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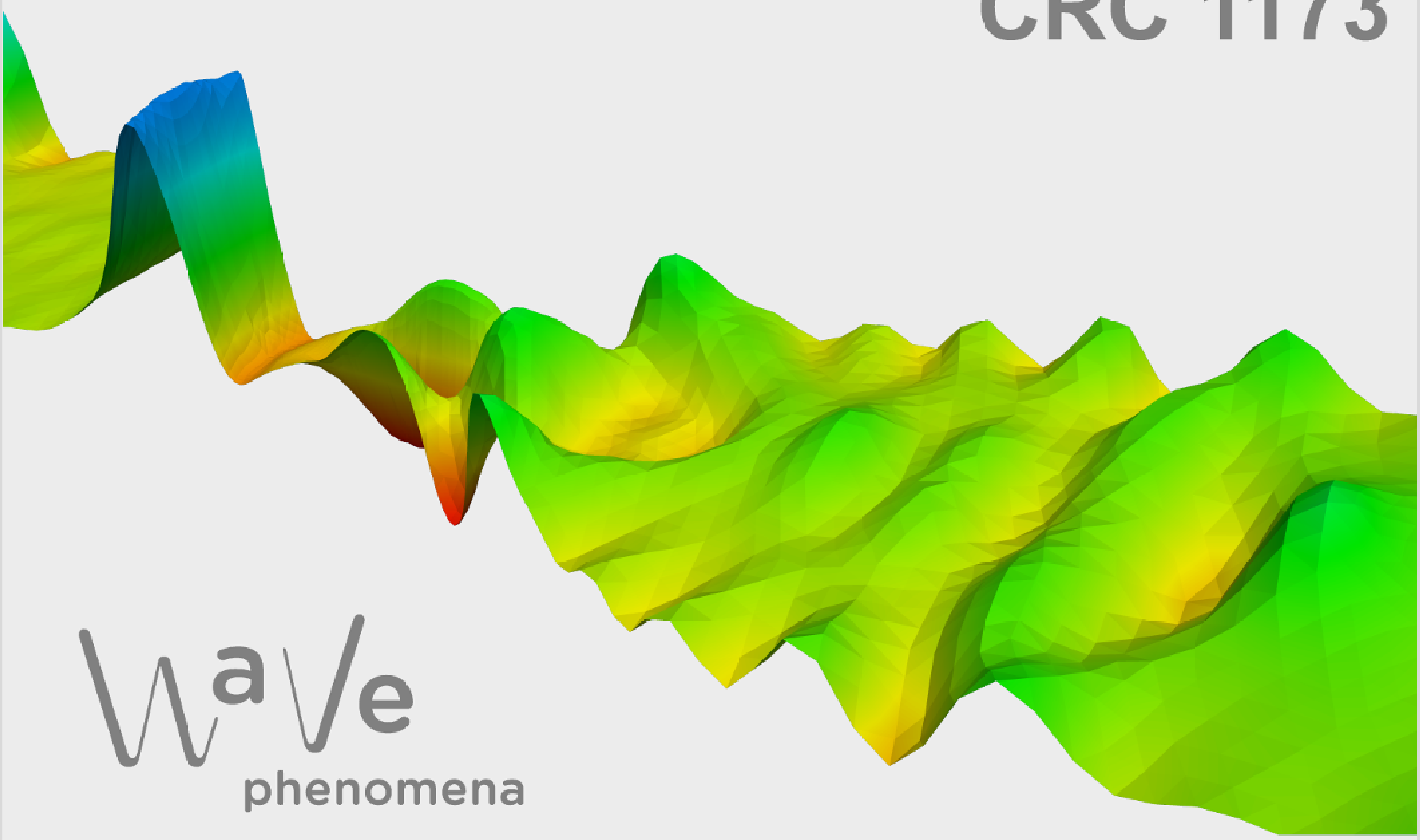

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A SIMPLE VARIATIONAL APPROACH TO NONLINEAR MAXWELL EQUATIONS

RAINER MANDEL[1]

Abstract. We show that nonlinear Maxwell Equations in $\mathbb{R}^3$ admit a convenient dual variational formulation that permits to prove the existence of ground states via standard variational methods.

In this note we provide a simple variational framework for Nonlinear Maxwell Equations of the form

$$\nabla \times \nabla \times E = f(x, E) \qquad \text{in } \mathbb{R}^3. \tag{1}$$

Here, $E : \mathbb{R}^3 \to \mathbb{R}^3$ stands for the electric field and the nonlinearity $f(x, E) \in \mathbb{R}^3$ represents the electric displacement field within the propagation medium, see [12, pp. 825-826]. In the past years, several existence results for nontrivial solutions of (1) have been found. All of them except [10, Theorem 3 (ii)] rely on variational methods so that $f(x, E) = \partial_E F(x, E)$ is assumed for some scalar-valued function F. These results may be separated into two classes: the first deals with cylindrically symmetric solutions that are divergence-free. For such functions the curl-curl operator acts like the classical Laplacian so that several analytic tools like the Rellich-Kondrachov theorem can be used. Of course, the nonlinearity f then needs to be cylindrically symmetric with respect to x as well. The first contribution in this direction is due to Azzollini, Benci, D'Aprile, Fortunato [2] and their existence results have been generalized in [3, 5, 9]. The second class of papers [12, 13] by Mederski and coauthors deals with $\mathbb{Z}^3$-periodic nonlinearities where a cylindrically symmetric ansatz does not make sense. The variational approach developed in these papers is very advanced and many difficulties have to be overcome to prove the existence of a nontrivial solution of (1) via some detailed analysis of generalized Palais-Smale sequences for the energy functional associated with (1). Our goal is to show that in some cases Nonlinear Maxwell Equations admit a much more convenient dual formulation that allows to prove existence results via standard methods of the calculus of variations. As a new feature, our result applies to nonlinearities $f(x, \cdot)$ that, contrary to all other results that we are aware of, need not satisfy any symmetry assumption with respect to x. On the other hand, our approach requires some decay of $f(x, E)$ as $|x| \to \infty$.

Our aim is to show that (1) has ground state solutions under the following assumptions:

(A) $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is a Carathéodory function with $f(x, E) = f_0(x, |E|)|E|^{-1}E$ where $s \mapsto s^{-1} f_0(x, s)$ is positive and increasing on $(0, \infty)$ and

$$\frac{1}{2} f_0(x, s)s - \int_0^s f_0(x, t)\, dt \geq \Gamma(x) \min\{s^p, s^q\} \geq c\, f_0(x, s)s$$

where $0 < c \leq \Gamma(x)(1 + |x|)^\alpha \leq C < \infty$ for $0 < \alpha < 3$ and $2 < p < 6, \max\{2, 6 - 2\alpha\} < q < \infty$.

Here, a ground state is a nontrivial critical point of the associated energy functional

$$I(E) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \times E|^2\, dx - \int_{\mathbb{R}^3} F(x, E)\, dx \qquad \text{where } F(x, E) := \int_0^{|E|} f_0(x, s)\, ds.$$

that has least energy among all nontrivial critical points. In view of assumption (A) the natural (smallest possible) function space for this functional is $H^1(\mathrm{curl}; \mathbb{R}^3) \cap \left[\Gamma^{-1/p} L^p(\mathbb{R}^3; \mathbb{R}^3) \cap \Gamma^{-1/q} L^q(\mathbb{R}^3; \mathbb{R}^3)\right]$. So our main result reads as follows:

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Theorem 1. *Assume that f satisfies (A). Then* (1) *has a ground state solution* $E \in H^1(\mathrm{curl};\mathbb{R}^3) \cap \big[\Gamma^{-1/p}L^p(\mathbb{R}^3;\mathbb{R}^3) \cap \Gamma^{-1/q}L^q(\mathbb{R}^3;\mathbb{R}^3)\big]$.

Remark 2.

(a) The result applies to the power-type nonlinearity $f(x,E) = (1+|x|)^{-\alpha}|E|^2E$ provided that $\alpha > 1$.

(b) The assumptions on Γ are imposed to ensure that certain powers of Γ are Muckenhoupt weights with suitable index, see Proposition 6. The lower bound for Γ seems to be a purely technical assumption, but this is not true. If Γ vanishes on an open set, then ground states do not exist because of a null sequence of (concentrating) gradient field solutions of (1). The argument is the same as in [11, Proposition 1].

(c) The existence of infinitely many other solutions is open. In the case of cylindrical symmetry this can be done by the Symmetric Mountain Pass Theorem for the functional J from (3) on the subspace of cylindrically symmetric functions, see [4, Theorem 2.5]. Whether or not cylindrical symmetry is motivated by some rearrangement principle, remains an open problem.

In the following the symbol $\lesssim$ stands for $\leq C$ for some positive number C, similar for $\gtrsim$. We write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. The exponents p', q' denote, as usual, the Hölder conjugates given by $\frac{1}{p}+\frac{1}{p'} = \frac{1}{q}+\frac{1}{q'} = 1$. We fix the standard norm $\|\cdot\|_r$ on $L^r(\mathbb{R}^3;\mathbb{R}^3)$. The Fourier transform of a tempered distribution f will occasionally be denoted by $\hat{f}$ or $\mathcal{F}(f)$.

1. The dual problem

We use a dual approach to prove Theorem 1. This means that instead of considering the electric field E as the unknown, we treat (1) as a variational problem for $P := f(x,E)$. The main advantage is that the vector field P is automatically divergence-free, which will allow us to use variational methods that almost ignore what happens on the irrotational part of the electric field E. The new task is to find a divergence-free vector field $P : \mathbb{R}^3 \to \mathbb{R}^3$ that solves the quasilinear equation

$$\nabla \times \nabla \times \big(\psi(x,P)\big) = P \qquad \text{in } \mathbb{R}^3 \tag{2}$$

where $\psi(x,\cdot)$ denotes the inverse of $f(x,\cdot)$. Assuming (A), the existence of such a function ψ will be verified in Proposition 9. We shall moreover prove that solutions of (2) can be obtained from critical points of the energy functional

$$J(P) = \int_{\mathbb{R}^3} \Psi(x,P)\,dx - \frac{1}{2}\int_{\mathbb{R}^3} (-\Delta)^{-1}P \cdot P\,dx. \tag{3}$$

where $\Psi(x,\cdot)$ denotes the primitive of $\psi(x,\cdot)$ with $\Psi(x,0) = 0$. So our aim is to provide a sufficient criterion ensuring that J has a ground state solution over a suitable function space, i.e., a nontrivial critical point having least energy among all nontrivial critical points of J. Later, in Section 2, we will show that this ground state solution of J gives rise to a ground state solution of I via $E = \psi(x,P)$.

Our analysis makes use of the following assumptions:

(B) $\psi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is a Carathéodory function with $\psi(x,P) = \psi_0(x,|P|)|P|^{-1}P$ where

$$z \mapsto \psi_0(x,z) \text{ is positive and increasing on } (0,\infty), \qquad z \mapsto z^{-1}\psi_0(x,z) \text{ is decreasing on } (0,\infty)$$

and, for Γ, α, p, q as in (A) and almost all $x \in \mathbb{R}$ and $z > 0$,

$$\int_0^z \psi_0(x,s)\,ds - \frac{1}{2}\psi_0(x,z)z \geq \max\{(\Gamma(x)^{-1}z)^{p'-1}, (\Gamma(x)^{-1}z)^{q'-1}\}z \geq c\psi_0(x,z)z$$

We will see later (Proposition 9) that assumption (B) is the natural counterpart of assumption (A). To benefit from the decay of Γ at infinity we set up our variational approach on the divergence-free part of the reflexive Banach space $Z := \Gamma^{1/p}L^{p'}(\mathbb{R}^3;\mathbb{R}^3) \cap \Gamma^{1/q}L^{q'}(\mathbb{R}^3;\mathbb{R}^3)$ with norm

$$\|P\| := \|\Gamma^{-\frac{1}{p}}P\|_{p'} + \|\Gamma^{-\frac{1}{q}}P\|_{q'}. \tag{4}$$

The corresponding dual space is $Z^* = \Gamma^{-1/p}L^p(\mathbb{R}^3;\mathbb{R}^3) + \Gamma^{-1/q}L^q(\mathbb{R}^3;\mathbb{R}^3)$ with norm

$$\|E\|_{Z^*} := \inf_{E_1+E_2=E} \|\Gamma^{\frac{1}{p}} E_1\|_p + \|\Gamma^{\frac{1}{q}} E_2\|_q. \tag{5}$$

We first introduce the Helmholtz Decomposition on Z that decomposes a vector field on $\mathbb{R}^3$ into its divergence-free (solenoidal) part belonging to X and its curl-free (irrotational) part in Y.

Proposition 3. *Then there are closed subspaces $X, Y \subset Z$ such that the Helmholtz Decomposition $Z = X \oplus Y$ holds with continuous projectors $\Pi : Z \to X$ and $\mathrm{id} - \Pi : Z \to Y$ such that $E = E_1 + E_2$ with $E_1 := \Pi E, E_2 := (\mathrm{id} - \Pi)E$ implies $\nabla \cdot E_1 = 0$ and $\nabla \times E_2 = 0$ in the distributional sense, in particular*

$$\nabla \times \nabla \times E_1 = -\Delta E_1, \qquad \nabla \times \nabla \times E_2 = 0. \tag{6}$$

Proof. For Schwartz functions $E \in \mathcal{S}(\mathbb{R}^3;\mathbb{R}^3)$ we define $\Pi E \in \mathcal{S}(\mathbb{R}^3;\mathbb{C}^3)$ via

$$\widehat{\Pi E}(\xi) := \hat{E}(\xi) - |\xi|^{-2}(\xi \cdot \hat{E}(\xi))\xi.$$

Then (6) follows from $\nabla \times \nabla \times E = -\Delta E + \nabla(\mathrm{div}(E))$. We verify the boundedness of $\Pi, \mathrm{id} - \Pi$ on Z. It suffices to show that all Riesz transforms $f \mapsto \mathcal{F}^{-1}(\xi_j|\xi|^{-1}\hat{f})$ are bounded on $\Gamma^{1/r}L^{r'}(\mathbb{R}^3)$ where $r \in \{p, q\}$. In [14, Corollary 2.2] it is shown that the Riesz transforms are bounded on the weighted Lebesgue space $L^{r'}_{\mathbb{R}^N}(\omega_r)$ provided that the function ω_r belongs to the class of Muckenhoupt weights $A_{r'}$, see [14, p.1239] for a definition. We use this for $\omega_r(x) := \Gamma(x)^{-1/(r-1)}$ because of $\Gamma^{1/r}L^{r'}(\mathbb{R}^3) = \omega_r^{-1/r'}L^{r'}(\mathbb{R}^3) = L^{r'}_{\mathbb{R}^3}(\omega_r)$ in the notation of [14]. From [8, Example 1.3] we get $\omega_p \in A_{p'}, \omega_q \in A_{q'}$ due to $0 < \alpha < 3$, which gives the boundedness of the Riesz transform as an operator on $\Gamma^{1/p}L^{p'}(\mathbb{R}^3)$ and on $\Gamma^{1/q}L^{q'}(\mathbb{R}^3)$, hence on Z. □

Since we are looking for divergence-free solutions P of (2), we set up our variational approach in the space X from the previous proposition. It is the subspace of divergence-free vector fields belonging to $Z = \Gamma^{1/p}L^{p'}(\mathbb{R}^3;\mathbb{R}^3) \cap \Gamma^{1/q}L^{q'}(\mathbb{R}^3;\mathbb{R}^3)$. To make sense of the functional J from (3) we first show that

$$J_1 : X \to \mathbb{R}, \quad P \mapsto \int_{\mathbb{R}^3} \Psi(x, P)\, dx \qquad \text{where } \Psi(x, P) := \int_0^{|P|} \psi_0(x, s)\, ds$$

defines a convex C^1-functional.

Proposition 4. *Assume (B). Then $J_1 \in C^1(X)$ is convex and there is $c > 0$ such that*

$$J_1'(P)[\tilde{P}] = \int_{\mathbb{R}^3} \psi(x, P) \cdot \tilde{P}\, dx, \qquad J_1(P) \geq c \min\{\|P\|^{p'}, \|P\|^{q'}\} \quad \textit{for } P, \tilde{P} \in X. \tag{7}$$

Proof. Assumption (B) yields the estimates

$$\begin{aligned}
J_1(P) &\lesssim \int_{\mathbb{R}^3} \int_0^{|P|} \max\{(\Gamma(x)^{-1}s)^{p'-1}, (\Gamma(x)^{-1}s)^{q'-1}\}\, ds\, dx \\
&\lesssim \int_{\mathbb{R}^3} \Gamma(x)^{1-p'}|P|^{p'} + \Gamma(x)^{1-q'}|P|^{q'}\, dx \\
&= \|\Gamma^{-\frac{1}{p}}P\|_{p'}^{p'} + \|\Gamma^{-\frac{1}{q}}P\|_{q'}^{q'} \\
&\overset{(4)}{\lesssim} \|P\|^{p'} + \|P\|^{q'}.
\end{aligned}$$

So $J_1 : X \to \mathbb{R}$ is well-defined. Similarly, one proves $J_1 \in C^1(X)$ and the formula for $J_1'(P)$ along the lines of the proof of [1, Theorem 2.19]. Since $z \mapsto \psi_0(x, z)$ is increasing by assumption (B), the function $\mathbb{R}^3 \to \mathbb{R}^3, P \mapsto \Psi(x, P)$ is convex for almost all $x \in \Omega$. This implies the convexity of J_1. To prove the inequality in (7) set $A(t) := t^{p'/q'} + t^{q'/p'}$ for $t > 0$. Then

$$\|P\|^{p'} + \|P\|^{q'} \overset{(4)}{\lesssim} \|\Gamma^{-\frac{1}{p}}P\|_{p'}^{p'} + \|\Gamma^{-\frac{1}{q}}P\|_{q'}^{p'} + \|\Gamma^{-\frac{1}{p}}P\|_{p'}^{q'} + \|\Gamma^{-\frac{1}{q}}P\|_{q'}^{q'}$$

$$\begin{aligned}
&\le \int_{\mathbb{R}^3} \Gamma(x)^{p'-1}|P|^{p'} + \Gamma(x)^{q'-1}|P|^{q'}\,dx \\
&+ \left(\int_{\mathbb{R}^3} \Gamma(x)^{p'-1}|P|^{p'}\,dx\right)^{\frac{q'}{p'}} + \left(\int_{\mathbb{R}^3} \Gamma(x)^{q'-1}|P|^{q'}\,dx\right)^{\frac{p'}{q'}} \\
&\lesssim A\left(\int_{\mathbb{R}^3} \Gamma(x)^{p'-1}|P|^{p'} + \Gamma(x)^{q'-1}|P|^{q'}\,dx\right).
\end{aligned}$$

This implies

$$\begin{aligned}
J_1(P) &= \int_{\mathbb{R}^3} \Psi(x,P)\,dx \\
&\overset{(B)}{\gtrsim} \int_{\mathbb{R}^3} \Gamma(x)^{p'-1}|P|^{p'} + \Gamma(x)^{q'-1}|P|^{q'}\,dx \\
&\gtrsim A^{-1}\left(\|P\|^{p'} + \|P\|^{q'}\right) \\
&\gtrsim \min\{(\|P\|^{p'} + \|P\|^{q'})^{\frac{p'}{q'}}, (\|P\|^{p'} + \|P\|^{q'})^{\frac{q'}{p'}}\} \\
&\gtrsim \min\{\|P\|^{p'}, \|P\|^{q'}\},
\end{aligned} \tag{8}$$

which proves (7). □

Next we show that the second part of the functional J from (3) is well-behaved. To this end, we investigate the mapping properties of the linear operator $(-\Delta)^{-1} : X \to X^*$ that is given by

$$(-\Delta)^{-1} f := \mathcal{F}^{-1}\left(|\xi|^{-2}\hat{f}(\xi)\right) = K * f \tag{9}$$

where $K(z) = (4\pi|z|)^{-1}$. Here the convolution respectively the Fourier multiplier $|\xi|^{-2}$ act componentwise on vector fields. We denote by X^* the subspace of divergence-free functions in Z^*.

Proposition 5. *Assume (B). Then the linear operator* $(-\Delta)^{-1} : X \to X^*, P \mapsto (-\Delta)^{-1}P$ *is compact.*

Proof. We have by Young's and Hölder's inequalities

$$\begin{aligned}
\|\Gamma^{\frac{1}{p}}(K\mathbb{1}_{|\cdot|\le 1} * P)\|_p &\le \|\Gamma\|_\infty^{\frac{1}{p}} \|K\mathbb{1}_{|\cdot|\le 1} * P\|_p \\
&\le \|\Gamma\|_\infty^{\frac{1}{p}} \|K\mathbb{1}_{|\cdot|\le 1}\|_{\frac{p}{2}} \|P\|_{p'} \\
&\le \|\Gamma\|_\infty^{\frac{2}{p}} \|K\mathbb{1}_{|\cdot|\le 1}\|_{\frac{p}{2}} \|\Gamma^{-\frac{1}{p}}P\|_{p'} \\
&\overset{(4)}{\le} \|\Gamma\|_\infty^{\frac{2}{p}} \|K\mathbb{1}_{|\cdot|\le 1}\|_{\frac{p}{2}} \|P\|
\end{aligned}$$

The prefactor is finite due to $\Gamma \in L^\infty(\mathbb{R}^3), |K(z)| \lesssim |z|^{-1}$ and $2 < p < 6$.

Since $q > 6 - 2\alpha$ and $\alpha > 0$, we may choose $\mu > 0$ according to $\frac{1}{q} - \frac{1}{6} < \frac{1}{\mu} < \frac{\alpha}{3q}$. Then $\alpha < 3$ implies $0 < \frac{1}{\mu} < \frac{1}{q}$. We may thus define $r, s \in (1,\infty)$ via $\frac{1}{r} := \frac{2}{q} - \frac{2}{\mu}, \frac{1}{s} := \frac{1}{\mu} + \frac{1}{q'}$. Then

$$\begin{aligned}
\|\Gamma^{\frac{1}{q}}(K\mathbb{1}_{|\cdot|\ge 1} * P)\|_q &\le \|\Gamma^{\frac{1}{q}}\|_\mu \|K\mathbb{1}_{|\cdot|\ge 1} * P\|_{\frac{q\mu}{\mu-q}} \\
&\le \|\Gamma^{\frac{1}{q}}\|_\mu \|K\mathbb{1}_{|\cdot|\ge 1}\|_r \|P\|_s \\
&\le \|\Gamma^{\frac{1}{q}}\|_\mu^2 \|K\mathbb{1}_{|\cdot|\ge 1}\|_r \|\Gamma^{-\frac{1}{q}}P\|_{q'} \\
&\overset{(4)}{\le} \|\Gamma^{\frac{1}{q}}\|_\mu^2 \|K\mathbb{1}_{|\cdot|\ge 1}\|_r \|P\|.
\end{aligned}$$

From $\Gamma(x) \sim (1+|x|)^{-\alpha}$ and $|K(z)| \sim |z|^{-1}$ we infer that the prefactor is finite if and only if $\mu\alpha/q > 3$ and $r > 3$. Both inequalities hold by our choice of μ. Hence, we may combine (5) with the previous two estimates to get

$$\|(-\Delta)^{-1}P\|_{Z^*} = \|K * P\|_{Z^*} \leq \left(\|\Gamma\|_\infty^{\frac{2}{p}} \|K\mathbb{1}_{|\cdot|\leq 1}\|_{\frac{p}{2}} + \|\Gamma^{\frac{1}{q}}\|_\mu^2 \|K\mathbb{1}_{|\cdot|\geq 1}\|_r \right) \|P\|,$$

which proves the boundedness of the operator from X to Z^*. Since $(-\Delta)^{-1}P$ is divergence-free (because so is P), we even conclude that $(-\Delta)^{-1} : X \to X^*$ is bounded. To prove the compactness, set $\chi_R = \chi(R^{-1}\cdot)$ where $\chi \in C_0^\infty(\mathbb{R}^3)$ is chosen such that $\chi(z) = 1$ for $|z| \leq 1$ and $\chi(z) = 0$ for $|z| \geq 2R$. The compactness of $\chi_R(-\Delta)^{-1}$ for any given $R > 0$ follows from local elliptic L^p-estimates, $p < 6$ and the Rellich-Kondrachov Theorem. Repeating the estimates from above one finds

$$\|(1-\chi_R)(-\Delta)^{-1}P\|_{Z^*} \leq \left(\|\Gamma\mathbb{1}_{\mathbb{R}^3\setminus B_R(0)}\|_\infty^{\frac{1}{p}} \|\Gamma\|_\infty^{\frac{1}{p}} \|K\mathbb{1}_{|\cdot|\leq 1}\|_{\frac{p}{2}} + \|\Gamma^{1/q}\mathbb{1}_{\mathbb{R}^3\setminus B_R(0)}\|_\mu \|\Gamma^{\frac{1}{q}}\|_\mu \|K\mathbb{1}_{|\cdot|\geq 1}\|_r \right) \|P\|.$$

Since the prefactor goes to zero as $R \to \infty$, we infer $(-\Delta)^{-1} = \lim_{R\to\infty} \chi_R(-\Delta)^{-1}$ with respect to the operator norm. Since the operators $\chi_R(-\Delta)^{-1} : X \to X^*$ are compact for all $R > 0$, we conclude that $(-\Delta)^{-1} : X \to X^*$ is compact as well. □

The previous two results imply the following:

Proposition 6. *Assume (B). Then $J \in C^1(X)$ and*

$$J'(P)[\tilde{P}] = \int_{\mathbb{R}^3} \psi(x,P)\cdot\tilde{P}\,dx - \int_{\mathbb{R}^3} (-\Delta)^{-1}P\cdot\tilde{P}\,dx$$

for all $P, \tilde{P} \in X$. In particular, $J'(P) = 0$ holds if and only if $\Pi(\psi(x,P)) = (-\Delta)^{-1}P$.

Note that the projector Π appears in the Euler-Lagrange equation because the test functions $\tilde{P} \in X$ are arbitrary only among the divergence-free vector fields. To prove the existence of ground states, we minimize J over the Nehari manifold

$$\begin{aligned}\mathcal{N} &= \{P \in X\setminus\{0\} : J'(P)[P] = 0\} \\ &= \left\{P \in X\setminus\{0\} : \int_{\mathbb{R}^3} \psi(x,P)\cdot P\,dx = \int_{\mathbb{R}^3} (-\Delta)^{-1}P\cdot P\,dx\right\}\end{aligned}$$

and define the corresponding min-max-level via

$$c_{\mathcal{N}} := \inf_{\mathcal{N}} J. \tag{10}$$

We first provide a suitable min-max characterization of this energy level with the aid of the fibering map.

Proposition 7. *Assume (B). Then*

$$c_{\mathcal{N}} = c \qquad \textit{where } c := \inf_{P\in X\setminus\{0\}} \max_{t>0} J(tP) \in (0,\infty) \tag{11}$$

and $\inf_{\mathcal{N}} J$ is attained at $P^* \in \mathcal{N}$ if and only if the infimum on the right is attained at tP^* for any $t > 0$.

Proof. Fix any $P \in X\setminus\{0\}$ and define the fibering map

$$\gamma(t) := J(tP) = \int_{\mathbb{R}^3} \Psi(x,tP)\,dx - \frac{t^2}{2}\int_{\mathbb{R}^3} (-\Delta)^{-1}P\cdot P\,dx.$$

By definition of $\mathcal{N}$ we have $tP \in \mathcal{N}$ if and only if $J'(tP)[P] = 0$, i.e., $\gamma'(t) = 0$. We claim that for any given $P \in X\setminus\{0\}$ there is precisely one such t. Indeed, $\gamma'(t_1) = \gamma'(t_2) = 0$ and $t_1 > t_2 > 0$ implies

$$0 = t_1^{-1}\gamma'(t_1) - t_2^{-1}\gamma'(t_2) = \int_{\mathbb{R}^3} \big(t_1^{-1}\psi_0(x,t_1|P|) - t_2^{-1}\psi_0(x,t_2|P|)\big)|P|\,dx < 0$$

in view of the monotonicity assumption on ψ_0 from (B), which is a contradiction. So γ has at most one critical point. On the other hand, γ has at least one critical point, a global maximizer, because of $\gamma(t) > \gamma(0) = 0$

for small $t>0$ and $\gamma(t)\to-\infty$ as $t\to\infty$. So we conclude that for any given $P\in X\setminus\{0\}$ there is a unique $t(P)>0$ such that $t(P)P\in\mathcal{N}$. This implies $c_{\mathcal{N}}=c$ and the claimed relation between the minimizers. It remains to show $c>0$.

For $P\in\mathcal{N}$ we have $J'(P)[P]=0, P\neq 0$. So we get as in the proof of (8)

$$\min\{\|P\|^{p'},\|P\|^{q'}\}\lesssim\int_{\mathbb{R}^3}\psi(x,P)\cdot P\,dx=\int_{\mathbb{R}^3}(-\Delta)^{-1}P\cdot P\,dx\leq\|(-\Delta)^{-1}\|\|P\|^2.$$

This and $p',q'<2$ implies $\|P\|\geq\kappa>0$ for some $\kappa>0$ which is independent of P. As in (8) this implies (using the first inequality in assumption (B))

$$J(P)=J(P)-\frac{1}{2}J'(P)[P]=\int_{\mathbb{R}^3}\Psi(x,P)-\frac{1}{2}\psi(x,P)\cdot P\,dx\gtrsim\min\{\|P\|^{p'},\|P\|^{q'}\}\gtrsim\min\{\kappa^{p'},\kappa^{q'}\}.$$

Since this holds for all $P\in\mathcal{N}$ we get $c>0$. □

Theorem 8. *Assume (B). Then $J\in C^1(X)$ admits a ground state.*

Proof. We prove that the min-max level c from (11) is attained at some function $P\in X\setminus\{0\}$. So let (P_n) be a minimizing sequence. After rescaling we may without loss of generality assume $\int_{\mathbb{R}^3}(-\Delta)^{-1}P_n\cdot P_n\,dx=1$. Then we get for any given $s>0$

$$c+o(1)=\sup_{t>0}J(tP_n)\geq J(sP_n)=\int_{\mathbb{R}^3}\Psi(x,sP_n)\,dx-\frac{s^2}{2}\qquad\text{as } n\to\infty.$$

This and the second estimate in (7) show that (P_n) is bounded in X and we may pass to a weakly convergent subsequence that we still denote by (P_n), so $P_n\rightharpoonup P_\star$ in X. Proposition 5 implies $(-\Delta)^{-1}P_n\to(-\Delta)^{-1}P_\star$ in X^* and thus $\int_{\mathbb{R}^3}(-\Delta)^{-1}P_\star\cdot P_\star\,dx=1$, whence

$$\begin{aligned}c&\geq\int_{\mathbb{R}^3}\Psi(x,sP_n)\,dx-\frac{s^2}{2}+o(1)\\&\geq\int_{\mathbb{R}^3}\Psi(x,sP_\star)\,dx-\frac{s^2}{2}\int_{\mathbb{R}^3}(-\Delta)^{-1}P_\star\cdot P_\star\,dx+o(1)\\&=J(sP_\star)+o(1).\end{aligned}$$

In the second inequality we exploited the weak lower semicontinuity (by convexity). Since the above estimate holds for all $s>0$ we conclude that $P_\star$ is a minimizer for the min-max-level c, so a multiple of it, say $P^\star:=t(P_\star)P_\star$, is a minimizer for $c_{\mathcal{N}}$ by Proposition 7. Using $J\in C^1(X)$ we get as in [15, Proposition 9] that the reduced functional $\tilde{J}(P):=\max_{t>0}J(tP)$ is continuously differentiable away from the origin, and $\tilde{J}'(P_\star)=0$ implies $J'(P^\star)=0$. Moreover, by construction of the Nehari manifold $\mathcal{N}$, no other nontrivial critical point of J attains a smaller energy level than $P^\star$, so $P^\star$ is a ground state. □

2. Proof of Theorem 1

We want to prove Theorem 1 using our analysis of the dual problem of the previous section. We first show that the latter applies under the assumptions of Theorem 1.

Proposition 9. *Assume that the function $f:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^3$ satisfies (A). Then there is a function $\psi:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^3$ satisfying (B) such that $f(x,\cdot)^{-1}=\psi(x,\cdot)$ for almost all $x\in\mathbb{R}^3$.*

Proof. By assumption (A) the function $z\mapsto f_0(x,z)$ is continuous and increasing on $[0,\infty)$ with $f(x,z)\to 0$ as $z\to 0$ and $f_0(x,z)\to+\infty$ as $z\to\infty$ for almost all $x\in\mathbb{R}^3$. Hence, there is an increasing inverse $\psi_0(x,\cdot)=f_0(x,\cdot)^{-1}:[0,\infty)\to[0,\infty)$. Then $f(x,\cdot)^{-1}=\psi(x,\cdot)$ where $\psi(x,P):=\psi_0(x,|P|)|P|^{-1}P$ is Carathéodory. Moreover, $\mathbb{R}_+\to\mathbb{R}_+$, $z\mapsto\frac{\psi_0(x,z)}{z}$ is decreasing because $\mathbb{R}_+\to\mathbb{R}_+$, $s\mapsto\frac{f_0(x,s)}{s}$ is increasing. So it remains to verify that ψ_0 satisfies the estimates from (B).

To this end fix $x \in \mathbb{R}^3$ such that $f_0(x,\cdot)^{-1} = \psi_0(x,\cdot)$ holds. In the following let $s, z \in [0,\infty)$ depend on each other via $s = \psi_0(x,z), z = f_0(x,s)$. Since (A) implies $|f_0(x,s)| \sim \Gamma(x)\min\{s^{p-1}, s^{q-1}\}$, we have

$$\psi_0(x,z) = s \sim \max\{(\Gamma(x)^{-1}z)^{p'-1}, (\Gamma(x)^{-1}z)^{q'-1}\}. \tag{12}$$

This shows that the second inequality in (B) hold. The Fenchel-Young identity shows that $f_0(x,\cdot)^{-1} = \psi_0(x,\cdot)$ satisfy

$$\int_0^z \psi_0(x,t)\,dt - \frac{1}{2}\psi_0(x,z)z = \frac{1}{2}f_0(x,s)s - \int_0^s f_0(x,t)\,dt.$$

This implies

$$\begin{aligned}\int_0^z \psi_0(x,t)\,dt - \frac{1}{2}\psi_0(x,z)z &= \frac{1}{2}sf_0(x,s) - \int_0^s f_0(x,t)\,dt \\ &\overset{(A)}{\geq} \Gamma(x)\min\{s^p, s^q\} \\ &\overset{(12)}{\gtrsim} \max\{(\Gamma(x)^{-1}z)^{p'-1}, (\Gamma(x)^{-1}z)^{q'-1}\},\end{aligned}$$

which finishes the proof. □

The above proposition shows that there is a function ψ such that our variational approach of the previous section applies. In particular, a ground state of J exists by Theorem 8. To show that this ground state of J produces a ground state of I via $E := \psi(x,P)$, we need the equivalence of the original problem and the dual problem. This is proved next.

Lemma 10. *Assume (A). Then* $I'(E) = 0, E \in H^1(\mathrm{curl};\mathbb{R}^3) \cap \left[\Gamma^{-1/p}L^p(\mathbb{R}^3;\mathbb{R}^3) + \Gamma^{-1/q}L^q(\mathbb{R}^3;\mathbb{R}^3)\right]$ *if and only if* $J'(P) = 0, P \in X$ *where* P, E *are related to each other via* $P = f(x,E)$, $E = \psi(x,P)$ *and* ψ *is given by Proposition 9.*

Proof. Assume first $I'(E) = 0$ for $E \in H^1(\mathrm{curl};\mathbb{R}^3) \cap (\Gamma^{-1/p}L^p(\mathbb{R}^3;\mathbb{R}^3) + \Gamma^{-1/q}L^q(\mathbb{R}^3;\mathbb{R}^3))$. Then $P = f(x,E)$ satisfies $\nabla\times\nabla\times E = P$ and thus $\nabla\cdot P = 0$ in the distributional sense. Combining this with $|P| = |f(x,E)| \sim \Gamma(x)\min\{|E|^{p-1}, |E|^{q-1}\}$ and $E \in \Gamma^{-1/p}L^p(\mathbb{R}^3;\mathbb{R}^3) + \Gamma^{-1/q}L^q(\mathbb{R}^3;\mathbb{R}^3)$, we infer that P is divergence-free and belongs to $\Gamma^{1/p}L^{p'}(\mathbb{R}^3;\mathbb{R}^3) \cap \Gamma^{1/q}L^{q'}(\mathbb{R}^3;\mathbb{R}^3)$, i.e., $P \in X$. Furthermore, $I'(E) = 0$ implies that $\nabla\times\nabla\times E = P$ holds in the weak sense. In view of $E = \psi(x,P)$ and Proposition 3 this means that $(-\Delta)\Pi(\psi(x,P)) = P$ holds in the weak sense. This implies $|\xi|^2\mathcal{F}(\Pi(\psi(x,P))) = \hat{P}$ in the sense of tempered distributions and thus, by (9), $\Pi(\psi(x,P)) = (-\Delta)^{-1}P$, which is finally equivalent to $J'(P) = 0$ by Proposition 6.

Now assume $P \in X$ and $J'(P) = 0$. Then $X \subset \Gamma^{1/p}L^{p'}(\mathbb{R}^3;\mathbb{R}^3) \cap \Gamma^{1/q}L^{q'}(\mathbb{R}^3;\mathbb{R}^3)$ and (12) gives $E \in \Gamma^{-1/p}L^p(\mathbb{R}^3;\mathbb{R}^3) + \Gamma^{-1/q}L^q(\mathbb{R}^3;\mathbb{R}^3)$. It remains to show $E \in H^1(\mathrm{curl};\mathbb{R}^3)$ because $J'(P) = 0$ then implies $I'(E) = 0$ using the reverse chain of implications from above. We have

$$\nabla\times E = \nabla\times\psi(x,P) = \nabla\times\Pi(\psi(x,P)) = \nabla\times(-\Delta)^{-1}P = \mathcal{F}^{-1}\left(-\frac{i\xi\times\hat{P}}{|\xi|^2}\right).$$

Standard mapping properties of Riesz transforms [7, Corollary 5.2.8] and Riesz potentials [6, Theorem 1.2.3] yield $\nabla\times E \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ provided that $P \in L^r(\mathbb{R}^3;\mathbb{R}^3)$ provided that $\frac{1}{r} - \frac{1}{2} = \frac{1}{3}$, i.e., $r = \frac{6}{5}$. So it remains to show

$$\Gamma^{1/p}L^{p'}(\mathbb{R}^3;\mathbb{R}^3) \cap \Gamma^{1/q}L^{q'}(\mathbb{R}^3;\mathbb{R}^3) \subset L^{6/5}(\mathbb{R}^3;\mathbb{R}^3). \tag{13}$$

In fact, if $6 - 2\alpha < q \leq 6$ then this follows from

$$\|f\|_{6/5} \leq \|\Gamma^{-1/q}f\|_{q'}\|\Gamma^{1/q}\|_{\frac{6q}{6-q}} \lesssim \|f\|.$$

If $q > 6$ choose $\theta \in (0,1)$ such that $\frac{\theta}{p'} + \frac{1-\theta}{q'} = \frac{5}{6}$. Then

$$\|f\|_{6/5} \leq \|\Gamma^{-1/q}f\|_{q'}^{1-\theta}\|\Gamma^{-1/p}f\|_{p'}^{\theta}\|\Gamma^{\frac{1-\theta}{q}+\frac{\theta}{p}}\|_\infty \lesssim \|f\|.$$

So (13) holds and we conclude $E \in H^1(\mathrm{curl};\mathbb{R}^3)$. □

Proof of Theorem 1: For f satisfying assumption (A) as in the theorem we define ψ by Proposition 9. Then $\psi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ satisfies assumption (B) and accordingly, by Theorem 8, the functional J has a ground state solution $P^* \in X \setminus \{0\}$. By Proposition 9, $I'(E) = 0$ with $E \in H^1(\mathrm{curl};\mathbb{R}^3) \cap \left[\Gamma^{-1/p} L^p(\mathbb{R}^3;\mathbb{R}^3) + \Gamma^{-1/q} L^q(\mathbb{R}^3;\mathbb{R}^3)\right]$ is equivalent to $J'(P) = 0$ with $P \in X$. Then [11, Theorem 15] shows that $E^\star(x) := \psi(x, P^\star(x))$ defines a ground state solution for I, which is all we had to prove. □

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