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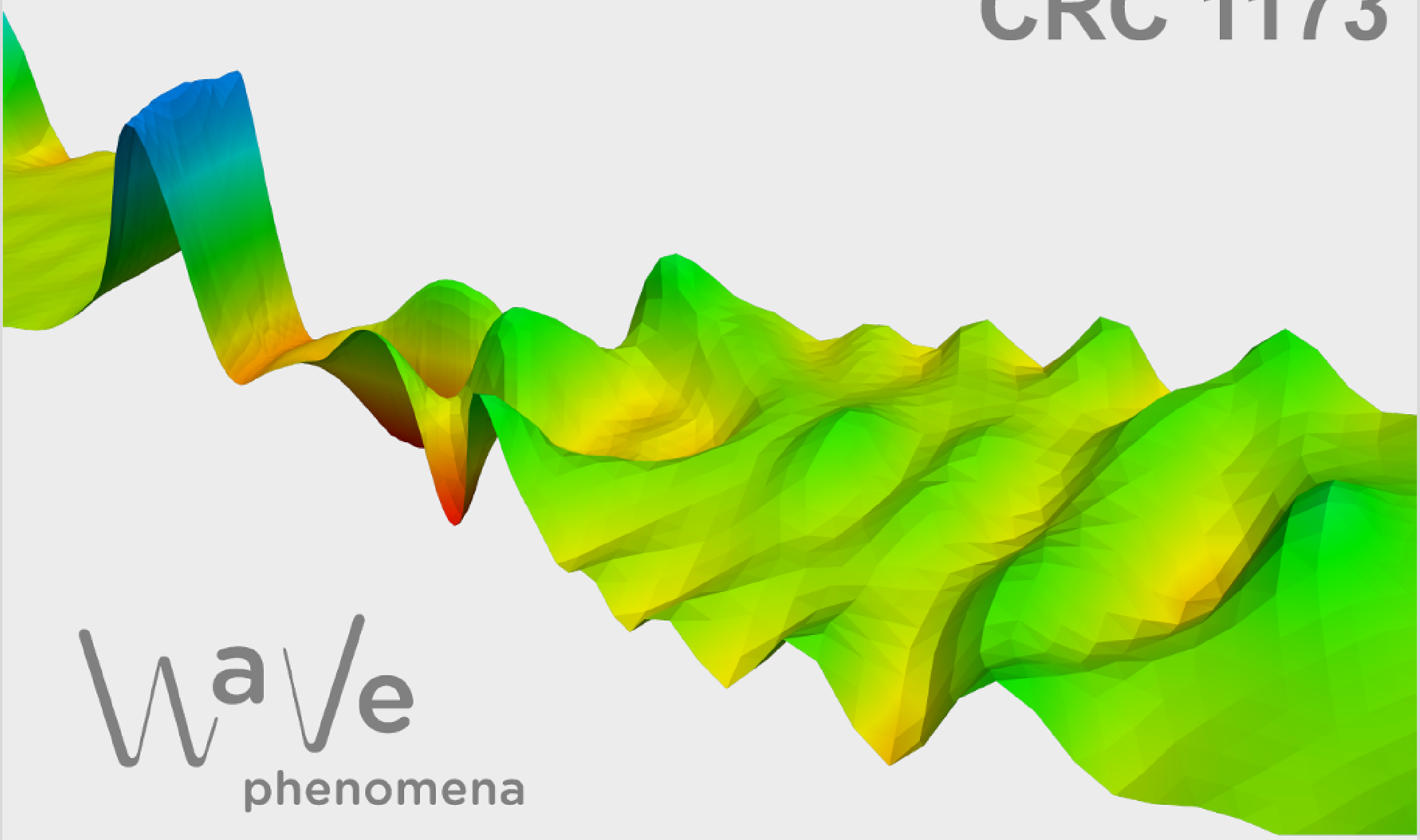

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STRICHARTZ ESTIMATES FOR MAXWELL EQUATIONS ON DOMAINS WITH PERFECTLY CONDUCTING BOUNDARY CONDITIONS

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Abstract. We consider Maxwell equations on a smooth domain with perfectly conducting boundary conditions in isotropic media in two and three dimensions. In the charge-free case we recover Strichartz estimates up to endpoints due to Blair–Smith–Sogge for wave equations on domains. For the proof we suitably extend Maxwell equations over the boundary, which introduces coefficients on the full space with codimension-1 Lipschitz singularity. We diagonalize this system to half-wave equations amenable to the results of Blair–Smith–Sogge. In case of non-vanishing charges, we quantify the defect to Strichartz estimates for wave equations on domains in terms of the charges.

1. Introduction

We discuss dispersive properties for Maxwell equations on bounded domains $\Omega \subseteq \mathbb{R}^3$ with compact boundary $\partial\Omega \in C^{\infty}$[1]. The system couples *electric* and *displacement field* $(\mathcal{E},\mathcal{D}) : \mathbb{R} \times \Omega \to \mathbb{R}^3 \times \mathbb{R}^3$ to *magnetic* and *magnetizing field* $(\mathcal{B},\mathcal{H}) : \mathbb{R} \times \Omega \to \mathbb{R}^3 \times \mathbb{R}^3$. The system of equations reads

$$\left\{ \begin{array}{lll} \partial_t \mathcal{D} &= \nabla \times \mathcal{H} - \mathcal{J}_e, & (t,x) \in \mathbb{R} \times \Omega, \\ \partial_t \mathcal{B} &= -\nabla \times \mathcal{E}, & \nabla \cdot \mathcal{D} = \rho_e, \ \nabla \cdot \mathcal{B} = 0 \end{array} \right. \tag{1}$$

with initial conditions $(\mathcal{E}(0),\mathcal{H}(0)) = (\mathcal{E}_0,\mathcal{H}_0)$. $\mathcal{J}_e : \mathbb{R}\times\Omega \to \mathbb{R}^3$ denotes the *electric current*, which is regarded as source term. We supplement the Maxwell system with pointwise time-independent material laws for isotropic media

$$\mathcal{D}(t,x) = \varepsilon(x)\mathcal{E}(t,x), \qquad \mathcal{B}(t,x) = \mu(x)\mathcal{H}(t,x) \tag{2}$$

with ε, $\mu \in C^\infty(\Omega;\mathbb{R}_{>0})$ denoting *permittivity* and *permeability*, which satisfy the uniform ellipticity conditions

$$\exists \lambda, \Lambda > 0 : \forall x \in \Omega : \lambda \leq \varepsilon(x), \mu(x) \leq \Lambda. \tag{3}$$

We further suppose that for some large $N \geq 2$[2] that

$$\varepsilon,\, \partial\varepsilon,\, \dots \partial^N \varepsilon \in C(\overline{\Omega}), \qquad \mu,\, \partial\mu,\, \dots, \partial^N \mu \in C(\overline{\Omega}). \tag{4}$$

Maxwell equations in media describe the electromagnetism of matter and are of great physical importance. We refer to the physics' literature for a detailed explanation (cf. [8, 14]). We also refer to the lecture notes surveying basic results by Schnaubelt [19].

[1]Certainly, the present arguments extend to $\partial\Omega \in C^N$ for N large enough corresponding to a generalization of the results due to Blair–Smith–Sogge [3] to the C^N-category. We are not attempting to minimize the required regularity.

[2]This constant is the regularity required for the metric such that the results of Blair–Smith–Sogge hold true. It is conceivable that $N = 2$ suffices, but this is currently unclear.

Let $\nu \in C^\infty(\partial\Omega,\mathbb{R}^3)$ denote the outer unit normal. Here we consider the *perfectly conducting boundary conditions*

$$[\mathcal{E}\times\nu]_{x\in\partial\Omega}=0,\qquad [\mathcal{B}\cdot\nu]_{x\in\partial\Omega}=0. \tag{5}$$

The boundary conditions of the perfect electric conductor are among the physically most relevant ones (cf. [22, 19]). We define surface charge and current by complementary boundary values of $\mathcal{D}$ and $\mathcal{H}$ (cf. [19, Eq. (2.3)]):

$$[\mathcal{D}\cdot\nu]_{x\in\partial\Omega}=\rho_\Sigma,\qquad [\mathcal{H}\times\nu]_{x\in\partial\Omega}=J_\Sigma. \tag{6}$$

Furthermore, we require the normal component of $\mathcal{J}_e$ to vanish at the boundary, which is physically sensible:

$$[\mathcal{J}_e\cdot\nu]_{x\in\partial\Omega}=0. \tag{7}$$

The Maxwell equations satisfy finite speed of propagation (see [22, Chapter 6]). Hence, in the interior of the domain we can use previously established results on the whole space for local-in-time results (see previous works by Dumas–Sueur [7] and the second author [17, 15]). Thus, it suffices to work close to the boundary, at which we resolve the Maxwell system in geodesic normal coordinates; see Section 2. At the boundary, we write the equation in geodesic normal coordinates to localize to the half-space $\mathbb{R}^3_{>0}=\{x\in\mathbb{R}^3 : x_3>0\}$. The cometric is given by

$$g^{-1}=\begin{pmatrix} g^{11} & g^{12} & 0\\ g^{21} & g^{22} & 0\\ 0 & 0 & 1\end{pmatrix}.$$

As short-hand notation, we write $\sqrt{g}:=\sqrt{\det g}$. This effectively gives rise to anisotropic permittivity $\sqrt{g}g^{-1}\varepsilon$ and permeability $\sqrt{g}g^{-1}\mu$:

$$\left\{\begin{array}{ll}\partial_t(\sqrt{g}g^{-1}\varepsilon\mathcal{E}) &= \nabla\times\mathcal{H},\quad (t,x)\in\mathbb{R}\times\mathbb{R}^3_{>0},\\ \partial_t(\sqrt{g}g^{-1}\mu\mathcal{H}) &= -\nabla\times\mathcal{E},\qquad (\mathcal{E}\times e_3)|_{x_3=0}=0,\ (\mathcal{H}\cdot e_3)|_{x_3=0}=0\end{array}\right. \tag{8}$$

with the divergence conditions now reading

$$\nabla\cdot(\sqrt{g}g^{-1}\varepsilon\mathcal{E})=\sqrt{g}\rho_e,\qquad \nabla\cdot(\sqrt{g}g^{-1}\varepsilon\mathcal{H})=0.$$

It is important to note that the boundary conditions (5) are respected by g^{-1}.

Note that taking time derivatives in (5) and plugging in (1) yields compatibility conditions. In order to maintain a less technical Introduction, we postpone the discussion of compatibility conditions to Section 2 after we have localised Maxwell's equations to (8). The second compatibility condition simplifies under the assumption

$$\partial\mu|_{x\in\partial\Omega}=0. \tag{9}$$

Spitz [23, 24] showed existence and local well-posedness in $H^3(\Omega)$ (also in the quasilinear case) provided that the compatibility conditions up to second order are satisfied. These are precisely the conditions, which are meaningful in the sense of traces. First, we tend to homogeneous solutions with $\mathcal{J}_e=0$. Accordingly, we let

$$\begin{aligned}\mathcal{H}^3(\Omega)=\{(\mathcal{E}_0,\mathcal{H}_0)\in H^3(\Omega)^2 \ :\ &(\mathcal{E}_0,\mathcal{H}_0)\text{ satisfies homogeneous}\\ &\text{compatibility conditions up to second order }\}.\end{aligned}$$

With the solutions existing, we can show Strichartz estimates for homogeneous solutions

$$\|(\mathcal{E},\mathcal{H})\|_{L_T^p L^q} \lesssim_T \|(\mathcal{E}_0,\mathcal{H}_0)\|_{H^\gamma(\Omega)} + \|\rho_e(0)\|_{H^{\gamma-1+\frac{1}{p}+\delta}(\Omega)} \tag{10}$$

for certain $2 \leq p, q \leq \infty$, $q < \infty$ with γ determined by scaling, and $\delta > 0$[3] such that

$$\gamma = 3\big(\frac{1}{2} - \frac{1}{q}\big) - \frac{1}{p}, \quad \delta < \frac{3}{q}. \tag{11}$$

All Strichartz estimates established in this paper are local in time. For $0 < T < \infty$, we write $L_T^p L_x^q(\Omega) = L_t^p([0,T], L^q(\Omega))$.

(10) is proved in two steps: First, we show

$$\|(\mathcal{E},\mathcal{H})\|_{L_T^p L^q(\Omega)} \lesssim \|(\mathcal{E},\mathcal{H})\|_{L_T^\infty H^{\gamma+\delta}(\Omega)} + \|\rho_e(0)\|_{H^{\gamma-1+\frac{1}{p}+\delta}(\Omega)}.$$

Then it suffices to prove energy estimates for homogeneous solutions for $0 \leq s \leq 3$:

$$\|(\mathcal{E},\mathcal{H})\|_{L_T^\infty H^s} \lesssim_T \|(\mathcal{E}_0,\mathcal{H}_0)\|_{H^s}.$$

Linearity and boundedness allows us to extend the linear solution mapping from the subspace $\mathcal{H}^3(\Omega)$ of $H^\gamma(\Omega)$ to its closure in the H^γ-norm:

$$\mathcal{H}^\gamma(\Omega) = \overline{\mathcal{H}^3(\Omega)}^{\|\cdot\|_{H^\gamma(\Omega)}}. \tag{12}$$

We denote Sobolev spaces (of real-valued functions) on Ω with Dirichlet boundary condition with $H_D^\gamma(\Omega)$; the Sobolev spaces with Neumann boundary conditions are denoted by $H_N^\gamma(\Omega)$.

Since we shall estimate the regularity of $(\mathcal{E}_0,\mathcal{H}_0)$ only in H^γ for $\gamma < \frac{3}{2}$, the compatibility conditions involving derivatives are not relevant. This means we actually only require the Dirichlet conditions for $\mathcal{H}^\gamma$, $\gamma < \frac{3}{2}$. We shall then recover inhomogeneous estimates by Duhamel's formula. Roughly speaking, $\mathcal{H}^\gamma(\Omega)$ is the Sobolev space with relevant compatibility conditions; see Proposition 2.1. For $\gamma < \frac{1}{2}$, this means there are no boundary conditions. For $\frac{1}{2} < \gamma < \frac{3}{2}$, we only have Dirichlet conditions. For $\frac{3}{2} < \gamma < \frac{5}{2}$, we have to take into account first order compatibility conditions, which imply Neumann boundary conditions for $\mathcal{H} \times \nu$.

On the full space, Maxwell equations with rough coefficients and also quasi-linear Maxwell equations were considered in [17] (the two-dimensional case) and the partially anisotropic case in three dimensions was analyzed in [15]. The fully anisotropic case in three dimensions was covered in [18]. In these works, it was pointed out how Maxwell equations (at least in the case of isotropic media) admit diagonalization to two degenerate half-wave equations and four non-degenerate half-wave equations. The contribution of the degenerate components, i.e., stationary solutions, is quantified by the charges. Here we extend Maxwell equations (8) over the boundary via suitable reflections to carry out the diagonalization afterwards. Since the coefficients of the cometric and the permittivity and permeability are extended evenly, the extension introduces a codimension-1 Lipschitz singularity. After paradifferential decomposition, we can still carry out the diagonalization to half-wave equations similar to the more regular case covered in [15] (see [17] for the previously established two-dimensional case). After diagonalization, we can

[3]Note that δ is chosen small enough such that boundary conditions are not relevant for the Sobolev space $H^{\gamma-1+\frac{1}{p}+\delta}$.

apply the Strichartz estimates for wave equations with structured Lipschitz singularity due to Blair–Smith–Sogge [3]. We find local-in-time Strichartz estimates for inhomogeneous Maxwell equations by Duhamel's formula:

$$\begin{aligned}\|(\mathcal{E},\mathcal{H})\|_{L^p([0,T],L^q(\Omega))} \lesssim_T \|(\mathcal{E},\mathcal{H})(0)\|_{\mathcal{H}^{\gamma+\delta}(\Omega)} + \|\mathcal{J}_e\|_{L^1(0,T;\mathcal{H}^{\gamma+\delta}(\Omega))} \\ + \|\rho_e(0)\|_{H^{\gamma-1+\frac{1}{p}+\delta}(\Omega)} + \|\nabla\cdot\mathcal{J}_e\|_{L^1_T H^{\gamma-1+\frac{1}{p}+\delta}(\Omega)}.\end{aligned} \tag{13}$$

The use of Duhamel's formula in $\mathcal{H}^{\gamma+\delta}$ requires us to impose Dirichlet boundary conditions on $\mathcal{J}_e$.

We digress for a moment to recall Strichartz estimates for the wave equation on domains: Strichartz estimates for wave equations on (general) manifolds with boundary for Dirichlet as well as Neumann boundary conditions were first investigated by Burq *et al.* [4, 5] and Blair–Smith–Sogge [3] based on the seminal contribution by Smith–Sogge [21] regarding spectral cluster estimates. Notably, there are more refined results and counterexamples on special domains due to Ivanovici *et al.* [11, 10, 12, 13]. For exterior convex domains, Smith–Sogge [20] recovered the Euclidean Strichartz estimates (local-in-time) much earlier by the Melrose–Taylor parametrix.

For Maxwell equations with perfectly conducting boundary conditions, we prove the following theorem:

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^3$ be a smooth domain with compact boundary and ε, $\mu \in C^\infty(\mathbb{R}^3;\mathbb{R}_{>0})$ satisfy* (3)*. Let $2 \leq p,q < \infty$, and let $(\mathcal{E},\mathcal{H}) : \mathbb{R}\times\Omega \to \mathbb{R}^3\times\mathbb{R}^3$ denote solutions to* (1) *with material laws* (2)*, which satisfy the perfectly conducting boundary conditions* (5)*. Then* (13) *holds with γ and δ given by* (11) *provided that*

$$\frac{3}{p}+\frac{2}{q} \leq 1. \tag{14}$$

Recall that the boundary conditions are indistinguishable at low regularities. We have $H^s_D(\Omega) = H^s(\Omega)$ for $s < 1/2$ and $H^s_N(\Omega) = H^s(\Omega)$ for $s < \frac{3}{2}$. Since we estimate $\mathcal{J}_e$ in Sobolev spaces with boundary conditions, we have to require

$$[\mathcal{J}_e]_{x\in\partial\Omega} = 0$$

for $\gamma \geq \frac{1}{2}$. Note that because $\gamma - 1 + \frac{1}{p} + \delta < \frac{1}{2}$ the boundary condition of ρ_e is not relevant.

We shall also discuss the two-dimensional case:

$$\left\{\begin{array}{llcl} \partial_t(\varepsilon\mathcal{E}) &= \nabla_\perp\mathcal{H} - \mathcal{J}_e, & (t,x) &\in& \mathbb{R}\times\Omega, \\ \partial_t(\mu\mathcal{H}) &= -(\nabla\times\mathcal{E})_3 = -(\partial_1\mathcal{E}_2 - \partial_2\mathcal{E}_1), & \nabla\cdot(\varepsilon\mathcal{E}) &=& \rho_e \end{array}\right. \tag{15}$$

with $\nabla_\perp = (\partial_2, -\partial_1)$. Here $\Omega \subseteq \mathbb{R}^2$ denotes a smooth domain in $\mathbb{R}^2$ with compact boundary, and $\mathcal{E} : \mathbb{R}\times\Omega \to \mathbb{R}^2$, $\mathcal{J}_e : \mathbb{R}\times\Omega \to \mathbb{R}^2$, $\mathcal{H} : \mathbb{R}\times\Omega \to \mathbb{R}$. We let $\varepsilon,\mu \in C^\infty(\Omega)$. We require $\varepsilon : \Omega \to \mathbb{R}$ and $\mu : \Omega \to \mathbb{R}$ to satisfy

$$\exists \lambda,\Lambda > 0 : \forall x \in \Omega : \lambda \leq \varepsilon(x), \mu(x) \leq \Lambda. \tag{16}$$

Like above, we require uniform bounds for finitely many derivatives up to the boundary for large $N \geq 2$:

$$\varepsilon, \partial\varepsilon, \ldots, \partial^N\varepsilon \in C(\overline{\Omega}), \quad \mu, \partial\mu, \ldots, \partial^N\mu \in C(\overline{\Omega}). \tag{17}$$

The perfectly conducting boundary condition for (15) is given by

$$[\mathcal{E}\wedge\nu]_{x\in\partial\Omega} = 0. \tag{18}$$

Spitz's local well-posedness in three dimensions descends to the two dimensional case. In the following we take into account boundary and compatibility conditions in $\mathcal{H}^3(\Omega)$ as we did in the three-dimensional case. We abuse notation and define $\mathcal{H}^\gamma(\Omega)$ as closure of $\mathcal{H}^3(\Omega)$ in the $H^\gamma(\Omega)$-topology like in (12). We prove the following:

Theorem 1.2. *Let $\Omega \subseteq \mathbb{R}^2$ be a smooth domain with compact boundary, $2 \leq p, q \leq \infty$, and suppose that*

$$\frac{3}{p} + \frac{1}{q} \leq \frac{1}{2}, \quad q < \infty, \quad \gamma = 2\big(\frac{1}{2} - \frac{1}{p}\big) - \frac{1}{q}, \quad 0 < \delta < \frac{1}{2}. \tag{19}$$

Suppose that $\varepsilon \in C^\infty(\Omega;\mathbb{R})$, $\mu \in C^\infty(\Omega;\mathbb{R})$ be like above and satisfy (16) *and* (17).

Then the following estimate holds for solutions to (15) *with initial data $(\mathcal{E}_0, \mathcal{H}_0) \in \mathcal{H}^\gamma(\Omega)$ satisfying boundary conditions* (18)*:*

$$\begin{aligned} \|(\mathcal{E},\mathcal{H})\|_{L_T^p L^q(\Omega)} &\lesssim_T \|(\mathcal{E}_0,\mathcal{H}_0)\|_{\mathcal{H}^{\gamma+\delta}} + \|\mathcal{J}_e\|_{L_T^1 \mathcal{H}^{\gamma+\delta}} \\ &\quad + \|\rho_e(0)\|_{H^{\gamma-1+\frac{1}{p}+\delta}(\Omega)} + \|\nabla\cdot\mathcal{J}_e\|_{L_T^1 H^{\gamma-1+\frac{1}{p}+\delta}(\Omega)}. \end{aligned}$$

Outline of the paper. In Section 2 we write Maxwell's equations with differential forms to facilitate change of variables. We use this to formulate Maxwell equations on the half-space. We reduce Strichartz estimates to homogeneous estimates for the reflected solutions. A key ingredient to conclude the proof are energy estimates. However, these we prove on the level of the original equations posed on domains in Section 3. In Section 4 we collect facts on pseudo-differential operators. In Section 5 we diagonalize three-dimensional Maxwell equations after localization to the half-space, in Section 6 we diagonalize two-dimensional Maxwell equations.

2. Maxwell equations on manifolds

To investigate the behavior of Maxwell equations under coordinate transformations, we set up Maxwell's equations on smooth Riemannian manifolds with boundary (M, g). In this context, the fields are given at any time $t \in \mathbb{R}$ as covectorfields $X(t) : M \to T^*M$, $X \in \{\mathcal{E}, \mathcal{D}, \mathcal{H}, \mathcal{B}, \mathcal{J}_e\}$, i.e., sections of the cotangential bundle. Permittivity and permeability are given by $\kappa(t) : M \to \mathrm{Sym}(T^*M \to T^*M)$, $\kappa \in \{\varepsilon, \mu\}$, and $\rho_e(t) : M \to \mathbb{R}$. Let $*, d : \Lambda T^*M \to \Lambda T^*M$ denote the Hodge dual and exterior derivative. We localize Maxwell equations to the half-space via geodesic normal coordinates. This facilitates to find compatibility conditions. This in turn allows us to find suitable extensions of the fields from the half-space to the full space. The extension respects the Sobolev regularity $0 \leq \gamma \leq 2$, which suffices for the presently considered Strichartz estimates, and the extended fields moreover satisfy Maxwell equations on the full space, albeit with coefficients with Lipschitz singularity. We first consider the more involved three-dimensional case and then shall be brief for the two-dimensional case.

2.1. **3d manifolds.**

With the aid of Hodge dual and the exterior derivative, we can write for the curl and divergence of vectorfields $F : \Omega \to \mathbb{R}^3$:

$$\nabla \times F = *dF, \qquad \nabla \cdot F = *d * F.$$

Consequently, the Maxwell system of equations reads

$$\left\{ \begin{array}{lll} \partial_t(\varepsilon\mathcal{E}) &= *d\mathcal{H} - \mathcal{J}_e, & *d*(\varepsilon\mathcal{E}) = \rho_e, \\ \partial_t(\mu\mathcal{H}) &= -*d\mathcal{E}, & (t,x) \in \mathbb{R}\times M. \end{array} \right. \tag{20}$$

Let $\# : TM \to T^*M$ and $\flat : T^*M \to TM$ denote the musical isomorphisms. The boundary conditions are given by

$$[(\mathcal{E}^\flat)_{||}]_{x\in\partial M} = 0, \qquad [(\mathcal{B}^\flat)_\perp]_{x\in\partial M} = 0 \tag{21}$$

We define surface current $\mathcal{J}_\Sigma$ and surface charges ρ_Σ on the boundary by

$$[(\mathcal{H}^\flat)_{||}]_{x\in\partial M} = [\mathcal{J}_\Sigma]_{x\in\partial M} \text{ and } [(\mathcal{D}^\flat)_\perp]_{x\in\partial M} = \rho_\Sigma. \tag{22}$$

2.1.1. *Finite speed of propagation.* In this section, we show how we can reduce the local-in-time analysis to charts. We recall the notion of finite speed of propagation. Let $(\mathcal{E},\mathcal{H})$ denote homogeneous solutions to

$$\left\{ \begin{array}{lll} \partial_t(\varepsilon\mathcal{E}) &= \nabla\times\mathcal{H}, & (t,x)\in\mathbb{R}\times\Omega, \\ \partial_t(\mu\mathcal{H}) &= -\nabla\times\mathcal{E}, & [\mathcal{E}\times\nu]_{x\in\partial\Omega} = 0,\ [\mathcal{H}\cdot\nu]_{x\in\partial\Omega} = 0. \end{array} \right.$$

For $X \subseteq \Omega$ let $\mathcal{N}_r(X) = \{x \in \Omega : \operatorname{dist}(x,X) < r\}$. By Maxwell equations having finite speed of propagation, we mean that there is $0 < c < \infty$ such that for $0 < t < \infty$ it holds

$$\operatorname{supp}_x((\mathcal{E},\mathcal{H})(t)) \subseteq \mathcal{N}_{ct}(\operatorname{supp}_x(\mathcal{E}_0,\mathcal{H}_0)).$$

We refer to [22, Theorem 6.1] for a more precise statement in terms of the backwards light cone.

Let $d : \Omega \to \mathbb{R}_{>0}$, $d(x) = \operatorname{dist}(x,\partial\Omega)$ denote the distance function away from the boundary, and $H_\tau = d^{-1}(\tau)$ denote corresponding level sets. By the implicit function theorem, H_τ is a smooth hypersurface with metric g_τ and we can write

$$g = d\tau^2 + g_\tau \text{ for } 0 \leq t \leq \tilde{\delta}.$$

By compactness of $\partial\Omega$, finitely many geodesics charts suffice to cover a set $\{x \in \Omega : d(x) < \varepsilon\}$ close to the boundary. Shrinking the charts allows us to restrict to local-in-time solutions, which do not leave the geodesic chart.

Regarding the interior part, we find T small enough such that $(\mathcal{E},\mathcal{H})(t)$ within $\Omega^{\text{int}} = \{x \in \Omega : d(x) > \varepsilon/2\}$ only depends on $\tilde{\Omega}^{\text{int}} = \{x \in \Omega : d(x) > \varepsilon/4\}$, and the solution does not reach the boundary for times $t \leq T$. This means we have

$$\|(\mathcal{E},\mathcal{H})\|_{L^p_T L^q(\Omega^{\text{int}})} \lesssim \|(\mathcal{E}_0,\mathcal{H}_0)\|_{H^s(\Omega)}.$$

2.1.2. *Geodesic normal coordinates.* Let $g = (g_{ij})$ denote the metric tensor and $g^{-1} = (g^{ij})$ the cometric. In this work, we only consider isotropic ε and μ on the original domain (Ω,δ^{ij}). We endow a chart in (Ω,δ^{ij}) with geodesic normal coordinates derived from the height function:

$$g^{ij} = dx_3^2 + r(x',x_3,(dx')^2).$$

The Hodge dual transforms by

$$*(dx^{i_1}\wedge\ldots\wedge dx^{i_k}) = \frac{\sqrt{g}}{(n-k)!} g^{i_1 j_1}\ldots g^{i_k j_k}\varepsilon_{j_1\ldots j_n} dx^{j_{k+1}}\wedge\ldots\wedge dx^{j_n}.$$

Above $\varepsilon_{j_1\ldots j_n}$ denotes the n-Levi–Civita tensor, i.e.,

$$\varepsilon_{j_1\ldots j_n} = \begin{cases} 1, & (j_1\ldots j_n) \text{ is an even permutation}, \\ -1, & (j_1\ldots j_n) \text{ is an odd permutation}, \\ 0, & (j_1\ldots j_n) \text{ is no permutation}. \end{cases}$$

and (g^{ij}) denotes the inverse metric. Recall that we let $\sqrt{g} = \sqrt{\det g}$. Consequently, we find in geodesic normal coordinates

$$*_g dA = \sqrt{g}\, ad(g^{-1}) \nabla \times A, \qquad *_g d *_g A = \frac{1}{\sqrt{g}} \nabla \cdot (\sqrt{g} g^{-1} (\varepsilon \mathcal{E}')).$$

In the above display $\mathrm{ad}(B)$ denotes the adjugate matrix, i.e.,

$$\mathrm{ad}(B) = ((-1)^{i+j} B_{ji})_{i,j}$$

with B_{ji} denoting the (j,i)-minor of B. By Cramer's rule, Maxwell equations become on the half-space $(t,x) \in \mathbb{R} \times \mathbb{R}^3_{>0}$:

$$\left\{ \begin{array}{llll} \partial_t(\varepsilon(x)\mathcal{E}') &= (\sqrt{g})^{-1} g \nabla \times \mathcal{H}' - \mathcal{J}_e', & \nabla \cdot \big(\sqrt{g} g^{-1} \mu \mathcal{H} \big) &= 0, \\ \partial_t(\mu(x)\mathcal{H}') &= -\big(\sqrt{g}\big)^{-1} g \nabla \times \mathcal{E}', & \frac{1}{\sqrt{g}} \nabla \cdot \big(\sqrt{g} g^{-1} \varepsilon \mathcal{E} \big) &= \rho_e'. \end{array} \right.$$

In a sense, $\sqrt{g} g^{-1} \varepsilon$ now plays the role of ε and $\sqrt{g} g^{-1} \mu$ the role of $\sqrt{g} g^{-1} \mu$. Also, we redefine $\rho_e' = \nabla \cdot (\sqrt{g} g^{-1} \varepsilon \mathcal{E})$, which does not effect regularity questions because $\sqrt{g}$ is smooth. Moreover, we write $\mathcal{J}_e' := \sqrt{g} g^{-1} \mathcal{J}_e'$. Below we shall see that this is consistent with the compatibility conditions. We rearrange the equations to

$$\left\{ \begin{array}{lll} \partial_t(\sqrt{g} g^{-1} \varepsilon \mathcal{E}') &= \nabla \times \mathcal{H}' - \mathcal{J}_e', & \nabla \cdot \big(\sqrt{g} g^{-1} \mu \mathcal{H} \big) = 0, \\ \partial_t(\sqrt{g} g^{-1} \mu \mathcal{H}') &= -\nabla \times \mathcal{E}', & \nabla \cdot \big(\sqrt{g} g^{-1} \varepsilon \mathcal{E} \big) = \rho_e'. \end{array} \right.$$

2.1.3. *Compatibility conditions.* On the half-space $x \in \mathbb{R}^3_{>0}$, the boundary conditions are given as follows:

$$[\mathcal{E}_1]_{x_3=0} = [\mathcal{E}_2]_{x_3=0} = [\mathcal{H}_3]_{x_3=0} = 0. \tag{23}$$

We call a relation

$$\mathrm{tr}(F(\partial^\alpha \mathcal{E}, \partial^\beta \mathcal{H})) = 0,$$

which follows from (23) by taking k time derivatives a compatibility condition of order k. Hence, (23) are of order zero. For (8), the tangential derivatives are ∂_t, ∂_1, and ∂_2, which allows for explicitly expressing the compatibitility conditions.

It is important to observe that the (possibly non-diagonal) metrical tensor only mixes the first and second component:

$$g^{-1} = \begin{pmatrix} g^{11} & g^{12} & 0 \\ g^{21} & g^{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We give the first order compatibility conditions in the homogeneous case: Applying tangential derivatives ∂_1, ∂_2 to $\mathcal{H}_3$ gives

$$[\partial_1 \mathcal{H}_3]_{x_3=0} = [\partial_2 \mathcal{H}_3]_{x_3=0} = 0.$$

The equation for the first and second component of the equation

$$\partial_t(\sqrt{g} g^{-1} \varepsilon \mathcal{E}) = \nabla \times \mathcal{H}$$

yields

$$[\partial_3 \mathcal{H}_1]_{x_3=0} = [\partial_3 \mathcal{H}_2]_{x_3=0} = 0.$$

Moreover, tangential derivatives ∂_1, ∂_2 applied to $\mathcal{E}_1$ and $\mathcal{E}_2$ and the charge condition yields

$$[\partial_1 \mathcal{E}_1]_{x_3=0} = [\partial_2 \mathcal{E}_2]_{x_3=0} = 0 \text{ and } [\nabla \cdot (\varepsilon \sqrt{g} g^{-1} \mathcal{E})]_{x_3=0} = \mathrm{tr}(\rho).$$

Let $\tilde{\varepsilon} = \sqrt{g} \varepsilon$. The above display becomes

$$[\partial_3(\tilde{\varepsilon} \mathcal{E}_3)]_{x_3=0} = \mathrm{tr}(\rho) \Leftrightarrow [(\partial_3 \tilde{\varepsilon}) \mathcal{E}_3]_{x_3=0} + [\tilde{\varepsilon} \partial_3 \mathcal{E}_3]_{x_3=0} = \mathrm{tr}(\rho).$$

This yields a Robin boundary condition for $\mathcal{E}_3$ in terms of $\mathrm{tr}(\rho)$ and ρ_Σ. If these are vanishing, we have Neumann boundary conditions for $\mathcal{E}_3$.

We extend the equations to the full space as follows: Reflect ε, μ, and g^{ij} evenly. Let

$$\tilde{\kappa}(x_1,x_2,x_3) = \begin{cases} \kappa(x_1,x_2,x_3), & x_3 \geq 0, \\ \kappa(x_1,x_2,-x_3), & x_3 < 0, \end{cases} \qquad \kappa \in \{\varepsilon, \mu, g^{ij}\}.$$

On the other hand, $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{H}_3$ are reflected oddly, and $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{E}_3$ are reflected evenly. $\mathcal{J}_{e1}$, $\mathcal{J}_{e2}$ are reflected oddly and $\mathcal{J}_{e3}$ evenly. Note that the boundary condition $[\mathcal{J}_e.\nu]_{x\in\partial\Omega} = 0$ would rather suggest odd extension, but for the considered regularity $\gamma < \frac{3}{2}$ this is not relevant. ρ_e is reflected oddly. Denoting the reflected quantities with $\tilde{X}$ of the original quantity X and $\sqrt{\tilde{g}} = \sqrt{\det \tilde{g}}$ the following system of equations holds on $\mathbb{R}^3$:

$$\left\{ \begin{array}{llr} \partial_t(\sqrt{\tilde{g}}\tilde{g}^{-1}\tilde{\varepsilon}\tilde{\mathcal{E}}) &= \nabla \times \tilde{\mathcal{H}} - \tilde{\mathcal{J}}_e, & \nabla \cdot (\sqrt{\det \tilde{g}}\tilde{g}^{-1}\tilde{\mu}\tilde{\mathcal{H}}) = 0, \\ \partial_t(\sqrt{\tilde{g}}\tilde{g}^{-1}\tilde{\mu}\tilde{\mathcal{H}}) &= -\nabla \times \tilde{\mathcal{E}}, & \nabla \cdot (\sqrt{\det \tilde{g}}\tilde{g}^{-1}\tilde{\varepsilon}\tilde{\mathcal{E}}) = \tilde{\rho}_e. \end{array} \right. \tag{24}$$

We give the compatibility conditions under assumptions (9):

$$[\mathcal{E}_0 \times \nu]_{x\in\partial\Omega} = 0, \quad [\mathcal{H}_0.\nu]_{x\in\partial\Omega} = 0, \tag{25}$$

$$[\partial_\nu \mathcal{H}_{\mathrm{tang}}]_{x\in\partial\Omega} = 0, \tag{26}$$

$$[\nabla_{\mathrm{tang}} \partial_\nu (\mathcal{E}.\nu)]_{x\in\partial\Omega} = 0. \tag{27}$$

We find the second compatibility condition by taking two time derivatives before changing to geodesic normal coordinates:

$$\partial_t^2(\varepsilon\mathcal{E}) = \nabla \times \big(-\frac{1}{\mu}\nabla \times \mathcal{E}\big) = O(\partial\mu\nabla\times\mathcal{E}) + \frac{1}{\mu}(\Delta\mathcal{E} - \nabla(\nabla\cdot\mathcal{E})). \tag{28}$$

Recall that we required (9)

$$\partial\mu|_{x\in\partial\Omega} = 0$$

to simplify the compatibility conditions. Hence, when taking the tangential trace in (28), the first expression vanishes. For the analysis of the second, we change to geodesic normal coordinates normalized to find

$$\Delta\mathcal{E} = \sum_{i,j=1}^{2} \partial_i g^{ij} \partial_j \mathcal{E} + \partial_3^2 \mathcal{E},$$
$$\nabla_{tang}(\nabla\cdot\mathcal{E}) = \nabla_{\mathrm{tang}}(\nabla_{\mathrm{tang}} \cdot (\tilde{g}\mathcal{E}_{\mathrm{tang}})) + \partial_3\mathcal{E}_3).$$

It turns out that the tangential part of the first expression is vanishing anyway, which follows from the local expansion into Dirichlet eigenfunctions. Since $[\nabla_{\mathrm{tang}}(\nabla_{\mathrm{tang}} \cdot (\tilde{g}\mathcal{E})_{\mathrm{tang}}] = 0$, we find that it suffices to require that $\mathcal{E}.\nu$ satisfies Neumann boundary conditions. As a consequence, the second compatibility condition will be satisfied.

Proposition 2.1. *Let $0 \leq \gamma \leq 3$ and $\mathcal{H}^\gamma(\Omega)$ be defined by (12) and suppose that (9) holds. Then, we have the following characterization:*

- $0 \leq \gamma < \frac{1}{2}$*: $\mathcal{H}^\gamma(\Omega) = H^\gamma(\Omega)$,*
- $\frac{1}{2} < \gamma < \frac{3}{2}$*: $\mathcal{H}^\gamma(\Omega) = \{(\mathcal{E}_0,\mathcal{H}_0) \in H^\gamma(\Omega) : (25)$ holds$\}$,*
- $\frac{3}{2} < \gamma < \frac{5}{2}$*: $\mathcal{H}^\gamma(\Omega) = \{(\mathcal{E}_0,\mathcal{H}_0) \in H^\gamma(\Omega) : (25)$ and (26) hold$\}$,*
- $\frac{5}{2} < \gamma \leq 3$*: $\mathcal{H}^\gamma(\Omega) = \{(\mathcal{E}_0,\mathcal{H}_0) \in H^\gamma(\Omega) : (25) - (27)$hold$\}$.*

For the proof we shall change to geodesic normal coordinates. In a chart endowed with geodesic coordinates, i.e., for Maxwell equations localized to the half-space, we have

$$[\mathcal{E}_1]_{x_3=0} = [\mathcal{E}_2]_{x_3=0} = [\mathcal{H}_3]_{x_3=0} = 0, \tag{29}$$

$$[\partial_3 \mathcal{H}_1]_{x_3=0} = [\partial_3 \mathcal{H}_2]_{x_3=0} = 0, \tag{30}$$

$$[\partial_1 \partial_3 \mathcal{E}_3]_{x_3=0} = [\partial_2 \partial_3 \mathcal{E}_3]_{x_3=0} = 0. \tag{31}$$

Proof of Proposition 2.1. Let $(\Omega_i, \varphi_i)_{i=1,\dots,n}$ denote a finite covering of a neighbourhood of the boundary with geodesic charts and $(\Omega_0, \varphi_0 = id)$ the trivial chart of the interior. We decompose $u_0 \in H^\gamma(\Omega)$ with a smooth partition of unity subordinate to $(\Omega_j)_{j=0,\dots,n}$, $1 = \sum_{i=1}^n \psi_i + \psi_0$ and write

$$u_0 = \sum_{i=1}^n \psi_i u_0 + \psi_0 u_0.$$

It suffices to show the claim for any $u_0^{(i)}$. Within Ω_i we can endow Ω with geodesic normal coordinates and it is enough to prove the claim for the transformed fields by invariance of Sobolev spaces under changes of coordinates. For Ω_0 this is trivial because there is no boundary. Within Ω_i we can use geodesic normal coordinates. Note that with

$$\mathcal{H}^3(\mathbb{R}^3_{>0}) = \{(\mathcal{E}_0, \mathcal{H}_0) \in H^3(\mathbb{R}^3_{>0})^2 : (29) - (31) \text{ holds}\},$$

we now have to show that

$$\overline{\mathcal{H}^3(\mathbb{R}^3_{>0})}^{\|\cdot\|_{H^s}} = \mathcal{H}^s(\mathbb{R}^3_{>0}) = \begin{cases} \{(\mathcal{E}_0, \mathcal{H}_0) \in H^s(\mathbb{R}^3_{>0})\}, & 0 \leq s < \frac{1}{2}, \\ \{(\mathcal{E}_0, \mathcal{H}_0) \in H^s(\mathbb{R}^3_{>0}) : (29) \text{ holds}\}, & \frac{1}{2} < s < \frac{3}{2}, \\ \{(\mathcal{E}_0, \mathcal{H}_0) \in H^s(\mathbb{R}^3_{>0}) : (29), (30) \text{ hold}\}, & \frac{3}{2} < s < \frac{5}{2}, \\ \{(\mathcal{E}_0, \mathcal{H}_0) \in H^s(\mathbb{R}^3_{>0}) : (29) - (31) \text{ hold }\}, & \frac{5}{2} < s \leq 3. \end{cases}$$

The limiting cases $s \in \mathbb{N} + \frac{1}{2}$, $n \in \mathbb{N}_0$ are excluded for the sake of simplicity. The inclusion

$$\overline{\mathcal{H}^3(\mathbb{R}^3_{>0})}^{\|\cdot\|_{H^s}} \subseteq \mathcal{H}^s(\mathbb{R}^3_{>0})$$

follows from the continuity of the trace. To show the reverse inclusion,

$$\mathcal{H}^s(\mathbb{R}^3_{>0}) \subseteq \overline{\mathcal{H}^3(\mathbb{R}^3_{>0})}^{\|\cdot\|_{H^s}},$$

we have to approximate elements in $(\mathcal{E}_0, \mathcal{H}_0) \in \mathcal{H}^s(\mathbb{R}^3_{>0})$ with elements in $\mathcal{H}^3$. For $0 \leq s < \frac{1}{2}$, we extend $(\mathcal{E}_0, \mathcal{H}_0)$ to the full space by reflecting $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{H}_3$ oddly and $\mathcal{E}_3$, $\mathcal{H}_1$, $\mathcal{H}_2$ evenly. Let $\overline{(\mathcal{E}_0, \mathcal{H}_0)}$ denote the extended datum. Recall the following: Since odd reflection is an extension for functions with vanishing boundary conditions, we find continuity of

$$\begin{aligned} \mathrm{ext}_D : H^s_0(\mathbb{R}^3_{>0}) &\to H^s(\mathbb{R}^3) \\ f &\mapsto \bar{f}_o \end{aligned}$$

with

$$\bar{f}_o(x) = \begin{cases} f(x), & x_3 > 0, \\ -f(-x), & x_3 < 0. \end{cases}$$

Likewise even reflection yields a continuous operator for Neumann functions for $0 \leq s \leq 2$:

$$\begin{aligned} \mathrm{ext}_N : H^s_N(\mathbb{R}^3_{>0}) &\to H^s(\mathbb{R}^3) \\ f &\mapsto \bar{f}_e \end{aligned}$$

with

$$\bar{f}_e(x) = \begin{cases} f(x), & x_3 > 0, \\ f(-x), & x_3 < 0. \end{cases}$$

Hence, $\overline{(\mathcal{E}_0, \mathcal{H}_0)} \in H^s(\mathbb{R}^3)$. We regularize the components as follows: Let $f_n = f * \varphi_3^n * \varphi_{12}^n$ with

$$(f * \varphi_3^n)(x_1, x_2, x_3) = \int_{\mathbb{R}} f(x_1, x_2, x_3 - y) \varphi_3^n(y) dy$$

and $\varphi_3^n(y) = n\psi(ny)$, $\psi \in C_c^\infty$, symmetric $\psi \geq 0$, and $\int_{\mathbb{R}} \psi(y) dy = 1$.

Secondly,

$$(g * \varphi_{12}^n)(x_1, x_2, x_3) = \int_{\mathbb{R}^2} g(x_1 - y, x_2 - y, x_3) \varphi_{12}^n(y) dy$$

and $\varphi_{12}^n(y) = n^2 \psi(ny_1)\psi(ny_2)$. We denote the component-wise regularized extension by $\overline{(\mathcal{E}_0, \mathcal{H}_0)}_n$. Clearly, $\overline{(\mathcal{E}_0, \mathcal{H}_0)}_n \in H^3(\mathbb{R}^3)$ and moreover,

$$\begin{cases} \overline{\mathcal{E}_0}_{n1}(x_1, x_2, 0) &= \overline{\mathcal{E}_0}_{n2}(x_1, x_2, 0) = \overline{\mathcal{H}_0}_n(x_1, x_2, 0) = 0, \\ \partial_3 \overline{\mathcal{H}_{03}}(x_1, x_2, 0) &= \partial_3 \overline{H_{03}}(x_1, x_2, 0) = 0, \\ \partial_3 \overline{\mathcal{E}_{03}}(x_1, x_2, 0) &= 0. \end{cases}$$

Thus, for the restricted function $\overline{(\mathcal{E}_0, \mathcal{H}_0)}_n|_{\mathbb{R}^3_{>0}}$ we find that the boundary conditions (29)-(31) are fulfilled. Since

$$\| \overline{(\mathcal{E}_0, \mathcal{H}_0)}_n - \overline{(\mathcal{E}_0, \mathcal{H}_0)} \|_{H^s(\mathbb{R}^3)} \to 0,$$

we infer that

$$\| \overline{(\mathcal{E}_0, \mathcal{H}_0)}_n|_{\mathbb{R}^3_{>0}} - (\mathcal{E}_0, \mathcal{H}_0) \|_{H^s(\mathbb{R}^3_{>0})} \to 0.$$

This yields the claim for $0 \leq s < \frac{1}{2}$, $\frac{1}{2} < s < \frac{3}{2}$, $\frac{3}{2} < s \leq 2$. For $s \in (2,3] \backslash \{5/2\}$, the preceding argument still yields

$$\| \overline{(\mathcal{E}_0, \mathcal{H}_0)}_n|_{\mathbb{R}^3_{>0}} - (\mathcal{E}_0, \mathcal{H}_0) \|_{H^2(\mathbb{R}^3_{>0})} \to 0.$$

To conclude the proof, it suffices to show that

$$\| \Delta(\overline{(\mathcal{E}_0, \mathcal{H}_0)}_n|_{\mathbb{R}^3_{>0}}) - \Delta(\mathcal{E}_0, \mathcal{H}_0) \|_{H^{s-2}(\mathbb{R}^3_{>0})} \to 0. \tag{32}$$

By the explicit form of our regularization, we have for $i = 1, 2, 3$

$$\partial_i^2 [\overline{(\mathcal{E}_0, \mathcal{H}_0)} * \varphi_3^n * \varphi_{12}^n] = (\overline{\partial_i^2 (\mathcal{E}_0, \mathcal{H}_0)}) * \varphi_3^n * \varphi_{12}^n.$$

Hence, (32) follows from applying the preceding argument to $\partial_i^2(\mathcal{E}_0, \mathcal{H}_0)$: We find

$$\| \overline{\partial_i^2 (\mathcal{E}_0, \mathcal{H}_0)_n} - \overline{\partial_i^2 (\mathcal{E}_0, \mathcal{H}_0)} \|_{H^{s-2}(\mathbb{R}^3)} \to 0$$

from continuity of the extensions in $H^{s-2}_{\mathbb{R}^3_{>0}}$, noting that two derivatives preserve the boundary conditions, which implies

$$\| \overline{\partial_i^2 (\mathcal{E}_0, \mathcal{H}_0)}_n|_{\mathbb{R}^3_{>0}} - \partial_i^2 (\mathcal{E}_0, \mathcal{H}_0) \|_{H^{s-2}(\mathbb{R}^3_{>0})} \to 0.$$

The proof is complete. $\square$

2.1.4. *Reductions for smooth coefficients.* As main step in the proof of Theorem 1.1, we show the following:

Proposition 2.2. *Let $\tilde{u} = (\tilde{\mathcal{E}}, \tilde{\mathcal{H}})$, and $\tilde{\varepsilon}$, $\tilde{\mu}$, $\tilde{g}$ be like in (24). Then the following estimates hold:*

$$\|\tilde{u}\|_{L^p L^q} \lesssim \|\tilde{u}\|_{L_T^\infty H^{\gamma+\delta}} + \|\tilde{\mathcal{J}}_e\|_{L_t^2 H^{\gamma+\delta}} + \|\tilde{\rho}\|_{L_T^\infty H^{\gamma-1+\frac{1}{p}+\delta}(\Omega)} \tag{33}$$

for $p, q \geq 2$, $q < \infty$, $\delta > 0$ satisfying the following

$$\frac{3}{p} + \frac{2}{q} \leq 1, \qquad \gamma = 3\Big(\frac{1}{2} - \frac{1}{q}\Big) - \frac{1}{p}, \qquad \delta < \frac{3}{q}.$$

Remark 2.3. Recall that ρ is reflected oddly. The Dirichlet condition is irrelevant for $s < \frac{1}{2}$, which is ensured with the condition on δ.

We conclude the section with the following:

Proposition 2.4. *Suppose that Proposition 2.2 holds true and the energy estimate*

$$\|u\|_{L_T^\infty \mathcal{H}^\gamma(\Omega)} \lesssim_T \|u\|_{\mathcal{H}^\gamma(\Omega)} \tag{34}$$

is valid for homogeneous solutions $u = (\mathcal{E}, \mathcal{H})$ to (1). Then, Theorem 1.1 follows.

Proof. First, we prove Theorem 1.1 for homogeneous solutions $u = (\mathcal{E}, \mathcal{H})$ with $\mathcal{J}_e = 0$. By virtue of the energy estimate (34), it suffices to show:

$$\|u\|_{L^p([0,T], L^q(\Omega))} \lesssim \|u\|_{L_T^\infty \mathcal{H}^\gamma} + \|\rho_e(0)\|_{H^{\gamma-1+\frac{1}{p}+\delta}(\Omega)}. \tag{35}$$

But for homogeneous solutions $u = (\mathcal{E}, \mathcal{H})$ to (20), the transformed and extended solutions $\tilde{u} = (\tilde{\mathcal{E}}, \tilde{\mathcal{H}})$ are likewise homogeneous and satisfy the following estimates by hypothesis:

$$\|\tilde{u}\|_{L^p L^q} \lesssim \|\tilde{u}\|_{L_T^\infty H^\gamma} + \|\tilde{\rho}_e(0)\|_{H^{\gamma-1+\frac{1}{p}+\delta}}. \tag{36}$$

But clearly, $\|u\|_{L^p L^q} \lesssim \|\tilde{u}\|_{L^p L^q}$ and

$$\|\tilde{u}(t)\|_{H^\gamma} + \|\tilde{\rho}_e(0)\|_{H^{\gamma-1+\frac{1}{p}+\delta}(\mathbb{R}^3)} \lesssim \|u(t)\|_{\mathcal{H}^\gamma} + \|\rho_e(0)\|_{H^{\gamma-1+\frac{1}{p}+\delta}(\Omega)}.$$

This reduces Theorem 1.1 to Proposition 2.2 for homogeneous solutions. Inhomogeneous solutions are covered by the energy estimate (34) and superposition. Indeed, suppose that (36) holds true. Let $(U(t))_{t\in\mathbb{R}}$ be the C_0-group of the Maxwell evolution in $L^2(\Omega)^6$ (cf. [9, Section 3.2]). Then, we can write the general solution by Duhamel's formula

$$u(t) = U(t)u_0 + \int_0^t U(t-s)(\tilde{\mathcal{P}}u)(s)ds.$$

We denote

$$\tilde{\mathcal{P}} = \begin{pmatrix} \partial_t & -\varepsilon^{-1}\nabla\times \\ \mu^{-1}\nabla\times & \partial_t \end{pmatrix}.$$

Changing to $\tilde{\mathcal{P}}$ is necessary as Duhamel's formula has to be applied in conservative form. By smoothness of the coefficients, this is admissible. The proof is complete. $\square$

2.2. **2d manifolds.** It is also useful to treat the two-dimensional case geometrically. In this case we rewrite (15) as

$$\left\{ \begin{array}{ll} \partial_t(\varepsilon(x)\mathcal{E}) & = *d\mathcal{H} - \mathcal{J}_e, \qquad *d*(\varepsilon\mathcal{E}) = \rho_e, \\ \partial_t(\mu(x)\mathcal{H}) & = - * d\mathcal{E}, \qquad (t,x) \in \mathbb{R} \times M \end{array} \right. \tag{37}$$

with $\mathcal{E}, \mathcal{J}_e(t) : M \to T^*M$ covectorfields and $\mathcal{H}(t) : M \to \mathbb{R}$ a zero-form. In (15) we have like above $M = (\Omega, \delta^{ij})$. The boundary condition is given by

$$[(\mathcal{E}^\flat)_{||}]_{x \in \partial M} = 0.$$

In the two-dimensional context, geodesic normal coordinates are given by

$$g^{ij} = g^{11}(x_1,x_2)dx_1^2 + dx_2^2.$$

Computing $*d$ and $*d*$ in these coordinates, we find

$$\left\{ \begin{array}{ll} \partial_t(\varepsilon(x)\mathcal{E}') & = (\sqrt{g})^{-1} g \nabla_\perp \mathcal{H}' - \mathcal{J}_e', \quad \frac{1}{\sqrt{g}} \nabla \cdot (\sqrt{g} g^{-1} \varepsilon \mathcal{E}') = \rho_e, \\ \partial_t(\mu(x)\mathcal{H}') & = -(\sqrt{g})^{-1}(\partial_1 \mathcal{E}_2' - \partial_2 \mathcal{E}_1'), \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^2_{>0}. \end{array} \right.$$

Above $\mathbb{R}^2_{>0} = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 > 0\}$ denotes the two-dimensional half-plane and $\nabla_\perp = (\partial_2, -\partial_1)$. The boundary condition reads

$$[\mathcal{E}_1]_{x_2=0} = 0.$$

We rewrite the system by redefining $\mathcal{J}_e := \sqrt{g} g^{-1} \mathcal{J}_e$, $\rho_e := \sqrt{g} \rho_e$ as

$$\left\{ \begin{array}{l} \partial_t(\sqrt{g} g^{-1} \varepsilon \mathcal{E}) = \nabla_\perp \mathcal{H} - \mathcal{J}_e, \quad \nabla \cdot (\sqrt{g} g^{-1} \varepsilon \mathcal{E}) = \rho_e, \\ \partial_t(\sqrt{g} \mu \mathcal{H}) = (\partial_1 \mathcal{E}_2 - \partial_2 \mathcal{E}_1), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2_{>0}. \end{array} \right.$$

Note that the components of $\mathcal{J}_e$ and $\mathcal{E}$ are respected by g^{-1}, which is diagonal. Let $\varepsilon' = \sqrt{g} g^{-1} \varepsilon$ for brevity. $\mathcal{E}_1$ is endowed with Dirichlet boundary conditions, we endow $\mathcal{H}$ with Neumann boundary conditions, which is a first order compatibility condition:

$$[\partial_2 \mathcal{H}]_{x_2=0} = 0.$$

For $\mathcal{E}$ we obtain from $[\partial_1 \mathcal{E}_1]_{x_2=0} = 0$ the following Robin boundary condition by considering the traces of the charges:

$$\partial_1(\varepsilon'_{11}\mathcal{E}_1) + \partial_2(\varepsilon'_{22}\mathcal{E}_2) = \rho_e \Rightarrow [(\partial_2 \varepsilon'_{22})\mathcal{E}_2] + [\varepsilon'_{22} \partial_2 \mathcal{E}_2] = \mathrm{tr}(\rho_e).$$

With $\gamma < \frac{3}{2}$ in the two-dimensional case, we choose even reflection for $\mathcal{E}_2$ such that the Robin condition is not relevant. In coordinate-free notation, we find the following compatibility conditions in the two-dimensional case:

$$[\mathcal{E} \wedge \nu]_{x \in \partial\Omega} = 0, \tag{38}$$

$$[\partial_\nu \mathcal{H}]_{x \in \partial\Omega} = 0, \tag{39}$$

$$[\partial_{\mathrm{tang}} \partial_\nu (\mathcal{E}.\nu)]_{x \in \partial\Omega} = 0. \tag{40}$$

Proposition 2.5. *Let $0 \leq \gamma \leq 3$ and $\mathcal{H}^\gamma(\Omega)$ be defined by $\mathcal{H}^\gamma(\Omega) = \overline{\mathcal{H}^3(\Omega)}$ and, if $\gamma > \frac{5}{2}$, we suppose that* (9) *holds. Then, we have the following characterization:*

- $0 \leq \gamma < \frac{1}{2}$: $\mathcal{H}^\gamma(\Omega) = H^\gamma(\Omega)$,
- $\frac{1}{2} < \gamma < \frac{3}{2}$: $\mathcal{H}^\gamma(\Omega) = \{(\mathcal{E}_0, \mathcal{H}_0) \in H^\gamma(\Omega) : (38)$ *holds* $\}$,
- $\frac{3}{2} < \gamma < \frac{5}{2}$: $\mathcal{H}^\gamma(\Omega) = \{(\mathcal{E}_0, \mathcal{H}_0) \in H^\gamma(\Omega) : (38)$ *and* (39) *hold* $\}$,
- $\frac{5}{2} < \gamma \leq 3$: $\mathcal{H}^\gamma(\Omega) = \{(\mathcal{E}_0, \mathcal{H}_0) \in H^\gamma(\Omega) : (38) - (40)$ *hold* $\}$.

The proof is omitted because it is essentially a special case of the proof of Proposition 2.1.

We extend the equations to the plane similar to the three-dimensional case: ε, μ, and g^{ij} are reflected evenly; $\mathcal{E}_1$ and ρ_e are reflected oddly corresponding to Dirichlet boundary conditions; $\mathcal{E}_2$ and $\mathcal{H}$ are reflected evenly. $\mathcal{J}_{ei}$ is reflected like $\mathcal{E}_i$. The extended functions are denoted with a $\tilde{}$. We find the following equations on $\mathbb{R}^2$:

$$\left\{ \begin{array}{ll} \partial_t(\tilde{\varepsilon}\sqrt{\tilde{g}}\tilde{g}^{-1}\tilde{\mathcal{E}}) &= \nabla_\perp \tilde{\mathcal{H}} - \tilde{\mathcal{J}}_e, \quad \nabla \cdot (\sqrt{\tilde{g}}\tilde{g}^{-1}\tilde{\varepsilon}\tilde{\mathcal{E}}) = \tilde{\rho}_e, \\ \partial_t(\tilde{\mu}\sqrt{\tilde{g}}\tilde{\mathcal{H}}) &= -(\partial_1 \tilde{\mathcal{E}}_2 - \partial_2 \tilde{\mathcal{E}}_1), \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^2. \end{array} \right. \tag{41}$$

For the proof of Theorem 1.2 it suffices to prove the following:

Proposition 2.6. *Let $\tilde{u} = (\tilde{\mathcal{E}}, \tilde{\mathcal{H}})$, and $(\tilde{\varepsilon}, \tilde{\mu}, \tilde{g})$ like in (41). Then the following estimate holds:*

$$\|\tilde{u}\|_{L^p L^q} \lesssim \|\tilde{u}\|_{L_T^\infty H^{\gamma+\delta}} + \|\tilde{\mathcal{J}}_e\|_{L_t^2 H^{\gamma+\delta}} + \|\tilde{\rho}_e\|_{L_T^\infty H^{\gamma - 1 + \frac{1}{p} + \delta}}$$

for $p, q \geq 2$, $q < \infty$, satisfying the following

$$\frac{3}{p} + \frac{1}{q} \leq \frac{1}{2}, \qquad \gamma = 2\big(\frac{1}{2} - \frac{1}{q}\big) - \frac{1}{p}, \qquad 0 < \delta < \frac{1}{2}.$$

We omit the proof of the following, which is analogous to Proposition 2.4:

Proposition 2.7. *Suppose that Proposition 2.2 holds true and the energy estimate*

$$\|u\|_{L_T^\infty \mathcal{H}^\gamma(\Omega)} \lesssim_T \|u(0)\|_{\mathcal{H}^\gamma(\Omega)}. \tag{42}$$

is valid for homogeneous solutions $u = (\mathcal{E}, \mathcal{H})$ to (15). Then, Theorem 1.2 follows.

3. Energy estimates

This section is devoted to the proof of energy estimates, i.e., a priori estimates for the Sobolev norm

$$\|(\mathcal{E}, \mathcal{H})\|_{L_T^\infty H^s(\Omega)} \lesssim \|(\mathcal{E}, \mathcal{H})(0)\|_{H^s(\Omega)} \tag{43}$$

for homogeneous solutions to Maxwell equations on domains with perfectly conducting boundary conditions. For the existence of sufficiently smooth solutions, which make the integration by parts argument licit, we again refer to Spitz's previous works [23, 24, 22] relying on the energy method. We stress that the a priori estimates in low regularity do not depend on the norms of the solution in high regularity.

It turns out that the L^2-norm of the solutions is approximately conserved and also in the quasilinear case we obtain a suitable quantification for a Grønwall argument. To estimate higher regularities, we differentiate the equation in time to see that the time-derivatives still satisfy a Maxwell-like equation. By comparing time and spatial derivatives via the equation, we see that the L^2-estimate for the time derivatives can be compared to a Sobolev regularity in space of the same order. Although the strategy is always the same, the arguments are slightly different in each instance, so we opt to give the proofs.

3.1. **The two-dimensional case.** We begin with the two-dimensional case:

$$\left\{ \begin{array}{ll} \partial_t(\varepsilon \mathcal{E}) &= \nabla_\perp \mathcal{H}, \qquad \varepsilon, \mu \in C^\infty(\Omega; \mathbb{R}_{>0}), \quad [\nu \wedge \mathcal{E}]_{x \in \partial\Omega} = 0, \\ \partial_t(\mu \mathcal{H}) &= -(\nabla \times \mathcal{E})_3, \quad \nabla \cdot (\varepsilon \mathcal{E}) = \rho_e. \end{array} \right. \tag{44}$$

ε, μ satisfy the uniform ellipticity condition (3). We prove the following:

Proposition 3.1. *Let $(\mathcal{E},\mathcal{H})$ be $\mathcal{H}^3$-solutions to* (44). *Then, for $s \in [0,2]$, we find* (43) *to hold.*

In the proof of Proposition 3.1, we need the following Helmholtz decomposition: As further preliminary, we prove the following Helmholtz decomposition on two-dimensional domains for vector fields with certain boundary conditions. For bounded domains this is [6, Proposition 6', Chapter IX, §1]:

Lemma 3.2. *Let $\Omega \subseteq \mathbb{R}^2$ be a smooth domain with compact boundary. Let $\mathcal{E} \in \mathcal{H}^3(\Omega;\mathbb{R}^2)$ be a vectorfield, which satisfies the boundary conditions:*

$$[\mathcal{E}_{||}]_{x \in \partial\Omega} = 0 \text{ and } [\partial_\nu \mathcal{E}_\perp]_{x\in\partial\Omega} = 0.$$

Then we have the equivalence of norms:

$$\|\mathcal{E}\|_{H^1(\Omega)} \sim \|(\nabla \times \mathcal{E})_3\|_{L^2(\Omega)} + \|\nabla \cdot \mathcal{E}\|_{L^2(\Omega)} + \|\mathcal{E}\|_{L^2(\Omega)}. \tag{45}$$

Proof. Note that the following estimate is immediate:

$$\|(\nabla \times \mathcal{E})_3\|_{L^2(\Omega)} + \|\nabla \cdot \mathcal{E}\|_{L^2(\Omega)} + \|\mathcal{E}\|_{L^2(\Omega)} \lesssim \|\mathcal{E}\|_{H^1(\Omega)}.$$

We turn to the reverse estimate. We resolve Ω in geodesic coordinates $g^{-1} = g_1 dx_1^2 + g_2 dx_2^2$ to the upper half-plane $\mathbb{R}^2_{>0}$ with $g_1 g_2 = 1$ such that $(\nabla \times \mathcal{E})_3 = \partial_1 \mathcal{E}_2 - \partial_2 \mathcal{E}_1$ and $\nabla \cdot \mathcal{E} = \partial_1(g_1 \mathcal{E}_1) + \partial_2(g_2 \mathcal{E}_2)$. We compute for the rotation

$$\begin{aligned} \|(\nabla \times \mathcal{E})_3\|^2_{L^2(\Omega)} &= \int_{\mathbb{R}^2_{>0}} (\partial_1 \mathcal{E}_2 - \partial_2 \mathcal{E}_1)(\partial_1 \mathcal{E}_2 - \partial_2 \mathcal{E}_1) \\ &= \int_{\mathbb{R}^2_{>0}} (\partial_1 \mathcal{E}_2)^2 dx + \int_{\mathbb{R}^2_{>0}} (\partial_2 \mathcal{E}_1)^2 - 2 \int_{\mathbb{R}^2_{>0}} \partial_2 \mathcal{E}_1 \partial_1 \mathcal{E}_2, \end{aligned}$$

and we find for the divergence

$$\begin{aligned} \|\nabla \cdot \mathcal{E}\|^2_{L^2(\Omega)} &= \int_{\mathbb{R}^2_{>0}} (\partial_1(g_1 \mathcal{E}_1) + \partial_2(g_2 \mathcal{E}_2))(\partial_1(g_1 \mathcal{E}_1) + \partial_2(g_2 \mathcal{E}_2)) \\ &= \int_{\mathbb{R}^2_{>0}} (\partial_1(g_1 \mathcal{E}_1))^2 + \int_{\mathbb{R}^2_{>0}} (\partial_2(g_2 \mathcal{E}_2))^2 + 2 \int_{\mathbb{R}^2_{>0}} \partial_1(g_1 \mathcal{E}_1) \partial_2(g_2 \mathcal{E}_2). \end{aligned}$$

For the third term we find

$$\int_{\mathbb{R}^2_{>0}} \partial_1(g_1 \mathcal{E}_1) \partial_2(g_2 \mathcal{E}_2) = \int_{\mathbb{R}^2_{>0}} \partial_1 \mathcal{E}_1 \partial_2 \mathcal{E}_2 + O(\partial \mathcal{E}.\mathcal{E})$$

and

$$\int_{\mathbb{R}^2_{>0}} \partial_1 \mathcal{E}_1 \partial_2 \mathcal{E}_2 = \int_{\mathbb{R}^2_{>0}} \partial_2 \mathcal{E}_1 \partial_1 \mathcal{E}_2$$

by integration by parts. The tangential derivative ∂_1 causes no boundary term, and the boundary term for the normal derivative ∂_2 vanishes because $\mathcal{E}_1|_{x_2=0} = 0$.

By ellipticity of the metric and an application of Young's inequality, we find for some $c, C > 0$ (depending only on ellipticity of g and $\|\partial g\|_{L^\infty}$):

$$\|(\nabla \times \mathcal{E})_3\|^2_{L^2(\Omega)} + \|\nabla \cdot \mathcal{E}\|^2_{L^2(\Omega)} \geq c \sum_{i,j=1}^{2} |\partial_i \mathcal{E}_j|^2 - C\|\mathcal{E}\|^2_{L^2},$$

which yields (45).

□

We are ready for the proof of the a priori estimates:

Proof of Proposition 3.1. Let

$$M(t) = \int_\Omega \mathcal{D}.\mathcal{E} + \mathcal{H}.\mathcal{B}\, dx$$

with $\mathcal{D} = \varepsilon \mathcal{E}$ and $\mathcal{B} = \mu \mathcal{H}$. We compute

$$\partial_t M(t) = 2\int_\Omega \nabla_\perp \mathcal{H}.\mathcal{E}\, dx - 2\int_\Omega \mathcal{H}(\partial_1 \mathcal{E}_2 - \partial_2 \mathcal{E}_1)\, dx.$$

By form invariance as argued in Section 2, we can suppose that $\Omega = \mathbb{R}^2_{>0}$, $\nu = e_2$. An integration by parts, using the boundary condition for the normal derivative ∂_2, gives $\partial_t M(t) = 0$.

The immediate consequence is an L^2-a priori estimate:

$$\|(\mathcal{E}, \mathcal{H})(t)\|_{L^2(\Omega)} \lesssim \|(\mathcal{E}, \mathcal{H})(0)\|_{L^2(\Omega)}. \tag{46}$$

For higher regularities, we consider time derivatives of (44). We denote $\partial_t A = \dot{A}$ and $\partial_t^2 A = \ddot{A}$ for $A \in \{\mathcal{E}, \mathcal{H}\}$. Taking one time derivative of (44) yields

$$\left\{ \begin{array}{llrcl} \partial_t(\varepsilon \dot{\mathcal{E}}) &= \nabla_\perp \dot{\mathcal{H}}, & \nabla \cdot (\varepsilon \dot{\mathcal{E}}) &=& 0, \\ \partial_t(\mu \dot{\mathcal{H}}) &= -(\partial_1 \dot{\mathcal{E}}_2 - \partial_2 \dot{\mathcal{E}}_1), & [\nu \wedge \dot{\mathcal{E}}]_{x \in \partial\Omega} &=& 0. \end{array} \right.$$

Hence, $(\dot{\mathcal{E}}, \dot{\mathcal{H}})$ solves (44), and we have the a priori estimates:

$$\|(\dot{\mathcal{E}}, \dot{\mathcal{H}})(t)\|_{L^2(\Omega)} \lesssim \|(\dot{\mathcal{E}}, \dot{\mathcal{H}})(0)\|_{L^2(\Omega)}.$$

Note that (again from (44) and ellipticity of ε and μ), we have

$$\|(\dot{\mathcal{E}}, \dot{\mathcal{H}})(t)\|_{L^2(\Omega)} \sim \|\mathcal{H}(t)\|_{\dot{H}^1(\Omega)} + \|\mathcal{E}(t)\|_{H_{curl}(\Omega)}.$$

To estimate the full H^1-norm, we observe that $\|(\mathcal{E}, \mathcal{H})(t)\|_{L^2}$ was estimated in the previous step and for $\|\mathcal{E}(t)\|_{H_{div}}$ we find from the condition on the charges

$$\varepsilon \nabla \cdot \mathcal{E} + (\nabla \varepsilon)\mathcal{E} = \rho_e.$$

The charges are conserved for homogeneous solutions and by (46) we find

$$\|\mathcal{E}(t)\|_{H_{div}(\Omega)} \lesssim \|\mathcal{E}(t)\|_{L^2(\Omega)} + \|\rho_e(t)\|_{L^2(\Omega)} \lesssim \|\mathcal{E}(0)\|_{L^2(\Omega)} + \|\rho_e(0)\|_{L^2(\Omega)}.$$

This yields

$$\|(\mathcal{E}, \mathcal{H})(t)\|_{H^1(\Omega)} \lesssim \|(\mathcal{E}, \mathcal{H})(0)\|_{H^1(\Omega)}.$$

Taking a second time derivative in (44), we find

$$\left\{ \begin{array}{llrcl} \partial_t(\varepsilon \ddot{\mathcal{E}}) &= \nabla_\perp \ddot{\mathcal{H}}, & \nabla \cdot (\varepsilon \ddot{\mathcal{E}}) &=& 0, \\ \partial_t(\mu \ddot{\mathcal{H}}) &= -(\partial_1 \ddot{\mathcal{E}}_2 - \partial_2 \ddot{\mathcal{E}}_1), & [\nu \wedge \ddot{\mathcal{E}}]_{x \in \partial\Omega} &=& 0. \end{array} \right.$$

We use L^2-conservation to find

$$\|(\ddot{\mathcal{E}}, \ddot{\mathcal{H}})(t)\|_{L^2} \lesssim \|(\ddot{\mathcal{E}}, \ddot{\mathcal{H}})(0)\|_{L^2}.$$

Clearly, from iterating (44), we have

$$\|(\ddot{\mathcal{E}},\ddot{\mathcal{H}})(0)\|_{L^2} \lesssim \|(\mathcal{E},\mathcal{H})(0)\|_{H^2}.$$

Secondly, we find

$$\|\ddot{\mathcal{E}}(t)\|_{L^2(\Omega)} \sim \|\nabla_\perp \dot{\mathcal{H}}(t)\|_{L^2} \text{ with } \nabla_\perp \dot{\mathcal{H}} = O(\partial \mu^{-1})(\partial_1 \mathcal{E}_2 - \partial_2 \mathcal{E}_1) + \mu^{-1}(\Delta \mathcal{E} - \nabla(\nabla\cdot\mathcal{E})).$$

This gives by the conservation of $\|(\ddot{\mathcal{E}},\ddot{\mathcal{H}})(t)\|_{L^2}$, the previous a priori estimate for the H^1-norm and conservation of charges:

$$\|\Delta\mathcal{E}(t)\|_{L^2} \lesssim \|\nabla_\perp \dot{\mathcal{H}}(t)\|_{L^2} + \|\rho_e(t)\|_{H^1} + \|(\mathcal{E},\mathcal{H})(t)\|_{H^1} \sim \|\ddot{\mathcal{E}}(t)\|_{L^2} + \|(\mathcal{E},\mathcal{H})(0)\|_{H^1} + \|\rho_e(0)\|_{H^1}.$$

For two time derivatives of $\mathcal{H}$ we find

$$\mu \ddot{\mathcal{H}} = \varepsilon^{-1}\Delta \mathcal{H} + O(\partial\varepsilon\, \partial\mathcal{H}).$$

It follows from the conservation of $\|(\ddot{\mathcal{E}},\ddot{\mathcal{H}}))(t)\|_{L^2}$ and the previously established a priori estimate for the H^1-norm:

$$\|\Delta\mathcal{H}(t)\|_{L^2} \lesssim \|\ddot{\mathcal{H}}(t)\|_{L^2} + \|\mathcal{H}(t)\|_{H^1} \lesssim \|(\mathcal{E},\mathcal{H})(0)\|_{H^2}.$$

$\square$

3.2. **The three-dimensional case.** Next, we extend the arguments to the three-dimensional case:

(47)

$$\begin{cases} \partial_t(\varepsilon\mathcal{E}) &= \nabla\times\mathcal{H}, \quad (t,x)\in\mathbb{R}\times\Omega;\ \nabla\cdot(\varepsilon\mathcal{E}) = \rho_e;\ \nabla\cdot\mathcal{B} = 0; \\ \partial_t(\mu\mathcal{H}) &= -\nabla\times\mathcal{E}, \quad [\mathcal{E}\times\nu]_{x\in\partial\Omega} = 0;\ [\nu\cdot\mathcal{B}]_{x\in\partial\Omega} = 0;\ \varepsilon,\mu\in C^\infty(\Omega;\mathbb{R}_{>0}). \end{cases}$$

Moreover, we suppose that (3), (4) holds for ε and μ. Local existence of $\mathcal{H}^3$-solutions was discussed in [23, 24]. We need the following Helmholtz decomposition. See again [6, Proposition 6', Chapter IX, §1] for bounded domains.

Lemma 3.3. *Let $\Omega \subseteq \mathbb{R}^3$ be a smooth domain with compact boundary. Let $\mathcal{E} \in H^3(\Omega;\mathbb{R}^3)$ be a vector field with $\mathcal{E} \in H_{curl}(\Omega)\cap H_{div}(\Omega)$. Suppose that either the tangential components satisfy Dirichlet boundary conditions and the normal component satisfies Neumann boundary conditions or vice versa. Then*

$$\|\mathcal{E}\|_{H^1(\Omega)} \sim \|\mathcal{E}\|_{H_{curl}(\Omega)} + \|\mathcal{E}\|_{H_{div}(\Omega)} + \|\mathcal{E}\|_{L^2(\Omega)}. \tag{48}$$

Proof. By density considerations we can suppose that $\mathcal{E}\in C^2(\Omega;\mathbb{R}^3)$. Note that clearly

$$\|\mathcal{E}\|_{H^1(\Omega)} \gtrsim \|\mathcal{E}\|_{H_{curl}(\Omega)} + \|\mathcal{E}\|_{H_{div}(\Omega)} + \|\mathcal{E}\|_{L^2(\Omega)}.$$

For the reverse inequality, we shall prove that

$$\|\mathcal{E}\|^2_{H_{curl}} + \|\mathcal{E}\|^2_{H_{div}} + C\|\mathcal{E}\|^2_{L^2(\Omega)} \geq c\|\mathcal{E}\|^2_{H^1(\Omega)}$$

for some $c, C > 0$.

To this end, we use integration by parts for the curl-operator:

$$\int_\Omega (\nabla\times u).v dx = \int_\Omega u.(\nabla\times v)dx + \int_{\partial\Omega} u\times v dS$$

provided that $u,v\in C^1(\Omega;\mathbb{R}^3)$. This is based on $\nabla\cdot(u\times v) = (\nabla\times u).v - u.(\nabla\times v)$ and the divergence theorem. Hence, we obtain

$$\int_\Omega (\nabla\times\mathcal{E}).(\nabla\times\mathcal{E})dx = \int_\Omega \mathcal{E}.(\nabla\times\nabla\times\mathcal{E})dx + \int_{\partial\Omega}(\mathcal{E}\times(\nabla\times\mathcal{E})).\nu dS.$$

The boundary term is of lower order, for which we have:

$$\int_{\partial\Omega} \mathcal{E} \times (\nabla \times \mathcal{E}).\nu dS = \int_{\partial\Omega} \partial_\nu(\mathcal{E}^2) - \mathcal{E}_\nu(\nabla \cdot \mathcal{E}) dS. \tag{49}$$

We find geodesic coordinates

$$\int_{\partial\Omega} \partial_\nu \mathcal{E}^2 = \int_{\partial\mathbb{R}^3_{>0}} \partial_3 \langle \mathcal{E}, g^{-1}\mathcal{E} \rangle dS.$$

Since g^{-1} separates tangential and normal components and by the assumed boundary conditions, we find that

$$\int_{\partial\mathbb{R}^3_{>0}} \partial_3 \langle \mathcal{E}, g^{-1}\mathcal{E} \rangle dS = \int_{\partial\mathbb{R}^3_{>0}} \langle \mathcal{E}_{tang}, (\partial_3 g^{-1}) \mathcal{E}_{tang} \rangle dS.$$

We estimate this by the Cauchy-Schwarz inequality, continuity of $\mathrm{tr} : B^{1/2}_{2,1}(\Omega) \to L^2(\partial\Omega)$ (see [1, Theorem 7.4.3] and [1, Remark 7.4.5]), and that $B^{1/2}_{2,1}(\Omega)$ is an interpolation space between $L^2(\Omega)$ and $H^1(\Omega)$ ([2, Theorem 6.4.5]):

$$\|\mathcal{E}_{tang}\|^2_{L^2(\partial\Omega)} \lesssim \|\mathcal{E}\|^2_{B^{1/2}_{2,1}(\Omega)} \lesssim \|\mathcal{E}\|_{L^2(\Omega)} \|\mathcal{E}\|_{H^1(\Omega)}.$$

Finally, by applying Young's inequality, we find

$$\int_{\partial\Omega} \partial_\nu \mathcal{E}^2 dS \leq \varepsilon \|\mathcal{E}\|^2_{H^1(\Omega)} + C_\varepsilon \|\mathcal{E}\|^2_{L^2(\Omega)}.$$

This suffices.

It turns out that by the assumed boundary conditions the second boundary term in (49) is always vanishing:

$$\int_{\partial\Omega} \mathcal{E}_\nu (\nabla \cdot \mathcal{E}) dS = 0.$$

This is clear if the normal component satisfies Dirichlet boundary conditions. If this is not the case, we can rewrite in geodesic coordinates

$$[\nabla \cdot \mathcal{E}]_{x \in \partial\Omega} = [\nabla_{tang} \mathcal{E}_{tang}]_{x \in \partial\Omega} + [\partial_3 \mathcal{E}_3]_{x \in \partial\Omega}.$$

The normal derivative of the normal component vanishes by Neumann conditions. The first expression vanishes since the derivatives are tangential, and we have Dirichlet boundary conditions on the tangential components.

We turn to the integral

$$\int_\Omega \mathcal{E}(\nabla \times \nabla \times \mathcal{E}) dx = -\int_\Omega \mathcal{E}.\Delta \mathcal{E} dx + \int_\Omega \mathcal{E} \nabla(\nabla \cdot \mathcal{E}) dx.$$

By the similar argument as for the boundary term handled above, we find

$$\int_\Omega \mathcal{E} \nabla (\nabla \cdot \mathcal{E}) dx = -\int_\Omega (\nabla \cdot \mathcal{E})^2 dx = -\|\mathcal{E}\|^2_{H_{div}}.$$

It remains to prove up to lower order terms:

$$-\int_\Omega \mathcal{E}.\Delta \mathcal{E} \gtrsim \sum_{i=1}^3 \int_\Omega |\nabla \mathcal{E}_i|^2 dx + O(\mathcal{E} \partial \mathcal{E}).$$

The lower order terms are handled like above:

$$\int_{\mathbb{R}^3_{>0}} \mathcal{E}.\partial \mathcal{E} dx \leq \varepsilon \|\mathcal{E}\|^2_{H^1} + C_\varepsilon \|\mathcal{E}\|^2_{L^2}.$$

For this reason, we can neglect this contribution.

We resolve in geodesic coordinates:

$$-\int_{\Omega} \mathcal{E}.\Delta \mathcal{E} dx = -\int_{\mathbb{R}^3_{>0}} \langle g^{-1} \mathcal{E}, \Delta_g \mathcal{E} \rangle dx.$$

For the normal component we find

$$-\int_{\mathbb{R}^3_{>0}} \mathcal{E}_3 \Delta_g \mathcal{E}_3 = -\int_{\mathbb{R}^3_{>0}} \mathcal{E}_3 \partial_i g^{ij} \partial_j \mathcal{E}_3 dx = \int_{\mathbb{R}^3_{>0}} \partial_i \mathcal{E}_3 g^{ij} \partial_j \mathcal{E}_3 \geq c \int_{\mathbb{R}^3_{>0}} |\nabla \mathcal{E}_3|^2$$

by the vanishing boundary term for the normal derivative and ellipticity of g.

We integrate by parts the tangential components to find up to lower order terms $O(\mathcal{E} \partial \mathcal{E})$:

$$\begin{aligned} &\int_{\mathbb{R}^3_{>0}} \begin{pmatrix} g^{11} \mathcal{E}_1 + g^{12} \mathcal{E}_2 \\ g^{21} \mathcal{E}_1 + g^{22} \mathcal{E}_2 \end{pmatrix} \cdot \begin{pmatrix} -\Delta_g \mathcal{E}_1 \\ -\Delta_g \mathcal{E}_2 \end{pmatrix} dx \\ &= \int_{\mathbb{R}^3_{>0}} g^{11} |\nabla_g \mathcal{E}_1|^2 + g^{12} \langle \nabla_g \mathcal{E}_2, \nabla_g \mathcal{E}_1 \rangle + g^{21} \langle \nabla_g \mathcal{E}_1, \nabla_g \mathcal{E}_2 \rangle + g^{22} |\nabla_g \mathcal{E}_2|^2 dx + O(\mathcal{E} \partial \mathcal{E}). \end{aligned}$$

Here we use the notation $\langle \nabla \mathcal{E}_i, g^{-1} \nabla \mathcal{E}_j \rangle = \langle \nabla_g \mathcal{E}_i, \nabla_g \mathcal{E}_j \rangle$.

By positive definiteness of g^{-1}, g^{11} and g^{22} must be positive. Hence,

$$\begin{aligned} &\int_{\mathbb{R}^3_{>0}} g^{11} |\nabla_g \mathcal{E}_1|^2 + g^{12} \langle \nabla_g \mathcal{E}_2, \nabla_g \mathcal{E}_1 \rangle + g^{21} \langle \nabla_g \mathcal{E}_1, \nabla_g \mathcal{E}_2 \rangle + g^{22} |\nabla_g \mathcal{E}_2|^2 dx + O(\mathcal{E} \partial \mathcal{E}) \\ &\geq \int_{\mathbb{R}^3_{>0}} \begin{pmatrix} |\nabla_g \mathcal{E}_1| \\ |\nabla_g \mathcal{E}_2| \end{pmatrix} \cdot \underbrace{\begin{pmatrix} g^{11} & -|g^{12}| \\ -|g^{21}| & g^{22} \end{pmatrix}}_{\tilde{g}} \begin{pmatrix} |\nabla_g \mathcal{E}_1| \\ |\nabla_g \mathcal{E}_2| \end{pmatrix} dx + O(\mathcal{E} \partial \mathcal{E}). \end{aligned}$$

Since the matrix $\tilde{g}$ is still positive definite, we conclude by ellipticity of g:

$$\begin{aligned} \int_{\mathbb{R}^3_{>0}} \langle \begin{pmatrix} |\nabla_g \mathcal{E}_1| \\ |\nabla_g \mathcal{E}_2| \end{pmatrix} . \tilde{g} \begin{pmatrix} |\nabla_g \mathcal{E}_1| \\ |\nabla_g \mathcal{E}_2| \end{pmatrix} \rangle dx &\geq c \int_{\mathbb{R}^3_{>0}} |\nabla_g \mathcal{E}_1|^2 + |\nabla_g \mathcal{E}_2|^2 dx \\ &\geq c' \int_{\mathbb{R}^3_{>0}} |\nabla \mathcal{E}_1|^2 + |\nabla \mathcal{E}_2|^2 dx. \end{aligned}$$

The proof is complete. □

We can now prove a priori estimates in the time-independent case:

Proposition 3.4. *For $s \in [0,2]$ we find the following estimate to hold for $\mathcal{H}^3$-solutions to (47):*

$$\|(\mathcal{E}, \mathcal{H})\|_{L^\infty H^s(\Omega)} \lesssim \|(\mathcal{E}, \mathcal{H})(0)\|_{H^s(\Omega)}. \tag{50}$$

Proof. We follow the argument from the two-dimensional case and begin with L^2-estimates. Let $M(t) = \int_\Omega \mathcal{D}.\mathcal{E} + \mathcal{H}.\mathcal{B} \, dx$. We have

$$\frac{dM}{dt} = 2 \int_\Omega \mathcal{E}.\nabla \times \mathcal{H} dx - 2 \int_\Omega \mathcal{H}.\nabla \times \mathcal{E} dx = 0$$

with the ultimate equality a consequence of the boundary conditions (after resolving (47) on $\mathbb{R}^3_{>0}$). This yields $\|(\mathcal{E}, \mathcal{H})(t)\|_{L^2} \sim \|(\mathcal{E}, \mathcal{H})(0)\|_{L^2}$, which is (50) for $s = 0$.

To prove (50) for $s = 1$, we consider one time derivative to find that $(\dot{\mathcal{E}}, \dot{\mathcal{H}})$ satisfies (47). Consequently, $\|(\dot{\mathcal{E}}, \dot{\mathcal{H}})(t)\|_{L^2} \sim \|(\dot{\mathcal{E}}, \dot{\mathcal{H}})(0)\|_{L^2}$ which yields by (47) that

$\|(\nabla\times\mathcal{E},\nabla\times\mathcal{H})(t)\|_{L^2}\sim\|(\nabla\times\mathcal{E},\nabla\times\mathcal{H})(0)\|_{L^2}$. By the Helmholtz decomposition, the defect to H^1 is precisely $\|(\nabla\cdot\mathcal{E},\nabla\cdot\mathcal{H})(t)\|_{L^2}$.

Here we use the divergence conditions:

$$\nabla\cdot(\varepsilon\mathcal{E})=(\nabla\varepsilon).\mathcal{E}+\varepsilon(\nabla\cdot\mathcal{E})=\rho_e,\quad \nabla\cdot(\mu\mathcal{H})=(\nabla\mu).\mathcal{H}+\mu(\nabla\cdot\mathcal{H})=0.$$

Since ρ_e is a conserved quantity, and we have already an a priori estimate for the L^2-norm, we have

$$\|\nabla\cdot\mathcal{E}(t)\|_{L^2}\lesssim\|\mathcal{E}(t)\|_{L^2}+\|\rho_e(t)\|_{L^2}\lesssim\|\mathcal{E}(0)\|_{L^2}+\|\rho_e(0)\|_{L^2}.$$

For $\|\mathcal{H}\|_{H_{div}}$ we can argue likewise. We obtain

$$\|(\mathcal{E},\mathcal{H})(t)\|_{H^1}\lesssim\|(\mathcal{E},\mathcal{H})(0)\|_{H^1}.$$

For the proof of (50) with $s=2$, we use that $(\ddot{\mathcal{E}},\ddot{\mathcal{H}})$ solves (47). We have

$$\begin{aligned}\partial_t^2\mathcal{E}(t)&=\frac{1}{\varepsilon}\nabla\times\partial_t\mathcal{H}(t)=-\frac{1}{\varepsilon}\nabla\times(\frac{1}{\mu}\nabla\times\mathcal{E})\\&=\frac{1}{\varepsilon\mu}\Delta\mathcal{E}+O(\|\mathcal{E}\|_{H^1})-\frac{1}{\varepsilon\mu}\nabla(\nabla\cdot\mathcal{E}),\end{aligned}$$

and from the divergence condition, we find

$$\nabla(\nabla\cdot\mathcal{E})(t)=(\nabla\varepsilon^{-1})\rho_e(t)+\varepsilon^{-1}\nabla\rho_e(t)+O(\|\mathcal{E}(t)\|_{H^1}).$$

This implies the estimate by conservation of charge and previously established a priori estimates

$$\begin{aligned}\|\mathcal{E}(t)\|_{\dot{H}^2}&\lesssim\|\partial_t^2\mathcal{E}(t)\|_{L^2}+\|\mathcal{E}(t)\|_{H^1}+\|\nabla(\nabla\cdot\mathcal{E})(t)\|_{L^2}\\&\lesssim\|(\partial_t^2\mathcal{E},\partial_t^2\mathcal{H})(0)\|_{L^2}+\|(\mathcal{E},\mathcal{H})(0)\|_{H^1}+\|\rho_e(0)\|_{H^1}.\end{aligned}$$

Similarly,

$$\begin{aligned}\partial_t^2\mathcal{H}(t)&=-\frac{1}{\mu}\nabla\times\partial_t\mathcal{E}(t)=-\frac{1}{\mu}\nabla\times\big(\frac{1}{\varepsilon}\nabla\times\mathcal{H})\\&=\frac{1}{\varepsilon\mu}\Delta\mathcal{H}-\frac{1}{\varepsilon\mu}\nabla(\nabla\cdot\mathcal{H})+O(\|\mathcal{H}\|_{H^1}).\end{aligned}$$

so that

$$\|\mathcal{H}(t)\|_{\dot{H}^2}\lesssim\|\partial_t^2\mathcal{H}(t)\|_{L^2}+\|\mathcal{H}(t)\|_{H^1}.$$

Similarly,

$$\|\partial_t^2\mathcal{H}(t)\|_{L^2}\lesssim\|\mathcal{H}(t)\|_{\dot{H}^2}+\|\mathcal{H}(t)\|_{H^1}.$$

The proof of (50) is complete for $s=2$. For non-integer s, we prove the claim by interpolation. $\square$

4. Preliminaries

In this section we collect facts on pseudo-differential operators, which we rely on in the remainder of the paper. We denote derivatives by

$$\partial_x^\alpha=\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\dots\partial_{x_m}^{\alpha_m}\text{ and }D_\xi^\alpha=\partial_\xi^\alpha/(i^{|\alpha|})\text{ for }\alpha\in\mathbb{N}_0^m.$$

Recall the Hörmander class of symbols:

$$S^m_{\rho,\delta}=\{a\in C^\infty(\mathbb{R}^m\times\mathbb{R}^m:|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|\lesssim_{\alpha,\beta}\langle\xi\rangle^{m-|\beta|\rho+|\alpha|\delta}\}$$

with $m \in \mathbb{R}$, $0 \leq \delta < \rho \leq 1$. The L^p-boundedness of symbols $a \in S^0_{1,\delta}$, $0 \leq \delta < \rho \leq 1$, is well-known. We use the following quantization:

$$a(x,D)f = (2\pi)^{-m} \int_{\mathbb{R}^m} e^{ix.\xi} a(x,\xi) \hat{f}(\xi) d\xi \qquad (f \in \mathcal{S}'(\mathbb{R}^m)).$$

We recall the composition of pseudo-differential operators.

Proposition 4.1 ([25, Prop. 0.3C])**.** *Given $P(x,\xi) \in S^{m_1}_{\rho_1,\delta_1}$, $Q(x,\xi) \in S^{m_2}_{\rho_2,\delta_2}$, suppose that*

$$0 \leq \delta_2 < \rho \leq 1 \text{ with } \rho = \min(\rho_1,\rho_2).$$

Then, $(P \circ Q)(x,D) \in OPS^{m_1+m_2}_{\rho,\delta}$ with $\delta = \max(\delta_1,\delta_2)$, and $P(x,D) \circ Q(x,D)$ satisfies the asymptotic expansion

$$(P \circ Q)(x,D) = \sum_\alpha \frac{1}{\alpha!} (D^\alpha_\xi P \partial^\alpha_x Q)(x,D) + R,$$

where $R : \mathcal{S}' \to C^\infty$ is a smoothing operator.

The following lemma will be useful:

Lemma 4.2 ([17, Lemma 2.3])**.** *Let $1 \leq p,q \leq \infty$, $s \geq 0$, and $a \in C^s_x C^\infty_c(\mathbb{R}^m \times \mathbb{R}^m)$ with $a(x,\xi) = 0$ for $\xi \notin B(0,2)$. Suppose that*

$$\sup_{x \in \mathbb{R}^m} \sum_{0 \leq |\alpha| \leq m+1} \| D^\alpha_\xi a(x,\cdot) \|_{L^1_\xi} \leq C.$$

Then the following estimate holds:

$$\| a(x,D) f \|_{L^p L^q} \lesssim C \| f \|_{L^p L^q}.$$

5. Diagonalizing reflected Maxwell equations

5.1. **Lipschitz coefficients.** The purpose of this section is to reduce the proof of Proposition 2.2 to Strichartz estimates for half-wave equations with metric $\frac{g^{ij}}{\varepsilon\mu}$. Here $\varepsilon, \mu, g^{ij} \in C^\infty(\mathbb{R}^3_{\geq 0})$ are extended evenly to the full space, introducing a Lipschitz-singularity of co-dimension 1. The following is due to Blair–Smith–Sogge [3]:

Theorem 5.1. *Let $d \geq 2$ and $(g^{ij})_{1 \leq i,j \leq d} \subseteq C^\infty(\mathbb{R}^d_{\geq 0})$ be uniformly elliptic. Let $u : [0,1] \times \mathbb{R}^d \to \mathbb{C}$. Then the following estimate holds:*

$$\| u \|_{L^p_t([0,1]) L^q_x(\mathbb{R}^d)} \lesssim \| u \|_{L^\infty_t H^\gamma(\mathbb{R}^3)} + \| (i\partial_t + D_{\tilde{g}}) u \|_{L^1_t H^\gamma}$$

with $\tilde{g}^{ij}$ denoting the even extension of g^{ij} and

$$D_{\tilde{g}} = Op\big(\tilde{g}^{ij} \xi_i \xi_j\big)^{\frac{1}{2}}$$

provided that $2 \leq p,q \leq \infty$ and γ satisfy

$$\frac{3}{p} + \frac{2}{q} \leq 1, \quad q < \infty, \quad \gamma = 3\big(\frac{1}{2} - \frac{1}{q}\big) - \frac{1}{p}.$$

The reduction to the above proceeds via diagonalization with pseudo-differential operators. However, the symbols are very rough, so extra care is required.

5.1.1. *Littlewood-Paley decomposition and frequency truncation.* We begin with a paradifferential decomposition. Recall that

$$\mathcal{P} = \begin{pmatrix} \sqrt{g} g^{-1} \varepsilon \partial_t & -\nabla \times \\ \nabla \times & \sqrt{g} g^{-1} \mu \partial_t \end{pmatrix}.$$

In the following we denote $u = (\mathcal{E}, \mathcal{H}) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ and omit the tilde for the reflected quantities to lighten the notation. Let $(S_\lambda)_{\lambda \in 2^{\mathbb{N}_0}}$ denote a family of inhomogeneous Littlewood-Paley projections for space-time frequencies and $(S'_\lambda)_{\lambda \in 2^{\mathbb{N}_0}}$, $(S^\tau_\lambda)_{\lambda \in 2^{\mathbb{N}_0}}$ projections for spatial or temporal frequencies, respectively. We define

$$\varepsilon' = \sqrt{g} g^{-1} \varepsilon, \quad \mu' = \sqrt{g} g^{-1} \mu, \quad \mathcal{P}_{<\lambda} = \begin{pmatrix} \varepsilon'_{<\lambda} \partial_t & -\nabla \times \\ \nabla \times & \mu'_{<\lambda} \partial_t \end{pmatrix}$$

through spatial frequency truncation: $\kappa_{<\lambda} = \sum_{\mu \leq \lambda/16} S'_\mu \kappa$ for $\kappa \in \{\varepsilon', \mu'\}$.

For the proof of Proposition 2.2 it suffices to prove the following estimate for frequency localized functions for $2^{\mathbb{N}_0} \ni \lambda \gg 1$: We can suppose that $\lambda \gg 1$ because low frequencies are easily estimated by Bernstein's inequality. Let $0 < \delta < \frac{3}{q}$.

$$\|S_{\{|\tau| \sim |\xi'|\}} u\|_{L^p L^q} \lesssim \|\langle \partial_t \rangle^{\gamma + \delta} u\|_{L^\infty_t L^2_x} + \|\langle \partial_t \rangle^\gamma \mathcal{P} u\|_{L^2_{t,x}}, \tag{51}$$

$$\|S_{\{|\tau| \gg |\xi'|\}} u\|_{L^p L^q} \lesssim \|\langle \partial_t \rangle^{\gamma - \frac{1}{2} + \delta} u\|_{L^2_{t,x}} + \|\langle \partial_t \rangle^{\gamma - \frac{1}{2} + \delta} \mathcal{P} u\|_{L^2_{t,x}}, \tag{52}$$

$$\begin{aligned} \|S_{\{|\tau| \ll |\xi'|\}} u\|_{L^p L^q} &\lesssim \|\langle D' \rangle^{\gamma - \frac{1}{2} + \delta} u\|_{L^2_{t,x}} + \|\langle \partial_t \rangle^{\gamma - \frac{1}{2} + \delta} u\|_{L^2_{t,x}} \\ &\quad + \|\langle D' \rangle^{\gamma - \frac{1}{2} + \delta} \mathcal{P} u\|_{L^2_{t,x}} + \|\rho_e\|_{L^\infty_t H^{\gamma - 1 + \frac{1}{p} + \delta}}. \end{aligned} \tag{53}$$

In the following we implicitly consider u compactly supported in $[0, T]$. This is strictly speaking not conserved by S_λ, but for $\lambda \gg 1$ up to Schwartz tails, which are neglected in the following. $S_{\{|\tau| \sim |\xi'|\}}$ denotes a space-time frequency projection to temporal frequencies comparable to spatial frequencies, $S_{\{|\tau| \gg |\xi'|\}}$ a space-time frequency projection to temporal frequencies $\{|\tau| \gtrsim 1\}$ and spatial frequencies $\{|\xi'| \ll |\tau|\}$. Correspondingly, $S_{\{|\xi'| \gg |\tau|\}}$ denotes a projection for spatial frequencies dominating temporal frequencies. Estimates (52) and (53) crucially rely on ellipticity of components of $\mathcal{P}$ after diagonalization. Since we can achieve estimates with regularity $\gamma - \frac{1}{2} < 1$, the commutator estimates for Lipschitz functions are applicable. We give the proof of (52) shortly using the ellipticity away from the characteristic surface. The proof of (51) is more involved and requires the use of the Strichartz estimates due to Blair–Smith–Sogge. However, if $\{|\tau| \sim |\xi'| \sim 1\}$, we can trade temporal for spatial frequencies.

Lemma 5.2. *Let $2^{\mathbb{N}_0} \ni \lambda \gg 1$, $2 \leq p, q < \infty$, and $\delta > 0$. The estimate*

$$\|S^\tau_\lambda S'_\lambda u\|_{L^p L^q} \lesssim \lambda^\gamma (\|S^\tau_\lambda S'_\lambda u\|_{L^\infty_t L^2_x} + \|\mathcal{P}_{<\lambda} S^\tau_\lambda S'_\lambda u\|_{L^2_{t,x}}) \tag{54}$$

implies

$$\|S_{\{|\tau| \sim |\xi'| \sim 1\}} u\|_{L^p L^q} \lesssim \|\langle \partial_t \rangle^{\gamma + \delta} u\|_{L^\infty_t L^2_x} + \|\langle \partial_t \rangle^\gamma \mathcal{P} u\|_{L^2_{t,x}}. \tag{55}$$

Proof. Littlewood-Paley decomposition and Minkowski's inequality give for $2 \leq p, q < \infty$

$$\|u\|_{L^p L^q} \lesssim \big(\sum_{\lambda \geq 1} \|S_\lambda u\|^2_{L^p L^q} \big)^{\frac{1}{2}}$$

which we can further decompose almost orthogonally into spatial and temporal frequencies. Summation of $\|\langle \partial_t \rangle^\gamma u\|_{L_t^\infty L_x^2}$ is clear. Note that the lack of almost orthogonality in $L_t^\infty L_x^2$ leads to the δ-loss in derivatives. Now we write

$$\mathcal{P}_{<\lambda} S'_\lambda v = S'_\lambda \mathcal{P}_{<\lambda} v + [\mathcal{P}_{<\lambda}, S'_\lambda] v$$

and note that

$$\|S_\lambda^\tau [\mathcal{P}_{<\lambda}, S'_\lambda] \langle \partial_t \rangle^\gamma u\|_{L^2_{t,x}} \lesssim \|S_\lambda^\tau \langle \partial_t \rangle^\gamma u\|_{L^2_{t,x}}$$

because $\|[\kappa_{<\lambda}, S'_\lambda]\|_{L_x^2 \to L_x^2} \lesssim \lambda^{-1}$ by a kernel estimate for $\kappa \in \{\varepsilon', \mu'\}$.

We write

$$S_\lambda^\tau S'_\lambda \mathcal{P}_{<\lambda} v = S_\lambda^\tau S'_\lambda \mathcal{P} v - S_\lambda^\tau S'_\lambda \mathcal{P}_{\gg \lambda} v - S_\lambda^\tau S'_\lambda \mathcal{P}_{\sim \lambda} v.$$

Clearly,

$$\|S_\lambda^\tau S'_\lambda \mathcal{P}_{\sim \lambda} v\|_{L^2_{t,x}} \lesssim \|S_\lambda^\tau v\|_{L^2_{t,x}}$$

and similarly, by a fixed-time estimate,

$$\|S_\lambda^\tau S'_\lambda (S'_{\gtrsim \lambda} \varepsilon \partial_t S'_{\gtrsim \lambda} v)\|_{L^2_{t,x}} \lesssim \lambda \|\varepsilon_{\gtrsim \lambda}\|_{L^\infty} \|S_\lambda^\tau v\|_{L^2_{t,x}} \lesssim \|\partial \varepsilon\|_{L^\infty} \|S_\lambda^\tau v\|_{L^2_{t,x}},$$

which estimates the second term. We remain with $S_\lambda^\tau S'_\lambda \mathcal{P} \langle \partial_t \rangle^\gamma u$ and conclude

$$\|S_\lambda^\tau \mathcal{P}_{<\lambda} S'_\lambda \langle \partial_t \rangle^\gamma u\|_{L^2_{t,x}} \lesssim \|S_\lambda^\tau S'_\lambda \mathcal{P} \langle \partial_t \rangle^\gamma u\|_{L^2_{t,x}} + \|S_\lambda^\tau \langle \partial_t \rangle^\gamma u\|_{L^2_{t,x}}.$$

This is the commutator argument for the Maxwell operator. After summing the Littlewood-Paley blocks, we obtain (55). $\square$

Similarly, for (52) it suffices to prove a frequency localized estimate.

Lemma 5.3. *Let $\lambda, \nu \in 2^{\mathbb{N}_0}$, $\lambda \ll \nu$. Let $2 \leq p, q < \infty$. The estimate*

$$\|S'_\nu S_\lambda^\tau u\|_{L^p L^q} \lesssim \nu^{\gamma - \frac{1}{2}} \|S'_\nu S_\lambda^\tau \mathcal{P}_{<\nu} u\|_{L^2_{t,x}} + \nu^{\gamma - 1 + \frac{1}{p}} (\|\rho'_{e\nu}\|_{L_t^\infty L_x^2} + \|\rho'_{m\nu}\|_{L_t^\infty L_x^2}) \tag{56}$$

with

$$\rho'_{e\nu} = \varepsilon'_{<\nu} S'_\nu \mathcal{E}, \qquad \rho'_{m\nu} = \mu'_{<\nu} S'_\nu \mathcal{H}$$

implies

(57)

$$\begin{aligned} \|S_{\{|\tau| \ll |\xi'|\}} u\|_{L^p L^q} &\lesssim \|\langle D' \rangle^{\gamma - \frac{1}{2} + \delta} u\|_{L^2_{t,x}} + \|\langle \partial_t \rangle^{\gamma - \frac{1}{2} + \delta} u\|_{L^2_{t,x}} + \|\langle D' \rangle^{\gamma - \frac{1}{2} + \delta} \mathcal{P} u\|_{L^2_{t,x}} \\ &\quad + \|\rho_e\|_{L_t^\infty H^{\gamma - 1 + \frac{1}{p} + \delta}} + \|\rho_m\|_{L_t^\infty H^{\gamma - 1 + \frac{1}{p} + \delta}} \end{aligned}$$

for $\delta > 0$.

Proof. We have to carry out the summation

$$\sum_{\substack{\nu \gg 1, \\ 1 \lesssim \lambda \ll \nu}} \nu^{\gamma - \frac{1}{2}} \|S_\lambda^\tau \mathcal{P}_{<\nu} S'_\nu u\|_{L^2_{t,x}} + \nu^{\gamma - 1 + \frac{1}{p}} (\|\rho'_{e\nu}\|_{L_t^\infty L_x^2} + \|\rho'_{m\nu}\|_{L_t^\infty L_x^2}).$$

For the Maxwell operator, we use that $\gamma - \frac{1}{2} < 1$. First, we note that

$$S_\lambda^\tau \mathcal{P}_{<\nu} S'_\nu u = \tilde{S}'_\nu S_\lambda^\tau \mathcal{P}_{<\nu} S'_\nu u.$$

Above and in the following $\tilde{S}'_\nu$ denotes a mildly enlarged spatial frequency projector. By $\mathcal{P} = \mathcal{P}_{<\lambda} + \mathcal{P}_{\sim \lambda} + \mathcal{P}_{\gg \lambda}$ and $\tilde{S}'_\nu \mathcal{P}_{\gg \nu} S'_\nu = 0$, we can write

$$\|S_\lambda^\tau \mathcal{P}_{<\nu} S'_\nu u\|_{L^2_{t,x}} \leq \|S_\lambda^\tau \tilde{S}'_\nu \mathcal{P} S'_\nu u\|_{L^2_{t,x}} + \|S_\lambda^\tau \tilde{S}'_\nu \mathcal{P}_{\sim \nu} S'_\nu u\|_{L^2_{t,x}}.$$

The latter term is clearly estimated by

$$\|S_\lambda^\tau \tilde{S}'_\nu \mathcal{P}_{\sim \nu} S'_\nu u\|_{L^2_{t,x}} \lesssim \|S'_\nu u\|_{L^2_{t,x}}.$$

For the first term, we write

(58)
$$\nu^{\gamma-\frac{1}{2}}\|S^\tau_\lambda \tilde{S}'_\nu \mathcal{P} S'_\nu u\|_{L^2_{t,x}} \lesssim \nu^{\gamma-\frac{1}{2}}\lambda\|S^\tau_\lambda \tilde{S}'_\nu [\varepsilon',S'_\nu]u\|_{L^2_{t,x}} + \nu^{\gamma-\frac{1}{2}}\lambda\|S^\tau_\lambda \tilde{S}'_\nu [\mu',S'_\nu]u\|_{L^2_{t,x}} + \|\langle D'\rangle^{\gamma-\frac{1}{2}} S^\tau_\lambda \tilde{S}'_\nu \mathcal{P}u\|_{L^2_{t,x}}$$

Furthermore,

(59)
$$\|S^\tau_\lambda \tilde{S}'_\nu [\varepsilon',S'_\nu]u\|_{L^2_{t,x}} = \|S^\tau_\lambda \tilde{S}'_\nu [\varepsilon',S'_\nu]\tilde{S}'_\nu u\|_{L^2_{t,x}} + \|S^\tau_\lambda \tilde{S}'_\nu [\varepsilon',S'_\nu]S'_{\ll \nu} u\|_{L^2_{t,x}} + \|S^\tau_\lambda \tilde{S}'_\nu [\varepsilon',S'_\nu]S'_{\gg \nu} u\|_{L^2_{t,x}}.$$

The estimate of the first term in (59) is straight-forward by the fixed-time commutator estimate $\|[\varepsilon',S'_\nu]\|_{L^2_x \to L^2_x} \lesssim \mu^{-1}$:

$$\sum_{\substack{\nu\gg 1,\\ 1\lesssim\lambda\ll\nu}} \nu^{\gamma-\frac{1}{2}}\lambda\|S^\tau_\lambda \tilde{S}'_\nu [\varepsilon',S'_\nu]\tilde{S}'_\nu u\|_{L^2_{t,x}} \lesssim \sum_{\substack{\nu\gg 1,\\ 1\lesssim\lambda\ll\nu}} \nu^{\gamma-\frac{1}{2}}\lambda\nu^{-1}\|\tilde{S}'_\nu u\|_{L^2_{t,x}} \lesssim \|\langle D'\rangle^{\gamma-\frac{1}{2}+\delta} u\|_{L^2_{t,x}}.$$

For the second term in (59) we note that

$$\begin{aligned}\sum_{\substack{\nu\gg 1,\\ 1\lesssim\lambda\ll\nu}} \nu^{\gamma-\frac{1}{2}}\lambda\|S^\tau_\lambda \tilde{S}'_\nu [\varepsilon',S'_\nu]S'_{\ll\nu} u\|_{L^2_{t,x}} &\lesssim \sum_{\substack{\nu\gg 1,\\ 1\lesssim\lambda\ll\nu}} \nu^{\gamma-\frac{1}{2}}\lambda\|\varepsilon'_{\sim\nu} S^\tau_\lambda S'_{\ll\nu} u\|_{L^2_{t,x}} \\ &\lesssim \sum_{\substack{\nu\gg 1,\\ 1\lesssim\lambda\ll\nu}} \nu^{\gamma-\frac{3}{2}}\lambda\|\partial\varepsilon\|_{L^\infty}\|S^\tau_\lambda u\|_{L^2_{t,x}} \\ &\lesssim \|\partial\varepsilon\|_{L^\infty}\|\langle\partial_t\rangle^{\gamma-\frac{1}{2}+\delta} u\|_{L^2_{t,x}}.\end{aligned}$$

For the third term in (59) we obtain similarly

$$\begin{aligned}\sum_{\substack{\nu\gg 1,\\ 1\lesssim\lambda\ll\nu}} \nu^{\gamma-\frac{1}{2}}\lambda\|S^\tau_\lambda \tilde{S}'_\nu [\varepsilon',S'_\nu]S'_{\gg\nu} u\|_{L^2_{t,x}} &\lesssim \sum_{\substack{\nu\gg 1,\\ 1\lesssim\lambda\ll\nu}} \nu^{\gamma-\frac{1}{2}}\lambda\|S^\tau_\lambda \varepsilon'_{\gg\nu} S'_{\gg\nu} u\|_{L^2_{t,x}} \\ &\lesssim \|\partial\varepsilon\|_{L^\infty}\|\langle\partial_t\rangle^{\gamma-\frac{1}{2}+\delta} u\|_{L^2_{t,x}}.\end{aligned}$$

Clearly, the second commutator in (58) can be handled likewise.

We turn to the charges: Recall that $\rho_e = \nabla\cdot(\varepsilon'\mathcal{E})$ with $\varepsilon' = \sqrt{g}g^{-1}\varepsilon$. Since we are working in geodesic normal coordinates, we have

$$\varepsilon' = \begin{pmatrix} \varepsilon'_{11} & \varepsilon'_{12} & 0 \\ \varepsilon'_{21} & \varepsilon'_{22} & 0 \\ 0 & 0 & \varepsilon'_{33} \end{pmatrix}.$$

To carry out the commutator argument, we separate

$$\begin{aligned}\rho'_{e\nu} &= \partial_1({\varepsilon'}^{11}_{<\nu} S'_\nu \mathcal{E}_1 + {\varepsilon'}^{21}_{<\nu} S'_\nu \mathcal{E}_2) + \partial_2({\varepsilon'}^{21}_{<\nu} S'_\nu \mathcal{E}_1 + {\varepsilon'}^{22}_{<\nu} S'_\nu \mathcal{E}_2) + \partial_3({\varepsilon'}^{33}_{<\nu} S'_\nu \mathcal{E}_3) \\ &= (\partial_1 {\varepsilon'}^{11}_{<\nu}) S'_\nu \mathcal{E}_1 + (\partial_1 {\varepsilon'}^{12}_{<\nu}) S'_\nu \mathcal{E}_2 + (\partial_2 {\varepsilon'}^{21}_{<\nu}) S'_\nu \mathcal{E}_1 + (\partial_2 {\varepsilon'}^{22}_{<\nu}) S'_\nu \mathcal{E}_2 + (\partial_3 {\varepsilon'}^{33}_{\nu}) \mathcal{E}_3 \\ &\quad + {\varepsilon'}^{11}_{<\nu} \partial_1 S'_\nu \mathcal{E}_1 + \varepsilon'^{12}_{<\nu} \partial_1 S'_\nu \mathcal{E}_2 + {\varepsilon'}^{21}_{<\nu} \partial_2 S'_\nu \mathcal{E}_1 + {\varepsilon'}^{22}_{<\nu} \partial_2 S'_\nu \mathcal{E}_2 + {\varepsilon'}^{33}_{<\nu} \partial_3 S'_\nu \mathcal{E}_3 \\ &=: {\rho'}^{(1)}_{e\nu} + {\rho'}^{(2)}_{e\nu}.\end{aligned}$$

We can estimate terms with derivative acting on ε' collected in ${\rho'}^{(1)}_{e\nu}$ directly by Lipschitz continuity. For example,

$$\nu^{\gamma-1+\frac{1}{p}}\|\partial_1 \varepsilon'_{<\nu} S'_\nu \mathcal{E}_1\|_{L^\infty_t L^2_{x'}} \lesssim \nu^{\gamma-\frac{1}{2}-\delta}\|S'_\nu \mathcal{E}\|_{L^\infty_t L^2_{x'}}.$$

The terms with derivative acting on $\mathcal{E}$ collected in $\rho'^{(2)}_{e\nu}$ are amenable to a commutator argument. Note that

$$\nu^{\gamma-1+\frac{1}{p}}\|\varepsilon'^{11}_{<\nu}\partial_1 S'_\nu \mathcal{E}_1\|_{L^2_x} = \nu^{\gamma-1+\frac{1}{p}}\|\tilde{S}'_\nu \varepsilon'^{11}_{<\nu}\partial_1 S'_\nu \mathcal{E}_1\|_{L^2_x}.$$

Since $\tilde{S}'_\nu \varepsilon'^{11}_{\gg\nu} S'_\nu = 0$, we can write

$$\nu^{\gamma-1+\frac{1}{p}}\|\tilde{S}'_\nu \varepsilon'^{11}_{<\nu}\partial_1 S'_\nu \mathcal{E}_1\|_{L^2_x} \leq \nu^{\gamma-1+\frac{1}{p}}\|\tilde{S}'_\nu \varepsilon'^{11}_{\sim\nu}\partial_1 S'_\nu \mathcal{E}_1\|_{L^2_x} + \nu^{\gamma-1+\frac{1}{p}}\|\tilde{S}'_\nu \varepsilon'^{11}\partial_1 S'_\nu \mathcal{E}_1\|_{L^2_x}.$$

The first expression is estimaetd by

$$\nu^{\gamma-1+\frac{1}{p}}\|\tilde{S}'_\nu \varepsilon'^{11}_{\sim\nu}\partial_1 S'_\nu \mathcal{E}_1\|_{L^2_x} \lesssim \|\partial \varepsilon'^{11}\|_{L^\infty_x} \nu^{\gamma-1+\frac{1}{p}}\|S'_\nu \mathcal{E}_1\|_{L^2_x},$$

which is more than enough. For $\gamma - 1 + \frac{1}{p} > 0$, we obtain by the Coifman–Meyer estimate for the second term:

$$\begin{aligned}
\sum_{\nu \geq 1} \|\langle D' \rangle^{\gamma-1+\frac{1}{p}} \tilde{S}'_\nu \varepsilon'^{11}\partial_1 S'_\nu \mathcal{E}_1\|_{L^2_x} &\leq \sum_{\nu\geq 1} (\nu^{-\delta} \|\langle D' \rangle^{\gamma-1+\frac{1}{p}+\delta} \tilde{S}'_\nu [\varepsilon'^{11}, S'_\nu]\partial_1 \mathcal{E}_1\|_{L^2_x} \\
&\quad + \|\langle D' \rangle^{\gamma-1+\frac{1}{p}+\delta} (\varepsilon'^{11}\partial_1 \mathcal{E}_1)\|_{L^2_x}) \\
&\lesssim \|\langle D' \rangle^{\gamma-1+\frac{1}{p}+\delta} \mathcal{E}\|_{L^2_x} + \|\langle D' \rangle^{\gamma-1+\frac{1}{p}+\delta} \varepsilon'^{11}\partial_1 \mathcal{E}_1\|_{L^2_x}.
\end{aligned}$$

Let

$$\rho'^{(2)}_e = \varepsilon'^{11}\partial_1 \mathcal{E}_1 + \varepsilon'^{12}\partial_1 \mathcal{E}_2 + \varepsilon'^{21}\partial_2 \mathcal{E}_1 + \varepsilon'^{22}\partial_2 \mathcal{E}_2 + \varepsilon'^{33}\partial_3 \mathcal{E}_3.$$

We obtain by the previous arguments:

$$\sum_\nu \nu^{\gamma-1+\frac{1}{p}} \|\rho'_{e\nu}\|_{L^\infty_t L^2_x} \lesssim \|\langle D' \rangle^{\gamma-1+\frac{1}{p}+\delta} \mathcal{E}\|_{L^\infty_t L^2_x} + \|\langle D' \rangle^{\gamma-1+\frac{1}{p}+\delta} \rho'^{(2)}_e\|_{L^\infty_t L^2_x}.$$

The first term is acceptable. We estimate the second term by oddness of the function $\rho'^{(2)}_e$ switching to the half-space:

$$\begin{aligned}
\|\langle D' \rangle^{\gamma-1+\frac{1}{p}+\delta} \rho'^{(2)}_e\|_{L^\infty_t L^2_x(\mathbb{R}^3)} &\lesssim \|\langle D' \rangle^{\gamma-1+\frac{1}{p}+\delta} \rho'^{(2)}_e\|_{L^\infty_t L^2_x(\mathbb{R}^3_+)} \\
&\lesssim \|\langle D' \rangle^{\gamma-1+\frac{1}{p}+\delta} \rho'_e\|_{L^\infty_t L^2_x(\mathbb{R}^3_+)} + \|\langle D' \rangle^{\gamma-1+\frac{1}{p}+\delta} \mathcal{E}\|_{L^\infty_t L^2_x(\mathbb{R}^3_+)}.
\end{aligned}$$

For the ultimate estimate we used smoothness of the coefficients and invariance of Sobolev functions under multiplication with smooth functions. We remark that the estimate is easier for $\gamma - 1 + \frac{1}{p} < 0$ because it is not necessary to switch between half-space and full space. The estimate for $\rho'_{m\nu}$ follows along the above lines. After summation of the Littlewood-Paley blocks, we obtain (57). $\square$

We turn to the proof of (52), which does not make use of the diagonalization of $\mathcal{P}$.

Proof of (52). Let $1 \ll \mu \ll \lambda$ and

$$\tilde{\mathcal{P}} = \begin{pmatrix} \partial_t & \varepsilon'^{-1} \nabla \times \\ -\mu'^{-1} \nabla \times & \partial_t \end{pmatrix}.$$

If $\{\lambda \sim |\tau| \gg |\xi'| \sim \mu\}$, the operator $\tilde{P}_{<\mu}$ (obtained from frequency truncation of ε'^{-1} and μ'^{-1}) is elliptic and gains one derivative. We estimate by Bernstein's inequality and ellipticity of $\tilde{\mathcal{P}}_{<\mu}$ (note that $\tilde{\mathcal{P}}_{<\mu}$ has Lipschitz coefficients):

$$\begin{aligned}\|S^\tau_\lambda S'_\mu u\|_{L^p L^q} &\lesssim \lambda^{\frac{1}{2}-\frac{1}{p}} \mu^{3\left(\frac{1}{2}-\frac{1}{q}\right)} \|S^\tau_\lambda S'_\mu u\|_{L^2_{t,x}} \\ &\lesssim \lambda^{-\frac{1}{2}-\frac{1}{p}} \mu^{3\left(\frac{1}{2}-\frac{1}{q}\right)} \|\tilde{\mathcal{P}}_{<\mu} S^\tau_\lambda S'_\mu u\|_{L^2_{t,x}} \\ &\lesssim \lambda^{-\frac{1}{2}-\frac{1}{p}} \mu^{\frac{1}{p}+\frac{1}{2}} \|S^\tau_\lambda \langle D'\rangle^{\gamma-\frac{1}{2}} \tilde{S}'_\mu \tilde{\mathcal{P}}_{<\mu} S'_\mu u\|_{L^2_{t,x}}.\end{aligned}$$

Above and in the following $\tilde{S}'_\mu$ denotes a mildly enlarged frequency projection around frequencies of size μ. Now we write again $\tilde{\mathcal{P}} = \tilde{\mathcal{P}}_{<\mu} + \tilde{\mathcal{P}}_{\sim\mu} + \tilde{\mathcal{P}}_{\gg\mu}$ and note that

$$\|S^\tau_\lambda \langle D'\rangle^{\gamma-\frac{1}{2}} \tilde{S}'_\mu \tilde{\mathcal{P}}_{\sim\mu} S'_\mu u\|_{L^2_{t,x}} \lesssim \mu^{\gamma-\frac{1}{2}} \|S'_\mu u\|_{L^2_{t,x}} \lesssim \|S^\tau_\lambda \langle D'\rangle^{\gamma-\frac{1}{2}} S'_\mu u\|_{L^2_{t,x}}.$$

Like above, $\tilde{S}'_\mu \tilde{\mathcal{P}}_{\gg\mu} S'_\mu = 0$ by impossible frequency interaction. Summation over μ and λ gives the acceptable contribution

$$\lesssim \|\langle \partial_t \rangle^{\gamma-\frac{1}{2}+\delta} u\|_{L^2_{t,x}}.$$

For $\tilde{\mathcal{P}}$ we use the estimate

$$\|[\kappa', S'_\mu]\|_{L^2_{x'} \to L^2_{x'}} \lesssim \mu^{-1}.$$

We have

$$\begin{aligned}&\mu^{\gamma-\frac{1}{2}} \|S^\tau_\lambda \tilde{S}'_\mu [\kappa', S'_\mu] \nabla \times A\|_{L^2_{t,x}} \\ &\lesssim \mu^{\gamma-\frac{1}{2}} \|S^\tau_\lambda \tilde{S}'_\mu [\kappa', S'_\mu] S'_{\lesssim\mu} \nabla \times A\|_{L^2_{t,x}} + \mu^{\gamma-\frac{1}{2}} \|S^\tau_\lambda \tilde{S}'_\mu [\kappa', S'_\mu] S'_{\gg\mu} \nabla \times A\|_{L^2_{t,x}} \\ &\lesssim \mu^{\gamma-\frac{1}{2}} \|S^\tau_\lambda A\|_{L^2_{t,x}} + \mu^{\gamma-\frac{1}{2}} \|S^\tau_\lambda \tilde{S}'_\mu (\kappa'_{\gg\mu} S'_{\gg\mu} \nabla \times A)\|_{L^2_{t,x}}.\end{aligned}$$

The first term is already acceptable. The second term is rewritten as

$$\tilde{S}'_\mu (\kappa'_{\gg\mu} S'_{\gg\mu} \partial S^\tau_\lambda A) = \tilde{S}'_\mu \partial (\kappa'_{\gg\mu} S'_{\gg\mu} S^\tau_\lambda A) - \tilde{S}'_\mu (\partial \kappa'_{\gg\mu} S'_{\gg\mu} S^\tau_\lambda A).$$

For the first term we find

$$\|\tilde{S}'_\mu \partial (\kappa'_{\gg\mu} S'_{\gg\mu} S^\tau_\lambda A)\|_{L^2_{t,x}} \lesssim \mu \|\kappa'_{\gg\mu}\|_{L^\infty_{x'}} \|S'_{\gg\mu} S^\tau_\lambda A\|_{L^2_{t,x'}} \lesssim \|\partial \kappa'\|_{L^\infty_{x'}} \|S^\tau_\lambda A\|_{L^2_{t,x}}.$$

This yields an acceptable contribution after summation over $\mu \ll \lambda$ and λ. Clearly,

$$\|\tilde{S}'_\mu (\partial \kappa'_{\gg\mu} S'_{\gg\mu} S^\tau_\lambda A\|_{L^2_{t,x}} \lesssim \|\partial \kappa'\|_{L^\infty} \|S^\tau_\lambda A\|_{L^2_{t,x}}.$$

This is likewise acceptable.

We summarize

$$\|S_{\{|\tau| \gg |\xi'| \gtrsim 1\}} u\|_{L^p L^q} \lesssim \|\langle \partial_t \rangle^{\gamma-\frac{1}{2}+\delta} u\|_{L^2_{t,x}} + \|\langle \partial_t \rangle^{\gamma-\frac{1}{2}+\delta} \mathcal{P} u\|_{L^2_{t,x}}. \tag{60}$$

This completes the proof.

□

With the estimates for different regions in phase space at hand, we can finish the proof of Proposition 2.2.

Conclusion of the Proof of Proposition 2.2. Taking (51)-(53) together, we find

$$\begin{aligned}\|u\|_{L^p L^q} \lesssim &\|\langle D'\rangle^\gamma u\|_{L^\infty_t L^2_x} + \|\langle \partial_t \rangle^{\gamma+\delta} u\|_{L^\infty_t L^2_x} \\ &+ \|\langle \partial_t \rangle^{\gamma+\delta} \mathcal{P} u\|_{L^2_{t,x}} + \|\langle D'\rangle^{\gamma+\delta} \mathcal{P} u\|_{L^2_{t,x}} \\ &+ \|\rho_e\|_{L^\infty_t H^{\gamma-1+\frac{1}{p}+\delta}}.\end{aligned}$$

By applying the estimate to homogeneous solutions, we obtain

$$\|u\|_{L^p L^q} \lesssim \|\langle D' \rangle^\gamma u\|_{L_t^\infty L_x^2} + \|\langle \partial_t \rangle^{\gamma+\delta} u\|_{L^2_{t,x}} + \|\rho_e\|_{L_t^\infty H^{\gamma-1+\frac{1}{p}+\varepsilon}}.$$

For homogeneous solutions, we can trade the time derivatives for spatial derivatives and by the energy estimates of Section 3, we obtain

$$\|u\|_{L^p L^q} \lesssim \|\langle D' \rangle^{\gamma+\delta} u(0)\|_{L_x^2} + \|\rho_e\|_{L_t^\infty H^{\gamma-1+\frac{1}{p}+\delta}}.$$

The conclusion follows from Duhamel's formula. □

The proofs of (51) and (53) make use of the diagonalization of $\mathcal{P}_{<\lambda}$ via pseudo-differential operators. This is carried out in the following. Let $h = \big(\det(g_{ij})\big)^{1/2}$ and denote $C(\xi')_{ij} = -\varepsilon_{ijk} \xi'_k$. The principal symbol (with rough coefficients) is given by

$$p(x,\xi) = i \begin{pmatrix} \xi_0 h g^{-1} \varepsilon & -C(\xi') \\ C(\xi') & h g^{-1} \mu \xi_0 \end{pmatrix}.$$

We consider as truncated operator $\mathcal{P}_\lambda$ the following: Let $g^{-1} = AA^t$ denote the factorization into Jacobians (which we also extend such that these are Lipschitz along the boundary). Let $A_{<\lambda}$ denote the truncation of spatial frequencies of A to frequencies less than $\lambda/8$. Let $h_{<\lambda} = \det(A_{<\lambda})$. We define

$$\mathcal{P}_\lambda = \begin{pmatrix} h_{<\lambda} A_{<\lambda} A^t_{<\lambda} \varepsilon_{<\lambda} \partial_t & -\nabla \times \\ \nabla \times & h_{<\lambda} A_{<\lambda} A^t_{<\lambda} \mu_{<\lambda} \partial_t \end{pmatrix}.$$

Observe that $\|(\mathcal{P} - \mathcal{P}_\lambda) S_\lambda u\|_{L^2} \lesssim \|S_\lambda u\|_{L^2}$. Note that in ρ'_e we can truncate h, A, A^t, and ε in frequencies because we can write the difference as a telescoping sum

$$\begin{aligned} &\|S_\lambda(\nabla \cdot (h A A^t \varepsilon \mathcal{E})) - S_\lambda \nabla \cdot (h_{<\lambda} A_{<\lambda} A^t_{<\lambda} \varepsilon_{<\lambda} \mathcal{E})\|_{L^2} \\ &= \|S_\lambda \nabla \cdot (h_{>\lambda} A A^t \varepsilon \mathcal{E} + h A_{>\lambda} A^t \varepsilon \mathcal{E} + \ldots)\|_{L^2}. \end{aligned}$$

For instance,

$$\|S_\lambda \nabla \cdot (h_{>\lambda} A A^t \varepsilon \mathcal{E})\|_{L^2} \lesssim \lambda \|h_{>\lambda}\|_{L^\infty} \|A\|_{L^\infty} \|A^t\|_{L^\infty} \|\varepsilon\|_{L^\infty} \|\mathcal{E}\|_{L^2}.$$

After these reductions, we are dealing with symbols in $S^1_{1,1}$, which is a borderline case for symbol composition. But the considered symbols $a \in S^i_{1,1}$ actually satisfy

$$|\partial_x a| \lesssim 1 \tag{61}$$

because the reflected Jacobians and coefficients are Lipschitz. This suffices for symbol composition to hold to first order. Accordingly, we make the following definition:

Definition 5.4. Let $k \in \mathbb{N}_0$. We define the symbol class

$$\tilde{S}^k_{1,1} = \{ a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d) \, : \, |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \lesssim \langle \xi \rangle^{k - |\beta| + (|\alpha|-1)_+} \}.$$

We have the following:

Lemma 5.5. *Let $m, n \in \mathbb{R}$, $a \in \tilde{S}^m_{1,1}$, $b \in \tilde{S}^n_{1,1}$. Then, we find the following estimate to hold:*

$$a(x,D) \circ b(x,D) = (ab)(x,D) + E$$

with $\|E\|_{2 \to 2} \lesssim 1$.

5.1.2. *Diagonalizing the principal symbol.* In the following we carry out the formal computation to find suitable conjugation matrices for the operator $\mathcal{P}_\lambda$. The aim is to prove the following proposition:

Proposition 5.6. *Let $2^{\mathbb{N}} \ni \lambda \gg \lambda_0$. There is a decomposition of phase space by projections*
$$S'_\lambda S_\lambda = S_{\lambda 1} + S_{\lambda 2} + S_{\lambda 3}$$
such that for every $i \in \{1,2,3\}$ there are $\mathcal{M}^i_\lambda \in OP\tilde{S}^0_{1,1}$, $\mathcal{N}^i_\lambda \in OP\tilde{S}^0_{1,1}$, and $\mathcal{D}^i_\lambda \in OP\tilde{S}^1_{1,1}$ such that
$$\mathcal{P}_\lambda S_{\lambda i} = \mathcal{M}^i_\lambda \mathcal{D}^i_\lambda \mathcal{N}^i_\lambda S_{\lambda i} + E^i_\lambda$$
with $\|E^i_\lambda\|_{2\to 2} \lesssim 1$ with implicit constant independent of λ.

Before we turn to the technical details, we carry out a formal diagonalization of
$$p(x,\xi) = i \begin{pmatrix} h_{<\lambda} A_{<\lambda} A^t_{<\lambda} \varepsilon_{<\lambda} \xi_0 & -C(\xi') \\ C(\xi') & h_{<\lambda} A_{<\lambda} A^t_{<\lambda} \mu_{<\lambda} \xi_0 \end{pmatrix}.$$
The symbol is in $\tilde{S}^1_{1,1}$. We diagonalize the principal symbol as follows:
$$p(x,\xi)\pi(x,\xi) = m(x,\xi) d(x,\xi) n(x,\xi) \pi(x,\xi)$$
with $m, n \in \tilde{S}^0_{1,1}$ and $d \in \tilde{S}^1_{1,1}$, and $\pi \in \tilde{S}^0_{1,1}$ denoting a projection to a region in phase space to be determined. In the first step, we write
$$\begin{aligned} &\begin{pmatrix} h_{<\lambda} A_{<\lambda} A^t_{<\lambda} \varepsilon_{<\lambda} \xi_0 & -C(\xi') \\ C(\xi') & h_{<\lambda} A_{<\lambda} A^t_{<\lambda} \mu_{<\lambda} \xi_0 \end{pmatrix} \\ &= \begin{pmatrix} A_{<\lambda} & 0 \\ 0 & A_{<\lambda} \end{pmatrix} \begin{pmatrix} h_{<\lambda} \varepsilon_{<\lambda} \xi_0 & -A^{-1}_{<\lambda} C(\xi') (A^t_{<\lambda})^{-1} \\ A^{-1}_{<\lambda} C(\xi') (A^t_{<\lambda})^{-1} & h_{<\lambda} \mu_{<\lambda} \xi_0 \end{pmatrix} \begin{pmatrix} A^t_{<\lambda} & 0 \\ 0 & A^t_{<\lambda} \end{pmatrix}. \end{aligned}$$
We recall the following:

Lemma 5.7. *Let $B \in \mathbb{C}^{3\times 3}$. The following identity holds:*
$$B^t C(\xi') B = C(adB \cdot \xi'). \tag{62}$$
In the above display adB denotes the adjugate matrix, i.e.,
$$adA = ((-1)^{i+j} A_{ji})_{i,j}$$
with A_{ji} denoting the (j,i)-minor of A.

This yields by the definition of the adjugate matrix, $h_{<\lambda}$, and using Cramer's rule
$$A^{-1}_{<\lambda} C(\xi') (A^t_{<\lambda})^{-1} = C(h_{<\lambda} A^t_{<\lambda} \xi').$$
We write
$$\begin{aligned} &\begin{pmatrix} h_{<\lambda} \varepsilon_{<\lambda} \xi_0 & -A^{-1}_{<\lambda} C(\xi') (A^t_{<\lambda})^{-1} \\ A^{-1}_{<\lambda} C(\xi') (A^t_{<\lambda})^{-1} & h_{<\lambda} \mu_{<\lambda} \xi_0 \end{pmatrix} \\ &= \begin{pmatrix} \varepsilon_{<\lambda} \xi_0 & -C(A^t_{<\lambda} \xi') \\ C(A^t_{<\lambda} \xi') & \mu_{<\lambda} \xi_0 \end{pmatrix} \begin{pmatrix} h_{<\lambda} & 0 \\ 0 & h_{<\lambda} \end{pmatrix} \\ &= \begin{pmatrix} \varepsilon^{\frac{1}{2}}_{<\lambda} & 0 \\ 0 & \mu^{\frac{1}{2}}_{<\lambda} \end{pmatrix} \begin{pmatrix} \xi_0 & -C\big(\frac{A^t_{<\lambda} \xi'}{(\varepsilon_{<\lambda} \mu_{<\lambda})^{\frac{1}{2}}}\big) \\ C\big(\frac{A^t_{<\lambda} \xi'}{(\varepsilon_{<\lambda} \mu_{<\lambda})^{\frac{1}{2}}}\big) & \xi_0 \end{pmatrix} \begin{pmatrix} \varepsilon^{\frac{1}{2}}_{<\lambda} & 0 \\ 0 & \mu^{\frac{1}{2}}_{<\lambda} \end{pmatrix} \begin{pmatrix} h_{<\lambda} & 0 \\ 0 & h_{<\lambda} \end{pmatrix}. \end{aligned}$$

Hence, we have reduced to diagonalizing

$$B = \begin{pmatrix} \xi_0 & -C(\tilde{\xi}') \\ C(\tilde{\xi}') & \xi_0 \end{pmatrix}. \tag{63}$$

This reflects invariance of pseudo-differential operators under change of coordinates. Since the symbols are very rough, we prefer to carry out the computation directly.

In [15, 16] the symbol was diagonalized in the more difficult case of partially anisotropic ε, i.e., ε having possibly two different eigenvalues. In this case, the resulting expressions are fairly complicated. We take the opportunity to point out a simplification for isotropic ε and μ. Write $\xi' = (\xi_1, \xi_2, \xi_3)$. We begin with computing the characteristic polynomial of p/i using the block matrix structure:

$$q(y) = \begin{vmatrix} y - \xi_0 & C(\xi') \\ -C(\xi') & y - \xi_0 \end{vmatrix} = \big| (y-\xi_0)^2 1_{3\times 3} + C^2(\xi') \big|.$$

Hence, we have reduced to computing the eigenvalues of $C^2(\xi')$. Note that

$$C^2(\xi') = \begin{pmatrix} \xi_2^2 + \xi_3^2 & -\xi_1 \xi_2 & -\xi_1 \xi_3 \\ -\xi_1 \xi_2 & \xi_1^2 + \xi_3^2 & -\xi_2 \xi_3 \\ -\xi_1 \xi_3 & -\xi_2 \xi_3 & \xi_1^2 + \xi_2^2 \end{pmatrix} = |\xi'|^2 1_{3 \times 3} - \xi \otimes \xi.$$

It follows that

$$r(\lambda, \xi') = \det(\lambda 1_{3\times 3} - C^2(\xi')) = (\lambda - \|\xi\|^2)^2 \lambda.$$

This gives for the characteristic polynomial q

$$q(\lambda) = (\lambda - \xi_0)^2 [(\lambda - (\xi_0 - \|\xi'\|))^2 (\lambda - (\xi_0 + \|\xi'\|))^2].$$

We conclude that the diagonalization is given by

$$d(x,\xi) = i(\xi_0, \xi_0, \xi_0 - \|\xi'\|, \xi_0 + \|\xi'\|, \xi_0 - \|\xi'\|, \xi_0 + \|\xi'\|). \tag{64}$$

In the following let $\xi_i^* = \frac{\xi_i}{\|\xi'\|}$ for $i = 1,2,3$. Eigenvectors of ξ_0 are clearly given by

$$\begin{pmatrix} \xi_1^* \\ \xi_2^* \\ \xi_3^* \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \\ \xi_1^* \\ \xi_2^* \\ \xi_3^* \end{pmatrix}.$$

Eigenvectors of $\xi_0 - \|\xi'\|$: We use the block matrix structure of $p(x,\xi)$. Let $v = (v_1, v_2)^t$ denote an eigenvector. We find the system of equations:

$$\begin{pmatrix} \|\xi'\| & C(\xi') \\ -C(\xi') & \|\xi'\| \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0.$$

Iterating the above in the non-trivial case $\xi' = 0$ yields the eigenvector equation for v_1:

$$\|\xi'\|^2 v_1 + C^2(\xi') v_1 = 0.$$

For this we find the zero-homogeneous eigenvectors:

$$\begin{pmatrix} 0 \\ -\xi_3^* \\ \xi_2^* \end{pmatrix}, \quad \begin{pmatrix} \xi_3^* \\ 0 \\ -\xi_1^* \end{pmatrix}, \quad \begin{pmatrix} -\xi_2^* \\ \xi_1^* \\ 0 \end{pmatrix}. \tag{65}$$

The system of equations from above yields

$$v_2 = \frac{C(\xi')}{\|\xi'\|} v_1.$$

This gives for v_2:

$$\begin{pmatrix} {\xi_2^*}^2 + {\xi_3^*}^2 \\ -\xi_1^*\xi_2^* \\ -\xi_1^*\xi_3^* \end{pmatrix}, \qquad \begin{pmatrix} -\xi_1^*\xi_2^* \\ {\xi_1^*}^2 + {\xi_3^*}^2 \\ -\xi_2^*\xi_3^* \end{pmatrix}, \qquad \begin{pmatrix} -\xi_1^*\xi_3^* \\ -\xi_2^*\xi_3^* \\ {\xi_1^*}^2 + {\xi_2^*}^2 \end{pmatrix}.$$

Eigenvectors of $\xi_0 + \|\xi'\|$: Again, we use the block matrix structure of $p(x,\xi)$, and let $v = (v_1, v_2)^t$ denote an eigenvector. This yields the system of equations:

$$\begin{pmatrix} -\|\xi'\| & C(\xi') \\ -C(\xi') & -\|\xi'\| \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0.$$

We find again for v_1

$$C^2(\xi') v_1 + \|\xi'\|^2 v_1 = 0,$$

and for v_2

$$v_2 = -\frac{C(\xi') v_1}{\|\xi'\|}.$$

Conjugation matrices: We choose conjugation matrices depending on a non-vanishing direction of ξ. In the following suppose that $|\xi_3^*| \gtrsim 1$. One choice of conjugation matrices according to (64) is given by choosing the first eigenvector in (65):

(66)

$$m_3(x,\xi) = \begin{pmatrix} \xi_1^* & 0 & 0 & 0 & \xi_3^* & \xi_3^* \\ \xi_2^* & 0 & -\xi_3^* & -\xi_3^* & 0 & 0 \\ \xi_3^* & 0 & \xi_2^* & \xi_2^* & -\xi_1^* & -\xi_1^* \\ 0 & \xi_1^* & {\xi_2^*}^2 + {\xi_3^*}^2 & -({\xi_2^*}^2 + {\xi_3^*}^2) & -\xi_1^*\xi_2^* & \xi_1^*\xi_2^* \\ 0 & \xi_2^* & -\xi_1^*\xi_2^* & \xi_1^*\xi_2^* & ({\xi_1^*}^2 + {\xi_3^*}^2) & -({\xi_1^*}^2 + {\xi_3^*}^2) \\ 0 & \xi_3^* & -\xi_1^*\xi_3^* & \xi_1^*\xi_3^* & -\xi_2^*\xi_3^* & \xi_2^*\xi_3^* \end{pmatrix}.$$

We have the following:

Lemma 5.8. *Let m_3 be given as in* (66)*. Then,*

$$\det m_3(x,\xi) = {\xi_3^*}^2. \tag{67}$$

Proof. By elementary column operations, that is adding and subtracting the third and fourth and fifth and sixth eigenvector, we compute the determinant to be

$$\begin{aligned} \det m_3 &= \begin{vmatrix} \xi_1^* & 0 & 0 & 0 & \xi_3^* & 0 \\ \xi_2^* & 0 & -\xi_3^* & 0 & 0 & 0 \\ \xi_3^* & 0 & \xi_2^* & 0 & -\xi_1^* & 0 \\ 0 & \xi_1^* & 0 & {\xi_2^*}^2 + {\xi_3^*}^2 & 0 & \xi_1^*\xi_2^* \\ 0 & \xi_2^* & 0 & -\xi_1^*\xi_2^* & 0 & -({\xi_1^*}^2 + {\xi_3^*}^2) \\ 0 & \xi_3^* & 0 & -\xi_1^*\xi_3^* & 0 & \xi_2^*\xi_3^* \end{vmatrix} \\ &= \begin{vmatrix} \xi_1^* & 0 & \xi_3^* \\ \xi_2^* & -\xi_3^* & 0 \\ \xi_3^* & \xi_2^* & -\xi_1^* \end{vmatrix} \begin{vmatrix} {\xi_2^*}^2 + {\xi_3^*}^2 & \xi_1^* & \xi_1^*\xi_2^* \\ -\xi_1^*\xi_2^* & \xi_2^* & -({\xi_1^*}^2 + {\xi_3^*}^2) \\ -\xi_1^*\xi_3^* & \xi_3^* & \xi_2^*\xi_3^* \end{vmatrix}. \end{aligned}$$

The ultimate line follows from permuting the columns and using block matrix structure. Then, it is straight-forward

$$\begin{vmatrix} \xi_1^* & 0 & \xi_3^* \\ \xi_2^* & -\xi_3^* & 0 \\ \xi_3^* & \xi_2^* & -\xi_1^* \end{vmatrix} = {\xi_1^*}^2 \xi_3^* + {\xi_2^*}^2 \xi_3^* + {\xi_3^*}^2 \xi_3^* = \xi_3^*.$$

Again by ${\xi_1^*}^2 + {\xi_2^*}^2 + {\xi_3^*}^2 = 1$, we find

$$\begin{aligned} \begin{vmatrix} {\xi_2^*}^2 + {\xi_3^*}^2 & \xi_1^* & \xi_1^* \xi_2^* \\ -\xi_1^* \xi_2^* & \xi_2^* & -({\xi_1^*}^2 + {\xi_3^*}^2) \\ -\xi_1^* \xi_3^* & \xi_3^* & \xi_2^* \xi_3^* \end{vmatrix} &= \begin{vmatrix} 1 - {\xi_1^*}^2 & \xi_1^* & \xi_1^* \xi_2^* \\ -\xi_1^* \xi_2^* & \xi_2^* & -({\xi_1^*}^2 + {\xi_3^*}^2) \\ -\xi_1^* \xi_3^* & \xi_3^* & \xi_2^* \xi_3^* \end{vmatrix} \\ &= \begin{vmatrix} 1 & \xi_1^* & \xi_1^* \xi_2^* \\ 0 & \xi_2^* & -({\xi_1^*}^2 + {\xi_3^*}^2) \\ 0 & \xi_3^* & \xi_2^* \xi_3^* \end{vmatrix} - \xi_1^* \begin{vmatrix} \xi_1^* & \xi_1^* & \xi_1^* \xi_2^* \\ \xi_2^* & \xi_2^* & -({\xi_1^*}^2 + {\xi_3^*}^2) \\ \xi_3^* & \xi_3^* & \xi_2^* \xi_3^* \end{vmatrix} \\ &= \xi_3^*. \end{aligned}$$

This finishes the proof. □

Likewise, we define m_1 and m_2 by choosing the non-trivial eigenvectors for $|\xi_i^*| \gtrsim 1$, which leads us to conjugation matrices with determinant

$$\det m_i(x, \xi) = {\xi_i^*}^2.$$

By elementary column operations, that is adding and subtracting the third and fourth and fifth and sixth eigenvector, the determinant is computed to be

$$\det m_3(x, \xi) = {\xi_3^*}^2.$$

We shall see that for $|\xi_3^*| \gtrsim 1$, we can choose the eigenvectors as an orthonormalbasis through linear combinations of the above. Let

$$w_1 = \begin{pmatrix} 0 \\ -\xi_3^* \\ \xi_2^* \\ {\xi_2^*}^2 + {\xi_3^*}^2 \\ -\xi_1^* \xi_2^* \\ -\xi_1^* \xi_3^* \end{pmatrix}, \qquad w_3 = \begin{pmatrix} \xi_3^* \\ 0 \\ -\xi_1^* \\ -\xi_1^* \xi_2^* \\ {\xi_1^*}^2 + {\xi_3^*}^2 \\ \xi_2^* \xi_3^* \end{pmatrix}.$$

We have $\|w_1\|^2 = 2({\xi_2^*}^2 + {\xi_3^*}^2)$, $\|w_3\|^2 = 2({\xi_1^*}^2 + {\xi_3^*}^2)$, and normalize $w_i' = w_i / \|w_i\|$. We compute

$$\langle w_1', w_3' \rangle = \frac{-\xi_1^* \xi_2^* (1 + (\xi_3^*)^2)}{2({\xi_2^*}^2 + {\xi_3^*}^2)^{\frac{1}{2}} ({\xi_1^*}^2 + {\xi_3^*}^2)^{\frac{1}{2}}}.$$

Now we consider $\tilde{w}_3 = w_3' - \langle w_1', w_3' \rangle w_1'$:

$$\tilde{w}_3 = \frac{1}{\sqrt{2}({\xi_1^*}^2 + {\xi_3^*}^2)} \begin{pmatrix} \xi_3^* \\ 0 \\ -\xi_1^* \\ -\xi_1^* \xi_2^* \\ {\xi_1^*}^2 + {\xi_3^*}^2 \\ \xi_2^* \xi_3^* \end{pmatrix} - \frac{\langle w_1', w_3' \rangle}{\sqrt{2}({\xi_2^*}^2 + {\xi_3^*}^2)^{\frac{1}{2}}} \begin{pmatrix} 0 \\ -\xi_3^* \\ \xi_2^* \\ {\xi_2^*}^2 + {\xi_3^*}^2 \\ -\xi_1^* \xi_2^* \\ -\xi_1^* \xi_3^* \end{pmatrix}.$$

Clearly, $\|\tilde{w}_3\|_2 \gtrsim 1$ for $|\xi_3^*| \gtrsim 1$. Hence, by renormalizing (and not changing notations for sake of brevity), we find

$$\tilde{w}_3 := \frac{\tilde{w}_3}{\|\tilde{w}_3\|_2}.$$

Similarly, consider

$$w_2 = \begin{pmatrix} 0 \\ -\xi_3^* \\ \xi_2^* \\ -({\xi_2^*}^2 + {\xi_3^*}^2) \\ \xi_1^* \xi_2^* \\ \xi_1^* \xi_3^* \end{pmatrix}, \qquad w_4 = \begin{pmatrix} \xi_3^* \\ 0 \\ -\xi_1^* \\ \xi_1^* \xi_2^* \\ -({\xi_1^*}^2 + {\xi_3^*}^2) \\ -\xi_1^* \xi_3^* \end{pmatrix}.$$

We compute

$$\|w_2\|_2^2 = 2({\xi_2^*}^2 + {\xi_3^*}^2), \qquad \|w_4\|_2^2 = 2({\xi_3^*}^2 + {\xi_1^*}^2),$$

which allows for renormalization $w_i' = w_i / \|w_i\|_2$. Now we consider $\tilde{w}_4 = w_4' - \langle w_2', w_4' \rangle \tilde{w}_2'$, which yields after an additional renormalization eigenvectors of $\xi_0 + \|\xi'\|$. We conclude that the matrix

$$\tilde{m}_3(x,\xi) = \begin{pmatrix} u_1 & u_2 & \tilde{w}_1 & \tilde{w}_2 & \tilde{w}_3 & \tilde{w}_4 \end{pmatrix}$$

consists of orthonormal eigenvectors to d as in (64) for $|\xi_3'| \gtrsim 1$. We summarize the accomplished diagonalization:

$$\begin{aligned} p(x,\xi) = &\begin{pmatrix} A_{<\lambda} & 0 \\ 0 & A_{<\lambda} \end{pmatrix} \begin{pmatrix} \varepsilon_{<\lambda}^{\frac{1}{2}} & 0 \\ 0 & \mu_{<\lambda}^{\frac{1}{2}} \end{pmatrix} \tilde{m}_i(x,\xi_0,\tilde{\xi}') d(x,\xi_0,\tilde{\xi}') \\ &\times \tilde{m}_i^t(x,\xi_0,\tilde{\xi}') \begin{pmatrix} \varepsilon_{<\lambda}^{\frac{1}{2}} & 0 \\ 0 & \mu_{<\lambda}^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} A_{<\lambda}^t & 0 \\ 0 & A_{<\lambda}^t \end{pmatrix} \text{ with } \tilde{\xi}' = \frac{A_{<\lambda}^t \xi'}{(\varepsilon_{<\lambda}\mu_{<\lambda})^{\frac{1}{2}}}. \end{aligned}$$

in the phase space region $|\tilde{\xi}_3^*| \gtrsim 1$. Note that there is always $i \in \{1,2,3\}$ such that

$$|\tilde{\xi}_i^*| \gtrsim 1.$$

We define phase-space projection operators by the function

$$\pi_3(x,\xi) = \chi(\lambda^{-1}\xi)\tilde{\chi}(\lambda^{-1}(A_{<\lambda}^t \xi')_3),$$

with χ, $\tilde{\chi} \in C_c^\infty$ suitable bump functions. The corresponding projections are denoted by $S_\lambda S_{\lambda 3}$. We let

$$N^3(x,\xi) = \tilde{m}_i^t(x,\xi_0,\tilde{\xi}') \begin{pmatrix} \varepsilon_{<\lambda}^{\frac{1}{2}}(x) & 0 \\ 0 & \mu_{<\lambda}^{\frac{1}{2}}(x) \end{pmatrix} \begin{pmatrix} A_{<\lambda}^t(x) & 0 \\ 0 & A_{<\lambda}^t(x) \end{pmatrix} \chi(\lambda^{-1}\xi')\tilde{\chi}(\lambda^{-1}(A_{<\lambda}^t \xi')_3),$$

and

$$D^3(x,\xi) = d(x,\xi_0,\tilde{\xi}')\chi(\lambda^{-1}\xi')\tilde{\chi}(\lambda^{-1}(A_{<\lambda}^t \xi')_3),$$

and

$$M^3(x,\xi) = \begin{pmatrix} A_{<\lambda}(x) & 0 \\ 0 & A_{<\lambda}(x) \end{pmatrix} \begin{pmatrix} \varepsilon_{<\lambda}^{\frac{1}{2}}(x) & 0 \\ 0 & \mu_{<\lambda}^{\frac{1}{2}} \end{pmatrix} \tilde{m}_i(x,\xi)\chi(\lambda^{-1}\xi')\tilde{\chi}(\lambda^{-1}(A_{<\lambda}^t \xi')_3).$$

The corresponding operators are defined by

$$\mathcal{M}_\lambda^3(x,D) = Op(M^3(x,\xi)),\ \mathcal{D}_\lambda^3(x,D) = Op(D^3(x,\xi)),\ \mathcal{N}_\lambda^3(x,D) = Op(N^3(x,\xi)).$$

By symbol composition, we can harmlessly insert frequency projectors after every factor. This makes the single factors bounded with symbols in $\tilde{S}^i_{1,1}$, $i \in \{0,1\}$. By Lemma 4.2, the claim follows, and the proof of Proposition 5.6 is complete. $\square$

5.1.3. *Conclusion of frequency localized estimate.* We have shown in Subsection 5.1.2 that after appropriate localization in phase space, the Maxwell system can be diagonalized to two degenerate and four non-degenerate half-wave equations. The degenerate equations correspond to stationary solutions, possibly induced by charges. We use this to finish the proof of Proposition 2.2 by showing the following estimates:

$$\|S^\tau_\lambda S'_\lambda u\|_{L^p L^q} \lesssim \lambda^\gamma (\|S^\tau_\lambda S'_\lambda u\|_{L^\infty_t L^2_x} + \|\mathcal{P}_{<\lambda} S^\tau_\lambda S'_\lambda u\|_{L^2_{t,x}}), \tag{68}$$

$$\begin{aligned} \|S'_\nu S^\tau_\lambda u\|_{L^p L^q} &\lesssim \nu^{\gamma - \frac{1}{2}} \|S'_\nu S^\tau_\lambda \mathcal{P}_{<\nu} u\|_{L^2_{t,x}} \\ &\quad + \nu^{\gamma - 1 + \frac{1}{p}} (\|\rho'_{e\nu}\|_{L^\infty_t L^2_x} + \|\rho'_{m\nu}\|_{L^\infty_t L^2_x}). \end{aligned} \tag{69}$$

To use the diagonalization, we need the following:

Lemma 5.9. *For $i \in \{1,2,3\}$ and $\lambda \gg 1$, we find the following estimates to hold:*

$$\begin{aligned} \|S_{\lambda i} u\|_{L^p L^q} &\lesssim \|\mathcal{N}^i_\lambda S_{\lambda i} u\|_{L^p L^q} + \lambda^{\gamma - \frac{1}{2}} \|S_{\lambda i} u\|_{L^2}, \\ \|S_{\lambda i} u\|_{L^p L^q} &\lesssim \|\mathcal{M}^i_\lambda S_{\lambda i} u\|_{L^2}. \end{aligned}$$

Proof. For the proof of the first estimate, we observe for the composed symbols of $\mathcal{M}^i_\lambda$ and $\mathcal{N}^i_\lambda$:

$$\begin{aligned} &\begin{pmatrix} A_{<\lambda} & 0 \\ 0 & A_{<\lambda} \end{pmatrix} \begin{pmatrix} \varepsilon^{\frac{1}{2}}_{<\lambda} & 0 \\ 0 & \mu_{<\lambda^{\frac{1}{2}}} \end{pmatrix} \tilde{m}_i(x,\xi_0,\tilde{\xi}') \tilde{m}^t_i(x,\xi_0,\tilde{\xi}') \\ &\times \begin{pmatrix} \varepsilon^{\frac{1}{2}}_{<\lambda} & 0 \\ 0 & \mu^{\frac{1}{2}}_{<\lambda} \end{pmatrix} \begin{pmatrix} A^t_{<\lambda} & 0 \\ 0 & A^t_{<\lambda} \end{pmatrix} = \begin{pmatrix} \varepsilon_{<\lambda} A_{<\lambda} A^t_{<\lambda} & 0 \\ 0 & \mu_{<\lambda} A_{<\lambda} A^t_{<\lambda} \end{pmatrix}. \end{aligned}$$

Hence, we find

$$\mathcal{M}^i_\lambda \mathcal{N}^i_\lambda S_{\lambda i} = \begin{pmatrix} \varepsilon_{<\lambda} A_{<\lambda} A^t_{<\lambda} & 0 \\ 0 & \mu_{<\lambda} A_{<\lambda} A^t_{<\lambda} \end{pmatrix} S_{\lambda i} + R_i(x,D)$$

with $\|R_i(x,D)\|_{L^2 \to L^2} \lesssim \lambda^{-1}$. This allows us to estimate

$$\begin{aligned} \|S_{\lambda i} u\|_{L^p L^q} &\lesssim \| \begin{pmatrix} \varepsilon_{<\lambda} A_{<\lambda} A^t_{<\lambda} & 0 \\ 0 & \mu_{<\lambda} A_{<\lambda} A^t_{<\lambda} \end{pmatrix} S_{\lambda i} u\|_{L^p L^q} \\ &\lesssim \|\mathcal{M}^i_\lambda \mathcal{N}^i_\lambda S_{\lambda i} u\|_{L^p L^q} + \|R^i(x,D) S_{\lambda i} u\|_{L^p L^q} \\ &\lesssim \|\mathcal{N}^i_\lambda S_{\lambda i} u\|_{L^p L^q} + \lambda^{\gamma - \frac{1}{2}} \|S_{\lambda i} u\|_{L^2} \end{aligned}$$

by Minkowski's inequality and Sobolev embedding.
For the proof of the second estimate, we argue similarly

$$\|S_{\lambda i} u\|_{L^2_{t,x}} \lesssim \|(1+R_i) S_{\lambda i} u\|_{L^2_{t,x}} = \|\mathcal{N}^i_\lambda \mathcal{M}^i_\lambda S_{\lambda i} u\|_{L^2_{t,x}} \lesssim \|\mathcal{M}^i_\lambda S_{\lambda i} u\|_{L^2_{t,x}}.$$

The proof is complete. $\square$

We can finally show (68) and (69):

Proof of (68). We split $S^\tau_\lambda S'_\lambda u = \sum_{i=1}^3 S^\tau_\lambda S_{\lambda i} u$ with $S^\tau_\lambda S_{\lambda i} u$ being amenable to the diagonalization of $\mathcal{P}$ provided by $\mathcal{M}^i_\lambda$ and $\mathcal{N}^i_\lambda$. We write

$$\|S^\tau_\lambda S_{\lambda i} u\|_{L^p L^q} \lesssim \|S^\tau_\lambda \mathcal{M}^i_\lambda \mathcal{N}^i_\lambda S_{\lambda i} u\|_{L^p L^q} + \|S^\tau_\lambda R(x,D') S'_\lambda u\|_{L^p L^q}.$$

Since $R(x,D')$ is smoothing of order -1, we can use Sobolev embedding to find

$$\begin{aligned}\|S^\tau_\lambda R(x,D') S'_\lambda u\|_{L^p L^q} &\lesssim \lambda^{\frac{1}{2}-\frac{1}{p}} \lambda^{3\left(\frac{1}{2}-\frac{1}{q}\right)-1} \|S^\tau_\lambda S'_\lambda u\|_{L^2_{t,x}} \\ &\lesssim \lambda^{-\varepsilon} \|\langle D'\rangle^\gamma S'_\lambda u\|_{L^2_{t,x}},\end{aligned}$$

which is acceptable. By Lemma 5.9, we have

$$\|S^\tau_\lambda \mathcal{M}^i_\lambda \mathcal{N}^i_\lambda S_{\lambda i} u\|_{L^p L^q} \lesssim \|S^\tau_\lambda \mathcal{N}^i_\lambda S_{\lambda i} u\|_{L^p L^q}.$$

We estimate the components $\|[S^\tau_\lambda \mathcal{N}^i_\lambda S_{\lambda i} u]_j\|_{L^p L^q}$ separately. The degenerate components $[\mathcal{D}_\lambda]_{jj}$, $j=1,2$, are elliptic. This yields by Sobolev embedding the estimate:

$$\begin{aligned}\|[\mathcal{N}^i_\lambda S^\tau_\lambda S'_\lambda u]_j\|_{L^p_t L^q_x} &\lesssim \lambda^{\gamma-\frac{1}{p}} \|[\mathcal{N}^i_\lambda S^\tau_\lambda S'_\lambda u]_j\|_{L^2_{t,x}} \\ &\lesssim \lambda^{\gamma-1+\frac{1}{p}} \|[\mathcal{D}_\lambda \mathcal{N}^i_\lambda S^\tau_\lambda S_{\lambda i} u]_j\|_{L^2_{t,x}} \\ &\lesssim \lambda^\gamma \|S^\tau_\lambda \mathcal{D}_\lambda \mathcal{N}^i_\lambda S_{\lambda i} u\|_{L^2_{t,x}}.\end{aligned}$$

Another application of Lemma 5.9 and Proposition 5.6 yields

$$\begin{aligned}\lambda^\gamma \|S^\tau_\lambda \mathcal{D}_\lambda \mathcal{N}^i_\lambda S_{\lambda i} u\|_{L^2_{t,x}} &\lesssim \lambda^\gamma \|S^\tau_\lambda \mathcal{M}^i_\lambda \mathcal{D}_\lambda \mathcal{N}^i_\lambda S_{\lambda i} u\|_{L^2_{t,x}} \\ &\lesssim \lambda^\gamma \|S^\tau_\lambda \mathcal{P}_\lambda u\|_{L^2_{t,x}} + \lambda^\gamma \|S^\tau_\lambda u\|_{L^2_{t,x}}.\end{aligned}$$

The non-degenerate components $j=3,\dots,6$ are estimated by [3, Eq. (2.1)]:

$$\|S^\tau_\lambda \mathcal{N}^i_\lambda S_{\lambda i} u\|_{L^p L^q} \lesssim \lambda^\gamma (\|S^\tau_\lambda \mathcal{N}^i_\lambda S_{\lambda i} u\|_{L^\infty_t L^2_x} + \|S^\tau_\lambda \mathcal{D}_\lambda \mathcal{N}_{\lambda i} S_{\lambda i} u\|_{L^2_{t,x}}).$$

By another application of Lemma 5.9 and Proposition 5.6, we find

$$\|S^\tau_\lambda \mathcal{N}^i_\lambda S_{\lambda i} u\|_{L^p L^q} \lesssim \lambda^\gamma (\|S^\tau_\lambda S'_\lambda u\|_{L^\infty_t L^2_x} + \|S^\tau_\lambda \mathcal{P}_\lambda S_\lambda u\|_{L^2_{t,x}}).$$

We passed from $S_{\lambda i}$ to S'_λ above by first order symbol composition. This finishes the proof. $\square$

Proof of (69). If $\{|\tau| \ll |\xi'|\}$ and $\{|\xi'| \gtrsim 1\}$, we see that after diagonalization, the operator $\mathcal{P}$ is elliptic up to the charges. Let $\lambda \sim |\tau| \ll |\xi'| \sim \nu$. We make an additional localization in phase space: $S'_\nu u = \sum_{i=1}^3 S_{\nu i} u$.

$$\|S^\tau_\lambda S_{\nu i} u\|_{L^p L^q} \lesssim \|S^\tau_\lambda \mathcal{M}^i_\nu \mathcal{N}^i_\nu S_{\nu i} u\|_{L^p L^q} + \|S^\tau_\lambda R(x,D') S_{\nu i} u\|_{L^p L^q}$$

with $R(x,D')$ being smoothing of order -1, we can use Sobolev embedding to find

$$\|S^\tau_\lambda R(x,D') S_{\nu i} u\|_{L^p L^q} \lesssim \lambda^{\frac{1}{2}-\frac{1}{p}} \mu^{3\left(\frac{1}{2}-\frac{1}{q}\right)-1} \|S^\tau_\lambda S'_\nu u\|_{L^2_{t,x}} \lesssim \|\langle D'\rangle^\gamma S'_\nu u\|_{L^2_{t,x}}.$$

By Lemma 5.9,

$$\|S^\tau_\lambda \mathcal{M}^i_\nu \mathcal{N}^i_\nu S_{\nu i} u\|_{L^p L^q} \lesssim \|S^\tau_\lambda \mathcal{N}^i_\nu S_{\nu i} u\|_{L^p L^q}.$$

For $[\mathcal{N}^i_\nu S_{\nu i} u]_j$ and $j=1,2$ we can use Sobolev embedding and definition of charges. For this purpose, recall the symbol of $\mathcal{N}^i_\nu$. With $\tilde{\xi}' = A^t_{<\nu} \xi'$, we find for $v \in \mathbb{C}^6$, $v = (v_1, v_2)^t$, with $v_i \in \mathbb{C}^3$:

$$[\tilde{m}^t_i(x,\xi_0,\tilde{\xi}') \begin{pmatrix} \varepsilon^{\frac{1}{2}}_{<\nu} & 0 \\ 0 & \mu^{\frac{1}{2}}_{<\nu} \end{pmatrix} \begin{pmatrix} A^t_{<\nu} & 0 \\ 0 & A^t_{<\nu} \end{pmatrix} v]_1 = \frac{(\xi')^t}{\mu^{\frac{1}{2}}_{<\nu} \|\xi'\|} A_{<\nu} A^t_{<\nu} v_1.$$

Moreover,

$$[\tilde{m}_i^t(x,\xi_0,\tilde{\xi}') \begin{pmatrix} \varepsilon_{<\nu}^{\frac{1}{2}} & 0 \\ 0 & \mu_{<\nu}^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} A_{<\nu}^t & 0 \\ 0 & A_{<\nu}^t \end{pmatrix} v]_2 = \frac{(\xi')^t}{\varepsilon_{<\nu}^{\frac{1}{2}} \|\xi'\|} A_{<\nu} A_{<\nu}^t v_2.$$

Consequently, we can write

$$[\mathcal{N}_\nu^i S_\nu u]_1 = \frac{1}{h_{<\nu} \varepsilon_{<\nu} \mu_{<\nu}^{\frac{1}{2}}} \frac{1}{|\nabla_{x'}|} \nabla \cdot (h_{<\nu} \varepsilon_{<\nu} A_{<\nu} A_{<\nu}^t \mathcal{E}) + R_1(x,D)\mathcal{E}$$

with $\|R_1\|_{L^2 \to L^2} \lesssim \nu^{-1}$. Therefore, the estimate for the first component follows from Sobolev embedding:

$$\| [\mathcal{N}_\nu^i S_{\nu i} u]_1 \|_{L^p L^q} \lesssim \nu^{\gamma - 1 + \frac{1}{p}} \big(\|S'_\nu \rho_e\|_{L^\infty L^2} + \|S'_\nu u\|_{L^\infty L^2} \big).$$

Similarly,

$$[\mathcal{N}_\nu^i S_{\nu i} u]_2 = \frac{1}{\varepsilon_{<\nu}^{\frac{1}{2}} h_{<\nu} \mu_{<\nu}^{\frac{1}{2}}} \frac{1}{|\nabla_{x'}|} \nabla \cdot (h_{<\nu} \mu_{<\nu} A_{<\nu} A_{<\nu}^t \mathcal{H}) + R_2(x,D)\mathcal{H}$$

with $\|R_2\|_{L^2 \to L^2} \lesssim \nu^{-1}$. We find by definition of $\rho'_{m\mu}$ and another Sobolev embedding yields

$$\| [\mathcal{N}_\nu^i S_{\nu i} u]_2 \|_{L^p L^q} \lesssim \nu^{\gamma - \frac{1}{2}} \|S_\nu u\|_{L^2}.$$

For the components $i = 3, \ldots, 6$ $[\mathcal{D}_\nu]_{ii}$ is elliptic:

$$\|S_\lambda^\tau \mathcal{N}_\nu S'_\nu u\|_{L^p L^q} \lesssim \nu^{3\left(\frac{1}{2} - \frac{1}{q}\right)} \lambda^{\frac{1}{2} - \frac{1}{p}} \nu^{-1} \|S_\lambda^\tau \mathcal{D}_\nu^i S'_{\nu i} u\|_{L^2_{t,x}}.$$

Consequently, we obtain

$$\|S_\lambda^\tau \mathcal{N}_\mu S'_\mu u\|_{L^p L^q} \lesssim \mu^{3\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{1}{p} - \frac{1}{2} + \varepsilon} \Big(\frac{\lambda}{\mu}\Big)^{\frac{1}{2} - \frac{1}{p}} \mu^{-\varepsilon} \|S_\lambda^\tau \mathcal{D}_\mu^i [\mathcal{N}_\mu S'_\mu u]_i\|_{L^2_{t,x}}.$$

By another application of Lemma 5.9 and Proposition 5.6, we conclude the proof. □

6. Diagonalizing reflected Maxwell equations in two dimensions

This section is devoted to the proof of Strichartz estimates in the two-dimensional case. We want to reduce to previously established results for half-wave equations either with structured Lipschitz coefficients or with metrical tensor satisfying $\|\partial_x \varepsilon\|_{L^2_T L^\infty} \lesssim 1$. For the diagonalization we can rely on results from [17, 16]. Interestingly, in the two-dimensional case, there are no symmetry assumptions on the permittivity (the permeability is scalar anyway) required for a diagonalization with L^p-bounded multipliers to hold. Thus, we simply redenote the permittivity and permeability decorated with the cometric and $\sqrt{g}$ by ε and μ to arrive at the Maxwell operator:

$$\mathcal{P} = \begin{pmatrix} \partial_t(\varepsilon^{11} \cdot) & 0 & -\partial_2 \\ 0 & \partial_t(\varepsilon^{22} \cdot) & \partial_1 \\ -\partial_2 & \partial_1 & \partial_t(\mu \cdot) \end{pmatrix}.$$

The principal symbol of $\mathcal{P}$ with rough coefficients is given by

$$\begin{aligned} p(x,\xi) &= i \begin{pmatrix} \xi_0 \varepsilon^{11} & 0 & -\xi_2 \\ 0 & \xi_0 \varepsilon^{22} & \xi_1 \\ -\xi_2 & \xi_1 & \xi_0 \mu \end{pmatrix} \\ &= i \begin{pmatrix} \xi_0 & 0 & -\xi_2/\mu \\ 0 & \xi_0 & \xi_1/\mu \\ -\xi_2 \varepsilon_{11} & \xi_1 \varepsilon_{22} & \xi_0 \end{pmatrix} \begin{pmatrix} \varepsilon^{11} & 0 & 0 \\ 0 & \varepsilon^{22} & 0 \\ 0 & 0 & \mu \end{pmatrix}. \end{aligned}$$

On the level of the equation, the above factorization corresponds to rewriting the equation in terms of $(\mathcal{D},\mathcal{B})$ instead of $(\mathcal{E},\mathcal{H})$. It turns out that this facilitates to find conjugation matrices. For the proof of Proposition 2.6 it suffices to show the following estimate for frequency localized functions for $1 \ll \lambda \in 2^{\mathbb{N}_0}$:

Proposition 6.1. *The following dyadic estimate holds:*
(70)

$$\|S'_\lambda S_\lambda u\|_{L_T^p L^q(\mathbb{R}^2)} \lesssim \lambda^\gamma (\|S_\lambda S'_\lambda u\|_{L_T^\infty L_x^2} + \|\mathcal{P}_\lambda S_\lambda S'_\lambda u\|_{L_{T,x}^2}) + \lambda^{\gamma - 1 + \frac{1}{p}} \|\rho'_{e\lambda}\|_{L_T^\infty L_x^2}$$

with $\rho'_{e\lambda} = \nabla \cdot (\varepsilon_{<\lambda} S'_\lambda \mathcal{E})$.

(70) handles the contribution of the phase space region $\{|\tau| \lesssim |\xi'|\}$. The commutator arguments to remove the frequency localization are easier than in three dimensions because $\gamma < 1$ and thus, omitted. The estimate for $\{|\tau| \gg |\xi'|\}$ follows from ellipticity of $\mathcal{P}$ in this region in phase space and is carried out like in three dimensions.

6.1. **Diagonalizing the principal symbol.** We use the diagonalization established in [17] (see also [16, Lemma 2.2]) to show the following:

Proposition 6.2. *Let* $2^{\mathbb{N}} \ni \lambda \gg \lambda_0$. *There are operators* $\mathcal{M}_\lambda \in OP\tilde{S}^0_{1,1}$, $\mathcal{N}_\lambda \in OP\tilde{S}^0_{1,1}$, *and* $\mathcal{D}_\lambda \in OP\tilde{S}^1_{1,1}$ *such that*

$$\mathcal{P}_\lambda S_\lambda S'_\lambda = \mathcal{M}_\lambda \mathcal{D}_\lambda \mathcal{N}_\lambda + E_\lambda$$

with $\|E_\lambda\|_{L^2 \to L^2} \lesssim 1$ *and implicit constant independent of* λ. *The principal symbols are given by*

$$\begin{aligned} m(x,\xi) &= \begin{pmatrix} \varepsilon_{22}\xi_1^* & -\xi_2^*/\mu & \xi_2^*/\mu \\ \varepsilon_{11}\xi_2^* & \xi_1^*/\mu & -\xi_1^*/\mu \\ 0 & -1 & -1 \end{pmatrix}, \\ n(x,\xi) &= \begin{pmatrix} \mu^{-1}\xi_1^* & \mu^{-1}\xi_2^* & 0 \\ \frac{-\xi_2^* \varepsilon_{11}}{2} & \frac{\xi_1^* \varepsilon_{22}}{2} & -\frac{1}{2} \\ \frac{\xi_2^* \varepsilon_{11}}{2} & \frac{-\xi_1^* \varepsilon_{22}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \varepsilon^{11} & 0 & 0 \\ 0 & \varepsilon^{22} & 0 \\ 0 & 0 & \mu \end{pmatrix}, \\ d(x,\xi) &= i\, diag(\xi_0, \xi_0 - \|\xi\|_{\varepsilon'}, \xi_0 + \|\xi\|_{\varepsilon'}) \end{aligned}$$

with $\|\xi\|^2_{\varepsilon'} = \langle \xi, \mu^{-1} \det(\varepsilon)^{-1} \varepsilon \xi \rangle$, $\xi^* = \xi / \|\xi\|$. *All coefficients in the above definitions are frequency truncated at* λ.

The diagonalization is substantially easier than in three dimensions because it does not require an additional localization in phase space.

6.2. **Conclusion of the proof.** To finish the proof of Theorem 1.2 like in Section 5, we have to check that the contribution of the charges is ameliorated like before:

Proposition 6.3. *With the notations from Proposition 6.2, the following estimate holds:*

$$\|\mathcal{N}_\lambda S_\lambda u\|_{L_t^p L_x^q} \lesssim \lambda^\gamma (\|\mathcal{N}_\lambda S_\lambda u\|_{L^2_{t,x}} + \|\mathcal{D}_\lambda \mathcal{N}_\lambda S_\lambda u\|_{L^2_{t,x}}) + \lambda^{\gamma - 1 + \frac{1}{p}} \|\rho_{e\lambda}\|_{L_t^\infty L_x^2}. \tag{71}$$

Proof. Again we show (71) componentwise. For the first component we have to use the divergence condition: We have

$$\begin{aligned} &[n(x,\xi)]_{11} = \frac{\xi_1 \varepsilon^{11}}{\mu \|\xi\|_{\varepsilon'}}, \quad [n(x,\xi)]_{12} = \frac{\xi_2 \varepsilon^{22}}{\mu \|\xi\|_{\varepsilon'}}, \\ &[n(x,\xi)]_{13} = 0. \end{aligned}$$

This gives

$$[\mathcal{N}_\lambda S_\lambda u]_1 = \frac{1}{\mu |\nabla_{\varepsilon'}|} [\nabla \cdot (\varepsilon S_\lambda u)] + R_1(x,D) \mathcal{E}$$

with $\|R_1\|_{2 \to 2} \lesssim \lambda^{-1}$. This yields the estimate for the first component by Sobolev embedding:

$$\|[\mathcal{N}_\lambda S_\lambda u]_1\|_{L^p L^q} \lesssim \lambda^{\gamma - 1 + \frac{1}{p}} (\|S_\lambda \rho_e\|_{L^\infty L^2} + \|S_\lambda u\|_{L^\infty L^2}).$$

The non-degenerate components are estimated by Theorem 5.1:

$$\|[\mathcal{N}_\lambda S_\lambda u]_i\|_{L^p L^q} \lesssim \lambda^\gamma (\|S_\lambda u\|_{L^\infty L^2} + \|\mathcal{D}^i_\lambda [\mathcal{N}_\lambda S_\lambda u]_i\|_{L^2}).$$

The proof is complete. □

We record the corresponding result of Lemma 5.9 to complete the proof of Theorem 1.2.

Lemma 6.4. *For $\lambda \gg 1$, we find the following estimates to hold:*

$$\begin{aligned} \|S_\lambda u\|_{L_t^p L_x^q} &\lesssim \|\mathcal{N}_\lambda S_\lambda u\|_{L_t^p L_x^q} + \lambda^{\gamma - \frac{1}{2}} \|S_\lambda u\|_{L^2_{t,x}}, \\ \|S_\lambda u\|_{L^2_{t,x}} &\lesssim \|\mathcal{M}_\lambda S_\lambda u\|_{L^2_{t,x}}. \end{aligned}$$

The lemma is proved like in the previous section.

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