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The massless three-loop Wilson coefficients for the deep-inelastic structure functions F_2, F_L, xF_3 and g_1

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Abstract: We calculate the massless unpolarized Wilson coefficients for deeply inelastic scattering for the structure functions $F_2(x, Q^2), F_L(x, Q^2), xF_3(x, Q^2)$ in the $\overline{\text{MS}}$ scheme and the polarized Wilson coefficients of the structure function $g_1(x, Q^2)$ in the Larin scheme up to three-loop order in QCD in a fully automated way based on the method of arbitrary high Mellin moments. We work in the Larin scheme in the case of contributing axial-vector couplings or polarized nucleons. For the unpolarized structure functions we compare to results given in the literature. The polarized three-loop Wilson coefficients are calculated for the first time. As a by-product we also obtain the quarkonic three-loop anomalous dimensions from the $O(1/\varepsilon)$ terms of the unrenormalized forward Compton amplitude. Expansions for small and large values of the Bjorken variable x are provided.

Keywords: Deep Inelastic Scattering or Small-x Physics, Higher-Order Perturbative Calculations, Renormalization Group, The Strong Coupling

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Contents

1 Introduction

The scaling violations of the deep-inelastic structure functions $F_k(x, Q^2)$ [1–13] in the massless limit and twist 2 approximation are described by those of the twist 2 parton distribution functions f_i and massless Wilson coefficients C_j in a perturbative expansion in the strong coupling constant $a_s(Q^2) = \alpha_s(Q^2)/(4\pi)$, where $Q^2 = -q^2$ denotes the space-like virtuality of the scattering process and $x = Q^2/(2p.q)$ is the Bjorken variable, with p the nucleon momentum. In the present paper we calculate the Wilson coefficients in the case of virtual photon exchange[1] in the unpolarized and polarized case to three-loop order, where the latter results are new. For the structure function xF_3 we consider the exchange of both the weak $W^\pm$ bosons. The present results are of importance for the experimental data analysis of the deep-inelastic scattering data at HERA [14] and future lepton-nucleon colliders, such as the EIC [15, 16] and the LHeC-project [17, 18] for precision measurements

[1] For the $\gamma - Z$ interference terms and pure Z-boson exchange structure functions the sets of Feynman diagrams are different in part and also additional axial-vector couplings contribute.

in the unpolarized and polarized case, to extract the parton distribution functions [19] and to measure the strong coupling constant at highest possible precision [20–23].

The calculation is based on massless virtual forward Compton amplitudes for on-shell partonic states, which are gauge-invariant quantities. The structure functions are described by

$$F_k(x,Q^2) = \sum_i C_{k,i}\left(x, \frac{Q^2}{\mu^2}\right) \otimes f_i\left(x,\mu^2\right) \tag{1.1}$$

with

$$C_{k,i}\left(x, \frac{Q^2}{\mu^2}\right) = \delta_{iq}\delta(1-x) + \sum_{l=1}^{\infty} a_s^l \sum_{n=0}^{l} \ln^n\left(\frac{Q^2}{\mu^2}\right) C_{k,i}^{(l,n)}(x), \tag{1.2}$$

where μ^2 denotes the factorization scale and $\otimes$ the Mellin convolution

$$A(x) \otimes B(x) = \int_0^1 dx_1 \int_0^1 dx_2 \delta(x - x_1 x_2) A(x_1) B(x_2). \tag{1.3}$$

The Mellin transform

$$\mathbf{M}[A(x)](N) = \int_0^1 dx x^{N-1} A(x) \tag{1.4}$$

diagonalizes the expression in eq. (1.3) into the product $\mathbf{M}[A](N) \cdot \mathbf{M}[B](N)$ and will therefore be often used in the subsequent calculation. The dependence on the factorization scale diminishes for higher and higher orders in the coupling constant. One may also derive representations in Mellin N space which are scheme-invariant order by order, see e.g. [24–27]. Here the preferred scale of the deep-inelastic process is $\mu^2 = Q^2$.

The massless Wilson coefficients for the structure functions $F_2(x,Q^2), F_L(x,Q^2)$ and $xF_3(x,Q^2)$ have been calculated at one– [28], two– [29–37] and three-loop order [38–40]. Furthermore, also their Mellin moments have been computed at 2– [41] and 3-loop order [42–45]. In the polarized massless case the corresponding 1- and 2-loop Wilson coefficients for the structure function $g_1(x,Q^2)$ have been obtained in [46–50].

In the present paper we recalculate for the first time independently the massless Wilson coefficients for the structure functions $F_2(x,Q^2)$ and $F_L(x,Q^2)$ as well as for the structure function $xF_3(x,Q^2) \equiv xF_3^{W^+}(x,Q^2) + xF_3^{W^-}(x,Q^2)$. We calculate newly those for the polarized structure function $g_1(x,Q^2)$ in the Larin scheme [51, 52]. The Wilson coefficients are computed using the method of the forward Compton amplitude, through which they are defined. Here the parameter $x = Q^2/(2p.q)$ is the variable on which the master integrals and the associated differential equations depend. The Mellin moments of the structure functions correspond to the expansion coefficients of the forward Compton amplitude in the unphysical limit $\omega = 1/x \ll 1$ which is frequently exploited to compute low Mellin moments. Therefore, the variable ω corresponds to the parameter t used in refs. [53, 54, 60] and the subsequent technical steps are quite similar to those in the calculation of the three-loop anomalous dimensions [53, 54]. One calls $(\Delta)\gamma_{qq}^{\pm,\mathrm{s},\mathrm{NS;PS}}$ and $(\Delta)\gamma_{qg}$ quarkonic anomalous dimensions, cf. [53–59].

Compared to [38, 40] we do not use special Ansätze for the structure of the solutions, or intermediary quantities, like linear combinations of shifted harmonic sums, cf. eq. (3.4) of [38], but derive general difference equations with rational coefficients in Mellin N space, which can have a much wider class of solutions. Equations of this kind may not even factorize to first order. We feel we cannot assume a representation in terms of harmonic sums a priori. Indeed, also elliptic structures can contribute in purely massless inclusive scattering cross sections from a certain loop order onward, cf. [61], where elliptic integrals start to contribute from three-loop order. In course of the present calculation it turns out that these recurrences factorize to first order. As such they lead to sum representations which need not to be harmonic sums, but can consist out of much more general sums, cf. [62–64]. Indeed, intermediately recurrences solve also in terms of cyclotomic harmonic sums [62] in the present case. Only in the end we see, that the solutions are spanned by nested harmonic sums.[2] Because of this, our method is much more general than those used in previous calculations. Further differences are the integration-by-parts (IBP) reduction of the scalar integrals and the way in which the integrals are calculated. Thus, the present calculation is technically different in many respects from the previous one for the Wilson coefficients of the structure functions $F_{2,L}$ and F_3. Agreement with previous results will therefore justify technical steps used in the earlier literature.

Furthermore, we represent all results by applying algebraic relations, known in explicit form since 2003 [66], constituting a further difference to earlier work. By using these relations the number of necessary special functions, both in N- and z space, is minimal. For analytic continuations to the complex N-plane this is of immediate importance, as this has to be performed for a much smaller number of special functions, here the nested harmonic sums; for a rigorous method cf. refs. [67, 68].

The paper is organized as follows. In section 2 we discuss the basic formalism, including the renormalization. We summarize the technical details of the calculation in section 3. In section 4 the Wilson coefficients for the structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$ are calculated up to three-loop order. Likewise, we compute in section 5 those for the structure function $xF_3(x, Q^2)$ to three-loop order. Finally, we present in section 6 the results of our calculation of the Wilson coefficients for the polarized structure function $g_1(x, Q^2)$ up to three-loop order.

We also remark that the quarkonic anomalous dimensions both in the unpolarized and the polarized case are obtained as a by-product of the present calculation from the $O(1/\varepsilon)$ terms of the unrenormalized Compton amplitude, and agree with the results in the literature. In section 7 we present the small-x and large-x expansions of the respective Wilson coefficients and compare to the large N_F behaviour predicted in the literature. Here N_F denotes the number of massless flavors. Section 8 contains the conclusions. To shorten the presentation we will identify the factorization scale $\mu^2 = Q^2$ and only print the Mellin N space representation for the complete expressions. In ancillary files to this paper, `Nspace.m` and `Zspace.m`, we present all Wilson coefficients in Mellin N and z-space in full form including their scale dependence. Here z denotes the parton momentum fraction in the nucleon, $z \in [0, 1]$, and is, for twist-2, identical to the Bjorken variable x. In an appendix we compute the Z-factor $Z_5^{\rm NS}(N)$, which allows the transformation between the Larin and M–scheme [52] for the non-singlet contributions in the polarized case, to three-loop order by using off-shell massless operator matrix elements.

[2] In the case of the massive 3-loop form factor, part of the first order factorizable contributions do lead to cyclotomic harmonic polylogarithms resp. cyclotomic harmonic sums, cf. [65].

2 Basic formalism

We consider the forward Compton amplitude for neutral current photon exchange in the unpolarized case for the deep-inelastic structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$, as well as for the structure functions $g_1(x, Q^2)$ in the polarized case in the twist-2 approximation to three-loop order. Furthermore, we study the charged current structure function $xF_3(x, Q^2)$.

The hadronic tensor has the principal structure [13, 69]

$$W_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) F_1\left(x, Q^2\right) + \frac{\hat{p}_\mu \hat{p}_\nu}{p.q} F_2(x, Q^2) - i\varepsilon_{\mu\nu\lambda\sigma} \frac{q^\lambda p^\sigma}{2p.q} F_3\left(x, Q^2\right)$$
$$+ i\varepsilon_{\mu\nu\lambda\sigma} \frac{q^\lambda S^\sigma}{p.q} g_1\left(x, Q^2\right) + \cdots . \tag{2.1}$$

Here the ellipses denote contributions from other structure functions not considered in the present paper, p denotes the nucleon 4-momentum and S the polarization vector of the nucleon, which can be taken in the longitudinal case in the nucleon rest frame

$$S_L = (M, 0, 0, 0)\,, \tag{2.2}$$

where only the energy component is non-vanishing and M denotes the nucleon mass. The vector $\hat{p}$ is given by

$$\hat{p} = p_\mu - \frac{p.q}{q^2} q_\mu\,. \tag{2.3}$$

All structure functions can be isolated by using corresponding projectors in $D = 4 + \varepsilon$ space-time dimensions. In the polarized case one furthermore considers the polarization asymmetry, i.e. the difference for $S = S_L$ and $-S_L$. In the following we will consider the structure functions $F_{2,L}(x, Q^2)$ and $g_1(x, Q^2)$ in the case of pure photon exchange, while $xF_3(x, Q^2)$ is measured for the arithmetic mean of $W^\pm$ boson exchange.

The associated forward Compton amplitudes are given by the Fourier transform of the time-ordered product of current operators

$$T_{\mu\nu} = i \int dz\; \mathrm{e}^{iqz} \langle p| T\left(J_\mu^\dagger(z) J_\nu(0)\right) |p\rangle\,, \tag{2.4}$$

and

$$W_{\mu\nu} = \frac{1}{\pi} \mathsf{Im} T_{\mu\nu}\,. \tag{2.5}$$

We consider the deep-inelastic scattering cross section for pure photon exchange [70]

$$\frac{d^2\sigma^{\gamma^*}(\lambda, \pm S_L)}{dxdy} = 2\pi S \frac{\alpha^2}{Q^4} \Bigg\{ y^2 2x F_1(x, Q^2) + 2\left(1 - y - \frac{xyM^2}{S}\right) F_2\left(x, Q^2\right)$$
$$\pm \left[-2\lambda y \left(2 - y - \frac{2xyM^2}{S}\right) x g_1\left(x, Q^2\right) + 8\lambda \frac{yx^2M^2}{S} g_2\left(x, Q^2\right)\right]\Bigg\}\,, \tag{2.6}$$

with λ the longitudinal charged lepton polarization, $y = Q^2/Sx$ a Bjorken variable, and S the cms energy squared. We will use the collinear parton model [71] in the present calculation, where the parton momentum is given by zp, with p the nucleon momentum, and $z \in [0,1]$. Due to this the Wilson coefficients of the structure function $g_2(x,Q^2)$ cannot be calculated. It requires the use of the covariant parton model [72–76] already at tree level.

One furthermore has

$$F_L\left(x,Q^2\right) = F_2\left(x,Q^2\right) - 2xF_1\left(x,Q^2\right) . \tag{2.7}$$

The structure functions are related to the parton densities for pure photon exchange at tree level as [13]

$$F_2\left(x,Q^2\right) = x\sum_{f=1}^{N_F} e_f^2\left[q_f\left(x,Q^2\right)+\bar{q}_f\left(x,Q^2\right)\right] , \tag{2.8}$$

$$g_1\left(x,Q^2\right) = \frac{1}{2}\sum_{f=1}^{N_F} e_f^2\left[\Delta q_f\left(x,Q^2\right)+\Delta\bar{q}_f\left(x,Q^2\right)\right] , \tag{2.9}$$

for pure photon exchange, with $q_i(\bar{q}_i)$ the unpolarized and $\Delta q_i(\Delta\bar{q}_i)$ the polarized parton densities and $F_L = 0$.

In the charged current case we consider the massless Wilson coefficients of the structure function $xF_3(x,Q^2)$. One should notice, that here also strange-charm transitions occur, which imply heavy flavor corrections [77–83]. They are of importance in quantitative analyses. Here we consider all contributing quark flavors as massless. For incoming charged leptons the scattering cross section reads, cf. [14, 84, 85],

$$\frac{d^2\sigma^{\rm cc}}{dxdy} = \frac{G_\mu^2 S}{8\pi}\left[\frac{M_W^2}{Q^2+M_W^2}\right]^2\left\{Y_+W_2\left(x,Q^2\right)+Y_-W_3\left(x,Q^2\right)-y^2W_L\left(x,Q^2\right)\right\} , \tag{2.10}$$

where G_μ denotes the Fermi constant and $Y_\pm = 1\pm(1-y)^2$ and both unpolarized leptons and nucleons are considered. The structure functions are given by

$$W_2\left(x,Q^2\right) = F_2^{{\rm cc},Q_l}\left(x,Q^2\right) , \tag{2.11}$$

$$W_3\left(x,Q^2\right) = -{\rm sign}\left(Q_l\right)xF_3^{{\rm cc},Q_l}\left(x,Q^2\right) , \tag{2.12}$$

where Q_l is the charge of the incoming charged lepton. At tree level these structure functions have the following quark flavor decomposition [84]

$$F_2^{{\rm cc},+}\left(x,Q^2\right) = 2x\sum_i\left[d_i\left(x,Q^2\right)+\bar{u}_i\left(x,Q^2\right)\right] , \tag{2.13}$$

$$F_2^{{\rm cc},-}\left(x,Q^2\right) = 2x\sum_i\left[u_i\left(x,Q^2\right)+\bar{d}_i\left(x,Q^2\right)\right] , \tag{2.14}$$

$$xF_3^{{\rm cc},+}\left(x,Q^2\right) = 2x\sum_i\left[d_i\left(x,Q^2\right)-\bar{u}_i\left(x,Q^2\right)\right] , \tag{2.15}$$

$$xF_3^{{\rm cc},-}\left(x,Q^2\right) = 2x\sum_i\left[u_i\left(x,Q^2\right)-\bar{d}_i\left(x,Q^2\right)\right] , \tag{2.16}$$

e.g. for four flavors,[3] and $W_L = F_L^{{\rm cc},Q_l}$ denotes the longitudinal structure function with $F_L^{{\rm cc},\pm} = F_2^{{\rm cc},\pm} - 2xF_1^{{\rm cc},\pm}$.

[3] More generally, one has to account for the Cabibbo-Kobayashi-Maskawa mixing, cf. e.g. [81], effectively redefining the down-quark densities in the charged current case.

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In the present paper we consider in the charged current case the flavor non-singlet structure function combination $xF_3^{\rm cc,+}(x,Q^2)+xF_3^{\rm cc,-}(x,Q^2)$. At the experimental side one might consider the scattering off deuteron targets, obeying the SU(2) isospin flavor symmetry

$$u\left(x,Q^2\right)=d\left(x,Q^2\right)\,, \qquad \bar{u}\left(x,Q^2\right)=\bar{d}\left(x,Q^2\right)\,, \tag{2.17}$$

after deuteron wave function corrections. Through this one obtains at tree level

$$F_2^{\mathrm{cc},d,+}=x[u+d+2s+\bar{u}+\bar{d}+2\bar{c}]\,, \tag{2.18}$$
$$F_2^{\mathrm{cc},d,-}=x[u+d+2c+\bar{u}+\bar{d}+2\bar{s}]\,, \tag{2.19}$$
$$F_2^{\mathrm{cc},d,+}+F_2^{\mathrm{cc},d,-}=2x\Sigma\,, \tag{2.20}$$
$$xF_3^{\mathrm{cc},d,+}=x[u+d+2s-\bar{u}-\bar{d}-2\bar{c}]\,, \tag{2.21}$$
$$xF_3^{\mathrm{cc},d,-}=x[u+d+2sc-\bar{u}-\bar{d}-2\bar{s}]\,, \tag{2.22}$$
$$\begin{aligned} xF_3^{\mathrm{cc},d,+}+xF_3^{\mathrm{cc},d,-} &= 2x[u+d-\bar{u}-\bar{d}+s-\bar{s}+c-\bar{c}]\,,\\ &\equiv 2x[u_v+d_v], \end{aligned} \tag{2.23}$$

and

$$xF_3=-\frac{8\pi}{Y_- G_\mu^2 S}\left[\frac{d^2\sigma^{\mathrm{cc},d,+}}{dxdy}-\frac{d^2\sigma^{\mathrm{cc},d,-}}{dxdy}\right]+\frac{Y_+}{Y_-}\left(F_2^{\mathrm{cc},d,+}-F_2^{\mathrm{cc},d,+}\right)-\frac{y^2}{Y_-}\left(W_L^{+,d}-W_L^{-d}\right). \tag{2.24}$$

Here the non-singlet combinations for F_2^d and $W_L^d \propto Y_+, y^2$ correspond to the quark flavor combination

$$f^{\mathrm{NS},+}=2x[(s-\bar{s})-(c-\bar{c})]\,, \tag{2.25}$$

which vanishes under the assumption of symmetric sea-quarks of the same flavor and one obtains the direct projection on xF_3^d by the weighted differential cross section difference. In other cases one has to perform fits in Bjorken y to separate the Y_- from the Y_+ and y^2 contributions.

The projectors of the hadronic tensor allowing to isolate the Wilson coefficients of the different structure functions are given by

$$P_L^{\mu\nu}=\frac{8x^3}{Q^2}p^\mu p^\nu\,, \tag{2.26}$$
$$P_2^{\mu\nu}=-\frac{2x}{D-2}\left(g^{\mu\nu}-(D-1)\frac{4x^2}{Q^2}p^\mu p^\nu\right)\,, \tag{2.27}$$
$$P_3^{\mu\nu}=\frac{-i}{(D-2)(D-3)}\frac{4x}{Q^2}\varepsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma\,, \tag{2.28}$$
$$P_{g_1}^{\mu\nu}=\frac{i}{(D-2)(D-3)}\frac{2x}{Q^2}\varepsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma\,, \tag{2.29}$$

applied to $W_{\mu\nu}$, cf. also [86]. The twist-2 contributions to the structure functions in the massless case are given by eq. (1.1).

The perturbative expansion of the structure functions

$$F_i\left(x,Q^2\right)=\sum_{k=0}^{\infty}a_s^k\left(Q^2\right)F_{i,k}(x) \tag{2.30}$$

obey the following renormalization group equation (RGE) in the massless limit [87, 88], see also [24],

$$[\mathcal{D} + \gamma_{J_1} + \gamma_{J_2} - n_\psi \gamma_\psi - n_A \gamma_A]\, F_i\left(N, Q^2\right) = 0 \tag{2.31}$$

with

$$\mathcal{D} = \mu^2 \frac{\partial}{\partial \mu^2} + \beta(a_s) \frac{\partial}{\partial a_s} \tag{2.32}$$

in Mellin space with $a_s = a_s(\mu^2)$, $\beta(a_s)$ the β function of Quantum Chromodynamics (QCD), and setting $\mu \equiv \mu_F = \mu_R$, where $\mu_{F,R}$ are the factorization and renormalization scale. γ_{J_i} are the anomalous dimensions of the currents forming the forward Compton amplitude and $n_{\psi(A)}$ and $\gamma_{\psi(A)}$ are the numbers of external quark (gluon fields) and their anomalous dimensions. One may now split eq. (2.31) for those of the renormalized Wilson coefficients and renormalized parton densities,

$$\sum_j \left[\mathcal{D}\left(\mu^2\right)\delta_{ij} + \gamma_{ij}^{\mathrm{S,NS}} - n_\psi \gamma_\psi - n_A \gamma_A\right] f_j\left(N, \mu^2\right) = 0\,, \tag{2.33}$$

$$\sum_j \left[\mathcal{D}(\mu^2)\delta_{ij} + \gamma_{J_1} + \gamma_{J_2} - \gamma_{ij}^{\mathrm{S,NS}}\right] C_i\left(N, \frac{Q^2}{\mu^2}\right) = 0\,. \tag{2.34}$$

In this way the Z factors of the massless Wilson coefficients and the massless parton densities are related.[4]

Before we turn to the renormalized Wilson coefficients we will study the unrenormalized ones, as those emerge in the calculation. They have the following representation. Here and in the following we work in Mellin N space.

$$\begin{aligned}
\hat{C}_{i,q}^{\mathrm{NS}} = 1 &+ \hat{a}_s \Big\{\frac{1}{\varepsilon} c_{i,q}^{\mathrm{NS},(1,-1)} + c_{i,q}^{\mathrm{NS},(1,0)} + \varepsilon c_{i,q}^{\mathrm{NS},(1,1)} + \varepsilon^2 c_{i,q}^{\mathrm{NS},(1,2)} + O(\varepsilon^3)\Big\} \\
&+ \hat{a}_s^2 \Big\{\frac{1}{\varepsilon^2} c_{i,q}^{\mathrm{NS},(2,-2)} + \frac{1}{\varepsilon} c_{i,q}^{\mathrm{NS},(2,-1)} + c_{i,q}^{\mathrm{NS},(2,0)} + \varepsilon c_{i,q}^{\mathrm{NS},(2,1)} + O(\varepsilon^2)\Big\} \\
&+ \hat{a}_s^3 \Big\{\frac{1}{\varepsilon^3} c_{i,q}^{\mathrm{NS},(3,-3)} + \frac{1}{\varepsilon^2} c_{i,q}^{\mathrm{NS},(3,-2)} + \frac{1}{\varepsilon} c_{i,q}^{\mathrm{NS},(3,-1)} + c_{i,q}^{\mathrm{NS},(3,0)} + O(\varepsilon)\Big\} + O\left(\hat{a}_s^4\right),
\end{aligned} \tag{2.35}$$

$$\begin{aligned}
\hat{C}_{i,q}^{\mathrm{PS}} = &\hat{a}_s^2 \Big\{\frac{1}{\varepsilon^2} c_{i,q}^{\mathrm{PS},(2,-2)} + \frac{1}{\varepsilon} c_{i,q}^{\mathrm{PS},(2,-1)} + c_{i,q}^{\mathrm{PS},(2,0)} + \varepsilon c_{i,q}^{\mathrm{PS},(2,1)} + O(\varepsilon^2)\Big\} \\
&+ \hat{a}_s^3 \Big\{\frac{1}{\varepsilon^3} c_{i,q}^{\mathrm{PS},(3,-3)} + \frac{1}{\varepsilon^2} c_{i,q}^{\mathrm{PS},(3,-2)} + \frac{1}{\varepsilon} c_{i,q}^{\mathrm{PS},(3,-1)} + c_{i,q}^{\mathrm{PS},(3,0)} + O(\varepsilon)\Big\} + O\left(\hat{a}_s^4\right),
\end{aligned} \tag{2.36}$$

$$\begin{aligned}
\hat{C}_{i,g} = &\hat{a}_s \Big\{\frac{1}{\varepsilon} c_{i,g}^{(1,-1)} + c_{i,g}^{(1,0)} + \varepsilon c_{i,g}^{(1,1)} + \varepsilon^2 c_{i,g}^{(1,2)} + O(\varepsilon^3)\Big\} \\
&+ \hat{a}_s^2 \Big\{\frac{1}{\varepsilon^2} c_{i,g}^{(2,-2)} + \frac{1}{\varepsilon} c_{i,g}^{(2,-1)} + c_{i,g}^{(2,0)} + \varepsilon c_{i,g}^{(2,1)} + O(\varepsilon^2)\Big\} \\
&+ \hat{a}_s^3 \Big\{\frac{1}{\varepsilon^3} c_{i,g}^{(3,-3)} + \frac{1}{\varepsilon^2} c_{i,g}^{(3,-2)} + \frac{1}{\varepsilon} c_{i,g}^{(3,-1)} + c_{i,g}^{(3,0)} + O(\varepsilon)\Big\} + O\left(\hat{a}_s^4\right),
\end{aligned} \tag{2.37}$$

[4] This is different in the massive case, cf. [89], and even in treating heavy quarks as light at asymptotic scales.

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with the bare coupling constant $\hat{a}_s$. These relations apply synonymously to the structure functions $i = F_2, F_3, g_1$.

For those of the structure function F_L one obtains

$$\begin{aligned}\hat{C}^{\rm NS}_{F_L,q} &= \hat{a}_s\Big\{c^{{\rm NS},(1,0)}_{F_L,q} + \varepsilon c^{{\rm NS},(1,1)}_{F_L,q} + \varepsilon^2 c^{{\rm NS},(1,2)}_{F_L,q} + O\left(\varepsilon^3\right)\Big\} \\ &+ \hat{a}_s^2\Big\{\frac{1}{\varepsilon}c^{{\rm NS},(2,-1)}_{F_L,q} + c^{{\rm NS},(2,0)}_{F_L,q} + \varepsilon c^{{\rm NS},(2,1)}_{F_L,q} + O\left(\varepsilon^2\right)\Big\} \\ &+ \hat{a}_s^3\Big\{\frac{1}{\varepsilon^2}c^{{\rm NS},(3,-2)}_{F_L,q} + \frac{1}{\varepsilon}c^{{\rm NS},(3,-1)}_{F_L,q} + c^{{\rm NS},(3,0)}_{F_L,q} + O(\varepsilon)\Big\} + O\left(\hat{a}_s^4\right), \end{aligned} \tag{2.38}$$

$$\begin{aligned}\hat{C}^{\rm PS}_{F_L,q} &= \hat{a}_s^2\Big\{\frac{1}{\varepsilon}c^{{\rm PS},(2,-1)}_{F_L,q} + c^{{\rm PS},(2,0)}_{F_L,q} + \varepsilon c^{{\rm PS},(2,1)}_{F_L,q} + O\left(\varepsilon^2\right)\Big\} \\ &+ \hat{a}_s^3\Big\{\frac{1}{\varepsilon^2}c^{{\rm PS},(3,-2)}_{F_L,q} + \frac{1}{\varepsilon}c^{{\rm PS},(3,-1)}_{F_L,q} + c^{{\rm PS},(3,0)}_{F_L,q} + O\left(\varepsilon\right)\Big\} + O\left(\hat{a}_s^4\right), \end{aligned} \tag{2.39}$$

$$\begin{aligned}\hat{C}_{F_L,g} &= \hat{a}_s\Big\{c^{(1,0)}_{F_L,g} + \varepsilon c^{(1,1)}_{F_L,g} + \varepsilon^2 c^{(1,2)}_{F_L,g} + O\left(\varepsilon^3\right)\Big\} \\ &+ \hat{a}_s^2\Big\{\frac{1}{\varepsilon}c^{(2,-1)}_{F_L,g} + c^{(2,0)}_{F_L,g} + \varepsilon c^{(2,1)}_{F_L,g} + O\left(\varepsilon^2\right)\Big\} \\ &+ \hat{a}_s^3\Big\{\frac{1}{\varepsilon^2}c^{(3,-2)}_{F_L,g} + \frac{1}{\varepsilon}c^{(3,-1)}_{F_L,g} + c^{(3,0)}_{F_L,g} + O(\varepsilon)\Big\} + O\left(\hat{a}_s^4\right). \end{aligned} \tag{2.40}$$

The renormalization proceeds in the following way. The coupling constant is renormalized in the $\overline{\mathsf{MS}}$ scheme

$$\hat{a}_s = a_s\left(1 + \frac{2\beta_0}{\varepsilon}a_s + \left[\frac{4\beta_0^2}{\varepsilon^2} + \frac{\beta_1}{\varepsilon}\right]a_s^2 + O\left(a_s^3\right)\right). \tag{2.41}$$

Here β_i denote the expansion coefficients of the QCD β-function, [90–100].

$$\beta_0(N_F) = \frac{11}{3}C_A - \frac{4}{3}T_F N_F\,, \tag{2.42}$$
$$\beta_1(N_F) = \frac{34}{3}C_A^2 - 4\left(\frac{5}{3}C_A + C_F\right)T_F N_F\,. \tag{2.43}$$

The color factors are $C_F = (N_C^2-1)/(2N_C), C_A = N_C, T_F = 1/2$ for SU(N_C) and $N_C = 3$ for QCD; N_F denotes the number of massless quark flavors. Later we will also need the color factor[5]

$$d_{abc}d^{abc} = \frac{(N_C^2-1)\,(N_C^2-4)}{16N_C}. \tag{2.44}$$

The dimension of the adjoint representation is $N_A = N_C^2 - 1$. We are carrying out a partial renormalization of the Wilson coefficient of the structure function xF_3 for the axial vector coupling, which is performed by multiplying the unrenormalized Wilson coefficient

[5] We follow the notation of COLOR, see ref. [101].

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by Z_A, [40], see also [51, 102],

$$Z_A = 1 + \frac{\hat{a}_s^2}{\varepsilon}\left[\frac{22}{3}C_AC_F - \frac{8}{3}C_FT_FN_F\right] + \hat{a}_s^3\left[\frac{1}{\varepsilon^2}\left(-\frac{484}{27}C_A^2C_F + \frac{352}{27}C_AC_FT_FN_F - \frac{64}{27}C_FT_F^2N_F^2\right) + \frac{1}{\varepsilon}\left(\frac{3578}{81}C_A^2C_F - \frac{308}{9}C_AC_F^2 - \frac{1664}{81}C_AC_FT_FN_F + \frac{64}{9}C_F^2T_FN_F + \frac{32}{81}C_FT_F^2N_F^2\right)\right]. \tag{2.45}$$

The non-singlet Wilson coefficients are renormalized via

$$C_{i,q}^{\rm NS} = Z_{qq}^{\rm NS}\hat{C}_{i,q}^{\rm NS} \tag{2.46}$$

and the singlet Wilson coefficients via

$$\begin{pmatrix} C_{i,q}^{\rm S} \\ C_{i,g} \end{pmatrix} = Z^{{\rm S}\,T} \cdot \begin{pmatrix} \hat{C}_{i,q}^{\rm S} \\ \hat{C}_{i,g} \end{pmatrix}, \tag{2.47}$$

with

$$Z_{qq}^{\rm NS} = 1 + a_s\frac{\gamma_{qq}^{(0),\rm NS}}{\varepsilon} + a_s^2\left[\frac{1}{\varepsilon^2}\left(\frac{1}{2}\gamma_{qq}^{(0),\rm NS\,2} + \beta_0\gamma_{qq}^{(0),\rm NS}\right) + \frac{1}{2\varepsilon}\gamma_{qq}^{(0),\rm NS}\right] + a_s^3\left[\frac{1}{\varepsilon^3}\left(\frac{1}{6}\gamma_{qq}^{(0),\rm NS\,3} + \beta_0\gamma_{qq}^{(0),\rm NS\,2} + \frac{4}{3}\beta_0^2\gamma_{qq}^{(0),\rm NS}\right) + \frac{1}{\varepsilon^2}\left(\frac{1}{2}\gamma_{qq}^{(0),\rm NS}\gamma_{qq}^{(1),\rm NS} + \frac{2}{3}\beta_0\gamma_{qq}^{(1),\rm NS} + \frac{2}{3}\beta_1\gamma_{qq}^{(0),\rm NS}\right) + \frac{1}{3\varepsilon}\gamma_{qq}^{(2),\rm NS}\right] + O\left(a_s^4\right), \tag{2.48}$$

$$Z_{ij}^{\rm S} = \delta_{ij} + a_s\frac{\gamma_{ij}^{(0)}}{\varepsilon} + a_s^2\left[\frac{1}{\varepsilon^2}\left(\frac{1}{2}\gamma_{il}^{(0)}\gamma_{lj}^{(0)} + \beta_0\gamma_{ij}^{(0)}\right) + \frac{1}{2\varepsilon}\gamma_{ij}^{(1)}\right] + a_s^3\left[\frac{1}{\varepsilon^3}\left(\frac{1}{6}\gamma_{il}^{(0)}\gamma_{lk}^{(0)}\gamma_{kj}^{(0)} + \beta_0\gamma_{il}^{(0)}\gamma_{lj}^{(0)} + \frac{4}{3}\beta_0^2\gamma_{ij}^{(0)}\right) + \frac{1}{\varepsilon^2}\left(\frac{1}{6}\gamma_{il}^{(1)}\gamma_{lj}^{(0)} + \frac{1}{3}\gamma_{il}^{(0)}\gamma_{lj}^{(1)} + \frac{2}{3}\beta_0\gamma_{ij}^{(1)} + \frac{2}{3}\beta_1\gamma_{ij}^{(0)}\right) + \frac{1}{3\varepsilon}\gamma_{ij}^{(2)}\right] + O\left(a_s^4\right). \tag{2.49}$$

The pole terms of the Wilson coefficients are predicted by the anomalous dimensions and the expansion coefficients of the unrenormalized Wilson coefficients in lower orders, where in the polarized case one has to refer to the corresponding polarized quantities, e.g. for the anomalous dimensions γ_{ij} to $\Delta\gamma_{ij}$, etc.

$$c_{i,q}^{{\rm NS},(1,-1)} = -\gamma_{qq}^{{\rm NS},(0)}, \tag{2.50}$$

$$c_{i,q}^{{\rm NS},(2,-2)} = \gamma_{qq}^{{\rm NS},(0)}\left(\beta_0 + \frac{1}{2}\gamma_{qq}^{{\rm NS},(0)}\right), \tag{2.51}$$

$$c_{i,q}^{{\rm NS},(2,-1)} = -c_{i,q}^{{\rm NS},(1,0)}\left(\gamma_{qq}^{{\rm NS},(0)} + 2\beta_0\right) - \frac{1}{2}\gamma_{qq}^{{\rm NS},(1)}, \tag{2.52}$$

$$c_{i,q}^{\mathrm{NS},(3,-3)} = -\gamma_{qq}^{\mathrm{NS},(0)} \left(\frac{4}{3}\beta_0^2 + \beta_0 \gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{6}\, \gamma_{qq}^{\mathrm{NS},(0)\,2} \right), \tag{2.53}$$

$$\begin{aligned} c_{i,q}^{\mathrm{NS},(3,-2)} &= 4\beta_0^2 c_{i,q}^{\mathrm{NS},(1,0)} + \frac{1}{3}\beta_1 \gamma_{qq}^{\mathrm{NS},(0)} \\ &\quad + 3\beta_0 c_{i,q}^{\mathrm{NS},(1,0)} \gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2} c_{i,q}^{\mathrm{NS},(1,0)}\, \gamma_{qq}^{\mathrm{NS},(0)\,2} + \frac{4}{3}\beta_0 \gamma_{qq}^{\mathrm{NS},(1)} + \frac{1}{2}\gamma_{qq}^{\mathrm{NS},(0)} \gamma_{qq}^{\mathrm{NS},(1)}, \end{aligned} \tag{2.54}$$

$$\begin{aligned} c_{i,q}^{\mathrm{NS},(3,-1)} &= -\beta_1 c_{i,q}^{\mathrm{NS},(1,0)} - 4\beta_0^2 c_{i,q}^{\mathrm{NS},(1,1)} - 4\beta_0 c_{i,q}^{\mathrm{NS},(2,0)} - 3\beta_0 c_{i,q}^{\mathrm{NS},(1,1)} \gamma_{qq}^{\mathrm{NS},(0)} - c_{i,q}^{\mathrm{NS},(2,0)} \gamma_{qq}^{\mathrm{NS},(0)} \\ &\quad - \frac{1}{2} c_{i,q}^{\mathrm{NS},(1,1)}\, \gamma_{qq}^{\mathrm{NS},(0)\,2} - \frac{1}{2} c_{i,q}^{\mathrm{NS},(1,0)} \gamma_{qq}^{\mathrm{NS},(1)} - \frac{1}{3}\gamma_{qq}^{\mathrm{NS},(2)}, \end{aligned} \tag{2.55}$$

$$c_{i,q}^{\mathrm{PS},(2,-2)} = \frac{1}{2}\gamma_{gq}^{(0)} \gamma_{qg}^{(0)}, \tag{2.56}$$

$$c_{i,q}^{\mathrm{PS},(2,-1)} = -\gamma_{gq}^{(0)} c_{i,g}^{(1,0)} - \frac{1}{2}\gamma_{qq}^{\mathrm{PS},(1)}, \tag{2.57}$$

$$c_{i,q}^{\mathrm{PS},(3,-3)} = -\beta_0 \gamma_{gq}^{(0)} \gamma_{qg}^{(0)} - \frac{1}{6}\gamma_{gg}^{(0)} \gamma_{gq}^{(0)} \gamma_{qg}^{(0)} - \frac{1}{3}\gamma_{qq}^{\mathrm{NS},(0)} \gamma_{gq}^{(0)} \gamma_{qg}^{(0)}, \tag{2.58}$$

$$\begin{aligned} c_{i,q}^{\mathrm{PS},(3,-2)} &= 3\beta_0 \gamma_{gq}^{(0)} c_{i,g}^{(1,0)} + \frac{1}{2}\gamma_{gg}^{(0)} \gamma_{gq}^{(0)} c_{i,g}^{(1,0)} + \frac{1}{2}\gamma_{gq}^{(0)} \gamma_{qg}^{(0)} c_{i,q}^{\mathrm{NS},(1,0)} + \frac{1}{6}\gamma_{gq}^{(1)} \gamma_{qg}^{(0)} + \frac{1}{3}\gamma_{gq}^{(0)} \gamma_{qg}^{(1)} \\ &\quad + \frac{1}{2}\gamma_{gq}^{(0)} \gamma_{qq}^{\mathrm{NS},(0)} c_{i,g}^{(1,0)} + \frac{4}{3}\beta_0 \gamma_{qq}^{\mathrm{PS},(1)} + \frac{1}{2}\gamma_{qq}^{\mathrm{NS},(0)} \gamma_{qq}^{\mathrm{PS},(1)}, \end{aligned} \tag{2.59}$$

$$\begin{aligned} c_{i,q}^{\mathrm{PS},(3,-1)} &= -4\beta_0 c_{i,q}^{\mathrm{PS},(2,0)} - 3\beta_0 \gamma_{gq}^{(0)} c_{i,g}^{(1,1)} - \gamma_{gq}^{(0)} c_{i,g}^{(2,0)} - \frac{1}{2}\gamma_{gg}^{(0)} \gamma_{gq}^{(0)} c_{i,g}^{(1,1)} - \frac{1}{2}\gamma_{gq}^{(1)} c_{i,g}^{(1,0)} \\ &\quad - \frac{1}{2}\gamma_{gq}^{(0)} \gamma_{qg}^{(0)} c_{i,q}^{\mathrm{NS},(1,1)} - \gamma_{qq}^{\mathrm{NS},(0)} c_{i,q}^{\mathrm{PS},(2,0)} - \frac{1}{2}\gamma_{gq}^{(0)} \gamma_{qq}^{\mathrm{NS},(0)} c_{i,g}^{(1,1)} - \frac{1}{2}\gamma_{qq}^{\mathrm{PS},(1)} c_{i,q}^{\mathrm{NS},(1,0)} \\ &\quad - \frac{1}{3}\gamma_{qq}^{\mathrm{PS},(2)}, \end{aligned} \tag{2.60}$$

and

$$c_{i,g}^{(1,-1)} = -\gamma_{qg}^{(0)}, \tag{2.61}$$

$$c_{i,g}^{(2,-2)} = \beta_0 \gamma_{qg}^{(0)} + \frac{1}{2}\gamma_{gg}^{(0)} \gamma_{qg}^{(0)} + \frac{1}{2}\gamma_{qg}^{(0)} \gamma_{qq}^{\mathrm{NS},(0)}, \tag{2.62}$$

$$c_{i,g}^{(2,-1)} = -2\beta_0 c_{i,g}^{(1,0)} - \gamma_{gg}^{(0)} c_{i,g}^{(1,0)} - \gamma_{qg}^{(0)} c_{i,q}^{\mathrm{NS},(1,0)} - \frac{1}{2}\gamma_{qg}^{(1)}, \tag{2.63}$$

$$\begin{aligned} c_{i,g}^{(3,-3)} &= -\frac{4}{3}\beta_0^2 \gamma_{qg}^{(0)} - \beta_0 \gamma_{gg}^{(0)} \gamma_{qg}^{(0)} - \frac{1}{6}\, \gamma_{gg}^{(0)\,2} \gamma_{qg}^{(0)} - \frac{1}{6}\gamma_{gq}^{(0)}\, \gamma_{qg}^{(0)\,2} - \beta_0 \gamma_{qg}^{(0)} \gamma_{qq}^{\mathrm{NS},(0)} \\ &\quad - \frac{1}{6}\gamma_{gg}^{(0)} \gamma_{qg}^{(0)} \gamma_{qq}^{\mathrm{NS},(0)} - \frac{1}{6}\gamma_{qg}^{(0)}\, \gamma_{qq}^{\mathrm{NS},(0)\,2}, \end{aligned} \tag{2.64}$$

$$\begin{aligned} c_{i,g}^{(3,-2)} &= 4\beta_0^2 c_{i,g}^{(1,0)} + 3\beta_0 c_{i,g}^{(1,0)} \gamma_{gg}^{(0)} + \frac{1}{2} c_{i,g}^{(1,0)}\, \gamma_{gg}^{(0)\,2} + \frac{1}{3}\beta_1 \gamma_{qg}^{(0)} + 3\beta_0 c_{i,q}^{\mathrm{NS},(1,0)} \gamma_{qg}^{(0)} \\ &\quad + \frac{1}{2} c_{i,q}^{\mathrm{NS},(1,0)} \gamma_{gg}^{(0)} \gamma_{qg}^{(0)} + \frac{1}{6}\gamma_{gg}^{(1)} \gamma_{qg}^{(0)} + \frac{1}{2} c_{i,g}^{(1,0)} \gamma_{gq}^{(0)} \gamma_{qg}^{(0)} + \frac{4}{3}\beta_0 \gamma_{qg}^{(1)} + \frac{1}{3}\gamma_{gg}^{(0)} \gamma_{qg}^{(1)} \\ &\quad + \frac{1}{2} c_{i,q}^{\mathrm{NS},(1,0)} \gamma_{qg}^{(0)} \gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{6}\gamma_{qg}^{(1)} \gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{3}\gamma_{qg}^{(0)} \gamma_{qq}^{\mathrm{NS},(1)} + \frac{1}{3}\gamma_{qg}^{(0)} \gamma_{qq}^{\mathrm{PS},(1)}, \end{aligned} \tag{2.65}$$

$$\begin{aligned} c_{i,g}^{(3,-1)} &= -\beta_1 c_{i,g}^{(1,0)} - 4\beta_0^2 c_{i,g}^{(1,1)} - 4\beta_0 c_{i,g}^{(2,0)} - 3\beta_0 c_{i,g}^{(1,1)} \gamma_{gg}^{(0)} - c_{i,g}^{(2,0)} \gamma_{gg}^{(0)} \\ &\quad - \frac{1}{2} c_{i,g}^{(1,1)}\, \gamma_{gg}^{(0)\,2} - \frac{1}{2} c_{i,g}^{(1,0)} \gamma_{gg}^{(1)} - 3\beta_0 c_{i,q}^{\mathrm{NS},(1,1)} \gamma_{qg}^{(0)} - c_{i,q}^{\mathrm{NS},(2,0)} \gamma_{qg}^{(0)} - c_{i,q}^{\mathrm{PS},(2,0)} \gamma_{qg}^{(0)} \\ &\quad - \frac{1}{2} c_{i,q}^{\mathrm{NS},(1,1)} \gamma_{gg}^{(0)} \gamma_{qg}^{(0)} - \frac{1}{2} c_{i,g}^{(1,1)} \gamma_{gq}^{(0)} \gamma_{qg}^{(0)} - \frac{1}{2} c_{i,q}^{\mathrm{NS},(1,0)} \gamma_{qg}^{(1)} - \frac{1}{3}\gamma_{qg}^{(2)} \\ &\quad - \frac{1}{2} c_{i,q}^{\mathrm{NS},(1,1)} \gamma_{qg}^{(0)} \gamma_{qq}^{\mathrm{NS},(0)}. \end{aligned} \tag{2.66}$$

For the longitudinal structure function F_L the poles are predicted by

$$c_{F_L,q}^{\mathrm{NS},(2,-1)} = -c_{F_L,q}^{\mathrm{NS},(1,0)}\left(\gamma_{qq}^{\mathrm{NS},(0)}+2\beta_0\right), \tag{2.67}$$

$$c_{F_L,q}^{\mathrm{NS},(3,-2)} = 4\beta_0^2 c_{F_L,q}^{\mathrm{NS},(1,0)}+3\beta_0 c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{NS},(0)}+\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,0)}\,\gamma_{qq}^{\mathrm{NS},(0)\,2}, \tag{2.68}$$

$$\begin{aligned}c_{F_L,q}^{\mathrm{NS},(3,-1)} =& -\beta_1 c_{F_L,q}^{\mathrm{NS},(1,0)}-4\beta_0^2 c_{F_L,q}^{\mathrm{NS},(1,1)}-4\beta_0 c_{F_L,q}^{\mathrm{NS},(2,0)}-3\beta_0 c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{qq}^{\mathrm{NS},(0)}-c_{F_L,q}^{\mathrm{NS},(2,0)}\gamma_{qq}^{\mathrm{NS},(0)}\\&-\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,1)}\,\gamma_{qq}^{\mathrm{NS},(0)\,2}-\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{NS},(1)},\end{aligned} \tag{2.69}$$

$$c_{F_L,q}^{\mathrm{PS},(2,-1)} = -\gamma_{gq}^{(0)}c_{F_L,g}^{(1,0)}, \tag{2.70}$$

$$c_{F_L,q}^{\mathrm{PS},(3,-2)} = 3\beta_0\gamma_{gq}^{(0)}c_{F_L,g}^{(1,0)}+\frac{1}{2}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)}c_{F_L,g}^{(1,0)}+\frac{1}{2}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}c_{F_L,q}^{\mathrm{NS},(1,0)}+\frac{1}{2}\gamma_{gq}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}c_{F_L,g}^{(1,0)}, \tag{2.71}$$

$$\begin{aligned}c_{F_L,q}^{\mathrm{PS},(3,-1)} =& -4\beta_0 c_{F_L,q}^{\mathrm{PS},(2,0)}-3\beta_0\gamma_{gq}^{(0)}c_{F_L,g}^{(1,1)}-\gamma_{gq}^{(0)}c_{F_L,g}^{(2,0)}-\frac{1}{2}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)}c_{F_L,g}^{(1,1)}-\frac{1}{2}\gamma_{gq}^{(1)}c_{F_L,g}^{(1,0)}\\&-\frac{1}{2}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}c_{F_L,q}^{\mathrm{NS},(1,1)}-\gamma_{qq}^{\mathrm{NS},(0)}c_{F_L,q}^{\mathrm{PS},(2,0)}-\frac{1}{2}\gamma_{gq}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}c_{F_L,g}^{(1,1)}-\frac{1}{2}\gamma_{qq}^{\mathrm{PS},(1)}c_{F_L,q}^{\mathrm{NS},(1,0)},\end{aligned} \tag{2.72}$$

$$c_{F_L,g}^{(2,-1)} = -2\beta_0 c_{F_L,g}^{(1,0)}-\gamma_{gg}^{(0)}c_{F_L,g}^{(1,0)}-\gamma_{qg}^{(0)}c_{F_L,q}^{\mathrm{NS},(1,0)}, \tag{2.73}$$

$$\begin{aligned}c_{F_L,g}^{(3,-2)} =& 4\beta_0^2 c_{F_L,g}^{(1,0)}+3\beta_0 c_{F_L,g}^{(1,0)}\gamma_{gg}^{(0)}+\frac{1}{2}c_{F_L,g}^{(1,0)}\,\gamma_{gg}^{(0)\,2}+3\beta_0 c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(0)}+\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)}\\&+\frac{1}{2}c_{F_L,g}^{(1,0)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}+\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)},\end{aligned} \tag{2.74}$$

$$\begin{aligned}c_{F_L,g}^{(3,-1)} =& -\beta_1 c_{F_L,g}^{(1,0)}-4\beta_0^2 c_{F_L,g}^{(1,1)}-4\beta_0 c_{F_L,g}^{(2,0)}-3\beta_0 c_{F_L,g}^{(1,1)}\gamma_{gg}^{(0)}-c_{F_L,g}^{(2,0)}\gamma_{gg}^{(0)}-\frac{1}{2}c_{F_L,g}^{(1,1)}\,\gamma_{gg}^{(0)\,2}\\&-\frac{1}{2}c_{F_L,g}^{(1,0)}\gamma_{gg}^{(1)}-3\beta_0 c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(0)}-c_{F_L,q}^{\mathrm{NS},(2,0)}\gamma_{qg}^{(0)}-c_{F_L,q}^{\mathrm{PS},(2,0)}\gamma_{qg}^{(0)}-\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)}\\&-\frac{1}{2}c_{F_L,g}^{(1,1)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}-\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(1)}-\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}.\end{aligned} \tag{2.75}$$

The renormalized Wilson coefficients for the structure functions F_2, F_3 and g_1 have the following structure

$$\begin{aligned}C_{i,q}^{\mathrm{NS}} =& 1+a_s\left\{\ln\left(\frac{Q^2}{\mu^2}\right)\left[-\frac{1}{2}\gamma_{qq}^{\mathrm{NS},(0)}\right]+c_{i,q}^{\mathrm{NS},(1,0)}\right\}+a_s^2\left\{\ln^2\left(\frac{Q^2}{\mu^2}\right)\left[\frac{1}{4}\beta_0\gamma_{qq}^{\mathrm{NS},(0)}\right.\right.\\&\left.+\frac{1}{8}\,\gamma_{qq}^{\mathrm{NS},(0)\,2}\right]-\ln\left(\frac{Q^2}{\mu^2}\right)\left[\beta_0 c_{i,q}^{\mathrm{NS},(1,0)}+\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{NS},(0)}+\frac{1}{2}\gamma_{qq}^{\mathrm{NS},(1)}\right]+2\beta_0 c_{i,q}^{\mathrm{NS},(1,1)}\\&\left.+c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qq}^{\mathrm{NS},(0)}+c_{i,q}^{\mathrm{NS},(2,0)}\right\}+a_s^3\left\{-\ln^3\left(\frac{Q^2}{\mu^2}\right)\left[\frac{1}{6}\beta_0^2\gamma_{qq}^{\mathrm{NS},(0)}+\frac{1}{8}\beta_0\,\gamma_{qq}^{\mathrm{NS},(0)\,2}\right.\right.\\&\left.+\frac{1}{48}\,\gamma_{qq}^{\mathrm{NS},(0)\,3}\right]+\ln^2\left(\frac{Q^2}{\mu^2}\right)\left[\beta_0^2 c_{i,q}^{\mathrm{NS},(1,0)}+\frac{1}{4}\beta_1\gamma_{qq}^{\mathrm{NS},(0)}+\frac{3}{4}\beta_0 c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{NS},(0)}\right.\\&\left.+\frac{1}{8}c_{i,q}^{\mathrm{NS},(1,0)}\,\gamma_{qq}^{\mathrm{NS},(0)\,2}+\frac{1}{2}\beta_0\gamma_{qq}^{\mathrm{NS},(1)}+\frac{1}{4}\gamma_{qq}^{\mathrm{NS},(0)}\gamma_{qq}^{\mathrm{NS},(1)}\right]-\ln\left(\frac{Q^2}{\mu^2}\right)\left[\beta_1 c_{i,q}^{\mathrm{NS},(1,0)}\right.\\&+4\beta_0^2 c_{i,q}^{\mathrm{NS},(1,1)}+2\beta_0 c_{i,q}^{\mathrm{NS},(2,0)}+3\beta_0 c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qq}^{\mathrm{NS},(0)}+\frac{1}{2}c_{i,q}^{\mathrm{NS},(2,0)}\gamma_{qq}^{\mathrm{NS},(0)}+\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,1)}\,\gamma_{qq}^{\mathrm{NS},(0)\,2}\\&\left.+\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{NS},(1)}+\frac{1}{2}\gamma_{qq}^{\mathrm{NS},(2)}\right]+\beta_1 c_{i,q}^{\mathrm{NS},(1,1)}+4\beta_0^2 c_{i,q}^{\mathrm{NS},(1,2)}+4\beta_0 c_{i,q}^{\mathrm{NS},(2,1)}\\&\left.+3\beta_0 c_{i,q}^{\mathrm{NS},(1,2)}\gamma_{qq}^{\mathrm{NS},(0)}+c_{i,q}^{\mathrm{NS},(2,1)}\gamma_{qq}^{\mathrm{NS},(0)}+\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,2)}\,\gamma_{qq}^{\mathrm{NS},(0)\,2}+\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qq}^{\mathrm{NS},(1)}+c_{i,q}^{\mathrm{NS},(3,0)}\right\}\\&+O\left(a_s^4\right),\end{aligned} \tag{2.76}$$

$$
\begin{aligned}
C_{i,q}^{\mathrm{PS}} = a_s^2 \Bigg\{ & \ln^2\left(\frac{Q^2}{\mu^2}\right)\left[\frac{1}{8}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}\right] - \ln\left(\frac{Q^2}{\mu^2}\right)\left[\frac{1}{2}\left(c_{i,g}^{(1,0)}\gamma_{gq}^{(0)}\right) + \frac{1}{2}\gamma_{qq}^{\mathrm{PS},(1)}\right] + c_{i,g}^{(1,1)}\gamma_{gq}^{(0)} \\
& + c_{i,q}^{\mathrm{PS},(2,0)} \Bigg\} + a_s^3 \Bigg\{ - \ln^3\left(\frac{Q^2}{\mu^2}\right)\left[\frac{1}{8}\left(\beta_0\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}\right) + \frac{1}{48}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{24}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}\right] \\
& + \ln^2\left(\frac{Q^2}{\mu^2}\right)\left[\frac{3}{4}\beta_0 c_{i,g}^{(1,0)}\gamma_{gq}^{(0)} + \frac{1}{8}c_{i,g}^{(1,0)}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)} + \frac{1}{8}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{8}\gamma_{gq}^{(1)}\gamma_{qg}^{(0)} \right.\\
& \left. + \frac{1}{8}\gamma_{gq}^{(0)}\gamma_{qg}^{(1)} + \frac{1}{8}c_{i,g}^{(1,0)}\gamma_{gq}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}\beta_0\gamma_{qq}^{\mathrm{PS},(1)} + \frac{1}{4}\gamma_{qq}^{\mathrm{NS},(0)}\gamma_{qq}^{\mathrm{PS},(1)}\right] - \ln\left(\frac{Q^2}{\mu^2}\right) \\
& \times\left[2\beta_0 c_{i,q}^{\mathrm{PS},(2,0)} + 3\beta_0 c_{i,g}^{(1,1)}\gamma_{gq}^{(0)} + \frac{1}{2}c_{i,g}^{(2,0)}\gamma_{gq}^{(0)} + \frac{1}{2}c_{i,g}^{(1,1)}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)} + \frac{1}{2}c_{i,g}^{(1,0)}\gamma_{gq}^{(1)} \right.\\
& \left. + \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{i,q}^{\mathrm{PS},(2,0)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}c_{i,g}^{(1,1)}\gamma_{gq}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{PS},(1)} + \frac{1}{2}\gamma_{qq}^{\mathrm{PS},(2)}\right] \\
& + 4\beta_0 c_{i,q}^{\mathrm{PS},(2,1)} + 3\beta_0 c_{i,g}^{(1,2)}\gamma_{gq}^{(0)} + c_{i,g}^{(2,1)}\gamma_{gq}^{(0)} + \frac{1}{2}c_{i,g}^{(1,2)}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)} + \frac{1}{2}c_{i,g}^{(1,1)}\gamma_{gq}^{(1)} \\
& + \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,2)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + c_{i,q}^{\mathrm{PS},(2,1)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}c_{i,g}^{(1,2)}\gamma_{gq}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qq}^{\mathrm{PS},(1)} + c_{i,q}^{\mathrm{PS},(3,0)} \Bigg\} \\
& + O\left(a_s^4\right), \qquad (2.77)
\end{aligned}
$$

$$
\begin{aligned}
C_{i,g} = a_s & \left\{ - \ln\left(\frac{Q^2}{\mu^2}\right)\left[\frac{1}{2}\gamma_{qg}^{(0)}\right] + c_{i,g}^{(1,0)}\right\} + a_s^2 \Bigg\{ \ln^2\left(\frac{Q^2}{\mu^2}\right)\left[\frac{1}{4}\beta_0\gamma_{qg}^{(0)} + \frac{1}{8}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)} \right.\\
& \left. + \frac{1}{8}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}\right] - \ln\left(\frac{Q^2}{\mu^2}\right)\left[\beta_0 c_{i,g}^{(1,0)} + \frac{1}{2}c_{i,g}^{(1,0)}\gamma_{gg}^{(0)} + \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(0)} + \frac{1}{2}\gamma_{qg}^{(1)}\right] \\
& + 2\beta_0 c_{i,g}^{(1,1)} + c_{i,g}^{(1,1)}\gamma_{gg}^{(0)} + c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(0)} + c_{i,g}^{(2,0)} \Bigg\} + a_s^3 \Bigg\{ - \ln^3\left(\frac{Q^2}{\mu^2}\right)\left[\frac{1}{6}\beta_0^2\gamma_{qg}^{(0)} \right.\\
& + \frac{1}{8}\beta_0\gamma_{gg}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{48}\gamma_{gg}^{(0)}{}^{2}\gamma_{qg}^{(0)} + \frac{1}{48}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}{}^{2} + \frac{1}{8}\beta_0\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{48}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} \\
& \left. + \frac{1}{48}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}{}^{2}\right] + \ln^2\left(\frac{Q^2}{\mu^2}\right)\left[\beta_0^2 c_{i,g}^{(1,0)} + \frac{3}{4}\beta_0 c_{i,g}^{(1,0)}\gamma_{gg}^{(0)} + \frac{1}{8}c_{i,g}^{(1,0)}\gamma_{gg}^{(0)}{}^{2} + \frac{1}{4}\beta_1\gamma_{qg}^{(0)} \right.\\
& + \frac{3}{4}\beta_0 c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(0)} + \frac{1}{8}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{8}\gamma_{gg}^{(1)}\gamma_{qg}^{(0)} + \frac{1}{8}c_{i,g}^{(1,0)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{2}\beta_0\gamma_{qg}^{(1)} + \frac{1}{8}\gamma_{gg}^{(0)}\gamma_{qg}^{(1)} \\
& \left. + \frac{1}{8}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{8}\gamma_{qg}^{(1)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{8}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(1)} + \frac{1}{8}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{PS},(1)}\right] \\
& - \ln\left(\frac{Q^2}{\mu^2}\right)\left[\left(\beta_1 c_{i,g}^{(1,0)}\right) + 4\beta_0^2 c_{i,g}^{(1,1)} + 2\beta_0 c_{i,g}^{(2,0)} + 3\beta_0 c_{i,g}^{(1,1)}\gamma_{gg}^{(0)} + \frac{1}{2}c_{i,g}^{(2,0)}\gamma_{gg}^{(0)} \right.\\
& + \frac{1}{2}c_{i,g}^{(1,1)}\gamma_{gg}^{(0)}{}^{2} + \frac{1}{2}c_{i,g}^{(1,0)}\gamma_{gg}^{(1)} + 3\beta_0 c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{i,q}^{\mathrm{NS},(2,0)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{i,q}^{\mathrm{PS},(2,0)}\gamma_{qg}^{(0)} \\
& \left. + \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{i,g}^{(1,1)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(1)} + \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}\gamma_{qg}^{(2)}\right] \\
& + \beta_1 c_{i,g}^{(1,1)} + 4\beta_0^2 c_{i,g}^{(1,2)} + 4\beta_0 c_{i,g}^{(2,1)} + 3\beta_0 c_{i,g}^{(1,2)}\gamma_{gg}^{(0)} + c_{i,g}^{(2,1)}\gamma_{gg}^{(0)} + \frac{1}{2}c_{i,g}^{(1,2)}\gamma_{gg}^{(0)}{}^{2} \\
& + \frac{1}{2}c_{i,g}^{(1,1)}\gamma_{gg}^{(1)} + 3\beta_0 c_{i,q}^{\mathrm{NS},(1,2)}\gamma_{qg}^{(0)} + c_{i,q}^{\mathrm{NS},(2,1)}\gamma_{qg}^{(0)} + c_{i,q}^{\mathrm{PS},(2,1)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,2)}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)} \\
& + \frac{1}{2}c_{i,g}^{(1,2)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(1)} + \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,2)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} + c_{i,g}^{(3,0)} \Bigg\} + O(a_s^4)\,. \qquad (2.78)
\end{aligned}
$$

Accordingly, one obtains for the structure function F_L,

$$
\begin{aligned}
C^{\rm NS}_{F_L,q} = {} & a_s\left\{c^{{\rm NS},(1,0)}_{F_L,q}\right\} + a_s^2\Bigg\{-\ln\left(\frac{Q^2}{\mu^2}\right)\left[\beta_0 c^{{\rm NS},(1,0)}_{F_L,q} + \frac{1}{2}c^{{\rm NS},(1,0)}_{F_L,q}\gamma^{{\rm NS},(0)}_{qq}\right] + 2\beta_0 c^{{\rm NS},(1,1)}_{F_L,q} \\
& + c^{{\rm NS},(1,1)}_{F_L,q}\gamma^{{\rm NS},(0)}_{qq} + c^{{\rm NS},(2,0)}_{F_L,q}\Bigg\} + a_s^3\Bigg\{\ln^2\left(\frac{Q^2}{\mu^2}\right)\left[\beta_0^2 c^{{\rm NS},(1,0)}_{F_L,q} + \frac{3}{4}\beta_0 c^{{\rm NS},(1,0)}_{F_L,q}\gamma^{{\rm NS},(0)}_{qq} + \right. \\
& \left. + \frac{1}{8}c^{{\rm NS},(1,0)}_{F_L,q}\,\gamma^{{\rm NS},(0)\,2}_{qq}\right] - \ln\left(\frac{Q^2}{\mu^2}\right)\left[\beta_1 c^{{\rm NS},(1,0)}_{F_L,q} + 4\beta_0^2 c^{{\rm NS},(1,1)}_{F_L,q} + 2\beta_0 c^{{\rm NS},(2,0)}_{F_L,q}\right. \\
& \left. + 3\beta_0 c^{{\rm NS},(1,1)}_{F_L,q}\gamma^{{\rm NS},(0)}_{qq} + \frac{1}{2}c^{{\rm NS},(2,0)}_{F_L,q}\gamma^{{\rm NS},(0)}_{qq} + \frac{1}{2}c^{{\rm NS},(1,1)}_{F_L,q}\,\gamma^{{\rm NS},(0)\,2}_{qq} + \frac{1}{2}c^{{\rm NS},(1,0)}_{F_L,q}\gamma^{{\rm NS},(1)}_{qq}\right] \\
& + \beta_1 c^{{\rm NS},(1,1)}_{F_L,q} + 4\beta_0^2 c^{{\rm NS},(1,2)}_{F_L,q} + 4\beta_0 c^{{\rm NS},(2,1)}_{F_L,q} + 3\beta_0 c^{{\rm NS},(1,2)}_{F_L,q}\gamma^{{\rm NS},(0)}_{qq} + c^{{\rm NS},(2,1)}_{F_L,q}\gamma^{{\rm NS},(0)}_{qq} \\
& + \frac{1}{2}c^{{\rm NS},(1,2)}_{F_L,q}\,\gamma^{{\rm NS},(0)\,2}_{qq} + \frac{1}{2}c^{{\rm NS},(1,1)}_{F_L,q}\gamma^{{\rm NS},(1)}_{qq} + c^{{\rm NS},(3,0)}_{F_L,q}\Bigg\} + O\left(a_s^4\right),
\end{aligned} \tag{2.79}
$$

$$
\begin{aligned}
C^{\rm PS}_{F_L,q} = {} & a_s^2\left\{-\ln\left(\frac{Q^2}{\mu^2}\right)\left[\frac{1}{2}\left(c^{(1,0)}_{F_L,g}\gamma^{(0)}_{gq}\right)\right] + c^{(1,1)}_{F_L,g}\gamma^{(0)}_{gq} + c^{{\rm PS},(2,0)}_{F_L,q}\right\} \\
& + a_s^3\Bigg\{\ln^2\left(\frac{Q^2}{\mu^2}\right)\left[\frac{3}{4}\beta_0 c^{(1,0)}_{F_L,g}\gamma^{(0)}_{gq} + \frac{1}{8}c^{(1,0)}_{F_L,g}\gamma^{(0)}_{gg}\gamma^{(0)}_{gq} + \frac{1}{8}c^{{\rm NS},(1,0)}_{F_L,q}\gamma^{(0)}_{gq}\gamma^{(0)}_{qg}\right. \\
& \left. + \frac{1}{8}c^{(1,0)}_{F_L,g}\gamma^{(0)}_{gq}\gamma^{{\rm NS},(0)}_{qq}\right] - \ln\left(\frac{Q^2}{\mu^2}\right)\left[2\beta_0 c^{{\rm PS},(2,0)}_{F_L,q} + 3\beta_0 c^{(1,1)}_{F_L,g}\gamma^{(0)}_{gq} + \frac{1}{2}c^{(2,0)}_{F_L,g}\gamma^{(0)}_{gq}\right. \\
& + \frac{1}{2}c^{(1,1)}_{F_L,g}\gamma^{(0)}_{gg}\gamma^{(0)}_{gq} + \frac{1}{2}c^{(1,0)}_{F_L,g}\gamma^{(1)}_{gq} + \frac{1}{2}c^{{\rm NS},(1,1)}_{F_L,q}\gamma^{(0)}_{gq}\gamma^{(0)}_{qg} + \frac{1}{2}c^{{\rm PS},(2,0)}_{F_L,q}\gamma^{{\rm NS},(0)}_{qq} \\
& \left. + \frac{1}{2}c^{(1,1)}_{F_L,g}\gamma^{(0)}_{gq}\gamma^{{\rm NS},(0)}_{qq} + \frac{1}{2}c^{{\rm NS},(1,0)}_{F_L,q}\gamma^{{\rm PS},(1)}_{qq}\right] + 4\beta_0 c^{{\rm PS},(2,1)}_{F_L,q} + 3\beta_0 c^{(1,2)}_{F_L,g}\gamma^{(0)}_{gq} \\
& + c^{(2,1)}_{F_L,g}\gamma^{(0)}_{gq} + \frac{1}{2}c^{(1,2)}_{F_L,g}\gamma^{(0)}_{gg}\gamma^{(0)}_{gq} + \frac{1}{2}c^{(1,1)}_{F_L,g}\gamma^{(1)}_{gq} + \frac{1}{2}c^{{\rm NS},(1,2)}_{F_L,q}\gamma^{(0)}_{gq}\gamma^{(0)}_{qg} \\
& + c^{{\rm PS},(2,1)}_{F_L,q}\gamma^{{\rm NS},(0)}_{qq} + \frac{1}{2}c^{(1,2)}_{F_L,g}\gamma^{(0)}_{gq}\gamma^{{\rm NS},(0)}_{qq} + \frac{1}{2}c^{{\rm NS},(1,1)}_{F_L,q}\gamma^{{\rm PS},(1)}_{qq} + c^{{\rm PS},(L)}_{q,(3,0)}\Bigg\} + O\left(a_s^4\right),
\end{aligned} \tag{2.80}
$$

$$
\begin{aligned}
C_{F_L,g} = {} & a_s\left\{c^{(1,0)}_{F_L,g}\right\} + a_s^2\Bigg\{-\ln\left(\frac{Q^2}{\mu^2}\right)\left[\beta_0 c^{(1,0)}_{F_L,g} + \frac{1}{2}c^{(1,0)}_{F_L,g}\gamma^{(0)}_{gg} + \frac{1}{2}c^{{\rm NS},(1,0)}_{F_L,q}\gamma^{(0)}_{qg}\right] \\
& + 2\beta_0 c^{(1,1)}_{F_L,g} + c^{(1,1)}_{F_L,g}\gamma^{(0)}_{gg} + c^{{\rm NS},(1,1)}_{F_L,q}\gamma^{(0)}_{qg} + c^{(2,0)}_{F_L,g}\Bigg\} + a_s^3\Bigg\{\ln^2\left(\frac{Q^2}{\mu^2}\right)\left[\beta_0^2 c^{(1,0)}_{F_L,g}\right. \\
& + \frac{3}{4}\beta_0 c^{(1,0)}_{F_L,g}\gamma^{(0)}_{gg} + \frac{1}{8}c^{(1,0)}_{F_L,g}\,\gamma^{(0)\,2}_{gg} + \frac{3}{4}\beta_0 c^{{\rm NS},(1,0)}_{F_L,q}\gamma^{(0)}_{qg} + \frac{1}{8}c^{{\rm NS},(1,0)}_{F_L,q}\gamma^{(0)}_{gg}\gamma^{(0)}_{qg} \\
& \left. + \frac{1}{8}c^{(1,0)}_{F_L,g}\gamma^{(0)}_{gq}\gamma^{(0)}_{qg} + \frac{1}{8}c^{{\rm NS},(1,0)}_{F_L,q}\gamma^{(0)}_{qg}\gamma^{{\rm NS},(0)}_{qq}\right] - \ln\left(\frac{Q^2}{\mu^2}\right)\left[\beta_1 c^{(1,0)}_{F_L,g} + 4\beta_0^2 c^{(1,1)}_{F_L,g}\right. \\
& + 2\beta_0 c^{(2,0)}_{F_L,g} + 3\beta_0 c^{(1,1)}_{F_L,g}\gamma^{(0)}_{gg} + \frac{1}{2}c^{(2,0)}_{F_L,g}\gamma^{(0)}_{gg} + \frac{1}{2}c^{(1,1)}_{F_L,g}\,\gamma^{(0)\,2}_{gg} + \frac{1}{2}c^{(1,0)}_{F_L,g}\gamma^{(1)}_{gg} \\
& + 3\beta_0 c^{{\rm NS},(1,1)}_{F_L,q}\gamma^{(0)}_{qg} + \frac{1}{2}c^{{\rm NS},(2,0)}_{F_L,q}\gamma^{(0)}_{qg} + \frac{1}{2}c^{{\rm PS},(2,0)}_{F_L,q}\gamma^{(0)}_{qg} + \frac{1}{2}c^{{\rm NS},(1,1)}_{F_L,q}\gamma^{(0)}_{gg}\gamma^{(0)}_{qg} \\
& \left. + \frac{1}{2}c^{(1,1)}_{F_L,g}\gamma^{(0)}_{gq}\gamma^{(0)}_{qg} + \frac{1}{2}c^{{\rm NS},(1,0)}_{F_L,q}\gamma^{(1)}_{qg} + \frac{1}{2}c^{{\rm NS},(1,1)}_{F_L,q}\gamma^{(0)}_{qg}\gamma^{{\rm NS},(0)}_{qq}\right] + \beta_1 c^{(1,1)}_{F_L,g} + 4\beta_0^2 c^{(1,2)}_{F_L,g} \\
& + 4\beta_0 c^{(2,1)}_{F_L,g} + 3\beta_0 c^{(1,2)}_{F_L,g}\gamma^{(0)}_{gg} + c^{(2,1)}_{F_L,g}\gamma^{(0)}_{gg} + \frac{1}{2}c^{(1,2)}_{F_L,g}\,\gamma^{(0)\,2}_{gg} + \frac{1}{2}c^{(1,1)}_{F_L,g}\gamma^{(1)}_{gg} \\
& + 3\beta_0 c^{{\rm NS},(1,2)}_{F_L,q}\gamma^{(0)}_{qg} + c^{{\rm NS},(2,1)}_{F_L,q}\gamma^{(0)}_{qg} + c^{{\rm PS},(2,1)}_{F_L,q}\gamma^{(0)}_{qg} + \frac{1}{2}c^{{\rm NS},(1,2)}_{F_L,q}\gamma^{(0)}_{gg}\gamma^{(0)}_{qg} \\
& + \frac{1}{2}c^{(1,2)}_{F_L,g}\gamma^{(0)}_{gq}\gamma^{(0)}_{qg} + \frac{1}{2}c^{{\rm NS},(1,1)}_{F_L,q}\gamma^{(1)}_{qg} + \frac{1}{2}c^{{\rm NS},(1,2)}_{F_L,q}\gamma^{(0)}_{qg}\gamma^{{\rm NS},(0)}_{qq} + c^{(3,0)}_{F_L,g}\Bigg\} + O\left(a_s^4\right).
\end{aligned} \tag{2.81}
$$

The Wilson coefficients for the different deep-inelastic structure functions are split w.r.t. the parton contents they are mapping to, the observables and the specific couplings of the gauge bosons exchanged, see ref. [103].

For pure photon exchange there are two contributions in the flavor non-singlet case. One corresponds to a through-flowing external fermion line to which the photons are coupling to, which is of relative weight $\mathsf{w}_1 = 1$. The second case corresponds to a photon coupling to one through-flowing line and another to a closed internal fermion line, with relative weight

$$\mathsf{w}_2 = \frac{\mathrm{tr}(\hat{Q}_f \lambda_\alpha)}{\mathrm{tr}(\hat{Q}_f^2 \lambda_\alpha)} \frac{1}{N_F} \sum_{f=1}^{N_F} e_f, \qquad \frac{\mathrm{tr}(\hat{Q}_f \lambda_\alpha)}{\mathrm{tr}(\hat{Q}_f^2 \lambda_\alpha)} = 3\,, \tag{2.82}$$

for $\mathrm{SU}(N_F)$ and λ_a the generalized Pauli-Gell-Mann matrices. e_f denotes the fermion charge and $\hat{Q}_f = \mathrm{diag}(2/3, -1/3, -1/3, 2/3, -1/3)$ the quark charge matrix of the electromagnetic current.

The flavor non-singlet structure functions for photon exchange read [37]

$$F_i^{\mathrm{NS},+}(x, Q^2) = \sum_{f=1}^{N_F} e_f^2 \left[C_{i,q}^{\mathrm{NS}} \left(x, \frac{Q^2}{\mu^2} \right) + \mathsf{w}_2 C_{i,q}^{d_{abc}} \left(x, \frac{Q^2}{\mu^2} \right) \right] \otimes f_{q,f}^{\mathrm{NS},+} \left(x, \mu^2 \right)\,, \tag{2.83}$$

with

$$\mathsf{w}_2 = \frac{\mathrm{tr}\left(\hat{Q}_f\right)}{N_F} \frac{\mathrm{tr}\left(\hat{Q}_f \lambda_\alpha\right)}{\mathrm{tr}\left(\hat{Q}_f^2 \lambda_\alpha\right)}, \qquad \mathsf{w}_2\,(N_F = 3) = 0\,, \qquad \mathsf{w}_2\,(N_F = 4) = \frac{1}{2}\,, \qquad \mathsf{w}_2\,(N_F = 5) = \frac{1}{5}\,. \tag{2.84}$$

The non-singlet distribution function is given by

$$f_{q,f}^{\mathrm{NS},+} \left(x, \mu^2 \right) = f_{q,f} \left(x, \mu^2 \right) + \bar{f}_{q,f} \left(x, \mu^2 \right) - \frac{1}{N_F} \Sigma(x, \mu^2)\,, \tag{2.85}$$

with the quark singlet distribution

$$\Sigma \left(x, \mu^2 \right) = \sum_{f=1}^{N_F} f_{q,f} \left(x, \mu^2 \right) + \bar{f}_{q,f} \left(x, \mu^2 \right)\,. \tag{2.86}$$

In the flavor singlet case the weight factor w_3 appears

$$\mathsf{w}_3 = \frac{1}{N_F} \frac{\left(\sum_{f=1}^{N_F} e_f \right)^2}{\sum_{f=1}^{N_F} e_f^2}\,, \qquad \mathsf{w}_3(N_F = 3) = 0\,, \qquad \mathsf{w}_3(N_F = 4) = \frac{1}{10}\,, \qquad \mathsf{w}_3(N_F = 5) = \frac{1}{55}\,, \tag{2.87}$$

because of the overall normalization to the sum of the quark charge squares for all diagrams in which the electromagnetic current couples to two different fermion lines.

The flavor singlet structure functions for photon exchange are given by [37]

$$F_i^{\mathrm{S},+}(x, Q^2) = \left(\frac{1}{N_F} \sum_{f=1}^{N_F} e_f^2 \right) \left[C_{i,q}^{\mathrm{S}} \left(x, \frac{Q^2}{\mu^2} \right) \otimes \Sigma(x, \mu^2) + C_{i,g} \left(x, \frac{Q^2}{\mu^2} \right) \otimes G \left(x, \mu^2 \right) \right]\,, \tag{2.88}$$

with

$$C_{i,q}^{\rm S} = C_{i,q}^{\rm NS} + \mathsf{w}_3 C_{i,q}^{d_{abc}} + C_{i,q}^{\rm PS}\,, \tag{2.89}$$

$$C_{i,g} = C_{i,g}^{a} + \mathsf{w}_3 C_{i,g}^{d_{abc}}\,, \tag{2.90}$$

and $G(x, \mu^2)$ denotes the gluon distribution and $C_{i,q}^{\rm S,1}$ the non-weighted contributions and $C_{i,q}^{\rm S,2}$ the one corresponding to the charge weight factor w_3. Synonymous relations apply to the structure function $g_1(x, Q^2)$ by replacing the unpolarized quantities by the corresponding polarized ones.

Usually the deep-inelastic structure functions are represented referring to $N_F = 3$ massless Wilson coefficients and massive Wilson coefficients [82, 83, 89, 104–120] due to charm and bottom quark corrections. The scaling violations of the heavy flavor Wilson coefficients are very different compared to those in the massless case.

The renormalization of the forward Compton amplitude is performed by renormalizing the strong coupling constant and removing the collinear singularities into the running of the parton distribution functions. In the case of the non-singlet structure function $xF_3(x, Q^2)$ the original calculation is performed in the Larin scheme and we finally switch to the $\overline{\mathsf{MS}}$ scheme by a finite renormalization. Here we refer to the $\overline{\mathsf{MS}}$ scheme only in the flavor non-singlet case, since there the corresponding Ward-identity is known. In the singlet case one sometimes uses a related scheme, called M–scheme, which has been introduced in ref. [52]. At 3-loop order we will not refer to the latter scheme for the Wilson coefficients in the singlet case.

In the case of the structure function $g_1(x, Q^2)$ this last step requires the knowledge of the polarized four-loop anomalous dimensions in the singlet case. Because of this we present the corresponding Wilson coefficients in the Larin scheme and switch to the $\overline{\mathsf{MS}}$ scheme in addition only for the non-singlet Wilson coefficient, since here the respective Ward identity is known in explicit form. Thereby, we also derive the Z-factor $Z_5^{\rm NS}(N, a_s)$ from the ratio of off-shell massless non-singlet operator matrix elements to three-loop order.

3 Details of the calculation

The Feynman diagrams for the different massless Wilson coefficients are generated by `QGRAF` [121] using the forward Compton amplitude.

We maintain all contributions up to the first order in the R_ξ gauge parameter ξ, which is canceling already for the unrenormalized result. For the Compton amplitude the corresponding crossing relations have to be observed [12, 70]. We decompose the Wilson coefficients into their flavor non-singlet $C_{i,q}^{\rm NS}$, pure singlet $C_{i,q}^{\rm PS}$ and gluonic $C_{i,g}$ parts. At one-loop order the contributions up to $O(\varepsilon^2)$ and two-loop order up to $O(\varepsilon)$ are required to extract the three-loop Wilson coefficients. We perform the calculation keeping the gauge parameter ξ to first order as a test for gauge invariance and show that the corresponding terms disappear. The Dirac and Lorentz algebra is performed by `FORM` [122, 123] and the color algebra is performed by using `Color` [101]. The crossing relations [12, 70] imply that for the Wilson coefficients contributing to the structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$

only the even moments contribute and to $xF_3(x, Q^2)$ and $g_1(x, Q^2)$ only the odd moments. The corresponding generating variable $\omega = 1/x$ appears therefore dominantly quadratically, except of a factor ω in the case of $xF_3(x, Q^2)$ and $g_1(x, Q^2)$.

The irreducible three-loop diagrams are reduced to 293 master integrals using the code `Crusher` [124] by applying the integration-by-parts relations [125–130]. The master integrals are mapped to ω^2 dependent structures and a sufficiently high number of their Mellin moments is calculated and inserted into the forward Compton amplitude. To generate a large number of Mellin moments we use the method of arbitrarily high moments of ref. [131], which allows to compute a much higher number of Mellin moments than possible using the algorithms of `Mincer` and `Matad` [132–134]. The required numbers of moments are summarized in table 1. For the gluonic and non-singlet structures we do not map to a manifestly ω^2 dependent structure a priori.

The number of moments in table 1 in the lower loop orders includes also the terms needed to renormalize the Wilson coefficients at three-loop order. As has been observed before in ref. [53] there is a small number of master-integral relations which are difficult to prove for vanishing powers in ω. We have verified them explicitly for their moments up to a much higher number than requested and to the respective contributing power in ε, cf. table 1. We determine recurrence relations for the corresponding color and ζ-value projections [135] of the Wilson coefficients using the method of arbitrary large moments [131] implemented within the package `SolveCoupledSystem` [136].

This is done by using the method of guessing [137, 138] and its implementation in `Sage` [139, 140]. For the calculation of the necessary initial values for the difference equations we use the results given in [130, 141–145]. In the three-loop case the calculation is based on 5000 even or odd moments. We have explicitly checked, that the other moments vanish, which is implied by the amplitude crossing relations. The difference equations are solved by using methods from difference field theory [146–158] implemented in the package `Sigma` [159, 160] utilizing functions from `HarmonicSums` [62–64, 66, 67, 161–169], to obtain the three-loop Wilson coefficients. The largest difference equation for the individual color, gauge parameter, and ζ_k-factors contributing in the present case has order $\mathsf{o} = 25$ and degree $\mathsf{d} = 778$ and needed 4305 moments. In parallel, we calculated the Wilson coefficients by using the differential equations for the master integrals directly in the variable ω, using the method presented in ref. [65]. The depth of the initial conditions in the dimensional parameter ε is the same as in the method described previously. The corresponding systems were decoupled using the formalisms of refs. [170, 171] implemented in the package `ORESYS` [172] and we further proceeded to find the N-space solution by using algorithms contained in the package `HarmonicSums`.

Comparing to the reconstruction of the anomalous dimensions and unpolarized Wilson coefficients out of their moments performed in ref. [138] in 2008 the largest difference equation had order $\mathsf{o} = 35$ and degree $\mathsf{d} = 938$ requiring 5114 moments. Here, however, even and odd moments had been used. The overall computation time using the automated chain of codes described amounted to about one year on `Intel(R) Xeon(R) CPU E5-2643 v4` processors, using also parallelization.

Working in the variable ω^2 rather than ω we obtain the results of the recurrences first for only the even or odd moments expressed in terms of cyclotomic harmonic sums at argument $2N$. For a systematic study of this class of nested sums see [62] in the present case. These objects can be algorithmically reduced to simple harmonic sums at argument N using `HarmonicSums` here.

Therefore, all Wilson coefficients can be finally expressed by harmonic sums [161, 162]

$$S_{b,\vec{a}}(N) = \sum_{k=1}^{N} \frac{(\mathrm{sign}(b))^k}{k^{|b|}} S_{\vec{a}}(k), \qquad S_\emptyset = 1, \qquad b, a_i \in \mathbb{Z}\backslash\{0\}, N \in \mathbb{N}\backslash\{0\}. \tag{3.1}$$

If it is clear from the context we will write $S_{\vec{a}}$ instead of $S_{\vec{a}}(N)$.

Their Mellin inversion to momentum fraction z-space

$$C(N) = \int_0^1 dz z^{N-1} \bar{C}(z) \tag{3.2}$$

can be performed using routines of the packages `HarmonicSums` and is expressed in terms of harmonic polylogarithms [163] given by

$$\mathrm{H}_{b,\vec{a}}(z) = \int_0^z dx f_b(x) \mathrm{H}_{\vec{a}}(x), \qquad \mathrm{H}_\emptyset = 1, \qquad b, a_i \in \{-1, 0, 1\}, \tag{3.3}$$

with the alphabet of letters

$$\mathfrak{A}_{\mathrm{H}} = \left\{ f_0(z) = \frac{1}{z}, \quad f_{-1}(z) = \frac{1}{1+z}, \quad f_1(z) = \frac{1}{1-z} \right\}. \tag{3.4}$$

In z-space one distinguishes three contributions to the individual Wilson coefficients because of their different treatment in Mellin convolutions,

$$\bar{C}(z) = \bar{C}^\delta(z) + \bar{C}^{\mathrm{plu}}(z) + \bar{C}^{\mathrm{reg}}(z), \tag{3.5}$$

where $\bar{C}^\delta(z) = c_0 \delta(1-z)$, $\bar{C}^{\mathrm{reg}}(z)$ is a regular function in $z \in [0,1]$ and $\bar{C}^{\mathrm{plu}}(z)$ denotes the remaining genuine $+$-distribution, the Mellin transformation of which is given by

$$C^{\mathrm{plu}}(N) = \int_0^1 dz \left(z^{N-1} - 1 \right) \bar{C}^{\mathrm{plu}}(z). \tag{3.6}$$

We will use this representation later on.

In the Mellin N space representation it can technically occur that there are factors of up to $1/(N-2)^2$ contributing and structures of $1/(N-1)$, in the cases that they are physically not allowed. However, these are all tractable poles, which can be shown by expanding at $N = 2$ and $N = 1$ using `HarmonicSums`. No Kronecker symbols have to be introduced for this case and the usual analytic continuation, described in ref. [67], can be applied to the N-space expressions directly.

The Wilson coefficients can be represented in N space by harmonic sums weighted by rational functions in N. Here the degree of the numerator needs not to be smaller than that of the denominator, which needs special care in performing the inverse Mellin transform to z-space. After applying the algebraic relations between harmonic sums, cf. e.g. [66], one

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Wilson coefficient	1 loop	2 loop	3 loop
$F_1^{\rm NS}$	126	1219	4305
$F_1^{\rm PS}$	0	374	1708
F_1^g	104	960	3534
$F_L^{\rm NS}$	48	560	2387
$F_L^{\rm PS}$	0	175	774
F_L^g	54	434	2046
$xF_3^{\rm NS}$	126	1219	4171
$g_1^{\rm NS}$	126	1219	4171
$g_1^{\rm PS}$	0	175	1458
g_1^g	84	1166	2998

Table 1. The necessary maximal number of non-vanishing even (resp. odd) Mellin moments for $F_1, F_L, (xF_3, g_1)$ to determine the Wilson coefficients.

obtains the following set of 60 harmonic sums

$$\{S_1; S_2, S_{-2}; S_3, S_{-3}, S_{2,1}, S_{-2,1}; S_4, S_{-4}, S_{-2,2}, S_{3,1}, S_{-3,1}, S_{2,1,1}, S_{-2,1,1}; S_5, S_{-5}, S_{-2,3}, S_{2,3}, S_{2,-3}, S_{-2,-3}, S_{2,2,1}, S_{-2,1,-2}, S_{-2,2,1} S_{4,1}, S_{-4,1}, S_{2,1,-2}, S_{3,1,1}, S_{-3,1,1}, S_{2,1,1,1}, S_{-2,1,1,1}; S_6, S_{-6}, S_{-3,3}, S_{4,2}, S_{4,-2}, S_{-4,2}, S_{-4,-2}, S_{5,1}, S_{-5,1}, S_{-2,2,-2}, S_{-2,2,2}, S_{2,-3,1}, S_{-2,3,1}, S_{-3,1,-2}, S_{-3,-2,1}, S_{-3,2,1}, S_{-4,1,1}, S_{2,3,1}, S_{3,1,-2}, S_{3,2,1}, S_{4,1,1}, S_{-2,-2,1,1}, S_{-2,1,1,2}, S_{-2,2,1,1}, S_{2,-2,1,1}, S_{2,2,1,1}, S_{3,1,1,1}, S_{-3,1,1,1}, S_{2,1,1,1,1}, S_{-2,1,1,1,1}\}. \tag{3.7}$$

One may, furthermore, apply also the structural relations [67, 68] and obtains the following 31 harmonic sums:

$$\{S_1, S_{-1,1}, S_{-2,1}, S_{-3,1}, S_{-4,1}, S_{-5,1}, S_{2,1}, S_{4,1}, S_{-1,1,1}, S_{2,1,1}, S_{1,2,-1}, S_{2,1,-1}, S_{-2,1,-2}, S_{2,1,-2}, S_{-3,1,1}, S_{3,1,1}, S_{-2,2,-2}, S_{-3,-2,1}, S_{3,1,-2}, S_{-4,1,1}, S_{4,1,1}, S_{-2,1,1,1}, S_{2,1,1,1}, S_{-3,1,1,1}, S_{-2,-2,1,1}, S_{-2,1,1,2}, S_{-2,2,1,1}, S_{2,-2,1,1}, S_{2,2,1,1}, S_{-2,1,1,1,1}, S_{2,1,1,1,1}\} \tag{3.8}$$

spanning all quantities. The representation in ref. [38] contains 167 harmonic sums for comparison.

In z space, the number of harmonic polylogarithms is, usually, higher if compared to the objects needed in Mellin N space. Here 68 harmonic polylogarithms of up to weight $\mathsf{w} = 5$, weighted by one more letter, contribute:

$$\Big\{{\rm H}_{-1}, {\rm H}_0, {\rm H}_1, {\rm H}_{0,-1}, {\rm H}_{0,1}, {\rm H}_{0,-1,-1}, {\rm H}_{0,-1,1}, {\rm H}_{0,0,-1}, {\rm H}_{0,0,1}, {\rm H}_{0,1,-1}, {\rm H}_{0,1,1}, {\rm H}_{0,-1,-1,-1}, {\rm H}_{0,-1,-1,1}, {\rm H}_{0,-1,0,1}, {\rm H}_{0,-1,1,-1}, {\rm H}_{0,-1,1,1}, {\rm H}_{0,0,-1,-1}, {\rm H}_{0,0,-1,1}, {\rm H}_{0,0,0,-1}, {\rm H}_{0,0,0,1}, {\rm H}_{0,0,1,-1}, {\rm H}_{0,0,1,1}, {\rm H}_{0,1,-1,-1}, {\rm H}_{0,1,-1,1}, {\rm H}_{0,1,1,-1}, {\rm H}_{0,1,1,1}, {\rm H}_{0,-1,-1,-1,-1}, {\rm H}_{0,-1,-1,-1,1}, {\rm H}_{0,-1,-1,0,1}, {\rm H}_{0,-1,-1,1,-1}, {\rm H}_{0,-1,-1,1,1}, {\rm H}_{0,-1,0,-1,-1}, {\rm H}_{0,-1,0,-1,1}, {\rm H}_{0,-1,0,1,-1}, {\rm H}_{0,-1,0,1,1}, {\rm H}_{0,-1,1,-1,-1}, {\rm H}_{0,-1,1,-1,1}, {\rm H}_{0,-1,1,0,1}, {\rm H}_{0,-1,1,1,-1},$$

$\mathrm{H}_{0,-1,1,1,1}, \mathrm{H}_{0,0,-1,-1,-1}, \mathrm{H}_{0,0,-1,-1,1}, \mathrm{H}_{0,0,-1,0,-1}, \mathrm{H}_{0,0,-1,0,1}, \mathrm{H}_{0,0,-1,1,-1}, \mathrm{H}_{0,0,-1,1,1}, \mathrm{H}_{0,0,0,-1,-1},$
$\mathrm{H}_{0,0,0,-1,1}, \mathrm{H}_{0,0,0,0,-1}, \mathrm{H}_{0,0,0,0,1}, \mathrm{H}_{0,0,0,1,-1}, \mathrm{H}_{0,0,0,1,1}, \mathrm{H}_{0,0,1,-1,-1}, \mathrm{H}_{0,0,1,-1,1}, \mathrm{H}_{0,0,1,0,-1}, \mathrm{H}_{0,0,1,0,1},$
$\mathrm{H}_{0,0,1,1,-1}, \mathrm{H}_{0,0,1,1,1}, \mathrm{H}_{0,1,-1,-1,-1}, \mathrm{H}_{0,1,-1,-1,1}, \mathrm{H}_{0,1,-1,1,-1}, \mathrm{H}_{0,1,-1,1,1}, \mathrm{H}_{0,1,0,1,-1}, \mathrm{H}_{0,1,0,1,1},$
$$\mathrm{H}_{0,1,1,-1,-1}, \mathrm{H}_{0,1,1,-1,1}, \mathrm{H}_{0,1,1,1,-1}, \mathrm{H}_{0,1,1,1,1}\Big\} \tag{3.9}$$

after algebraic reduction. In comparison, the results of [38, 40] were represented by 128 harmonic polylogarithms.

4 The three-loop Wilson coefficients for the structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$

Before we present the three-loop Wilson coefficients we would like to remark that the unrenormalized three-loop forward Compton amplitudes depend on the complete anomalous dimensions up to two-loop order, cf. e.g. [60], and contain the following three-loop anomalous dimensions in their pole terms of $O(1/\varepsilon)$: $\gamma_{qq}^{(2),\mathrm{NS}}, \gamma_{qq}^{(2),\mathrm{PS}}, \Delta\gamma_{qq}^{(2),\mathrm{NS}}, \Delta\gamma_{qq}^{(2),\mathrm{PS}}, \gamma_{qg}^{(2)}$ and $\Delta\gamma_{qg}^{(2)}$. We confirm the previous results given in refs. [53–59, 106, 173–176].

We now turn to the unpolarized three-loop Wilson coefficients for neutral-current deep-inelastic scattering for pure virtual photon exchange, $F_2(x, Q^2)$ and $F_L(x, Q^2)$. We have calculated the different Wilson coefficients contributing to these structure functions along the lines being described in sections 2 and 3. These are the Wilson coefficients

$$C_{2,q}^{\mathrm{NS}} = 1 + a_s C_{2,q}^{(1),\mathrm{NS}} + a_s^2 C_{2,q}^{(2),\mathrm{NS}} + a_s^3 \left[C_{2,q}^{(3),\mathrm{NS}} + \mathsf{w}_2 C_{2,q}^{(3),d_{abc},\mathrm{NS}}\right], \tag{4.1}$$

$$C_{2,q}^{\mathrm{PS}} = a_s^2 C_{2,q}^{(2),\mathrm{PS}} + a_s^3 C_{2,q}^{(3),\mathrm{PS}}, \tag{4.2}$$

$$C_{2,g} = a_s C_{2,g}^{(1)} + a_s^2 C_{2,g}^{(2)} + a_s^3 \left[C_{2,g}^{(3)} + \mathsf{w}_3 C_{2,g}^{(3),d_{abc}}\right] \tag{4.3}$$

and

$$C_{L,q}^{\mathrm{NS}} = a_s C_{L,q}^{(1),\mathrm{NS}} + a_s^2 C_{L,q}^{(2),\mathrm{NS}} + a_s^3 \left[C_{L,q}^{(3),\mathrm{NS}} + \mathsf{w}_2 C_{L,q}^{(3),d_{abc},\mathrm{NS}}\right], \tag{4.4}$$

$$C_{L,q}^{\mathrm{PS}} = a_s^2 C_{L,q}^{(2),\mathrm{PS}} + a_s^3 C_{L,q}^{(3),\mathrm{PS}}, \tag{4.5}$$

$$C_{L,g} = a_s C_{L,g}^{(1)} + a_s^2 C_{L,g}^{(2)} + a_s^3 \left[C_{L,g}^{(3)} + \mathsf{w}_3 C_{L,g}^{(3),d_{abc}}\right], \tag{4.6}$$

where we have split off the d_{abc} terms. In the structure of the following expressions there are some evanescent poles at $N = 1, 2$, which are, however, all tractable. Therefore, the analytic continuation [67] to $N \in \mathbb{N}$ can be carried out directly.

Despite the fact that the above equations contain terms in which the power of N is larger in the numerator than the denominator, the whole expressions behave maximally $\propto \ln(N)/N$ for large values of N for the Wilson coefficient of F_2 and $\propto \ln(N)/N^2$ for that of F_L. Our results agree with those of [38] and the previous one- and two-loop results, cf. section 1. We have expressed them in terms of a much smaller number of basic functions by using the algebraic relations both in N- and z-space, [66].

Despite being the first independent recalculation of these quantities using quite different techniques we present the results on request only in an ancillary file, but do not print the

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results. This is done although having provided a more compact form of the results by eliminating algebraic dependencies inside previous three-loop results. The ancillary files of the results are electronically readable. For the convenience of readers who are more used to texts we mention the following opportunity. One may automatically produce the respective formulae in loading the package Sigma.m [159, 160] into a Mathematica session and import the respective Wilson coefficients of the structure functions F_2, F_L and xF_3 from the ancillary files Nspace.m or Zspace.m, and apply then the command

```
SigmaToTeX[expr,PolynomialSize → 500,DropArguments → True,
InsertLinebreaks → 45]
```

and copy the result as plain text into an executable LaTeX environment. We apologize for this inconvenience.

The mathematical structure of the final expressions is in some way similar to the Wilson coefficients of the polarized structure function $g_1(x, Q^2)$ given in section 6.

5 The three-loop Wilson coefficients for the structure function $xF_3(x, Q^2)$

In order to renormalize the Wilson coefficient for the structure function $xF_3(x, Q^2)$ we have to renormalize the coupling and the axial-vector coupling. Furthermore, we have to remove the collinear singularities to arrive at $C^{(3)}_{\hat{F}_3,q}$, see section 2. This quantity still requires the finite renormalization to restore the Ward identities which is performed by

$$C^{(3)}_{F_3,q} = Z_5 C^{(3)}_{\hat{F}_3,q}, \tag{5.1}$$

with, [40],

$$\begin{aligned}
Z_5 = 1 &+ a_s C_F \Big[-4 - 10\varepsilon + \varepsilon^2 (-22 + 2\zeta_2)\Big] + a_s^2 \Big[-\frac{107}{9} C_A C_F + 22 C_F^2 + \frac{4}{9} C_F T_F N_F \\
&+ \varepsilon \left(\frac{662}{27} C_F T_F N_F + C_F^2 (132 - 48\zeta_3) + C_A C_F \left(-\frac{7229}{54} + 48\zeta_3\right)\right)\Big] \\
&+ a_s^3 \Big[\frac{208}{81} C_F T_F^2 N_F^2 + C_A C_F^2 \left(\frac{5834}{27} - 160\zeta_3\right) + 2 C_F^2 T_F N_F \left(-\frac{62}{27} - \frac{32\zeta_3}{3}\right) \\
&+ 2 C_A C_F T_F N_F \left(\frac{356}{81} + \frac{32\zeta_3}{3}\right) + C_A^2 C_F \left(-\frac{2147}{27} + 56\zeta_3\right) + C_F^3 \left(-\frac{370}{3} + 96\zeta_3\right)\Big].
\end{aligned} \tag{5.2}$$

One obtains

$$C^{\rm NS}_{F_3,q} = 1 + a_s C^{(1),{\rm NS}}_{F_3,q} + a_s^2 C^{(2),{\rm NS}}_{F_3,q} + a_s^3 \left[C^{(3),{\rm NS}}_{F_3,q} + {\sf w}_2 C^{(3),d_{abc},{\rm NS}}_{F_3,q}\right]. \tag{5.3}$$

Our results agree with the results of [40] at three-loop oder and previous results at one- and two-loop order, see section 1. They are given, together with the one- and two-loop results, in the ancillary files to the present paper in N- and z-space, again after applying the algebraic relations between the special functions, in computer-readable form.

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6 The three-loop Wilson coefficients for the structure function $g_1(x, Q^2)$

The 3-loop Wilson coefficients for the polarized structure function $g_1(x,Q^2)$ have not been calculated before and we are presenting the results in explicit form. The Wilson coefficients read

$$\Delta C_{g_1,q}^{\rm NS} = 1 + a_s \Delta C_{g_1,q}^{(1)} + a_s^2 \Delta C_{g_1,q}^{(2),\rm NS} + a_s^3 [\Delta C_{g_1,q}^{(3),\rm NS} + \mathsf{w}_2 \Delta C_{g_1,q}^{(3),d_{abc},\rm NS}], \tag{6.1}$$
$$\Delta C_{g_1,q}^{\rm PS} = a_s^2 \Delta C_{g_1,q}^{(2),\rm PS} + a_s^3 \Delta C_{g_1,q}^{(3),\rm PS}, \tag{6.2}$$
$$\Delta C_{g_1,g} = a_s \Delta C_{g_1,g}^{(1)} + a_s^2 \Delta C_{g_1,g}^{(2)} + a_s^3 [\Delta C_{g_1,g}^{(3)} + \mathsf{w}_3 \Delta C_{g_1,g}^{(3),d_{abc}}]. \tag{6.3}$$

We calculate them first in the Larin scheme. Up to 2-loop order and for the non-singlet case at three-loop order one may transform them to the M–scheme, since the respective transformation relations are known.[6] For completeness we list also our results at one- and two-loop order. At one-loop order the Wilson coefficients read

$$\Delta C_{g_1,q}^{(1),\rm L} = C_F \Biggl\{ \frac{2 - 5N - 6N^2 - 9N^3}{N^2(1+N)} + \frac{(-2+3N+3N^2)}{N(1+N)} S_1 + 2S_1^2 - 2S_2 \Biggr\}, \tag{6.4}$$
$$\Delta C_{g_1,q}^{(1),\rm M} = \Delta C_{g_1,q}^{(1),\rm L} - z_{qq}^{(1)}, \tag{6.5}$$
$$\Delta C_{g_1,g}^{(1)} = T_F N_F (N-1) \Biggl\{ -\frac{4(N-1)}{N^2(1+N)} - \frac{4}{N(1+N)} S_1 \Biggr\}. \tag{6.6}$$

The two-loop results both in the Larin (L) and M–scheme are given by

$$\begin{aligned}
\Delta C_{g_1,q}^{(2),\rm NS,L} = {} & C_F \Biggl\{ T_F N_F \Biggl[\frac{P_{251}}{54N^3(1+N)^3} + \Biggl(-\frac{2P_{239}}{27N^2(1+N)^2} + \frac{8}{3} S_2 \Biggr) S_1 - \frac{8}{9} S_1^3 \\
& - \frac{2\bigl(-6+29N+29N^2\bigr)}{9N(1+N)} S_1^2 + \frac{2\bigl(-6+85N+85N^2\bigr)}{9N(1+N)} S_2 - \frac{88}{9} S_3 + \frac{16}{3} S_{2,1} \Biggr] \\
& + C_A \Biggl[\frac{P_{253}}{216(N-1)N^4(1+N)^4(2+N)} + \Biggl(-\frac{2\bigl(6+11N+11N^2\bigr)}{3N(1+N)} S_2 \\
& + \frac{P_{242}}{54N^3(1+N)^3} + 24S_3 - 16S_{2,1} - 16S_{-2,1} - 48\zeta_3 \Biggr) S_1 + \frac{22}{9} S_1^3 + \Biggl(4S_2 \\
& + \frac{-66+367N+367N^2}{18N(1+N)} \Biggr) S_1^2 + \frac{\bigl(66-929N-2134N^2-1067N^3\bigr)}{18N(1+N)^2} S_2 - 4S_2^2 \\
& + \frac{2\bigl(-36+121N+121N^2\bigr)}{9N(1+N)} S_3 - 8S_4 + \Biggl(\frac{4P_{165}}{(N-1)N(1+N)^2(2+N)} \\
& - \frac{8}{N(1+N)} S_1 + 16S_1^2 - 8S_2 \Biggr) S_{-2} - 12S_{-2}^2 + \Biggl(\frac{8}{N(1+N)} - 8S_1 \Biggr) S_{-3} \\
& - 20S_{-4} - \frac{4\bigl(-6+11N+11N^2\bigr)}{3N(1+N)} S_{2,1} - 24S_{3,1} + 8S_{-2,2} + 16S_{-3,1} + 24S_{2,1,1} \\
& + \frac{6(1+3N)(2+3N)}{N(1+N)} \zeta_3 \Biggr] \Biggr\} + C_F^2 \Biggl[\frac{P_{256}}{8(N-1)N^4(1+N)^4(2+N)} +
\end{aligned}$$

[6] In the non-singlet case it is even the $\overline{\mathsf{MS}}$ scheme.

$$
\begin{aligned}
&+\frac{P_{246}}{2N^2(1+N)^2}S_2+\Biggl(\frac{P_{248}}{2N^3(1+N)^3}-\frac{2(-2+3N)(5+3N)S_2}{N(1+N)}-24S_3+16S_{2,1}\\
&+32S_{-2,1}+48\zeta_3\Biggr)S_1+\Biggl(\frac{P_{245}}{2N^2(1+N)^2}-20S_2\Biggr)S_1^2+\frac{2\bigl(-2+3N+3N^2\bigr)}{N(1+N)}S_1^3\\
&+2S_1^4+6S_2^2-\frac{2\bigl(-2+9N+9N^2\bigr)S_3}{N(1+N)}+12S_4+\Biggl(\frac{16}{N(1+N)}S_1\\
&-\frac{8P_{165}}{(N-1)N(1+N)^2(2+N)}-32S_1^2+16S_2\Biggr)S_{-2}+24S_{-2}^2+16\Biggl(-\frac{1}{N(1+N)}\\
&+S_1\Biggr)S_{-3}+40S_{-4}+\frac{4\bigl(-2+3N+3N^2\bigr)}{N(1+N)}S_{2,1}+40S_{3,1}-16S_{-2,2}-32S_{-3,1}\\
&-24S_{2,1,1}-72\zeta_3\Biggr], \qquad (6.7)
\end{aligned}
$$

$$
\begin{aligned}
\Delta C_{g_1,q}^{(2),\mathrm{NS,M}}&=\Delta C_{g_1,q}^{(2),\mathrm{NS,L}}-\Bigl(\Delta C_{g1,q}^{(1),\mathrm{NS,L}}-z_{qq}^{(1)}\Bigr)z_{qq}^{(1)}-z_{qq}^{(2),\mathrm{NS}}\\
&=\Delta C_{g_1,q}^{(2),\mathrm{NS,L}}+C_F\Biggl\{-T_FN_F\frac{16\bigl(-3-N+5N^2\bigr)}{9N^2(1+N)^2}+C_A\Biggl[\frac{4P_{247}}{9N^3(1+N)^3}\\
&+\frac{16}{N(1+N)}S_{-2}\Biggr]\Biggr\}+C_F^2\Biggl[-\frac{8\bigl(11+16N+11N^2\bigr)}{N(1+N)^3}+\frac{8(-4+3N)}{N^2(1+N)}S_1\\
&+\frac{16}{N(1+N)}S_1^2-\frac{32}{N(1+N)}S_2-\frac{32}{N(1+N)}S_{-2}\Biggr], \qquad (6.8)
\end{aligned}
$$

$$
\begin{aligned}
\Delta C_{g_1,q}^{(2),\mathrm{PS,L}}&=C_FT_FN_F\Biggl[\frac{4P_{252}}{(N-1)N^4(1+N)^4(2+N)}+\frac{8(2+N)\bigl(2+N-N^2+2N^3\bigr)}{N^3(1+N)^3}S_1\\
&+\frac{4(N-1)(2+N)}{N^2(1+N)^2}S_1^2-\frac{4(N-1)(2+N)}{N^2(1+N)^2}S_2-\frac{64}{(N-1)N(1+N)(2+N)}\\
&\times S_{-2}\Biggr], \qquad (6.9)
\end{aligned}
$$

$$
\Delta C_{g_1,q}^{(2),\mathrm{PS,M}}=\Delta C_{g_1,q}^{(2),\mathrm{NS,L}}-z_{qq}^{(2),\mathrm{PS}}=\Delta C_{g_1,q}^{(2),\mathrm{NS,L}}+C_FT_FN_F\frac{8(2+N)\bigl(-1-N+N^2\bigr)}{N^3(1+N)^3}, \qquad (6.10)
$$

$$
\begin{aligned}
\Delta C_{g_1,g}^{(2)}&=C_AT_FN_F\Biggl[-\frac{4P_{255}}{(N-1)N^4(1+N)^4(2+N)^2}+\Biggl(-\frac{4P_{250}}{N^3(1+N)^3(2+N)}\\
&+\frac{20(N-1)}{N(1+N)}S_2\Biggr)S_1-\frac{8\bigl(3-N^2+N^3\bigr)}{N^2(1+N)^2}S_1^2+\frac{8\bigl(3-2N+N^2+N^3\bigr)}{N^2(1+N)^2}S_2\\
&-\frac{4(N-1)}{3N(1+N)}S_1^3+\frac{8\bigl(2+5N+5N^2\bigr)}{3N(1+N)(2+N)}S_3+\Biggl(\frac{16P_{163}}{(N-1)N(1+N)^2(2+N)^2}\\
&+\frac{16\bigl(2+N+N^2\bigr)}{N(1+N)(2+N)}S_1\Biggr)S_{-2}-\frac{16\bigl(-4+N+N^2\bigr)}{N(1+N)(2+N)}S_{-3}-\frac{16(N-1)}{N(1+N)}S_{2,1}\\
&-\frac{64}{N(1+N)(2+N)}S_{-2,1}-\frac{24\bigl(2+N+N^2\bigr)}{N(1+N)(2+N)}\zeta_3\Biggr]\\
&+C_FT_FN_F\Biggl[-\frac{2P_{254}}{(N-1)N^4(1+N)^4(2+N)^2}+\Biggl(\frac{4P_{249}}{N^3(1+N)^3(2+N)}\\
&+\frac{12(N-1)}{N(1+N)}S_2\Biggr)S_1-\frac{2(N-1)\bigl(-8+3N+9N^2\bigr)}{N^2(1+N)^2}S_1^2-\frac{20(N-1)}{3N(1+N)}S_1^3+
\end{aligned}
$$

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$$+\frac{2\left(4-19N-10N^2+9N^3\right)}{N^2(1+N)^2}S_2-\frac{64\left(1+N+N^2\right)}{3N(1+N)(2+N)}S_3$$
$$+\left(-\frac{128}{N(1+N)(2+N)}S_1+\frac{16\left(10+N+N^2\right)}{(N-1)(2+N)^2}\right)S_{-2}-\frac{64}{N(1+N)(2+N)}S_{-3}$$
$$+\frac{16(N-1)}{N(1+N)}S_{2,1}+\frac{128}{N(1+N)(2+N)}S_{-2,1}+\frac{48\left(2+N+N^2\right)}{N(1+N)(2+N)}\zeta_3\Bigg], \qquad (6.11)$$

with the polynomials

$$P_{163}=N^5+3N^4-3N^3-9N^2-8N-8, \qquad (6.12)$$
$$P_{165}=2N^5+6N^4+N^3-6N^2+11N+10, \qquad (6.13)$$
$$P_{245}=-27N^4-42N^3-47N^2+24, \qquad (6.14)$$
$$P_{246}=95N^4+178N^3+123N^2+40N-16, \qquad (6.15)$$
$$P_{247}=103N^4+140N^3+58N^2+21N+36, \qquad (6.16)$$
$$P_{239}=247N^4+548N^3+223N^2+30N+72, \qquad (6.17)$$
$$P_{248}=-51N^6-131N^5-163N^4-81N^3+26N^2-8N-16, \qquad (6.18)$$
$$P_{249}=N^6+5N^5+3N^4-13N^3-20N^2+12N+16, \qquad (6.19)$$
$$P_{250}=7N^6+16N^5-11N^4-18N^3+6N^2-44N-32, \qquad (6.20)$$
$$P_{251}=1371N^6+3429N^5+3437N^4+655N^3-940N^2+72N+360, \qquad (6.21)$$
$$P_{242}=3155N^6+9951N^5+9867N^4+3473N^3+546N^2-72N-432, \qquad (6.22)$$
$$P_{252}=6N^8+20N^7+2N^6-36N^5-43N^4-62N^3-47N^2+12N+20, \qquad (6.23)$$
$$P_{253}=-16395N^{10}-74235N^9-111388N^8-28126N^7+111413N^6+125177N^5$$
$$+41930N^4+12464N^3+23688N^2+3600N-5184, \qquad (6.24)$$
$$P_{254}=2N^{10}+18N^9+26N^8-42N^7-253N^6-563N^5-633N^4-181N^3+126N^2$$
$$+4N-40, \qquad (6.25)$$
$$P_{255}=4N^{10}+18N^9-2N^8-74N^7+5N^6+141N^5+99N^4+171N^3+122N^2$$
$$-44N-56, \qquad (6.26)$$
$$P_{256}=331N^{10}+1451N^9+2196N^8+606N^7-2293N^6-2697N^5-2506N^4$$
$$-2336N^3-1232N^2+112N+224, \qquad (6.27)$$

and the coefficients $z_{ij}^{(l)}$ given in [54, 55]. Up to two-loop order, the Wilson coefficients were given in different schemes [47, 49, 50, 119]. In particular the non-singlet Wilson coefficients are also given in the $\overline{\mathsf{MS}}$ scheme [40, 120]. Our results agree with the results given in different form in refs. [46–48] and [49].

We now turn to the Wilson coefficients at three-loop order. The non-singlet Wilson coefficient is given by[7]

$$\Delta C_{g_1,q}^{\mathrm{NS},(3)}=$$
$$C_F^2\Bigg\{C_A\Bigg[-\frac{20\zeta_5P_{754}}{3N(1+N)}+\frac{8S_{-2,1,-2}P_{758}}{3N(1+N)}+\frac{16S_{-2,3}P_{760}}{9N(1+N)}+\frac{36\zeta_2^2P_{764}}{5N^2(1+N)^2}-$$

[7]The polynomials P_1 to P_{742} mostly describe the different parts of the Wilson coefficients of F_2, F_L and xF_3, given in the ancillary files to the present paper.

$$-\frac{4S_5P_{766}}{9N(1+N)}-\frac{4S_{2,1,1}P_{776}}{3N^2(1+N)^2}+\frac{S_2^2P_{788}}{27N^2(1+N)^2}-\frac{4S_{-3,1}P_{815}}{9(-1+N)N^2(1+N)^2(2+N)}$$
$$+\frac{8S_{-2,2}P_{821}}{27(-1+N)N^2(1+N)^2(2+N)}+\frac{4S_4P_{837}}{27(-1+N)N^2(1+N)^2(2+N)}$$
$$+\frac{4S_{3,1}P_{838}}{27(-1+N)N^2(1+N)^2(2+N)}-\frac{8S_{-2,1,1}P_{839}}{27(-1+N)N^2(1+N)^2(2+N)}$$
$$+\frac{\zeta_3P_{850}}{3(-1+N)N^3(1+N)^3(2+N)}+\frac{P_{884}}{648(-1+N)^2N^6(1+N)^6(2+N)}$$
$$+\Bigg(\frac{2\zeta_3P_{828}}{3(-1+N)N^2(1+N)^2(2+N)}-\frac{2S_3P_{820}}{27(-1+N)N^2(1+N)^2(2+N)}$$
$$+\frac{8S_{-2,1}P_{836}}{27(-1+N)N^2(1+N)^2(2+N)}+\frac{P_{878}}{648(-1+N)^2N^5(1+N)^5(2+N)}$$
$$+\frac{4S_{2,1}P_{782}}{9N^2(1+N)^2}+\Bigg(\frac{P_{843}}{162N^3(1+N)^3(2+N)}-\frac{2960}{3}S_3+\frac{160}{3}S_{2,1}-\frac{15104}{3}S_{-2,1}$$
$$+480\zeta_3\Bigg)S_2+\frac{8\big(15+236N+236N^2\big)S_2^2}{9N(1+N)}+\frac{8\big(-99+734N+734N^2\big)S_4}{9N(1+N)}$$
$$+256S_5+\frac{2080}{3}S_{2,3}+3920S_{2,-3}-\frac{8\big(-1212+869N+869N^2\big)S_{3,1}}{9N(1+N)}+240S_{4,1}$$
$$-\frac{8\big(1662+61N+61N^2\big)S_{-2,2}}{9N(1+N)}-3840S_{-2,3}-\frac{32\big(107+50N+50N^2\big)S_{-3,1}}{3N(1+N)}$$
$$+64S_{-2,-3}-\frac{4000}{3}S_{-4,1}-\frac{40\big(6+13N+13N^2\big)S_{2,1,1}}{3N(1+N)}-\frac{11552}{3}S_{2,1,-2}-\frac{32}{3}S_{2,2,1}$$
$$+1984S_{3,1,1}-\frac{32\big(-1020+239N+239N^2\big)S_{-2,1,1}}{9N(1+N)}-\frac{11744}{3}S_{-2,1,-2}-\frac{12224}{3}S_{-2,2,1}$$
$$-\frac{6784}{3}S_{-3,1,1}-128S_{2,1,1,1}+11776S_{-2,1,1,1}+560\zeta_5\Bigg)S_1+\Bigg(-\frac{4S_2P_{786}}{9N^2(1+N)^2}$$
$$+\frac{P_{868}}{36(-1+N)N^4(1+N)^4(2+N)}-\frac{40}{3}S_2^2+\frac{4\big(-744+733N+733N^2\big)S_3}{9N(1+N)}-88S_4$$
$$+\frac{16\big(30+29N+29N^2\big)S_{2,1}}{9N(1+N)}-\frac{2464}{3}S_{3,1}+\frac{208\big(-54+19N+19N^2\big)S_{-2,1}}{9N(1+N)}$$
$$+\frac{7216}{3}S_{-2,2}+\frac{6304}{3}S_{-3,1}+80S_{2,1,1}-\frac{16256}{3}S_{-2,1,1}+\frac{4\big(94+35N+35N^2\big)\zeta_3}{N(1+N)}\Bigg)S_1^2$$
$$+\Bigg(\frac{P_{842}}{54N^3(1+N)^3}-\frac{4\big(108+491N+491N^2\big)S_2}{27N(1+N)}+\frac{1648}{9}S_3-\frac{320}{9}S_{2,1}+1120S_{-2,1}$$
$$-96\zeta_3\Bigg)S_1^3+\Bigg(\frac{-110+433N+433N^2}{9N(1+N)}+8S_2\Bigg)S_1^4+\frac{44}{9}S_1^5+\Bigg(-\frac{616}{3}S_4+880S_{-3,1}$$
$$+\frac{P_{873}}{324(-1+N)N^4(1+N)^4(2+N)}-\frac{16\big(-1143+1159N+1159N^2\big)S_3}{27N(1+N)}$$
$$-\frac{3760}{3}S_{-2,2}-\frac{20\big(4+9N+9N^2\big)S_{2,1}}{3N(1+N)}+\frac{3136}{3}S_{3,1}-\frac{8\big(-446+79N+79N^2\big)S_{-2,1}}{3N(1+N)}$$
$$-\frac{544}{3}S_{2,1,1}+\frac{12736}{3}S_{-2,1,1}-\frac{12\big(18+77N+77N^2\big)\zeta_3}{N(1+N)}\Bigg)S_2+\frac{64}{3}S_2^3$$
$$+\Bigg(\frac{P_{849}}{81(-1+N)N^3(1+N)^3(2+N)}-\frac{1072}{9}S_{2,1}+2656S_{-2,1}+256\zeta_3\Bigg)S_3+$$

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$$+\frac{3392}{9}S_3^2+224S_6+\Big(-\frac{16\zeta_3P_{753}}{3N(1+N)}+\frac{4S_2P_{832}}{27(-1+N)N^2(1+N)^2(2+N)}$$

$$-\frac{8S_{-2,1}P_{759}}{3N(1+N)}-\frac{8S_3P_{768}}{27N(1+N)}-\frac{2P_{876}}{81(-1+N)^2N^4(1+N)^4(2+N)}$$

$$+\Big(-\frac{4P_{867}}{81(-1+N)^2N^3(1+N)^3(2+N)}+\frac{8\big(24+185N+185N^2\big)S_2}{3N(1+N)}+608S_3$$

$$+\frac{11264}{3}S_{2,1}+\frac{1312}{3}S_{-2,1}+1248\zeta_3\Big)S_1+\Big(-\frac{4P_{823}}{9(-1+N)N^2(1+N)^2(2+N)}$$

$$-112S_2\Big)S_1^2-\frac{16\big(81+139N+139N^2\big)S_1^3}{27N(1+N)}+32S_1^4-\frac{32\big(150+71N+71N^2\big)S_{2,1}}{9N(1+N)}$$

$$+\frac{128}{3}S_2^2+2224S_4-3136S_{3,1}-1728S_{-2,2}-2048S_{-3,1}+4480S_{-2,1,1}+\frac{288}{5}\zeta_2^2\Big)S_{-2}$$

$$+\Big(\frac{2P_{818}}{3(-1+N)N^2(1+N)^2(2+N)}+\frac{4\big(-222+161N+161N^2\big)S_1}{3N(1+N)}+392S_1^2$$

$$-168S_2\Big)S_{-2}^2-\frac{32}{3}S_{-2}^3+\Big(\frac{4P_{859}}{81(-1+N)N^3(1+N)^3(2+N)}+\Big(\frac{2176}{3}S_2$$

$$+\frac{4P_{822}}{27(-1+N)N^2(1+N)^2(2+N)}\Big)S_1+\frac{8\big(1044+335N+335N^2\big)S_1^2}{9N(1+N)}-\frac{2416}{3}S_1^3$$

$$-\frac{4\big(-678+773N+773N^2\big)S_2}{9N(1+N)}-2720S_3+\Big(\frac{8\big(-212+N+N^2\big)}{3N(1+N)}+\frac{4064S_1}{3}\Big)S_{-2}$$

$$-80S_{2,1}+3584S_{-2,1}-64\zeta_3\Big)S_{-3}-1184S_{-3}^2+\Big(\frac{2P_{840}}{27(-1+N)N^2(1+N)^2(2+N)}$$

$$-\frac{4\big(-126+31N+31N^2\big)S_1}{9N(1+N)}-456S_1^2+\frac{1400}{3}S_2+960S_{-2}\Big)S_{-4}+\Big(-\frac{4P_{765}}{9N(1+N)}$$

$$+\frac{3920S_1}{3}\Big)S_{-5}-\frac{544}{3}S_{-6}+\Big(\frac{4P_{846}}{9N^3(1+N)^3(2+N)}-288\zeta_3\Big)S_{2,1}+40S_{2,1}^2$$

$$+\frac{16\big(-303+431N+431N^2\big)S_{2,3}}{9N(1+N)}-\frac{4\big(2898+487N+487N^2\big)S_{2,-3}}{9N(1+N)}$$

$$-\frac{4\big(-162+1807N+1807N^2\big)S_{4,1}}{9N(1+N)}+552S_{4,2}-2080S_{4,-2}+\frac{5888}{3}S_{5,1}$$

$$+\Big(\frac{8P_{855}}{81(-1+N)N^3(1+N)^3(2+N)}+96S_{2,1}-640\zeta_3\Big)S_{-2,1}-2688S_{-2,1}^2$$

$$+\frac{16(-1+N)(2+N)S_{-2,-3}}{N(1+N)}+3456S_{-3,3}-\frac{8\big(-484+189N+189N^2\big)S_{-4,1}}{3N(1+N)}$$

$$-2384S_{-4,2}-1216S_{-4,-2}+288S_{-5,1}+\frac{8\big(654+293N+293N^2\big)S_{2,1,-2}}{9N(1+N)}$$

$$-\frac{4\big(-4+65N+65N^2\big)S_{2,2,1}}{3N(1+N)}-\frac{1760}{3}S_{2,3,1}+\frac{32\big(-387+361N+361N^2\big)S_{3,1,1}}{9N(1+N)}$$

$$-2160S_{2,-3,1}+2784S_{3,1,-2}+48S_{3,2,1}+\frac{320}{3}S_{4,1,1}+\frac{16\big(390+133N+133N^2\big)S_{-2,2,1}}{9N(1+N)}$$

$$+704S_{-2,2,2}+64S_{-2,2,-2}+2816S_{-2,3,1}+\frac{16\big(-40+127N+127N^2\big)S_{-3,1,1}}{3N(1+N)}+$$

$$
\begin{aligned}
&+960S_{-3,1,-2}-112S_{-3,2,1}-704S_{-3,-2,1}+5504S_{-4,1,1}+\frac{32\big(6+13N+13N^2\big)S_{2,1,1,1}}{3N(1+N)}\\
&+\frac{208}{3}S_{2,2,1,1}-1792S_{2,-2,1,1}-\frac{6368}{3}S_{3,1,1,1}+\frac{160\big(-180+23N+23N^2\big)S_{-2,1,1,1}}{9N(1+N)}\\
&-288S_{-2,1,1,2}+192S_{-2,2,1,1}-1408S_{-2,-2,1,1}-896S_{-3,1,1,1}+\frac{640}{3}S_{2,1,1,1,1}\\
&-9728S_{-2,1,1,1,1}\Bigg]\\
&+T_FN_F\Bigg[-\frac{128S_{-2,1}P_{770}}{81N^2(1+N)^2}-\frac{16S_{2,1}P_{789}}{9N^2(1+N)^2(2+N)}+\frac{4S_3P_{792}}{81N^2(1+N)^2(2+N)}\\
&+\frac{16\zeta_3P_{817}}{9(-1+N)N^2(1+N)^2(2+N)}+\frac{P_{879}}{162(-1+N)^2N^5(1+N)^5(2+N)}\\
&+\Bigg(\frac{2S_2P_{793}}{81N^2(1+N)^2(2+N)}+\frac{P_{871}}{162(-1+N)N^4(1+N)^4(2+N)}-\frac{752}{9}S_2^2\\
&-\frac{8\big(-210+61N+61N^2\big)S_3}{27N(1+N)}-\frac{1664}{9}S_4-\frac{16\big(-10+13N+13N^2\big)S_{2,1}}{9N(1+N)}+\frac{1088}{9}S_{3,1}\\
&-\frac{512\big(-3+10N+10N^2\big)S_{-2,1}}{27N(1+N)}+\frac{256}{9}S_{-2,2}+\frac{512}{3}S_{-3,1}+\frac{320}{3}S_{2,1,1}+\frac{2048}{9}S_{-2,1,1}\\
&+\frac{64}{5}\zeta_2^2-\frac{16\big(42+73N+73N^2\big)\zeta_3}{9N(1+N)}\Bigg)S_1+\Bigg(\frac{16\big(-23+82N+82N^2\big)S_2}{9N(1+N)}\\
&+\frac{P_{805}}{9N^3(1+N)^3}-\frac{464}{9}S_3-\frac{352}{9}S_{2,1}-\frac{1280}{9}S_{-2,1}-32\zeta_3\Bigg)S_1^2+\Bigg(-\frac{2P_{783}}{27N^2(1+N)^2}\\
&+\frac{832S_2}{27}\Bigg)S_1^3-\frac{20\big(-2+7N+7N^2\big)S_1^4}{9N(1+N)}-\frac{16}{9}S_1^5+\Bigg(\frac{P_{844}}{81N^3(1+N)^3(2+N)}\\
&+\frac{4976}{27}S_3+32S_{2,1}+\frac{256}{3}S_{-2,1}+\frac{352}{3}\zeta_3\Bigg)S_2-\frac{4\big(-210+167N+167N^2\big)S_2^2}{27N(1+N)}\\
&-\frac{16\big(-3+170N+170N^2\big)S_4}{27N(1+N)}+\frac{2368}{9}S_5\\
&+\Bigg(\frac{32P_{866}}{81(-1+N)^2N^3(1+N)^3(2+N)}+\Bigg(\frac{128P_{806}}{81(-1+N)N^2(1+N)^2(2+N)}\\
&-\frac{512}{3}S_2\Bigg)S_1+\frac{256\big(-1+5N+5N^2\big)S_1^2}{9N(1+N)}\\
&+\frac{1280}{27}S_1^3-\frac{128\big(-9+20N+20N^2\big)S_2}{27N(1+N)}+\frac{3712}{27}S_3+\frac{512}{9}S_{2,1}+\frac{896}{3}\zeta_3\Bigg)S_{-2}\\
&+\Bigg(-\frac{32\big(-2+13N+13N^2\big)}{3N(1+N)}-\frac{256}{3}S_1\Bigg)S_{-2}^2-\Bigg(\frac{64P_{795}}{81(-1+N)N^2(1+N)^2(2+N)}\\
&+\frac{512\big(-3+5N+5N^2\big)S_1}{27N(1+N)}+\frac{896}{9}S_1^2-\frac{640}{9}S_2-\frac{512}{3}S_{-2}\Bigg)S_{-3}+\Bigg(-\frac{256}{9}S_1\\
&-\frac{32\big(42+197N+197N^2\big)}{27N(1+N)}\Bigg)S_{-4}-\frac{16\big(48+431N+431N^2\big)S_{3,1}}{27N(1+N)}+\frac{3008}{9}S_{-5}\\
&-\frac{1984}{9}S_{2,3}+\frac{512}{9}S_{2,-3}+\frac{3008}{9}S_{4,1}+\frac{128\big(-3+10N+10N^2\big)S_{-2,2}}{27N(1+N)}-\frac{896}{9}S_{-2,3}+
\end{aligned}
$$

$$+\frac{128\big(-3+10N+10N^2\big)S_{-3,1}}{9N(1+N)}+\frac{16\big(-4+5N+5N^2\big)S_{2,1,1}}{3N(1+N)}-\frac{512}{9}S_{2,1,-2}$$
$$+\frac{64}{3}S_{2,2,1}-\frac{2944}{9}S_{3,1,1}+\frac{256\big(-3+10N+10N^2\big)S_{-2,1,1}}{27N(1+N)}+\frac{512}{3}S_{-2,1,-2}$$
$$-\frac{512}{9}S_{-2,2,1}-\frac{512}{3}S_{-3,1,1}-\frac{256}{3}S_{2,1,1,1}-\frac{1024}{9}S_{-2,1,1,1}-\frac{16\big(2+3N+3N^2\big)\zeta_2^2}{5N(1+N)}\Bigg]\Bigg\}$$
$$+C_F\Bigg\{C_AT_FN_F\Bigg[-\frac{8S_{3,1}P_{773}}{27(-1+N)N(1+N)(2+N)}+\frac{4S_4P_{787}}{27(-1+N)N(1+N)(2+N)}$$
$$+\frac{64S_{-2,1}P_{770}}{81N^2(1+N)^2}+\frac{8S_{2,1}P_{791}}{27N^2(1+N)^2(2+N)}-\frac{8S_3P_{830}}{27(-1+N)N^2(1+N)^2(2+N)}$$
$$-\frac{4\zeta_3P_{834}}{27(-1+N)N^2(1+N)^2(2+N)}+\frac{P_{881}}{729(-1+N)^2N^5(1+N)^5(2+N)}$$
$$+\Bigg(-\frac{32S_3P_{772}}{27(-1+N)N(1+N)(2+N)}+\frac{16\zeta_3P_{778}}{27(-1+N)N(1+N)(2+N)}$$
$$+\frac{8S_2P_{790}}{27N^2(1+N)^2(2+N)}-\frac{4P_{872}}{729(-1+N)N^4(1+N)^4(2+N)}+\frac{224}{9}S_2^2+\frac{1312}{9}S_4$$
$$+\frac{16\big(-13+40N+40N^2\big)S_{2,1}}{9N(1+N)}-96S_{3,1}+\frac{256\big(-3+10N+10N^2\big)S_{-2,1}}{27N(1+N)}$$
$$-\frac{128}{9}S_{-2,2}-\frac{256}{3}S_{-3,1}-\frac{160}{3}S_{2,1,1}-\frac{1024}{9}S_{-2,1,1}-\frac{64}{5}\zeta_2^2\Bigg)S_1+\Bigg(-\frac{2P_{841}}{81N^3(1+N)^3}$$
$$+\frac{8\big(11+2N+2N^2\big)S_2}{9N(1+N)}-\frac{16}{9}S_3+\frac{304}{9}S_{2,1}+\frac{640}{9}S_{-2,1}+32\zeta_3\Bigg)S_1^2+\Bigg(-\frac{224}{27}S_2$$
$$-\frac{16\big(-33+194N+194N^2\big)}{81N(1+N)}\Bigg)S_1^3-\frac{88}{27}S_1^4+\Bigg(\frac{2P_{848}}{81N^3(1+N)^3(2+N)}-\frac{1360}{27}S_3$$
$$-16S_{2,1}-\frac{128}{3}S_{-2,1}-\frac{352}{3}\zeta_3\Bigg)S_2-\frac{4\big(16+109N+109N^2\big)S_2^2}{9N(1+N)}-\frac{1312}{9}S_5$$
$$+\Bigg(-\frac{16P_{866}}{81(-1+N)^2N^3(1+N)^3(2+N)}+\Bigg(-\frac{64P_{806}}{81(-1+N)N^2(1+N)^2(2+N)}$$
$$+\frac{256}{3}S_2\Bigg)S_1-\frac{128\big(-1+5N+5N^2\big)S_1^2}{9N(1+N)}-\frac{640}{27}S_1^3+\frac{64\big(-9+20N+20N^2\big)S_2}{27N(1+N)}$$
$$-\frac{1856}{27}S_3-\frac{256}{9}S_{2,1}-\frac{448}{3}\zeta_3\Bigg)S_{-2}+\Bigg(\frac{16\big(-2+13N+13N^2\big)}{3N(1+N)}+\frac{128}{3}S_1\Bigg)S_{-2}^2$$
$$+\Bigg(\frac{32P_{795}}{81(-1+N)N^2(1+N)^2(2+N)}+\frac{256\big(-3+5N+5N^2\big)S_1}{27N(1+N)}+\frac{448}{9}S_1^2-\frac{320}{9}S_2$$
$$-\frac{256}{3}S_{-2}\Bigg)S_{-3}+\Bigg(\frac{16\big(42+197N+197N^2\big)}{27N(1+N)}+\frac{128}{9}S_1\Bigg)S_{-4}-\frac{1504}{9}S_{-5}+\frac{320}{3}S_{2,3}$$
$$-\frac{256}{9}S_{2,-3}-\frac{64\big(-3+10N+10N^2\big)S_{-2,2}}{27N(1+N)}-\frac{64\big(-3+10N+10N^2\big)S_{-3,1}}{9N(1+N)}$$
$$+\frac{448}{9}S_{-2,3}-\frac{1120}{9}S_{4,1}-\frac{32\big(-3+8N+8N^2\big)S_{2,1,1}}{9N(1+N)}+\frac{256}{9}S_{2,1,-2}-\frac{64}{3}S_{2,2,1}$$
$$+\frac{1696}{9}S_{3,1,1}-\frac{128\big(-3+10N+10N^2\big)S_{-2,1,1}}{27N(1+N)}-\frac{256}{3}S_{-2,1,-2}+\frac{256}{9}S_{-2,2,1}+$$

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$$+\frac{256}{3}S_{-3,1,1}+\frac{64}{9}S_{2,1,1,1}+\frac{512}{9}S_{-2,1,1,1}+\frac{16\big(2+3N+3N^2\big)\zeta_2^2}{5N(1+N)}\Bigg]$$

$$+T_F^2N_F^2\Bigg[-\frac{8S_2P_{784}}{81N^2(1+N)^2}-\frac{2P_{861}}{729N^4(1+N)^4}+\Bigg(-\frac{16\big(-6+29N+29N^2\big)S_2}{27N(1+N)}$$

$$+\frac{8P_{835}}{729N^3(1+N)^3}+\frac{128}{27}S_3+\frac{128}{27}\zeta_3\Bigg)S_1+\Bigg(\frac{8P_{775}}{81N^2(1+N)^2}-\frac{32S_2}{9}\Bigg)S_1^2$$

$$+\frac{16\big(-6+29N+29N^2\big)S_1^3}{81N(1+N)}+\frac{16}{27}S_1^4+\frac{80}{9}S_2^2+\frac{32\big(-6+247N+247N^2\big)S_3}{81N(1+N)}$$

$$-\frac{992}{27}S_4-\frac{1216}{27}S_{2,1}+\frac{256}{9}S_{3,1}-\frac{128}{9}S_{2,1,1}-\frac{32\big(2+3N+3N^2\big)\zeta_3}{27N(1+N)}\Bigg]$$

$$+C_A^2\Bigg[-\frac{4S_{-2,1,-2}P_{745}}{3N(1+N)}-\frac{4S_{-2,3}P_{761}}{9N(1+N)}+\frac{2S_5P_{763}}{9N(1+N)}-\frac{12\zeta_2^2P_{764}}{5N^2(1+N)^2}+\frac{S_2^2P_{779}}{9N^2(1+N)^2}$$

$$-\frac{8S_{-2,2}P_{811}}{27(-1+N)N^2(1+N)^2(2+N)}+\frac{4S_{-3,1}P_{813}}{9(-1+N)N^2(1+N)^2(2+N)}$$

$$-\frac{4S_{3,1}P_{814}}{27(-1+N)N^2(1+N)^2(2+N)}+\frac{8S_{-2,1,1}P_{824}}{27(-1+N)N^2(1+N)^2(2+N)}$$

$$-\frac{2S_4P_{831}}{9(-1+N)N^2(1+N)^2(2+N)}+\frac{2\zeta_3P_{860}}{27(-1+N)N^3(1+N)^3(2+N)}$$

$$+\frac{P_{882}}{5832(-1+N)^2N^6(1+N)^6(2+N)}+\Bigg(-\frac{8S_{-2,1}P_{825}}{27(-1+N)N^2(1+N)^2(2+N)}$$

$$+\frac{4S_3P_{829}}{27(-1+N)N^2(1+N)^2(2+N)}-\frac{8\zeta_3P_{833}}{27(-1+N)N^2(1+N)^2(2+N)}$$

$$-\frac{4S_{2,1}P_{781}}{9N^2(1+N)^2}+\frac{P_{877}}{729(-1+N)N^5(1+N)^5(2+N)}+\Bigg(\frac{P_{845}}{27N^3(1+N)^3(2+N)}$$

$$+\frac{1184}{3}S_3-32S_{2,1}+1376S_{-2,1}-64\zeta_3\Bigg)S_2-\frac{4\big(-6+145N+145N^2\big)S_2^2}{9N(1+N)}$$

$$-\frac{4\big(-162+893N+893N^2\big)S_4}{9N(1+N)}-1040S_{2,-3}+\frac{16\big(-73+48N+48N^2\big)S_{3,1}}{3N(1+N)}$$

$$+\frac{448}{3}S_5-\frac{1120}{3}S_{2,3}-\frac{176}{3}S_{4,1}+\frac{32\big(105+11N+11N^2\big)S_{-2,2}}{9N(1+N)}+\frac{2912}{3}S_{-2,3}$$

$$-\frac{64}{3}S_{-2,-3}+\frac{64\big(12+11N+11N^2\big)S_{-3,1}}{3N(1+N)}+\frac{1184}{3}S_{-4,1}+\frac{8\big(12+55N+55N^2\big)S_{2,1,1}}{3N(1+N)}$$

$$+992S_{2,1,-2}+64S_{2,2,1}-\frac{2816}{3}S_{3,1,1}+\frac{128\big(-69+22N+22N^2\big)S_{-2,1,1}}{9N(1+N)}$$

$$+\frac{3424}{3}S_{-2,1,-2}+\frac{3200}{3}S_{-2,2,1}+\frac{1600}{3}S_{-3,1,1}+128S_{2,1,1,1}-3200S_{-2,1,1,1}+240\zeta_5\Bigg)S_1$$

$$+\Bigg(\frac{2S_2P_{774}}{9N^2(1+N)^2}+\frac{P_{863}}{162N^4(1+N)^4}-\frac{8}{3}S_2^2+\frac{4\big(240+11N+11N^2\big)S_3}{9N(1+N)}-72S_4$$

$$-\frac{4\big(12+209N+209N^2\big)S_{2,1}}{9N(1+N)}+\frac{1168}{3}S_{3,1}-\frac{352\big(-9+5N+5N^2\big)S_{-2,1}}{9N(1+N)}$$

$$-\frac{1888}{3}S_{-2,2}-512S_{-3,1}-32S_{2,1,1}+\frac{4480}{3}S_{-2,1,1}-\frac{16\big(4+11N+11N^2\big)\zeta_3}{N(1+N)}\Bigg)S_1^2+$$

$$
+\left(\frac{-726+4649N+4649N^2}{81N(1+N)}+\frac{616}{27}S_2-\frac{640}{9}S_3+\frac{32}{9}S_{2,1}-320S_{-2,1}\right)S_1^3+\frac{121}{27}S_1^4
$$

$$
+\left(\frac{P_{864}}{54N^4(1+N)^4(2+N)}+\frac{296\big(-18+13N+13N^2\big)S_3}{27N(1+N)}+\frac{424}{3}S_4\right.
$$

$$
+\frac{16\big(1+3N+3N^2\big)S_{2,1}}{N(1+N)}-\frac{1168}{3}S_{3,1}+\frac{16\big(-57+22N+22N^2\big)S_{-2,1}}{3N(1+N)}+352S_{-2,2}
$$

$$
\left.-\frac{1040}{3}S_{-3,1}+\frac{272}{3}S_{2,1,1}-1088S_{-2,1,1}+\frac{32\big(1+18N+18N^2\big)\zeta_3}{N(1+N)}\right)S_2-\frac{56}{9}S_2^3
$$

$$
+\left(\frac{2P_{862}}{81(-1+N)N^3(1+N)^3(2+N)}+\frac{880}{9}S_{2,1}-\frac{2080}{3}S_{-2,1}-128\zeta_3\right)S_3-\frac{1280}{9}S_3^2
$$

$$
+\frac{80}{9}S_6+\left(\frac{4S_{-2,1}P_{752}}{3N(1+N)}+\frac{4\zeta_3P_{756}}{3N(1+N)}+\frac{4S_3P_{767}}{27N(1+N)}-\frac{4S_2P_{785}}{27N^2(1+N)^2}\right.
$$

$$
+\frac{2P_{875}}{81(-1+N)^2N^4(1+N)^4(2+N)}+\left(\frac{4P_{858}}{81(-1+N)N^3(1+N)^3(2+N)}\right.
$$

$$
\left.-\frac{32\big(-3+22N+22N^2\big)S_2}{3N(1+N)}-\frac{320}{3}S_3-960S_{2,1}-160S_{-2,1}-320\zeta_3\right)S_1
$$

$$
+\left(\frac{4P_{780}}{9N^2(1+N)^2}-32S_2\right)S_1^2+\frac{1760}{27}S_1^3-\frac{16}{3}S_2^2+\frac{32\big(27+22N+22N^2\big)S_{2,1}}{9N(1+N)}
$$

$$
\left.-\frac{1648}{3}S_4+768S_{3,1}+\frac{1280}{3}S_{-2,2}+\frac{1600}{3}S_{-3,1}-\frac{3584}{3}S_{-2,1,1}-\frac{96}{5}\zeta_2^2\right)S_{-2}
$$

$$
+\left(-\frac{8P_{800}}{3(-1+N)N^2(1+N)^2(2+N)}-\frac{8\big(-38+47N+47N^2\big)S_1}{3N(1+N)}-\frac{400}{3}S_1^2\right.
$$

$$
\left.+\frac{208}{3}S_2\right)S_{-2}^2+\frac{32}{9}S_{-2}^3+\left(-\frac{2P_{857}}{81(-1+N)N^3(1+N)^3(2+N)}\right.
$$

$$
+\left(-\frac{4P_{819}}{27(-1+N)N^2(1+N)^2(2+N)}-144S_2\right)S_1-\frac{16\big(117+77N+77N^2\big)S_1^2}{9N(1+N)}
$$

$$
+\frac{544}{3}S_1^3+\frac{40\big(-27+22N+22N^2\big)S_2}{9N(1+N)}+\frac{2080}{3}S_3+\left(\frac{8\big(40+13N+13N^2\big)}{3N(1+N)}\right.
$$

$$
\left.-\frac{832}{3}S_1\right)S_{-2}+\frac{80}{3}S_{2,1}-\frac{3136}{3}S_{-2,1}+32\zeta_3\Big)S_{-3}+\frac{1072}{3}S_{-3}^2+\left(\frac{320}{3}S_1^2\right.
$$

$$
\left.-\frac{4P_{826}}{27(-1+N)N^2(1+N)^2(2+N)}-\frac{8\big(12+53N+53N^2\big)S_1}{9N(1+N)}-\frac{320}{3}S_2-\frac{736}{3}S_{-2}\right)
$$

$$
\times S_{-4}+\left(\frac{2P_{762}}{9N(1+N)}-208S_1\right)S_{-5}+\frac{544}{9}S_{-6}+\left(-\frac{4P_{847}}{27N^3(1+N)^3(2+N)}\right.
$$

$$
\left.+96\zeta_3\right)S_{2,1}-\frac{16\big(-35+52N+52N^2\big)S_{2,3}}{3N(1+N)}+\frac{8\big(369+106N+106N^2\big)S_{2,-3}}{9N(1+N)}
$$

$$
-\frac{56}{3}S_{2,1}^2+\frac{8\big(33+367N+367N^2\big)S_{4,1}}{9N(1+N)}-\frac{952}{3}S_{4,2}+\frac{1600}{3}S_{4,-2}-\frac{2144}{3}S_{5,1}
$$

$$
+\left(-\frac{4P_{856}}{81(-1+N)N^3(1+N)^3(2+N)}-32S_{2,1}+192\zeta_3\right)S_{-2,1}+\frac{32S_{-2,-3}}{3N(1+N)}+
$$

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$$+\frac{2240}{3}S_{-2,1}^2-\frac{2848}{3}S_{-3,3}+\frac{8\big(-140+57N+57N^2\big)S_{-4,1}}{3N(1+N)}+\frac{2096}{3}S_{-4,2}$$
$$-224S_{-5,1}+\frac{4\big(-84+607N+562N^2\big)S_{2,1,1}}{9N(1+N)}-\frac{16\big(63+44N+44N^2\big)S_{2,1,-2}}{9N(1+N)}$$
$$+\frac{1216}{3}S_{-4,-2}+\frac{4\big(-24+29N+29N^2\big)S_{2,2,1}}{3N(1+N)}+\frac{640}{3}S_{2,3,1}+\frac{2096}{3}S_{2,-3,1}$$
$$-\frac{16\big(-264+287N+287N^2\big)S_{3,1,1}}{9N(1+N)}-\frac{1952}{3}S_{3,1,-2}-\frac{208}{3}S_{3,2,1}+\frac{320}{3}S_{4,1,1}$$
$$-\frac{64\big(21+11N+11N^2\big)S_{-2,2,1}}{9N(1+N)}-\frac{832}{3}S_{-2,2,2}-\frac{64}{3}S_{-2,2,-2}-\frac{1984}{3}S_{-2,3,1}$$
$$-\frac{352\big(-1+2N+2N^2\big)S_{-3,1,1}}{3N(1+N)}-320S_{-3,1,-2}+\frac{112}{3}S_{-3,2,1}+\frac{704}{3}S_{-3,-2,1}$$
$$-\frac{4480}{3}S_{-4,1,1}-\frac{16\big(36+11N+11N^2\big)S_{2,1,1,1}}{9N(1+N)}-\frac{272}{3}S_{2,2,1,1}+384S_{2,-2,1,1}$$
$$+\frac{2720}{3}S_{3,1,1,1}-\frac{64\big(-117+22N+22N^2\big)S_{-2,1,1,1}}{9N(1+N)}+96S_{-2,1,1,2}-64S_{-2,2,1,1}$$
$$+\frac{1408}{3}S_{-2,-2,1,1}+384S_{-3,1,1,1}-\frac{640}{3}S_{2,1,1,1,1}+2560S_{-2,1,1,1,1}$$
$$+\frac{5(-5+N)(6+N)\big(8+N+N^2\big)\zeta_5}{3N(1+N)}\Bigg]\Bigg\}$$
$$+C_F^3\Bigg\{\frac{S_2^2P_{743}}{3N^2(1+N)^2}-\frac{16S_{-2,1,-2}P_{746}}{3N(1+N)}-\frac{16S_{-2,3}P_{748}}{3N(1+N)}+\frac{8S_5P_{755}}{3N(1+N)}+\frac{20\zeta_5P_{757}}{3N(1+N)}$$
$$-\frac{24\zeta_2^2P_{764}}{5N^2(1+N)^2}+\frac{8S_{2,1,1}P_{769}}{N^2(1+N)^2}-\frac{8S_{-3,1}P_{798}}{3(-1+N)N^2(1+N)^2(2+N)}$$
$$-\frac{16S_{-2,2}P_{801}}{3(-1+N)N^2(1+N)^2(2+N)}-\frac{8S_{3,1}P_{810}}{3(-1+N)N^2(1+N)^2(2+N)}$$
$$+\frac{16S_{-2,1,1}P_{812}}{3(-1+N)N^2(1+N)^2(2+N)}-\frac{2S_4P_{816}}{3(-1+N)N^2(1+N)^2(2+N)}$$
$$-\frac{4\zeta_3P_{853}}{3(-1+N)N^3(1+N)^3(2+N)}+\frac{P_{883}}{24(-1+N)^2N^6(1+N)^6(2+N)}$$
$$+\Bigg(\frac{2S_3P_{804}}{3(-1+N)N^2(1+N)^2(2+N)}-\frac{16S_{-2,1}P_{809}}{3(-1+N)N^2(1+N)^2(2+N)}$$
$$-\frac{4S_{2,1}P_{771}}{3N^2(1+N)^2}-\frac{8\zeta_3P_{803}}{(N-1)N^2(1+N)^2(2+N)}+\frac{P_{880}}{24(N-1)^2N^5(1+N)^5(2+N)}$$
$$+\Bigg(\frac{P_{827}}{6N^3(1+N)^3}+\frac{976}{3}S_3-\frac{64}{3}S_{2,1}+\frac{13696}{3}S_{-2,1}-416\zeta_3\Bigg)S_2$$
$$+\frac{2\big(-54+25N+25N^2\big)S_2^2}{N(1+N)}-\frac{4\big(-2+85N+85N^2\big)S_4}{N(1+N)}-\frac{2560}{3}S_5-64S_{2,3}$$
$$-3680S_{2,-3}+\frac{8\big(-106+109N+109N^2\big)S_{3,1}}{N(1+N)}-\frac{16\big(-274+9N+9N^2\big)S_{-2,2}}{3N(1+N)}$$
$$-\frac{1696}{3}S_{4,1}+\frac{11392}{3}S_{-2,3}-\frac{128}{3}S_{-2,-3}+\frac{64\big(59+6N+6N^2\big)S_{-3,1}}{3N(1+N)}+1088S_{-4,1}$$
$$-\frac{24\big(-2+3N+3N^2\big)S_{2,1,1}}{N(1+N)}+\frac{11200}{3}S_{2,1,-2}+\frac{64\big(-52+7N+7N^2\big)S_{-2,1,1}}{N(1+N)}$$
$$-\frac{1984}{3}S_{3,1,1}+3264S_{-2,1,-2}+\frac{11648}{3}S_{-2,2,1}+\frac{7168}{3}S_{-3,1,1}-10752S_{-2,1,1,1}-1120\zeta_5-$$

$$
\begin{aligned}
&-\frac{160}{3}S_{2,2,1}\Big)S_1+\Big(\frac{2S_2P_{777}}{3N^2(1+N)^2}+\frac{P_{870}}{12(N-1)N^4(1+N)^4(2+N)}+108S_2^2\\
&-\frac{4\big(-46+75N+75N^2\big)S_3}{N(1+N)}+344S_4+\frac{24\big(-2+3N+3N^2\big)S_{2,1}}{N(1+N)}+336S_{3,1}\\
&-\frac{32\big(-34+3N+3N^2\big)S_{-2,1}}{N(1+N)}-\frac{6880}{3}S_{-2,2}-\frac{6464}{3}S_{-3,1}-48S_{2,1,1}+4864S_{-2,1,1}\\
&-\frac{352\zeta_3}{N(1+N)}\Big)S_1^2+\Big(\frac{P_{794}}{6N^3(1+N)^3}-\frac{12\big(-6+7N+7N^2\big)S_2}{N(1+N)}-48S_3+32S_{2,1}\\
&-960S_{-2,1}+96\zeta_3\Big)S_1^3+\Big(\frac{P_{744}}{N^2(1+N)^2}-36S_2\Big)S_1^4+\frac{2\big(-2+3N+3N^2\big)S_1^5}{N(1+N)}\\
&+\frac{4}{3}S_1^6+\Big(\frac{P_{869}}{12(-1+N)N^4(1+N)^4(2+N)}+\frac{4\big(-398+513N+513N^2\big)S_3}{3N(1+N)}+40S_4\\
&-\frac{8\big(-4+15N+15N^2\big)S_{2,1}}{3N(1+N)}-432S_{3,1}-\frac{16\big(218+9N+9N^2\big)S_{-2,1}}{3N(1+N)}+\frac{3296}{3}S_{-2,2}\\
&-\frac{1120}{3}S_{-3,1}+\frac{272}{3}S_{2,1,1}-\frac{12416}{3}S_{-2,1,1}+\frac{32\big(5+12N+12N^2\big)\zeta_3}{N(1+N)}\Big)S_2-\frac{148}{9}S_2^3\\
&+\Big(\frac{P_{854}}{3(-1+N)N^3(1+N)^3(2+N)}+\frac{64}{3}S_{2,1}-\frac{7616}{3}S_{-2,1}-\frac{320}{3}\zeta_3\Big)S_3-\frac{512}{3}S_3^2\\
&-\frac{1664}{9}S_6+\Big(\frac{16\zeta_3P_{747}}{3N(1+N)}+\frac{16S_3P_{749}}{3N(1+N)}-\frac{8S_2P_{802}}{3(-1+N)N^2(1+N)^2(2+N)}\\
&+\frac{16S_{-2,1}P_{750}}{3N(1+N)}+\frac{4P_{874}}{3(-1+N)^2N^4(1+N)^4(2+N)}-\Big(\frac{2368}{3}S_3+\frac{11008}{3}S_{2,1}\\
&+\frac{8P_{865}}{3(N-1)^2N^3(1+N)^3(2+N)}+\frac{16\big(16+3N+3N^2\big)S_2}{N(1+N)}+\frac{704}{3}S_{-2,1}+1216\zeta_3\Big)S_1\\
&+\Big(\frac{8P_{807}}{3(-1+N)N^2(1+N)^2(2+N)}+352S_2\Big)S_1^2-\frac{96\big(-1+N+N^2\big)S_1^3}{N(1+N)}-64S_1^4\\
&-64S_2^2-\frac{6752}{3}S_4+\frac{64\big(32+9N+9N^2\big)S_{2,1}}{3N(1+N)}+3200S_{3,1}+\frac{5248}{3}S_{-2,2}+\frac{5888}{3}S_{-3,1}\\
&-\frac{12544}{3}S_{-2,1,1}-\frac{192}{5}\zeta_2^2\Big)S_{-2}+\Big(-\frac{752}{3}S_1^2-\frac{4P_{799}}{3(-1+N)N^2(1+N)^2(2+N)}\\
&+\frac{176}{3}S_2+\frac{8\big(70+27N+27N^2\big)S_1}{3N(1+N)}\Big)S_{-2}^2+\frac{64}{9}S_{-2}^3+\Big(\Big(-\frac{2624}{3}S_2\\
&+\frac{8P_{797}}{3(-1+N)N^2(1+N)^2(2+N)}\Big)S_1-\frac{8P_{852}}{3(-1+N)N^3(1+N)^3(2+N)}\\
&-\frac{16\big(64+3N+3N^2\big)S_1^2}{N(1+N)}+\frac{2656}{3}S_1^3+\frac{8\big(-46+111N+111N^2\big)S_2}{3N(1+N)}+\frac{8000}{3}S_3\\
&+\Big(-\frac{16\big(-44+9N+9N^2\big)}{N(1+N)}-1600S_1\Big)S_{-2}+\frac{160}{3}S_{2,1}-\frac{8960}{3}S_{-2,1}\Big)S_{-3}\\
&+\frac{2816}{3}S_{-3}^2+\Big(-\frac{4P_{796}}{3(-1+N)N^2(1+N)^2(2+N)}+\frac{8\big(-26+81N+81N^2\big)S_1}{3N(1+N)}\\
&+\frac{1456}{3}S_1^2-\frac{1520}{3}S_2-\frac{2816}{3}S_{-2}\Big)S_{-4}+\Big(\frac{8P_{751}}{3N(1+N)}-\frac{5344S_1}{3}\Big)S_{-5}+\frac{1088}{9}S_{-6}+
\end{aligned}
$$

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$$+\left(-\frac{4P_{808}}{3N^3(1+N)^3}+192\zeta_3\right)S_{2,1}-\frac{64}{3}S_{2,1}^2-\frac{32\big(-13+17N+17N^2\big)S_{2,3}}{N(1+N)}$$
$$+\frac{8\big(158+7N+7N^2\big)S_{2,-3}}{N(1+N)}+\frac{8\big(-38+135N+135N^2\big)S_{4,1}}{3N(1+N)}-\frac{1088}{3}S_{4,2}$$
$$+\frac{6080}{3}S_{4,-2}-1120S_{5,1}+\left(\frac{16P_{851}}{3(-1+N)N^3(1+N)^3(2+N)}-64S_{2,1}+512\zeta_3\right)$$
$$\times S_{-2,1}-\frac{32\big(-2+3N+3N^2\big)S_{-2,-3}}{3N(1+N)}+\frac{16\big(-68+25N+25N^2\big)S_{-4,1}}{N(1+N)}$$
$$-\frac{9344}{3}S_{-3,3}+\frac{5920}{3}S_{-4,2}+\frac{2432}{3}S_{-4,-2}-\frac{16\big(134+39N+39N^2\big)S_{2,1,-2}}{3N(1+N)}$$
$$+320S_{-5,1}+\frac{80S_{2,2,1}}{3N(1+N)}+\frac{352}{3}S_{2,3,1}-\frac{16\big(-206+195N+195N^2\big)S_{3,1,1}}{3N(1+N)}$$
$$+\frac{4576}{3}S_{2,-3,1}-\frac{8896}{3}S_{3,1,-2}+\frac{64}{3}S_{3,2,1}+\frac{128}{3}S_{4,1,1}-\frac{32\big(74+15N+15N^2\big)S_{-2,2,1}}{3N(1+N)}$$
$$-\frac{896}{3}S_{-2,2,2}-\frac{128}{3}S_{-2,2,-2}-\frac{8960}{3}S_{-2,3,1}-\frac{32\big(4+39N+39N^2\big)S_{-3,1,1}}{3N(1+N)}$$
$$-640S_{-3,1,-2}+\frac{224}{3}S_{-3,2,1}+\frac{1408}{3}S_{-3,-2,1}-\frac{15104}{3}S_{-4,1,1}+\frac{64}{3}S_{2,2,1,1}$$
$$+2048S_{2,-2,1,1}+704S_{3,1,1,1}-\frac{192\big(-16+N+N^2\big)S_{-2,1,1,1}}{N(1+N)}+192S_{-2,1,1,2}$$
$$-128S_{-2,2,1,1}+\frac{2816}{3}S_{-2,-2,1,1}+256S_{-3,1,1,1}+9216S_{-2,1,1,1,1}+\frac{7168}{3}S_{-2,1}^2\Bigg\} \tag{6.28}$$

with

$$P_{743}=-617N^4-1180N^3-485N^2+90N+52, \tag{6.29}$$
$$P_{744}=-9N^4-12N^3-9N^2+22N+20, \tag{6.30}$$
$$P_{745}=N^4+2N^3-293N^2-294N+356, \tag{6.31}$$
$$P_{746}=N^4+2N^3-120N^2-121N+258, \tag{6.32}$$
$$P_{747}=N^4+2N^3+15N^2+14N+60, \tag{6.33}$$
$$P_{748}=N^4+2N^3+15N^2+14N+194, \tag{6.34}$$
$$P_{749}=N^4+2N^3+24N^2+23N+68, \tag{6.35}$$
$$P_{750}=N^4+2N^3+42N^2+41N+10, \tag{6.36}$$
$$P_{751}=N^4+2N^3+90N^2+89N+166, \tag{6.37}$$
$$P_{752}=N^4+2N^3+111N^2+110N+36, \tag{6.38}$$
$$P_{753}=N^4+2N^3+115N^2+114N+54, \tag{6.39}$$
$$P_{754}=N^4+2N^3+195N^2+194N-156, \tag{6.40}$$
$$P_{755}=N^4+2N^3+204N^2+203N+40, \tag{6.41}$$
$$P_{756}=N^4+2N^3+215N^2+214N+48, \tag{6.42}$$
$$P_{757}=N^4+2N^3+303N^2+302N-144, \tag{6.43}$$
$$P_{758}=2N^4+4N^3-413N^2-415N+614, \tag{6.44}$$

$$P_{759} = 2N^4 + 4N^3 + 153N^2 + 151N + 46, \tag{6.45}$$
$$P_{760} = 3N^4 + 6N^3 + 100N^2 + 97N + 495, \tag{6.46}$$
$$P_{761} = 3N^4 + 6N^3 + 155N^2 + 152N + 408, \tag{6.47}$$
$$P_{762} = 3N^4 + 6N^3 + 1321N^2 + 1318N - 252, \tag{6.48}$$
$$P_{763} = 3N^4 + 6N^3 + 1435N^2 + 1432N - 480, \tag{6.49}$$
$$P_{764} = 5N^4 + 10N^3 + 9N^2 + 4N + 4, \tag{6.50}$$
$$P_{765} = 6N^4 + 12N^3 + 1591N^2 + 1585N + 246, \tag{6.51}$$
$$P_{766} = 6N^4 + 12N^3 + 1871N^2 + 1865N - 216, \tag{6.52}$$
$$P_{767} = 9N^4 + 18N^3 + 925N^2 + 916N + 144, \tag{6.53}$$
$$P_{768} = 18N^4 + 36N^3 + 1141N^2 + 1123N + 756, \tag{6.54}$$
$$P_{769} = 29N^4 + 51N^3 + 16N^2 - 20N - 6, \tag{6.55}$$
$$P_{770} = 83N^4 + 166N^3 + 239N^2 + 192N + 63, \tag{6.56}$$
$$P_{771} = 85N^4 + 134N^3 + 57N^2 - 80N - 52, \tag{6.57}$$
$$P_{772} = 97N^4 + 194N^3 - 121N^2 - 218N - 6, \tag{6.58}$$
$$P_{773} = 131N^4 + 262N^3 - 23N^2 - 154N + 216, \tag{6.59}$$
$$P_{774} = 147N^4 + 294N^3 + 38N^2 - 121N + 6, \tag{6.60}$$
$$P_{775} = 235N^4 + 524N^3 + 211N^2 + 30N + 72, \tag{6.61}$$
$$P_{776} = 277N^4 + 536N^3 + 200N^2 - 143N - 36, \tag{6.62}$$
$$P_{777} = 291N^4 + 488N^3 + 217N^2 - 304N - 256, \tag{6.63}$$
$$P_{778} = 361N^4 + 722N^3 - 433N^2 - 794N - 72, \tag{6.64}$$
$$P_{779} = 423N^4 + 792N^3 + 413N^2 + 44N - 90, \tag{6.65}$$
$$P_{780} = 536N^4 + 1075N^3 + 601N^2 + 62N + 162, \tag{6.66}$$
$$P_{781} = 536N^4 + 1087N^3 + 414N^2 - 149N + 6, \tag{6.67}$$
$$P_{782} = 575N^4 + 1084N^3 + 420N^2 - 365N - 150, \tag{6.68}$$
$$P_{783} = 683N^4 + 1510N^3 + 479N^2 + 68N + 292, \tag{6.69}$$
$$P_{784} = 1055N^4 + 2164N^3 + 1031N^2 + 30N + 72, \tag{6.70}$$
$$P_{785} = 1108N^4 + 2261N^3 + 1099N^2 - 126N + 288, \tag{6.71}$$
$$P_{786} = 1124N^4 + 2224N^3 + 847N^2 - 331N - 57, \tag{6.72}$$
$$P_{787} = 1687N^4 + 3374N^3 - 1657N^2 - 3344N + 372, \tag{6.73}$$
$$P_{788} = 4513N^4 + 9026N^3 + 2815N^2 - 1698N + 612, \tag{6.74}$$
$$P_{789} = 112N^5 + 313N^4 + 188N^3 - 73N^2 + 8N + 52, \tag{6.75}$$
$$P_{790} = 328N^5 + 1513N^4 + 2150N^3 + 959N^2 - 102N - 120, \tag{6.76}$$
$$P_{791} = 1103N^5 + 4010N^4 + 4309N^3 + 1096N^2 - 60N + 240, \tag{6.77}$$
$$P_{792} = 7241N^5 + 26102N^4 + 28297N^3 + 8428N^2 - 540N - 384, \tag{6.78}$$

$$P_{793} = 9187N^5+33508N^4+33155N^3+2138N^2+2844N+8136, \tag{6.79}$$
$$P_{794} = -279N^6-705N^5-663N^4-159N^3-402N^2-816N-304, \tag{6.80}$$
$$P_{795} = 25N^6+75N^5-161N^4-303N^3+901N^2+705N+54, \tag{6.81}$$
$$P_{796} = 33N^6+9N^5-669N^4-1433N^3-1180N^2+368N-8, \tag{6.82}$$
$$P_{797} = 34N^6+114N^5+695N^4+1472N^3+1051N^2-430N-632, \tag{6.83}$$
$$P_{798} = 36N^6+138N^5+969N^4+2040N^3+1355N^2-682N-400, \tag{6.84}$$
$$P_{799} = 37N^6+49N^5+161N^4+491N^3+242N^2-188N-216, \tag{6.85}$$
$$P_{800} = 49N^6+148N^5+110N^4-4N^3-24N^2-61N-74, \tag{6.86}$$
$$P_{801} = 61N^6+190N^5+530N^4+926N^3+617N^2-320N-276, \tag{6.87}$$
$$P_{802} = 74N^6+206N^5+5N^4-328N^3-203N^2+94N+8, \tag{6.88}$$
$$P_{803} = 79N^6+215N^5+73N^4-143N^3-52N^2+36N-96, \tag{6.89}$$
$$P_{804} = 81N^6+103N^5-651N^4-1335N^3-570N^2+748N+568, \tag{6.90}$$
$$P_{805} = 83N^6-293N^5+15N^4-23N^3+170N^2+1040N+704, \tag{6.91}$$
$$P_{806} = 83N^6+249N^5+164N^4-33N^3+14N^2-63N-90, \tag{6.92}$$
$$P_{807} = 96N^6+250N^5+49N^4-300N^3-333N^2+78N+16, \tag{6.93}$$
$$P_{808} = 100N^6+349N^5+301N^4+31N^3+79N^2+128N+76, \tag{6.94}$$
$$P_{809} = 158N^6+454N^5+651N^4+780N^3+471N^2-290N-496, \tag{6.95}$$
$$P_{810} = 173N^6+466N^5-273N^4-1264N^3-472N^2+462N+380, \tag{6.96}$$
$$P_{811} = 182N^6+528N^5+2678N^4+5724N^3+4385N^2-1383N-1746, \tag{6.97}$$
$$P_{812} = 208N^6+622N^5+1151N^4+1664N^3+1113N^2-598N-704, \tag{6.98}$$
$$P_{813} = 248N^6+771N^5-1216N^4-4599N^3-3616N^2+828N+672, \tag{6.99}$$
$$P_{814} = 379N^6+1065N^5-3293N^4-9309N^3-5402N^2+2304N+3240, \tag{6.100}$$
$$P_{815} = 388N^6+1128N^5-5339N^4-15318N^3-11297N^2+3702N+2544, \tag{6.101}$$
$$P_{816} = 403N^6+1191N^5+633N^4-599N^3-440N^2-116N-16, \tag{6.102}$$
$$P_{817} = 417N^6+900N^5-253N^4-1298N^3-458N^2+20N-192, \tag{6.103}$$
$$P_{818} = 429N^6+1233N^5+1041N^4+459N^3+50N^2-676N-808, \tag{6.104}$$
$$P_{819} = 676N^6+2109N^5-2858N^4-10617N^3-5747N^2+2811N+3258, \tag{6.105}$$
$$P_{820} = 781N^6+1227N^5-5909N^4-14895N^3-12548N^2+5496N+8568, \tag{6.106}$$
$$P_{821} = 913N^6+2766N^5+10126N^4+19782N^3+14323N^2-5646N-5976, \tag{6.107}$$
$$P_{822} = 1046N^6+3192N^5-11971N^4-34482N^3-20953N^2+9492N+12204, \tag{6.108}$$
$$P_{823} = 1360N^6+3972N^5+1355N^4-3874N^3-2955N^2+310N-600, \tag{6.109}$$
$$P_{824} = 1472N^6+4425N^5+7064N^4+9099N^3+6692N^2-3048N-4968, \tag{6.110}$$
$$P_{825} = 1540N^6+4629N^5+4822N^4+3087N^3+1945N^2-1785N-3870, \tag{6.111}$$
$$P_{826} = 1540N^6+4701N^5+1444N^4-5937N^3-7331N^2-1671N-522, \tag{6.112}$$

$$P_{827}=1545N^6+4511N^5+3537N^4+737N^3+1222N^2+2384N+784, \quad (6.113)$$
$$P_{828}=1873N^6+5331N^5+2385N^4-3299N^3-2470N^2+44N-1368, \quad (6.114)$$
$$P_{829}=1964N^6+5856N^5+68N^4-10116N^3-6055N^2+1461N+1314, \quad (6.115)$$
$$P_{830}=2251N^6+6552N^5+1826N^4-6688N^3-3669N^2+292N-132, \quad (6.116)$$
$$P_{831}=2284N^6+6951N^5+3143N^4-5233N^3-2970N^2-53N-450, \quad (6.117)$$
$$P_{832}=2882N^6+8592N^5+2333N^4-10050N^3-5899N^2+1926N-1080, \quad (6.118)$$
$$P_{833}=3098N^6+9240N^5+3107N^4-8898N^3-5665N^2-18N-1188, \quad (6.119)$$
$$P_{834}=3153N^6+7047N^5-1489N^4-9635N^3-5120N^2-148N-720, \quad (6.120)$$
$$P_{835}=4357N^6+16149N^5+16977N^4+9091N^3+3798N^2-1404N-1944, \quad (6.121)$$
$$P_{836}=4502N^6+13344N^5+15503N^4+13194N^3+8129N^2-6180N-12204, \quad (6.122)$$
$$P_{837}=4591N^6+13935N^5+6934N^4-8871N^3-5873N^2-1464N-612, \quad (6.123)$$
$$P_{838}=4645N^6+13341N^5-4124N^4-31599N^3-16541N^2+8430N+8568, \quad (6.124)$$
$$P_{839}=4816N^6+14448N^5+24487N^4+33174N^3+23401N^2-11478N-16272, \quad (6.125)$$
$$P_{840}=6457N^6+18885N^5-245N^4-36645N^3-39944N^2-3372N-2160, \quad (6.126)$$
$$P_{841}=7531N^6+23619N^5+23253N^4+7825N^3+2064N^2+1080N-252, \quad (6.127)$$
$$P_{842}=8425N^6+26643N^5+24471N^4+6857N^3+2520N^2+1484N-864, \quad (6.128)$$
$$P_{843}=-116957N^7-556705N^6-889137N^5-480947N^4-48370N^3-93780N^2 \\ -76536N+2592, \quad (6.129)$$
$$P_{844}=-28023N^7-129045N^6-212881N^5-149635N^4-36908N^3-4628N^2 \\ -17376N-9216, \quad (6.130)$$
$$P_{845}=-3221N^7-19897N^6-43683N^5-43139N^4-15736N^3+3828N^2+3720N \\ +720, \quad (6.131)$$
$$P_{846}=1511N^7+6361N^6+8811N^5+3500N^4-61N^3+1842N^2+1772N+552, \quad (6.132)$$
$$P_{847}=3691N^7+16568N^6+25248N^5+13189N^4+236N^3+1188N^2+1860N \\ +360, \quad (6.133)$$
$$P_{848}=32287N^7+161273N^6+288291N^5+220675N^4+64886N^3+4416N^2 \\ +1188N-792, \quad (6.134)$$
$$P_{849}=-97249N^8-360322N^7-305212N^6+226586N^5+292481N^4-76528N^3 \\ -50508N^2+100320N+63072, \quad (6.135)$$
$$P_{850}=-5021N^8-15158N^7-6690N^6+17488N^5+14927N^4-3058N^3+1392N^2 \\ +6920N+4560, \quad (6.136)$$
$$P_{851}=25N^8+105N^7+101N^6+52N^5+1332N^4+2547N^3+890N^2-932N \\ -664, \quad (6.137)$$
$$P_{852}=79N^8+323N^7+315N^6+22N^5+1546N^4+3031N^3+1088N^2-1076N-720, \quad (6.138)$$

$$P_{853}=307N^8+1174N^7+1142N^6-608N^5-2451N^4-2370N^3-250N^2$$
$$+888N+824, \tag{6.139}$$
$$P_{854}=575N^8+2226N^7+1974N^6-1440N^5+15N^4+5006N^3+1172N^2-3320N$$
$$-1984, \tag{6.140}$$
$$P_{855}=1288N^8+4747N^7+10549N^6+7126N^5-99269N^4-195599N^3-68574N^2$$
$$+70380N+51624, \tag{6.141}$$
$$P_{856}=1963N^8+7582N^7+13276N^6+8530N^5-63305N^4-126830N^3-44544N^2$$
$$+45216N+33696, \tag{6.142}$$
$$P_{857}=3761N^8+15422N^7+6284N^6-22930N^5+78905N^4+182846N^3+61872N^2$$
$$-51408N-36288, \tag{6.143}$$
$$P_{858}=4258N^8+16762N^7+18841N^6-1469N^5-28625N^4-39305N^3-12672N^2$$
$$+15102N+12636, \tag{6.144}$$
$$P_{859}=5894N^8+24143N^7+14789N^6-22336N^5+120647N^4+264683N^3$$
$$+91248N^2-80460N-55728, \tag{6.145}$$
$$P_{860}=17529N^8+56958N^7+34840N^6-52470N^5-72773N^4-21696N^3-4220N^2$$
$$-10440N-7344, \tag{6.146}$$
$$P_{861}=28551N^8+92280N^7+118370N^6+46548N^5-11961N^4+10748N^3$$
$$+12600N^2-14112N-9936, \tag{6.147}$$
$$P_{862}=43949N^8+169784N^7+155918N^6-99922N^5-174643N^4-20848N^3$$
$$+15276N^2-15534N-7884, \tag{6.148}$$
$$P_{863}=50689N^8+208318N^7+323478N^6+236812N^5+86401N^4+16554N^3$$
$$-2304N^2-2772N-540, \tag{6.149}$$
$$P_{864}=-71503N^9-425688N^8-980826N^7-1101180N^6-615171N^5$$
$$-147456N^4-4292N^3+4428N^2+2940N+72, \tag{6.150}$$
$$P_{865}=24N^9+82N^8-N^7-166N^6+590N^5+262N^4-1321N^3-1098N^2$$
$$+52N+424, \tag{6.151}$$
$$P_{866}=261N^9+783N^8-127N^7-1760N^6-493N^5-1289N^4-1879N^3$$
$$-758N^2+114N-36, \tag{6.152}$$
$$P_{867}=7868N^9+22794N^8+4185N^7-36138N^6-70242N^5-28434N^4$$
$$+88933N^3+85194N^2-6336N-36720, \tag{6.153}$$
$$P_{868}=-5563N^{10}-22141N^9-36762N^8-34822N^7+5681N^6+48719N^5$$
$$+42980N^4+28996N^3+19056N^2-7936N-10560, \tag{6.154}$$
$$P_{869}=-4253N^{10}-19223N^9-23706N^8+10042N^7+39207N^6+16757N^5$$
$$-6072N^4+1800N^3+6168N^2-528N-1760, \tag{6.155}$$

$$\begin{aligned}
P_{870} = {} & 561N^{10}+2523N^9+4538N^8+4510N^7+1005N^6-5025N^5-8992N^4 \\
& -7960N^3-8936N^2-1968N+1312, && (6.156)\\
P_{871} = {} & 6009N^{10}+21117N^9+38586N^8+68306N^7+61705N^6+28925N^5 \\
& -2468N^4-31900N^3+38584N^2+73824N+29088, && (6.157)\\
P_{872} = {} & 80453N^{10}+440578N^9+771262N^8+284116N^7-538565N^6-656852N^5 \\
& -328702N^4-44730N^3+83592N^2+19008N-16848, && (6.158)\\
P_{873} = {} & 463143N^{10}+2173209N^9+3165914N^8+374654N^7-3058933N^6 \\
& -2659411N^5-640204N^4-29428N^3-117840N^2+26496N+53568, && (6.159)\\
P_{874} = {} & 245N^{11}+944N^{10}+786N^9-975N^8-1959N^7-1998N^6-1756N^5 \\
& -695N^4+748N^3+228N^2-112N-64, && (6.160)\\
P_{875} = {} & 10683N^{11}+42732N^{10}+25931N^9-78468N^8-83703N^7-19116N^6 \\
& -61407N^5-68436N^4-10052N^3+14856N^2-468N-648, && (6.161)\\
P_{876} = {} & 27981N^{11}+110952N^{10}+73084N^9-183261N^8-220299N^7 \\
& -92178N^6-170226N^5-155637N^4+92N^3+35868N^2-3960N-3024, && (6.162)\\
P_{877} = {} & 599375N^{12}+3815193N^{11}+8947106N^{10}+8383052N^9-1037899N^8 \\
& -9404623N^7-8442036N^6-2234074N^5+1430550N^4+814536N^3 \\
& -384156N^2-321408N-50544, && (6.163)\\
P_{878} = {} & -458487N^{13}-2189691N^{12}-3476151N^{11}-706147N^{10} \\
& +5181991N^9+6607155N^8-353013N^7-6891201N^6-4645260N^5 \\
& +669484N^4+2725928N^3+920640N^2-844704N-521856, && (6.164)\\
P_{879} = {} & -3069N^{13}+2556N^{12}+79368N^{11}+221328N^{10}+43770N^9-547648N^8 \\
& -435404N^7-15640N^6-194549N^5-563692N^4-68324N^3+170136N^2 \\
& +10848N-26784, && (6.165)\\
P_{880} = {} & 3003N^{13}+13255N^{12}+27059N^{11}+32543N^{10}+397N^9-54447N^8-26095N^7 \\
& +141989N^6+192852N^5+62644N^4-92576N^3-56336N^2+29952N \\
& +20672, && (6.166)\\
P_{881} = {} & 428649N^{13}+1845381N^{12}+2026189N^{11}-1935731N^{10}-5520669N^9 \\
& -1634037N^8+3712287N^7+2628591N^6+266216N^5+1213892N^4 \\
& +461088N^3-542160N^2-82944N+119232, && (6.167)\\
P_{882} = {} & -5729259N^{15}-30802914N^{14}-53071856N^{13}-478888N^{12} \\
& +104531506N^{11}+96490716N^{10}-31073520N^9-90914040N^8-63564367N^7 \\
& -54366650N^6-31332904N^5+18604224N^4+17762832N^3-2637792N^2 \\
& -4240512N-559872, && (6.168)
\end{aligned}$$

$$P_{883} = -7255N^{15} - 40527N^{14} - 54850N^{13} + 77690N^{12} + 228204N^{11} + 45864N^{10}$$
$$- 240778N^9 - 240422N^8 - 203989N^7 - 270809N^6 - 72500N^5 + 144732$$
$$N^4 + 110144N^3 - 17904N^2 - 37248N - 10176, \tag{6.169}$$
$$P_{884} = 494694N^{15} + 2631933N^{14} + 3703662N^{13} - 3856542N^{12} - 13210842N^{11}$$
$$- 3078616N^{10} + 14028102N^9 + 13282506N^8 + 9865380N^7 + 14654523N^6$$
$$+ 7299204N^5 - 6176300N^4 - 4977096N^3 + 807456N^2 + 1390176N$$
$$+ 300672. \tag{6.170}$$

The transition from the Larin scheme to the $\overline{\mathsf{MS}}$ scheme is performed by

$$\Delta C_{g_1,q}^{\mathrm{NS},(3),\mathrm{M}}(N, a_s) = Z_5^{-1}(N, a_s)\, \Delta C_{g_1,q}^{\mathrm{NS},(3),\mathrm{L}}(N, a_s) \tag{6.171}$$

and the Z-factor $Z_5^{-1}(N, a_s)$ providing the finite renormalization is calculated in appendix A.

Comparing (6.171) to the non-singlet Wilson coefficient of $F_3(x, Q^2)$,[8] without the d_{abc} term one obtains

$$\Delta C_{g_1,q}^{\mathrm{NS},(\mathrm{k}),\mathrm{M}} = C_{F_3,q}^{\mathrm{NS},(\mathrm{k})}, \quad \text{for } k = 1, 2, 3 \tag{6.172}$$

in the $\overline{\mathsf{MS}}$ scheme.

The function $\Delta C_{g_1,q}^{d_{abc},(3)}$ is given by

$$\begin{aligned}
\Delta C_{g_1,q}^{d_{abc},(3)} = \frac{d_{abc}d^{abc}}{N_C} N_F \Bigg\{ & \frac{64 P_{896}}{(N-1)N(1+N)(2+N)} - \frac{512 P_{897}}{(N-1)N(1+N)(2+N)} S_{-2,1} \\
& - \frac{256 P_{898}}{(N-1)N^2(1+N)^2(2+N)} S_4 + \frac{512 P_{898}}{(N-1)N^2(1+N)^2(2+N)} S_{3,1} \\
& + \frac{32 P_{900}}{3(N-1)N(1+N)(2+N)} \zeta_3 + \Bigg(-\frac{128\big(-16+5N+5N^2\big)}{(N-1)N(1+N)(2+N)} \\
& - \frac{256 P_{898}}{(N-1)N^2(1+N)^2(2+N)} S_3 - \frac{1024}{(N-1)(2+N)} S_{-2,1} \\
& - \frac{1024\big(2+N+N^2\big)}{(N-1)N^2(1+N)^2(2+N)} \zeta_3 \Bigg) S_1 - \frac{128\big(2-N+2N^3+N^4\big)}{(N-1)N(1+N)(2+N)} S_3 \\
& + \Bigg(-\frac{64 P_{899}}{(N-1)N(1+N)(2+N)} - \frac{256\big(-1+2N+2N^2\big)}{(N-1)N(1+N)(2+N)} S_1 \\
& + \frac{1024}{N(1+N)} \zeta_3 \Bigg) S_{-2} + 64 S_{-2}^2 + \Bigg(\frac{256 P_{897}}{(N-1)N(1+N)(2+N)} \\
& + \frac{512}{(N-1)(2+N)} S_1 \Bigg) S_{-3} + \frac{64\big(6+N+N^2\big)}{(N-1)(2+N)} S_{-4} - \frac{1024}{(N-1)(2+N)} S_{-2,2} \\
& - \frac{512}{N(1+N)} S_{-2,3} - \frac{1024}{(N-1)(2+N)} S_{-3,1} + \frac{512}{N(1+N)} S_{-4,1} \\
& + \frac{2048}{(N-1)(2+N)} S_{-2,1,1} - \frac{1280(N-2)(3+N)}{3N(1+N)} \zeta_5 \Bigg\},
\end{aligned} \tag{6.173}$$

[8] See the ancillary files to the present paper.

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with

$$P_{896} = N^4 + 2N^3 - 7N^2 - 8N + 30, \tag{6.174}$$
$$P_{897} = N^4 + 2N^3 - 4N^2 - 5N + 2, \tag{6.175}$$
$$P_{898} = N^4 + 2N^3 - N^2 - 2N - 4, \tag{6.176}$$
$$P_{899} = 5N^4 + 10N^3 - 19N^2 - 24N + 10, \tag{6.177}$$
$$P_{900} = 7N^4 + 14N^3 - 19N^2 - 26N + 216. \tag{6.178}$$

The Wilson coefficient $\Delta C_{g_1,q}^{d_{abc},(3)}$ obeys

$$\Delta C_{g_1,q}^{d_{abc},(3)}(N=1) = 0 \tag{6.179}$$

and does not contribute to the Bjorken sum rule [102, 177–179]. Three and four-index d_{abcd} structures do, however, contribute at four-loop order. Let us also present this term in z-space. One obtains

$$\Delta C_{g_1,q}^{d_{abc},(3),\delta(1-z)}(z) = \frac{d_{abc}d^{abc}N_F}{N_C}\left[64+160\zeta_2+\frac{224}{3}\zeta_3-\frac{32}{5}\zeta_2^2-\frac{1280}{3}\zeta_5\right]\delta(1-z) \tag{6.180}$$

$$\Delta C_{g_1,q}^{d_{abc},(3),+}(z) = 0 \tag{6.181}$$

$$\begin{aligned}
\Delta C_{g_1,q}^{d_{abc},(3),reg}(z) = \frac{d_{abc}d^{abc}N_F}{N_C}\Biggl\{&-768(1-z)+\frac{1}{1+z}\Biggl[\frac{1}{1-z}\Biggl(32\bigl(2+z+3z^2-3z^3+z^4\bigr)\mathrm{H}_0^2\\
&-\frac{64}{3}z^4\mathrm{H}_0^3+\frac{128}{3}\bigl(-6-11z+3z^2+11z^3+6z^4\bigr)\mathrm{H}_0\zeta_2\Biggr)+64\bigl(-3\\
&+3z+11z^2\bigr)\mathrm{H}_0-128\bigl(1+6z+z^2\bigr)\mathrm{H}_{0,1}+64\bigl(-5+5z+3z^2\\
&+z^3\bigr)\zeta_2\Biggr]+(1-z)\Biggl[-896\mathrm{H}_1+\frac{64}{3}\mathrm{H}_0^2\mathrm{H}_1+\frac{64\bigl(1+7z+z^2\bigr)}{3z}\mathrm{H}_0^2\mathrm{H}_1^2\\
&-\frac{128}{3}\mathrm{H}_0\mathrm{H}_{0,1}-\frac{256\bigl(1+7z+z^2\bigr)}{3z}\mathrm{H}_0\mathrm{H}_1\mathrm{H}_{0,1}+\frac{128\bigl(1+7z+z^2\bigr)}{3z}\mathrm{H}_{0,1}^2\\
&+\frac{256\bigl(1+7z+z^2\bigr)}{3z}\mathrm{H}_0\mathrm{H}_{0,1,1}-\frac{256\bigl(1+7z+z^2\bigr)}{3z}\mathrm{H}_{0,0,1,1}+\frac{512}{3}z\mathrm{H}_{0,-1,0,1}\\
&+\frac{256\bigl(1+7z+z^2\bigr)}{3z}\mathrm{H}_0\mathrm{H}_1\zeta_2-\frac{256\bigl(1+7z+z^2\bigr)}{3z}\mathrm{H}_{0,1}\zeta_2-\frac{512}{3}z\mathrm{H}_{0,-1}\zeta_2\\
&-\frac{512\bigl(1+4z+z^2\bigr)}{3z}\mathrm{H}_1\zeta_3\Biggr]+(1+z)\Biggl[\frac{(1-z)^2}{z}\bigl(64\mathrm{H}_{-1}^2\mathrm{H}_0-128\mathrm{H}_{-1}\mathrm{H}_{0,-1}\\
&+128\mathrm{H}_{0,-1,-1}\bigr)+\frac{512(1-z)}{3z}\bigl(\mathrm{H}_{0,-1,0,1}-\mathrm{H}_{0,-1}\zeta_2\bigr)+\frac{64\bigl(1-8z+z^2\bigr)}{z}\\
&\times\mathrm{H}_{-1}\mathrm{H}_0-\frac{64\bigl(3+4z+3z^2\bigr)}{3z}\mathrm{H}_{-1}\mathrm{H}_0^2+\frac{128\bigl(1-z+z^2\bigr)}{3z}\mathrm{H}_{-1}^2\mathrm{H}_0^2\\
&+256\mathrm{H}_{-1}\mathrm{H}_0^2\mathrm{H}_{0,1}-\frac{128\bigl(3+14z+3z^2\bigr)}{3z}\mathrm{H}_{-1}\mathrm{H}_{0,1}+\frac{512\bigl(1-z+z^2\bigr)}{3z}\\
&\times\mathrm{H}_{-1}^2\mathrm{H}_{0,1}-256\mathrm{H}_0\mathrm{H}_{0,1}^2-\frac{64\bigl(1-8z+z^2\bigr)}{z}\mathrm{H}_{0,-1}-\frac{512\bigl(1-z+z^2\bigr)}{3z}\\
&\mathrm{H}_{-1}\mathrm{H}_0H_{0,-1}+\frac{128}{3}(7+3z)\mathrm{H}_{0,0,1}-1024\mathrm{H}_{-1}\mathrm{H}_0\mathrm{H}_{0,0,1}-\frac{512\bigl(1-z+z^2\bigr)}{3z}\times
\end{aligned}$$

$$
\begin{aligned}
&\times \mathrm{H}_{-1}\mathrm{H}_{0,0,1}+\frac{512\left(1-z+z^2\right)}{3z}\mathrm{H}_{-1}\mathrm{H}_{0,0,-1}+\frac{128\left(3+14z+3z^2\right)}{3z}\mathrm{H}_{0,1,-1}\\
&+256(\mathrm{H}_{0,1,1}-\mathrm{H}_{0,1,-1})\mathrm{H}_0^2-\frac{1024\left(1-z+z^2\right)}{3z}\mathrm{H}_{-1}\mathrm{H}_{0,1,-1}-256\mathrm{H}_{0,-1,1}\mathrm{H}_0^2\\
&+\frac{128\left(3+14z+3z^2\right)}{3z}\mathrm{H}_{0,-1,1}-\frac{1024\left(1-z+z^2\right)}{3z}\mathrm{H}_{-1}\mathrm{H}_{0,-1,1}\\
&+\frac{512\left(1-z+z^2\right)}{3z}\mathrm{H}_0\mathrm{H}_{0,-1,-1}+1024\mathrm{H}_{-1}\mathrm{H}_{0,0,0,1}+\frac{512\left(1-z+z^2\right)}{3z}\\
&\times \mathrm{H}_{0,0,1,-1}+1024\mathrm{H}_0\mathrm{H}_{0,0,1,-1}+\frac{512\left(1-z+z^2\right)}{3z}\mathrm{H}_{0,0,-1,1}+1024\mathrm{H}_0\mathrm{H}_{0,0,-1,1}\\
&-\frac{512\left(1-z+z^2\right)}{3z}\mathrm{H}_{0,0,-1,-1}+\frac{1024\left(1-z+z^2\right)}{3z}\mathrm{H}_{0,1,-1,-1}+512\mathrm{H}_0\\
&\times \mathrm{H}_{0,-1,0,1}+\frac{1024\left(1-z+z^2\right)}{3z}\mathrm{H}_{0,-1,1,-1}+\frac{1024\left(1-z+z^2\right)}{3z}\mathrm{H}_{0,-1,-1,1}\\
&+1536\mathrm{H}_{0,0,0,1,1}-1024\mathrm{H}_{0,0,0,1,-1}-1024\mathrm{H}_{0,0,0,-1,1}+1024\mathrm{H}_{0,0,1,0,1}\\
&-512\mathrm{H}_{0,0,-1,0,1}+\frac{512\left(1-z+z^2\right)}{3z}\mathrm{H}_{-1}\mathrm{H}_0\zeta_2+256\mathrm{H}_{-1}\mathrm{H}_0^2\zeta_2\\
&+\frac{64\left(9+22z+9z^2\right)}{3z}\mathrm{H}_{-1}\zeta_2-\frac{512\left(1-z+z^2\right)}{3z}\mathrm{H}_{-1}^2\zeta_2+512\mathrm{H}_0\mathrm{H}_{0,1}\zeta_2\\
&-512\mathrm{H}_0\mathrm{H}_{0,-1}\zeta_2-1024\mathrm{H}_{0,0,1}\zeta_2+512\mathrm{H}_{0,0,-1}\zeta_2-\frac{2048}{5}\mathrm{H}_{-1}\zeta_2^2\\
&-512\mathrm{H}_{-1}\mathrm{H}_0\zeta_3+\frac{256\left(1-z+z^2\right)}{z}\mathrm{H}_{-1}\zeta_3-512\mathrm{H}_{0,1}\zeta_3+512\mathrm{H}_{0,-1}\zeta_3\Big]\\
&-\frac{128}{3}z(6+z)\mathrm{H}_0^2\mathrm{H}_{0,1}-\frac{128\left(-3-4z-6z^2+10z^3\right)}{3(1-z)z}\mathrm{H}_0\mathrm{H}_{0,-1}\\
&-\frac{256}{3}z^2\mathrm{H}_0^2\mathrm{H}_{0,-1}+\frac{512}{3}z(6+z)\mathrm{H}_0\mathrm{H}_{0,0,1}+\frac{128\left(-3-z-13z^2+3z^3\right)}{3z}\\
&\times \mathrm{H}_{0,0,-1}+\frac{1024}{3}z^2\mathrm{H}_0\mathrm{H}_{0,0,-1}-1024z\mathrm{H}_{0,0,0,1}-512z^2\mathrm{H}_{0,0,0,-1}\\
&-128z(2+z)\mathrm{H}_0^2\zeta_2+\frac{64(1-z)(1+z)^2}{z}\mathrm{H}_1\zeta_2+\frac{512}{5}z(4+z)\zeta_2^2\\
&+\left(-\frac{64}{3}(79-25z+15z^2)+256\zeta_2\right)\zeta_3-\frac{256}{3}(-6+z)z\mathrm{H}_0\zeta_3+2560\zeta_5\\
&-\left(\frac{256\mathrm{H}_{0,0,-1}-192\zeta_3}{1-z}\right)\Big\}. \qquad (6.182)
\end{aligned}
$$

Illustrating three-loop heavy flavor effects on the polarized non-singlet structure function $g_1^{\rm NS}(x,Q^2)$ in ref. [120] it was assumed that there is no contribution by $\Delta C_{g_1,q}^{d_{abc},(3)}$, despite this function is only known now. This is correct for the three massless quark flavors dealt with there, but changes if four or five massless flavors are considered at very high virtualities or energies. One may generally assume some suppression due to (6.179), but there is a finite contribution. Note that $\Delta C_{g_1,q}^{d_{abc},(3)}$ is independent of the scheme employed for the treatment of γ_5, since its contribution is finite.

The other Wilson coefficients are calculated in the Larin scheme. The pure-singlet Wilson coefficient at three-loops reads

$$
\Delta C_{g_1,q}^{\mathrm{PS},(3),\mathrm{L}} = C_F\Big\{T_F^2N_F^2\Big[\frac{128\zeta_3 P_{902}}{9(N-1)N^2(1+N)^2(2+N)}-\frac{32P_{936}}{243(N-1)^2N^5(1+N)^5(2+N)}+
$$

$$
\begin{aligned}
&+\left(-\frac{32(2+N)P_{919}}{81N^4(1+N)^4}-\frac{32(N-1)(2+N)S_2}{9N^2(1+N)^2}\right)S_1+\frac{64(N-1)(2+N)S_1^3}{27N^2(1+N)^2}\\
&+\frac{16(2+N)\big(21+17N-6N^2+16N^3\big)S_1^2}{27N^3(1+N)^3}+\frac{16(2+N)\big(21+5N-24N^2+46N^3\big)}{27N^3(1+N)^3}\\
&\times S_2-\frac{160(N-1)(2+N)S_3}{27N^2(1+N)^2}+\frac{512\big(-5-12N+5N^2\big)S_{-2}}{9(N-1)^2N(1+N)^2(2+N)}\\
&-\frac{1024S_{-3}}{3(N-1)N(1+N)(2+N)}\Bigg]\\
&+{\color{blue}C_AT_FN_F}\Bigg[\frac{64S_{-3,1}P_{901}}{3(N-1)N^2(1+N)^2(2+N)}-\frac{20S_4P_{903}}{3(N-1)N^2(1+N)^2(2+N)}\\
&-\frac{16S_{-2,2}P_{905}}{(N-1)N^2(1+N)^2(2+N)}+\frac{16S_{2,1}P_{906}}{3N^3(1+N)^3}+\frac{64S_{-2,1,1}P_{907}}{3(N-1)N^2(1+N)^2(2+N)}\\
&+\frac{8S_1^3P_{910}}{27N^3(1+N)^3}-\frac{8S_{-4}P_{915}}{3(N-1)N^2(1+N)^2(2+N)}-\frac{16S_{-2,1}P_{923}}{3(N-1)N^3(1+N)^3(2+N)}\\
&+\frac{8S_3P_{927}}{27(N-1)N^3(1+N)^3(2+N)}+\frac{8P_{938}}{243(N-1)^2N^6(1+N)^6(2+N)}\\
&-\left(\frac{8S_2P_{913}}{9N^3(1+N)^3}+\frac{16S_{-2,1}P_{911}}{3(N-1)N^2(1+N)^2(2+N)}-\frac{8S_3P_{912}}{9(N-1)N^2(1+N)^2(2+N)}\right.\\
&\left.-\frac{8P_{934}}{81(N-1)N^5(1+N)^5(2+N)}\right)S_1+\left(\frac{4P_{926}}{27N^4(1+N)^4}-\frac{80(N-1)(2+N)S_2}{3N^2(1+N)^2}\right)\\
&\times S_1^2+\frac{28(N-1)(2+N)S_1^4}{9N^2(1+N)^2}-\frac{16\big(5-2N-2N^2\big)S_2^2}{3N^2(1+N)^2}-\left(\frac{16\big(22+N+N^2\big)S_1^2}{3N^2(1+N)^2}\right.\\
&+\frac{16S_1P_{922}}{3(N-1)N^3(1+N)^3(2+N)}+\frac{16P_{933}}{9(N-1)^2N^4(1+N)^4(2+N)}\\
&\left.-\frac{32\big(8+N+N^2\big)S_2}{3N^2(1+N)^2}\right)S_{-2}-\frac{16\big(-2+3N+3N^2\big)S_{-2}^2}{3N^2(1+N)^2}-\frac{4S_2P_{928}}{27N^4(1+N)^4}\\
&+\left(\frac{8P_{925}}{3(N-1)N^3(1+N)^3(2+N)}+\frac{8S_1P_{916}}{3(N-1)N^2(1+N)^2(2+N)}\right)S_{-3}\\
&+\frac{320\big(2+N+N^2\big)S_{3,1}}{(N-1)N^2(1+N)^2(2+N)}+\frac{32(N-1)(2+N)S_{2,1,1}}{3N^2(1+N)^2}-\frac{96(N-1)(2+N)\zeta_2^2}{5N^2(1+N)^2}\\
&+\left(-\frac{16S_1P_{908}}{3(N-1)N^2(1+N)^2(2+N)}-\frac{16P_{924}}{9(N-1)N^3(1+N)^3(2+N)}\right)\zeta_3\Bigg]\Bigg\}\\
&+{\color{blue}C_F^2T_FN_F}\Bigg\{-\frac{64S_{3,1}P_{904}}{(N-1)N^2(1+N)^2(2+N)}+\frac{8S_4P_{909}}{3(N-1)N^2(1+N)^2(2+N)}\\
&-\frac{8S_3P_{920}}{9(N-1)N^3(1+N)^3(2+N)}-\frac{4S_2P_{931}}{3(N-1)N^4(1+N)^4(2+N)}\\
&-\frac{8P_{937}}{3(N-1)^2N^6(1+N)^6(2+N)}+\left(\frac{16S_3P_{914}}{9(N-1)N^2(1+N)^2(2+N)}\right.\\
&-\frac{8S_2P_{917}}{3N^3(1+N)^3}+\frac{8P_{935}}{3(N-1)^2N^5(1+N)^5(2+N)}\\
&\left.+\frac{256\big(2+N+N^2\big)S_{-2,1}}{(N-1)N^2(1+N)^2(2+N)}\right)S_1+
\end{aligned}
$$

$$+\left(\frac{4P_{932}}{3(N-1)N^4(1+N)^4(2+N)}-\frac{112(N-1)(2+N)S_2}{3N^2(1+N)^2}\right)S_1^2$$
$$+\frac{16(2+N)\big(22+2N-15N^2+27N^3\big)S_1^3}{9N^3(1+N)^3}+\frac{68(N-1)(2+N)S_2^2}{3N^2(1+N)^2}$$
$$+\Bigg(-\frac{32P_{929}}{(N-1)^2N^3(1+N)^3(2+N)}-\frac{32S_1P_{930}}{(N-1)^2N^3(1+N)^3(2+N)}$$
$$+\frac{128\big(-4+N+N^2\big)S_1^2}{(N-1)N^2(1+N)^2(2+N)}-\frac{128\big(-4+N+N^2\big)S_2}{(N-1)N^2(1+N)^2(2+N)}\Bigg)S_{-2}$$
$$+\left(\frac{64P_{918}}{(N-1)N^3(1+N)^2(2+N)}-\frac{128(N-2)(3+N)S_1}{(N-1)N^2(1+N)^2(2+N)}\right)S_{-3}$$
$$-\frac{128\big(-6+5N+5N^2\big)S_{-4}}{(N-1)N^2(1+N)^2(2+N)}-\frac{64\big(4+3N-3N^2+2N^3\big)S_{2,1}}{3N^3(1+N)^3}$$
$$-\frac{256\big(6-N-2N^2+N^3\big)S_{-2,1}}{(N-1)N^3(1+N)^2(2+N)}+\frac{256\big(-2+3N+3N^2\big)S_{-2,2}}{(N-1)N^2(1+N)^2(2+N)}$$
$$+\frac{1024\big(-1+N+N^2\big)S_{-3,1}}{(N-1)N^2(1+N)^2(2+N)}-\frac{32(N-1)(2+N)S_{2,1,1}}{3N^2(1+N)^2}$$
$$+\frac{96(N-1)(2+N)\zeta_2^2}{5N^2(1+N)^2}-\frac{1024S_{-2,1,1}}{(N-1)N(1+N)(2+N)}+\frac{44(N-1)(2+N)S_1^4}{9N^2(1+N)^2}$$
$$+\left(\frac{32P_{921}}{3(N-1)N^3(1+N)^3(2+N)}-\frac{128\big(7+4N+4N^2\big)S_1}{3N^2(1+N)^2}\right)\zeta_3\Bigg\}, \tag{6.183}$$

with the polynomials

$$P_{901}=N^4+2N^3-63N^2-64N+28, \tag{6.184}$$
$$P_{902}=N^4+2N^3-39N^2-40N+4, \tag{6.185}$$
$$P_{903}=N^4+2N^3-15N^2-16N+124, \tag{6.186}$$
$$P_{904}=N^4+2N^3+5N^2+4N+20, \tag{6.187}$$
$$P_{905}=N^4+2N^3+73N^2+72N-20, \tag{6.188}$$
$$P_{906}=N^4+10N^3-5N^2+18N+12, \tag{6.189}$$
$$P_{907}=5N^4+10N^3+93N^2+88N-4, \tag{6.190}$$
$$P_{908}=7N^4+14N^3-81N^2-88N+100, \tag{6.191}$$
$$P_{909}=19N^4+38N^3-105N^2-124N+556, \tag{6.192}$$
$$P_{910}=23N^4+N^3-107N^2+167N+222, \tag{6.193}$$
$$P_{911}=25N^4+50N^3+137N^2+112N+60, \tag{6.194}$$
$$P_{912}=37N^4+74N^3-375N^2-412N-44, \tag{6.195}$$
$$P_{913}=37N^4+77N^3-97N^2+151N+246, \tag{6.196}$$
$$P_{914}=43N^4+86N^3+87N^2+44N+316, \tag{6.197}$$
$$P_{915}=47N^4+94N^3-393N^2-440N+308, \tag{6.198}$$
$$P_{916}=63N^4+126N^3+127N^2+64N+4, \tag{6.199}$$
$$P_{917}=65N^4+86N^3-49N^2+66N+96, \tag{6.200}$$

$$P_{918} = N^5+2N^4+7N^3-6N+12, \tag{6.201}$$
$$P_{919} = 161N^5+23N^4+55N^3+55N^2-12N-18, \tag{6.202}$$
$$P_{920} = 3N^6+27N^5+157N^4+461N^3+1212N^2+380N+64, \tag{6.203}$$
$$P_{921} = 3N^6+33N^5+71N^4+115N^3+294N^2+4N-88, \tag{6.204}$$
$$P_{922} = 10N^6+24N^5+7N^4-30N^3-135N^2-64N+92, \tag{6.205}$$
$$P_{923} = 21N^6+34N^5-191N^4-248N^3-40N^2-184N-160, \tag{6.206}$$
$$P_{924} = 58N^6+237N^5-811N^4-1741N^3+873N^2+652N-852, \tag{6.207}$$
$$P_{925} = 63N^6+138N^5+23N^4+228N^3+308N^2-456N-240, \tag{6.208}$$
$$P_{926} = 79N^6+78N^5-1526N^4-1632N^3-794N^2-3351N-1710, \tag{6.209}$$
$$P_{927} = 199N^6+588N^5+20N^4-208N^3+3177N^2+1612N-1068, \tag{6.210}$$
$$P_{928} = 953N^6+2298N^5-664N^4-2490N^3+980N^2-1455N-1422, \tag{6.211}$$
$$P_{929} = N^7+N^6-10N^5-14N^4-35N^3+9N^2+20N-4, \tag{6.212}$$
$$P_{930} = N^7+2N^6+20N^5+10N^4-25N^3-92N^2-12N+32, \tag{6.213}$$
$$P_{931} = 56N^8+205N^7+154N^6+40N^5-84N^4-977N^3-942N^2+308N+472, \tag{6.214}$$
$$P_{932} = 156N^8+501N^7+164N^6-374N^5-242N^4-1139N^3-958N^2+524N \\ +600, \tag{6.215}$$
$$P_{933} = 45N^9+57N^8+386N^7+232N^6-1896N^5-2008N^4+937N^3+63N^2 \\ -228N-84, \tag{6.216}$$
$$P_{934} = 2701N^{10}+11348N^9+6606N^8-13281N^7+8301N^6+20247N^5+980N^4 \\ +32572N^3+10626N^2-17460N-10800, \tag{6.217}$$
$$P_{935} = 96N^{11}+302N^{10}-111N^9-890N^8-689N^7+706N^6+2477N^5+2544N^4 \\ -497N^3-1854N^2+324N+664, \tag{6.218}$$
$$P_{936} = 2042N^{11}+6095N^{10}-2575N^9-15042N^8-2412N^7+15972N^6+22057N^5 \\ +18358N^4+2452N^3-8427N^2+468N+2484, \tag{6.219}$$
$$P_{937} = 260N^{13}+1033N^{12}+579N^{11}-2050N^{10}-2768N^9-582N^8+146N^7-680N^6 \\ +752N^5+1237N^4-765N^3-950N^2+356N+360, \tag{6.220}$$
$$P_{938} = 30520N^{13}+123245N^{12}+50588N^{11}-274955N^{10}-241017N^9+129972N^8 \\ +331898N^7+404113N^6+59461N^5-137557N^4-25818N^3+116766N^2 \\ +1080N-29160. \tag{6.221}$$

Finally, the Wilson coefficient $\Delta C_{g_1,g}^{(3)}$ is given by

$$\Delta C_{g_1,g}^{(3)} = \Delta C_{g_1,g}^{(3),a} + \Delta C_{g_1,g}^{(3),d_{abc}}, \tag{6.222}$$

with

$$\Delta C_{g_1,g}^{(3),a} = C_F \Big\{ T_F^2 N_F^2 \Big[-\frac{256 P_{941}}{9N(1+N)^2(2+N)^2} S_{2,1} - \frac{64 P_{1002}}{9(N-1)N^3(1+N)^3(2+N)} \zeta_3 \\ + \frac{16(N-1)P_{946}}{81N^3(1+N)^3} S_1^3 + \frac{32 P_{1007}}{81N^3(1+N)^3(2+N)^2} S_3 - \frac{4 P_{1029}}{27N^4(1+N)^4(2+N)^2} \times$$

$$\times S_2 + \frac{P_{1045}}{243(N-1)^2N^6(1+N)^6(2+N)^2} + \Biggl(-\frac{16P_{1003}}{27N^3(1+N)^3(2+N)^2}$$

$$\times S_2 - \frac{8P_{1039}}{243(N-1)N^5(1+N)^5(2+N)^2} + \frac{32\bigl(74+35N+35N^2\bigr)}{27N(1+N)(2+N)}S_3$$

$$-\frac{128\bigl(2+N+N^2\bigr)}{3N(1+N)(2+N)}S_{2,1} - \frac{512}{3N(1+N)(2+N)}S_{-2,1}$$

$$+\frac{64\bigl(-46+5N+5N^2\bigr)}{9N(1+N)(2+N)}\zeta_3\Biggr)S_1 + \frac{68(N-1)}{27N(1+N)}S_1^4 + \Biggl(\frac{4P_{1025}}{81N^4(1+N)^4(2+N)}$$

$$-\frac{8\bigl(-62+7N+7N^2\bigr)}{9N(1+N)(2+N)}S_2\Biggr)S_1^2 + \frac{4\bigl(-226+17N+17N^2\bigr)}{9N(1+N)(2+N)}S_2^2$$

$$-\frac{56\bigl(14+17N+17N^2\bigr)}{9N(1+N)(2+N)}S_4 + \Biggl(-\frac{64P_{960}}{9(N-1)N^2(1+N)^2(2+N)^2}S_1$$

$$-\frac{32P_{1028}}{9(N-1)^2N^3(1+N)^3(2+N)^2} + \frac{512}{3N(1+N)(2+N)}S_1^2$$

$$-\frac{512}{3N(1+N)(2+N)}S_2\Biggr)S_{-2} + \Biggl(\frac{128P_{961}}{9(N-1)N^2(1+N)^2(2+N)^2}$$

$$-\frac{256}{N(1+N)(2+N)}S_1\Biggr)S_{-3} - \frac{256S_{-2}^2}{3N(1+N)(2+N)} - \frac{1024}{3N(1+N)(2+N)}S_{-4}$$

$$+\frac{64\bigl(-10+3N+3N^2\bigr)}{3N(1+N)(2+N)}S_{3,1} - \frac{512(-3+5N)}{9N^2(1+N)(2+N)}S_{-2,1}$$

$$+\frac{512}{3N(1+N)(2+N)}S_{-2,2} + \frac{1024}{3N(1+N)(2+N)}S_{-3,1} + \frac{64\bigl(10+N+N^2\bigr)}{3N(1+N)(2+N)}$$

$$\times S_{2,1,1}\Biggr] + C_AT_FN_F\Biggl[-\frac{32P_{965}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-2,1,1}$$

$$-\frac{2P_{949}}{9N^2(1+N)^2(2+N)}S_2^2 + \frac{16P_{966}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-2,2}$$

$$+\frac{16P_{967}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-3,1} - \frac{16P_{978}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{3,1}$$

$$+\frac{4P_{997}}{9(N-1)N^2(1+N)^2(2+N)^2}S_4 + \frac{8P_{1006}}{9N^3(1+N)^3(2+N)^2}S_{2,1}$$

$$-\frac{16P_{1017}}{9(N-1)N^3(1+N)^3(2+N)^2}S_{-2,1} - \frac{8P_{1023}}{9(N-1)N^3(1+N)^3(2+N)^2}\zeta_3$$

$$-\frac{8P_{1027}}{81(N-1)N^3(1+N)^3(2+N)^2}S_3 + \frac{P_{1047}}{972(N-1)^2N^6(1+N)^6(2+N)^3}$$

$$+\Biggl(\frac{8P_{944}}{3N^2(1+N)^2(2+N)}S_{2,1} - \frac{32P_{959}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-2,1}$$

$$-\frac{8P_{992}}{27(N-1)N^2(1+N)^2(2+N)^2}S_3 - \frac{8P_{999}}{9(N-1)N^2(1+N)^2(2+N)^2}\zeta_3$$

$$+\frac{4P_{1010}}{27N^3(1+N)^3(2+N)^2}S_2 + \frac{2P_{1043}}{243(N-1)^2N^5(1+N)^5(2+N)^3}$$

$$-\frac{80(N-1)}{N(1+N)}S_2^2 + \frac{8\bigl(-382+47N+47N^2\bigr)}{3N(1+N)(2+N)}S_4 + \frac{64\bigl(130+7N+7N^2\bigr)}{3N(1+N)(2+N)}S_{3,1}$$

$$+\frac{128\bigl(-54+5N+5N^2\bigr)}{N(1+N)(2+N)}S_{-2,2} + \frac{16\bigl(-1378+113N+113N^2\bigr)}{3N(1+N)(2+N)}S_{-3,1} -$$

$$
\begin{aligned}
&-\frac{80(N-1)}{N(1+N)}S_{2,1,1}-\frac{32\big(-1214+127N+127N^2\big)}{3N(1+N)(2+N)}S_{-2,1,1}-\frac{96(N-1)\zeta_2^2}{5N(1+N)}\bigg)S_1\\
&+\bigg(\frac{4P_{948}}{9N^2(1+N)^2(2+N)}S_2+\frac{P_{1030}}{81(N-1)N^4(1+N)^4(2+N)^2}\\
&-\frac{8\big(310+13N+13N^2\big)}{3N(1+N)(2+N)}S_3-\frac{32(N-1)}{N(1+N)}S_{2,1}+\frac{32\big(-286+35N+35N^2\big)}{3N(1+N)(2+N)}\\
&\times S_{-2,1}+\frac{32\big(166+19N+19N^2\big)}{3N(1+N)(2+N)}\zeta_3\bigg)S_1^2+\bigg(-\frac{4P_{1001}}{81N^3(1+N)^3(2+N)}\\
&+\frac{136(N-1)}{3N(1+N)}S_2\bigg)S_1^3+\bigg(\frac{P_{1035}}{27(N-1)N^4(1+N)^4(2+N)^2}+\frac{8\big(898+7N+7N^2\big)}{3N(1+N)(2+N)}\\
&\times S_3-\frac{128\big(-72+7N+7N^2\big)}{N(1+N)(2+N)}S_{-2,1}-\frac{96\big(-2+3N+3N^2\big)}{N(1+N)(2+N)}\zeta_3\bigg)S_2\\
&-\frac{2\big(369-23N-96N^2+206N^3\big)}{27N^2(1+N)^2}S_1^4+\frac{256\big(11-N-N^2\big)}{3N(1+N)(2+N)}S_5\\
&+\bigg(-\frac{8P_{981}}{3(N-1)N^2(1+N)^2(2+N)^2}S_2\\
&+\frac{8P_{985}}{3(N-1)N^2(1+N)^2(2+N)^2}S_1^2-\frac{8P_{1037}}{9(N-1)^2N^4(1+N)^4(2+N)^3}\\
&+\bigg(-\frac{16P_{1031}}{9(N-1)^2N^3(1+N)^3(2+N)^3}-\frac{48\big(-18+N+N^2\big)}{N(1+N)(2+N)}S_2\bigg)S_1\\
&-\frac{16\big(178-17N-17N^2\big)}{9N(1+N)(2+N)}S_1^3-\frac{64\big(2+35N+35N^2\big)}{9N(1+N)(2+N)}\\
&\times S_3+\frac{64\big(-146+13N+13N^2\big)}{N(1+N)(2+N)}S_{2,1}-\frac{1024\big(-4+N+N^2\big)}{N(1+N)(2+N)}\zeta_3\\
&-\frac{128(N-1)}{3N(1+N)}S_{-2,1}\bigg)S_{-2}+\bigg(\frac{32P_{971}}{3(N-1)N^2(1+N)^2(2+N)^2}+\frac{256\big(-11+N+N^2\big)}{3N(1+N)(2+N)}\\
&\times S_1\bigg)S_{-2}^2+\bigg(-\frac{16P_{977}}{3(N-1)N^2(1+N)^2(2+N)^2}S_1+\frac{8P_{1022}}{9(N-1)N^3(1+N)^3(2+N)^2}\\
&-\frac{16\big(-490+41N+41N^2\big)}{3N(1+N)(2+N)}S_1^2+\frac{128\big(-107+10N+10N^2\big)}{3N(1+N)(2+N)}S_2\\
&-\frac{32\big(22+5N+5N^2\big)}{N(1+N)(2+N)}S_{-2}\bigg)S_{-3}+\bigg(\frac{8P_{980}}{3(N-1)N^2(1+N)^2(2+N)^2}\\
&-\frac{16\big(-198+19N+19N^2\big)}{N(1+N)(2+N)}S_1\bigg)S_{-4}-\frac{16\big(562+31N+31N^2\big)}{3N(1+N)(2+N)}S_{-5}\\
&+\frac{64\big(-38+N+N^2\big)}{N(1+N)(2+N)}S_{2,3}-\frac{64\big(-134+13N+13N^2\big)}{3N(1+N)(2+N)}S_{4,1}\\
&+\frac{160\big(-86+7N+7N^2\big)}{3N(1+N)(2+N)}S_{2,-3}-\frac{256\big(-50+N+N^2\big)}{3N(1+N)(2+N)}S_{-2,3}\\
&+\frac{256\big(-7+N+N^2\big)}{N(1+N)(2+N)}S_{-4,1}-\frac{16\big(128+67N-26N^2+5N^3\big)}{3N(1+N)^2(2+N)}S_{2,1,1}-
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{64\big(-434+37N+37N^2\big)}{3N(1+N)(2+N)}S_{2,1,-2}-\frac{96\big(46+N+N^2\big)}{N(1+N)(2+N)}S_{3,1,1}\\
&-\frac{16(N-1)}{3N(1+N)}S_{2,2,1}-\frac{64\big(-118+11N+11N^2\big)}{3N(1+N)(2+N)}S_{-2,1,-2}-\frac{2560\big(-11+N+N^2\big)}{3N(1+N)(2+N)}\\
&\times S_{-2,2,1}-\frac{160\big(-58+5N+5N^2\big)}{N(1+N)(2+N)}S_{-3,1,1}+\frac{320\big(-178+17N+17N^2\big)}{3N(1+N)(2+N)}S_{-2,1,1,1}\\
&+\frac{160(N-1)}{N(1+N)}S_{2,1,1,1}+\frac{96(N-1)\big(5-3N-3N^2\big)}{5N^2(1+N)^2}\zeta_2^2-\frac{80\big(-118+N+N^2\big)}{N(1+N)(2+N)}\\
&\times\zeta_5\Big]\Big\}+C_AT_F^2N_F^2\Big\{\frac{128P_{954}}{27N^2(1+N)^2(2+N)^2}S_{2,1}+\frac{32P_{955}}{9(N-1)N^2(1+N)^2(2+N)}\zeta_3\\
&-\frac{64P_{956}}{81N^2(1+N)^2(2+N)^2}S_3+\frac{8P_{1041}}{243(N-1)^2N^5(1+N)^5(2+N)^2}\\
&-\frac{16P_{1004}}{81N^3(1+N)^3(2+N)^2}S_2+\Big[\frac{8P_{1033}}{243(N-1)N^4(1+N)^4(2+N)^2}\\
&-\frac{32P_{957}}{27N^2(1+N)^2(2+N)^2}S_2+\frac{32\big(-110+19N+19N^2\big)}{27N(1+N)(2+N)}S_3+\frac{256\big(1+N+N^2\big)}{9N(1+N)(2+N)}\\
&\times S_{2,1}+\frac{256}{3N(1+N)(2+N)}S_{-2,1}-\frac{32\big(-62+13N+13N^2\big)}{9N(1+N)(2+N)}\zeta_3\Big]S_1\\
&+\Big[\frac{16P_{996}}{81N^3(1+N)^3(2+N)}-\frac{8\big(-14+19N+19N^2\big)}{9N(1+N)(2+N)}S_2\Big]S_1^2+\frac{28(N-1)}{27N(1+N)}S_1^4\\
&+\frac{32\big(27-22N-9N^2+19N^3\big)}{81N^2(1+N)^2}S_1^3+\frac{4\big(34+31N+31N^2\big)}{9N(1+N)(2+N)}S_2^2+\frac{56\big(10+7N+7N^2\big)}{9N(1+N)(2+N)}\\
&\times S_4+\Big[-\frac{64P_{964}}{9(N-1)N^2(1+N)^2(2+N)^2}S_1-\frac{32P_{1021}}{81(N-1)^2N^2(1+N)^3(2+N)^2}\\
&-\frac{32\big(6+N+N^2\big)}{3N(1+N)(2+N)}S_1^2+\frac{32\big(6+N+N^2\big)}{3N(1+N)(2+N)}S_2\Big]S_{-2}+\frac{128}{3N(1+N)(2+N)}S_{-2}^2\\
&+\Big[\frac{128P_{972}}{27(N-1)N^2(1+N)^2(2+N)^2}+\frac{128\big(1+N+N^2\big)}{3N(1+N)(2+N)}S_1\Big]S_{-3}\\
&-\frac{64\big(-38+7N+7N^2\big)}{9N(1+N)(2+N)}S_{-4}-\frac{128\big(-13+5N+5N^2\big)}{9N(1+N)(2+N)}S_{3,1}+\frac{256(-3+5N)}{9N^2(1+N)(2+N)}\\
&\times S_{-2,1}-\frac{256}{3N(1+N)(2+N)}S_{-2,2}-\frac{512}{3N(1+N)(2+N)}S_{-3,1}\\
&-\frac{64\big(16+N+N^2\big)}{9N(1+N)(2+N)}S_{2,1,1}\Big\}+C_A^2T_FN_F\Big\{\frac{P_{939}}{9N^2(1+N)^2(2+N)}S_2^2\\
&+\frac{16P_{943}}{9N^2(1+N)^2(2+N)}S_{2,1,1}-\frac{32P_{983}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-2,1,1}\\
&+\frac{16P_{984}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-2,2}+\frac{16P_{986}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-3,1}\\
&+\frac{8P_{987}}{9(N-1)N^2(1+N)^2(2+N)^2}S_{3,1}-\frac{2P_{998}}{9(N-1)N^2(1+N)^2(2+N)^2}S_4\\
&-\frac{8P_{1008}}{27N^3(1+N)^3(2+N)^2}S_{2,1}+\frac{4P_{1014}}{9(N-1)N^3(1+N)^3(2+N)^2}\zeta_3-
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{32P_{1015}}{9(N-1)N^3(1+N)^3(2+N)^2}S_{-2,1}+\frac{8P_{1026}}{81(N-1)N^3(1+N)^3(2+N)^2}S_3\\
&-\frac{4P_{1046}}{243(N-1)^2N^6(1+N)^6(2+N)^3}+\Bigg[-\frac{16P_{945}}{9N^2(1+N)^2(2+N)}S_{2,1}\\
&+\frac{16P_{982}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-2,1}+\frac{8P_{994}}{9(N-1)N^2(1+N)^2(2+N)^2}\zeta_3\\
&-\frac{8P_{995}}{27(N-1)N^2(1+N)^2(2+N)^2}S_3+\frac{8P_{1009}}{27N^3(1+N)^3(2+N)^2}S_2\\
&-\frac{4P_{1044}}{243(N-1)^2N^5(1+N)^5(2+N)^3}-\frac{124(N-1)}{3N(1+N)}S_2^2-\frac{48(N-1)}{N(1+N)}S_4\\
&-\frac{32\big(70+N+N^2\big)}{3N(1+N)(2+N)}S_{3,1}-\frac{80\big(-106+5N+5N^2\big)}{3N(1+N)(2+N)}S_{-3,1}-\frac{16(N-1)}{3N(1+N)}S_{2,1,1}\\
&-\frac{32\big(-226+5N+5N^2\big)}{3N(1+N)(2+N)}S_{-2,2}-\frac{32\big(374+5N+5N^2\big)}{3N(1+N)(2+N)}S_{-2,1,1}+\frac{96(N-1)}{5N(1+N)}\zeta_2^2\Bigg]\\
&\times S_1+\Bigg[\frac{2P_{950}}{9N^2(1+N)^2(2+N)}S_2-\frac{2P_{1036}}{81(N-1)N^4(1+N)^4(2+N)^2}\\
&-\frac{8\big(-154+29N+29N^2\big)}{3N(1+N)(2+N)}S_3+\frac{32(N-1)}{3N(1+N)}S_{2,1}+\frac{32\big(22+3N+3N^2\big)}{N(1+N)(2+N)}S_{-2,1}\\
&+\frac{16\big(-190+17N+17N^2\big)}{3N(1+N)(2+N)}\zeta_3\Bigg]S_1^2+\Bigg[-\frac{8P_{1000}}{81N^3(1+N)^3(2+N)}+\frac{128(N-1)}{9N(1+N)}S_2\Bigg]\\
&\times S_1^3+\frac{\big(-528+221N+216N^2-293N^3\big)}{27N^2(1+N)^2}S_1^4-\frac{4(N-1)}{3N(1+N)}S_1^5\\
&+\Bigg[\frac{2P_{1034}}{81(N-1)N^4(1+N)^4(2+N)^2}-\frac{8\big(658+31N+31N^2\big)}{9N(1+N)(2+N)}S_3+\frac{48\big(2+N+N^2\big)}{N(1+N)(2+N)}\\
&\times\zeta_3+\frac{128\big(-27+2N+2N^2\big)}{N(1+N)(2+N)}S_{-2,1}\Bigg]S_2+\frac{32\big(-26+7N+7N^2\big)}{3N(1+N)(2+N)}S_5\\
&+\Bigg[\frac{8P_{975}}{3(N-1)N^2(1+N)^2(2+N)^2}S_1^2-\frac{8P_{979}}{3(N-1)N^2(1+N)^2(2+N)^2}S_2\\
&+\frac{8P_{1040}}{81(N-1)^2N^4(1+N)^4(2+N)^3}+\Bigg(\frac{16P_{1032}}{9(N-1)^2N^3(1+N)^3(2+N)^3}\\
&-\frac{16\big(50+11N+11N^2\big)}{3N(1+N)(2+N)}S_2\Bigg)S_1-\frac{16\big(-26+N+N^2\big)}{3N(1+N)(2+N)}S_1^3+\frac{128\big(-5+4N+4N^2\big)}{3N(1+N)(2+N)}\\
&\times S_3+\frac{1024\big(11-N-N^2\big)}{3N(1+N)(2+N)}S_{2,1}+\frac{128(N-1)}{3N(1+N)}S_{-2,1}-\frac{128\big(7-2N-2N^2\big)}{N(1+N)(2+N)}\\
&\times\zeta_3\Bigg]S_{-2}+\Bigg[-\frac{8P_{976}}{3(N-1)N^2(1+N)^2(2+N)^2}-\frac{16\big(-18+N+N^2\big)}{N(1+N)(2+N)}S_1\Bigg]S_{-2}^2\\
&+\Bigg[-\frac{8P_{991}}{3(N-1)N^2(1+N)^2(2+N)^2}S_1-\frac{16P_{1024}}{27(N-1)N^3(1+N)^3(2+N)^2}\\
&-\frac{16\big(122+29N+29N^2\big)}{3N(1+N)(2+N)}S_1^2-\frac{32\big(-64+9N+9N^2\big)}{N(1+N)(2+N)}S_2\\
&+\frac{32\big(10+13N+13N^2\big)}{3N(1+N)(2+N)}S_{-2}\Bigg]S_{-3}+\Bigg[-\frac{8P_{989}}{9(N-1)N^2(1+N)^2(2+N)^2}+
\end{aligned}
$$

$$+\frac{128\left(-13+2N+2N^2\right)}{N(1+N)(2+N)}S_1\Bigg]S_{-4}+\frac{16\left(238+N+N^2\right)}{3N(1+N)(2+N)}S_{-5}$$
$$-\frac{48\left(-14+N+N^2\right)}{N(1+N)(2+N)}S_{2,3}+\frac{128\left(34+N+N^2\right)}{3N(1+N)(2+N)}S_{2,-3}-\frac{256\left(19+N+N^2\right)}{3N(1+N)(2+N)}S_{-2,3}$$
$$+\frac{16\left(-146+19N+19N^2\right)}{3N(1+N)(2+N)}S_{4,1}-\frac{32\left(-142+35N+35N^2\right)}{3N(1+N)(2+N)}S_{-4,1}$$
$$+\frac{1024\left(-11+N+N^2\right)}{3N(1+N)(2+N)}S_{2,1,-2}+\frac{176(N-1)}{3N(1+N)}S_{2,2,1}+\frac{1152}{N(1+N)(2+N)}S_{3,1,1}$$
$$-\frac{64\left(22+7N+7N^2\right)}{3N(1+N)(2+N)}S_{-2,1,-2}+\frac{64\left(-178+17N+17N^2\right)}{3N(1+N)(2+N)}S_{-2,2,1}$$
$$+\frac{32\left(-130+17N+17N^2\right)}{N(1+N)(2+N)}S_{-3,1,1}-\frac{64\left(-322+17N+17N^2\right)}{3N(1+N)(2+N)}S_{-2,1,1,1}$$
$$+\frac{48(N-1)\left(-8+3N+3N^2\right)}{5N^2(1+N)^2}\zeta_2^2+\frac{160\left(-14+N+N^2\right)}{N(1+N)(2+N)}\zeta_5\Bigg\}$$
$$+C_F^2T_FN_F\Bigg\{\frac{P_{940}}{3N^2(1+N)^2(2+N)}S_2^2+\frac{64P_{952}}{3(N-1)N(1+N)^2(2+N)^2}S_{-2,1,1}$$
$$-\frac{16P_{953}}{3N^3(1+N)^2(2+N)}S_{2,1}-\frac{64P_{962}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-2,2}$$
$$+\frac{32P_{968}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{3,1}-\frac{32P_{969}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-3,1}$$
$$-\frac{2P_{988}}{3(N-1)N^2(1+N)^2(2+N)^2}S_4+\frac{4P_{1005}}{9(N-1)N^3(1+N)^2(2+N)^2}S_3$$
$$+\frac{32P_{1013}}{3(N-1)N^3(1+N)^3(2+N)^2}S_{-2,1}+\frac{16P_{1019}}{3(N-1)N^3(1+N)^3(2+N)^2}\zeta_3$$
$$+\frac{P_{1042}}{12(N-1)N^6(1+N)^6(2+N)^2}+\Bigg[\frac{32P_{942}}{3N^2(1+N)^2(2+N)}S_{2,1}$$
$$-\frac{64P_{951}}{3(N-1)N(1+N)^2(2+N)^2}S_{-2,1}+\frac{16P_{990}}{3(N-1)N^2(1+N)^2(2+N)^2}\zeta_3$$
$$+\frac{2P_{974}}{3N^3(1+N)^3(2+N)}S_2-\frac{8P_{993}}{9(N-1)N^2(1+N)^2(2+N)^2}S_3$$
$$-\frac{2P_{1038}}{3(N-1)N^5(1+N)^5(2+N)^2}-\frac{8\left(-766+95N+95N^2\right)}{3N(1+N)(2+N)}S_4-\frac{188(N-1)}{3N(1+N)}$$
$$\times S_2^2-\frac{32\left(262+13N+13N^2\right)}{3N(1+N)(2+N)}S_{3,1}+\frac{128\left(14+N+N^2\right)}{N(1+N)(2+N)}S_{-2,2}$$
$$+\frac{64\left(70+13N+13N^2\right)}{3N(1+N)(2+N)}S_{-3,1}+\frac{256(N-1)}{3N(1+N)}S_{2,1,1}+\frac{128\left(-98+N+N^2\right)}{3N(1+N)(2+N)}S_{-2,1,1}$$
$$\Bigg]S_1+\Bigg[\frac{2P_{947}}{3N^2(1+N)^2(2+N)}S_2+\frac{48\left(18-N-N^2\right)}{N(1+N)(2+N)}S_3$$
$$+\frac{P_{1011}}{3N^4(1+N)^4(2+N)}+\frac{64(N-1)}{3N(1+N)}S_{2,1}-\frac{256\left(-14+N+N^2\right)}{3N(1+N)(2+N)}S_{-2,1}$$
$$-\frac{128\left(14+5N+5N^2\right)}{3N(1+N)(2+N)}\zeta_3\Bigg]S_1^2+\Bigg[-\frac{2P_{958}}{9N^3(1+N)^3}+\frac{472(N-1)}{9N(1+N)}S_2\Bigg]S_1^3-$$

$$
\begin{aligned}
&-\frac{20(N-1)}{3N(1+N)}S_1^5-\frac{(N-1)\big(-226+97N+249N^2\big)}{9N^2(1+N)^2}S_1^4+\Bigg[\frac{P_{1018}}{3N^4(1+N)^4(2+N)}\\
&+\frac{16\big(-1462+83N+83N^2\big)}{9N(1+N)(2+N)}S_3+\frac{256\big(-38+N+N^2\big)}{3N(1+N)(2+N)}S_{-2,1}+\frac{192(N-1)}{N(1+N)}\zeta_3\Bigg]S_2\\
&+\frac{64\big(-82+17N+17N^2\big)}{3N(1+N)(2+N)}S_5+\Bigg[\frac{8P_{1020}}{3(N-1)N^2(1+N)^4(2+N)^2}\\
&+\Bigg(\frac{32P_{1012}}{3(N-1)N^3(1+N)^3(2+N)^2}-\frac{640(N-1)}{3N(1+N)}S_2\Bigg)S_1+\frac{640(N-1)}{9N(1+N)}S_1^3\\
&-\frac{64\big(-3+10N+26N^2+6N^3\big)}{3N^2(1+N)^2(2+N)}S_1^2+\frac{64\big(-5+N+13N^2+6N^3\big)}{3N^2(1+N)^2(2+N)}S_2\\
&+\frac{512\big(-5+7N+7N^2\big)}{9N(1+N)(2+N)}S_3+\frac{3072}{N(1+N)(2+N)}S_{2,1}-\frac{128(N-1)}{3N(1+N)}S_{-2,1}\\
&+\frac{512\big(-9+2N+2N^2\big)}{N(1+N)(2+N)}\zeta_3\Bigg]S_{-2}+\Bigg[-\frac{32P_{973}}{3(N-1)N^2(1+N)^2(2+N)^2}\\
&-\frac{256\big(-8+N+N^2\big)}{3N(1+N)(2+N)}S_1\Bigg]S_{-2}^2+\Bigg[\frac{32P_{963}}{3(N-1)N^2(1+N)^2(2+N)^2}S_1\\
&-\frac{16P_{1016}}{3(N-1)N^3(1+N)^3(2+N)^2}-\frac{64\big(2+11N+11N^2\big)}{3N(1+N)(2+N)}S_1^2-\frac{64\big(-74+N+N^2\big)}{3N(1+N)(2+N)}\\
&\times S_2+\frac{512\big(-1+N+N^2\big)}{N(1+N)(2+N)}S_{-2}\Bigg]S_{-3}+\Bigg[-\frac{32P_{970}}{3(N-1)N^2(1+N)^2(2+N)^2}\\
&-\frac{512S_1}{N(1+N)(2+N)}\Bigg]S_{-4}+\frac{32\big(10+67N+67N^2\big)}{3N(1+N)(2+N)}S_{-5}-\frac{128(-4+N)(5+N)}{N(1+N)(2+N)}\\
&\times S_{2,3}+\frac{192\big(6+N+N^2\big)}{N(1+N)(2+N)}S_{2,-3}+\frac{32\big(-98+13N+13N^2\big)}{N(1+N)(2+N)}S_{4,1}\\
&-\frac{64\big(-2+25N+25N^2\big)}{3N(1+N)(2+N)}S_{-2,3}-\frac{64\big(2+3N+3N^2\big)}{N(1+N)(2+N)}S_{-4,1}-\frac{160(N-1)}{3N(1+N)}S_{2,2,1}\\
&-\frac{32\big(-8-5N+19N^2+12N^3\big)}{3N^2(1+N)^2(2+N)}S_{2,1,1}-\frac{128\big(70+N+N^2\big)}{3N(1+N)(2+N)}S_{2,1,-2}\\
&+\frac{4608}{N(1+N)(2+N)}S_{3,1,1}-\frac{128\big(22+N+N^2\big)}{3N(1+N)(2+N)}S_{-2,1,-2}-\frac{3072}{N(1+N)(2+N)}S_{-2,2,1}\\
&-\frac{64\big(142+N+N^2\big)}{3N(1+N)(2+N)}S_{-3,1,1}-\frac{160(N-1)}{N(1+N)}S_{2,1,1,1}+\frac{128\big(146-N-N^2\big)}{3N(1+N)(2+N)}\\
&\times S_{-2,1,1,1}+\frac{48(N-1)\big(-2+3N+3N^2\big)}{5N^2(1+N)^2}\zeta_2^2-\frac{160\big(66+N+N^2\big)}{N(1+N)(2+N)}\zeta_5\Bigg\}
\end{aligned}
\tag{6.223}
$$

and

$$P_{939}=-305N^4-514N^3-19N^2+142N-1416, \tag{6.224}$$
$$P_{940}=-117N^4+14N^3+791N^2+676N-116, \tag{6.225}$$
$$P_{941}=2N^4+8N^3-6N^2-35N-20, \tag{6.226}$$
$$P_{942}=3N^4+15N^3+14N^2+2N+8, \tag{6.227}$$
$$P_{943}=11N^4-14N^3+148N^2+191N-48, \tag{6.228}$$

$$P_{944} = 17N^4 - 20N^3 + 51N^2 + 112N - 36, \tag{6.229}$$
$$P_{945} = 44N^4 + 61N^3 + 55N^2 + 74N - 6, \tag{6.230}$$
$$P_{946} = 124N^4 + 233N^3 + 49N^2 - 60N + 54, \tag{6.231}$$
$$P_{947} = 177N^4 + 146N^3 - 731N^2 - 492N + 324, \tag{6.232}$$
$$P_{948} = 349N^4 + 680N^3 - 550N^2 - 245N + 1038, \tag{6.233}$$
$$P_{949} = 368N^4 + 1036N^3 - 1145N^2 - 1741N + 954, \tag{6.234}$$
$$P_{950} = 497N^4 + 646N^3 - 785N^2 + 482N + 1848, \tag{6.235}$$
$$P_{951} = 6N^5 + 12N^4 + 65N^3 + 138N^2 + 225N + 130, \tag{6.236}$$
$$P_{952} = 12N^5 + 36N^4 + 191N^3 + 318N^2 + 369N + 226, \tag{6.237}$$
$$P_{953} = 16N^5 - 47N^3 + 19N^2 - 4N + 12, \tag{6.238}$$
$$P_{954} = 19N^5 + 76N^4 + 45N^3 - 52N^2 + 20N + 36, \tag{6.239}$$
$$P_{955} = 42N^5 + 15N^4 + 23N^3 - 141N^2 - 153N + 70, \tag{6.240}$$
$$P_{956} = 53N^5 + 158N^4 + 9N^3 - 119N^2 + 34N - 108, \tag{6.241}$$
$$P_{957} = 67N^5 + 277N^4 + 216N^3 - 145N^2 + 44N + 180, \tag{6.242}$$
$$P_{958} = 127N^5 + N^4 - 421N^3 + 267N^2 - 114N - 436, \tag{6.243}$$
$$P_{959} = 3N^6 - 14N^5 - 551N^4 - 838N^3 - 644N^2 - 572N - 264, \tag{6.244}$$
$$P_{960} = 3N^6 + 9N^5 - N^4 + 7N^3 + 82N^2 + 20N + 24, \tag{6.245}$$
$$P_{961} = 3N^6 + 9N^5 + 49N^4 + 77N^3 - 28N^2 - 50N + 84, \tag{6.246}$$
$$P_{962} = 6N^6 + 24N^5 + 113N^4 + 158N^3 + 159N^2 + 112N + 4, \tag{6.247}$$
$$P_{963} = 6N^6 + 36N^5 + 147N^4 + 158N^3 + 87N^2 + 78N + 64, \tag{6.248}$$
$$P_{964} = 8N^6 + 27N^5 + 18N^4 - 25N^3 - 80N^2 - 44N + 24, \tag{6.249}$$
$$P_{965} = 9N^6 + 71N^5 + 981N^4 + 1675N^3 + 1936N^2 + 1224N - 136, \tag{6.250}$$
$$P_{966} = 9N^6 + 95N^5 + 955N^4 + 1477N^3 + 2132N^2 + 1620N - 528, \tag{6.251}$$
$$P_{967} = 9N^6 + 119N^5 + 929N^4 + 1279N^3 + 2328N^2 + 2016N - 920, \tag{6.252}$$
$$P_{968} = 12N^6 + 50N^5 + 565N^4 + 958N^3 + 433N^2 + 86N + 488, \tag{6.253}$$
$$P_{969} = 12N^6 + 60N^5 + 261N^4 + 314N^3 + 267N^2 + 222N + 16, \tag{6.254}$$
$$P_{970} = 15N^6 + 23N^5 - 101N^4 - 131N^3 - 32N^2 - 82N + 20, \tag{6.255}$$
$$P_{971} = 15N^6 + 47N^5 + 107N^4 + 139N^3 + 108N^2 + 24N - 8, \tag{6.256}$$
$$P_{972} = 19N^6 + 63N^5 - 28N^4 - 154N^3 - 63N^2 - 35N - 18, \tag{6.257}$$
$$P_{973} = 21N^6 + 57N^5 + 29N^4 + 3N^3 + 126N^2 + 76N - 24, \tag{6.258}$$
$$P_{974} = 23N^6 - 201N^5 - 955N^4 - 583N^3 + 828N^2 - 48N - 536, \tag{6.259}$$
$$P_{975} = 23N^6 + 61N^5 + 93N^4 - 117N^3 - 692N^2 - 256N + 600, \tag{6.260}$$
$$P_{976} = 27N^6 + 87N^5 + 179N^4 + 213N^3 + 94N^2 - 40N + 16, \tag{6.261}$$
$$P_{977} = 30N^6 + 163N^5 + 788N^4 + 865N^3 + 1322N^2 + 1408N - 544, \tag{6.262}$$
$$P_{978} = 33N^6 + 113N^5 + 815N^4 + 1319N^3 + 1140N^2 + 708N + 1632, \tag{6.263}$$

$$P_{979} = 35N^6+93N^5-17N^4-97N^3-238N^2-320N+256, \quad (6.264)$$
$$P_{980} = 39N^6+13N^5-439N^4-421N^3-2156N^2-2412N+1344, \quad (6.265)$$
$$P_{981} = 39N^6+161N^5+45N^4-345N^3+600N^2+908N-832, \quad (6.266)$$
$$P_{982} = 60N^6+126N^5-491N^4-724N^3-271N^2-584N-420, \quad (6.267)$$
$$P_{983} = 60N^6+132N^5-367N^4-652N^3-711N^2-782N+16, \quad (6.268)$$
$$P_{984} = 66N^6+140N^5-337N^4-550N^3-835N^2-1024N+236, \quad (6.269)$$
$$P_{985} = 69N^6+203N^5-153N^4+21N^3+1536N^2+308N-1408, \quad (6.270)$$
$$P_{986} = 72N^6+148N^5-307N^4-448N^3-959N^2-1266N+456, \quad (6.271)$$
$$P_{987} = 85N^6+243N^5+443N^4+425N^3+2592N^2+2644N+3072, \quad (6.272)$$
$$P_{988} = 93N^6+39N^5+3465N^4+7681N^3+5118N^2+924N+3416, \quad (6.273)$$
$$P_{989} = 143N^6+231N^5+298N^4+1495N^3-4263N^2-6796N+3708, \quad (6.274)$$
$$P_{990} = 147N^6+453N^5+811N^4+755N^3+606N^2+500N+760, \quad (6.275)$$
$$P_{991} = 176N^6+390N^5-829N^4-1288N^3-429N^2-1208N-268, \quad (6.276)$$
$$P_{992} = 205N^6+225N^5-7345N^4-15529N^3-14712N^2-4748N-9936, \quad (6.277)$$
$$P_{993} = 213N^6+703N^5+3205N^4+4849N^3+3122N^2+1340N+2120, \quad (6.278)$$
$$P_{994} = 233N^6+591N^5-713N^4-683N^3+7632N^2+5372N-4656, \quad (6.279)$$
$$P_{995} = 263N^6+771N^5-848N^4-1349N^3+9819N^2+7532N-1932, \quad (6.280)$$
$$P_{996} = 274N^6+624N^5-278N^4-615N^3+625N^2+108N-540, \quad (6.281)$$
$$P_{997} = 434N^6+810N^5+5245N^4+10912N^3+3351N^2-2956N+16764, \quad (6.282)$$
$$P_{998} = 485N^6+975N^5+2077N^4+4777N^3+666N^2-3820N+13848, \quad (6.283)$$
$$P_{999} = 641N^6+1995N^5+3601N^4+5437N^3+11178N^2+6116N-1320, \quad (6.284)$$
$$P_{1000} = 764N^6+1896N^5-592N^4-1797N^3+413N^2-3762N-3600, \quad (6.285)$$
$$P_{1001} = 1721N^6+3837N^5+29N^4+909N^3+4838N^2-5394N-5508, \quad (6.286)$$
$$P_{1002} = 30N^7+51N^6-5N^5-173N^4-63N^3-22N^2-118N+12, \quad (6.287)$$
$$P_{1003} = 140N^7+643N^6+902N^5+N^4-676N^3+94N^2+264N-216, \quad (6.288)$$
$$P_{1004} = 200N^7+1357N^6+2729N^5+1069N^4-481N^3+1174N^2-324N-1080, \quad (6.289)$$
$$P_{1005} = 293N^7+513N^6+2089N^5+4287N^4+11362N^3+9848N^2+3496N-784, \quad (6.290)$$
$$P_{1006} = 383N^7+1702N^6+689N^5-4466N^4-4516N^3-176N^2-312N-336, \quad (6.291)$$
$$P_{1007} = 388N^7+1781N^6+1660N^5-2725N^4-5366N^3-2458N^2-192N-216, \quad (6.292)$$
$$P_{1008} = 1067N^7+5191N^6+5663N^5-2933N^4-2176N^3+4672N^2-288N-1008, \quad (6.293)$$
$$P_{1009} = 1427N^7+6712N^6+6878N^5-5021N^4-4822N^3+280N^2-12132N-8856, \quad (6.294)$$
$$P_{1010} = 1471N^7+7367N^6+12949N^5+9107N^4+6508N^3+4334N^2-12144N -10440, \quad (6.295)$$
$$P_{1011} = -205N^8-904N^7+88N^6+3218N^5+2309N^4+254N^3-944N^2-2584N -1424, \quad (6.296)$$

$$P_{1012} = 13N^8+50N^7+104N^6+130N^5+127N^4+136N^3+100N^2+36N-24, \tag{6.297}$$
$$P_{1013} = 35N^8+142N^7+16N^6-424N^5+341N^4+1418N^3+704N^2+24N+48, \tag{6.298}$$
$$P_{1014} = 39N^8+390N^7-2192N^6-4350N^5+10577N^4+8384N^3-28776N^2-9800N \\ +13632, \tag{6.299}$$
$$P_{1015} = 42N^8+141N^7+284N^6+351N^5-1843N^4-2925N^3-730N^2-1272N \\ -960, \tag{6.300}$$
$$P_{1016} = 71N^8+290N^7+108N^6-624N^5+437N^4+1838N^3+384N^2 \\ -248N+48, \tag{6.301}$$
$$P_{1017} = 90N^8+417N^7-155N^6-1779N^5+7189N^4+14142N^3+6376N^2+4776N \\ +3504, \tag{6.302}$$
$$P_{1018} = 93N^8+1048N^7+2200N^6+1622N^5+1891N^4+890N^3-936N^2+376N \\ +624, \tag{6.303}$$
$$P_{1019} = 114N^8+345N^7-222N^6-1074N^5-1960N^4-3335N^3-2280N^2-68N \\ +416, \tag{6.304}$$
$$P_{1020} = 153N^8+725N^7+380N^6-2494N^5-3431N^4-399N^3-510N^2-1672N \\ -1008, \tag{6.305}$$
$$P_{1021} = 191N^8+684N^7-217N^6-1949N^5-1246N^4+2305N^3+4728N^2+1120N \\ -432, \tag{6.306}$$
$$P_{1022} = 240N^8+1089N^7-917N^6-6465N^5+6181N^4+19980N^3+8596N^2+3840N \\ +3168, \tag{6.307}$$
$$P_{1023} = 339N^8+843N^7-2092N^6-2274N^5+1147N^4-14645N^3-26082N^2+116N \\ +6360, \tag{6.308}$$
$$P_{1024} = 430N^8+1951N^7-340N^6-7385N^5+2891N^4+11059N^3+970N^2+8136N \\ +3888, \tag{6.309}$$
$$P_{1025} = 1565N^8+5906N^7+3476N^6-7570N^5-4705N^4+152N^3-8328N^2+3888, \tag{6.310}$$
$$P_{1026} = 1973N^8+5921N^7+1402N^6-1417N^5+14062N^4+21596N^3+41243N^2 \\ +14832N-14076, \tag{6.311}$$
$$P_{1027} = 4724N^8+15473N^7+3469N^6-22813N^5+48721N^4+140300N^3+112538N^2 \\ +15108N-6480, \tag{6.312}$$
$$P_{1028} = 19N^9+63N^8+192N^7+162N^6-513N^5-513N^4+230N^3-720N^2 \\ -216N+144, \tag{6.313}$$
$$P_{1029} = 651N^9+3056N^8+1492N^7-12986N^6-24959N^5-15422N^4-6056N^3 \\ -5680N^2+1296N+2592, \tag{6.314}$$
$$P_{1030} = -12043N^{10}-52199N^9-68592N^8-22926N^7+115557N^6+458949N^5 \\ +428806N^4-302176N^3-337584N^2+149040N+140832, \tag{6.315}$$

$$P_{1031} = 63N^{10}+291N^9+581N^8+1646N^7+2335N^6+4289N^5+13213N^4+5150N^3 -12616N^2+552N+5232, \tag{6.316}$$

$$P_{1032} = 226N^{10}+1061N^9+860N^8-1831N^7-2444N^6+2045N^5+8496N^4+6381N^3 -4438N^2-2268N+2280, \tag{6.317}$$

$$P_{1033} = 5399N^{10}+21865N^9+4191N^8-66642N^7-36909N^6+68193N^5+32287N^4 +4016N^3+45360N^2+3888N-19440, \tag{6.318}$$

$$P_{1034} = 5459N^{10}+35950N^9+52398N^8-48294N^7-52359N^6+143592N^5+37432N^4 -26596N^3+134946N^2-70632N-87480, \tag{6.319}$$

$$P_{1035} = 9429N^{10}+33109N^9-26912N^8-199686N^7-184155N^6+5745N^5+39702N^4 +97328N^3+107792N^2-12816N-35424, \tag{6.320}$$

$$P_{1036} = 13789N^{10}+60242N^9+31062N^8-128262N^7-48825N^6+174492N^5 +110408N^4+144844N^3+66474N^2-183816N-115992, \tag{6.321}$$

$$P_{1037} = 205N^{12}+1122N^{11}-2485N^{10}-21156N^9-27453N^8+17874N^7+42893N^6 -5568N^5-25832N^4+2784N^3+10512N^2+576N-384, \tag{6.322}$$

$$P_{1038} = 254N^{12}+1391N^{11}+413N^{10}-8542N^9-8007N^8+22639N^7+38167N^6 +10124N^5-9715N^4-4980N^3+2736N^2+3344N+1328, \tag{6.323}$$

$$P_{1039} = 1538N^{12}+8703N^{11}+20491N^{10}+20280N^9-17340N^8-20277N^7 +105025N^6+77580N^5-83254N^4+66642N^3+131004N^2-18792N -42768, \tag{6.324}$$

$$P_{1040} = 3805N^{12}+23880N^{11}+31020N^{10}-63109N^9-153501N^8-33090N^7+131168N^6 +144597N^5-48240N^4-172070N^3+49356N^2+44280N+10800, \tag{6.325}$$

$$P_{1041} = 4920N^{13}+22305N^{12}+218N^{11}-95044N^{10}-22165N^9+174177N^8+27618N^7 -156150N^6-99101N^5-98972N^4-72698N^3+67716N^2+20376N-22032, \tag{6.326}$$

$$P_{1042} = 617N^{14}+4175N^{13}+14349N^{12}+30747N^{11}+2959N^{10}-140311N^9-226729N^8 -109247N^7-131196N^6-283508N^5-248768N^4-18848N^3+45408N^2 +6528N-2944, \tag{6.327}$$

$$P_{1043} = 32968N^{14}+214291N^{13}+230166N^{12}-1212607N^{11}-2507302N^{10}+1955271N^9 +5076722N^8-7562245N^7-17306682N^6-3182438N^5+7389328N^4+339888N^3 -2572992N^2+515808N+673920, \tag{6.328}$$

$$P_{1044} = 34763N^{14}+211157N^{13}+229596N^{12}-789770N^{11}-1434236N^{10}+1014978N^9 +2087956N^8-1947938N^7-3974499N^6-2470711N^5+66548N^4+2887812N^3 +1290168N^2-982368N-702432, \tag{6.329}$$

$$P_{1045} = -10701N^{15}-42606N^{14}-112294N^{13}-235574N^{12}-177548N^{11}-133306N^{10} -388110N^9+1317318N^8+3159337N^7+1521824N^6+1661012N^5+2875000N^4 -65712N^3-1912320N^2+39744N+466560, \tag{6.330}$$

$$P_{1046} = 23379N^{16}+176637N^{15}+336908N^{14}-418342N^{13}-1616541N^{12}+367147N^{11} +3663481N^{10}+1241253N^{9}-2594354N^{8}-1895204N^{7}+1153767N^{6} +3040589N^{5}-340384N^{4}-2307312N^{3}-1311840N^{2}+714096N+513216, \quad (6.331)$$

$$P_{1047} = 295317N^{16}+2034216N^{15}+4013522N^{14}-2161486N^{13}-8694264N^{12} +30856978N^{11}+88306930N^{10}+17465238N^{9}-110878589N^{8}-41117522N^{7} +85727340N^{6}+17006960N^{5}-58178704N^{4}-9306528N^{3}+13384512N^{2} -1838592N-3027456. \quad (6.332)$$

$$\Delta C_{g_1,g}^{(3),d_{abc}} = \frac{d_{abc}d^{abc}N_F^2}{N_A}\Bigg\{-\frac{128(N-2)(3+N)P_{1048}}{45N(1+N)(2+N)}S_5 - \frac{64(N-2)(3+N)P_{1048}}{9N(1+N)(2+N)}\zeta_5 + \frac{128P_{1049}}{(N-1)N(1+N)(2+N)^2}S_3 + \frac{32P_{1052}}{45(N-1)N(1+N)(2+N)^2} - \frac{256P_{1053}}{45N^2(1+N)^2(2+N)}S_{-2,1} - \frac{512P_{1055}}{3(N-1)N^2(1+N)^2(2+N)^2}S_4 + \frac{1024P_{1055}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{3,1} - \frac{64P_{1058}}{45(N-1)N^2(1+N)^2(2+N)^2}\zeta_3 + \Bigg[-\frac{128\big(-4+N+N^2\big)P_{1050}}{(N-1)N^2(1+N)^2(2+N)^2}S_3 - \frac{64P_{1051}}{45(N-1)N(1+N)(2+N)^2} + \frac{512P_{1056}}{3(N-1)N^2(1+N)^2(2+N)^2}\zeta_3 + \frac{512\big(-4+N+N^2\big)}{N(1+N)(2+N)}S_4 - \frac{1024\big(-4+N+N^2\big)S_{3,1}}{N(1+N)(2+N)} + \frac{512\big(-8+3N+3N^2\big)}{3N(1+N)(2+N)}S_{-2,1}\Bigg]S_1 + \Bigg[\frac{128\big(-19+3N+3N^2\big)}{3N(1+N)(2+N)} + \frac{256\big(-4+N+N^2\big)}{N(1+N)(2+N)}S_3 - \frac{512\big(-4+N+N^2\big)}{N(1+N)(2+N)} \times\zeta_3\Bigg]S_1^2 - \frac{768\big(-4+N+N^2\big)}{N(1+N)(2+N)}S_2S_3 + \Bigg[-\frac{64P_{1054}}{45(N-1)N(1+N)(2+N)^2} + \frac{256P_{1057}}{45(N-1)N^2(1+N)^2(2+N)^2}S_1 - \frac{256\big(-8+3N+3N^2\big)}{3N(1+N)(2+N)}S_1^2 - \frac{256}{45}(N-1)\big(-21+N+N^2\big)[S_3+S_{-2,1}+\zeta_3]\Bigg]S_{-2} - \frac{128}{2+N}S_{-2}^2 + \Bigg[\frac{128P_{1053}}{45N^2(1+N)^2(2+N)} - \frac{256\big(-8+3N+3N^2\big)}{3N(1+N)(2+N)}S_1\Bigg]S_{-3} - \frac{256\big(-4+3N+3N^2\big)}{3N(1+N)(2+N)}S_{-4} - \frac{128}{45}(N-1)\big(-21+N+N^2\big)S_{-5} + \frac{768\big(-4+N+N^2\big)}{N(1+N)(2+N)}[S_{2,3}-S_{4,1}] + \frac{256\big(-8+3N+3N^2\big)}{3N(1+N)(2+N)}S_{-2,2} + \frac{256}{45}(N-1)\big(-21+N+N^2\big)[S_{-2,3}+S_{-2,1,-2}] + \frac{256\big(-8+3N+3N^2\big)}{3N(1+N)(2+N)}S_{-3,1} + \frac{1536\big(-4+N+N^2\big)}{N(1+N)(2+N)}S_{3,1,1} - \frac{512\big(-8+3N+3N^2\big)}{3N(1+N)(2+N)}S_{-2,1,1}\Bigg\} \quad (6.333)$$

with

$$P_{1048} = N^4+2N^3-16N^2-17N+120, \quad (6.334)$$

$$P_{1049} = 5N^4+10N^3-16N^2-21N+10, \quad (6.335)$$

$$P_{1050} = 5N^4+10N^3+3N^2-2N-4, \tag{6.336}$$
$$P_{1051} = 46N^4+92N^3+1049N^2+1003N-3342, \tag{6.337}$$
$$P_{1052} = 65N^4+130N^3-593N^2-658N+5232, \tag{6.338}$$
$$P_{1053} = N^6+3N^5+24N^4+43N^3-142N^2-163N-156, \tag{6.339}$$
$$P_{1054} = 2N^6+6N^5+315N^4+620N^3-1391N^2-1700N+60, \tag{6.340}$$
$$P_{1055} = 3N^6+9N^5-4N^4-23N^3-14N^2-N+12, \tag{6.341}$$
$$P_{1056} = 9N^6+27N^5-13N^4-71N^3-26N^2+14N+24, \tag{6.342}$$
$$P_{1057} = 45N^6+135N^5-140N^4-505N^3+35N^2+310N+156, \tag{6.343}$$
$$P_{1058} = 2N^8+8N^7+815N^6+2417N^5-1509N^4-7037N^3-1780N^2 +2140N+624. \tag{6.344}$$

At one- and two-loop order ΔC_g is the same in the Larin and M–scheme [52]. The above Wilson coefficients complete the massless parts of the polarized single and two mass heavy flavor Wilson coefficients still missing in refs. [116–120].

7 The small- and large x expansions of the Wilson coefficients

We will represent the small x and large x behaviour of the different Wilson coefficients first in Mellin N space, where the former corresponds to the expansion around $N = 0$ and $N = 1$, cf. [180–182], while the latter corresponds to the expansion in the limit $N \to \infty$. Finally, we also consider the large N_F limit.

The non-singlet Wilson coefficients receive their leading small x contributions from the expansion around $N = 0$, retaining the pole terms. The singlet ones also receive contributions from the pole terms at $N = 1$. This is due to the Mellin transforms

$$\frac{1}{(N-1)^k} = \frac{(-1)^{k-1}}{(k-1)!}\mathbf{M}\left[\frac{\ln^{k-1}(x)}{x}\right](N) \tag{7.1}$$
$$\frac{1}{N^k} = \frac{(-1)^{k-1}}{(k-1)!}\mathbf{M}[\ln^{k-1}(x)](N). \tag{7.2}$$

In the large x region we will retain all terms up to $\propto 1/N$ in the asymptotic expansion. Here, terms of the kind

$$L_a^l \equiv (\ln(N) + \gamma_E)^l \tag{7.3}$$

contribute, which we will replace by

$$\ln(N) + \gamma_E = S_1(N) - \frac{1}{2N} + O\left(\frac{1}{N^2}\right), \tag{7.4}$$

to derive in a more direct way the inverse Mellin transforms. One obtains asymptotically up to terms of $O(L_a^k/N^2)$, $\quad k \geq 0$, and smaller

$$\mathbf{M}^{-1}[S_1(N)](x) \simeq -\frac{1}{1-x} + 1, \tag{7.5}$$
$$\mathbf{M}^{-1}[S_1^2(N)](x) \simeq \frac{2\ln(1-x)}{1-x} - 2\ln(1-x) - \zeta_2\delta(1-x) + 1, \tag{7.6}$$

$$\mathbf{M}^{-1}[S_1^3(N)](x) \simeq -\frac{3\ln^2(1-x)}{1-x} + \frac{3\zeta_2}{1-x} + 3\ln^2(1-x) - 3\ln(1-x) - 2\zeta_3\delta(1-x) - 3\zeta_2, \tag{7.7}$$

$$\mathbf{M}^{-1}[S_1^4(N)](x) \simeq \frac{4\ln^3(1-x)}{1-x} - \frac{12\zeta_2\ln(1-x)}{1-x} + \frac{8\zeta_3}{1-x} - 4\ln^3(1-x) + 6\ln^2(1-x) + 12\zeta_2\ln(1-x) + \frac{3}{5}\zeta_2^2\delta(1-x) - 6\zeta_2 - 8\zeta_3, \tag{7.8}$$

$$\mathbf{M}^{-1}[S_1^5(N)](x) \simeq -\frac{5\ln^4(1-x)}{1-x} + \frac{30\zeta_2\ln^2(1-x)}{1-x} - \frac{40\zeta_3\ln(1-x)}{1-x} - \frac{3\zeta_2^2}{1-x} + 5\ln^4(1-x) - 10\ln^3(1-x) - 30\zeta_2\ln^2(1-x) + (30\zeta_2 + 40\zeta_3) \times \ln(1-x) + (20\zeta_2\zeta_3 - 24\zeta_5)\delta(1-x) + 3\zeta_2^2 - 20\zeta_3, \tag{7.9}$$

$$\mathbf{M}^{-1}[S_1^6(N)](x) \simeq \frac{6\ln^5(1-x)}{1-x} - \frac{60\zeta_2\ln^3(1-x)}{1-x} + \frac{120\zeta_3\ln^2(1-x)}{1-x} + \frac{18\zeta_2^2\ln(1-x)}{1-x} -(120\zeta_2\zeta_3 - 144\zeta_5)\frac{1}{1-x} - 6\ln^5(1-x) + 15\ln^4(1-x) + 60\zeta_2\ln^3(1-x) - (90\zeta_2 + 120\zeta_3)\ln^2(1-x) + (120\zeta_3 - 18\zeta_2^2)\ln(1-x) + \left(40\zeta_3^2 - \frac{45}{7}\zeta_2^3\right)\delta(1-x) + 9\zeta_2^2 + 120\zeta_2\zeta_3 - 144\zeta_5, \tag{7.10}$$

and

$$\mathbf{M}^{-1}\left[\frac{S_1(N)}{N}\right](x) \simeq -\ln(1-x), \tag{7.11}$$

$$\mathbf{M}^{-1}\left[\frac{S_1^2(N)}{N}\right](x) \simeq \ln^2(1-x) - \zeta_2, \tag{7.12}$$

$$\mathbf{M}^{-1}\left[\frac{S_1^3(N)}{N}\right](x) \simeq -\ln^3(1-x) + 3\zeta_2\ln(1-x) - 2\zeta_3, \tag{7.13}$$

$$\mathbf{M}^{-1}\left[\frac{S_1^4(N)}{N}\right](x) \simeq \ln^4(1-x) - 6\zeta_2\ln^2(1-x) + 8\zeta_3\ln(1-x) + \frac{3}{5}\zeta_2^2, \tag{7.14}$$

$$\mathbf{M}^{-1}\left[\frac{S_1^5(N)}{N}\right](x) \simeq -\ln^5(1-x) + 10\zeta_2\ln^3(1-x) - 20\zeta_3\ln^2(1-x) - 3\zeta_2^2\ln(1-x) + 20\zeta_2\zeta_3 - 24\zeta_5, \tag{7.15}$$

$$\mathbf{M}^{-1}\left[\frac{S_1^6(N)}{N}\right](x) \simeq \ln^6(1-x) - 15\zeta_2\ln^4(1-x) + 40\zeta_3\ln^3(1-x) + 9\zeta_2^2\ln^2(1-x) + (144\zeta_5 - 120\zeta_2\zeta_3)\ln(1-x) - \frac{45}{7}\zeta_2^3 + 40\zeta_3^2. \tag{7.16}$$

We will now present the asymptotic behaviour of the Wilson coefficients in the small x and large x region. For the pure singlet and gluonic Wilson coefficients of the structure function $g_1(x, Q^2)$ we refer to the Larin scheme.

7.1 The small x limit

In this section we will show the small x expansion of the Wilson coefficients calculated above. The non-singlet unpolarized Wilson coefficients have leading singularities at $N = 0$, $\propto \ln^k(x)$ and the unpolarized pure singlet and gluonic Wilson coefficients have their leading singularity at $N = 1$, i.e. terms as $\ln(x)/x$ and $1/x$. Here we also list the logarithmic contributions up to the constant terms. One obtains

$$C_{F_2,q}^{(1),\mathrm{NS}} \simeq C_F[3 - 2\ln(x)], \tag{7.17}$$

$$C_{F_2,q}^{(2),\rm NS} \simeq C_F\Bigg\{T_FN_F\left[\frac{10}{3}\ln^2(x)+8\ln(x)+\frac{178}{27}-\frac{16}{3}\zeta_2\right]+C_A\left[-\frac{55}{6}\ln^2(x)-32\ln(x)\right.$$
$$\left.-\frac{1693}{54}+\frac{44}{3}\zeta_2-12\zeta_3\right]\Bigg\}+C_F^2\Bigg\{-\frac{5}{3}\ln^3(x)+9\ln^2(x)+(17+24\zeta_2)\ln(x)-8\zeta_2$$
$$+\frac{3}{2}+56\zeta_3\Bigg\}, \tag{7.18}$$

$$C_{F_2,q}^{(3),\rm NS} \simeq -\frac{1}{2}C_F^3\ln^5(x)+\left[\frac{67}{12}C_F^3+C_F^2\left(-\frac{1001}{108}C_A+\frac{91}{27}T_FN_F\right)\right]\ln^4(x)$$
$$+\Bigg[C_F\left(\frac{2024}{81}C_AT_FN_F-\frac{368}{81}T_F^2N_F^2+\left(-\frac{2783}{81}+20\zeta_2\right)C_A^2\right)+C_F^2\left(\frac{10}{27}N_FT_F\right.$$
$$\left.+\left(-\frac{835}{54}-64\zeta_2\right)C_A\right)+C_F^3\left(5+\frac{262}{3}\zeta_2\right)\Bigg]\ln^3(x)+\Bigg[C_F\left(-\frac{1984}{81}T_F^2N_F^2\right.$$
$$\left.+\left(\frac{14392}{81}-16\zeta_2\right)C_AT_FN_F+\left(-\frac{23062}{81}+84\zeta_2\right)C_A^2\right)+C_F^2\Bigg(T_FN_F\Bigg(-\frac{2630}{81}$$
$$-\frac{532}{9}\zeta_2\Bigg)+C_A\left(\frac{17315}{162}-\frac{265\zeta_2}{9}-64\zeta_3\right)\Bigg)+\Bigg(-\frac{113}{6}+\frac{299}{3}\zeta_2$$
$$+\frac{646}{3}\zeta_3\Bigg)C_F^3\Bigg]\ln^2(x)+\Bigg[C_F\Bigg[T_F^2N_F^2\left(-\frac{4816}{81}+\frac{64}{3}\zeta_2\right)+C_A^2\Bigg(-\frac{78338}{81}+\frac{3058}{9}\zeta_2$$
$$+32\zeta_3-24\zeta_2^2\Bigg)+C_AT_FN_F\left(\frac{14152}{27}-\frac{1376}{9}\zeta_2+\frac{256}{3}\zeta_3\right)\Bigg]+C_F^2\Bigg[T_FN_F\Bigg($$
$$-\frac{2999}{81}-\frac{964}{9}\zeta_2-264\zeta_3\Bigg)+C_A\left(\frac{106801}{324}+\frac{599}{9}\zeta_2+\frac{418}{3}\zeta_3+\frac{656}{15}\zeta_2^2\right)\Bigg]$$
$$+C_F^3\left(\frac{1619}{12}+\frac{764}{3}\zeta_2+154\zeta_3-\frac{556}{3}\zeta_2^2\right)\Bigg]\ln(x)$$
$$+\Bigg[C_F^2\Bigg(C_A\left(\frac{193961}{648}+\frac{6014}{81}\zeta_2+\frac{13189}{27}\zeta_3-\frac{3434}{45}\zeta_2^2+520\zeta_2\zeta_3+\frac{1970}{3}\zeta_5\right)$$
$$+T_FN_F\left(-\frac{2881}{162}+\frac{944}{81}\zeta_2-\frac{11804}{27}\zeta_3-\frac{848}{45}\zeta_2^2\right)\Bigg)+C_F\Bigg(C_A^2\Bigg(-\frac{1779023}{1458}$$
$$+\frac{14917}{27}\zeta_2-\frac{1960}{27}\zeta_3-\frac{148}{3}\zeta_2^2-\frac{436}{3}\zeta_2\zeta_3-\frac{152}{3}\zeta_5\Bigg)+T_F^2N_F^2\Bigg(-\frac{44680}{729}$$
$$+\frac{928\zeta_2}{27}+\frac{128\zeta_3}{27}\Bigg)+C_AT_FN_F\left(\frac{448438}{729}-\frac{9040}{27}\zeta_2+\frac{704}{15}\zeta_2^2+\frac{2824}{27}\zeta_3\right)\Bigg)$$
$$+C_F^3\left(\frac{5603}{24}+\frac{533}{3}\zeta_2+\frac{1730}{3}\zeta_3-\frac{68}{15}\zeta_2^2-904\zeta_2\zeta_3-872\zeta_5\right)\Bigg], \tag{7.19}$$

$$C_{F_2,q}^{(3,d_{abc})} \simeq 0, \tag{7.20}$$

$$C_{F_2,q}^{(2),\rm PS} \simeq C_FT_FN_F\left[\frac{1}{x}\left(\frac{688}{27}-\frac{32}{3}\zeta_2\right)+\frac{20}{3}\ln^3(x)-2\ln^2(x)+(112-32\zeta_2)\ln(x)\right.$$
$$\left.+12-16\zeta_3\right], \tag{7.21}$$

$$
\begin{aligned}
C_{F_2,q}^{(3),\mathrm{PS}} \simeq\; & C_A C_F T_F N_F\left[-\frac{78976}{243}+\frac{832}{9}\zeta_2-\frac{256}{9}\zeta_3\right]\frac{\ln(x)}{x}+\left\{C_F\left[T_F^2N_F^2\left(\frac{88448}{729}-\frac{128}{9}\zeta_2\right.\right.\right.\\
&\left.\left.+\frac{512}{27}\zeta_3\right)+C_AT_FN_F\left(-\frac{1942568}{729}+\frac{30080}{81}\zeta_2+\frac{1504}{9}\zeta_3+\frac{7936}{45}\zeta_2^2\right)\right]\\
&\left.+C_F^2T_FN_F\left(\frac{2180}{27}-32\zeta_2-\frac{1600}{9}\zeta_3+\frac{384}{5}\zeta_2^2\right)\right\}\frac{1}{x}+\Big[-4C_FC_AT_FN_F\\
&+4C_F^2T_FN_F\Big]\ln^5(x)+\left[-\frac{110}{9}C_F^2T_FN_F+C_F\left(\frac{854}{27}C_AT_FN_F-\frac{184}{27}T_F^2N_F^2\right)\right]\\
&\times\ln^4(x)+\left[C_F\left(-\frac{3568}{81}T_F^2N_F^2+C_AT_FN_F\left(-\frac{16984}{81}+\frac{320}{9}\zeta_2\right)\right)\right.\\
&\left.+C_F^2T_FN_F\left(\frac{572}{3}-\frac{1328}{9}\zeta_2\right)\right]\ln^3(x)+\left[C_F\left(T_F^2N_F^2\left(-\frac{22912}{81}+\frac{64}{9}\zeta_2\right)\right.\right.\\
&\left.+C_AT_FN_F\left(\frac{101192}{81}-\frac{920}{9}\zeta_2-\frac{1024}{3}\zeta_3\right)\right)+C_F^2T_FN_F\left(-\frac{218}{3}-\frac{512}{3}\zeta_2\right.\\
&\left.\left.-\frac{640}{3}\zeta_3\right)\right]\ln^2(x)+\left[C_F\left(C_AT_FN_F\left(-\frac{304304}{243}+\frac{30872}{27}\zeta_2-\frac{640}{9}\zeta_3-\frac{752}{5}\zeta_2^2\right)\right.\right.\\
&\left.+T_F^2N_F^2\left(-\frac{198560}{243}+\frac{3392}{27}\zeta_2-\frac{64}{9}\zeta_3\right)\right)+C_F^2T_FN_F\left(692-2200\zeta_2-\frac{160}{3}\zeta_3\right.\\
&\left.\left.+\frac{2896}{15}\zeta_2^2\right)\right]\ln(x)+C_F\left[C_AT_FN_F\left(\frac{1102604}{243}-\frac{71896}{81}\zeta_2-\frac{32864}{27}\zeta_3+\frac{1856}{15}\zeta_2^2\right.\right.\\
&\left.\left.+592\zeta_2\zeta_3-1240\zeta_5\right)+T_F^2N_F^2\left(-\frac{225376}{243}+\frac{6176}{81}\zeta_2+\frac{1312}{27}\zeta_3+\frac{224}{15}\zeta_2^2\right)\right]\\
&+C_F^2T_FN_F\left(-248-\frac{4936}{3}\zeta_2-880\zeta_3+\frac{736}{5}\zeta_2^2+288\zeta_2\zeta_3+352\zeta_5\right),
\end{aligned} \tag{7.22}
$$

$$
C_{F_2,g}^{(1)} \simeq -4T_FN_F(\ln(x)+1), \tag{7.23}
$$

$$
\begin{aligned}
C_{F_2,g}^{(2)} \simeq\; & C_FT_FN_F\left[-\frac{10}{3}\ln^3(x)-3\ln^2(x)-32(1-\zeta_2)\ln(x)-86+32\zeta_2+64\zeta_3\right]\\
&+C_AT_FN_F\left[\frac{1}{x}\left(\frac{688}{27}-\frac{32}{3}\zeta_2\right)+\frac{20}{3}\ln^3(x)-2\ln^2(x)+(116-16\zeta_2)\ln(x)+30\right.\\
&\left.+16\zeta_2+8\zeta_3\right],
\end{aligned} \tag{7.24}
$$

$$
\begin{aligned}
C_{F_2,g}^{(3),a} \simeq\; & C_A^2T_FN_F\left[-\frac{78976}{243}+\frac{832}{9}\zeta_2-\frac{256}{9}\zeta_3\right]\frac{\ln(x)}{x}+\left\{C_F\left[C_AT_FN_F\left(\frac{2180}{27}-32\zeta_2\right.\right.\right.\\
&\left.\left.-\frac{1600}{9}\zeta_3+\frac{384}{5}\zeta_2^2\right)+T_F^2N_F^2\left(\frac{181472}{729}-\frac{2048}{27}\zeta_2+\frac{512}{27}\zeta_3\right)\right]\\
&+C_AT_F^2N_F^2\left(-\frac{2288}{729}+\frac{640}{27}\zeta_2+\frac{256}{27}\zeta_3\right)+C_A^2T_FN_F\left(-\frac{2004664}{729}+\frac{32192}{81}\zeta_2\right.\\
&\left.\left.+\frac{7936}{45}\zeta_2^2+\frac{4384}{27}\zeta_3\right)\right\}\frac{1}{x}+\left[-4C_A^2T_FN_F-C_F^2T_FN_F+C_F\left(2C_AT_FN_F\right.\right.\\
&\left.\left.+4T_F^2N_F^2\right)\right]\ln^5(x)+\left[\frac{359}{27}C_A^2T_FN_F+\frac{11}{9}C_F^2T_FN_F-\frac{4}{27}C_AT_F^2N_F^2+\right.
\end{aligned}
$$

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$$
\begin{aligned}
&+C_F\left(-\frac{409}{27}C_AT_FN_F+\frac{818}{27}T_F^2N_F^2\right)\Bigg]\ln^4(x)+\Bigg[-\frac{1064}{81}C_AT_F^2N_F^2+C_F\Bigg(\\
&C_AT_FN_F\left(\frac{4589}{81}-\frac{656}{9}\zeta_2\right)+T_F^2N_F^2\left(\frac{35564}{81}-\frac{176}{3}\zeta_2\right)\Bigg)+C_A^2T_FN_F\Bigg(\\
&-\frac{27338}{81}+\frac{56}{3}\zeta_2\Bigg)+C_F^2T_FN_F\left(-\frac{178}{9}+\frac{524}{9}\zeta_2\right)\Bigg]\ln^3(x)+\Bigg[C_F\Bigg(C_AT_FN_F\Bigg(\\
&-\frac{20815}{81}+\frac{284}{3}\zeta_2-\frac{320}{3}\zeta_3\Bigg)+T_F^2N_F^2\left(\frac{156500}{81}-\frac{872}{3}\zeta_2-\frac{304}{3}\zeta_3\right)\Bigg)\\
&+C_AT_F^2N_F^2\left(-\frac{1024}{81}-\frac{16}{3}\zeta_2\right)+C_A^2T_FN_F\left(\frac{28580}{81}-32\zeta_2-\frac{1136}{3}\zeta_3\right)\\
&+C_F^2T_FN_F\left(-88+\frac{284}{3}\zeta_2+284\zeta_3\right)\Bigg]\ln^2(x)+\Bigg[C_F\Bigg(T_F^2N_F^2\Bigg(\frac{1683904}{243}\\
&-\frac{17632}{9}\zeta_2-\frac{5072}{9}\zeta_3+\frac{224}{5}\zeta_2^2\Bigg)+C_AT_FN_F\Bigg(-\frac{147806}{243}-\frac{1516}{9}\zeta_2+\frac{2632}{9}\zeta_3\\
&-\frac{2176}{15}\zeta_2^2\Bigg)\Bigg)+C_A^2T_FN_F\left(-\frac{935806}{243}+\frac{10912}{9}\zeta_2-\frac{2432}{15}\zeta_2^2-\frac{832}{9}\zeta_3\right)\\
&+C_AT_F^2N_F^2\left(-\frac{53512}{243}+\frac{112}{9}\zeta_2-\frac{256}{9}\zeta_3\right)+C_F^2T_FN_F\Bigg(\frac{368}{3}+\frac{550}{3}\zeta_2+600\zeta_3\\
&-\frac{56}{15}\zeta_2^2\Bigg)\Bigg]\ln(x)+C_F\Bigg[T_F^2N_F^2\Bigg(\frac{2179673}{243}-\frac{226144}{81}\zeta_2-\frac{41744}{27}\zeta_3-\frac{3632}{45}\zeta_2^2\\
&+128\zeta_2\zeta_3-96\zeta_5\Bigg)+C_AT_FN_F\Bigg(-\frac{2310157}{972}+\frac{91424}{81}\zeta_2+\frac{1840}{27}\zeta_3-\frac{13064}{45}\zeta_2^2\\
&-352\zeta_2\zeta_3+\frac{1852}{3}\zeta_5\Bigg)\Bigg]+C_A^2T_FN_F\Bigg[\frac{359827}{243}-\frac{4138}{9}\zeta_2-\frac{7348}{9}\zeta_3+\frac{908}{15}\zeta_2^2\\
&+\frac{1288}{3}\zeta_2\zeta_3-\frac{5224}{3}\zeta_5\Bigg]+C_F^2T_FN_F\Bigg[\frac{1925}{4}-\frac{572}{3}\zeta_2+\frac{4478}{3}\zeta_3+\frac{2176}{15}\zeta_2^2\\
&-320\zeta_2\zeta_3-248\zeta_5\Bigg]+C_AT_F^2N_F^2\Bigg[-\frac{52712}{243}-\frac{400}{9}\zeta_2-\frac{1072}{9}\zeta_3+\frac{224}{15}\zeta_2^2\Bigg],
\end{aligned}
\tag{7.25}
$$

$$
C_{F_2,g}^{(3),d_{abc}} \simeq 0, \tag{7.26}
$$

$$
C_{F_L,q}^{(1),\mathrm{NS}} \simeq 0, \tag{7.27}
$$

$$
C_{F_L,q}^{(2),\mathrm{NS}} \simeq C_F\left[-C_A\frac{44}{3}+T_FN_F\frac{16}{3}\right]-4C_F^2\Big[2\ln(x)-3\Big], \tag{7.28}
$$

$$
\begin{aligned}
C_{F_L,q}^{(3),\mathrm{NS}} \simeq\ & -\frac{20}{3}C_F^3\ln^3(x)+\Big[32C_F^3-66C_F^2C_A+24C_F^2T_FN_F\Big]\ln^2(x)+\Bigg[C_F\Bigg(\frac{704}{9}C_AT_FN_F\\
&-\frac{128}{9}T_F^2N_F^2+\left(-\frac{968}{9}+120\zeta_2\right)C_A^2\Bigg)+C_F^2\left(\frac{448}{9}T_FN_F-\left(\frac{1832}{9}+384\zeta_2\right)C_A\right)\\
&+(168+416\zeta_2)C_F^3\Bigg]\ln(x)+C_F\Bigg[-\frac{1216}{27}T_F^2N_F^2+\left(\frac{8368}{27}-32\zeta_2\right)C_AT_FN_F\\
&+\left(-\frac{13060}{27}+244\zeta_2\right)C_A^2\Bigg]+C_F^2\Bigg(\left(-\frac{2288}{27}-\frac{64}{3}\zeta_2\right)T_FN_F+\Bigg(\frac{5800}{27}-\frac{1696}{3}\zeta_2\\
&-96\zeta_3\Bigg)C_A\Bigg)+\Big(608\zeta_2+288\zeta_3\Big)C_F^3,
\end{aligned}
\tag{7.29}
$$

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$$C_{F_L,q}^{(3),d_{abc}} \simeq 0, \tag{7.30}$$

$$C_{F_L,q}^{(2),\mathrm{PS}} \simeq 32 C_F T_F N_F \left[-\frac{1}{9x} + \ln(x) \right], \tag{7.31}$$

$$\begin{aligned}
C_{F_L,q}^{(3),\mathrm{PS}} \simeq{}& C_F \Biggl\{ T_F^2 N_F^2 \left[\frac{1}{x}\left(\frac{13568}{81} - \frac{256}{9}\zeta_2 \right) - \frac{128}{3}\ln^2(x) - \frac{2048}{9}\ln(x) - \frac{11264}{27} \right] \\
&+ C_A T_F N_F \left[\frac{1}{x}\left(-\frac{28768}{27} + 224\zeta_2 + \frac{640}{3}\zeta_3 \right) - \frac{160}{3}\ln^3(x) + \frac{496}{3}\ln^2(x) \right. \\
&\left. + \left(-\frac{800}{9} + \frac{1}{x}\left(-\frac{4352}{27} + \frac{128}{3}\zeta_2 \right) + 224\zeta_2 \right)\ln(x) + \frac{35200}{27} + 64\zeta_2 - 704\zeta_3 \right] \Biggr\} \\
&+ C_F^2 T_F N_F \left[\frac{1}{x}\left(\frac{3584}{27} - \frac{64}{3}\zeta_2 - \frac{256}{3}\zeta_3 \right) - 576\zeta_2 + \left(160 - 512\zeta_2 \right)\ln(x) \right. \\
&\left. - 16\ln^2(x) + \frac{160}{3}\ln^3(x) \right],
\end{aligned} \tag{7.32}$$

$$C_{F_L,g}^{(1)} \simeq 0, \tag{7.33}$$

$$C_{F_L,g}^{(2)} \simeq 32 C_A T_F N_F \left[-\frac{1}{9x} + \ln(x) \right] - 16 C_F T_F N_F (\ln(x) + 1), \tag{7.34}$$

$$\begin{aligned}
C_{F_L,g}^{(3),a} \simeq{}& C_A^2 T_F N_F \left[-\frac{4352}{27} + \frac{128}{3}\zeta_2 \right] \frac{\ln(x)}{x} + \Biggl\{ C_F \left[T_F^2 N_F^2 \left(\frac{7744}{81} - \frac{256}{9}\zeta_2 \right) + C_A T_F N_F \Biggl(\right. \\
&\left. \frac{3584}{27} - \frac{64}{3}\zeta_2 - \frac{256}{3}\zeta_3 \Biggr) \right] + C_A T_F^2 N_F^2 \left(\frac{3232}{27} - \frac{128}{9}\zeta_2 \right) + C_A^2 T_F N_F \Biggl(-\frac{88192}{81} \\
&+ \frac{2080}{9}\zeta_2 + \frac{640}{3}\zeta_3 \Biggr) \Biggr\} \frac{1}{x} + \left[-\frac{160}{3} C_A^2 T_F N_F - \frac{40}{3} C_F^2 T_F N_F + C_F \Biggl(\frac{80}{3} C_A T_F N_F \right. \\
&\left. + \frac{160}{3} T_F^2 N_F^2 \Biggr) \right] \ln^3(x) + \left[\frac{56}{3} C_A^2 T_F N_F - 20 C_F^2 T_F N_F + \frac{32}{3} C_A T_F^2 N_F^2 \right. \\
&\left. + C_F \left(-\frac{152}{3} C_A T_F N_F + \frac{928}{3} T_F^2 N_F^2 \right) \right] \ln^2(x) + \left[-\frac{320}{9} C_A T_F^2 N_F^2 + C_F \Biggl(\right. \\
&C_A T_F N_F \left(-\frac{616}{9} - 288\zeta_2 \right) + T_F^2 N_F^2 \left(\frac{13664}{9} - 256\zeta_2 \right) \Biggr) + C_F^2 T_F N_F \Biggl(32 \\
&\left. + 192\zeta_2 \Biggr) + C_A^2 T_F N_F \left(-\frac{7280}{9} + 240\zeta_2 \right) \right] \ln(x) + C_F \left[T_F^2 N_F^2 \Biggl(\frac{20864}{9} \right. \\
&\left. - \frac{1664}{3}\zeta_2 - 128\zeta_3 \Biggr) + C_A T_F N_F \left(-\frac{2776}{9} - \frac{512}{3}\zeta_2 + 64\zeta_3 \right) \right] + C_A^2 T_F N_F \Biggl(\\
&\frac{5168}{27} + 200\zeta_2 - 704\zeta_3 \Biggr) + C_F^2 T_F N_F \left(-24 + 256\zeta_2 + 384\zeta_3 \right) \\
&- \frac{3520}{27} C_A T_F^2 N_F^2,
\end{aligned} \tag{7.35}$$

$$C_{F_L,g}^{(3),d_{abc}} \simeq 0, \tag{7.36}$$

$$C_{F_3,q}^{(1),\mathrm{NS}} \simeq C_F [1 - 2\log(x)], \tag{7.37}$$

$$C_{F_3,q}^{(2),\mathrm{NS}} \simeq C_F \Biggl\{ C_A \Biggl[-2\ln^3(x) - \frac{103}{6}\ln^2(x) + \left(-\frac{122}{3} + 8\zeta_2 \right)\ln(x) - \frac{1327}{54} + \frac{80}{3}\zeta_2 +$$

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$$+8\zeta_3\Big]+T_FN_F\Big[\frac{10}{3}\ln^2(x)+\frac{40}{3}\ln(x)+\frac{262}{27}-\frac{16}{3}\zeta_2\Big]\Big\}+C_F^2\Big[\frac{7}{3}\ln^3(x)$$
$$+21\ln^2(x)+(21+8\zeta_2)\ln(x)-\frac{59}{2}-20\zeta_2+16\zeta_3\Big], \tag{7.38}$$

$$C^{(3),\mathrm{NS}}_{F_3,q}\simeq C_F^2\Big\{T_FN_F\Big[-\frac{31}{9}\ln^4(x)-\frac{4358}{81}\ln^3(x)+\Big(-\frac{5374}{27}-\frac{124}{3}\zeta_2\Big)\ln^2(x)+\Big(-\frac{271}{81}$$
$$-\frac{1436}{27}\zeta_2-\frac{2056}{9}\zeta_3\Big)\ln(x)+\frac{4123}{54}+\frac{8344}{81}\zeta_2+\frac{752}{9}\zeta_2^2-\frac{9884}{27}\zeta_3\Big]$$
$$+C_A\Big[-\frac{29}{15}\ln^5(x)+\frac{43}{36}\ln^4(x)+\Big(\frac{24245}{162}+\frac{220}{3}\zeta_2\Big)\ln^3(x)+\Big(\frac{13297}{54}+401\zeta_2$$
$$+328\zeta_3\Big)\ln^2(x)+\Big(-\frac{404119}{324}+\frac{16273}{27}\zeta_2+\frac{9206}{9}\zeta_3+\frac{3416}{15}\zeta_2^2\Big)\ln(x)-\frac{496687}{216}$$
$$+\frac{54412}{81}\zeta_2+\frac{1252}{45}\zeta_2^2+\Big(-\frac{1865}{27}-648\zeta_2\Big)\zeta_3+\frac{5306}{3}\zeta_5\Big]\Big\}$$
$$+C_F\Big\{T_F^2N_F^2\Big[-\frac{368}{81}\ln^3(x)-\frac{2848}{81}\ln^2(x)+\Big(-\frac{6736}{81}+\frac{64}{3}\zeta_2\Big)\ln(x)-\frac{35896}{729}$$
$$+\frac{1312}{27}\zeta_2+\frac{128}{27}\zeta_3\Big]+C_AT_FN_F\Big[\frac{92}{27}\ln^4(x)+\frac{1544}{27}\ln^3(x)+\Big(\frac{8432}{27}-\frac{224}{9}\zeta_2\Big)$$
$$\times\ln^2(x)+\Big(\frac{52168}{81}-\frac{6224}{27}\zeta_2+\frac{608}{9}\zeta_3\Big)\ln(x)+\frac{433462}{729}-\frac{37480}{81}\zeta_2+\frac{1288}{27}\zeta_3$$
$$-\frac{64}{15}\zeta_2^2\Big]+C_A^2\Big[\frac{2}{5}\ln^5(x)-\frac{166}{27}\ln^4(x)+\Big(-\frac{9799}{81}-\frac{64}{9}\zeta_2\Big)\ln^3(x)+\Big(-\frac{38642}{81}$$
$$+\frac{226}{9}\zeta_2-\frac{212}{3}\zeta_3\Big)\ln^2(x)+\Big(-\frac{53650}{81}+\frac{11260}{27}\zeta_2-\frac{416}{5}\zeta_2^2-\frac{1696}{9}\zeta_3\Big)\ln(x)$$
$$-\frac{1010495}{1458}+\frac{52214}{81}\zeta_2-\frac{1258}{15}\zeta_2^2+\Big(\frac{9932}{27}+\frac{292}{3}\zeta_2\Big)\zeta_3-\frac{896}{3}\zeta_5\Big]\Big\}$$
$$+C_F^3\Big[\frac{53}{30}\ln^5(x)+\frac{89}{12}\ln^4(x)+\Big(-\frac{49}{3}-\frac{710}{9}\zeta_2\Big)\ln^3(x)+\Big(\frac{1085}{6}-\frac{1373}{3}\zeta_2$$
$$-286\zeta_3\Big)\ln^2(x)+\Big(\frac{5553}{4}-\frac{2308}{3}\zeta_2-\frac{1978}{3}\zeta_3-\frac{4748}{15}\zeta_2^2\Big)\ln(x)+\frac{71777}{24}$$
$$-\frac{2561}{3}\zeta_2-\frac{2248}{15}\zeta_2^2+\Big(\frac{710}{3}+\frac{1384}{3}\zeta_2\Big)\zeta_3-2104\zeta_5\Big], \tag{7.39}$$

$$C^{(3),d_{abc}}_{F_3,q}\simeq\frac{d_{abc}d^{abc}N_F}{N_C}\Big[-\frac{32}{15}\ln^5(x)+\Big(-\frac{704}{9}+\frac{256}{3}\zeta_2\Big)\ln^3(x)+\Big(-1600-\frac{2176}{5}\zeta_2^2$$
$$+\frac{1024}{3}\zeta_2+\frac{1088}{3}\zeta_3\Big)\ln(x)+\Big(-1056+96\zeta_2+\frac{1216}{3}\zeta_3\Big)\ln^2(x)-\frac{11456}{3}$$
$$+2560\zeta_2-\frac{3104}{15}\zeta_2^2+\Big(\frac{3904}{3}-\frac{1856\zeta_2}{3}\Big)\zeta_3-\frac{3584}{3}\zeta_5\Big], \tag{7.40}$$

$$\Delta C^{(1),\mathrm{NS,L}}_{g_1,q}\simeq -C_F(7+2\ln(x)), \tag{7.41}$$

$$\Delta C^{(2),\mathrm{NS,L}}_{g_1,q}\simeq C_F\Big\{C_A\Big[-2\ln^3(x)-\frac{151}{6}\ln^2(x)+\Big(-\frac{238}{3}+8\zeta_2\Big)\ln(x)-\frac{6391}{54}+\frac{128}{3}\zeta_2+8\zeta_3\Big]+$$

$$+T_F N_F\left[\frac{10}{3}\ln^2(x)+\frac{56}{3}\ln(x)+\frac{502}{27}-\frac{16}{3}\zeta_2\right]\Bigg\}+C_F^2\left[\frac{7}{3}\ln^3(x)+21\ln^2(x)\right.$$
$$\left.+(21+8\zeta_2)\ln(x)+\frac{117}{2}-20\zeta_2+16\zeta_3\right], \tag{7.42}$$

$$\tag{7.43}$$
$$\Delta C_{g_1,q}^{(2),\mathrm{PS,L}} \simeq C_F T_F N_F\left[\frac{20}{3}\ln^3(x)+58\ln^2(x)+(172-32\zeta_2)\ln(x)+280-96\zeta_2-16\zeta_3\right],$$

$$\tag{7.44}$$
$$\Delta C_{g_1,q}^{(3),\mathrm{PS,L}} \simeq C_F\Bigg\{T_F^2N_F^2\left[-\frac{184}{27}\ln^4(x)-\frac{7312}{81}\ln^3(x)+\left(-\frac{37744}{81}+\frac{64}{9}\zeta_2\right)\ln^2(x)\right.$$
$$\left.+\left(-\frac{245504}{243}+\frac{1088}{27}\zeta_2-\frac{64}{9}\zeta_3\right)\ln(x)-\frac{292640}{243}+\frac{14528}{81}\zeta_2-\frac{32}{27}\zeta_3+\frac{224}{15}\zeta_2^2\right]$$
$$+C_F T_F N_F\left[4\ln^5(x)+\frac{632}{9}\ln^4(x)+\left(\frac{5156}{9}-\frac{1328}{9}\zeta_2\right)\ln^3(x)+\left(2708\right.\right.$$
$$\left.-\frac{4096\zeta_2}{3}-\frac{256\zeta_3}{3}\right)\ln^2(x)+\left(\frac{23360}{3}-5064\zeta_2-224\zeta_3+\frac{3856}{15}\zeta_2^2\right)\ln(x)$$
$$\left.+\frac{31712}{3}-6856\zeta_2+\left(-\frac{6160}{3}+32\zeta_2\right)\zeta_3+\frac{15008}{15}\zeta_2^2+352\zeta_5\right]$$
$$+C_A T_F N_F\left[4\ln^5(x)+\frac{2450}{27}\ln^4(x)+\left(\frac{67472}{81}-\frac{688}{9}\zeta_2\right)\ln^3(x)+\left(\frac{325580}{81}\right.\right.$$
$$\left.-\frac{6992\zeta_2}{9}-352\zeta_3\right)\ln^2(x)+\left(\frac{2395192}{243}-\frac{74464}{27}\zeta_2-\frac{21136}{9}\zeta_3+\frac{3088}{15}\zeta_2^2\right)\ln(x)$$
$$\left.+\frac{2166976}{243}-\frac{315832}{81}\zeta_2+\left(-\frac{93968}{27}+\frac{1808}{3}\zeta_2\right)\zeta_3+\frac{10312}{15}\zeta_2^2-264\zeta_5\right]\Bigg\}, \tag{7.45}$$
$$\Delta C_{g_1,g}^{(1)} \simeq 4T_F N_F[3+\ln(x)], \tag{7.46}$$
$$\Delta C_{g_1,g}^{(2)} \simeq C_F T_F N_F\left[\frac{10}{3}\ln^3(x)+41\ln^2(x)+(160-32\zeta_2)\ln(x)+134-88\zeta_2\right]$$
$$+C_A T_F N_F\left[\frac{28}{3}\ln^3(x)+90\ln^2(x)+(304-64\zeta_2)\ln(x)+484-168\zeta_2\right.$$
$$\left.-48\zeta_3\right], \tag{7.47}$$
$$\Delta C_{g_1,g}^{(3)} \simeq T_F N_F\Bigg\{\left[\frac{44}{5}C_A^2-\frac{23}{45}C_F^2+C_F\left(\frac{146}{45}C_A-4T_F N_F\right)\right]\ln^5(x)+\left[\frac{1627}{9}C_A^2-21C_F^2\right.$$
$$\left.-\frac{68}{9}C_A T_F N_F+C_F\left(\frac{2143}{27}C_A-\frac{2654}{27}T_F N_F\right)\right]\ln^4(x)+\left[-\frac{12056}{81}C_A T_F N_F\right.$$
$$+C_F\left(C_A\left(\frac{61699}{81}-\frac{976}{9}\zeta_2\right)+T_F N_F\left(-\frac{79220}{81}+\frac{176}{3}\zeta_2\right)\right)+C_A^2\left(\frac{132454}{81}\right.$$
$$\left.\left.-\frac{656}{3}\zeta_2\right)+C_F^2\left(-\frac{602}{3}-\frac{332}{9}\zeta_2\right)\right]\ln^3(x)+\left[C_F\left(T_F N_F\left(-\frac{422240}{81}+\frac{2600}{3}\zeta_2\right.\right.\right.$$
$$\left.\left.+\frac{304}{3}\zeta_3\right)+C_A\left(\frac{221032}{81}-1188\zeta_2+\frac{824}{3}\zeta_3\right)\right)+C_A T_F N_F\left(-\frac{30296}{27}+80\zeta_2\right)+$$

$$+C_A^2\left(\frac{236002}{27}-\frac{6844}{3}\zeta_2-\frac{2216}{3}\zeta_3\right)+C_F^2\left(-\frac{877}{3}-\frac{824}{3}\zeta_2-252\zeta_3\right)\Bigg]\ln^2(x)$$
$$+\Bigg[C_F\Bigg(T_FN_F\left(-\frac{3711364}{243}+\frac{37264\zeta_2}{9}+\frac{10064\zeta_3}{9}-\frac{224\zeta_2^2}{5}\right)+C_A\Bigg(\frac{728285}{243}$$
$$-\frac{46892}{9}\zeta_2+\frac{2944}{15}\zeta_2^2+\frac{27296}{9}\zeta_3\Bigg)\Bigg)+C_A^2\Bigg(\frac{6175024}{243}-\frac{25564}{3}\zeta_2+\frac{9968}{15}\zeta_2^2$$
$$-\frac{52064}{9}\zeta_3\Bigg)+C_F^2\left(\frac{2755}{3}-\frac{1450}{3}\zeta_2-\frac{5632}{3}\zeta_3+\frac{5464}{15}\zeta_2^2\right)$$
$$+C_AT_FN_F\left(-\frac{970424}{243}+\frac{2144}{3}\zeta_2+\frac{1408}{9}\zeta_3\right)\Bigg]\ln(x)+C_F^2\Bigg(\frac{39635}{12}-1280\zeta_2$$
$$+\frac{2494}{3}\zeta_3+\frac{9568}{15}\zeta_2^2+\frac{640}{3}\zeta_2\zeta_3-\frac{11032}{3}\zeta_5\Bigg)+C_A^2\Bigg(\frac{2218820}{81}-\frac{1035208}{81}\zeta_2$$
$$-\frac{83068}{9}\zeta_3+\frac{86584}{45}\zeta_2^2+1544\zeta_2\zeta_3-664\zeta_5\Bigg)+C_AT_FN_F\Bigg(-\frac{495448}{81}+\frac{135320}{81}\zeta_2$$
$$+\frac{5696}{9}\zeta_3-\frac{1376}{45}\zeta_2^2\Bigg)+C_F\Bigg[T_FN_F\Bigg(-\frac{1456493}{81}+\frac{536656}{81}\zeta_2+\frac{75680}{27}\zeta_3$$
$$-\frac{22288}{45}\zeta_2^2-128\zeta_2\zeta_3+96\zeta_5\Bigg)+C_A\Bigg(-\frac{367691}{324}-\frac{488996}{81}\zeta_2-\frac{7432}{27}\zeta_3$$
$$+\frac{72272}{45}\zeta_2^2-\frac{1936}{3}\zeta_2\zeta_3+\frac{12844}{3}\zeta_5\Bigg)\Bigg]\Bigg\}, \tag{7.48}$$
$$\Delta C_{g_1,g}^{(3),d_{abc}} \simeq \frac{d_{abc}d^{abc}N_F^2}{N_A}\left[-\frac{13952}{15}-\frac{64}{3}\zeta_2+640\zeta_3+\frac{8192}{75}\zeta_2^2\right]. \tag{7.49}$$

Note that the expansion has to be performed in z-space, since the Mellin space expression is either applicable for even or odd moments. There are predictions of the leading series small x behaviour of the Wilson coefficients of the structure functions $F_2(x,Q^2)$ and $F_L(x,Q^2)$ from ref. [183], to which we agree to three-loop order except of the overall sign. A small x expansion of the non-singlet Wilson coefficient of the structure function $g_1(x,Q^2)$ has been considered in [184] based on [185], however, in a different not fully specified scheme.

7.2 The large x limit

In the large x limit one obtains the following leading term behaviour for the Wilson coefficients calculated in the present paper. Here we dropped the terms $O(L_a^k/N^2),\ \ k\geq 0$.

$$C_{F_2,q}^{(1),\rm NS} \simeq C_F\left[2S_1^2+3S_1-9-2\zeta_2+\frac{9}{N}\right], \tag{7.50}$$
$$C_{F_2,q}^{(2),\rm NS} \simeq C_F\Bigg\{T_FN_F\left[-\frac{8}{9}S_1^3-\frac{58}{9}S_1^2-\left(\frac{494}{27}-\frac{8}{3}\zeta_2+\frac{52}{3N}\right)S_1+\frac{457}{18}+\frac{170}{9}\zeta_2+\frac{8}{9}\zeta_3-\frac{484}{9N}\right]$$
$$+C_A\Bigg[\frac{22}{9}S_1^3+\left(\frac{367}{18}-4\zeta_2\right)S_1^2+\left(\frac{3155}{54}-\frac{22}{3}\zeta_2-40\zeta_3+\frac{161}{3N}-\frac{16}{N}\zeta_2\right)S_1-\frac{5465}{72}$$
$$-\frac{1139}{18}\zeta_2+\frac{464}{9}\zeta_3+\frac{51}{5}\zeta_2^2+\frac{1202}{9N}+\frac{8}{N}\zeta_2-\frac{24}{N}\zeta_3\Bigg]\Bigg\}+C_F^2\Bigg\{2S_1^4+6S_1^3$$
$$+\left(-\frac{27}{2}+\frac{30}{N}-4\zeta_2\right)S_1^2+\left(-\frac{51}{2}-18\zeta_2+24\zeta_3-\frac{11}{N}+\frac{32\zeta_2}{N}\right)S_1+\frac{331}{8}$$
$$+\frac{111}{2}\zeta_2-66\zeta_3+\frac{4}{5}\zeta_2^2-\frac{95}{N}-\frac{38}{N}\zeta_2+\frac{48}{N}\zeta_3\Bigg\}, \tag{7.51}$$

$$
\begin{aligned}
C^{(3),\mathrm{NS}}_{F_2,q} \simeq\; & C_F\Big\{T_F^2N_F^2\Big[\frac{16}{27}S_1^4+\frac{464}{81}S_1^3+\Big(\frac{1880}{81}-\frac{32}{9}\zeta_2+\frac{208}{9N}\Big)S_1^2+\Big(\frac{34856}{729}-\frac{464}{27}\zeta_2 \\
& +\frac{256}{27}\zeta_3+\frac{4192}{27N}\Big)S_1-\frac{19034}{243}-\frac{8440}{81}\zeta_2+\frac{320}{81}\zeta_3-\frac{1168}{135}\zeta_2^2+\frac{24608}{81N}-\frac{208}{9N}\zeta_2\Big] \\
& +C_AT_FN_F\Big[-\frac{88}{27}S_1^4+\Big(-\frac{3104}{81}+\frac{32}{9}\zeta_2\Big)S_1^3+\Big(-\frac{15062}{81}-\frac{1252}{9N}+\frac{112}{3}\zeta_2 \\
& +16\zeta_3+\frac{32}{N}\zeta_2\Big)S_1^2+\Big(-\frac{321812}{729}+\frac{10528}{81}\zeta_2+\frac{3952}{27}\zeta_3-\frac{256}{15}\zeta_2^2-\frac{29908}{27N}+\frac{488}{3N}\zeta_2 \\
& +\frac{512}{3N}\zeta_3\Big)S_1+\frac{142883}{243}+\frac{66662}{81}\zeta_2-\frac{42836}{81}\zeta_3+\frac{328}{135}\zeta_2^2-\frac{128}{9}\zeta_2\zeta_3+\frac{16}{3}\zeta_5 \\
& -\frac{55130}{27N}+\frac{2140\zeta_2}{27N}+\frac{2552\zeta_3}{9N}+\frac{96\zeta_2^2}{5N}\Big]+C_A^2\Big[\frac{121}{27}S_1^4+\Big(\frac{4649}{81}-\frac{88}{9}\zeta_2\Big)S_1^3 \\
& +\Big(\frac{50689}{162}-\frac{778}{9}\zeta_2-132\zeta_3+\frac{88}{5}\zeta_2^2+\frac{2011}{9N}-\frac{178}{3N}\zeta_2-\frac{32}{N}\zeta_3\Big)S_1^2+\Big(\frac{599375}{729} \\
& -\frac{18179}{81}\zeta_2-\frac{6688}{9}\zeta_3+\frac{212}{15}\zeta_2^2+\frac{176}{3}\zeta_2\zeta_3+232\zeta_5+\frac{50410}{27N}-\frac{568}{N}\zeta_2-\frac{904}{3N}\zeta_3 \\
& +\frac{128}{5N}\zeta_2^2\Big)S_1-\frac{1909753}{1944}-\frac{78607}{54}\zeta_2+\frac{115010}{81}\zeta_3+\frac{13151}{135}\zeta_2^2+\frac{3496}{9}\zeta_2\zeta_3-\frac{416}{3}\zeta_5 \\
& -\frac{12016}{315}\zeta_2^3-\frac{248}{3}\zeta_3^2+\frac{534221}{162N}+\frac{4399}{27N}\zeta_2+\frac{518}{15N}\zeta_2^2-\frac{12844}{9N}\zeta_3-\frac{112}{N}\zeta_2\zeta_3 \\
& -\frac{160}{N}\zeta_5\Big]\Big\}+C_F^2\Big\{T_FN_F\Big[-\frac{16}{9}S_1^5-\frac{140}{9}S_1^4+\Big(-\frac{1366}{27}+\frac{64\zeta_2}{9}-\frac{1504}{27N}\Big)S_1^3 \\
& +\Big(\frac{83}{9}+\frac{224}{3}\zeta_2+\frac{16}{9}\zeta_3-\frac{602}{3N}-\frac{64}{N}\zeta_2\Big)S_1^2+\Big(\frac{2003}{54}+\frac{4354}{27}\zeta_2-\frac{40}{9}\zeta_3-\frac{16}{3}\zeta_2^2 \\
& +\frac{19690}{81N}-\frac{1456}{9N}\zeta_2-\frac{640}{3N}\zeta_3\Big)S_1-\frac{341}{18}-\frac{10733}{27}\zeta_2+\frac{21532}{27}\zeta_3-\frac{21604}{135}\zeta_2^2 \\
& -\frac{80}{3}\zeta_2\zeta_3-\frac{1568}{9}\zeta_5+\frac{47101}{54N}+\frac{10610\zeta_2}{27N}-\frac{16136\zeta_3}{27N}+\frac{192\zeta_2^2}{5N}\Big]+C_A\Big[\frac{44}{9}S_1^5 \\
& +\Big(\frac{433}{9}-8\zeta_2\Big)S_1^4+\Big(\frac{8425}{54}-\frac{284}{9}\zeta_2-80\zeta_3+\frac{4460}{27N}-\frac{32}{N}\zeta_2\Big)S_1^3+\Big(-\frac{5563}{36} \\
& -\frac{592}{3}\zeta_2+\frac{142}{5}\zeta_2^2+\frac{640}{9}\zeta_3+\frac{3127}{6N}-\frac{8}{3N}\zeta_2+\frac{80}{N}\zeta_3\Big)S_1^2+\Big(-\frac{16981}{24}-\frac{28495}{54}\zeta_2 \\
& +752\zeta_3+\frac{299}{3}\zeta_2^2+96\zeta_2\zeta_3+120\zeta_5-\frac{152317}{162N}+\frac{7532}{9N}\zeta_2-\frac{1496}{3N}\zeta_3+\frac{672}{5N}\zeta_2^2\Big)S_1 \\
& +\frac{191545}{108}\zeta_2-\frac{49346}{27}\zeta_3+\frac{11419}{27}\zeta_2^2-\frac{3896}{9}\zeta_5-828\zeta_2\zeta_3-\frac{23098}{315}\zeta_2^3+\frac{536}{3}\zeta_3^2 \\
& -\frac{751331}{216N}-\frac{88043}{54N}\zeta_2+\frac{78052}{27N}\zeta_3-\frac{1721}{5N}\zeta_2^2+\frac{432}{N}\zeta_2\zeta_3+\frac{1200}{N}\zeta_5+\frac{9161}{12}\Big]\Big\} \\
& +C_F^3\Big[\frac{4}{3}S_1^6+6S_1^5+\Big(-9-4\zeta_2+\frac{42}{N}\Big)S_1^4+\Big(-\frac{93}{2}-36\zeta_2+48\zeta_3+\frac{14}{N}+\frac{64}{N}\zeta_2\Big) \\
& \times S_1^3+\Big(\frac{187}{4}+66\zeta_2-60\zeta_3+\frac{8}{5}\zeta_2^2-\frac{1237}{6N}-\frac{4}{N}\zeta_2-\frac{32}{N}\zeta_3\Big)S_1^2+\Big(\frac{1001}{8}+\frac{579}{2}\zeta_2 \\
& -346\zeta_3+84\zeta_2^2-80\zeta_2\zeta_3-240\zeta_5+\frac{119}{6N}+\frac{984}{N}\zeta_3-\frac{998}{3N}\zeta_2-\frac{1856}{5N}\zeta_2^2\Big)S_1-\frac{7255}{24}-
\end{aligned}
$$

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$$- \frac{6197}{12}\zeta_2 - 411\zeta_3 - \frac{1791}{5}\zeta_2^2 + 556\zeta_2\zeta_3 + 1384\zeta_5 + \frac{8144}{315}\zeta_2^3 - \frac{176}{3}\zeta_3^2 + \frac{16889}{24N}$$
$$+ \frac{5029}{6N}\zeta_2 - \frac{650}{N}\zeta_3 + \frac{1780}{3N}\zeta_2^2 - \frac{416}{N}\zeta_2\zeta_3 - \frac{1760}{N}\zeta_5\Bigg], \quad (7.52)$$

$$C_{F_2,q}^{(3),d_{abc}} \simeq \frac{d_{abc}d^{abc}N_F}{N_C}\Bigg[64 + 160\zeta_2$$
$$+ \frac{224}{3}\zeta_3 - \frac{32}{5}\zeta_2^2 - \frac{1280}{3}\zeta_5 - \frac{128}{N} - \frac{448}{N}\zeta_2 + \frac{64}{5N}\zeta_2^2 - \frac{704}{N}\zeta_3 + \frac{128}{N}\zeta_2\zeta_3$$
$$+ \frac{1280}{N}\zeta_5\Bigg], \quad (7.53)$$

$$C_{F_2,q}^{(2),\mathrm{PS}} \simeq 0, \quad (7.54)$$

$$C_{F_2,q}^{(3),\mathrm{PS}} \simeq 0, \quad (7.55)$$

$$C_{F_2,g}^{(1)} \simeq -4T_F N_F \frac{1}{N}[S_1 + 1], \quad (7.56)$$

$$C_{F_2,g}^{(2)} \simeq \frac{1}{N}\Bigg\{C_A T_F N_F\Bigg[-\frac{4}{3}S_1^3 - 8S_1^2 + \Big(-28 + 12\zeta_2\Big)S_1 - 16 - \frac{92}{3}\zeta_3\Bigg]$$
$$+ C_F T_F N_F\Bigg[-\frac{20}{3}S_1^3 - 18S_1^2 + \Big(4 + 12\zeta_2\Big)S_1 - 4 + 10\zeta_2 + \frac{176}{3}\zeta_3\Bigg]\Bigg\}, \quad (7.57)$$

$$C_{F_2,g}^{(3),a} \simeq \frac{1}{N}\Bigg\{C_F\Bigg\{T_F^2 N_F^2\Bigg[\frac{68}{27}S_1^4 + \frac{1984}{81}S_1^3 + \Big(\frac{6260}{81} - \frac{56}{9}\zeta_2\Big)S_1^2 + \Big(-\frac{12304}{243} - \frac{1952}{27}\zeta_2$$
$$- \frac{224}{27}\zeta_3\Big)S_1 - \frac{1189}{27} - \frac{188}{3}\zeta_2 - \frac{16672}{81}\zeta_3 + \frac{1028}{45}\zeta_2^2\Bigg] + C_A T_F N_F\Bigg[-\frac{412}{27}S_1^4$$
$$+ \Big(-\frac{6884}{81} + \frac{272}{9}\zeta_2\Big)S_1^3 + \Big(-\frac{12043}{81} + \frac{568}{9}\zeta_2 + \frac{104}{3}\zeta_3\Big)S_1^2 + \Big(\frac{65936}{243} + \frac{7396}{27}\zeta_2$$
$$- \frac{10796}{27}\zeta_3 + \frac{976}{15}\zeta_2^2\Big)S_1 + \frac{32813}{108} + \frac{1321}{3}\zeta_2 - \frac{11908}{81}\zeta_3 - \frac{1172}{9}\zeta_2^2 + \frac{352}{9}\zeta_2\zeta_3$$
$$+ \frac{1496}{3}\zeta_5\Bigg]\Bigg\} + C_A T_F^2 N_F^2\Bigg[\frac{28}{27}S_1^4 + \frac{608}{81}S_1^3 + \Big(\frac{4384}{81} - \frac{104}{9}\zeta_2\Big)S_1^2 + \Big(\frac{43192}{243}$$
$$- \frac{1376}{27}\zeta_2 + \frac{32}{27}\zeta_3\Big)S_1 + \frac{13120}{81} - \frac{16}{9}\zeta_2 + \frac{17824}{81}\zeta_3 - \frac{4}{5}\zeta_2^2\Bigg] + C_F^2 T_F N_F\Bigg[-\frac{20}{3}S_1^5$$
$$- \frac{83}{3}S_1^4 + \Big(-\frac{254}{9} + \frac{152}{9}\zeta_2\Big)S_1^3 + \Big(-\frac{205}{3} + 118\zeta_2 + \frac{32}{3}\zeta_3\Big)S_1^2 + \Big(-\frac{508}{3}$$
$$- 54\zeta_2 + \frac{2072}{3}\zeta_3 - \frac{2732}{15}\zeta_2^2\Big)S_1 + \frac{617}{12} - 173\zeta_2 + \frac{5564}{9}\zeta_3 + \frac{193}{5}\zeta_2^2 - \frac{1040}{9}\zeta_2\zeta_3$$
$$- \frac{2384}{3}\zeta_5\Bigg] + C_A^2 T_F N_F\Bigg[-\frac{4}{3}S_1^5 - \frac{293}{27}S_1^4 + \Big(-\frac{6112}{81} + \frac{152}{9}\zeta_2\Big)S_1^3 + \Big(-\frac{27578}{81}$$
$$+ \frac{718}{9}\zeta_2 + \frac{272}{3}\zeta_3\Big)S_1^2 + \Big(-\frac{139052}{243} + \frac{5992}{27}\zeta_2 + \frac{3368}{27}\zeta_3 - \frac{136}{3}\zeta_2^2\Big)S_1 - \frac{31172}{81}$$
$$- \frac{478}{9}\zeta_2 - \frac{10988}{81}\zeta_3 + \frac{163}{5}\zeta_2^2 - \frac{344}{9}\zeta_2\zeta_3 - 16\zeta_5\Bigg]\Bigg\}, \quad (7.58)$$

$$C_{F_2,g}^{(3),d_{abc}} \simeq \frac{d_{abc}d^{abc}N_F^2}{N_A}\frac{1}{N}\Bigg[\Big(128 + 128\zeta_2 - 256\zeta_3\Big)S_1^2 + \Big(-64 - 128\zeta_2 - \frac{1536}{5}\zeta_2^2 +$$

$$+768\zeta_3\Big)S_1+80+256\zeta_2-512\zeta_3+\frac{1728}{5}\zeta_2^2-640\zeta_5\Big], \quad (7.59)$$

$$C_{F_L,q}^{(1),\rm NS} \simeq C_F\frac{4}{N}, \quad (7.60)$$

$$C_{F_L,q}^{(2),\rm NS} \simeq \frac{1}{N}\Big\{C_F\Big[T_FN_F\Big(-\frac{16S_1}{3}-\frac{152}{9}\Big)+C_A\Big(\Big(\frac{92}{3}-16\zeta_2\Big)S_1+\frac{430}{9}+16\zeta_2 -24\zeta_3\Big)\Big]+C_F^2\Big[8S_1^2+\Big(-36+32\zeta_2\Big)S_1-34-40\zeta_2+48\zeta_3\Big]\Big\}, \quad (7.61)$$

$$C_{F_L,q}^{(3),\rm NS} \simeq \frac{1}{N}\Big\{C_F\Big\{T_F^2N_F^2\Big[\frac{64}{9}S_1^2+\frac{1216}{27}S_1+\frac{6496}{81}-\frac{64}{9}\zeta_2\Big]+C_AT_FN_F\Big[\Big(-\frac{640}{9}+32\zeta_2\Big) \times S_1^2+\Big(-\frac{12880}{27}+\frac{640\zeta_2}{9}+\frac{512\zeta_3}{3}\Big)S_1-\frac{42976}{81}-64\zeta_2+\frac{128}{3}\zeta_3+\frac{96}{5}\zeta_2^2\Big] +C_A^2\Big[\Big(\frac{1276}{9}-56\zeta_2-32\zeta_3\Big)S_1^2+\Big(\frac{25756}{27}-\frac{3008\zeta_2}{9}+\frac{128\zeta_2^2}{5}-\frac{880\zeta_3}{3}\Big)S_1 +\frac{67312}{81}+\frac{3820}{9}\zeta_2-\frac{1240}{3}\zeta_3-112\zeta_2\zeta_3-160\zeta_5\Big]\Big\}+C_F^2\Big\{T_FN_F\Big[-\frac{128}{9}S_1^3 +\Big(\frac{184}{9}-64\zeta_2\Big)S_1^2+\Big(\frac{9472}{27}-\frac{1088\zeta_2}{9}-\frac{640\zeta_3}{3}\Big)S_1+\frac{2312}{9}\zeta_2-\frac{2656}{9}\zeta_3+158 +\frac{192}{5}\zeta_2^2\Big]+C_A\Big[\Big(-\frac{32732}{27}+\frac{6640}{9}\zeta_2-\frac{472}{3}\zeta_3+\frac{672}{5}\zeta_2^2\Big)S_1+\Big(-\frac{530}{9} +80\zeta_2+80\zeta_3\Big)S_1^2+\Big(\frac{640}{9}-32\zeta_2\Big)S_1^3-\frac{5255}{6}-\frac{11806}{9}\zeta_2+\frac{10832}{9}\zeta_3 -436\zeta_2^2+432\zeta_2\zeta_3+1200\zeta_5\Big]\Big\}+C_F^3\Big\{8S_1^4+\Big(-72+64\zeta_2\Big)S_1^3+\Big(-34+16\zeta_2 -32\zeta_3\Big)S_1^2+\Big(264-232\zeta_2+816\zeta_3-\frac{1856}{5}\zeta_2^2\Big)S_1+\frac{1937}{6}+506\zeta_2-200\zeta_3 +688\zeta_2^2-416\zeta_2\zeta_3-1760\zeta_5\Big\}\Big\}, \quad (7.62)$$

$$C_{F_L,q}^{(3),d_{abc}} \simeq \frac{d_{abc}d^{abc}N_F}{N_C}\frac{1}{N}\Big[-128-448\zeta_2-704\zeta_3+\frac{64}{5}\zeta_2^2+128\zeta_2\zeta_3+1280\zeta_5\Big], \quad (7.63)$$

$$C_{F_L,q}^{(2),\rm PS} \simeq 0, \quad (7.64)$$

$$C_{F_L,q}^{(3),\rm PS} \simeq 0, \quad (7.65)$$

$$C_{F_L,g}^{(1)} \simeq 0, \quad (7.66)$$

$$C_{F_L,g}^{(2)} \simeq 0, \quad (7.67)$$

$$C_{F_L,g}^{(3),a} \simeq 0, \quad (7.68)$$

$$C_{F_L,g}^{(3),d_{abc}} \simeq 0, \quad (7.69)$$

$$C_{F_3,q}^{(1),\rm NS} \simeq C_F\Big[2S_1^2+3S_1-9-2\zeta_2+\frac{5}{N}\Big], \quad (7.70)$$

$$C_{F_3,q}^{(2),\rm NS} \simeq C_F\Big\{T_FN_F\Big[-\frac{8}{9}S_1^3-\frac{58}{9}S_1^2+\Big(-\frac{494}{27}+\frac{8}{3}\zeta_2-\frac{12}{N}\Big)S_1+\frac{457}{18}+\frac{170}{9}\zeta_2+\frac{8}{9}\zeta_3-$$

$$
\begin{aligned}
&-\frac{332}{9N}\bigg] + C_A\bigg[\frac{22}{9}S_1^3 + \left(\frac{367}{18} - 4\zeta_2\right)S_1^2 + \left(\frac{3155}{54} + \frac{23}{N} - \frac{22\zeta_2}{3} - 40\zeta_3\right)S_1 \\
&- \frac{5465}{72} - \frac{1139}{18}\zeta_2 + \frac{464}{9}\zeta_3 + \frac{51}{5}\zeta_2^2 + \frac{772}{9N} - \frac{8\zeta_2}{N}\bigg]\bigg\} + C_F^2\bigg[2S_1^4 + 6S_1^3 \\
&+ \left(-\frac{27}{2} - 4\zeta_2 + \frac{22}{N}\right)S_1^2 + \left(-\frac{51}{2} - 18\zeta_2 + 24\zeta_3 + \frac{25}{N}\right)S_1 + \frac{331}{8} + \frac{111}{2}\zeta_2 \\
&- 66\zeta_3 + \frac{4}{5}\zeta_2^2 - \frac{61}{N} + \frac{2\zeta_2}{N}\bigg], \qquad (7.71)
\end{aligned}
$$

$$
\begin{aligned}
C_{F_3,q}^{(3),\mathrm{NS}} \simeq\; & C_F^2\bigg\{C_A\bigg[\frac{44}{9}S_1^5 + \left(\frac{433}{9} - 8\zeta_2\right)S_1^4 + \left(\frac{8425}{54} - \frac{284\zeta_2}{9} - 80\zeta_3 + \frac{2540}{27N}\right)S_1^3 \\
&+ \left(-\frac{5563}{36} - \frac{592}{3}\zeta_2 + \frac{640}{9}\zeta_3 + \frac{142}{5}\zeta_2^2 + \frac{10441}{18N} - \frac{248}{3N}\zeta_2\right)S_1^2 + \bigg(-\frac{16981}{24} - \frac{28495}{54}\zeta_2 \\
&+ 752\zeta_3 + \frac{299}{3}\zeta_2^2 + 96\zeta_2\zeta_3 + 120\zeta_5 + \frac{44075}{162N} + \frac{892}{9N}\zeta_2 - \frac{1024}{3N}\zeta_3\bigg)S_1 + \frac{9161}{12} \\
&+ \frac{191545}{108}\zeta_2 - \frac{49418}{27}\zeta_3 + \frac{11419}{27}\zeta_2^2 - 828\zeta_2\zeta_3 - \frac{3896}{9}\zeta_5 - \frac{23098}{315}\zeta_2^3 + \frac{536}{3}\zeta_3^2 \\
&- \frac{562151}{216N} - \frac{17207}{54N}\zeta_2 + \frac{45556}{27N}\zeta_3 + \frac{459}{5N}\zeta_2^2\bigg] + T_F N_F\bigg[-\frac{16}{9}S_1^5 - \frac{140}{9}S_1^4 \\
&+ \left(-\frac{1366}{27} - \frac{1120}{27N} + \frac{64}{9}\zeta_2\right)S_1^3 + \left(\frac{83}{9} - \frac{1990}{9N} + \frac{224}{3}\zeta_2 + \frac{16}{9}\zeta_3\right)S_1^2 \\
&+ \left(\frac{2003}{54} - \frac{8726}{81N} + \frac{4354}{27}\zeta_2 - \frac{368}{9N}\zeta_2 - \frac{16}{3}\zeta_2^2 - \frac{40}{9}\zeta_3\right)S_1 - \frac{341}{18} - \frac{10733}{27}\zeta_2 \\
&+ \frac{21532}{27}\zeta_3 - \frac{21604}{135}\zeta_2^2 - \frac{80}{3}\zeta_2\zeta_3 - \frac{1568}{9}\zeta_5 + \frac{38569}{54N} + \frac{3674}{27N}\zeta_2 - \frac{8168}{27N}\zeta_3\bigg]\bigg\} \\
&+ C_F\bigg\{T_F^2 N_F^2\bigg[\frac{16}{27}S_1^4 + \frac{464}{81}S_1^3 + \left(\frac{1880}{81} - \frac{32}{9}\zeta_2 + \frac{16}{N}\right)S_1^2 + \bigg(\frac{34856}{729} - \frac{464}{27}\zeta_2 \\
&+ \frac{256}{27}\zeta_3 + \frac{992}{9N}\bigg)S_1 - \frac{19034}{243} - \frac{8440}{81}\zeta_2 + \frac{320}{81}\zeta_3 - \frac{1168}{135}\zeta_2^2 + \frac{18112}{81N} - \frac{16}{N}\zeta_2\bigg] \\
&+ C_A^2\bigg[\frac{121}{27}S_1^4 + \left(\frac{4649}{81} - \frac{88}{9}\zeta_2\right)S_1^3 + \bigg(\frac{50689}{162} - \frac{778}{9}\zeta_2 - 132\zeta_3 + \frac{88}{5}\zeta_2^2 \\
&+ \frac{245}{3N} - \frac{10}{3N}\zeta_2\bigg)S_1^2 + \bigg(\frac{599375}{729} - \frac{18179}{81}\zeta_2 - \frac{6688}{9}\zeta_3 + \frac{212}{15}\zeta_2^2 + \frac{176}{3}\zeta_2\zeta_3 \\
&+ 232\zeta_5 + \frac{8218}{9N} - \frac{2104}{9N}\zeta_2 - \frac{8}{N}\zeta_3\bigg)S_1 - \frac{1909753}{1944} - \frac{78607}{54}\zeta_2 + \frac{115064}{81}\zeta_3 \\
&+ \frac{13151}{135}\zeta_2^2 + \frac{3496}{9}\zeta_2\zeta_3 - \frac{416}{3}\zeta_5 - \frac{12016}{315}\zeta_2^3 - \frac{248}{3}\zeta_3^2 + \frac{133199}{54N} - \frac{7061}{27N}\zeta_2 \\
&- \frac{9124}{9N}\zeta_3 + \frac{518}{15N}\zeta_2^2\bigg] + C_A T_F N_F\bigg[-\frac{88}{27}S_1^4 + \left(-\frac{3104}{81} + \frac{32\zeta_2}{9}\right)S_1^3 + \bigg(-\frac{15062}{81} \\
&- \frac{68}{N} + \frac{112\zeta_2}{3} + 16\zeta_3\bigg)S_1^2 + \bigg(-\frac{321812}{729} - \frac{1892}{3N} + \frac{10528}{81}\zeta_2 \\
&+ \frac{824}{9N}\zeta_2 - \frac{256}{15}\zeta_2^2 + \frac{3952}{27}\zeta_3\bigg)S_1 + \frac{142883}{243} - \frac{122414}{81N} + \frac{66662}{81}\zeta_2 - \frac{42836}{81}\zeta_3 + \frac{328}{135}\zeta_2^2 -
\end{aligned}
$$

$$- \frac{128}{9}\zeta_2\zeta_3 + \frac{16}{3}\zeta_5 + \frac{3868\zeta_2}{27N} + \frac{2168\zeta_3}{9N}\Bigg]\Bigg\} + C_F^3\Bigg\{\frac{4}{3}S_1^6 + 6S_1^5 + \Bigg(-\frac{93}{2} - 36\zeta_2$$

$$+ 48\zeta_3 + \frac{86}{N}\Bigg)S_1^3 + \Bigg(-9 - 4\zeta_2 + \frac{34}{N}\Bigg)S_1^4 + \Bigg(\frac{187}{4} + 66\zeta_2 - 60\zeta_3 + \frac{8}{5}\zeta_2^2$$

$$- \frac{1033}{6N} - \frac{20}{N}\zeta_2\Bigg)S_1^2 + \Bigg(\frac{1001}{8} + \frac{579}{2}\zeta_2 - 346\zeta_3 + 84\zeta_2^2 - 80\zeta_2\zeta_3 - 240\zeta_5$$

$$- \frac{1465}{6N} - \frac{302}{3N}\zeta_2 + \frac{168}{N}\zeta_3\Bigg)S_1 - \frac{7255}{24} - \frac{6197}{12}\zeta_2 - \frac{1225}{3}\zeta_3 - \frac{1791}{5}\zeta_2^2$$

$$+ 556\zeta_2\zeta_3 + 1384\zeta_5 + \frac{8144}{315}\zeta_2^3 - \frac{176}{3}\zeta_3^2 + \frac{3047}{8N} + \frac{1993}{6N}\zeta_2 - \frac{450}{N}\zeta_3 - \frac{284}{3N}\zeta_2^2\Bigg\}, \tag{7.72}$$

$$C_{F_3,q}^{(3),d_{abc}} \simeq 0, \tag{7.73}$$

and

$$\Delta C_{g_1,q}^{(1),\mathrm{NS,L}} \simeq C_F\Bigg[2S_1^2 + 3S_1 - 9 - 2\zeta_2 + \frac{5}{N}\Bigg], \tag{7.74}$$

$$\Delta C_{g_1,q}^{(2),\mathrm{NS,L}} \simeq C_F\Bigg\{T_F N_F\Bigg[-\frac{58}{9}S_1^2 - \frac{8}{9}S_1^3 + \Bigg(-\frac{494}{27} + \frac{8\zeta_2}{3} - \frac{12}{N}\Bigg)S_1 + \frac{457}{18} + \frac{170}{9}\zeta_2 + \frac{8}{9}\zeta_3$$

$$- \frac{332}{9N}\Bigg] + C_A\Bigg[\frac{22}{9}S_1^3 + \Bigg(\frac{367}{18} - 4\zeta_2\Bigg)S_1^2 + \Bigg(\frac{3155}{54} + \frac{23}{N} - \frac{22\zeta_2}{3} - 40\zeta_3\Bigg)S_1$$

$$- \frac{5465}{72} - \frac{1139}{18}\zeta_2 + \frac{464}{9}\zeta_3 + \frac{51}{5}\zeta_2^2 + \frac{772}{9N} - \frac{8}{N}\zeta_2\Bigg]\Bigg\} + C_F^2\Bigg[2S_1^4 + 6S_1^3$$

$$+ \Bigg(-\frac{27}{2} + \frac{22}{N} - 4\zeta_2\Bigg)S_1^2 + \Bigg(-\frac{51}{2} + \frac{25}{N} - 18\zeta_2 + 24\zeta_3\Bigg)S_1 + \frac{331}{8} + \frac{111}{2}\zeta_2$$

$$- 66\zeta_3 + \frac{4}{5}\zeta_2^2 - \frac{61}{N} + \frac{2\zeta_2}{N}\Bigg], \tag{7.75}$$

$$\Delta C_{g_1,q}^{(2),\mathrm{PS,L}} \simeq 0, \tag{7.76}$$

$$\Delta C_{g_1,q}^{(3),\mathrm{PS,L}} \simeq 0, \tag{7.77}$$

$$\Delta C_{g_1,g}^{(1)} \simeq -\frac{4}{N}T_F N_F(1 + S_1), \tag{7.78}$$

$$\Delta C_{g_1,g}^{(2)} \simeq \frac{1}{N}\Bigg\{C_A T_F N_F\Bigg[-\frac{4}{3}S_1^3 - 8S_1^2 + \Bigg(-28 + 12\zeta_2\Bigg)S_1 - 16 - \frac{92}{3}\zeta_3\Bigg]$$

$$+ C_F T_F N_F\Bigg[-\frac{20}{3}S_1^3 - 18S_1^2 + \Bigg(4 + 12\zeta_2\Bigg)S_1 - 4 + 10\zeta_2 + \frac{176}{3}\zeta_3\Bigg]\Bigg\}, \tag{7.79}$$

$$\Delta C_{g_1,g}^{(3)} \simeq \frac{1}{N}T_F N_F\Bigg\{-\Bigg[\frac{4}{3}C_A^2 + \frac{20}{3}C_F^2\Bigg]S_1^5 + \Bigg[-\frac{293}{27}C_A^2 - \frac{83}{3}C_F^2 + \frac{28}{27}C_A T_F N_F$$

$$+ C_F\Bigg(-\frac{412}{27}C_A + \frac{68}{27}T_F N_F\Bigg)\Bigg]S_1^4 + \Bigg[\frac{608}{81}C_A T_F N_F + C_F\Bigg(\frac{1984}{81}T_F N_F + C_A\Bigg($$

$$- \frac{6884}{81} + \frac{272}{9}\zeta_2\Bigg)\Bigg) + C_A^2\Bigg(-\frac{6112}{81} + \frac{152}{9}\zeta_2\Bigg) + C_F^2\Bigg(-\frac{254}{9} + \frac{152}{9}\zeta_2\Bigg)\Bigg]S_1^3$$

$$+ \Bigg[C_F\Bigg(T_F N_F\Bigg(\frac{6260}{81} - \frac{56}{9}\zeta_2\Bigg) + C_A\Bigg(-\frac{12043}{81} + \frac{568}{9}\zeta_2 + \frac{104}{3}\zeta_3\Bigg)\Bigg) + C_A T_F N_F \times$$

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$$
\times \left(\frac{4384}{81} - \frac{104}{9}\zeta_2\right) + C_F^2\left(-\frac{205}{3} + 118\zeta_2 + \frac{32}{3}\zeta_3\right) + C_A^2\left(-\frac{27578}{81} + \frac{718}{9}\zeta_2
+ \frac{272}{3}\zeta_3\right)\Bigg] S_1^2 + \Bigg[C_F\Bigg(C_A\left(\frac{65936}{243} + \frac{7396}{27}\zeta_2 - \frac{10796}{27}\zeta_3 + \frac{976}{15}\zeta_2^2\right)
+ T_F N_F\left(-\frac{12304}{243} - \frac{1952}{27}\zeta_2 - \frac{224}{27}\zeta_3\right)\Bigg) + C_A T_F N_F\left(\frac{43192}{243} - \frac{1376}{27}\zeta_2
+ \frac{32}{27}\zeta_3\right) + C_A^2\left(-\frac{139052}{243} + \frac{5992}{27}\zeta_2 + \frac{3368}{27}\zeta_3 - \frac{136}{3}\zeta_2^2\right) + C_F^2\Bigg(-\frac{508}{3} - 54\zeta_2
+ \frac{2072}{3}\zeta_3 - \frac{2732}{15}\zeta_2^2\Bigg)\Bigg] S_1 + C_F\Bigg\{C_A\Bigg[\frac{32813}{108} + \frac{1321}{3}\zeta_2 - \frac{11908}{81}\zeta_3 - \frac{1172}{9}\zeta_2^2
+ \frac{352}{9}\zeta_2\zeta_3 + \frac{1496}{3}\zeta_5\Bigg] + T_F N_F\left(-\frac{1189}{27} - \frac{188}{3}\zeta_2 - \frac{16672}{81}\zeta_3 + \frac{1028}{45}\zeta_2^2\right)\Bigg\}
+ C_F^2\left[\frac{617}{12} - 173\zeta_2 + \frac{5564}{9}\zeta_3 + \frac{193}{5}\zeta_2^2 - \frac{1040}{9}\zeta_2\zeta_3 - \frac{2384}{3}\zeta_5\right] + C_A^2\Bigg[-\frac{31172}{81}
- \frac{478}{9}\zeta_2 - \frac{10988}{81}\zeta_3 + \frac{163}{5}\zeta_2^2 - \frac{344}{9}\zeta_2\zeta_3 - 16\zeta_5\Bigg] + C_A T_F N_F\Bigg[\frac{13120}{81} - \frac{16}{9}\zeta_2
+ \frac{17824}{81}\zeta_3 - \frac{4}{5}\zeta_2^2\Bigg]\Bigg\}, \qquad (7.80)
$$

$$
\Delta C_{g_1,g}^{(3),d_{abc}} \simeq \frac{1}{N}\frac{d_{abc}d^{abc}N_F^2}{N_A}\Bigg\{\Big[128 + 128\zeta_2 - 256\zeta_3\Big] S_1^2 + 48 + \Bigg[-64
- 128\zeta_2 + 768\zeta_3 - \frac{1536}{5}\zeta_2^2\Bigg] S_1 + 256\zeta_2 - 512\zeta_3 + \frac{1728}{5}\zeta_2^2 - 640\zeta_5\Bigg\}. \qquad (7.81)
$$

7.3 The large N_F limit

There are also some predictions on the large N_F behaviour of deep-inelastic Wilson coefficients. The $O(C_F T_F^2 N_F^2)$ contribution to the flavor non-singlet Wilson coefficient of the structure function $F_2(x, Q^2)$ agrees with eq. (15a) of [186], which can be brought into a more compact form.

For the structure function $F_L^{\rm NS}(N, a_s)$ the generating functional of the Wilson coefficients reads [187].

$$
G_L(g) = C_F \sum_{k=0}^{\infty} \frac{1}{k!}\frac{\partial^k}{\partial x^k}\left[\frac{g e^{5/3x}\Gamma(x+N)}{(1-x)(2-x)(1+x+n)\Gamma(1+x)\Gamma(N)}\right]_{x=(4/3)T_F N_F g}. \qquad (7.82)
$$

The $O(a_s^k)$ Wilson coefficient is given by its kth expansion coefficient in the variable g, $f_L^{(k)}(g = a_s)$ to be multiplied by $(-1)^{k+1}$. This prediction agrees with the corresponding contributions in our direct calculation to three loop order and implies a modification of [187].

8 Conclusions

We have calculated the unpolarized massless three-loop Wilson coefficients of the structure functions $F_2(x, Q^2)$ and $F_L(x, Q)$ for pure photon exchange and for the charged current structure function $xF_3(x, Q^2)$ to three-loop order. In the polarized case, we calculated the Wilson coefficients to the structure function $g_1(x, Q^2)$ to three-loop order in the Larin scheme and for the non-singlet case also in the $\overline{\mathsf{MS}}$ scheme. We applied the forward Compton amplitude for the respective scattering cross section and followed other technical calculation steps having been described in great detail in our previous calculations of the three-loop anomalous dimensions in refs. [53, 54]. By using the method of arbitrarily high Mellin moments [131] we have obtained a sufficiently large basis to compute the Wilson coefficients without any reference to further structural assumptions. In parallel, we also used the method of differential equations [65] since we face only first order factorizable problems, which, even further, relate to harmonic polylogarithms only in the final results. This method works for any basis of master integrals. The Wilson coefficients depend on 60 harmonic sums, weighted by rational functions in the Mellin variable N, after applying the algebraic relations, and on 31 harmonic sums by also applying the structural relations. The results in z-space are spanned by only 68 harmonic polylogarithms.

Our calculation of the Wilson coefficients of the structure functions F_2, F_L and xF_3 is technically very different from those in [38, 40], the results of which we confirm in a first independent calculation. We also confirm the former one- and two-loop results, also for the structure function $g_1(x, Q^2)$.

The three-loop results for the structure function $g_1(x, Q^2)$ are new. Concerning the polarized case we would like to remark that one may perfectly work in the Larin scheme, expressing the Wilson coefficients, the anomalous dimensions and the parton densities at the starting scale in this scheme, representing the polarized deep-inelastic structure functions or any other observables. For convenience, we also provide the expansions of the Wilson coefficients in the small x and large x region. The latter information may be of relevance studying the high energy Regge limit and the soft region, respectively. We also checked some predictions in the large N_F limit. The present results are of importance for the application in experimental and phenomenological analyzes of precision deep-inelastic data and in particular for precision measurements of the strong coupling constant.

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A $Z_5^{\rm NS}(N)$

In the following we calculate the function $Z_5^{\rm NS}(N)$ needed to transform the non-singlet Wilson coefficient of the structure function g_1 from the Larin into the $\overline{\mathsf{MS}}$ scheme to three-loop order. To two-loop order it has been calculated in [52] in unrenormalized form.[9] In [60] we provided the expansion coefficients needed to two-loop order at the required depth in ε for the present calculation in the unrenormalized case.

[9]Note that eq. (A11) in ref. [52] contains typographical errors.

The practical approach consists in deriving $Z_5^{\rm NS}(N)$ in renormalized form, since it provides a finite renormalization. It is given by

$$Z_5^{\rm NS}(N) = \frac{A_{qq}^{\rm NS,phys,ac,ren}}{A_{qq}^{\rm NS,phys,Larin,ren}} = 1 + a_s z_5^{(1),\rm NS} + a_s^2 z_5^{(2),\rm NS} + a_s^3 z_5^{(3),\rm NS} + O(\hat{a}_s^4), \tag{A.1}$$

where the OMEs are calculated for anticommuting γ_5 (ac) and in the Larin scheme. The gauge parameter ξ is defined in [53]. Note that in the Larin scheme one has to add the physical and EOM expansion coefficients projected in ref. [60], although the two projections are orthogonal.

For odd integers N it is given by

$$z_5^{(1),\rm NS,odd} = -C_F \frac{8}{N(1+N)} \tag{A.2}$$

$$\begin{aligned} z_5^{(2),\rm NS,odd} &= C_F T_F N_F \frac{16\big(-3-N+5N^2\big)}{9N^2(1+N)^2} + C_F^2\Big[\frac{16(1+2N)}{N^2(1+N)^2}S_1 + \frac{16}{N(1+N)}S_2 \\ &+ \frac{8Q_2}{N^3(1+N)^3} + \frac{32}{N(1+N)}S_{-2}\Big] - C_A C_F\Big[\frac{4Q_5}{9N^3(1+N)^3} + \frac{16}{N(1+N)}S_{-2}\Big], \end{aligned} \tag{A.3}$$

$$\begin{aligned} z_5^{(3),\rm NS,odd} &= C_F T_F^2 N_F^2 \frac{128Q_1}{81N^3(1+N)^3} + C_A\Big\{C_F T_F N_F\Big[-\frac{32Q_3}{9N^3(1+N)^3}S_1 + \frac{16Q_{13}}{81N^4(1+N)^4} \\ &- \frac{64}{3N(1+N)}S_3 + \frac{128\big(-3+4N+10N^2\big)}{27N^2(1+N)^2}S_{-2} - \frac{128}{9N(1+N)}S_{-3} \\ &- \frac{256}{9N(1+N)}S_{-2,1} + \frac{128}{3N(1+N)}\zeta_3\Big] + C_F^2\Big[\frac{64Q_4}{27N^3(1+N)^3}S_2 \\ &+ \frac{8Q_{15}}{27(N-1)N^5(1+N)^5(2+N)} + \Big(-\frac{16Q_{11}}{27N^4(1+N)^4} - \frac{512}{3N(1+N)}S_3 \\ &- \frac{3584}{3N(1+N)}S_{-2,1}\Big)S_1 - \frac{32\big(-30-7N+5N^2\big)}{9N^2(1+N)^2}S_3 - \frac{320}{3N(1+N)}S_4 \\ &+ \Big(\frac{32Q_{12}}{27(N-1)N^3(1+N)^3(2+N)} + \frac{64\big(-10+19N+11N^2\big)S_1}{3N^2(1+N)^2} \\ &+ \frac{256}{3N(1+N)}S_2\Big)S_{-2} - \frac{512}{3N(1+N)}S_{-2}^2 + \Big(-\frac{64\big(78-N+5N^2\big)}{9N^2(1+N)^2} \\ &+ \frac{768}{N(1+N)}S_1\Big)S_{-3} + \frac{1472}{3N(1+N)}S_{-4} - \frac{256\big(-33+16N+13N^2\big)}{9N^2(1+N)^2}S_{-2,1} \\ &+ \frac{1024}{3N(1+N)}S_{3,1} - \frac{3712}{3N(1+N)}S_{-2,2} - \frac{1280}{N(1+N)}S_{-3,1} + \frac{7168}{3N(1+N)}S_{-2,1,1} \\ &- \frac{96\big(-2+5N+5N^2\big)}{N^2(1+N)^2}\zeta_3\Big]\Big\} + C_F^2 T_F N_F\Big[\frac{8Q_7}{27N^4(1+N)^4} + \frac{256\big(3+N-5N^2\big)}{27N^2(1+N)^2} \\ &\times S_2 + \frac{128\big(12+17N-14N^3+3N^4\big)}{27N^3(1+N)^3}S_1 + \frac{512}{9N(1+N)}S_3 + \frac{256}{9N(1+N)}S_{-3} \\ &- \frac{256\big(-3+4N+10N^2\big)}{27N^2(1+N)^2}S_{-2} + \frac{512}{9N(1+N)}S_{-2,1} - \frac{128}{3N(1+N)}\zeta_3\Big] \\ &+ C_A^2 C_F\Big[-\frac{4Q_{16}}{81(N-1)N^5(1+N)^5(2+N)} + \Big(\frac{16Q_9}{9N^4(1+N)^4} + \frac{256}{3N(1+N)}S_3 + \end{aligned}$$

$$
\begin{aligned}
&+\frac{1024}{3N(1+N)}S_{-2,1}\Big)S_1-\frac{16\big(8+N+N^2\big)}{3N^2(1+N)^2}S_3+\frac{256}{3N(1+N)}S_4\\
&+\Big(-\frac{64Q_{10}}{27(N-1)N^3(1+N)^3(2+N)}-\frac{256\big(-1+N+N^2\big)}{3N^2(1+N)^2}S_1\Big)S_{-2}\\
&+\frac{64}{N(1+N)}S_{-2}^2+\Big(\frac{64\big(21+N+N^2\big)}{9N^2(1+N)^2}-\frac{512}{3N(1+N)}S_1\Big)S_{-3}-\frac{320}{3N(1+N)}S_{-4}\\
&-\frac{512}{3N(1+N)}S_{3,1}+\frac{128\big(-21+10N+10N^2\big)}{9N^2(1+N)^2}S_{-2,1}+\frac{1024}{3N(1+N)}S_{-2,2}\\
&+\frac{1024}{3N(1+N)}S_{-3,1}-\frac{2048}{3N(1+N)}S_{-2,1,1}+\frac{32\big(-2+5N+5N^2\big)}{N^2(1+N)^2}\zeta_3\Big]\\
&+C_F^3\Big[-\frac{8Q_{14}}{3(N-1)N^5(1+N)^5(2+N)}+\Big(\frac{16Q_8}{3N^4(1+N)^4}-\frac{256(1+2N)}{3N^2(1+N)^2}S_2\\
&+\frac{1024}{N(1+N)}S_{-2,1}\Big)S_1-\frac{128\big(1+3N+3N^2\big)}{3N^3(1+N)^3}S_1^2-\frac{32\big(3+11N-4N^2+4N^3\big)}{3N^2(1+N)^3}S_2\\
&-\frac{128}{3N(1+N)}S_2^2-\frac{64\big(2+5N+N^2\big)S_3}{3N^2(1+N)^2}-\frac{128}{3N(1+N)}S_4\\
&+\Big(-\frac{64Q_6}{3(N-1)N^3(1+N)^3(2+N)}-\frac{128\big(-2+11N+3N^2\big)}{3N^2(1+N)^2}S_1\\
&-\frac{512}{3N(1+N)}S_2\Big)S_{-2}+\frac{256}{3N(1+N)}S_{-2}^2+\Big(\frac{128\big(12-N+N^2\big)}{3N^2(1+N)^2}\\
&-\frac{2560}{3N(1+N)}S_1\Big)S_{-3}-\frac{1664}{3N(1+N)}S_{-4}+\frac{512\big(-4+2N+N^2\big)}{3N^2(1+N)^2}S_{-2,1}\\
&-\frac{512}{3N(1+N)}S_{3,1}+\frac{3328}{3N(1+N)}S_{-2,2}+\frac{3584}{3N(1+N)}S_{-3,1}-\frac{2048}{N(1+N)}S_{-2,1,1}\\
&+\frac{64\big(-2+5N+5N^2\big)}{N^2(1+N)^2}\zeta_3\Big] \qquad\qquad (A.4)
\end{aligned}
$$

with

$$Q_1=N^4+12N^3+7N^2-4N-3, \qquad (A.5)$$

$$Q_2=2N^4+N^3+8N^2+5N+2, \qquad (A.6)$$

$$Q_3=3N^4+6N^3+5N^2+2N+2, \qquad (A.7)$$

$$Q_4=85N^4+104N^3+13N^2-6N+18, \qquad (A.8)$$

$$Q_5=103N^4+140N^3+58N^2+21N+36, \qquad (A.9)$$

$$Q_6=N^6-3N^5+9N^3-33N^2-6N+8, \qquad (A.10)$$

$$Q_7=17N^6+207N^5-685N^4-691N^3-312N^2+80N+48, \qquad (A.11)$$

$$Q_8=22N^6+50N^5+41N^4-132N^3-153N^2-104N-40, \qquad (A.12)$$

$$Q_9=24N^6+72N^5+44N^4-32N^3-35N^2-7N-12, \qquad (A.13)$$

$$Q_{10}=85N^6+222N^5-38N^4-336N^3-92N^2+69N+36, \qquad (A.14)$$

$$Q_{11}=165N^6-185N^5-1034N^4-1285N^3-895N^2-726N-396, \qquad (A.15)$$

$$Q_{12}=349N^6+861N^5-152N^4-1263N^3-665N^2+222N+216, \qquad (A.16)$$

$$Q_{13}=485N^6+643N^5+253N^4+85N^3+326N^2-96N-144, \qquad (A.17)$$

$$
\begin{aligned}
Q_{14}=&\,24N^{10}+99N^9+259N^8+308N^7-186N^6-853N^5-1153N^4-82N^3\\
&+344N^2+328N+144, \qquad (A.18)
\end{aligned}
$$

$$Q_{15} = 845N^{10}+3292N^9+7545N^8+11366N^7-121N^6-19168N^5-19017N^4 \\ -2522N^3+5420N^2+4008N+1440, \quad \text{(A.19)}$$

$$Q_{16} = 6087N^{10}+24679N^9+32532N^8+11838N^7-18471N^6-38727N^5 \\ -37968N^4-12190N^3+11772N^2+8352N+1728. \quad \text{(A.20)}$$

To derive the above relations one has to apply the relations [135]

$$S_{n_1,\ldots,n_p}\left(\frac{N}{2}\right) = 2^{n_1+\ldots+n_p-p}\sum_{\pm} S_{\pm n_1,\ldots,\pm n_p}(N). \quad \text{(A.21)}$$

References

[1] S.D. Drell and J.D. Walecka, *Electrodynamic Processes with Nuclear Targets*, *Annals Phys.* **28** (1964) 18 [INSPIRE].

[2] T. De Forest Jr. and J.D. Walecka, *Electron scattering and nuclear structure*, *Adv. Phys.* **15** (1966) 1 [INSPIRE].

[3] E. Derman, *Parity Violation in Polarized Electron-Deuteron Scattering Without the Parton Model*, *Phys. Rev. D* **19** (1979) 133 [INSPIRE].

[4] A.J. Buras, *Asymptotic Freedom in Deep Inelastic Processes in the Leading Order and Beyond*, *Rev. Mod. Phys.* **52** (1980) 199 [INSPIRE].

[5] E. Reya, *Perturbative Quantum Chromodynamics*, *Phys. Rept.* **69** (1981) 195 [INSPIRE].

[6] F. Eisele, *High-energy neutrino interactions*, *Rept. Prog. Phys.* **49** (1986) 233 [INSPIRE].

[7] M. Diemoz, F. Ferroni and E. Longo, *Nucleon Structure Functions From Neutrino Scattering*, *Phys. Rept.* **130** (1986) 293 [INSPIRE].

[8] T. Sloan, R. Voss and G. Smadja, *The Quark Structure of the Nucleon from the CERN Muon Experiments*, *Phys. Rept.* **162** (1988) 45 [INSPIRE].

[9] S.R. Mishra and F. Sciulli, *Deep inelastic lepton-nucleon scattering*, *Ann. Rev. Nucl. Part. Sci.* **39** (1989) 259 [INSPIRE].

[10] B. Lampe and E. Reya, *Spin physics and polarized structure functions*, *Phys. Rept.* **332** (2000) 1 [hep-ph/9810270] [INSPIRE].

[11] A. Deur, S.J. Brodsky and G.F. De Téramond, *The Spin Structure of the Nucleon*, arXiv:1807.05250 [INSPIRE].

[12] H.D. Politzer, *Asymptotic Freedom: An Approach to Strong Interactions*, *Phys. Rept.* **14** (1974) 129 [INSPIRE].

[13] J. Blümlein, *The Theory of Deeply Inelastic Scattering*, *Prog. Part. Nucl. Phys.* **69** (2013) 28 [arXiv:1208.6087] [INSPIRE].

[14] J. Blümlein, M. Klein, T. Naumann and T. Riemann, *Structure Functions, Quark Distributions and Λ_{QCD} at HERA*, in *Proceedings, HERA Workshop, Hamburg, F.R. Germany, October 12-14, 1987. Volume 1*, R.D. Peccei ed., DESY (1988), pp. 66-106 [INSPIRE].

[15] D. Boer et al., *Gluons and the quark sea at high energies: Distributions, polarization, tomography*, arXiv:1108.1713 [INSPIRE].

[16] R. Abdul Khalek et al., *Science Requirements and Detector Concepts for the Electron-Ion Collider*: EIC Yellow Report, *Nucl. Phys. A* **1026** (2022) 122447 [arXiv:2103.05419] [INSPIRE].

[17] LHeC Study Group collaboration, *A Large Hadron Electron Collider at CERN: Report on the Physics and Design Concepts for Machine and Detector*, *J. Phys. G* **39** (2012) 075001 [arXiv:1206.2913] [INSPIRE].

[18] LHeC and FCC-he Study Group collaborations, *The Large Hadron-Electron Collider at the HL-LHC*, *J. Phys. G* **48** (2021) 110501 [arXiv:2007.14491] [INSPIRE].

[19] A. Accardi et al., *A Critical Appraisal and Evaluation of Modern PDFs*, *Eur. Phys. J. C* **76** (2016) 471 [arXiv:1603.08906] [INSPIRE].

[20] *Workshop on Precision Measurements of α_S*, Munich Germany, February 9–11 2011 [arXiv:1110.0016] [INSPIRE].

[21] S. Moch et al., *High precision fundamental constants at the TeV scale*, arXiv:1405.4781 [INSPIRE].

[22] S. Alekhin, J. Blümlein and S.O. Moch, *α_s from global fits of parton distribution functions*, *Mod. Phys. Lett. A* **31** (2016) 1630023 [INSPIRE].

[23] D. d'Enterria et al., *The strong coupling constant: State of the art and the decade ahead*, arXiv:2203.08271 [INSPIRE].

[24] J. Blümlein, V. Ravindran and W.L. van Neerven, *On the Drell-Levy-Yan relation to $O(\alpha_s^2)$*, *Nucl. Phys. B* **586** (2000) 349 [hep-ph/0004172] [INSPIRE].

[25] J. Blümlein and A. Guffanti, *Scheme-invariant NNLO evolution for unpolarized DIS structure functions*, *Nucl. Phys. B Proc. Suppl.* **152** (2006) 87 [hep-ph/0411110] [INSPIRE].

[26] J. Blümlein, H. Böttcher and A. Guffanti, *Non-singlet QCD analysis of deep inelastic world data at $O(\alpha_s^3)$*, *Nucl. Phys. B* **774** (2007) 182 [hep-ph/0607200] [INSPIRE].

[27] J. Blümlein and M. Saragnese, *The N^3LO scheme-invariant QCD evolution of the non-singlet structure functions $F_2^{\rm NS}(x,Q^2)$ and $g_1^{\rm NS}(x,Q^2)$*, *Phys. Lett. B* **820** (2021) 136589 [arXiv:2107.01293] [INSPIRE].

[28] W. Furmanski and R. Petronzio, *Lepton-Hadron Processes Beyond Leading Order in Quantum Chromodynamics*, *Z. Phys. C* **11** (1982) 293 [INSPIRE].

[29] W.L. van Neerven and E.B. Zijlstra, *Order α_S^2 contributions to the deep inelastic Wilson coefficient*, *Phys. Lett. B* **272** (1991) 127 [INSPIRE].

[30] E.B. Zijlstra and W.L. van Neerven, *Contribution of the second order gluonic Wilson coefficient to the deep inelastic structure function*, *Phys. Lett. B* **273** (1991) 476 [INSPIRE].

[31] E.B. Zijlstra and W.L. van Neerven, *Order α_S^2 correction to the structure function $F_3(x,Q^2)$ in deep inelastic neutrino-hadron scattering*, *Phys. Lett. B* **297** (1992) 377 [INSPIRE].

[32] D.I. Kazakov and A.V. Kotikov, *Total α_s Correction to Deep Inelastic Scattering Cross-section Ratio, $R = \sigma_L/\sigma_T$ in* QCD. *Calculation of Longitudinal Structure Function*, *Nucl. Phys. B* **307** (1988) 721 [*Erratum ibid.* **345** (1990) 299] [INSPIRE].

[33] D.I. Kazakov, A.V. Kotikov, G. Parente, O.A. Sampayo and J. Sanchez Guillen, *Complete quartic α_s^2 correction to the deep inelastic longitudinal structure function F_L in QCD*, *Phys. Rev. Lett.* **65** (1990) 1535 [*Erratum ibid.* **65** (1990) 2921] [INSPIRE].

[34] J. Sanchez Guillen, J. Miramontes, M. Miramontes, G. Parente and O.A. Sampayo, *Next-to-leading order analysis of the deep inelastic $R = \sigma_L/\sigma_T$*, Nucl. Phys. B **353** (1991) 337 [INSPIRE].

[35] S. Moch and J.A.M. Vermaseren, *Deep inelastic structure functions at two loops*, Nucl. Phys. B **573** (2000) 853 [hep-ph/9912355] [INSPIRE].

[36] J. Blümlein, A. De Freitas, C.G. Raab and K. Schönwald, *The unpolarized two-loop massive pure singlet Wilson coefficients for deep-inelastic scattering*, Nucl. Phys. B **945** (2019) 114659 [arXiv:1903.06155] [INSPIRE].

[37] E.B. Zijlstra and W.L. van Neerven, *$O(\alpha_S^2)$ QCD corrections to the deep inelastic proton structure functions F_2 and F_L*, Nucl. Phys. B **383** (1992) 525 [INSPIRE].

[38] J.A.M. Vermaseren, A. Vogt and S. Moch, *The Third-order QCD corrections to deep-inelastic scattering by photon exchange*, Nucl. Phys. B **724** (2005) 3 [hep-ph/0504242] [INSPIRE].

[39] S. Moch, M. Rogal and A. Vogt, *Differences between charged-current coefficient functions*, Nucl. Phys. B **790** (2008) 317 [arXiv:0708.3731] [INSPIRE].

[40] S. Moch, J.A.M. Vermaseren and A. Vogt, *Third-order QCD corrections to the charged-current structure function F_3*, Nucl. Phys. B **813** (2009) 220 [arXiv:0812.4168] [INSPIRE].

[41] S.A. Larin and J.A.M. Vermaseren, *Two Loop QCD Corrections to the Coefficient Functions of the Deep Inelastic Structure Functions F_2 and F_L*, Z. Phys. C **57** (1993) 93 [INSPIRE].

[42] S.A. Larin, T. van Ritbergen and J.A.M. Vermaseren, *The Next next-to-leading QCD approximation for nonsinglet moments of deep inelastic structure functions*, Nucl. Phys. B **427** (1994) 41 [INSPIRE].

[43] A. Retey and J.A.M. Vermaseren, *Some higher moments of deep inelastic structure functions at next-to-next-to-leading order of perturbative QCD*, Nucl. Phys. B **604** (2001) 281 [hep-ph/0007294] [INSPIRE].

[44] J. Blümlein and J.A.M. Vermaseren, *The 16th moment of the non-singlet structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$ to $O(\alpha_S^3)$*, Phys. Lett. B **606** (2005) 130 [hep-ph/0411111] [INSPIRE].

[45] S. Moch and M. Rogal, *Charged current deep-inelastic scattering at three loops*, Nucl. Phys. B **782** (2007) 51 [arXiv:0704.1740] [INSPIRE].

[46] J. Kodaira, S. Matsuda, T. Muta, K. Sasaki and T. Uematsu, *QCD Effects in Polarized Electroproduction*, Phys. Rev. D **20** (1979) 627 [INSPIRE].

[47] E.B. Zijlstra and W.L. van Neerven, *$O(\alpha_s^2)$ corrections to the polarized structure function $g_1(x, Q^2)$*, Nucl. Phys. B **417** (1994) 61 [*Erratum ibid.* **426** (1994) 245] [*Erratum ibid.* **773** (2007) 105] [*Erratum ibid.* **501** (1997) 599] [INSPIRE].

[48] W.L. van Neerven, FORTRAN-*code for the massless polarized 2-loop Wilson coefficient*, (2003).

[49] A. Vogt, S. Moch, M. Rogal and J.A.M. Vermaseren, *Towards the NNLO evolution of polarised parton distributions*, Nucl. Phys. B Proc. Suppl. **183** (2008) 155 [arXiv:0807.1238] [INSPIRE].

[50] J. Blümlein, C. Raab and K. Schönwald, *The Polarized Two-Loop Massive Pure Singlet Wilson Coefficient for Deep-Inelastic Scattering*, Nucl. Phys. B **948** (2019) 114736 [arXiv:1904.08911] [INSPIRE].

[51] S.A. Larin, *The Renormalization of the axial anomaly in dimensional regularization*, Phys. Lett. B **303** (1993) 113 [hep-ph/9302240] [INSPIRE].

[52] Y. Matiounine, J. Smith and W.L. van Neerven, *Two loop operator matrix elements calculated up to finite terms for polarized deep inelastic lepton-hadron scattering*, *Phys. Rev. D* **58** (1998) 076002 [hep-ph/9803439] [INSPIRE].

[53] J. Blümlein, P. Marquard, C. Schneider and K. Schönwald, *The three-loop unpolarized and polarized non-singlet anomalous dimensions from off shell operator matrix elements*, *Nucl. Phys. B* **971** (2021) 115542 [arXiv:2107.06267] [INSPIRE].

[54] J. Blümlein, P. Marquard, C. Schneider and K. Schönwald, *The three-loop polarized singlet anomalous dimensions from off-shell operator matrix elements*, *JHEP* **01** (2022) 193 [arXiv:2111.12401] [INSPIRE].

[55] S. Moch, J.A.M. Vermaseren and A. Vogt, *The Three-Loop Splitting Functions in QCD: The Helicity-Dependent Case*, *Nucl. Phys. B* **889** (2014) 351 [arXiv:1409.5131] [INSPIRE].

[56] S. Moch, J.A.M. Vermaseren and A. Vogt, *The Three loop splitting functions in QCD: The Nonsinglet case*, *Nucl. Phys. B* **688** (2004) 101 [hep-ph/0403192] [INSPIRE].

[57] A. Vogt, S. Moch and J.A.M. Vermaseren, *The Three-loop splitting functions in QCD: The Singlet case*, *Nucl. Phys. B* **691** (2004) 129 [hep-ph/0404111] [INSPIRE].

[58] J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel and C. Schneider, *The three-loop splitting functions $P_{qg}^{(2)}$ and $P_{gg}^{(2,N_F)}$*, *Nucl. Phys. B* **922** (2017) 1 [arXiv:1705.01508] [INSPIRE].

[59] A. Behring et al., *The Polarized Three-Loop Anomalous Dimensions from On-Shell Massive Operator Matrix Elements*, *Nucl. Phys. B* **948** (2019) 114753 [arXiv:1908.03779] [INSPIRE].

[60] J. Blümlein, P. Marquard, C. Schneider and K. Schönwald, *The two-loop massless off-shell QCD operator matrix elements to finite terms*, *Nucl. Phys. B* **980** (2022) 115794 [arXiv:2202.03216] [INSPIRE].

[61] B. Mistlberger, *Higgs boson production at hadron colliders at N^3LO in QCD*, *JHEP* **05** (2018) 028 [arXiv:1802.00833] [INSPIRE].

[62] J. Ablinger, J. Blümlein and C. Schneider, *Harmonic Sums and Polylogarithms Generated by Cyclotomic Polynomials*, *J. Math. Phys.* **52** (2011) 102301 [arXiv:1105.6063] [INSPIRE].

[63] J. Ablinger, J. Blümlein and C. Schneider, *Analytic and Algorithmic Aspects of Generalized Harmonic Sums and Polylogarithms*, *J. Math. Phys.* **54** (2013) 082301 [arXiv:1302.0378] [INSPIRE].

[64] J. Ablinger, J. Blümlein, C.G. Raab and C. Schneider, *Iterated Binomial Sums and their Associated Iterated Integrals*, *J. Math. Phys.* **55** (2014) 112301 [arXiv:1407.1822] [INSPIRE].

[65] J. Ablinger, J. Blümlein, P. Marquard, N. Rana and C. Schneider, *Automated Solution of First Order Factorizable Systems of Differential Equations in One Variable*, *Nucl. Phys. B* **939** (2019) 253 [arXiv:1810.12261] [INSPIRE].

[66] J. Blümlein, *Algebraic relations between harmonic sums and associated quantities*, *Comput. Phys. Commun.* **159** (2004) 19 [hep-ph/0311046] [INSPIRE].

[67] J. Blümlein, *Structural Relations of Harmonic Sums and Mellin Transforms up to Weight $w = 5$*, *Comput. Phys. Commun.* **180** (2009) 2218 [arXiv:0901.3106] [INSPIRE].

[68] J. Blümlein, *Structural Relations of Harmonic Sums and Mellin Transforms at Weight $w = 6$*, *Clay Math. Proc.* **12** (2010) 167 [arXiv:0901.0837] [INSPIRE].

[69] J. Blümlein and A. Tkabladze, *Target mass corrections for polarized structure functions and new sum rules*, *Nucl. Phys. B* **553** (1999) 427 [hep-ph/9812478] [INSPIRE].

[70] J. Blümlein and N. Kochelev, *On the twist -2 and twist - three contributions to the spin dependent electroweak structure functions*, Nucl. Phys. B **498** (1997) 285 [hep-ph/9612318] [INSPIRE].

[71] R.P. Feynman, *Photon-Hadron Interactions*, Addison-Wesley, Reading, MA, U.S.A. (1972).

[72] P.V. Landshoff and J.C. Polkinghorne, *Models for hadronic and leptonic processes at high-energy*, Phys. Rept. **5** (1972) 1 [INSPIRE].

[73] J.D. Jackson, G.G. Ross and R.G. Roberts, *Polarized Structure Functions in the Parton Model*, Phys. Lett. B **226** (1989) 159 [INSPIRE].

[74] R.G. Roberts and G.G. Ross, *Quark model description of polarized deep inelastic scattering and the prediction of G_2*, Phys. Lett. B **373** (1996) 235 [hep-ph/9601235] [INSPIRE].

[75] J. Blümlein and N. Kochelev, *On the twist-2 contributions to polarized structure functions and new sum rules*, Phys. Lett. B **381** (1996) 296 [hep-ph/9603397] [INSPIRE].

[76] J. Blümlein, V. Ravindran and W.L. van Neerven, *Twist-2 heavy flavor contributions to the structure function $G_2(x,Q^2)$*, Phys. Rev. D **68** (2003) 114004 [hep-ph/0304292] [INSPIRE].

[77] M. Glück, S. Kretzer and E. Reya, *The Strange sea density and charm production in deep inelastic charged current processes*, Phys. Lett. B **380** (1996) 171 [*Erratum ibid.* **405** (1997) 391] [hep-ph/9603304] [INSPIRE].

[78] J. Blümlein, A. Hasselhuhn, P. Kovacikova and S. Moch, *$O(\alpha_s)$ Heavy Flavor Corrections to Charged Current Deep-Inelastic Scattering in Mellin Space*, Phys. Lett. B **700** (2011) 294 [arXiv:1104.3449] [INSPIRE].

[79] M. Buza and W.L. van Neerven, *$O(\alpha_S^2)$ contributions to charm production in charged current deep inelastic lepton-hadron scattering*, Nucl. Phys. B **500** (1997) 301 [hep-ph/9702242] [INSPIRE].

[80] J. Blümlein, G. Falcioni and A. De Freitas, *The Complete $O(\alpha_s^2)$ Non-Singlet Heavy Flavor Corrections to the Structure Functions $g_{1,2}^{ep}(x,Q^2)$, $F_{1,2,L}^{ep}(x,Q^2)$, $F_{1,2,3}^{\nu(\bar{\nu})}(x,Q^2)$ and the Associated Sum Rules*, Nucl. Phys. B **910** (2016) 568 [arXiv:1605.05541] [INSPIRE].

[81] J. Blümlein, A. Hasselhuhn and T. Pfoh, *The $O(\alpha_s^2)$ heavy quark corrections to charged current deep-inelastic scattering at large virtualities*, Nucl. Phys. B **881** (2014) 1 [arXiv:1401.4352] [INSPIRE].

[82] A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel and C. Schneider, *$O(\alpha_s^3)$ heavy flavor contributions to the charged current structure function $xF_3(x,Q^2)$ at large momentum transfer*, Phys. Rev. D **92** (2015) 114005 [arXiv:1508.01449] [INSPIRE].

[83] A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, A. von Manteuffel and C. Schneider, *Asymptotic 3-loop heavy flavor corrections to the charged current structure functions $F_L^{W^+-W^-}(x,Q^2)$ and $F_2^{W^+-W^-}(x,Q^2)$*, Phys. Rev. D **94** (2016) 114006 [arXiv:1609.06255] [INSPIRE].

[84] A. Arbuzov, D.Y. Bardin, J. Blümlein, L. Kalinovskaya and T. Riemann, *Hector 1.00: A Program for the calculation of QED, QCD and electroweak corrections to e p and lepton$^\pm$ N deep inelastic neutral and charged current scattering*, Comput. Phys. Commun. **94** (1996) 128 [hep-ph/9511434] [INSPIRE].

[85] M. Schmitz, *Neutrinophysik*, Teubner, Stuttgart, Germany (1997) [DOI].

[86] C. Itzykson ans J.-B. Zuber, *Quantum Field Theory*, McGraw-Hill, New York, U.S.A. (1980).

[87] K. Symanzik, *Small distance behavior in field theory and power counting*, Commun. Math. Phys. **18** (1970) 227 [INSPIRE].

[88] C.G. Callan Jr., *Broken scale invariance in scalar field theory*, Phys. Rev. D **2** (1970) 1541 [INSPIRE].

[89] I. Bierenbaum, J. Blümlein and S. Klein, *Mellin Moments of the $O(\alpha_s^3)$ Heavy Flavor Contributions to unpolarized Deep-Inelastic Scattering at $Q^2 \gg m^2$ and Anomalous Dimensions*, Nucl. Phys. B **820** (2009) 417 [arXiv:0904.3563] [INSPIRE].

[90] D.J. Gross and F. Wilczek, *Ultraviolet Behavior of Nonabelian Gauge Theories*, Phys. Rev. Lett. **30** (1973) 1343 [INSPIRE].

[91] H.D. Politzer, *Reliable Perturbative Results for Strong Interactions?*, Phys. Rev. Lett. **30** (1973) 1346 [INSPIRE].

[92] W.E. Caswell, *Asymptotic Behavior of Nonabelian Gauge Theories to Two Loop Order*, Phys. Rev. Lett. **33** (1974) 244 [INSPIRE].

[93] D.R.T. Jones, *Two Loop Diagrams in Yang-Mills Theory*, Nucl. Phys. B **75** (1974) 531 [INSPIRE].

[94] O.V. Tarasov, A.A. Vladimirov and A.Y. Zharkov, *The Gell-Mann-Low Function of QCD in the Three Loop Approximation*, Phys. Lett. B **93** (1980) 429 [INSPIRE].

[95] S.A. Larin and J.A.M. Vermaseren, *The Three loop QCD β-function and anomalous dimensions*, Phys. Lett. B **303** (1993) 334 [hep-ph/9302208] [INSPIRE].

[96] T. van Ritbergen, J.A.M. Vermaseren and S.A. Larin, *The Four loop β-function in quantum chromodynamics*, Phys. Lett. B **400** (1997) 379 [hep-ph/9701390] [INSPIRE].

[97] M. Czakon, *The four-loop QCD β-function and anomalous dimensions*, Nucl. Phys. B **710** (2005) 485 [hep-ph/0411261] [INSPIRE].

[98] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, *Five-Loop Running of the QCD coupling constant*, Phys. Rev. Lett. **118** (2017) 082002 [arXiv:1606.08659] [INSPIRE].

[99] F. Herzog, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, *The five-loop β-function of Yang-Mills theory with fermions*, JHEP **02** (2017) 090 [arXiv:1701.01404] [INSPIRE].

[100] T. Luthe, A. Maier, P. Marquard and Y. Schröder, *The five-loop β-function for a general gauge group and anomalous dimensions beyond Feynman gauge*, JHEP **10** (2017) 166 [arXiv:1709.07718] [INSPIRE].

[101] T. van Ritbergen, A.N. Schellekens and J.A.M. Vermaseren, *Group theory factors for Feynman diagrams*, Int. J. Mod. Phys. A **14** (1999) 41 [hep-ph/9802376] [INSPIRE].

[102] S.A. Larin and J.A.M. Vermaseren, *The α_S^3 corrections to the Bjorken sum rule for polarized electroproduction and to the Gross-Llewellyn Smith sum rule*, Phys. Lett. B **259** (1991) 345 [INSPIRE].

[103] S.A. Larin, P. Nogueira, T. van Ritbergen and J.A.M. Vermaseren, *The Three loop QCD calculation of the moments of deep inelastic structure functions*, Nucl. Phys. B **492** (1997) 338 [hep-ph/9605317] [INSPIRE].

[104] J. Ablinger, J. Blümlein, S. Klein, C. Schneider and F. Wißbrock, *The $O(\alpha_s^3)$ Massive Operator Matrix Elements of $O(n_f)$ for the Structure Function $F_2(x, Q^2)$ and Transversity*, Nucl. Phys. B **844** (2011) 26 [arXiv:1008.3347] [INSPIRE].

[105] J. Blümlein, A. Hasselhuhn, S. Klein and C. Schneider, *The $O(\alpha_s^3 n_f T_F^2 C_{A,F})$ Contributions to the Gluonic Massive Operator Matrix Elements*, Nucl. Phys. B **866** (2013) 196. [INSPIRE].

[106] J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel and C. Schneider, *The 3-loop pure singlet heavy flavor contributions to the structure function $F_2(x, Q^2)$ and the anomalous dimension*, Nucl. Phys. B **890** (2014) 48 [arXiv:1409.1135] [INSPIRE].

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[107] J. Ablinger et al., *The 3-Loop Non-Singlet Heavy Flavor Contributions and Anomalous Dimensions for the Structure Function $F_2(x,Q^2)$ and Transversity*, *Nucl. Phys. B* **886** (2014) 733 [arXiv:1406.4654] [INSPIRE].

[108] J. Ablinger et al., *The $O(\alpha_s^3 T_F^2)$ Contributions to the Gluonic Operator Matrix Element*, *Nucl. Phys. B* **885** (2014) 280 [arXiv:1405.4259] [INSPIRE].

[109] A. Behring, I. Bierenbaum, J. Blümlein, A. De Freitas, S. Klein and F. Wißbrock, *The logarithmic contributions to the $O(\alpha_s^3)$ asymptotic massive Wilson coefficients and operator matrix elements in deeply inelastic scattering*, *Eur. Phys. J. C* **74** (2014) 3033 [arXiv:1403.6356] [INSPIRE].

[110] J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, C. Schneider and F. Wißbrock, *The Transition Matrix Element $A_{gq}(N)$ of the Variable Flavor Number Scheme at $O(\alpha_s^3)$*, *Nucl. Phys. B* **882** (2014) 263 [arXiv:1402.0359].

[111] J. Blümlein et al., *Heavy Flavor Wilson Coefficients in Deep-Inelastic Scattering: Recent Results*, *PoS* **QCDEV2017** (2017) 031 [arXiv:1711.07957] [INSPIRE].

[112] J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, C. Schneider and F. Wißbrock, *Three Loop Massive Operator Matrix Elements and Asymptotic Wilson Coefficients with Two Different Masses*, *Nucl. Phys. B* **921** (2017) 585 [arXiv:1705.07030] [INSPIRE].

[113] J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, C. Schneider and K. Schönwald, *The Two-mass Contribution to the Three-Loop Gluonic Operator Matrix Element $A_{gg,Q}^{(3)}$*, *Nucl. Phys. B* **932** (2018) 129 [arXiv:1804.02226] [INSPIRE].

[114] J. Ablinger, J. Blümlein, A. De Freitas, C. Schneider and K. Schönwald, *The two-mass contribution to the three-loop pure singlet operator matrix element*, *Nucl. Phys. B* **927** (2018) 339 [arXiv:1711.06717] [INSPIRE].

[115] A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, K. Schönwald and C. Schneider, *The polarized transition matrix element $A_{gq}(N)$ of the variable flavor number scheme at $O(\alpha_s^3)$*, *Nucl. Phys. B* **964** (2021) 115331 [arXiv:2101.05733] [INSPIRE].

[116] J. Ablinger et al., *The three-loop single mass polarized pure singlet operator matrix element*, *Nucl. Phys. B* **953** (2020) 114945 [arXiv:1912.02536] [INSPIRE].

[117] J. Ablinger, J. Blümlein, A. De Freitas, M. Saragnese, C. Schneider and K. Schönwald, *The three-loop polarized pure singlet operator matrix element with two different masses*, *Nucl. Phys. B* **952** (2020) 114916 [arXiv:1911.11630] [INSPIRE].

[118] J. Ablinger et al., *The two-mass contribution to the three-loop polarized gluonic operator matrix element $A_{gg,Q}^{(3)}$*, *Nucl. Phys. B* **955** (2020) 115059 [arXiv:2004.08916] [INSPIRE].

[119] J. Blümlein, A. De Freitas, M. Saragnese, C. Schneider and K. Schönwald, *Logarithmic contributions to the polarized $O(\alpha_s^3)$ asymptotic massive Wilson coefficients and operator matrix elements in deeply inelastic scattering*, *Phys. Rev. D* **104** (2021) 034030 [arXiv:2105.09572] [INSPIRE].

[120] A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel and C. Schneider, *The 3-Loop Non-Singlet Heavy Flavor Contributions to the Structure Function $g_1(x,Q^2)$ at Large Momentum Transfer*, *Nucl. Phys. B* **897** (2015) 612 [arXiv:1504.08217] [INSPIRE].

[121] P. Nogueira, *Automatic Feynman graph generation*, *J. Comput. Phys.* **105** (1993) 279 [INSPIRE].

[122] J.A.M. Vermaseren, *New features of FORM*, math-ph/0010025 [INSPIRE].

[123] M. Tentyukov and J.A.M. Vermaseren, *The Multithreaded version of FORM*, *Comput. Phys. Commun.* **181** (2010) 1419 [hep-ph/0702279] [INSPIRE].

[124] P. Marquard and D. Seidel, *The* `Crusher` *algorithm*, unpublished.

[125] J. Lagrange, *Nouvelles recherches sur la nature et la propagation du son*, Miscellanea Taurinensia **1–2**, Oeuvres (1760–1761), p. 263.

[126] C.F. Gauß, *Theoria attractionis corporum sphaeroidicorum ellipticorum homogeneorum methodo novo tractate*, Commentationes societas scientiarum Gottingensis recentiores III, Werke Bd. **V** (1813), pp. 5–7.

[127] G. Green, *Essay on the Mathematical Theory of Electricity and Magnetism*, Green Papers, Nottingham (1828), pp. 1.

[128] M. Ostrogradsky, *Première note sur la théorie de la chaleur*, Mémoires de l'Académie impériale des sciences de St. Pétersbourg, series **6** 1: 129–133 (presented: November 5, 1828 ; published: 1831).

[129] S. Laporta, *High precision calculation of multiloop Feynman integrals by difference equations*, *Int. J. Mod. Phys. A* **15** (2000) 5087 [hep-ph/0102033] [INSPIRE].

[130] K.G. Chetyrkin and F.V. Tkachov, *Integration by Parts: The Algorithm to Calculate β-functions in 4 Loops*, *Nucl. Phys. B* **192** (1981) 159 [INSPIRE].

[131] J. Blümlein and C. Schneider, *The Method of Arbitrarily Large Moments to Calculate Single Scale Processes in Quantum Field Theory*, *Phys. Lett. B* **771** (2017) 31 [arXiv:1701.04614] [INSPIRE].

[132] S.G. Gorishnii, S.A. Larin, L.R. Surguladze and F.V. Tkachov, *Mincer: Program for Multiloop Calculations in Quantum Field Theory for the Schoonschip System*, *Comput. Phys. Commun.* **55** (1989) 381 [INSPIRE].

[133] S.A. Larin, F.V. Tkachov and J.A.M. Vermaseren, *The FORM version of MINCER*, NIKHEF, Amsterdam, The Netherlands (1991) [INSPIRE].

[134] M. Steinhauser, *MATAD: A Program package for the computation of MAssive TADpoles*, *Comput. Phys. Commun.* **134** (2001) 335 [hep-ph/0009029] [INSPIRE].

[135] J. Blümlein, D.J. Broadhurst and J.A.M. Vermaseren, *The Multiple Zeta Value Data Mine*, *Comput. Phys. Commun.* **181** (2010) 582 [arXiv:0907.2557] [INSPIRE].

[136] J. Blümlein, P. Marquard and C. Schneider, *A refined machinery to calculate large moments from coupled systems of linear differential equations*, in *14th International Symposium on Radiative Corrections: Application of Quantum Field Theory to Phenomenology*, Avignon France, September 9–13 2019 [DOI] [arXiv:1912.04390] [INSPIRE].

[137] M. Kauers, *Guessing Handbook*, JKU Linz, Technical Report RISC 09.

[138] J. Blümlein, M. Kauers, S. Klein and C. Schneider, *Determining the closed forms of the $O(a_s^3)$ anomalous dimensions and Wilson coefficients from Mellin moments by means of computer algebra*, *Comput. Phys. Commun.* **180** (2009) 2143 [arXiv:0902.4091] [INSPIRE].

[139] *Sage*, http://www.sagemath.org/.

[140] M. Kauers, M. Jaroschek and F. Johansson, *Computer Algebra and Polynomials*, J. Gutierrez, J. Schicho, Josef, M. Weimann, eds., Lecture Notes in Computer Science **8942**, Springer, Berlin, Germany (2015), p. 105–125 [DOI] [arXiv:1306.4263].

[141] F.V. Tkachov, *A Theorem on Analytical Calculability of Four Loop Renormalization Group Functions*, *Phys. Lett. B* **100** (1981) 65 [INSPIRE].

[142] F.V. Tkachov, *An algorithm for calculating multiloop integrals*, *Theor. Math. Phys.* **56** (1983) 866 [INSPIRE].

[143] T. Gehrmann, E.W.N. Glover, T. Huber, N. Ikizlerli and C. Studerus, *Calculation of the quark and gluon form factors to three loops in QCD*, *JHEP* **06** (2010) 094 [arXiv:1004.3653] [INSPIRE].

[144] R.N. Lee, A.V. Smirnov and V.A. Smirnov, *Dimensional recurrence relations: an easy way to evaluate higher orders of expansion in* ϵ, *Nucl. Phys. B Proc. Suppl.* **205–206** (2010) 308 [arXiv:1005.0362] [INSPIRE].

[145] R.N. Lee and V.A. Smirnov, *Analytic* ϵ*-expansions of Master Integrals Corresponding to Massless Three-Loop Form Factors and Three-Loop* $g-2$ *up to Four-Loop Transcendentality Weight*, *JHEP* **02** (2011) 102 [arXiv:1010.1334] [INSPIRE].

[146] M. Karr, *Summation in finite terms*, *Journal of the ACM* **28** (1981) 305.

[147] M. Bronstein, *On solutions of linear ordinary difference equations in their coefficient field*, *J. Symb. Comput.* **29** (2000) 841.

[148] C. Schneider, *Symbolic Summation in Difference Fields*, Ph.D. Thesis, RISC, Johannes Kepler University (2001).

[149] C. Schneider, *A Collection of Denominator Bounds to Solve Parameterized Linear Difference Equations in* $\Pi\Sigma$*-Extensions*, *An. Univ. Timisoara Ser. Mat.-Inform.* **42** (2004) 163.

[150] C. Schneider, *Solving parameterized linear difference equations in terms of indefinite nested sums and products*, *J. Differ. Equations Appl.* **11** (2005) 799.

[151] C. Schneider, *Degree Bounds to Find Polynomial Solutions of Parameterized Linear Difference Equations in* $\Pi\Sigma$*-Fields* *Appl. Algebra Engrg. Comm. Comput.* **16** (2005) 1.

[152] C. Schneider, *Simplifying sums in* $\Pi\Sigma^*$*-extensions* *J. Algebra* Appl. **6** (2007) 415.

[153] C. Schneider, *A Symbolic Summation Approach to Find Optimal Nested Sum Representations*, *Clay Math. Proc.* **12** (2010) 285 [arXiv:0904.2323] [INSPIRE].

[154] C. Schneider, *Parameterized Telescoping Proves Algebraic Independence of Sums*, arXiv:0808.2596 [INSPIRE].

[155] C. Schneider, *Fast Algorithms for Refined Parameterized Telescoping in Difference Fields*, in *Computer Algebra and Polynomials*, J. Gutierrez, J. Schicho, M. Weimann eds., Lecture Notes in Computer Science **8942**, Springer (2015), p.157 [DOI] [arXiv:1307.7887].

[156] C. Schneider, *A difference ring theory for symbolic summation*, *J. Symb. Comput.* **72** (2016) 82.

[157] C. Schneider, *Summation theory II: Characterizations of* $R\Pi\Sigma^*$*-extensions and algorithmic aspects*, *J. Symb. Comput.* **80** (2017) 616.

[158] S.A. Abramov, M. Bronstein, M. Petkovšek and C. Schneider, *On rational and hypergeometric solutions of linear ordinary difference equations in* $\Pi\Sigma^*$*-field extensions*, *J. Symb. Comput.* **107** (2021) 23.

[159] C. Schneider, *Symbolic summation assists combinatorics*, *Sém. Lothar. Combin.* **56** (2007) 1.

[160] C. Schneider, *Simplifying Multiple Sums in Difference Fields*, in *Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions*, C. Schneider and J. Blümlein eds, Texts and Monographs in Symbolic Computation Springer, Wien (2013), pp. 325–360 [DOI] [arXiv:1304.4134].

[161] J.A.M. Vermaseren, *Harmonic sums, Mellin transforms and integrals*, *Int. J. Mod. Phys. A* **14** (1999) 2037 [hep-ph/9806280] [INSPIRE].

[162] J. Blümlein and S. Kurth, *Harmonic sums and Mellin transforms up to two loop order*, *Phys. Rev. D* **60** (1999) 014018 [hep-ph/9810241] [INSPIRE].

[163] E. Remiddi and J.A.M. Vermaseren, *Harmonic polylogarithms*, *Int. J. Mod. Phys. A* **15** (2000) 725 [hep-ph/9905237] [INSPIRE].

[164] J. Ablinger, J. Blümlein and C. Schneider, *Generalized Harmonic, Cyclotomic, and Binomial Sums, their Polylogarithms and Special Numbers*, *J. Phys. Conf. Ser.* **523** (2014) 012060 [arXiv:1310.5645] [INSPIRE].

[165] J. Ablinger, *The package HarmonicSums: Computer Algebra and Analytic aspects of Nested Sums*, *PoS* **LL2014** (2014) 019 [arXiv:1407.6180] [INSPIRE].

[166] J. Ablinger, *A Computer Algebra Toolbox for Harmonic Sums Related to Particle Physics*, MSc Thesis, Johannes Kepler University of Linz (2009) [arXiv:1011.1176] [INSPIRE].

[167] J. Ablinger, *Computer Algebra Algorithms for Special Functions in Particle Physics*, Ph.D. Thesis, Johannes Kepler University of Linz, (2012). arXiv:1305.0687 [INSPIRE].

[168] J. Ablinger, *Computing the inverse mellin transform of holonomic sequences using kovacic's algorithm*, in *Proceedings of 13th International Symposium on Radiative Corrections (Applications of Quantum Field Theory to Phenomenology) — PoS(RADCOR2017)*, Sissa Medialab, march 2018, [DOI] [arXiv:1801.01039].

[169] J. Ablinger, *Discovering and Proving Infinite Pochhammer Sum Identities*, arXiv:1902.11001 [INSPIRE].

[170] B. Zürcher, *Rationale Normalformen von pseudo-linearen Abbildungen*, MSc Thesis, Mathematik, ETH Zürich (1994).

[171] C. Schneider, A. De Freitas and J. Blümlein, *Recent Symbolic Summation Methods to Solve Coupled Systems of Differential and Difference Equations*, *PoS* **LL2014** (2014) 017 [arXiv:1407.2537] [INSPIRE].

[172] S. Gerhold, *Uncoupling systems of linear Ore operator equations*, Master's thesis, RISC, Johannes Kepler University, Linz, Austria (2002).

[173] C. Anastasiou, C. Duhr, F. Dulat, F. Herzog and B. Mistlberger, *Higgs Boson Gluon-Fusion Production in QCD at Three Loops*, *Phys. Rev. Lett.* **114** (2015) 212001 [arXiv:1503.06056] [INSPIRE].

[174] C. Duhr, F. Dulat and B. Mistlberger, *Drell-Yan Cross Section to Third Order in the Strong Coupling Constant*, *Phys. Rev. Lett.* **125** (2020) 172001 [arXiv:2001.07717] [INSPIRE].

[175] C. Duhr, F. Dulat and B. Mistlberger, *Charged current Drell-Yan production at N^3LO*, *JHEP* **11** (2020) 143 [arXiv:2007.13313] [INSPIRE].

[176] T. Gehrmann, A. von Manteuffel and T.-Z. Yang, *Renormalization of twist-two operators in QCD and its application to singlet splitting functions*, *PoS* **LL2022** (2022) 063 [arXiv:2207.10108] [INSPIRE].

[177] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, *Adler Function, Bjorken Sum Rule, and the Crewther Relation to Order α_s^4 in a General Gauge Theory*, *Phys. Rev. Lett.* **104** (2010) 132004 [arXiv:1001.3606] [INSPIRE].

[178] S.A. Larin, *The singlet contribution to the Bjorken sum rule for polarized deep inelastic scattering*, *Phys. Lett. B* **723** (2013) 348 [arXiv:1303.4021] [INSPIRE].

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[179] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, *Massless Propagators, $R(s)$ and Multiloop QCD*, Nucl. Part. Phys. Proc. **261–262** (2015) 3 [arXiv:1501.06739] [INSPIRE].

[180] J. Blümlein and A. Vogt, *On the behavior of nonsinglet structure functions at small x*, Phys. Lett. B **370** (1996) 149 [hep-ph/9510410] [INSPIRE].

[181] J. Blümlein and A. Vogt, *The Singlet contribution to the structure function $g_1(x, Q^2)$ at small x*, Phys. Lett. B **386** (1996) 350 [hep-ph/9606254] [INSPIRE].

[182] J. Blümlein and A. Vogt, *The Evolution of unpolarized singlet structure functions at small x*, Phys. Rev. D **58** (1998) 014020 [hep-ph/9712546] [INSPIRE].

[183] S. Catani and F. Hautmann, *High-energy factorization and small x deep inelastic scattering beyond leading order*, Nucl. Phys. B **427** (1994) 475 [hep-ph/9405388] [INSPIRE].

[184] Y. Kiyo, J. Kodaira and H. Tochimura, *Does leading ln x resummation predict the rise of g_1 at small x?*, Z. Phys. C **74** (1997) 631 [hep-ph/9701365] [INSPIRE].

[185] J. Bartels, B.I. Ermolaev and M.G. Ryskin, *Nonsinglet contributions to the structure function $g - 1$ at small x*, Z. Phys. C **70** (1996) 273 [hep-ph/9507271] [INSPIRE].

[186] L. Mankiewicz, M. Maul and E. Stein, *Perturbative part of the nonsinglet structure function F_2 in the large N_F limit*, Phys. Lett. B **404** (1997) 345 [hep-ph/9703356] [INSPIRE].

[187] J.A. Gracey, *Large N_f methods for computing the perturbative structure of deep inelastic scattering*, in *4th International Workshop on Software Engineering and Artificial Intelligence for High-energy and Nuclear Physics*, Pisa Italy, 3–8 April 1995 [hep-ph/9509276] [INSPIRE].