((\boldsymbol{h} \cdot \mathbf{grad}\,\varphi) \mathbf{grad}\, v + (\boldsymbol{h} \cdot \mathbf{grad}\, v)\, \mathbf{grad}\,\varphi - (\mathbf{grad}\,\varphi \cdot \mathbf{grad}\, v) \boldsymbol{h}\Big) \mathrm{d}x \\
&\quad + \int_{B_R} \omega^2 \mu_0 \varepsilon^2 \operatorname{div}(\varphi v \boldsymbol{h}) \mathrm{d}\boldsymbol{x} + \left\langle \varepsilon_0\, \gamma_D^+ (\boldsymbol{h} \cdot \mathbf{grad}\,\varphi)\,,\, \frac{\partial v}{\partial \boldsymbol{\nu}}\bigg|^+ \right\rangle_{\partial D} - \left\langle \varepsilon_1\, \gamma_D^- (\boldsymbol{h} \cdot \mathbf{grad}\,\varphi)\,,\, \frac{\partial v}{\partial \boldsymbol{\nu}}\bigg|^- \right\rangle_{\partial D} .
\end{aligned}
$$

Applying the divergence theorem to the integrals on the right hand side gives

$$
\begin{aligned}
&\int_{B_R} \varepsilon \Big(\mathbf{grad}\,\varphi^\top \left(J_{\boldsymbol{h}} + J_{\boldsymbol{h}}^\top - \operatorname{div}(\boldsymbol{h}) I\right) \mathbf{grad}\, v + \omega^2 \mu_0 \varepsilon \operatorname{div}(\boldsymbol{h}) \varphi v\Big)\, \mathrm{d}\boldsymbol{x} \\
&\quad - \int_{B_R} \varepsilon \left(\mathbf{grad}\,(\mathbf{grad}\,\varphi \cdot \boldsymbol{h}) \cdot \mathbf{grad}\, v - \omega^2 \mu_0 \varepsilon\, \mathbf{grad}\,\varphi \cdot \boldsymbol{h} v\right) \mathrm{d}\boldsymbol{x} \\
&= \left\langle \varepsilon_0 \left(h_{\boldsymbol{\nu}}\, \gamma_D^+\, \mathbf{grad}\,\varphi - \frac{\partial \varphi}{\partial \boldsymbol{\nu}}\bigg|^+ \boldsymbol{h} \right) ,\, \gamma_D^+\, \mathbf{grad}\, v \right\rangle_{\partial D} \\
&\quad - \left\langle \varepsilon_1 \left(h_{\boldsymbol{\nu}}\, \gamma_D^-\, \mathbf{grad}\,\varphi - \frac{\partial \varphi}{\partial \boldsymbol{\nu}}\bigg|^- \boldsymbol{h} \right) ,\, \gamma_D^-\, \mathbf{grad}\, v \right\rangle_{\partial D} \\
&\quad - \omega^2 \mu_0 \left\langle \left[\varepsilon^2\right]_{\partial D} h_{\boldsymbol{\nu}}\, \gamma_D \varphi\,,\, \gamma_D v \right\rangle_{\partial D} .
\end{aligned}
$$

We now decompose $\boldsymbol{h}$ into its tangential and normal component of ∂D, $\boldsymbol{h} = \boldsymbol{h}_{\boldsymbol{\tau}} + h_{\boldsymbol{\nu}} \boldsymbol{\nu}$. Since φ, $v \in H^1(B_R)$, the tangential components of their gradients do not jump across ∂D. Hence we have

$$
\begin{aligned}
&\int_{B_R} \varepsilon \Big(\mathbf{grad}\,\varphi^\top \left(J_{\boldsymbol{h}} + J_{\boldsymbol{h}}^\top - \operatorname{div}(\boldsymbol{h}) I\right) \mathbf{grad}\, v + \omega^2 \mu_0 \varepsilon \operatorname{div}(\boldsymbol{h}) \varphi v\Big)\, \mathrm{d}\boldsymbol{x} \\
&\quad - \int_{B_R} \varepsilon \left(\mathbf{grad}\,(\mathbf{grad}\,\varphi \cdot \boldsymbol{h}) \cdot \mathbf{grad}\, v - \omega^2 \mu_0 \varepsilon\, \mathbf{grad}\,\varphi \cdot \boldsymbol{h} v\right) \mathrm{d}\boldsymbol{x} \\
&= \left\langle [\varepsilon]_{\partial D}\, h_{\boldsymbol{\nu}}\ \mathbf{Grad}\,\varphi - \left[\varepsilon \frac{\partial \varphi}{\partial \boldsymbol{\nu}}\right]_{\partial D} \boldsymbol{h}_{\boldsymbol{\tau}},\, \mathbf{Grad}\, v \right\rangle_{\partial D} - \omega^2 \mu_0 \left\langle \left[\varepsilon^2\right]_{\partial D} h_{\boldsymbol{\nu}}\, \gamma_D \varphi\,,\, \gamma_D v \right\rangle_{\partial D} .
\end{aligned}
$$

Completely analogous calculations may be carried out for the components of $\boldsymbol{A}$ instead of φ. Combining these equations, we have

$$
\begin{aligned}
&a\left(\begin{pmatrix} \boldsymbol{A}' \\ \varphi' \end{pmatrix}, \begin{pmatrix} \boldsymbol{V} \\ v \end{pmatrix} \right) \\
&= \left\langle [\varepsilon]_{\partial D}\, h_{\boldsymbol{\nu}}\ \mathbf{Grad}\,\varphi - \left[\varepsilon \frac{\partial \varphi}{\partial \boldsymbol{\nu}}\right]_{\partial D} \boldsymbol{h}_{\boldsymbol{\tau}},\, \mathbf{Grad}\, v \right\rangle_{\partial D} - \omega^2 \mu_0 \left\langle \left[\varepsilon^2\right]_{\partial D} h_{\boldsymbol{\nu}}\, \gamma_D \varphi\,,\, \gamma_D v \right\rangle_{\partial D} \\
&\quad - \frac{1}{\mu_0} \sum_{j=1}^{3} \left\langle \left[\frac{\partial \boldsymbol{A}_j}{\partial \boldsymbol{\nu}}\right]_{\partial D} \boldsymbol{h}_{\boldsymbol{\tau}},\, \mathbf{Grad}\, \boldsymbol{V}_j \right\rangle_{\partial D} - \omega^2 \left\langle [\varepsilon]_{\partial D}\, h_{\boldsymbol{\nu}}\, \gamma_D \boldsymbol{A}\,,\, \gamma_D \boldsymbol{V} \right\rangle_{\partial D} \\
&\quad - \mathrm{i}\omega \left\langle [\varepsilon]_{\partial D}\, \gamma_D \varphi\,,\, \boldsymbol{\nu}^\top \left(\operatorname{div}(\boldsymbol{h}) I - J_{\boldsymbol{h}}\right) \boldsymbol{V} \right\rangle_{\partial D} - \mathrm{i}\omega \left\langle [\varepsilon]_{\partial D}\, \boldsymbol{\nu}^\top \left(\operatorname{div}(\boldsymbol{h}) I - J_{\boldsymbol{h}}\right) \boldsymbol{A}\,,\, \gamma_D v \right\rangle_{\partial D} \\
&\quad - \mathrm{i}\omega \left\langle \left[\varepsilon\, \gamma_D\, \mathbf{grad}\,\varphi \cdot \boldsymbol{h}\right]_{\partial D}\,,\, \boldsymbol{\nu}^\top \boldsymbol{V} \right\rangle_{\partial D} - \mathrm{i}\omega \left\langle \left[\varepsilon\, \boldsymbol{\nu}^\top J_{\boldsymbol{A}} \boldsymbol{h}\right]_{\partial D}\,,\, \gamma_D v \right\rangle_{\partial D} .
\end{aligned}
$$

We now use the transmission conditions (19), (20) to replace the normal derivatives of φ and $\boldsymbol{A}_j$. Subsequently, partial integration is applied to all surface gradients. Combining (36)–(39) yields

$$\mathrm{Div}(A_{\boldsymbol{\nu}}\boldsymbol{h}_{\boldsymbol{\tau}}) - A_{\boldsymbol{\nu}}\,\mathrm{div}(\boldsymbol{h}) + \boldsymbol{\nu}^\top J_{\boldsymbol{h}}\boldsymbol{A} - \boldsymbol{\nu}^\top J_{\boldsymbol{A}}\boldsymbol{h} = \boldsymbol{A}_{\boldsymbol{\tau}}\cdot\mathbf{Grad}(h_{\boldsymbol{\nu}}) - h_{\boldsymbol{\nu}}\,\mathrm{div}(A_{\boldsymbol{\nu}}\boldsymbol{\nu}).$$

Furthermore, we have

$$\begin{aligned}\sum_{j=1}^{3}\mathrm{Div}\left(\varphi\boldsymbol{\nu}_j\boldsymbol{h}_{\boldsymbol{\tau}}\right)\boldsymbol{V}_j &= \varphi\,\mathrm{Div}(\boldsymbol{h}_{\boldsymbol{\tau}})\boldsymbol{\nu}^\top\boldsymbol{V} + \varphi\boldsymbol{h}_{\boldsymbol{\tau}}^\top J_{\boldsymbol{\nu}}\boldsymbol{V} + \boldsymbol{h}_{\boldsymbol{\tau}}^\top\,\mathbf{Grad}(\varphi)\boldsymbol{\nu}^\top\boldsymbol{V},\\ \boldsymbol{\nu}^\top J_{\boldsymbol{h}}\boldsymbol{V} &= \mathbf{grad}(h_{\boldsymbol{\nu}})\cdot\boldsymbol{V} - \boldsymbol{h}_{\boldsymbol{\tau}}^\top J_{\boldsymbol{\nu}}\boldsymbol{V},\end{aligned}$$

which leads to

$$\begin{aligned}&\sum_{j=1}^{3}\mathrm{Div}(\varphi\boldsymbol{\nu}_j\boldsymbol{h}_\tau)\boldsymbol{V}_j - \varphi\boldsymbol{\nu}^\top\boldsymbol{V}\,\mathrm{div}(\boldsymbol{h}) + \varphi\nu^\top J_{\boldsymbol{h}}\boldsymbol{V} - (\mathbf{grad}(\varphi)\cdot\boldsymbol{h})\nu^\top\boldsymbol{V}\\ &\quad= \varphi\,\mathbf{Grad}(h_{\boldsymbol{\nu}})\cdot\boldsymbol{V} - h_{\boldsymbol{\nu}}\,\mathrm{div}(\varphi\boldsymbol{\nu})\boldsymbol{\nu}^\top\boldsymbol{V}.\end{aligned}$$

Therefore, we finally obtain

$$\begin{aligned}&a\left(\begin{pmatrix}\boldsymbol{A}'\\ \varphi'\end{pmatrix},\begin{pmatrix}\boldsymbol{V}\\ v\end{pmatrix}\right)\\ &= \left\langle\left[\mathrm{i}\omega\varepsilon\left(\boldsymbol{A}_{\boldsymbol{\tau}}\cdot\mathbf{Grad}(h_{\boldsymbol{\nu}}) - h_{\boldsymbol{\nu}}\,\mathrm{div}(\boldsymbol{A}_{\boldsymbol{\nu}}\boldsymbol{\nu})\right) - \varepsilon\,\mathrm{Div}(h_{\boldsymbol{\nu}}\,\mathbf{Grad}\,\varphi) - \varepsilon^2\omega^2\mu_0 h_{\boldsymbol{\nu}}\varphi\right]_{\partial D}, \gamma_D v\right\rangle_{\partial D}\\ &\quad+ \left\langle\left[\mathrm{i}\omega\mu_0\varepsilon\left(\varphi\,\mathbf{Grad}(h_{\boldsymbol{\nu}}) - h_{\boldsymbol{\nu}}\,\mathrm{div}(\varphi\boldsymbol{\nu})\boldsymbol{\nu}\right) - \omega^2\mu_0\varepsilon h_\nu\boldsymbol{A}\right]_{\partial D}^\top, \gamma_D\boldsymbol{V}\right\rangle_{\partial D},\end{aligned}$$

i.e. $\boldsymbol{A}', \varphi'$ is a weak solution of the exterior boundary value problem as asserted. ■