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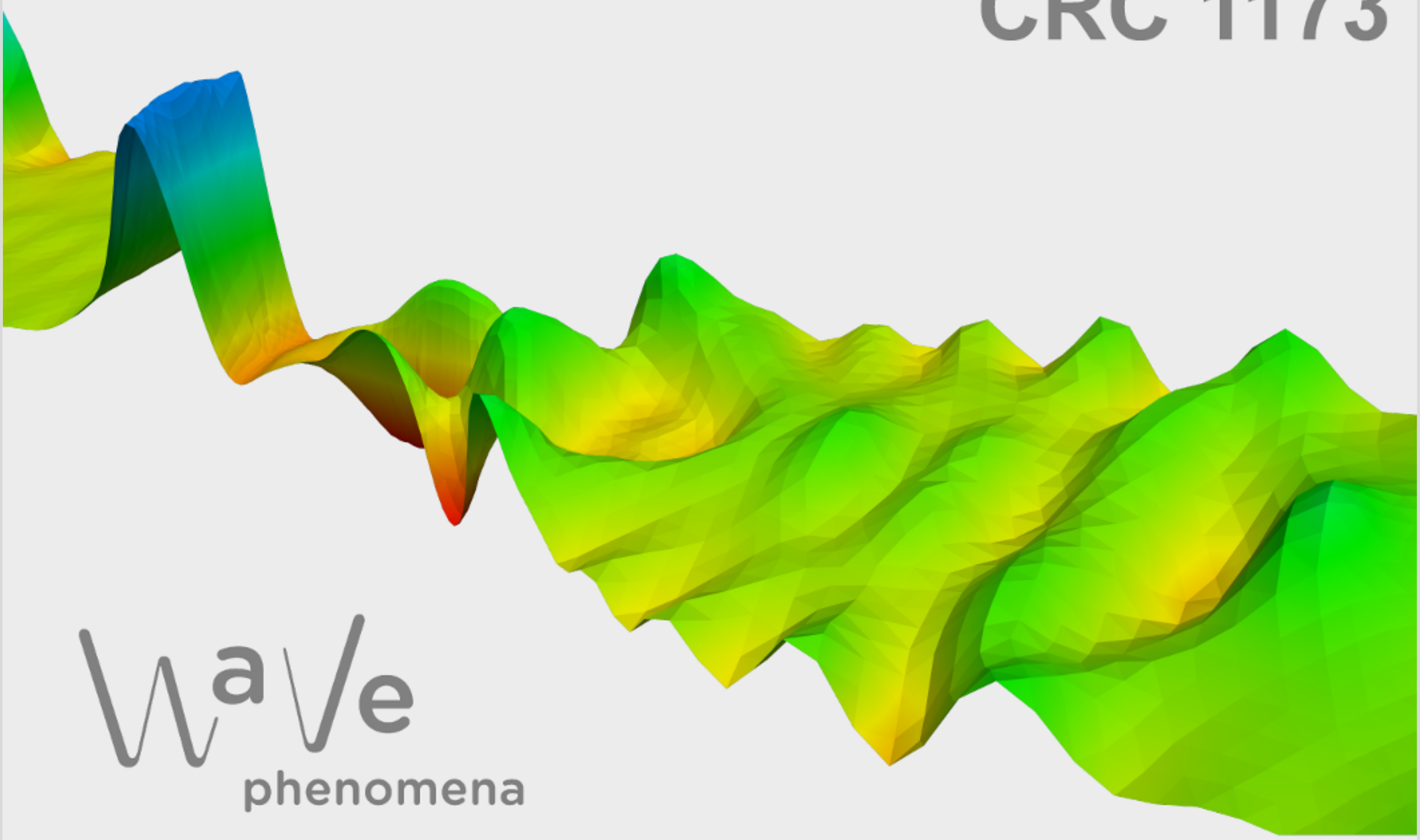

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ERROR ANALYSIS OF THE IMPLICIT EULER SCHEME FOR THE MAXWELL–KERR SYSTEM

ROLAND SCHNAUBELT

Abstract. We establish first-order convergence of the implicit Euler scheme for the quasilinear Maxwell equations with Kerr-type material laws. We only impose regularity assumption which are in accordance with the newly established wellposed theory for the PDE system. In recent literure CFL conditions had to be imposed on full discretizations of this system even for implicit time integration schemes. In our results on the semi-discretization, the time step size is only restricted by the $\mathcal{H}^3$-norm r_0 of the initial fields, and the solutions of the scheme are bounded by $c(r_0)$. We thus expect to obtain full discretization results without CFL condition in future work. The estimates are shown by an intricate iterative procedure inspired by the methods used in the wellposedness theory of the PDE.

1. Introduction

The Maxwell equations are the fundamental laws of electromagnetic theory. In media, they contain constitutive relations which describe the response of the material to the electromagnetic fields. In this work we focus on nonlinear instantaneous relations for which the Maxwell equations become a quasilinear hyperbolic system. On domains $G \subseteq \mathbb{R}^3$ with the standard boundary conditions of a perfect conductor, only recently a comprehensive wellposedness theory in the Sobolev space $\mathcal{H}^3(G)$ for the quasilinear Maxwell system has been established in [24] and [25]. The numerical approximation of these equations is a formidable task since they form a nonlinear, highly coupled 6×6-system on a 3D domain. Explicit time integration schemes suffer from severe CFL conditions and require very regular solutions for a rigorous error analysis. Only very recently, for the semi-implicit Euler and midpoint rules and the exponential Euler method, error estimates for the full discretization were shown under an improved CFL condition in [10], [18], and [19]. In the present paper we analyze the implicit Euler scheme without space discretization and show first-order convergence under the regularity conditions of [24] and [25]. In our main results the time step size $\tau > 0$ is only restricted by the $\mathcal{H}^3$-norm r_0 of the initial fields and the approximations are bounded by $c(r_0)$ uniformly in τ, so that we expect

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to obtain error estimates for full discretizations without a CFL-condition in future research.

We study the quasilinear Maxwell system

$$\begin{aligned}\partial_t(\varepsilon(E)E) &= \operatorname{curl} H - \sigma(E)E, \qquad t \geq 0,\ x \in G,\\ \partial_t(\mu H) &= -\operatorname{curl} E, \qquad t \geq 0,\ x \in G,\\ E \times \nu &= 0, \qquad t \geq 0,\ x \in \partial G,\\ E(0) = E_0, &\quad H(0) = H_0, \qquad x \in G,\end{aligned} \tag{1.1}$$

on a bounded open set $G \subseteq \mathbb{R}^3$ with a C^5-boundary and outer unit normal ν. Here, $E(t,x) \in \mathbb{R}^3$ and $H(t,x) \in \mathbb{R}^3$ are the electric and magnetic fields, respectively, $\varepsilon(x,E) \in \mathbb{R}_+ = (0,\infty)$ is the permittivity, $\sigma(x,E) \in \mathbb{R}$ the conductivity, $\mu(x) \in \mathbb{R}_+$ the permeability. State-independent μ are typically considered in nonlinear optics, see [1], [21]. We treat the isotropic material laws

$$\varepsilon(x,\xi) = \varepsilon_{\text{lin}}(x) + \varepsilon_{\text{nl}}(x)\phi_e(|\xi|^2), \qquad \sigma(x,\xi) = \sigma_{\text{lin}}(x) + \sigma_{\text{nl}}(x)\phi_s(|\xi|^2),$$

with smooth scalar coefficients satisfying $\varepsilon_{\text{lin}}, \mu \geq 2\eta$ for some $\eta \in \mathbb{R}_+$ and $\phi_e(0) = 0$. Then we can find a number $\kappa > 0$, see (2.4), such that $\varepsilon(x,\xi) \geq \eta$ if $|\xi| < \kappa$, where $\kappa = \infty$ if $\varepsilon_{\text{nl}}, \phi_e \geq 0$. This condition yields the strict hyperbolicity of the system. This size restriction has to be imposed on the initial field E_0. A prototypical case for the above constitutive relations is the Kerr law with $\phi_e(s) = s$, see [1], [7].

The paper [24] provides unique solutions $u = (E,H)$ of (1.1) in the space $\mathcal{G}^3 = \bigcap_{j=0}^3 C^j([0,T_0], \mathcal{H}^{3-j}(G))$, depending continously on $u_0 = (E_0, H_0)$ in $\mathcal{H}^3(G)$. The data u_0 have to satisfy certain compatibility conditions, see (2.6), which are necessary for the existence of a solution in $\mathcal{G}^3$. The existence time $T_0 > 0$ can be bounded from below by a positive number depending on the $\mathcal{H}^3$-norm of u_0. Actually, [24] treats anisotropic material laws, which are far more general than in (1.1) and lead to nonlinear state-dependent compatibility conditions. These conditions become linear for our material laws, namely

$$\operatorname{tr}_{\text{ta}} E_0 = 0, \quad \operatorname{tr}_{\text{ta}} \operatorname{curl} H_0 = 0, \quad \operatorname{tr}_{\text{ta}} \operatorname{curl}\left(\tfrac{1}{\mu}\operatorname{curl} E_0\right) = 0. \tag{1.2}$$

This simplification is shown in Lemma 2.1 and heavily exploits the structure of the laws. As in [3], [10] and [19], we restrict to this case in order to focus on the main error estimates here. Below we discuss the possibility for extensions in future work.

The approach of [24] and [25] is based on energy methods adapted to the Maxwell system. The standard energy estimate indicates that one has to control $\partial_t u$ uniformly in $x \in G$ for solving (1.1). This corresponds to the blow-up condition in $W^{1,\infty}$ proved in [24]. In L^2-based integer Sobolev spaces one thus has to work in a regularity level as above, since $\mathcal{H}^2(G) \hookrightarrow C(\overline{G})$. Compared to [24] and [25], the general theory of quasilinear symmetric hyperbolic systems yields less precise results in Sobolev spaces of higher order (treating a much larger class of problems though), see e.g. [9].

The recent works [4], [11] and [12] analyzed (semi-)implicit Euler, implicit Runge–Kutta schema and exponential integrators in the class of quasilinear hyperbolic evolution equations taken from [15] and [22], which involves weighted scalar products that are also used in the present paper. Analogous results were obtained in [16] for the original class introduced by Kato in [13], see also the earlier contribution [2]. In the framework of [15] and [22], but not in that of [13], one can treat the Maxwell system on the full space $G = \mathbb{R}^3$ or with (unphysical) Dirichlet boundary conditions. Moreover, for a quasilinear 1D wave equation with periodic boundary conditions, a trigonometric integrator was studied in [8] and error estimates for the full discretization with a Fourier spectral method were established. Space discretizations for the quasilinear Westervelt equation from nonlinear acoustics were treated in [23], for instance.

However, the settings of [13], [15] and [22] do not cover the Maxwell system with the standard boundary conditions of a perfect conductor, as in (1.1). These conditions are excluded by a condition in Kato's work that provides an ismorphism allowing one to transfer energy estimates from the L^2- to the $\mathcal{H}^3$-level. In [24] and [25] this step is performed in a more PDE-type approach using the structure of the Maxwell system, as explained below.

In a next step, the papers [10], [18] and [19] presented a uniform error analysis for a large class of space discretizations combined with Runge–Kutta methods or the semi- and fully implicit midpoint rules as time discretizations. The analysis is performed within Kato's framework from [15] and [22], but without assuming the existence of the isomorphism mentioned above. Instead, the existence of a solution to the evoluton equation in a space like $\mathcal{G}^3$ is required, which is guaranteed by [24] for the Maxwell system (1.1). The proofs in [10], [18] and [19] rely on a sophisticated iterative argument using the regularity of the solution and inverse estimates for the space discretizations. However, here one needs a restriction of the time step size $\tau > 0$ compared to the space discretization parameter $h > 0$, namely $\tau \leq ch^\beta$ for $\beta > \frac{5}{4}$, which improves on results for the elastic wave equation in [20].

By the same approach, in [19] one obtains a CFL condition with $\beta > \frac{3}{4}$ for the Westervelt equation in 3D from nonlinear acoustics. In the very recent contribution [3] this exponent was improved to $\beta > \frac{1}{4}$, exploiting additional boundedness assumptions on derivatives of the solution to the PDE, see also [5] for related work in the linear non-autonomous case.

As a main novelty, in this paper we use for the first time the methods of [24] and [25] in numerical analysis. Adapting them to the time-discrete situation, we establish a priori estimaates for linearized problems and set up fixed-point arguments based on this estimates. As in [11], we use the implicit Euler scheme

$$\begin{aligned} \Lambda(u_{n+1})(u_{n+1} - u_n) &= \tau M u_{n+1} + \tau Q(u_{n+1}) u_{n+1}, \qquad 0 \leq n \leq N, \\ B u_{n+1} &= 0, \end{aligned} \tag{1.3}$$

setting $\Lambda(u) = \operatorname{diag}(\varepsilon_{\mathrm{d}}(E), \mu)$, $Q(u) = \operatorname{diag}(-\sigma(E), 0)$, and $Bu = (E \times \nu)\restriction_{\partial G}$ for $u = (E, H)$, as well as

$$M = \begin{pmatrix} 0 & \operatorname{curl} \\ -\operatorname{curl} & 0 \end{pmatrix}.$$

Here, $\varepsilon_{\mathrm{d}}(x, E)$ is an (invertible) matrix given in (2.2) such that $\partial_t(\varepsilon(E)E) = \varepsilon_{\mathrm{d}}(E)\partial_t E$. To solve the recursion (1.3), one freezes a sequence (v_n) from a suitable fixed-point space, see (2.19), in the nonlinearities which leads to the linearized Euler scheme

$$\begin{aligned} \Lambda(v_{n+1})(u_{n+1}-u_n) &= \tau M u_{n+1} + \tau Q(v_{n+1})u_{n+1}, \qquad 0 \le n \le N, \\ B u_{n+1} &= 0. \end{aligned} \tag{1.4}$$

This recursion can be solved in the space of $u_n \in \mathcal{H}^3(G)$ satisfying the compatibiltiy conditions (1.2) by means of the resolvents of the (frozen-time) operator $A_{n+1} = \Lambda(v_{n+1})^{-1}(M + Q(v_{n+1}))$ endowed with a suitable domain. The necessary mapping properties of these resolvents follow from the main results of [25] and our Lemma 2.1. This is the core step where we use the special structure of our Kerr-type laws. For more general material laws one only obtains $\mathcal{H}^1$-solutions for (1.4) by means of these frozen-time resolvents, because of the state dependent compatiblity conditions. In this general case, one would then have to show the needed $\mathcal{H}^3$-regularity using the a priori estimates discussed below and adapt regularization arguments from [25] to the time-discrete setting.

We make use of the difference quotients $\mathrm{d}_\tau u_n = \frac{1}{\tau}(u_n - u_{n-1})$. In the main step of our analyis we show in Proposition 4.2 that the $\mathcal{H}^{3-j}$-norms of $\mathrm{d}_\tau^j u_n$ for $j \in \{0,1,2,3\}$ and $n \in \{0, \cdots, N\}$ are bounded by a constant $c(R, T_0)$, where R is larger than the $\mathcal{H}^{3-j}$-norms of $\mathrm{d}_\tau^j v_n$ and τN is smaller than the existence time T_0 of the solution u to (1.1).[1] This estimate is proved in Section 4 in several steps. In Lemma 4.1 we first show a basic energy estimate in L^2, proceeding as in [11] in Kato's framework from [15] and [22]. However, we have to include nontrivial boundary terms $Bu_{n+1} = \varphi_n$ in view of error terms arising later. One next splits the solution u_n in a part with support off the boundary ∂G and one close to it. To the equation for the interior part, one applies third-order tupels ∂^α of spatial derivatives and difference quotients. The differentiated fields $\partial^\alpha u_n$ can then estimated in L^2 by means of the basic energy estimate. Here and below various commutator terms appear which are treated as inhomogeneities in the energy estimate, see (4.6). In this interior case, all boundary terms vanish.

The part near ∂G is estimated by intricate iteration steps. Here we use ideas from [17] which avoid the lengthy localization procedure of [25]. However, in the time-discrete case we have to modify the arguments considerably. First, we differentiate the recursion in tangential directions and apply d_τ^j, leading to commutator terms in (4.10) also at ∂G. One has to be careful when estimating these terms in order to obtain constants that depend on the norms of v_n in (1.4) in a way fitting to the fixed-point argument. By means of the energy estimate we then bound tangential derivatives and difference quotients of u_n in L^2.

The normal derivatives produce error terms at the boundary which cannot be handled in this way. One thus proceeds differently and uses the equation (1.4) and its tangetially differentiated version (4.10) which give rather complicated expressions of the curl and the divergence of u_n. These can then be used to control the normal derivatives iteratively in quite delicate estimates. In the end we put together the various steps in a discrete Gronwall argument.

[1]Throughout, we write $c(\alpha, \dots)$ for a generic constant that depends on positive numbers $\alpha, \dots$ non-decreasingly and is independent of other relevant quantities, in particluar of τ.

The problem (1.3) is then solved in fixed-point argument which is inspired by the arguments in [24]. It is crucial that one can fix a radius R for the fixed-point space which only depends on the $\mathcal{H}^3$-norm r_0 of u_0. This is feasible due to the precise form of the a priori estimate in Proposition 4.2. As in the existence result for the PDE, one has to choose a possibly small existence time $T \le T_0$ which only depends on r_0. Analogously, the time step size τ and the solutions are bounded by numbers depending on r_0, see Theorem 5.1.

In Theorem 5.2 we then show first-order convergence of the implicit Euler scheme (1.3) in L^2 for data (E_0, H_0) in $\mathcal{H}^3$ satisfying the compatibility conditions (1.2) and the hyperbolicity condition $\|E_0\|_{L^\infty} < \kappa$, see (2.4). The proof of this result is similar those in [11], based on the estimates in our Theorem 5.1.

In the next two sections we introduce our setting and some basic tools. Section 4 is devoted to the proof of the higher-order energy estimates. In the last section we solve the scheme and show its convergence.

2. The Maxwell system and the Euler scheme

We assume that the coefficients of the Maxwell equations satisfy

$$\begin{aligned}
&\varepsilon(x,\xi) = \varepsilon_{\mathrm{lin}}(x) + \varepsilon_{\mathrm{nl}}(x)\phi_e(|\xi|^2), \quad \sigma(x,\xi) = \sigma_{\mathrm{lin}}(x) + \sigma_{\mathrm{nl}}(x)\phi_s(|\xi|^2), \qquad (2.1)\\
&\varepsilon_{\mathrm{lin}}, \varepsilon_{\mathrm{nl}}, \mu, \sigma_{\mathrm{lin}}, \sigma_{\mathrm{nl}} \in C^3(\overline{G}, \mathbb{R}), \quad \phi_e, \phi_s \in C^4(\mathbb{R}_{\ge 0}, \mathbb{R}), \quad \phi_e(0) = 0, \quad \varepsilon_{\mathrm{lin}}, \mu \ge 2\eta,
\end{aligned}$$

for $x \in \overline{G}$, $\xi \in \mathbb{R}^3$, and some $\eta \in \mathbb{R}_+$. For $\phi_e(s) = s$ one obtains the well-known *Kerr law* $\varepsilon(E)E = \varepsilon_{\mathrm{lin}}E + \varepsilon_{\mathrm{nl}}|E|^2E$. We define

$$\varepsilon_{\mathrm{d}}(\cdot,\xi) = \varepsilon(\cdot,\xi) + 2\varepsilon_{\mathrm{nl}}\phi_e'(|\xi|^2)\xi\xi^\top, \qquad \sigma_{\mathrm{d}}(\cdot,\xi) = \sigma(\cdot,\xi) + 2\sigma_{\mathrm{nl}}\phi_s'(|\xi|^2)\xi\xi^\top, \quad (2.2)$$

and abbreviate $\varepsilon_1(x,\xi) = 2\varepsilon_{\mathrm{nl}}(x)\phi_e'(|\xi|^2)$ and $\sigma_1(x,\xi) = 2\sigma_{\mathrm{nl}}(x)\phi_s'(|\xi|^2)$. Observe that $\varepsilon_{\mathrm{d}}(x,\xi)$ is symmetric.

Because of $\partial_t(\varepsilon(E)E) = \varepsilon_{\mathrm{d}}(E)\partial_t E$, the Maxwell system (1.1) is equivalent to

$$\begin{aligned}
\varepsilon_{\mathrm{d}}(E)\partial_t E &= \operatorname{curl} H - \sigma(E)E - J, \qquad t \ge 0,\ x \in G,\\
\mu\partial_t H &= -\operatorname{curl} E, \qquad t \ge 0,\ x \in G, \qquad (2.3)\\
E \times \nu &= 0, \qquad t \ge 0,\ x \in \partial G,\\
E(0) &= E_0, \quad H(0) = H_0, \qquad x \in G,
\end{aligned}$$

where we include the current density $J(t,x) \in \mathbb{R}^3$. In our main result we restrict to the case $J = 0$, but in the analysis commutator terms appear that will be treated as inhomogeneities. The differentiated version (2.3) suits better for energy estimates,

To invert ε and ε_{d}, we fix a number $\kappa \in (0,\infty]$ such that

$$|\xi| < \kappa \quad \Longrightarrow \quad \forall x \in \overline{G}: \ \ \varepsilon(x,\xi) \ge \eta, \quad \varepsilon(x,\xi) + \varepsilon_1(x,\xi)|\xi|^2 \ge \eta. \qquad (2.4)$$

If $\varepsilon_{\mathrm{nl}}\phi_e, \varepsilon_1 \ge 0$, one can simply take $\kappa = \infty$. Otherwise we may choose a number $\kappa \in \mathbb{R}_+$ such that

$$\max_{0 \le s \le \kappa} \|\varepsilon_{\mathrm{nl}}\|_\infty(|\phi_e(s^2)| + 2|\phi_e'(s^2)|s^2) \le \eta.$$

For $|\xi| < \kappa$ we have the inverse

$$\varepsilon_{\mathrm{d}}(\cdot,\xi)^{-1} = \frac{1}{\varepsilon(\cdot,\xi)} - \frac{\varepsilon_1(\cdot,\xi)}{\varepsilon(\cdot,\xi)(\varepsilon(\cdot,\xi)+\varepsilon_1(\cdot,\xi)|\xi|^2)}\,\xi\xi^\top =: \frac{1}{\varepsilon(\cdot,\xi)} - a(\cdot,\xi)\xi\xi^\top. \quad (2.5)$$

Concerning the tangential trace in the boundary condition of (2.3), we recall that the linear map $\mathrm{tr}_{\mathrm{ta}} : \varphi \mapsto (\varphi \times \nu)\restriction_{\partial G}$ from $\mathcal{H}(\mathrm{curl}) \cap C(\overline{G})$ to $C(\partial G)$ can be extended to a continuous operator from $\mathcal{H}(\mathrm{curl})$ to $\mathcal{H}^{-1/2}(\partial G)$. Its kernel is the closure $\mathcal{H}_0(\mathrm{curl})$ of the test functions in

$$\mathcal{H}(\mathrm{curl}) := \{\varphi \in L^2(G) \mid \mathrm{curl}\,\varphi \in L^2(G)\}.$$

Here (and often below) we write $L^2(G)$ instead of $L^2(G)^3$ or $L^2(G)^6$ etc., sometimes also omitting the spatial domain, and $\mathcal{H}^s$ denotes the (fractional) Sobolev space on an open subset of $\mathbb{R}^m$ or its (at least Lipschitz) boundary.

Occasionally we use the rotated tangential trace $\mathrm{Tr}_{\mathrm{ta}}\,\varphi = \nu \times \mathrm{tr}_{\mathrm{ta}}\,\varphi$. Moreover, the normal trace $(\nu\cdot\varphi)\restriction_{\partial G}$ can be extended to continuous map $\mathrm{tr}_{\mathrm{no}}$ from $\mathcal{H}(\mathrm{div})$ onto $\mathcal{H}^{-1/2}(\partial G)$, where $\mathcal{H}(\mathrm{div}) := \{\varphi \in L^2(G) \mid \mathrm{div}\,\varphi \in L^2(G)\}$. We note that the full trace is decomposed as $\mathrm{tr}\,\varphi = \mathrm{Tr}_{\mathrm{ta}}\,\varphi + (\mathrm{tr}_{\mathrm{no}}\,\varphi)\nu$.

We want to obtain solutions $u = (E, H) = (u^1, u^2)$ of (2.3) in $\mathcal{G}^3(I)$ for some interval $I \subseteq [0,\infty)$ with $0 \in I$. Here we employ the space

$$\mathcal{G}^k = \mathcal{G}^k(I) = \bigcap\nolimits_{j=0}^{k} C^j(I, \mathcal{H}^{k-j}(G)^6)$$

which is endowed with its canonical norm if the interval I is compact. (Throughout, we write $\xi = (\xi^1, \xi^2) \in \mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$.) To this aim, the data (E_0, H_0) and J must belong to $\mathcal{H}^3(G)$ and $\mathcal{H}^3((0,T) \times G)$, respectively. Moreover, we can differentiate the boundary condition in (2.3) twice in time at $t = 0$ and infer that the *compatibility conditions*

$$\mathrm{tr}_{\mathrm{ta}}\,E_k = 0 \qquad \text{for} \quad \partial_t^k E(0) =: E_k \quad \text{and} \quad k \in \{0,1,2\}, \qquad (2.6)$$

have to hold on ∂G. In general, for $k \in \{1,2\}$ these equations lead to nonlinear conditions on E_0 and H_0, see [24], which would make the following analysis much more difficult.

Under our hypotheses, the conditions (2.6) actually turn out to be linear. To see this fact, by means of (2.3) we first compute

$$\begin{aligned}
\partial_t E(0) &= \varepsilon_{\mathrm{d}}(E_0)^{-1}\big(\mathrm{curl}\,H_0 - \sigma(E_0)E_0 - J_0\big),\\
\partial_t^2 E(0) &= -\varepsilon_{\mathrm{d}}(E_0)^{-1}\big(\mathrm{curl}(\tfrac{1}{\mu}\,\mathrm{curl}\,E_0) + J_1 + \sigma_{\mathrm{d}}(E_0)E_1\big)\\
&\quad - \big[a_1 + a_2 E_0 E_0^\top + a(E_0)(E_0 E_1^\top + E_1 E_0^\top)\big]\big[\mathrm{curl}\,H_0 - J_0 - \sigma(E_0)E_0\big],
\end{aligned} \qquad (2.7)$$

where we set $J_0 = J(0)$ and $J_1 = \partial_t J(0)$. The scalar scalar functions a_k depend on (E_0, E_1) and arise from differentiating $\varepsilon_{\mathrm{d}}(E(t))^{-1}$ in time at $t = 0$. In the analysis, one linearizes (2.3) to

$$\begin{aligned}
\varepsilon_{\mathrm{d}}(v^1)\partial_t E &= \mathrm{curl}\,H - \sigma(v^1)E - J, \qquad t \geq 0,\ x \in G,\\
\mu\partial_t H &= -\,\mathrm{curl}\,E, \qquad t \geq 0,\ x \in G,\\
E \times \nu &= 0, \qquad t \geq 0,\ x \in \partial G,\\
E(0) &= E_0, \quad H(0) = H_0, \qquad x \in G,
\end{aligned} \qquad (2.8)$$

by inserting a function $v \in \mathcal{G}^3$ into the nonlinear terms. The solution of (2.8) is still denoted by $u = (E, H)$. Setting $v_0 = v^1(0)$ and $v_1 = \partial_t v^1(0)$, we obtain

$$\partial_t E(0) = \varepsilon_{\mathrm{d}}(v_0)^{-1}\big(\operatorname{curl} H_0 - \sigma(v_0)E_0 - J_0\big), \tag{2.9}$$
$$\partial_t^2 E(0) = -\varepsilon_{\mathrm{d}}(v_0)^{-1}\big(\operatorname{curl}(\tfrac{1}{\mu}\operatorname{curl} E_0) + J_1 + \sigma(v_0)E_1 + 2\sigma_{\mathrm{nl}}\phi'(|v_0|^2)v_0^\top v_1 E_0\big)$$
$$- \big(a_1(v_0, v_1) + a_2(v_0, v_1)v_0 v_0^\top + a(v_0)(v_0 v_1^\top + v_1 v_0^\top)\big]\big[\operatorname{curl} H_0 - J_0 - \sigma(v_0)E_0\big].$$

We write $\xi \cdot \zeta = \xi^\top \zeta$ for the scalar product in $\mathbb{R}^m$. We can now describe the compatibility conditions both for the nonlinear and the linear case.

Lemma 2.1. *Let* (2.1) *be true and* $v_0, E_0, H_0 \in \mathcal{H}^3(G)$, $v_1, J_0 \in \mathcal{H}^2(G)$ *and* $J_1 \in \mathcal{H}^1(G)$ *satisfy*

$$\operatorname{tr}_{\mathrm{ta}} E_0 = \operatorname{tr}_{\mathrm{ta}} v_0 = \operatorname{tr}_{\mathrm{ta}} J_0 = \operatorname{tr}_{\mathrm{ta}} v_1 = \operatorname{tr}_{\mathrm{ta}} J_1 = 0, \qquad \operatorname{tr}_{\mathrm{ta}} \operatorname{curl} H_0 = 0,$$
$$\operatorname{tr}_{\mathrm{ta}} \operatorname{curl}(\tfrac{1}{\mu}\operatorname{curl} E_0) = 0.$$

We then obtain $\operatorname{tr}_{\mathrm{ta}} E_1 = \operatorname{tr}_{\mathrm{ta}} E_2 = 0$, *where* $E_k = \partial_t^k E(0)$ *are defind by* (2.7) *or* (2.9). *If* $v_0 = v^1(0)$ *and* $v_1 = \partial_t v^1(0)$ *for some* $v \in \mathcal{G}^3([0,T])$ *and analogously for* $J \in \mathcal{H}^3((0,T) \times G)$, *then* $\operatorname{tr}_{\mathrm{ta}} f = 0$ *implies* $\operatorname{tr}_{\mathrm{ta}} f_1 = 0$ *for* $f \in \{v, J\}$.

Proof. The last assertion follows from the continuity of the trace. For the first one, we observe that $(\xi\xi^\top\zeta) \times \nu = (\xi \cdot \zeta)\, \xi \times \nu$ for $\xi, \zeta \in \mathbb{R}^3$. This fact yields

$$(\varepsilon_{\mathrm{d}}(v_0)^{-1}\varphi) \times \nu = \tfrac{1}{\varepsilon}\varphi \times \nu - a(v_0)(v_0 \cdot \varphi)\, v_0 \times \nu \tag{2.10}$$

by (2.5), and hence $\operatorname{tr}_{\mathrm{ta}} E_1 = 0$. Similarly, one shows that $\operatorname{tr}_{\mathrm{ta}} E_2 = 0$. □

In view of the above lemma, Theorem 3.3 in [24] yields a unique, maximally defined solution $u = (E, H) \in \mathcal{G}^3([0, \bar{t}))$ of (2.3) (and (1.1)) provided that (2.1) is true, the initial fields $u_0 = (E_0, H_0) \in \mathcal{H}^3(G)$ satisfy $\|E_0\|_\infty < \kappa$ and

$$\operatorname{tr}_{\mathrm{ta}} E_0 = 0, \quad \operatorname{tr}_{\mathrm{ta}} \operatorname{curl} H_0 = 0, \quad \operatorname{tr}_{\mathrm{ta}} \operatorname{curl}(\tfrac{1}{\mu}\operatorname{curl} E_0) = 0, \tag{2.11}$$

and the current $J \in \mathcal{H}^3((0,b) \times G)$ fulfills $\operatorname{tr}_{\mathrm{ta}} J = 0$, where $b > 0$ is arbitrary. Moreover, the maximal existence time $\bar{t} = \bar{t}(u_0, J) \in (0, \infty]$ is larger than a positive number depending only $\|u_0\|_{\mathcal{H}^3}$, $\|J\|_{\mathcal{H}^3}$ and $\kappa - \|E_0\|_\infty$, it is characterized by a blow-up condition in $W^{1,\infty}(G)$, and solutions depend continuously on the data. See Theorem 3.3 in [24] for precise statements. Lemma 2.1 with $v^1 = E$ shows that the compatibility conditions (2.11) are true for all times, since we have $\operatorname{tr}_{\mathrm{ta}} \partial_t E = \operatorname{tr}_{\mathrm{ta}} \partial_t J = 0$.

We include the compatibility conditions (2.11) in the state spaces, setting $\mathcal{H}^0_{\mathrm{cc}}(G) = L^2(G)$ and

$$\mathcal{H}^1_{\mathrm{cc}}(G) = \{\varphi \in \mathcal{H}^1(G)^6 \mid \operatorname{tr}_{\mathrm{ta}} \varphi^1 = 0\},$$
$$\mathcal{H}^2_{\mathrm{cc}}(G) = \{\varphi \in \mathcal{H}^2(G)^6 \mid \operatorname{tr}_{\mathrm{ta}} \varphi^1 = 0,\ \operatorname{tr}_{\mathrm{ta}} \operatorname{curl} \varphi^2 = 0\}, \tag{2.12}$$
$$\mathcal{H}^3_{\mathrm{cc}}(G) = \{\varphi \in \mathcal{H}^3(G)^6 \mid \operatorname{tr}_{\mathrm{ta}} \varphi^1 = 0,\ \operatorname{tr}_{\mathrm{ta}} \operatorname{curl} \varphi^2 = 0,\ \operatorname{tr}_{\mathrm{ta}} \operatorname{curl}(\tfrac{1}{\mu}\operatorname{curl} \varphi^1) = 0\}.$$

We also need the linearized problem (2.8) for a constant-in-time function $v^1 \in \mathcal{H}^3(G)^3$. Theorem 1.1 of [25] and Lemma 2.1 then show that the solutions of (2.8) generate a C_0-semigroup on $\mathcal{H}^k_{\mathrm{cc}}(G)^6$ for $k \in \{0,1,2,3\}$. The estimates in Theorem 1.1 of [25] also imply that these semigroups have exponential bounds which are unform for v^1 with $\|v^1\|_{\mathcal{H}^3} \le R$.

Since v^1 does not depend on time here, the case $k = 0$ already follows from standard semigroup theory using the generator

$$A = \begin{pmatrix} -\varepsilon_{\mathrm{d}}(v^1)^{-1}\sigma(v^1) & \varepsilon_{\mathrm{d}}(v^1)^{-1}\operatorname{curl} \\ -\mu^{-1}\operatorname{curl} & 0 \end{pmatrix}, \qquad \mathrm{D}(A) = \mathcal{H}_0(\operatorname{curl}) \times \mathcal{H}(\operatorname{curl}).$$

Observe that the off-diagonal part of A is skew-adjoint on $L^2(G)$ endowed with the equivalent scalar product for the weight $(\varepsilon_{\mathrm{d}}(v^1), \mu)$ and that the diagonal part is bounded on this space. By Paragraph II.2.3 of [6], the semigroups on $\mathcal{H}^k_{\mathrm{cc}}(G)$ are generated by the restrictions of A to

$$\mathrm{D}(A, \mathcal{H}^k_{\mathrm{cc}}) = \{\varphi \in \mathcal{H}^k_{\mathrm{cc}}(G) \mid A\varphi \in \mathcal{H}^k_{\mathrm{cc}}(G)\} \tag{2.13}$$

for $k \in \{1, 2, 3\}$, with $\mathrm{D}(A, \mathcal{H}^0_{\mathrm{cc}}) = \mathrm{D}(A)$. Lemma 3.1 yields the equivalence

$$A\varphi \in \mathcal{H}^k(G) \iff \operatorname{curl}\varphi^i \in \mathcal{H}^k(G),\ i \in \{1, 2\}. \tag{2.14}$$

As in Lemma 2.1 one can see that $\varphi \in \mathrm{D}(A, \mathcal{H}^k_{\mathrm{cc}})$ has to satisfy the boundary conditions from $\mathcal{H}^{k+1}_{\mathrm{cc}}(G)$ if $k \le 2$, whereas $\mathrm{D}(A, \mathcal{H}^3_{\mathrm{cc}})$ involves conditions depending on v^1 that are not needed below. Later on we denote the restrictions of A also by A.

We want to approximate the maximal solution $u \in \mathcal{G}^3([0, \bar{t}))$ of (2.3) for initial fields $u_0 = (E_0, H_0) \in \mathcal{H}^3_{\mathrm{cc}}(G)$ with $\|E_0\|_{L^\infty} < \kappa$, see (2.4), and a current $J \in \mathcal{H}^3((0, b) \times G)$ with $\operatorname{tr}_{\mathrm{ta}} J = 0$, where $b > 0$ is arbitrary. We assume that (2.1) holds. For the approximation we use the implicit Euler scheme

$$\begin{aligned} \Lambda(u_{n+1})(u_{n+1} - u_n) &= \tau M u_{n+1} + \tau Q(u_{n+1})u_{n+1} + \tau f_n, \quad 0 \le n \le N, \\ B u_{n+1} &= 0, \end{aligned} \tag{2.15}$$

for $n \in \mathbb{N}_0$ and the time step size $\tau > 0$, where we set

$$\begin{aligned} &\Lambda(u) = \begin{pmatrix} \varepsilon_{\mathrm{d}}(u^1) & 0 \\ 0 & \mu \end{pmatrix}, \quad M = \begin{pmatrix} 0 & \operatorname{curl} \\ -\operatorname{curl} & 0 \end{pmatrix}, \quad Q(u) = \begin{pmatrix} -\sigma(u^1) & 0 \\ 0 & 0 \end{pmatrix}, \\ &B = \begin{pmatrix} \operatorname{tr}_{\mathrm{ta}} & 0 \end{pmatrix}, \quad f_n = \begin{pmatrix} -J(n\tau) \\ 0 \end{pmatrix}. \end{aligned} \tag{2.16}$$

In the following we deal with sequences (w_n), where $n_0 \le n \le N$ or $n_0 \le n < \infty$ for some $N \in \mathbb{N}$ and $n_0 \in \{-3, -2, -1, 0\}$. We fix a time $T_0 < \bar{t}$ and take N with $\tau N \le T \le T_0$, where $T > 0$ is chosen later. Moreover, the difference quotient and the backward shift are given by

$$\mathrm{d}_\tau w_n = \tfrac{1}{\tau}(w_n - w_{n-1}) \quad \text{ and } \quad S w_n = w_{n-1} \quad \text{ for } n > n_0.$$

In our analysis we will have to work with functions such as $\mathrm{d}_\tau^j u_0$. To make this possible, first the given sequence $(f_n)_{n\ge 0}$ is extended to $f_{-k} \in \mathcal{H}^{3-k}_{\mathrm{cc}}(G)$ for $k \in \{1, 2, 3\}$. We then iteratively define

$$\begin{aligned} &u_{-k} = u_{-k+1} - \tau\Lambda(u_{-k+1})^{-1}(M u_{-k+1} + Q(u_{-k+1})u_{-k+1} + f_{-k}) \\ &u_{-3} = u_0 - 3u_{-1} + 3u_{-2} \end{aligned} \tag{2.17}$$

for $k \in \{1, 2\}$. So we extend (2.15) backwards in two steps. In the third step we are not able to guarantuee the invertibility of $\Lambda(u_{-2})$. But u_{-3} is only needed to determine $\mathrm{d}_\tau^3 u_0$, and by our choice we simply set it to 0. We state the basic properties of the extended initial fields in Lemma 3.3, noting already that $u_{-k} \in \mathcal{H}^{3-k}_{\mathrm{cc}}(G)$ and $\|u^1_{-1}\|_\infty < \kappa$ if $0 < \tau \le \tilde\tau_0(r_0)$.

We also record a possible extension of $(f_n)_{n\ge 0}$ to $n \ge -3$ using only f_0, f_1 and $\mathrm{d}_\tau^2 f_2 =: a$, namely

$$f_{-1} \coloneqq 2f_0 - f_1 + \tau^2 \mathrm{d}_\tau^2 f_2, \quad f_{-2} \coloneqq 3f_0 - 2f_1 + 3\tau^2 \mathrm{d}_\tau^2 f_2, \quad f_{-3} \coloneqq 4f_0 - 3f_1 + 6\tau^2 \mathrm{d}_\tau^2 f_2.$$

Beyond the terms detemined by $(f_n)_{n\ge 0}$, we thus obtain the additional (iterated) difference quotients

$$\mathrm{d}_\tau^3 f_2 = \mathrm{d}_\tau^3 f_1 = \mathrm{d}_\tau^3 f_0 = 0, \quad \mathrm{d}_\tau^2 f_1 = \mathrm{d}_\tau^2 f_0 = a, \quad \mathrm{d}_\tau f_0 = -\mathrm{d}_\tau f_2,$$

also given by f_0, f_1 and $\mathrm{d}_\tau^2 f_2$.

Similar as in [11], we solve the recursion (2.15) by a fixed-point argument, freezing fields v^1_{n+1} in the maps Λ and Q. For sequences (w_n) in $\mathcal{H}^k(G)^6$ with $k \in \{0,1,2,3\}$ and $\tau > 0$, we define

$$z_n^{(k)} = z_n^{w,(k)} = \max_{0\le j\le k} \|\mathrm{d}_\tau^j w_n\|^2_{\mathcal{H}^{k-j}}, \qquad z_n = z_n^w = z_n^{w,(3)}, \tag{2.18}$$

for $n \in \mathbb{N}_0$ with $n \le N$. These quantities are used throughout the paper. To compute $\mathrm{d}_\tau^j w_n$ for $0 \le n < j$ we need the vectors w_{-1}, w_{-2} and w_{-3} which are considered to be given. For the solutions u_n and for the fields v_n inserted in Λ and Q, we use the extended initial data u_{-1}, u_{-2} and u_{-3} from (2.17). These extra vectors do not enter in the linearized recursion (2.21). We only use them to estimate the first and second iteration step in the same way as the later ones, thus avoiding case distinctions. Moreover, for the estimates only v_{n+1} with $n \ge 0$ will be relevant, so that $v_{-3} = u_{-3}$ is not used here.

Take $R, T, \tau > 0$ with $T \le T_0$ and N be the largest integer with $\tau N \le T$. In our main results, first R will be fixed according to the norm of u_0 in $\mathcal{H}^3$. The time horizon T and the step size τ then have to be smaller than some numbers depending on R. Let κ be given by (2.4), and fix κ' with $\|E_0\|_{L^\infty} < \kappa' < \kappa$. We introduce the space

$$\begin{aligned} \mathcal{E} = \mathcal{E}(R,T,\tau) = \big\{ (v_n)_{-3\le n\le N} \,\big|\, \forall\, n \ge 0:\ v_n \in \mathcal{H}^3_{\mathrm{cc}}(G),\ z_n^v \le R^2,\ \|v_n^1\|_{L^\infty} \le \kappa', \\ v_{-k} = u_{-k} \in \mathcal{H}^{3-k}_{\mathrm{cc}}(G) \text{ for } k \in \{0,1,2,3\} \big\}, \end{aligned} \tag{2.19}$$

We will also use numbers $r^2 \ge \max_n z_n^{v,(2)}$ in the proof. One can control r by R, T, and $\|v_0\|_{\mathcal{H}^3}$ as we will see in (5.2). But in our estimates it is more convenient to use r separately.

Given $(v_n)_{-3\le n\le N} \in \mathcal{E}$, for $n \ge 0$ we now define

$$\Lambda_n = \Lambda(v_n), \qquad Q_n = Q(v_n), \qquad A_n = \Lambda_n^{-1}(M + Q_n), \tag{2.20}$$

with domain $\mathrm{D}(A_n, \mathcal{H}^k_{\mathrm{cc}}) = \{\varphi \in \mathcal{H}^k_{\mathrm{cc}}(G) \,|\, A_n\varphi \in \mathcal{H}^k_{\mathrm{cc}}(G)\}$, see (2.13). The linearized version of (2.15) then reads as

$$\begin{aligned} \Lambda_{n+1}(u_{n+1} - u_n) &= \tau M u_{n+1} + \tau Q_{n+1} u_{n+1} + \tau f_n, \qquad 0 \le n \le N, \\ B u_{n+1} &= 0, \end{aligned} \tag{2.21}$$

where $u_0 \in \mathcal{H}^3_{\mathrm{cc}}(G)^6$ is given. Let $f_n \in \mathcal{H}^2_{\mathrm{cc}}(G)$ for all $n \in \mathbb{N}_0$ which means that $J_n \in \mathcal{H}^2(G)^3$ and $\mathrm{tr}_{\mathrm{ta}}\, J_n = 0$ Then also $\Lambda_{n+1}^{-1} f_n$ belongs to $\mathcal{H}^2_{\mathrm{cc}}(G)$ by (2.10). In general, such an implication is not true for $\mathcal{H}^3_{\mathrm{cc}}(G)$ unfortunately.

For this reason we let $f_n = 0$ in the main results, but we keep non-zero f_n in our presentation until Section 5, since such terms appear in our analysis as error terms (without causing troubles because of the missing boundary condition).

By the above mentioned results from [25], we can solve (2.21) via

$$u_{n+1} = (I - \tau A_{n+1})^{-1}(u_n + \tau \Lambda_{n+1}^{-1} f_n) \tag{2.22}$$

for $0 \le n \le N-1$, provided that $\tau \le \tau_0(R) < \omega_0(R)^{-1}$ for the common growth bound of the semigroups generated by A_{n+1} for data v_n with $z_n^v \le R^2$. The number $\tau_0(R)$ will be replaced by smaller ones later on. The restriction $\tau \le \tau_0(R)$ is tacitly assumed below. If $f_n = 0$, the fields u_{n+1} and $A_{n+1}u_{n+1}$ belong to $\mathcal{H}^3_{\mathrm{cc}}(G)$ and $\operatorname{curl} u^i_{n+1}$ to $\mathcal{H}^3(G)$ for all n and $i \in \{1,2\}$, see (2.14). Otherwise they are only contained in $\mathcal{H}^2_{\mathrm{cc}}(G)$ and $\mathcal{H}^2(G)$, respectively,

3. Auxiliary results

We first note that $\mathcal{H}^2(G)$ and $\mathcal{H}^3(G)$ are Banach algebras. The following calculus results can be shown as Lemma 2.1 of [25] and Lemma 2.1 of [24].

Lemma 3.1. *a) Let $3 \ge k \ge \max\{j,2\}$ and $j \in \{0,1,2,3\}$. We then have the product estimates*

$$\|\varphi\psi\|_{\mathcal{H}^j} \le c\|\varphi\|_{\mathcal{H}^k}\|\psi\|_{\mathcal{H}^j},$$

where φ and ψ can be scalar-, vector- or matrix-valued.

b) If $\varphi \in \mathcal{H}^k(G)$ is matrix-valued with $\varphi = \varphi^\top \ge \eta > 0$, then $\varphi^{-1} \in \mathcal{H}^k(G)$ with norm bounded by $c(\|\varphi\|_{\mathcal{H}^k})\|\varphi\|_{\mathcal{H}^k}$.

c) Let $a \in C^3(\overline{G} \times \mathbb{R}^m)$, $\varphi \in \mathcal{H}^3(G)^m$ with norm less that R, and $\chi, \psi \in \mathcal{H}^2(G)^m$ with norm less than r. We then have

$$\|a(\varphi)\|_{\mathcal{H}^3} \le c(R)(1 + \|\varphi\|^3_{\mathcal{H}^3}), \qquad \|a(\psi) - a(\chi)\|_{\mathcal{H}^2} \le c(r)\|\psi - \chi\|_{\mathcal{H}^2}.$$

To work with the difference quotient d_τ, we will use the following observations. Let (a_n) and (b_n) be sequences such that a product $a_n b_k$ is defined. We start with discrete product formulas:

$$\begin{aligned}
\mathrm{d}_\tau(a_n b_n) &= \tfrac{1}{\tau}(a_n b_n \mp a_n b_{n-1} - a_{n-1}b_{n-1}) = a_n \mathrm{d}_\tau b_n + \mathrm{d}_\tau a_n b_{n-1} \\
&= a_n \mathrm{d}_\tau b_n + \mathrm{d}_\tau a_n S b_n \ (= \mathrm{d}_\tau a_n b_n + S a_n \mathrm{d}_\tau b_n), \\
\mathrm{d}^2_\tau(a_n b_n) &= a_n \mathrm{d}^2_\tau b_n + 2\mathrm{d}_\tau a_n S \mathrm{d}_\tau b_n + \mathrm{d}^2_\tau a_n S^2 b_n, \\
\mathrm{d}^3_\tau(a_n b_n) &= a_n \mathrm{d}^3_\tau b_n + 3\mathrm{d}_\tau a_n S \mathrm{d}^2_\tau b_n + 3\mathrm{d}^2_\tau a_n S^2 \mathrm{d}_\tau b_n + \mathrm{d}^3_\tau a_n S^3 b_n.
\end{aligned} \tag{3.1}$$

We further need chain rules for d_τ.

Lemma 3.2. *Let (2.1) be true and take $(v_n), (\overline{v}_n) \in \mathcal{E}(R)$ with $z_n^{v,(2)}, z_n^{\overline{v},(2)} \le r^2$. Let $B_n \in \{\Lambda_n, \Lambda_n^{-1}, Q_n\}$ be given by (2.20) and define $\overline{B}_n$ analogously for $\overline{v}_n$. We then have*

$$\|\mathrm{d}^k_\tau B_{n+1}\|_{\mathcal{H}^{3-k}} \le c(r)\sum_{i=1}^k \|S^k \mathrm{d}^i_\tau v_{n+1+i}\|_{\mathcal{H}^{3-i}} \le c(r)R,$$

$$\|\mathrm{d}^j_\tau(B_{n+1} - \overline{B}_{n+1})\|_{\mathcal{H}^{2-j}} \le c(r)\sum_{i=1}^j \|S^j \mathrm{d}^i_\tau(v_{n+1+i} - \overline{v}_{n+1+i})\|_{\mathcal{H}^{2-i}}$$

for $k \in \{1,2,3\}$, $j \in \{1,2\}$, and $n \le N-1$.

Proof. Let $\beta \in \{\varepsilon_{\mathrm{d}}^{-1}, \sigma\}$, $\tau > 0$, and $n \in \mathbb{N}_0$. We ignore that only v_n^1 appears in the nonlinearities. We first compute

$$\tfrac{1}{\tau}(B_{n+1} - B_n) = \int_0^1 \beta'(\delta_{n+1}(s))\tfrac{1}{\tau}(v_{n+1} - v_n)\,\mathrm{d}s, \tag{3.2}$$

where $\delta_{n+1}(s) \coloneqq v_n + s(v_{n+1} - v_n)$. Lemma 3.1 then implies

$$\|\mathrm{d}_\tau B_{n+1}\|_{\mathcal{H}^2} \leq c(r)\|\mathrm{d}_\tau v_{n+1}\|_{\mathcal{H}^2} \leq c(r)R.$$

Setting $\overline{\delta}_n(s,s') = \delta_n(s) + s'(\delta_{n+1}(s) - \delta_n(s))$ and using (3.2), we calculate

$$\begin{aligned}
\mathrm{d}_\tau^2 B_{n+1} &= \tfrac{1}{\tau^2}\big(B_{n+1} - B_n - (B_n - B_{n-1})\big) \\
&= \frac{1}{\tau^2}\int_0^1 \big[\beta'(\delta_{n+1}(s))(v_{n+1} - v_n \mp (v_n - v_{n-1})) - \beta'(\delta_n(s))(v_n - v_{n-1})\big]\,\mathrm{d}s \\
&= \int_0^1 \beta'(\delta_{n+1}(s))\mathrm{d}_\tau^2 v_{n+1}\,\mathrm{d}s + \int_0^1\int_0^1 \beta''(\overline{\delta}_n(s,s'))[\mathrm{d}_\tau v_n, \mathrm{d}_\tau\delta_{n+1}(s)]\,\mathrm{d}s'\,\mathrm{d}s.
\end{aligned}$$

Since $\mathrm{d}_\tau\delta_{n+1}(s) = (1-s)\mathrm{d}_\tau v_n + s\mathrm{d}_\tau v_{n+1}$, from Lemma 3.1 we also infer

$$\|\mathrm{d}_\tau^2 B_{n+1}\|_{\mathcal{H}^1} \leq c(r)(\|\mathrm{d}_\tau^2 v_{n+1}\|_{\mathcal{H}^1} + \|\mathrm{d}_\tau v_n\|_{\mathcal{H}^2}) \leq c(r)R.$$

Finally, set $\hat{\delta}_n(s,s',s'') = \overline{\delta}_{n-1}(s,s') + s''(\overline{\delta}_n(s,s') - \overline{\delta}_{n-1}(s,s'))$. Proceeding as above, we write

$$\begin{aligned}
\mathrm{d}_\tau^3 B_{n+1} &= \tfrac{1}{\tau^3}(B_{n+1} - 2B_n + B_{n-1}) - \tfrac{1}{\tau^3}(B_n - 2B_{n-1} + B_{n-2}) \\
&= \int_0^1 \beta'(\delta_{n+1}(s))\mathrm{d}_\tau^3 v_{n+1}\,\mathrm{d}s + \int_0^1\int_0^1 \beta''(\overline{\delta}_n(s,s'))[\mathrm{d}_\tau^2 v_n, \mathrm{d}_\tau\delta_{n+1}(s)]\,\mathrm{d}s'\,\mathrm{d}s \\
&\quad + \int_0^1\int_0^1 \beta''(\overline{\delta}_n(s,s'))[\tfrac{1}{\tau}(\mathrm{d}_\tau v_n - \mathrm{d}_\tau v_{n-1}), \mathrm{d}_\tau\delta_{n+1}(s)]\,\mathrm{d}s'\,\mathrm{d}s \\
&\quad + \int_0^1\int_0^1 \beta''(\overline{\delta}_n(s,s'))[\mathrm{d}_\tau v_{n-1}, \tfrac{1}{\tau}(\mathrm{d}_\tau\delta_{n+1}(s) - \mathrm{d}_\tau\delta_n(s))]\,\mathrm{d}s'\,\mathrm{d}s \\
&\quad + \int_0^1\int_0^1\int_0^1 \beta'''(\hat{\delta}_n(s,s',s''))[\mathrm{d}_\tau v_{n-1}, \mathrm{d}_\tau\delta_n(s), \mathrm{d}_\tau\overline{\delta}_n(s,s')]\,\mathrm{d}s''\,\mathrm{d}s'\,\mathrm{d}s.
\end{aligned}$$

Observe that $\mathrm{d}_\tau^2\delta_{n+1}(s) = (1-s)\mathrm{d}_\tau^2 v_n + s\mathrm{d}_\tau^2 v_{n+1}$. By Sobolev's embdding, we conclude

$$\|\mathrm{d}_\tau^3 B_{n+1}\|_{\mathcal{H}^2} \leq c(r)\big(\|\mathrm{d}_\tau^3 v_{n+1}\|_{L^2} + \|\mathrm{d}_\tau^2 v_n\|_{\mathcal{H}^1} + \|\mathrm{d}_\tau v_{n-1}\|_{\mathcal{H}^2}\big) \leq c(r)R.$$

The second assertion is shown similarly. □

In our estimates the quantity z_0^u at initial time will appear. By the next lemma we can bound it by the $\mathcal{H}^3$-norm of u_0, which is considered as given.

Lemma 3.3. *Let (2.1) be true, $u_0 \in \mathcal{H}^3_{\mathrm{cc}}(G)$ and $f_{-j} \in \mathcal{H}^{3-j}_{\mathrm{cc}}(G)$ with norms bounded by r_0 and $\|u_0^1\|_\infty < \kappa$, u_{-k} be given by (2.17), and $j \in \{0,1,2,3\}$. Then there is a number $\tilde{\tau}_0(r_0) > 0$ such that for $0 < \tau \leq \tilde{\tau}_0(r_0)$ we have $\|u_{-1}\|_\infty < \kappa'$ and $\mathrm{d}_\tau^j u_k$ belongs to $\mathcal{H}^{3-j}_{\mathrm{cc}}(G)$ with norm bounded by $c(r_0)$.*

Proof. Recall that $\Lambda_{-k} = \Lambda(u_{-k})$ and $Q_{-k} = Q(u_{-k})$ for $k \in \{0,1,2,3\}$. The κ-bound and the claim for $j = 0$ follows from (2.17), Lemma 3.1, and (2.10). Equations (2.17) and (3.1) yield

$$\begin{aligned}
&\mathrm{d}_\tau u_0 = \Lambda_0^{-1}(Mu_0 + Q_0 u_0 + f_{-1}), \qquad \mathrm{d}_\tau u_{-1} = \Lambda_{-1}^{-1}(Mu_{-1} + Q_{-1}u_{-1} + f_{-2}),\\
&\mathrm{d}_\tau^2 u_0 = \mathrm{d}_\tau \Lambda_0^{-1}\big((M + Q_{-1})u_{-1} + f_{-2}\big) + \Lambda_0^{-1}\big((M\mathrm{d}_\tau u_0 + \mathrm{d}_\tau(Q_0 u_0) + \mathrm{d}_\tau f_{-1}\big).
\end{aligned}$$

Lemma 3.1 and (3.2) then imply the asserted estimates in $\mathcal{H}^{3-j}(G)$. Using (2.10), one checks the compatibility conditions for $\mathrm{d}_\tau u_0$ and $\mathrm{d}_\tau u_{-1}$ and then for $\mathrm{d}_\tau^2 u_0$. We have $\mathrm{d}_\tau^3 u_0 = 0$ by the definition in (2.17). $\square$

We want to avoid the sophisticated localization arguments from [24] and [25]. To this aim, we use adapted coordinates as in [17].

For a fixed distance $\varrho > 0$, on the collar $\Gamma_\varrho = \{x \in \overline{G} \,|\, \mathrm{dist}(x, \partial G) < \varrho\}$ we can find C^4-functions $\theta_1, \theta_2, \nu : \Gamma_\varrho \to \mathbb{R}^3$ such that the vectors $\{\theta_1(x), \theta_2(x), \nu(x)\}$ form a basis of $\mathbb{R}^3$ for each point $x \in \Gamma_\varrho$ and ν extends the outer unit normal at ∂G. Hence, θ_1 and θ_2 span the tangential planes at ∂G. For $\xi, \zeta \in \{\theta_1, \theta_2, \nu\}$, $v \in \mathbb{R}^3$ and $a \in \mathbb{R}^{3\times 3}$, we set

$$\partial_\xi = \sum\nolimits_j \xi_j \partial_j, \quad v_\xi = v \cdot \xi, \quad v^\xi = v_\xi \xi, \quad v^\theta = v_{\theta_1}\theta_1 + v_{\theta_2}\theta_2, \quad a_{\xi\zeta} = \xi^\top a \zeta.$$

Later we will apply these operations also to $\mathbb{R}^6$-valued function $v = (v^1, v^2)$ setting, e.g., $v_\xi = (v_\xi^1, v_\xi^2)$. We state calculus formulas needed below, where it is always assumed that the functions involved are sufficiently regular. We can switch between the derivatives of the coefficient v_ξ and the component v^ξ up to a zero-order term since

$$\partial_\zeta v^\xi = \partial_\zeta v_\xi \xi + v_\xi \partial_\zeta \xi.$$

The commutator of tangential derivatives and traces

$$\partial_{\theta_i} \operatorname{tr_{ta}} v = \partial_{\theta_i}(v \times \nu) = \operatorname{tr_{ta}} \partial_{\theta_i} v + v \times \partial_{\theta_i}\nu \qquad \text{on } \ \partial G$$

is also of lower order. Similarly, the directional derivatives commute

$$\partial_\xi \partial_\zeta v = \sum\nolimits_{j,k} \xi_j \partial_j(\zeta_k \partial_k v) = \partial_\zeta \partial_\xi v + \sum\nolimits_{j,k} \xi_j \partial_j \zeta_k \, \partial_k v - \zeta_k \partial_k \xi_j \, \partial_j v$$

up to a first-order operator with bounded coefficients.

The gradient of a scalar function φ is expanded as

$$\nabla\varphi = \sum\nolimits_\xi \xi \cdot \nabla\varphi \, \xi = \sum\nolimits_\xi \xi \partial_\xi \varphi,$$

so that $\partial_j = \sum_\xi \xi_j \partial_\xi$ for $j \in \{1,2,3\}$. Recall that $\mathrm{curl} = \mathrm{J}_1\partial_1 + \mathrm{J}_2\partial_2 + \mathrm{J}_3\partial_3$ for the matrices

$$\mathrm{J}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \mathrm{J}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad \mathrm{J}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We thus obtain

$$\mathrm{curl} = \sum\nolimits_j \mathrm{J}_j \partial_j = \sum\nolimits_{j,\xi} \mathrm{J}_j \xi_j \partial_\xi =: \sum\nolimits_\xi \mathrm{J}(\xi)\partial_\xi.$$

Since the kernel of $\mathrm{J}(\nu)$ is spanned by ν, we can write $\mathrm{J}(\nu)v = \mathrm{J}(\nu)v^\theta$, and the restriction of $\mathrm{J}(\nu)$ to $\mathrm{span}\{\theta_1, \theta_2\}$ has an inverse $\mathrm{R}(\nu)$.

We now provide the tools needed for the apriori estimates in the next section. We first isolate the normal derivative of the tangential components of v in the equation $\operatorname{curl} v = f$. From the expansion

$$\operatorname{curl} v = \mathrm{J}(\nu)(\partial_\nu v)^\theta + \mathrm{J}(\theta_1)\partial_{\theta_1} v + \mathrm{J}(\theta_2)\partial_{\theta_2} v,$$

we derive

$$\partial_\nu v^\theta = \sum\nolimits_i (\partial_\nu \theta_i \, v_{\theta_i} + \theta_i \partial_\nu \theta_i \cdot v) + \mathrm{R}(\nu)\Big(f - \sum\nolimits_i \mathrm{J}(\theta_i)\partial_{\theta_i} v\Big) \tag{3.3}$$

where the first sum only contains zero-order terms.

In order to recover the normal derivative of the normal component of v, we resort to the divergence operator. The divergence of a vector field v can be expressed as

$$\operatorname{div} v = \sum\nolimits_j \partial_j \sum\nolimits_\xi v_\xi \xi_j = \sum\nolimits_\xi \big(\partial_\xi v_\xi + \operatorname{div}(\xi) v_\xi\big).$$

Letting $\varphi = \operatorname{div}(av)$ for a matrix-valued function a, we derive

$$\begin{aligned}
\operatorname{div}(av) &= \sum\nolimits_{\xi,\zeta} \partial_\xi (\xi^\top a \zeta v_\zeta) + \sum\nolimits_\xi \operatorname{div}(\xi)\, \xi^\top a v \\
&= \sum\nolimits_{\xi,\zeta} (a_{\xi\zeta} \partial_\xi v_\zeta + \partial_\xi a_{\xi\zeta} v_\zeta) + \sum\nolimits_\xi \operatorname{div}(\xi)\, \xi^\top a v, \\
a_{\nu\nu} \partial_\nu v_\nu &= \varphi - \sum_{(\xi,\zeta)\neq(\nu,\nu)} a_{\xi\zeta} \partial_\xi v_\zeta - \sum\nolimits_{\xi,\zeta} \partial_\xi a_{\xi\zeta} v_\zeta - \sum\nolimits_\xi \operatorname{div}(\xi)\, \xi^\top a v. \qquad (3.4)
\end{aligned}$$

where the sums in the last line contain the tangential derivatives and the normal derivatives of tangential components of v plus zero-order terms.

4. The core a priori estimates

In this section we estimate fields $u_n \in \mathcal{H}^3_{\mathrm{cc}}(G)$, see (2.12), solving the linearized implicit Euler scheme (2.21) for given (v_n) in $\mathcal{E}(R, T, \tau)$ and data $u_0 \in \mathcal{H}^3_{\mathrm{cc}}(G)$ and $f_{-k} \in \mathcal{H}^{3-k}_{\mathrm{cc}}(G)$ with norms bounded by r_0, $\|E_0\|_{L^\infty} < \kappa$, u_{-k} be given by (2.17), and $f_n \in \mathcal{H}^2_{\mathrm{cc}}(G) \cap \mathcal{H}^3(G)$, where $k \in \{1, 2, 3\}$ and $n \geq 0$. Here we let $R > 0$, $0 < T \leq T_0$, and $0 < \tau \leq \max\{\tau_0(R), \tilde{\tau}_0(r_0)\}$, see (2.19), Lemma 3.3, and the comments after (2.16) and (2.22). We further fix a number $r^2 \geq \max_{n \geq -1} z_n^{v,(2)}$. Let N be the largest integer with $N\tau \leq T$. Recall that in the homogeneous case $f_n = 0$ such u_n exist and are given by (2.22).

We proceed in several steps to control the quantity $z_n = z_n^u$ from (2.18). First, we establish the basic energy inequality on the L^2-level also allowing for inhomogeneities at the boundary, as it is needed to deal with error terms later on. For higher-order estimates we use differentiated versions of (2.21). In a second step, we treat the part of u_n localized off the boundary by means of the energy estimate. This step is easier since these functions vanish near the boundary. Third, the part of u_n near the boundary is handled by a intricate recursive argument using the adapted derivatives from the previous section. Tangential derivatives and difference quotients in time can be bounded via the energy estimate and a careful analysis of the error terms. The normal erivatives are recovered from the (differentiated) equation and formulas (3.3) and (3.4).

4.1. **The basic energy inequality.** We consider the linear problem

$$\begin{aligned}\Lambda_{n+1}(u_{n+1}-u_n) &= \tau M u_{n+1} + \tau Q_{n+1} u_{n+1} + \tau f_n, \qquad n \in \mathbb{N}_0,\\ B u_{n+1} &= \varphi_n,\end{aligned} \tag{4.1}$$

where M and B are given by (2.16). Compared to (2.21), we allow for non-zero boundary data and weaken the assumptions on the cofficients and data, namely

$$\begin{aligned} u_0 \in \mathcal{H}(\mathrm{curl})^2, \qquad f_n \in L^2(G)^6, \qquad \varphi_n \in \mathcal{H}^{\frac{1}{2}}_{\mathrm{ta}}(\partial G),\\ \Lambda_n, Q_n \in L^\infty(G, \mathbb{R}^{6\times 6}), \qquad \Lambda_n = \Lambda_n^\top \geq \eta,\end{aligned} \tag{4.2}$$

for $n \in \mathbb{N}_0$ and some $\eta \in \mathbb{R}_+$, where we set

$$\mathcal{H}^s_{\mathrm{ta}}(\partial G) = \{\varphi \in \mathcal{H}^s(\partial G)^3 \,|\, \varphi \cdot \nu = 0\}, \qquad s \geq 0.$$

We employ weights in the spatial variables to use the formal symmetry of M. In the context of quasilinear problems, such weights have been used in continuous time at least since [14], see also [11] in the time-discrete case. Moreover, we introduce decaying weights in (discrete) time for notational convenience. (The latter play a smaller role than in continuous time, compare e.g. [25].) For $n \in \mathbb{N}_0$, $\gamma = \mathrm{e}^{c\tau} \geq 1$ and $v \in L^2(G)$, let

$$\|v\|^2_{n,\gamma} = (\gamma^{-n}\Lambda_n v | \gamma^{-n} v) = \int_G \gamma^{-2n} \Lambda_n v \cdot v \,\mathrm{d}x, \qquad \|v\|_n \coloneqq \|v\|_{n,1}.$$

For $n \leq N$ and γ in compact intervals these norms are uniformly equivalent to the usual L^2-norm. We also define

$$z^{v,(k)}_{n,\gamma} = z^{(k)}_{n,\gamma} = \max_{0\leq j\leq k} \gamma^{-2n} \|\mathrm{d}^j_\tau v_n\|^2_{\mathcal{H}^{k-j}}, \qquad z^v_{n,\gamma} = z^{v,(3)}_{n,\gamma},$$

cf. (2.18). In the following, $z^{(k)}_{n,\gamma}$ and $z^{(k)}_n$ refer to the solutions u_n.

We state our basic energy estimate.

Lemma 4.1. *Assume that (4.2) holds and that $\|Q_n\|_\infty, \|\Lambda_n\|_\infty \leq r'$ and $\|\mathrm{d}_\tau \Lambda_{n+1}\|_\infty \leq R'$ for all $n \in \mathbb{N}_0$ and some $R', r' \geq 0$. Let $u_n \in \mathcal{H}(\mathrm{curl})^2$ solve (4.1). Take $0 < \tau \leq \tau_0'(r') \coloneqq \frac{1}{2}\eta(2r'+1)^{-1}$. Set $C = \eta^{-1}(R' + 4r' + 2)$ and $\gamma = \mathrm{e}^{\tau C/2}$. We then obtain*

$$\|u_{n+1}\|^2_{n+1} \leq \mathrm{e}^{C(n+1)\tau} \|u_0\|^2_0 + \tau \sum_{k=0}^n e^{C(n+1-k)\tau} (\|f_k\|^2_{L^2} + 2\langle \varphi_k, \mathrm{Tr}_{\mathrm{ta}}\, u^2_{k+1}\rangle),$$

$$\|u_{n+1}\|^2_{n+1,\gamma} \leq \|u_0\|^2_0 + \tau \sum_{k=0}^n \big(\|f_k\|^2_{L^2,\gamma} + c\gamma \|\varphi_k\|_{\mathcal{H}^{\frac{1}{2}},\gamma} \|\mathrm{Tr}_{\mathrm{ta}}\, u^2_{k+1}\|_{\mathcal{H}^{-\frac{1}{2}},\gamma}\big),$$

where the brackets denote the duality $\mathcal{H}^{\frac{1}{2}}_{\mathrm{ta}}(\partial G) \times \mathcal{H}^{-\frac{1}{2}}_{\mathrm{ta}}(\partial G)$.

Proof. Multiplying (4.1) by u_{n+1} and integrating in x, we compute

$$\begin{aligned}\|u_{n+1}\|^2_{n+1} &= (\Lambda_{n+1} u_{n+1} | u_{n+1})\\ &= (\Lambda^{\frac{1}{2}}_{n+1} u_n | \Lambda^{\frac{1}{2}}_{n+1} u_{n+1}) + \tau(M u_{n+1}|u_{n+1}) + \tau(Q_{n+1}u_{n+1}|u_{n+1}) + \tau(f_n|u_{n+1})\\ &\leq \tfrac{1}{2}((\Lambda_{n+1} \mp \Lambda_n) u_n | u_n) + \tfrac{1}{2}(\Lambda_{n+1} u_{n+1}|u_{n+1}) + \tau\langle \varphi_n, \mathrm{Tr}_{\mathrm{ta}}\, u^2_{n+1}\rangle\\ &\qquad + \tau(Q_{n+1}u_{n+1}|u_{n+1}) + \tau(f_n|u_{n+1}),\end{aligned}$$

by means of the Cauchy–Schwarz inequality and the integration by parts formula for curl. We absorb the second summand on the right-hand side and employ the bounds on the coefficients, obtaining

$$\|u_{n+1}\|_{n+1}^2 \le \frac{\tau R'}{\eta}\|u_n\|_n^2 + \|u_n\|_n^2 + \frac{2\tau r'}{\eta}\|u_{n+1}\|_{n+1}^2 + \frac{\tau}{\eta}\|u_{n+1}\|_{n+1}^2 \\ + \tau\|f_n\|_{L^2}^2 + 2\tau\langle\varphi_n, \operatorname{Tr_{ta}} u_{n+1}^2\rangle.$$

We now choose $0 < \tau \le \tau_0'(r') := \frac{1}{2}\eta(2r'+1)^{-1}$ to absorb the terms with u_{n+1} by the left-hand side. Using $(1-s)^{-1} \le \mathrm{e}^{2s}$ for $0 < s \le 1/2$, it follows

$$\|u_{n+1}\|_{n+1}^2 \le \mathrm{e}^{\eta^{-1}(4r'+2)\tau}\Big(\mathrm{e}^{\tau R'/\eta}\|u_n\|_n^2 + \tau\|f_n\|_{L^2}^2 + 2\tau\langle\varphi_n, \operatorname{Tr_{ta}} u_{n+1}^2\rangle\Big). \tag{4.3}$$

Let $C = \eta^{-1}(R' + 4r' + 2)$. An iteration of (4.3) yields

$$\|u_{n+1}\|_{n+1}^2 \le \mathrm{e}^{C(n+1)\tau}\|u_0\|_0^2 + \tau\sum_{k=0}^{n} e^{C(n+1-k)\tau}(\|f_k\|_{L^2}^2 + 2\langle\varphi_k, \operatorname{Tr_{ta}} u_{k+1}^2\rangle),$$

which is the first asertion. The second one follows immediately. □

We replace $\tau_0(R)$ and $\tilde\tau_0(r_0)$ by

$$\tau_1(R) := \min\{\tau_0(R), \tilde\tau_0(r_0), \tau_0'(r'), 1\}, \tag{4.4}$$

see the comments after (2.22) and Lemma 3.3. Note that $1 \le \gamma \le c(R)$ which will be used below often without further notice.

4.2. **The interior estimate.** Let again (2.1) be true and $u_n \in \mathcal{H}^3_{\mathrm{cc}}(G)$ solve (2.21) for $(v_n) \in \mathcal{E}(R, T, \tau)$ as at the beginning of this section. We fix a function $\chi \in C^4(\overline{G})$ with support in Γ_ϱ such that $0 \le \chi \le 1$ and $\chi = 1$ on $\Gamma_{\varrho/2}$. We set $\tilde\chi = 1 - \chi \in C_c^4(G)$. For any scalar map $\vartheta \in C^4(\overline{G}, \mathbb{R})$, we obtain the discrete linearized problem

$$\Lambda_{n+1}(\vartheta u_{n+1} - \vartheta u_n) = \tau M(\vartheta u_{n+1}) + \tau Q_{n+1}(\vartheta u_{n+1}) + \tau\begin{bmatrix} -\nabla\vartheta\times u_{n+1}^2 \\ \nabla\vartheta\times u_{n+1}^1\end{bmatrix} + \tau\vartheta f_n, \\ B(\vartheta u_{n+1}) = \vartheta\varphi_n, \qquad 0 \le n \le N, \tag{4.5}$$

for solutions u_n of (4.1). The coefficients are again given by (2.20) for a sequence $(v_n)_{0\le n\le N}$ from $\mathcal{E} = \mathcal{E}(R, T, \tau)$ defined in (2.19). The solutions are computed in (2.22) if $\varphi_n = 0$ and τ is less or equal $\tau_1(R)$ from (4.4).

We now use (4.5) for $\tilde u_n := \tilde\chi u_n$, setting also $\tilde f_n = \tilde\chi f_n$ and

$$\tilde z_n = \max_{0\le j\le 3}\|\mathrm{d}_\tau^j \tilde u_n\|_{\mathcal{H}^{3-j}}^2, \qquad \tilde z_{n,\gamma} = \max_{0\le j\le 3}\gamma^{-2n}\|\mathrm{d}_\tau^j \tilde u_n\|_{\mathcal{H}^{3-j}}^2$$

for $\gamma = \gamma(R) \ge 1$ from Lemma 4.1. Let $\alpha \in \mathbb{N}_0^4$ be a multi-index with $0 < l := |\alpha| \le 3$ whose component α_0 refers to the difference quotient d_τ and the others to the spatial derivatives ∂_j. We calculate

$$\Lambda_{n+1}(\partial^\alpha\tilde u_{n+1} - \partial^\alpha\tilde u_n) = \tau M\partial^\alpha\tilde u_{n+1} + \tau Q_{n+1}\partial^\alpha\tilde u_{n+1} + \tau\partial^\alpha\tilde f_n + \tau\tilde f_{n,\alpha}, \tag{4.6} \\ B\tilde u_{n+1} = 0, \qquad 0 \le n \le N,$$

where we set

$$\tilde f_{n,\alpha} = \sum_{0\le\beta\le\alpha,\beta_0=0} \binom{\alpha}{\beta} \begin{pmatrix} -\partial^\beta\nabla\tilde\chi\times\partial^{\alpha-\beta}u^2_{n+1} \\ \partial^\beta\nabla\tilde\chi\times\partial^{\alpha-\beta}u^1_{n+1}\end{pmatrix}$$
$$\qquad - \sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\big(\partial^\beta\Lambda_{n+1}\,\partial^{\alpha-\beta}S^{\beta_0}\mathrm{d}_\tau\tilde u_{n+1} - \partial^\beta Q_{n+1}\,\partial^{\alpha-\beta}S^{\beta_0}\tilde u_{n+1}\big).$$

Here we have used formulas (3.1) and recall that $v_{-k}=u_{-k}$ are given by (2.17) for $k\in\{1,2,3\}$. Since $(v_n)\in\mathcal{E}$, Lemmas 3.1 and 3.2 yield

$$\|\tilde f_{n,\alpha}\|^2_{L^2} \le c z^{(l)}_{n+1} + c(R)\sum_{j=0}^{3}\tilde z^{(l)}_{n+1-j}. \tag{4.7}$$

We sum over α and use Lemmas 3.3 and 4.1, obtaining

$$\tilde z^{(l)}_{n+1,\gamma} \le c(r_0)z_0 + \tau\sum_{k=0}^{n}\big(c(R)\tilde z^{(l)}_{k+1,\gamma} + c(R)z^{(l)}_{k+1,\gamma} + cz^{f,(l)}_{k,\gamma}\big). \tag{4.8}$$

The above estimate will be used later on, but we also note an immediate consequence. For $\tau\le C(R)/2$, Gronwall's inequality implies

$$\tilde z^{(l)}_{n+1,\gamma} \le \mathrm{e}^{2n\tau}\Big(c(r_0)z_0 + \tau\sum_{k=0}^{n}\big(c(R)z^{(l)}_{k+1,\gamma} + cz^{f,(l)}_{k,\gamma}\big)\Big). \tag{4.9}$$

4.3. **The tangential estimate near the boundary.** We turn to the functions $\hat u_n := \chi u_n$ and $\hat f_n := \chi f_n$ supported in Γ_ϱ. The multi-index $\alpha\in\mathbb{N}_0^4$ with $l=|\alpha|\le 3$ now refers to d_τ, ∂_{θ_1}, ∂_{θ_2} and ∂_ν from Section 3. Here, we write d^α instead of ∂^α and $\mathrm{d}^\alpha_{\mathrm{ta}}$ if $\alpha_3=0$. Let $\alpha'=(\alpha_1,\alpha_2,\alpha_3)$. As in (4.6) we obtain

$$\Lambda_{n+1}(\mathrm{d}^\alpha\hat u_{n+1} - \mathrm{d}^\alpha\hat u_n) = \tau M\mathrm{d}^\alpha\hat u_{n+1} + \tau Q_{n+1}\mathrm{d}^\alpha\hat u_{n+1} + \tau\mathrm{d}^\alpha\hat f_n + \tau\hat f_{n,\alpha}, \tag{4.10}$$
$$B\mathrm{d}^\alpha\tilde u_{n+1} = [\mathrm{d}^\alpha,\mathrm{tr}_{\mathrm{ta}}]\hat u^1_{n+1} =: \varphi_{n,\alpha}, \qquad 0\le n\le N,$$

where we set

$$\hat f_{n,\alpha} = [\mathrm{d}^\alpha, M]\hat u_{n+1} + \sum_{0\le\beta\le\alpha,\beta_0=0}\binom{\alpha}{\beta}\begin{pmatrix} -\mathrm{d}^\beta\nabla\chi\times\mathrm{d}^{\alpha-\beta}u^2_{n+1} \\ \mathrm{d}^\beta\nabla\chi\times\mathrm{d}^{\alpha-\beta}u^1_{n+1}\end{pmatrix} \tag{4.11}$$
$$\qquad - \sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\big(\mathrm{d}^\beta\Lambda_{n+1}\,\mathrm{d}^{\alpha-\beta}S^{\beta_0}\mathrm{d}_\tau\hat u_{n+1} - \mathrm{d}^\beta Q_{n+1}\,\mathrm{d}^{\alpha-\beta}S^{\beta_0}\hat u_{n+1}\big).$$

The two commutators $[\cdot,\cdot]$ have order $|\alpha'|-1$ at the boundary and $|\alpha'|$ in the domain, respectively, and they are 0 if $\alpha'=0$. As in (4.7), we thus deduce

$$\|\hat f_{n,\alpha}\|^2_{L^2} \le cz^{(l)}_{n+1} + \delta_\alpha c(R)\sum_{j=0}^{l}\hat z^{(l)}_{n+1-j}, \tag{4.12}$$
$$\|\varphi_{n,\alpha}\|^2_{\mathcal{H}^{\frac12}(\partial G)} \le c\delta_{\alpha'}\|\mathrm{tr}\,\hat u_{n+1}\|^2_{\mathcal{H}^{|\alpha'|-\frac12}(\partial G)} \le c\delta_{\alpha'}\|\hat u_{n+1}\|^2_{\mathcal{H}^{|\alpha'|}(G)} \le c\delta_{\alpha'}\hat z^{(l)}_{n+1},$$

where we use also the standard trace estimate and set $\delta_a = 0$ if $a = 0$ and $\delta_a = 1$ otherwise. Lemmas 3.3 and 4.1 then imply

$$\sum_{\substack{|\alpha|\le l\\ \alpha_3=0}} \|\mathrm{d}^\alpha \hat{u}_{n+1}\|_{n+1,\gamma}^2 \le c(r_0)\hat{z}_0 + \tau \sum_{k=0}^{n} \big[c(R)\hat{z}_{k+1,\gamma}^{(l)} + c(R) z_{k+1,\gamma}^{(l)} + c z_{k,\gamma}^{f,(l)}\big]. \quad (4.13)$$

4.4. **Normal derivatives near the boundary.** We now let $0 \le l = |\alpha| \le 2$. Before we can tackle the estimates for the normal derivatives, we collect a few formulas describing curl and divergence of $\hat{u}_{n+1}$.

1) First, equation (4.10) yields

$$M\mathrm{d}^\alpha \hat{u}_{n+1} = \Lambda_{n+1}\mathrm{d}^\alpha \mathrm{d}_\tau \hat{u}_{n+1} - Q_{n+1}\mathrm{d}^\alpha \hat{u}_{n+1} - \mathrm{d}^\alpha \hat{f}_n - \hat{f}_{n,\alpha}. \quad (4.14)$$

From formula (3.3) we then deduce

$$\begin{aligned}\partial_\nu(\mathrm{d}^\alpha \hat{u}_{n+1})^\theta &= \mathcal{R}(\nu)\big[\Lambda_{n+1}\mathrm{d}^\alpha \mathrm{d}_\tau \hat{u}_{n+1} - Q_{n+1}\mathrm{d}^\alpha \hat{u}_{n+1} - \mathrm{d}^\alpha \hat{f}_n - \hat{f}_{n,\alpha}\big] \quad (4.15)\\ &\quad -\sum\nolimits_i \big[\mathcal{R}(\nu)\mathcal{J}(\theta_i)\partial_{\theta_i}\mathrm{d}^\alpha \hat{u}_{n+1} - \partial_\nu\theta_i(\mathrm{d}^\alpha \hat{u}_{n+1})_{\theta_i} - \theta_i\partial_\nu\theta_i\cdot \mathrm{d}^\alpha \hat{u}_{n+1}\big],\end{aligned}$$

where we set

$$\mathcal{R}(\nu) = \begin{bmatrix}\mathrm{R}(\nu) & 0\\ 0 & -\mathrm{R}(\nu)\end{bmatrix}, \quad \mathcal{J}(\theta_i) = \begin{bmatrix}\mathrm{J}(\theta_i) & 0\\ 0 & -\mathrm{J}(\theta_i)\end{bmatrix}, \quad \mathrm{Div} = \begin{bmatrix}\mathrm{div} & 0\\ 0 & \mathrm{div}\end{bmatrix}.$$

We stress that there is no factor τ on the right so that one cannot simply use Gronwall's inequality. In fact, most of the summands on the right-hand side of (4.15) will be treated by means of previous steps in an iterative argument. Here terms proportional to $\hat{z}_0$ appear. In the fixed-point argument of Theorem 5.1 it will be crucial that we do not have contributions of the form $c(R)\hat{z}_0$. In addition, when estimating the error terms $\hat{f}_{n,\alpha}$ from (4.11), summands containing $\hat{z}_{n+1}^{(l)}$ appear if one wants to avoid a prefactor $c(R)$. To simplify the iteration, we refine the estimate so that we can absorb these contributions of highest order.

Let $\delta > 0$. For $\beta = \alpha$ and $l = |\alpha| = 1$, Hölder's und Sobolev's inequality and interpolation yield

$$\begin{aligned}\|\mathrm{d}^\beta \Lambda_k\, S^{\beta_0}\mathrm{d}_\tau \hat{u}_k\|_{L^2} &\le c(r)\|\mathrm{d}v_k\, S^{\beta_0}\mathrm{d}_\tau \hat{u}_k\|_{L^2} \le c(r)\|\mathrm{d}v_k\|_{L^6}\,\|S^{\beta_0}\mathrm{d}_\tau \hat{u}_k\|_{L^3}\\ &\le c(r)\|S^{\beta_0}\mathrm{d}_\tau \hat{u}_k\|_{\mathcal{H}^{\frac12}} \le \delta S^{\beta_0}\hat{z}_k^{(l+1)} + c(\delta,r)S^{\beta_0}\hat{z}_k^{(l)}. \quad (4.16)\end{aligned}$$

Let $l = |\alpha| = 2$. For $|\beta| = 1$ we obtain a term of the form $\Lambda_k' \mathrm{d}v_k S^{\beta_0}\mathrm{d}\mathrm{d}_\tau \hat{u}_k$, and for $\beta = \alpha$ one has

$$\Lambda_k' \mathrm{d}^2 v_k S^{\beta_0}\mathrm{d}_\tau \hat{u}_k + \Lambda_k'' \mathrm{d}v_k \mathrm{d}v_k S^{\beta_0}\mathrm{d}_\tau \hat{u}_k,$$

where $\Lambda_k' = (\partial_v\Lambda)(v_k)$, $\Lambda_k'' = (\partial_v^2\Lambda)(v_k)$, and we use a somewhat informal notation. In both cases the squared L^2-norm can be bounded by $\delta S^{\beta_0}\hat{z}_k^{(l+1)} + c(\delta,r)S^{\beta_0}\hat{z}_k^{(l)}$ as above. We thus deduce

$$\begin{aligned}\|\partial_\nu(\mathrm{d}^\alpha \hat{u}_{n+1})^\theta\|_{L^2}^2 &\le c(r)\big(\|\mathrm{d}_{\mathrm{ta}}\mathrm{d}^\alpha \hat{u}_{n+1}\|_{L^2}^2 + \|\mathrm{d}^\alpha \hat{u}_{n+1}\|_{L^2}^2 + \|\mathrm{d}^\alpha \hat{f}_n\|_{L^2}^2\big)\\ &\quad + \delta_\alpha \sum_{j=0}^{l}(\delta \hat{z}_{n+1-j}^{(l+1)} + c(\delta,r)\hat{z}_{n+1-j}^{(l)}) + c z_{n+1}^{(l)}. \quad (4.17)\end{aligned}$$

2) For the normal component of the normal derivative, we use the divergence of the fields. From (4.10) we infer

$$\begin{aligned}
&\operatorname{Div}(\Lambda_{n+1}\mathrm{d}^\alpha\hat{u}_{n+1}) \qquad\qquad (4.18)\\
&= \operatorname{Div}(\Lambda_n\mathrm{d}^\alpha\hat{u}_n) + \tau\operatorname{Div}\big(\mathrm{d}_\tau\Lambda_{n+1}\mathrm{d}^\alpha\hat{u}_n + Q_{n+1}\mathrm{d}^\alpha\hat{u}_{n+1}\big) + \tau\operatorname{Div}(\mathrm{d}^\alpha\hat{f}_n + \hat{f}_{n,\alpha})\\
&= \operatorname{Div}(\Lambda_0\mathrm{d}^\alpha\hat{u}_0) + \tau\sum_{k=0}^{n}\big[\operatorname{Div}(\mathrm{d}_\tau\Lambda_{k+1}\mathrm{d}^\alpha\hat{u}_k + Q_{k+1}\mathrm{d}^\alpha\hat{u}_{k+1}) + \operatorname{Div}(\mathrm{d}^\alpha\hat{f}_k + \hat{f}_{k,\alpha})\big].
\end{aligned}$$

Set $\lambda_n = \operatorname{diag}(\varepsilon_{\mathrm{d}}(v_n^1)_{\nu\nu}, \mu) \geq \eta$. Equations (4.18) and (3.4) then yield

$$\begin{aligned}
&\partial_\nu(\mathrm{d}^\alpha\hat{u}_{n+1})_\nu \qquad\qquad (4.19)\\
&= \lambda_{n+1}^{-1}\Big[\operatorname{Div}(\Lambda_0\mathrm{d}^\alpha\hat{u}_0) + \tau\sum_{k=0}^{n}\big[\operatorname{Div}(\mathrm{d}_\tau\Lambda_{k+1}\mathrm{d}^\alpha\hat{u}_k + Q_{k+1}\mathrm{d}^\alpha\hat{u}_{k+1}) + \operatorname{Div}\mathrm{d}^\alpha\hat{f}_k\\
&\quad + \operatorname{Div}\hat{f}_{k,\alpha}\big] - \sum_{(\xi,\zeta)\neq(\nu,\nu)}(\Lambda_{n+1})_{\xi\zeta}\partial_\xi(\mathrm{d}^\alpha\hat{u}_{n+1})_\zeta - \sum_{\xi,\zeta}\partial_\xi(\Lambda_{n+1})_{\xi\zeta}(\mathrm{d}^\alpha\hat{u}_{n+1})_\zeta\\
&\quad - \sum_{\xi,i}\operatorname{div}(\xi)\xi^\top(\Lambda_{n+1}\mathrm{d}^\alpha\hat{u}_{n+1})^i\Big].
\end{aligned}$$

The three last terms will be treated by previous steps in the iteration argument. The first of these summands contains tangential derivatives and the tangential component of the normal derivative. The penultimate term is of lower order, but one has to be careful not to produce a pre-factor $c(R)$ in the calculations. As in (4.16), we thus compute

$$\|\partial_\xi(\Lambda_{n+1})_{\xi\zeta}(\mathrm{d}^\alpha\hat{u}_{n+1})_\zeta\|_{L^2} \leq \delta\hat{z}_{n+1}^{(l+1)} + c(\delta,r)\hat{z}_{n+1}^{(l)}.$$

We pass to squares when estimating (4.19). Concerning the sum, we note that

$$\Big(\tau\sum_{k=0}^{n}a_k\Big)^2 \leq n\tau^2\sum_{k=0}^{n}a_k^2 \leq \tau T\sum_{k=0}^{n}a_k^2$$

by Hölder's inequality and $n\tau \leq N\tau \leq T$. Combined with (4.12) and Lemmas 3.1 and 3.3, formulas (4.19) and (4.17) then lead to

$$\begin{aligned}
\|\partial_\nu\mathrm{d}^\alpha\hat{u}_{n+1}\|_{L^2}^2 &\leq c(r_0)\hat{z}_0^{(l+1)} + \tau T\sum_{k=0}^{n}\Big(c(R)(\hat{z}_k^{(l+1)} + \hat{z}_{k+1}^{(l)}) + \delta_\alpha c(R)\sum_{j=0}^{l}\hat{z}_{k+1-j}^{(l+1)}\\
&\quad + c(\hat{z}_k^{f,(l+1)} + z_{k+1}^{(l+1)})\Big) + c(r)\big(\|\mathrm{d}_{\mathrm{ta}}\mathrm{d}^\alpha\hat{u}_{n+1}\|_{L^2}^2 + \|\mathrm{d}^\alpha\hat{u}_{n+1}\|_{L^2}^2\\
&\quad + \|\mathrm{d}^\alpha\hat{f}_n\|_{L^2}^2 + z_{n+1}^{(l)}\big) + \delta_\alpha\sum_{j=0}^{l}(\delta\hat{z}_{n+1-j}^{(l+1)} + c(\delta,r)\hat{z}_{n+1-j}^{(l)})\\
&\quad + \delta\hat{z}_{n+1}^{(l+1)} + c(\delta,r)\hat{z}_{n+1}^{(l)}. \qquad\qquad (4.20)
\end{aligned}$$

We can multiply this inequality by γ^{-2n-2} with $\gamma = \gamma(R) \geq 1$ from Lemma 4.1 to obtain the weigthed quantities $\hat{z}_{n+1,\gamma}^{(l)}$ etc.

4.5. **Conclusion of the higher-order estimates.** We note that $z_{n,\gamma}^{(l)} \leq c(\tilde{z}_{n,\gamma}^{(l)} + \hat{z}_{n,\gamma}^{(l)}) \leq c z_{n,\gamma}^{(l)}$, which will be used without further notice.

1) We start with the first-order term $z_{n,\gamma}^{(1)}$ which is a bit easier since some commutators do not appear. Lemma 4.1, estimates (4.8) and (4.13) with $|\alpha| = 1$, and inequality (4.20) with $\alpha = 0$ imply

$$
\begin{aligned}
z_{n+1,\gamma}^{(1)} &\leq c(r) z_0 + \tau(1+T) \sum_{k=0}^{n} \big(c(R) z_{k+1,\gamma}^{(1)} + c(r) z_{k,\gamma}^{f,(1)} \big) + c(r) z_{n,\gamma}^{f,(0)} \\
&\qquad + c(r)\delta z_{n+1}^{(1)} + c(\delta, r) z_{n+1}^{(0)}, \\
z_{n+1,\gamma}^{(1)} &\leq c(r) z_0 + \tau(1+T) \sum_{k=0}^{n} \big(c(R) z_{k+1,\gamma}^{(1)} + c(r) z_{k,\gamma}^{f,(1)} \big) + c(r) z_{n,\gamma}^{f,(0)}, \qquad (4.21)
\end{aligned}
$$

where we have chosen a small $\delta > 0$ to obtain in the last line. Recalling $T \leq T_0$ and decreasing $\tau_1(R) > 0$ if needed, by means of Gronwall's inequality we infer

$$
z_n^{(1)} \leq \gamma^{2n} \mathrm{e}^{2n\tau} \Big(c(r) z_0 + c(r) z_{n-1,\gamma}^{f,(0)} + \tau(1+T_0) \sum_{k=0}^{n} \gamma^{-2k} c(r) z_{k-1}^{f,(1)} \Big). \qquad (4.22)
$$

2) Employing the previous step instead of Lemma 4.1, we now deal with $z_{n,\gamma}^{(2)}$. We first bound $\mathrm{d}_{\mathrm{ta}} \hat{u}_{n+1}$ in $\mathcal{H}^1$ and $\mathrm{d}_\tau \mathrm{d}_{\mathrm{ta}} \hat{u}_{n+1}$ in L^2. To this aim, we let $\alpha_3 = 0$ and use (4.13) with $|\alpha| = 2$, (4.20) with $|\alpha| = 1$, as well as (4.21), obtaining

$$
\begin{aligned}
&\|\mathrm{d}_{\mathrm{ta}} \hat{u}_{n+1}\|_{\mathcal{H}^1,\gamma}^2 + \|\mathrm{d}_\tau \mathrm{d}_{\mathrm{ta}} \hat{u}_{n+1}\|_{L^2,\gamma}^2 \qquad (4.23) \\
&\quad \leq c(r)(z_0 + z_{n,\gamma}^{f,(1)}) + \tau(1+T) \sum_{k=0}^{n} \big(c(R)(\hat{z}_{k+1,\gamma}^{(2)} + z_{k+1,\gamma}^{(2)}) + c(r) z_{k,\gamma}^{f,(2)} \big) \\
&\qquad + c(r) \sum_{j=0}^{1} (\delta \hat{z}_{n+1-j,\gamma}^{(2)} + c_\delta \hat{z}_{n+1-j,\gamma}^{(1)}).
\end{aligned}
$$

We absorb the term with δ below. We still have to bound $\partial_\nu^2 \hat{u}_{n+1}$ in L^2. This is done via (4.20) with $\alpha = e_4$. Combined with (4.23), we derive

$$
\begin{aligned}
\hat{z}_{n+1,\gamma}^{2} &\leq c(r)(z_0 + z_{n,\gamma}^{f,(1)}) + \tau(1+T) \sum_{k=0}^{n} \big(c(R)(\hat{z}_{k+1,\gamma}^{(2)} + z_{k+1,\gamma}^{(2)}) + c(r) z_{k,\gamma}^{f,(2)} \big) \\
&\qquad + c(r) \sum_{j=0}^{1} (\delta \hat{z}_{n+1-j,\gamma}^{(2)} + c_\delta \hat{z}_{n+1-j,\gamma}^{(1)}).
\end{aligned}
$$

To absorb also the term $\hat{z}_{n,\gamma}^{(2)}$ in the last line, we define

$$
\hat{Z}_{n,\gamma}^{(l)} = \max_{0 \leq m \leq n} \hat{z}_{m,\gamma}^{(l)}, \qquad \hat{Z}_{n,\gamma} = \hat{Z}_{n,\gamma}^{(3)}, \qquad \hat{Z}_n^{(l)} = Z_{n,1}^{(l)},
$$

and analogously for z_n, z_n^f etc.. We obtain

$$
\begin{aligned}
\hat{Z}_{n+1,\gamma}^{(2)} &\leq c(r)(z_0 + Z_{n,\gamma}^{f,(1)}) + \tau(1+T) \sum_{k=0}^{n} \big(c(R)(\hat{z}_{k+1,\gamma}^{(2)} + z_{k+1,\gamma}^{(2)}) + c(r) z_{k,\gamma}^{f,(2)} \big) \\
&\qquad + c(r)\delta \hat{Z}_{n+1,\gamma}^{(2)} + c(\delta, r) Z_{n+1,\gamma}^{(1)}).
\end{aligned}
$$

We can now absorb the penultimate summand taking a small $\delta = \delta(r) > 0$. The last term is then dominated by the first line due to (4.21). Also using $\hat{z}^{(2)}_{n+1,\gamma} \le \hat{Z}^{(2)}_{n+1,\gamma}$, we derive

$$\hat{z}^{(2)}_{n+1,\gamma} \le c(r)(z_0 + Z^{f,(1)}_{n,\gamma}) + \tau(1+T)\sum_{k=0}^{n} \big(c(R)(\hat{z}^{(2)}_{k+1,\gamma} + z^{(2)}_{k+1,\gamma}) + c(r) z^{f,(2)}_{k,\gamma}\big)$$

Together with (4.8), it follows

$$z^{(2)}_{n+1,\gamma} \le c(r)(z_0 + Z^{f,(1)}_{n,\gamma}) + \tau(1+T)\sum_{k=0}^{n} \big(c(R) z^{(2)}_{k+1,\gamma} + c(r) z^{f,(2)}_{k,\gamma}\big). \tag{4.24}$$

Possibly decreasing $\tau_1(R)$ in (4.4), we further deduce

$$z^{(2)}_{n} \le \gamma^{2n} \mathrm{e}^{2n\tau} \Big(c(r) z_0 + c(r) Z^{f,(1)}_{n-1,\gamma} + \tau(1+T_0)\sum_{k=0}^{n} \gamma^{-2k} c(r) z^{f,(2)}_{k-1}\Big). \tag{4.25}$$

3) We finally tackle $z^{(3)}_{n,\gamma}$. Here we first employ (4.13) with $|\alpha| = 3$, and then apply (4.20) with $|\alpha| = 2$ iteratively for $\alpha_3 = 0, 1, 2$, also invoking (4.24). In this way, (4.13), (4.20), and (4.24) first yield

$$\begin{aligned} &\|\mathrm{d}^2_{\mathrm{ta}} \hat{u}_{n+1}\|^2_{\mathcal{H}^1,\gamma} + \|\mathrm{d}_\tau \mathrm{d}^2_{\mathrm{ta}} \hat{u}_{n+1}\|^2_{L^2,\gamma} \\ &\quad \le c(r)(z_0 + Z^{f,(2)}_{n,\gamma}) + \tau(1+T)\sum_{k=0}^{n} \big(c(R)(\hat{z}^{(3)}_{k+1,\gamma} + z^{(3)}_{k+1,\gamma}) + c(r) z^{f,(3)}_{k,\gamma}\big) \\ &\qquad + c(r)\sum_{j=0}^{2} (\delta \hat{z}^{(3)}_{n+1-j,\gamma} + c_\delta \hat{z}^{(2)}_{n+1-j,\gamma}) \end{aligned} \tag{4.26}$$

with $\alpha_3 = 0$ in (4.20). Combining (4.20) for $\alpha_3 = 1$ with (4.26) and (4.24), we next see that one can add $\|\partial^2_\nu \mathrm{d}_{\mathrm{ta}} \hat{u}_{n+1}\|^2_{L^2}$ to the left-hand side of (4.26). In the same way the missing term $\partial^3_\nu \hat{u}_{n+1}$ is estimated in L^2, arriving at the final bound near ∂G:

$$\begin{aligned} \hat{z}^{(3)}_{n+1,\gamma} &\le c(r)(z_0 + Z^{f,(2)}_{n,\gamma}) + \tau(1+T)\sum_{k=0}^{n} \big(c(R)(\hat{z}^{(3)}_{k+1,\gamma} + z^{(3)}_{k+1,\gamma}) + c(r) z^{f,(3)}_{k,\gamma}\big) \\ &\quad + c(r)\sum_{j=0}^{2} (\delta \hat{z}^{(3)}_{n+1-j,\gamma} + c_\delta \hat{z}^{(2)}_{n+1-j,\gamma}). \end{aligned} \tag{4.27}$$

Fixing a number $\delta = \delta(r) > 0$, the first term in the last line can be absorbed by the left. By (4.24) the last summand in the inequality (4.27) is bounded by its first line. It follows

$$\hat{z}^{(3)}_{n+1,\gamma} \le c(r)(z_0 + Z^{f,(2)}_{n,\gamma}) + \tau(1+T)\sum_{k=0}^{n} \big(c(R)(\hat{z}^{(3)}_{k+1,\gamma} + z^{(3)}_{k+1,\gamma}) + c(r) z^{f,(3)}_{k,\gamma}\big).$$

Together with the interior estimate (4.8), we conclude

$$z^{(3)}_{n+1,\gamma} \le c(r)(z_0 + Z^{f,(2)}_{n,\gamma}) + \tau(1+T)\sum_{k=0}^{n} \big(\overline{c}(R) z^{(3)}_{k+1,\gamma} + c(r) z^{f,(3)}_{k,\gamma}\big). \tag{4.28}$$

We now fix our maximal time step size as

$$0 < \tau \le \tau_2(R) := \min\{\tau_1(R), (2(1+T_0)\overline{c}(R))^{-1}\} \tag{4.29}$$

see the line before (4.25). We recall the notation $\gamma = \mathrm{e}^{C(R)\tau}$ in Lemma 4.1, the definition (2.19), and the comments after (2.16) and (2.22). The discrete Gronwall inequality and (4.28), now easily yield the core a priori estimate.

Proposition 4.2. *Let (2.1) be true and $u_n \in \mathcal{H}^3_{\mathrm{cc}}(G)$ solve the linearized implicit Euler scheme (2.21) for given $u_0 \in \mathcal{H}^3_{\mathrm{cc}}(G)$ with $\|u_0^1\|_{L^\infty} < \kappa$ and (v_n) in $\mathcal{E}(R,T,\tau)$, where $R > 0$, $0 < T \le T_0$ and $0 < \tau \le \tau_2(R)$. Let N be the largest integer with $N\tau \le T$ and $r^2 \ge \max_{n\ge 1} z_n^{v,(2)}$. For $n \in \{0,\dots,N\}$ we then have*

$$z_n \le \mathrm{e}^{2(C(R)+1)n\tau}\Big(c(r)z_0 + c(r)Z^{f,(2)}_{n-1} + \tau c(r)(1+T_0)\sum_{k=0}^{n-1} \mathrm{e}^{-2C(R)k} z_k^f\Big). \tag{4.30}$$

5. Construction of the scheme and error analysis

In the next result we construct a time discretized approximation of a solution $u = (E,H) \in \mathcal{G}^3([0,T_0])$ of (2.3) with the material laws (2.1) and $J = 0$. We use the Euler scheme (2.15) with operators defined in (2.16), where we let $f_n = 0$. There is a maximal bound $\tau_2(r_0)$ on the time step size, but it only depends on the norm $\|u_0\|_{\mathcal{H}^3} \le r_0$ of the initial value. The additional condition $\|E_0\|_{L^\infty} < \kappa$ ensuring invertibility of $\varepsilon_{\mathrm{d}}(E_0)$, disappears (i.e., $\kappa = \infty$) if the coefficents in (2.1) and (2.2) have a good sign, see (2.4). We also show that the solution is bounded in $\mathcal{H}^3$ by a constant R only depending on r_0. In this sense the scheme is (unconditionally) stable.

Theorem 5.1. *Let (2.1) be true, $u_0 = (E_0, h_0) \in \mathcal{H}^3(G)$ satisfy $\|E_0\|_{L^\infty} < \kappa$ and (2.11). Fix $r_0 \ge \|u_0\|_{\mathcal{H}^3}$. Then there is a number $R = R(r_0) > 0$, a time horizon $T_3(R) > 0$ and a maximal step size $\tau_2(R) > 0$, see (5.1), (5.3) and (4.29), such that for $0 < \tau \le \tau_2(R)$, $\tau N \le T$ and $0 \le n \le N$ we have a unique solution $(u_n)_{0\le n\le N}$ in $\mathcal{E}(R,T,\tau)$ of (2.15). The solution satisfies the bound (4.30) with $z_k^f = 0 = Z_k^{f,(2)}$ uniformly in τ.*

Proof. The solution is constructed by a fixed-point argument on the space $\mathcal{E} = \mathcal{E}(R,T,\tau)$ given by (2.19), $0 < T \le T_0$ and $0 < \tau \le \tau_2(R)$ with $\tau_2(R)$ from (4.29). Let N be the largest integer with $\tau N \le T$. Below these numbers are chosen depending on r_0.

We have $z_0 \le c_0(r_0)$ by Lemma 3.3. Set $r_0^* = c_0(r_0) + 1$ and take

$$R^2 := 2c^*(r_0^*)c_0(r_0), \tag{5.1}$$

with $c^*(r)$ and $C^*(r)$ being the maxima of the constants $c(r)$ and $C(R)$, respctively, in (4.25) and (4.30). We equip $\mathcal{E}$ with the the metric induced by $\max_n z_n^{(2)}$, namely

$$\mathrm{d}(v,\overline{v}) = \max_{0\le n\le N}\max_{0\le j\le 2} \|\mathrm{d}_\tau^j(v_n - \overline{v}_n)\|_{\mathcal{H}^{2-j}}.$$

(Recall that $v_{-k} = u_{-k}$ for $k \in \{0,1,2,3\}$ with u_{-k} from (2.17).) It is then straightforward to check that $\mathcal{E}$ is complete.

Take $v = (v_n)_{0\le n\le N} \in \mathcal{E}$. We then obtain

$$v_n = v_0 + \tau \sum_{k=1}^{n} \mathrm{d}_\tau v_k \tag{5.2}$$

for $1 \le n \le N$, and thus $\|v_n\|_{\mathcal{H}^2} \le r_0 + N\tau R \le r_0 + TR$. Applying d_τ, it follows $\|\mathrm{d}_\tau^j v_n\|_{\mathcal{H}^{3-j}} \le c_0(r_0) + TR$ for $j \in \{0,1\}$. Taking $T \le T_1(R) \coloneqq \min\{T_0, \frac{1}{R}\}$, we infer $\max_{n\ge 1} z_n^{v,(2)} \le r^2 \coloneqq (c_0(r_0)+1)^2 = (r_0^*)^2$. We define $\Phi(v) = \Phi_{u_0}(v)$ by

$$[\Phi_{u_0}(v)]_{n+1} \coloneqq \prod_{k=0}^{n} (I - \tau A_{k+1})^{-1} u_0 = (I - \tau A_{n+1})^{-1} \cdot \ldots \cdot (I - \tau A_1)^{-1} u_0$$

for $n \le N-1$, which is well-defined because of $0 < \tau \le \tau_2(R) < 1/\omega_0(R)$, see (2.22) and the text following it. The sequence $(u_{n+1}) = \Phi(v)$ in $\mathcal{H}^3_{\mathrm{cc}}(G)$ solves the linearized recursion (2.21) for v. To simplify notation we write

$$\Pi_{n,k} = \prod_{j=k}^{n} (I - \tau A_{j+1})^{-1}.$$

Let $T \le T_2(R) \coloneqq \min\{T_1(R), \ln(2)(2C^*(R)+2)^{-1}\}$. Estimate (4.30) then shows that

$$z_{n+1} \le \exp\big(2(C(R)+1)T\big) c(r) c_0(r_0) \le R^2.$$

The restriction $\|u_n^1\|_{L^\infty} \le \kappa'$ then follows as in (5.2), replacing $T_2(R)$ by $T_2'(R) \coloneqq \min\{T_2(R), (\kappa' - \|E_0\|_{L^\infty})(c_S R)^{-1}\}$, where c_S is the norm of the embedding $\mathcal{H}^2(G) \hookrightarrow C(\overline{G})$. Hence, Φ maps $\mathcal{E}$ into itself.

To show the strict contractivity of Φ, we let $\overline{v} \in \mathcal{E}$ and set $w = \Phi(v) - \Phi(\overline{v})$, $\overline{\Lambda}_n = \Lambda(\overline{v}_n)$ etc.. We compute

$$w_{n+1} = \tau \sum_{k=0}^{n} \Pi_{n,k} \big[(\Lambda_{k+1}^{-1} - \overline{\Lambda}_{k+1}^{-1}) M + \Lambda_{k+1}^{-1} Q_{k+1} - \overline{\Lambda}_{k+1}^{-1} \overline{Q}_{k+1} \big] \overline{\Pi}_{k,0} u_0$$

The term $\varphi_0 = [\cdots]\overline{\Pi}_{k,0} u_0$ belongs to $\mathcal{H}^2_{\mathrm{cc}}(G)$ by Lemmas 2.1 and 3.1. Observe that $\varphi_n = \Pi_{n,k}\varphi_0$ solves (2.21) for v_n with $f_n = 0$, starting time k and initial value φ_0 We can thus apply (4.25) to φ_n and (4.30) to $\overline{\Pi}_{k,0} u_0$. Using also Lemmas 3.1 and 3.2, we deduce

$$\begin{aligned} \mathrm{d}(\Phi(v) - \Phi(\overline{v})) &\le (n+1)\tau c(R) \exp\big((C^*(R)+1)T\big) c_0(r_0) \mathrm{d}(v, \overline{v}) \\ &\le 2T c_1(R) \mathrm{d}(v, \overline{v}) \le \tfrac{1}{2} \mathrm{d}(v, \overline{v}) \end{aligned}$$

if we let

$$0 < T \le T_3(R) \coloneqq \min\{T_2'(R), (4c_1(R))^{-1}\}. \tag{5.3}$$

As a result, we have unique fixed point $u \in \mathcal{E}(R, T, \tau)$ of $u = \Phi(u)$, which then solves (2.15) with $f_n = 0$. $\square$

We can now proceed as in [11] to show convergence of the scheme. Let (2.1) be true and $u_0 = (E_0, h_0) \in \mathcal{H}^3_{\mathrm{cc}}(G)$ fulfill $\|E_0\|_{L^\infty} < \kappa$. Then we have the solution $u \in \mathcal{G}^3([0, T_0])$ of (1.1) (or (2.3) with $J = 0$) satisfying $\|u\|_{\mathcal{G}^3([0,T_0])} \le \hat{R}$. Moreover, Theorem 5.1 provides the unique solution $u_n \in \mathcal{H}^3_{\mathrm{cc}}(G)$, $0 \le n \le N$, of (2.15) with $f_n = 0$ subject to $z_n \le R^2$ and $\|u_n^1\|_\infty \le \kappa'$, where $N\tau \le T \le T_0$. We set $t_n = n\tau$ and

$$\begin{aligned}
&\Lambda_n = \Lambda(u_n^1), \quad Q_n = Q(u_n^1), \quad A_n = \Lambda_n^{-1}(M + Q_n),\\
&\hat{\Lambda}_n = \Lambda(u^1(t_n)), \quad \hat{Q}_n = Q(u^1(t_n)), \quad \hat{A}_n = \hat{\Lambda}_n^{-1}(M + \hat{Q}_n),\\
&\hat{u}_n = u(t_n), \quad e_n = u_n - \hat{u}_n, \quad \delta_{n+1} = \int_{t_n}^{t_{n+1}} \partial_t^2 u(t)(t_n - t)\,\mathrm{d}t.
\end{aligned}$$

for $n \le N$, cf. (2.16). We analyze the error e_n. Note that

$$u(t_{n+1}) = u(t_n) + \tau\partial_t u(t_{n+1}) + \delta_{n+1} = \hat{u}_n + \tau\hat{A}_{n+1}u_{n+1} + \delta_{n+1}.$$

Substracting this equation from (2.15), i.e., $u_{n+1} = u_n + \tau A_{n+1}u_{n+1}$, we obtain the error equation

$$\begin{aligned}
e_{n+1} &= e_n + \tau(A_{n+1}u_{n+1} - \hat{A}_{n+1}\hat{u}_{n+1}) - \delta_{n+1}\\
&= e_n + \tau A_{n+1}e_{n+1} + \tau(A_{n+1} - \hat{A}_{n+1})\hat{u}_{n+1} - \delta_{n+1}. \qquad (5.4)
\end{aligned}$$

We can now show our main convergence result, bounding the L^2-error in first order by an energy-type estimate.

Theorem 5.2. *Let (2.1) be true, $u_0 = (E_0, h_0) \in \mathcal{H}^3_{\mathrm{cc}}(G)$ satisfy $\|E_0\|_{L^\infty} < \kappa$, and let $u \in \mathcal{G}^3([0,T_0])$ solve (1.1). Fix $\hat{R} \ge \|u\|_{\mathcal{G}^3([0,T_0])}$ and $r_0 \ge \|u_0\|_{\mathcal{H}^3}$, and define $R = R(r_0) > 0$, $T_3(R) > 0$ and $\tau_2(R)$ by (5.1), (5.3) and (4.29). Let $0 < T \le T_3(R)$, $0 < \tau \le \tau_2(R)$, $\tau N \le T$, and $(u_n)_{0\le n\le N}$ be the solution of the Euler scheme (2.15) with $f_n = 0$. For $0 \le n \le N$ we then obtain*

$$\|u_n - u(n\tau)\|^2_{L^2(G)} \le c(r)\mathrm{e}^{n\tau\overline{c}(\overline{R})}\tau^2 \int_0^T \|\partial_t^2 u(t)\|^2_{L^2(G)}\,\mathrm{d}t,$$

where $\overline{R} \coloneqq \max\{R, \hat{R}\}$, $r^2 \ge \max_{n\ge0} z_n^{u,(2)}$ and $R^2 \ge \max_{n\ge0} z_n^{u,(3)}$, see (2.18).

Proof. Set $\|u\|_{\mathcal{G}^2([0,T_0])} = \hat{r}$ and $\overline{r} = \max\{r, \hat{r}\}$, We integrate (5.4) against $\Lambda_{n+1}e_{n+1}$ obtaining

$$\begin{aligned}
(\Lambda_{n+1}^{\frac12}e_{n+1}|\Lambda_{n+1}^{\frac12}e_{n+1}) &= (\Lambda_{n+1}^{\frac12}e_n|\Lambda_{n+1}^{\frac12}e_{n+1}) + \tau((M+Q_{n+1})e_{n+1}|e_{n+1}) \qquad (5.5)\\
&\quad + \tau((A_{n+1}-\hat{A}_{n+1})\hat{u}_{n+1}|\Lambda_{n+1}e_{n+1}) - (\delta_{n+1}|\Lambda_{n+1}e_{n+1}).
\end{aligned}$$

Observe that $\|\Lambda_{n+1}^{-1} - \hat{\Lambda}_{n+1}^{-1}\|_{L^2} \le c(r)\|e_{n+1}\|_{L^2}$ and analogously for $Q_{n+1} - \hat{Q}_{n+1}$. Using (the proof of) Lemma 3.1 of [11], one shows

$$\|\Lambda_{n+1}^{\frac12} - \Lambda_n^{\frac12}\|_{L^\infty} \le c(r)\|u_n - u_{n+1}\|_{L^\infty} \le \tau c(r)\|A_{n+1}u_{n+1}\|_{\mathcal{H}^2} \le \tau c(R).$$

We now subtract $(\Lambda_n^{\frac12}e_n|\Lambda_{n+1}^{\frac12}e_{n+1})$ from (5.5) and use the above observations and the skew-adjointness of M on $\mathcal{H}^1_{\mathrm{cc}}(G)$. It follows

$$\begin{aligned}
\big(\Lambda_{n+1}^{\frac12}e_{n+1} - \Lambda_n^{\frac12}e_n\big|\Lambda_{n+1}^{\frac12}e_{n+1}\big) &\le \tau c(R)\|e_n\|_{L^2}\|e_{n+1}\|_{L^2} + \tau c(r)\|e_{n+1}\|^2_{L^2}\\
&\quad + \tau c(\overline{r})\|\hat{u}_{n+1}\|_{W^{1,\infty}}\|e_{n+1}\|^2_{L^2}\\
&\quad + \tau c(r)(\|\tau^{-1}\delta_{n+1}\|^2_{L^2} + \|e_{n+1}\|^2_{L^2})\\
&\le \tau c(\overline{R})(\|e_n\|^2_{L^2} + \|e_{n+1}\|^2_{L^2}) + \tau c(r)\|\tfrac{1}{\tau}\delta_{n+1}\|^2_{L^2}.
\end{aligned}$$

Because of the Cauchy–Schwartz inequality and $e_0 = 0$, the left-hand side is bounded from below by

$$\sum_{k=0}^{n-1} \big(\Lambda_{k+1}^{\frac{1}{2}} e_{k+1} - \Lambda_k^{\frac{1}{2}} e_k \big| \Lambda_{k+1}^{\frac{1}{2}} e_{k+1}\big) \geq \sum_{k=0}^{n-1} \big(\|e_{k+1}\|_{k+1}^2 - \tfrac{1}{2}\|e_{k+1}\|_{k+1}^2 - \tfrac{1}{2}\|e_k\|_k^2\big)$$
$$= \|e_n\|_n^2.$$

Together we have show

$$\|e_n\|_{L^2}^2 \leq \tau \sum_{k=0}^{n-1} \big(\overline{c}(\overline{R})\|e_{k+1}\|_{L^2}^2 + c(r)\|\tfrac{1}{\tau}\delta_{k+1}\|_{L^2}^2\big)$$

For $0 < \tau \leq (2\overline{c}(\overline{R}))^{-1}$, the discrete Gronwall inequality now yields

$$\|e_n\|_{L^2}^2 \leq c(r)\mathrm{e}^{n\tau\overline{c}(\overline{R})}\tau \sum_{k=0}^{n-1} \|\tfrac{1}{\tau}\delta_{k+1}\|_{L^2}^2 \leq c(r)\mathrm{e}^{n\tau\overline{c}(\overline{R})}\tau^2 \int_0^{n\tau} \|\partial_t^2 u(t)\|_{L^2}^2 \,\mathrm{d}t. \quad \square$$

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