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IIB matrix model, bosonic master field, and emergent spacetime

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The IIB matrix model has been suggested as a particular formulation of nonperturbative superstring theory (M-theory). It has now been realized that an emerging classical spacetime may reside in its large-N master field. This bosonic master field can, in principle, give rise to Minkowski and Robertson-Walker spacetimes. The outstanding task is to solve the bosonic master-field equation, which is essentially an algebraic equation. In this article, we present new results for the $(D, N) = (10, 4)$ bosonic master-field equation of the IIB matrix model, where D is the number of bosonic matrices and N the matrix size. We also give, in a self-contained appendix, explicit results for critical points of the effective bosonic action. The main physics application of the (dimensionless) IIB matrix model may be in providing a (conformal) phase that replaces the Friedmann big bang singularity.

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1. Introduction

One of the great questions of physics is: how did the Universe start? Or, at a more technical level: what replaces the big bang singularity of the Friedmann cosmology [1]? The Friedmann big bang singularity would have infinite energy density and curvature, but it is clear that Nature somehow will prevent these infinities from occurring.

Recall that general relativity [2] and the standard model of elementary particle physics [3] are the current underlying theories of the standard Friedmann cosmology [1]. Hence, the answer to the second question above, most likely, requires an extended theory, which incorporates general relativity and the standard model. In this article, we will consider superstrings [4] as such a candidate theory. We will need, in fact, a nonperturbative formulation of superstring theory.

The final formulation of nonperturbative superstring theory, also known as M-theory [5, 6], is still incomplete. The IKKT matrix model [7] is one suggestion. As that matrix model reproduces the basic structure of the light-cone string field theory of type-IIB superstrings, the model is also known as the IIB matrix model [8]. It may, therefore, be a worthwhile undertaking to investigate the IIB matrix model thoroughly.

A few years ago, we started thinking about how the IIB matrix model could, in principle, produce a new phase replacing the Friedmann big bang singularity, with an emerging classical spacetime and emergent matter. But how the spacetime, in particular, emerged from the matrix model was far from clear. We then realized that the large-N master field of Witten [9–12] could play a crucial role and we showed that, in principle, the master field of the IIB matrix model could give rise to the points and the metric of an emerging classical spacetime [13]. This discussion was, however, based on *assumed* master-field matrices and we really need to *calculate* them. Our work of the last year has focussed on that task.

For the first conceptual phase of our endeavours, with a main paper [13] and a cosmology follow-up paper [14] available, we have already written a review [15]. For the second calculational phase, with three technical papers [16–18] available, we now wish to present a corresponding review that focuses on the main results, while leaving out the technicalities. We also take the opportunity to present some new numerical results.

The outline of our paper is as follows. In Sec. 2, we review the main points of the IIB matrix model. In Sec. 3, we introduce the bosonic master field and the equation which determines it. In Sec. 4, we present numerical solutions for the $(D, N) = (10, 4)$ case, both with and without dynamical fermions (here, D is the number of bosonic matrices and N the matrix size). In Sec. 5, we give some closing remarks, also about getting a "tamed" big bang from the IIB matrix model. There are furthermore five appendices with technical details and additional results. In App. A, we present a few new results in favor of the crucial property of large-N factorization in the IIB matrix model. In App. B, we give the explicit choice for the $SU(4)$ generators used in our calculations. In App. C, we list one particular realization of pseudorandom numbers entering the $(D, N) = (10, 4)$ master-field equation, which is essentially an algebraic equation. In App. D, we give the obtained coefficients of an approximate numerical solution of the full $(D, N) = (10, 4)$ algebraic equation. In App. E, we look for solutions of another algebraic equation, the stationarity equation from the effective bosonic action, and present two nontrivial critical points for the case $(D, N) = (3, 3)$.

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2. IIB matrix model

2.1 Partition function

Let us recall the definition of the IIB matrix model [7, 8]. We essentially take over the conventions and notations of Ref. [19], except that we write A^μ for the bosonic matrices with a directional index μ running over $\{1, \ldots, D\}$ for $D = 10$. These bosonic matrices A^μ, as well as the fermionic matrices Ψ_α, are $N \times N$ traceless Hermitian matrices. The partition function of the supersymmetric IIB matrix model (IIB-m-m) for $N \geq 2$ is then defined as follows [7, 8, 19]:

$$Z^F_{D,\,N} = \int \prod_{c=1}^{g} \prod_{\mu=1}^{D} dA^c_\mu \; e^{-\,S_{\text{bos}}[A]} \left(\int \prod_{c=1}^{g} \prod_{\alpha=1}^{\mathcal{N}} d\Psi^c_\alpha \; e^{-\,S_{\text{ferm}}[A,\,\Psi]} \right)^F , \tag{2.1a}$$

$$S_{\text{bos}}[A] = -\frac{1}{2}\,\text{Tr}\Big(\big[A^\mu,\,A^\nu\big]\,\big[A^\mu,\,A^\nu\big]\Big), \tag{2.1b}$$

$$S_{\text{ferm}}[A,\,\Psi] = -\text{Tr}\Big(\Psi_\alpha\;(C\,\Gamma^\mu)_{\alpha\beta}\;\big[A^\mu,\,\Psi_\beta\big]\Big), \tag{2.1c}$$

$$A_\mu = A^c_\mu\, t_c\,, \quad \Psi_\alpha = \Psi^c_\alpha\, t_c\,, \quad t_c \in \text{su}(N)\,, \tag{2.1d}$$

$$\text{Tr}\,\big(t_c \cdot t_d\big) = \frac{1}{2}\,\delta_{cd}\,, \tag{2.1e}$$

$$g \equiv N^2 - 1\,, \tag{2.1f}$$

$$\mathcal{N} = 2\,\big(D-2\big) = 2,\,4,\,16\,, \quad \text{for}\;\; D = 3,\,4,\,10\,, \tag{2.1g}$$

$$F \in \big\{0,\,1\big\}\,, \tag{2.1h}$$

$$\big\{D,\,F\big\}\Big|^{(\text{IIB-m-m})} = \big\{10,\,1\big\}\,, \tag{2.1i}$$

where repeated Greek indices are summed over (just as with an implicit Euclidean "metric") and F is an on/off parameter to include ($F = 1$) or exclude ($F = 0$) dynamic-fermion effects. The commutators entering (2.1b) and (2.1c) are defined by $[X,\,Y] \equiv X \cdot Y - Y \cdot X$ for square matrices X and Y of equal dimension. The fermionic action (2.1c) contains also the charge conjugation matrix C and the Γ^μ for $D = 10$ are Weyl-projected "gamma" matrices, $\Gamma^\mu = \Sigma^\mu$. These 16×16 matrices Σ^μ have been given explicitly in Refs. [18, 20] and their notation recalls the 2×2 Pauli matrices σ^μ of the four-dimensional case with 4×4 Dirac matrices γ^μ in the chiral representation.

The expansions (2.1d), with real coefficients A^c_μ and real Grassmannian coefficients Ψ^c_α, use the $N \times N$ traceless Hermitian $SU(N)$ generators t_c with normalization (2.1e) and structure constants f^{abc} as given by (A.7b) in App. A.3. (The matrix model for $D = 6$ has Weyl spinors without Majorana condition, while the model has Majorana spinors for $D = 3, 4$ and Majorana–Weyl spinors for $D = 10$; see App. 4. A of Ref. [4] for a brief review of supersymmetric Yang–Mills gauge theory in $D = 3,\,4,\,6,\,10$ spacetime dimensions.) Equation (2.1g) corrects the corresponding equation in Ref. [17].

The action of the model (2.1) is invariant under the following global gauge transformation:

$$A^\mu \;\rightarrow\; \Omega\, A^\mu\, \Omega^\dagger\,, \quad \Psi_\alpha \rightarrow \Omega\, \Psi_\alpha\, \Omega^\dagger\,, \quad \Omega \in SU(N)\,. \tag{2.2}$$

In addition, there is an $SO(D)$ rotation invariance and supersymmetry [7, 8]. Note that, as it stands, the model variables A^c_μ and Ψ^c_α in (2.1) are dimensionless (see Sec. 3.3 for further remarks).

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The Gaussian-type integrals over the Grassmann variables Ψ^c_α in (2.1a) can be performed analytically, so that the partition function reduces to a purely bosonic integral,

$$Z^F_{D,\,N} = \int \prod_{c=1}^{g} \prod_{\mu=1}^{D} dA^c_\mu \; e^{-\,S_{\text{bos}}[A]} \left(\mathcal{P}_{D,\,N}[A]\right)^F = \int \prod_{c=1}^{g} \prod_{\mu=1}^{D} dA^c_\mu \; e^{-\,S^F_{\text{eff},\,D,\,N}[A]}\,, \tag{2.3a}$$

$$S^F_{\text{eff},\,D,\,N}[A] = S_{\text{bos}}[A] - F\,\log \mathcal{P}_{D,\,N}[A]\,. \tag{2.3b}$$

The obtained quantity $\mathcal{P}_{D,\,N}[A]$ corresponds, in fact, to the following Pfaffian [19, 20]:

$$\mathcal{P}_{D,\,N}[A] = \text{Pf}\left[\mathcal{M}\,(A)\right]\,, \tag{2.4a}$$

$$\mathcal{M}_{a\alpha\,,\,b\beta} = -i\, f_{abc}\,\left(C\,\Gamma_\mu\right)_{\alpha\beta}\, A^c_\mu \equiv \mathcal{M}_{\underline{A}\,,\,\underline{B}}\,, \tag{2.4b}$$

with Lie-algebra indices a, b, c running over $\{1, \ldots, g\}$, spinorial indices α, β running over $\{1, \ldots, \mathcal{N}\}$, and the directional index μ being summed over $\{1, \ldots, D\}$, where the pair of indices $a\alpha$ gives a combined index $\underline{A}$ and the pair $b\beta$ a combined index $\underline{B}$. Note that the matrix $\mathcal{M}_{\underline{A}\,,\,\underline{B}}$ is antisymmetric, because f_{abc} is antisymmetric in the indices a, b and $\left(C\,\Gamma_\mu\right)_{\alpha\beta}$ symmetric in the indices α, β. Recall that the Pfaffian can be defined as a sum over permutations [19] or as a sum involving the completely antisymmetric Levi–Civita symbol ϵ normalized to unity [18]. This last definition of the Pfaffian of a $(2n) \times (2n)$ skew-symmetric matrix $S = (s_{ij})$ reads

$$\text{Pf}[S] \equiv \frac{1}{2^n\, n!}\; \epsilon_{i_1 j_1 i_2 j_2 \cdots i_n j_n}\; s_{i_1 j_1}\, s_{i_2 j_2} \cdots s_{i_n j_n}\,, \tag{2.5}$$

with implicit summations over repeated indices. An example is given by the 2×2 skew-symmetric matrix $T = a\, i\, \sigma^2 = \left\{\{0,\, a\},\, \{-a,\, 0\}\right\}$, which has the Pfaffian $\text{Pf}[T] = a$.

From (2.4) and (2.5), it is clear that the Pfaffian $\mathcal{P}_{D,\,N}[A]$ is a homogenous polynomial in the bosonic coefficients A^c_μ, where the order K is given by the following expression:

$$K = \frac{1}{2}\,\mathcal{N}\, g = \left(D - 2\right)\left(N^2 - 1\right), \quad \text{for} \;\; D = 3,\, 4,\, 10\,. \tag{2.6}$$

Further discussion of the Pfaffian appears in, e.g., Refs. [19–21] and the $D = 10$ Pfaffian has been detailed in App. A of Ref. [18].

As mentioned above, the partition function of the genuine IIB matrix model [7, 8] has the following parameters in (2.1):

$$D = 10\,, \quad F = 1\,, \quad N \gg 1\,, \tag{2.7}$$

and there are now two supersymmetry transformations (see Sec. 3.3 below). The large-N limit may require further discussion, but, at this moment, we just consider N to be large and finite (for exploratory numerical results, see, e.g., Refs. [22–24] and references therein).

2.2 Bosonic observables and expectation values

As our main interest is in the possible recovery of an emerging classical spacetime, we primarily consider the bosonic observable

$$w^{\mu_1 \cdots \mu_m} \;\equiv\; \frac{1}{N}\,\text{Tr}\left(A^{\mu_1} \cdots A^{\mu_m}\right), \tag{2.8}$$

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which is invariant under (2.2) and where the $1/N$ prefactor on the right-hand side is solely for convenience. Now, arbitrary strings of these bosonic observables have expectation values

$$\begin{aligned}&\langle w^{\mu_1\,\ldots\,\mu_m}\, w^{\nu_1\,\ldots\,\nu_n}\,\ldots\, w^{\omega_1\,\ldots\,\omega_z}\rangle^F_{D,\,N}\\ &= \frac{1}{Z^F_{D,\,N}}\int dA\,\left(w^{\mu_1\,\ldots\,\mu_m}\, w^{\nu_1\,\ldots\,\nu_n}\,\ldots\, w^{\omega_1\,\ldots\,\omega_z}\right) e^{-\,S^F_{\text{eff},\,D,\,N}}\,,\end{aligned} \tag{2.9}$$

where "dA" is a short-hand notation of the measure appearing in (2.1a) and $Z^F_{D,\,N}$ is defined by the integral (2.3a).

In closing, we emphasize that the IIB matrix model is relatively straightforward to formulate, having only a finite number of matrices, but hard to evaluate and interpret.

3. Bosonic master field and master field equation

3.1 Bosonic master field

The expectation values (2.9), at large values of N, have a remarkable factorization property:

$$\langle w^{\mu_1\,\ldots\,\mu_m}\, w^{\nu_1\,\ldots\,\nu_n}\,\ldots\, w^{\omega_1\,\ldots\,\omega_z}\rangle^F_{D,\,N} \;\overset{N}{=}\; \langle w^{\,\mu_1\,\ldots\,\mu_m}\rangle^F_{D,\,N}\,\langle w^{\,\nu_1\,\ldots\,\nu_n}\rangle^F_{D,\,N}\,\ldots\,\langle w^{\,\omega_1\,\ldots\,\omega_z}\rangle^F_{D,\,N}\,, \tag{3.1}$$

where the equality holds to leading order in N. Evidence for the factorization property (3.1) in the context of the IIB matrix model has been presented in Ref. [25]. See also App. A here for further results in support of large-N factorization.

According to Witten [9], the factorization (3.1) implies that the path integrals (2.9) are saturated by a single configuration, which has been called the "master field" [10] and whose matrices will be denoted by $\widehat{A}^{\,\mu}$. To leading order in N, the expectation values (2.9) are then given by the bosonic master-field matrices $\widehat{A}^{\,\mu}$ in the following way:

$$\langle w^{\mu_1\,\ldots\,\mu_m}\, w^{\nu_1\,\ldots\,\nu_n}\,\ldots\, w^{\omega_1\,\ldots\,\omega_z}\rangle^F_{D,\,N} \;\overset{N}{=}\; \widehat{w}^{\,\mu_1\,\ldots\,\mu_m}\,\widehat{w}^{\,\nu_1\,\ldots\,\nu_n}\,\ldots\,\widehat{w}^{\,\omega_1\,\ldots\,\omega_z}, \tag{3.2a}$$

$$\widehat{w}^{\,\mu_1\,\ldots\,\mu_m} \equiv \frac{1}{N}\,\mathrm{Tr}\left(\widehat{A}^{\,\mu_1}\,\ldots\,\widehat{A}^{\,\mu_m}\right), \tag{3.2b}$$

where the master-field matrices $\widehat{A}^{\,\mu}$ have an implicit dependence on the model parameters D, N, and F. Note that, for simplicity, we speak about a single master field but there can be many [12].

The task at hand is to obtain an equation which governs these master-field matrices $\widehat{A}^{\,\mu}$ and to find solutions of that equation.

3.2 Bosonic master-field equation

Introducing N random constants $\widehat{p}_k$ and the $N\times N$ diagonal matrix

$$D_{(\widehat{p})}(\tau) \equiv \mathrm{diag}\left(e^{i\,\widehat{p}_1\,\tau},\;\ldots,\;e^{i\,\widehat{p}_N\,\tau}\right), \tag{3.3}$$

the bosonic master-field matrices take the following "quenched" form [26]:

$$\widehat{A}^{\,\mu} \;=\; D_{(\widehat{p})}(\tau_{\text{eq}})\cdot\widehat{a}^{\,\mu}\cdot D^{-1}_{(\widehat{p})}(\tau_{\text{eq}})\,, \tag{3.4a}$$

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for a sufficiently large value of τ_{eq} (see below for further explanations). The τ-independent matrix $\widehat{a}^{\,\mu}$ in (3.4a) is determined by the equation [11, 13]

$$\frac{d}{d\tau}\left[D_{(\widehat{p})}(\tau)\cdot\widehat{a}^{\,\mu}\cdot D^{-1}_{(\widehat{p})}(\tau)\right]_{\tau=0} \;=\; -\,\frac{\delta S^{F}_{\text{eff},\,D,\,N}\left[\,\widehat{a}\,\right]}{\delta\,\widehat{a}_{\mu}} + \widehat{\eta}^{\,\mu}\,. \tag{3.4b}$$

All matrix indices have been suppressed in the three equations above and $S^{F}_{\text{eff},\,D,\,N}[\,\widehat{a}\,]$ is given by (2.3b).

It is instructive to write out (3.4b) explicitly and to add matrix indices $\{k,\,l\}$ running over $\{1,\,\ldots\,,\,N\}$. It is then clear that the equation is algebraic. In fact, this algebraic equation for D traceless Hermitian matrices $\widehat{a}^{\,\mu}$ of dimension $N\times N$ reads

$$i\left(\widehat{p}_k-\widehat{p}_l\right)\widehat{a}^{\,\mu}_{kl}+\delta_{\lambda\nu}\left[\widehat{a}^{\,\lambda},\,\left[\widehat{a}^{\,\nu},\,\widehat{a}^{\,\mu}\right]\right]_{kl}-F\,\frac{1}{\mathcal{P}_{D,\,N}\left(\widehat{a}\right)}\,\frac{\partial\,\mathcal{P}_{D,\,N}\left(\widehat{a}\right)}{\partial\left(\delta_{\mu\nu}\,\widehat{a}^{\,\nu}_{lk}\right)}-\widehat{\eta}^{\,\mu}_{kl}=0\,, \tag{3.5}$$

where the Pfaffian $\mathcal{P}_{D,\,N}\left(\widehat{a}\right)$ is a homogeneous polynomial of order K from (2.6) and directional indices $\mu,\,\nu$ run over $\{1,\,\ldots\,,\,D\}$, with the repeated indices λ,ν implicitly summed over. The $N\times N$ matrices $\widehat{\eta}^{\,\mu}$ are also traceless and Hermitian.

The algebraic equation (3.5) has two types of constants: the master momenta $\widehat{p}_k$ (fixed random numbers from a uniform distribution with a cutoff) and the master noise matrices $\widehat{\eta}^{\,\mu}_{kl}$ (fixed random numbers from a Gaussian distribution). Very briefly, the meaning of these two types of random numbers can be explained as follows.

The dimensionless time τ is the fictitious Langevin time of stochastic quantization, with a Gaussian noise term η in the differential equation,

$$\frac{dA^{\mu}(\tau)}{d\tau}=-\,\frac{\delta S_{\text{eff}}(\tau)}{\delta A_{\mu}(\tau)}+\eta^{\mu}(\tau)\,. \tag{3.6}$$

The τ evolution drives the system to equilibrium at $\tau=\tau_{\text{eq}}$ and the resulting configuration $A^{\mu}(\tau_{\text{eq}})$ corresponds to the master field $\widehat{A}^{\,\mu}$. For large N, the τ-dependence of the Langevin noise matrices $\eta^{\mu}_{k\,l}(\tau)$ can be quenched [26] by use of the master momenta $\widehat{p}_k$ and the same holds for the corresponding bosonic variables $A^{\mu}_{k\,l}(\tau)$. We then have at $\tau\geq\tau_{\text{eq}}$:

$$\eta^{\mu}_{k\,l}(\tau) \;=\; e^{i\,\widehat{p}_k\,\tau}\;\widehat{\eta}^{\,\mu}_{k\,l}\;e^{-i\,\widehat{p}_l\,\tau}\,, \tag{3.7a}$$

$$A^{\mu}_{k\,l}(\tau) \;=\; e^{i\,\widehat{p}_k\,\tau}\;\widehat{a}^{\,\mu}_{k\,l}\;e^{-i\,\widehat{p}_l\,\tau}\,, \tag{3.7b}$$

where the matrices $\widehat{\eta}^{\,\mu}$ and $\widehat{a}^{\,\mu}$ on the right-hand sides have no dependence on τ. Using the quenched behavior (3.7) with master momenta $\widehat{p}_k$, the Langevin differential equation (3.6) reduces to the algebraic equation (3.5). Most importantly, the random constants $\widehat{p}_k$ and $\widehat{\eta}^{\,\mu}_{k\,l}$ in (3.5) can be *fixed once and for all,* provided N is large enough. See Refs. [11, 13] for further discussion and references.

Incidentally, we have also considered a simplified version of the algebraic equation (3.5), with all constants $\widehat{p}_k$ and $\widehat{\eta}^{\,\mu}_{kl}$ set to zero. This simplified equation corresponds, in fact, to the stationarity equation of the effective bosonic action. Some of its solutions, critical points, will be presented in App. E (related critical points have been used in a recent paper [27]).

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Remark that, as the Pfaffian (2.4) is a K-th order homogeneous polynomial denoted symbolically by $P_K[A]$ with K given by (2.6), the basic structure of the algebraic equation (3.5) is as follows:

$$P_1^{(\widehat{p})}\,[\widehat{a}\,] + P_3\,[\widehat{a}\,] + F\,\frac{P_{K-1}\,[\widehat{a}\,]}{P_K\,[\widehat{a}\,]} + P_0^{(\widehat{\eta})}\,[\widehat{a}\,] = 0\,, \tag{3.8}$$

where only the on/off constant F is shown explicitly and where the suffixes on P_1 and P_0 indicate their respective dependence on the master momenta $\widehat{p}_k$ and the master noise $\widehat{\eta}^{\,\mu}_{kl}$. If we multiply (3.8) by $P_K\,[\widehat{a}\,]$, we get a homogeneous polynomial equation of order $K+3$.

In order to obtain the component equations [labelled by an index c running over $1, \ldots, (N^2-1)$ and an index μ running over $1, \ldots, D$], we matrix multiply (3.5) by t_c, take the trace, and multiply the result by two. There are then $D\,g = D\,\left(N^2-1\right)$ coupled algebraic equations for an equal number of unknowns $\{\widehat{a}^{\,1}_1, \ldots, \widehat{a}^{\,g}_D\}$.

It appears impossible to obtain a *general* solution of these algebraic equations. We will look for solutions of these coupled algebraic equations with a *specific* realization of the random constants $\widehat{p}_k$ and $\widehat{\eta}^{\,\mu}_{kl}$ [this procedure suffices for large N, as mentioned a few lines below (3.7b)]. This is still a formidable problem for large values of N. At this moment, we are able to get explicit results only for very small values of N.

The solution of (3.5) is given by D traceless matrices $\widehat{a}^{\,\mu}$ of dimension $N \times N$ and the number of real variables reads

$$N_{\text{real-var}} = D\,\left(N^2-1\right) \times \begin{cases} 2\,, & \text{for } F = 1 \text{ and } N \geq 4\,, \\ 1\,, & \text{otherwise}\,, \end{cases} \tag{3.9}$$

where the extra factor of 2 comes from having a complex Pfaffian [20], so that the solution has complex coefficients and non-Hermitian matrices (see Sec. 4.2 for further discussion). In case complex coefficients are needed, we define

$$\widehat{a}^{\,c}_\mu = \widehat{r}^{\,c}_\mu + i\,\widehat{s}^{\,c}_\mu\,, \tag{3.10}$$

with a Lie-algebra index c and real numbers $\widehat{r}^{\,c}_\mu$ and $\widehat{s}^{\,c}_\mu$.

3.3 Conceptual and technical remarks

Before we set out on obtaining solutions of the algebraic equation (3.5), we have two conceptual remarks and one technical remark:

- there are no $\hbar$'s and G_N's in the IIB matrix model (2.1), which is just a "statistical" theory, and we will have to identify the emerging quantum effects and gravity;
- there is a symmetry argument [8] for A_μ having the dimension of length (that is, not the dimension of inverse length or "momentum");
- in our subsequent calculations, the Pfaffian term in the algebraic equation (3.5) for $F = 1$ will be obtained *exactly* (different from the calculations of, e.g., Ref. [24]).

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The meaning of the first and last remarks is clear, but let us expand on the second remark.

The action $S = S_{\text{bos}} + S_{\text{ferm}}$ from (2.1) has two fermionic symmetries (labeled by $i = 1, 2$), which are given by Eqs. (1.2), (1.4), and (1.5) in Ref. [8]. (Incidentally, a useful discussion of the invariance properties of supersymmetric Yang–Mills gauge theory in $D = 3, 4, 6, 10$ spacetime dimensions can be found in App. 4. A of Ref. [4].) With two infinitesimal fermionic parameters ϵ_1 and ϵ_2, the two symmetry generators $\widetilde{Q}^{(i)}$ have the following commutation relations [8]:

$$\left[\bar{\epsilon}_1\, \widetilde{Q}^{(i)},\, \bar{\epsilon}_2\, \widetilde{Q}^{(j)}\right] = -2\, \bar{\epsilon}_1\, \gamma_\mu\, \epsilon_2\, \mathcal{P}^\mu\, \delta^{i\,j}\,, \tag{3.11}$$

where $\mathcal{P}^\mu$ is the generator of the following bosonic transformation (similar to a translation):

$$\delta A_\mu = \xi_\mu\, \mathbb{1}_N\,, \tag{3.12}$$

for D infinitesimal constants ξ_μ and the $N \times N$ unit matrix $\mathbb{1}_N$. The transformation (3.12) obviously leaves the action $S = S_{\text{bos}} + S_{\text{ferm}}$ from (2.1) invariant, as $\mathbb{1}_N$ trivially commutes with the matrices A^μ and Ψ_α.

The structure of the commutation relations (3.11) is precisely that of a ten-dimensional $\mathcal{N} = 2$ spacetime supersymmetry and we may interpret $\mathcal{P}^\mu$ as a "momentum" with the dimension of an inverse length. In turn, this implies that the A_μ matrices (and, *a fortiori*, their eigenvalues) have the dimension of length. [Let us clarify the reason for calling $\mathcal{P}^\mu$ a "momentum": freely introducing $\hbar$'s and c's, the Hamiltonian of a supersymmetric theory is the anticommutator of two supercharges Q (cf. Sec. 5.2.2 of Ref. [4]) and the energy corresponds to the 0-component of the momentum vector ($E = c\, p^0$), so that the identification of $\mathcal{P}^\mu$ in (3.11) with a momentum vector makes sense.]

It is, in principle, easy to give the bosonic matrices A_μ of the IIB matrix model the dimension of length. For this, we replace the actions S_{bos} and S_{ferm} in the exponentials of (2.1a) by S_{bos}/ℓ^4 and S_{ferm}/ℓ^4, for a model length scale ℓ. But it is very well possible that the correct version of the IIB matrix model has no such *ad hoc* length scale. Then, the matrices A^μ and Ψ_α are dimensionless and the theory has, most likely, conformal symmetry. In this case, we only notice the *structure* of the commutation relations (3.11), with all quantities being dimensionless (the matrices A^μ must later get their dimension of length dynamically). See Sec. 5 for further comments relating to a "tamed" big bang [28] and subsequent conformal symmetry breaking [29–31].

As a last preliminary remark, it may be helpful to recall the heuristics [15] for obtaining an emerging classical spacetime from the bosonic master-field matrices:

- the expectation values $\langle w^{\mu_1 \cdots \mu_m} \cdots w^{\omega_1 \cdots \omega_z} \rangle$ from (2.9), infinitely many numbers, correspond to a large part of the information content of the IIB matrix model (but, of course, not all the information);
- that very same information is contained in the master-field matrices $\widehat{A}^\mu$, as these matrices give, to leading order in N, identical numbers from the products $\widehat{w}^{\,\mu_1 \cdots \mu_m} \cdots \widehat{w}^{\,\omega_1 \cdots \omega_z}$, where $\widehat{w}$ is the observable w evaluated for the $\widehat{A}^\mu$, according to (3.2b);
- from these master-field matrices $\widehat{A}^\mu$, it then appears possible to extract the points and the metric of an emergent classical spacetime with a dimensionality less than or equal to D (recall that the original matrices A^μ were merely integration variables).

One suggestion for an extraction procedure of points and metric has been presented in Ref. [13] and was reviewed in the appendices of Ref. [15].

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4. Numerical solutions for (D, N) = (10, 4)

4.1 Diagnostic quantities

Approximate numerical solutions of the full $(D,\, N) = (10,\, 4)$ bosonic master-field equation (3.5) with $F = 1$ have been obtained and will be discussed shortly. With these approximate numerical solutions, the complex residues of the 150 component equations eq-$\widehat{a}_\mu^c$ are computed (they all vanish for a perfect solution). The quantity MaxAbsRes is the maximum of the absolute values of these residues and the function f_{penalty} is the sum of their squared absolute values. Explicitly, these quantities are defined by

$$f_{\text{penalty}} \equiv \sum_{\mu=1}^{10} \sum_{c=1}^{15} \left|\text{eq-}\widehat{a}_\mu^c\right|^2 , \tag{4.1a}$$

$$\text{MaxAbsRes} \equiv \max\left\{\left|\text{eq-}\widehat{a}_1^1\right|,\, \ldots\,,\, \left|\text{eq-}\widehat{a}_{10}^{15}\right|\right\} . \tag{4.1b}$$

We get the expression eq-$\widehat{a}_\mu^c$ from (3.5) by performing a matrix multiplication with t_c, taking the trace, and multiplying the result by two [here, the t_c are the $SU(4)$ generators given in App. B].

Following the discussion of a previous paper [16], we will first consider the absolute values of the entries in the 4×4 matrix $\widehat{a}_1$, calculate the average band-diagonal value from 3+4+3 entries and the average off-band-diagonal value from 3+3 entries, and get the ratio R_1 of the average band-diagonal value over the average off-band-diagonal value. For the $\mu = 2,\, \ldots\,,\, 10$ matrices $\widehat{a}_\mu$, we then follow the same procedure and get the ratios $R_2,\, \ldots\,,\, R_{10}$.

In order to avoid any confusion, we give the general definition of the ratio R for an arbitrary symmetric 4×4 matrix M with nonnegative entries $m[i,\, j]$:

$$R \equiv \frac{\left(\sum_{i=1}^{4} m[i,\, i] + 2\, \sum_{j=1}^{3} m[j,\, j+1]\right) \Big/ 10}{\left(2\, m[1,\, 3] + 2\, m[1,\, 4] + 2\, m[2,\, 4]\right) \Big/ 6} , \tag{4.2}$$

where the symmetry of M has been used to simplify the expression.

4.2 Numerical results from the full algebraic equation

In Ref. [18], we obtained approximate numerical solutions of the full $(D,\, N) = (10,\, 4)$ bosonic master-field equation (3.5) with $F = 1$ and the particular realization (the "κ-realization") of the pseudorandom numbers as given in App. C. In that paper, we used a self-made Random-Step (RS) routine for Mathematica 12.1 (cf. Ref. [32]), which could be partially parallelized.

A short description of this RS routine is as follows. Given the function value f_{penalty} at a starting point in the 300-dimensional configuration space (with a trivial Euclidean metric), the routine calculates the f_{penalty} values at $N_{\text{r-p}}$ random points on a sphere centered at this starting point with radius $r_{\text{r-p}}$ and moves to the point on the sphere with the lowest f_{penalty} value if that value is better than the starting-point value, otherwise the routine takes $N_{\text{r-p}}$ other random points, etc., etc. Choosing $N_{\text{r-p}}$ as an integer multiple of the number of kernels available, the random points can be calculated in parallel (we have used, for example, $N_{\text{r-p}} = 12$ for the 6 kernels of our processor). The

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Table 1: Approximate numerical solutions of the full $(D,\, N) = (10,\, 4)$ bosonic master-field equation (3.5) with $F = 1$ and pseudorandom constants given by (C.1) and (C.2). Shown are the values of the penalty function (4.1a), the maximum of the absolute values of the residues (4.1b), and 4 out of 300 coefficients defined by (4.3). The numerical calculations used a self-made Random-Step (RS) routine, which could be partially parallelized. The results of the first five rows have already been reported in Ref. [18] and the result of the last row is new. The calculations for these results took about 6 months, using a Lenovo T15p notebook with an Intel Core i7-10750H processor running Mathematica 12.1.

f_{penalty}	MaxAbsRes	$\{\widehat{r}_1^{\,1},\, \widehat{s}_1^{\,1},\, \widehat{r}_{10}^{\,15},\, \widehat{s}_{10}^{\,15}\}$	method
1375.200	6.81	$\{-0.219000, -0.279000, -0.434500, -0.175000\}$	RS
422.468	4.56	$\{0.0294495, -0.395907, -0.371477, 0.0285794\}$	RS
310.932	3.46	$\{0.0299337, -0.378790, -0.368679, 0.00749545\}$	RS
209.330	2.83	$\{0.0124611, -0.388325, -0.366319, 0.0159236\}$	RS
108.094	1.83	$\{-0.0255319, -0.470109, -0.338870, 0.0613333\}$	RS
33.8776	0.957	$\{-0.0692546, -0.586430, -0.350589, -0.0458760\}$	RS

Table 2: Ratios R_μ, as defined by (4.2), calculated with the absolute values of the entries in the transformed matrices (4.6) from the approximate numerical solutions of Table 1. The $\mu = 1$ matrix has been diagonalized and the corresponding ratio R_1 is trivially infinite, as the off-diagonal matrix elements are zero. The ten ratios are presented in two batches of five, in order to facilitate comparison with Fig. 2.

f_{penalty}	$\{R_1,\, R_2,\, R_3,\, R_4,\, R_5\}$	$\{R_6,\, R_7,\, R_8,\, R_9,\, R_{10}\}$
1375.200	$\{\infty, 0.786, 1.10, 2.01, 0.915\}$	$\{0.460, 0.919, 1.10, 0.781, 0.590\}$
422.468	$\{\infty, 0.933, 0.637, 1.67, 0.398\}$	$\{0.466, 2.31, 2.59, 0.757, 0.648\}$
310.932	$\{\infty, 0.971, 0.631, 1.28, 0.506\}$	$\{1.11, 1.81, 1.08, 0.552, 0.668\}$
209.330	$\{\infty, 1.12, 0.651, 1.15, 0.495\}$	$\{1.11, 1.22, 0.885, 0.626, 0.727\}$
108.094	$\{\infty, 1.26, 0.780, 1.13, 0.522\}$	$\{1.10, 1.20, 0.934, 0.590, 0.697\}$
33.8776	$\{\infty, 1.22, 0.822, 1.17, 0.540\}$	$\{0.913, 1.31, 1.07, 0.589, 0.617\}$

RS routine is robust, but requires fine-tuning of the radius of the sphere and number of random points.

With these approximate numerical solutions, we compute the complex residues of the 150 component equations eq-$\widehat{a}_\mu^{\,c}$ (they all vanish for a perfect solution) and calculate the two diagnostics as defined by (4.1). An approximate numerical solution is determined by 300 real numbers and characterized by the corresponding real number f_{penalty}. These ten 4×4 matrices are

$$\widehat{a}_\mu \,\big|_{\kappa\text{-num-sol}} = \left(\widehat{r}_\mu^{\,c} + i\, \widehat{s}_\mu^{\,c}\right) t_c \,, \tag{4.3}$$

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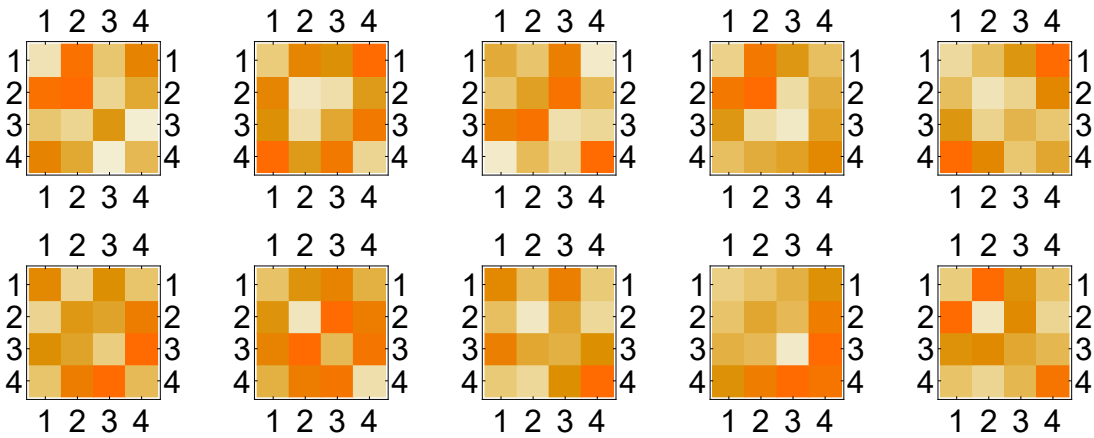

Figure 1: Results from the full $(D, N) = (10, 4)$ bosonic master-field equation (3.5) with $F = 1$ and pseudorandom constants given by (C.1) and (C.2). Shown are the density plots of $\text{Abs}\big[\widehat{a}^{\,\mu}_{\kappa\text{-}\underline{\text{num-sol-HERM}}}\big]$ from the approximate solution with $f_{\text{penalty}} = 33.8776$, as given in Table 1. These matrices are defined by (4.5). The panels on the top row are for $\mu = 1, \ldots, 5$ and those on the bottom row for $\mu = 6, \ldots, 10$.

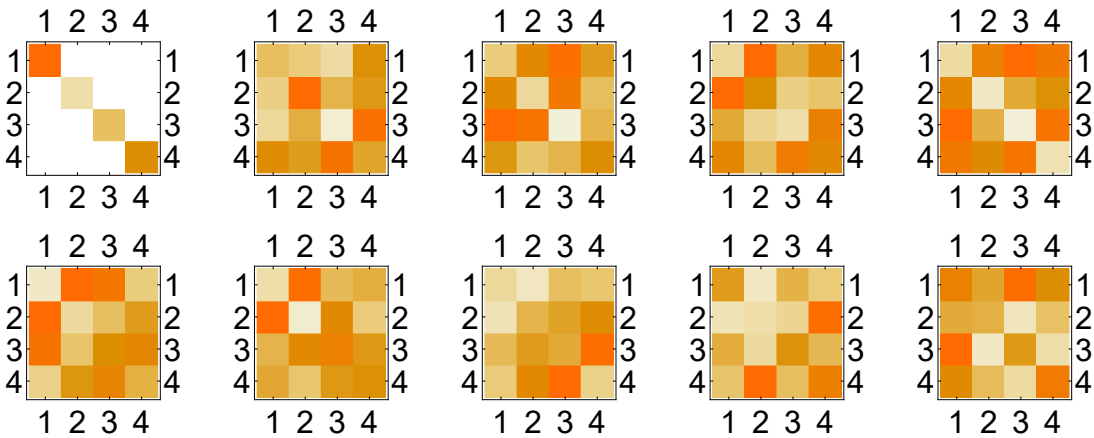

Figure 2: Density plots of $\text{Abs}\big[\widehat{a}'^{\,\mu}_{\kappa\text{-}\underline{\text{num-sol-HERM}}}\big]$ from the matrices of Fig. 1 with a change of basis to diagonalize and order one of them ($\mu = 1$).

with an implicit sum over the Lie-algebra index c and real numbers $\widehat{r}^{\,c}_{\mu}$ and $\widehat{s}^{\,c}_{\mu}$. A selection of approximate numerical solutions is shown in Table 1. Specifically, we will discuss the approximate numerical solution from the last row of Table 1 with $f_{\text{penalty}} = 33.8776$ and denote this particular solution by "κ-num-sol". We, then, have 300 real numbers defining the following matrices:

$$\widehat{a}^{\,\mu}_{\kappa\text{-}\underline{\text{num-sol}}}\,, \quad \text{for} \quad \mu = 1, \ldots, 10\,. \tag{4.4}$$

Specifically, the 300 real numbers for (4.4) are listed in App. D.

With complex coefficients $\widehat{a}^{\,c}_{\mu}$, these (approximate) master-field matrices are no longer Hermitian. The situation is perhaps analogous to that of pairs of complex saddle-points appearing for a real problem. Our interpretation is that these (approximate) master-field matrices carry *information*

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both in their Hermitian and anti-Hermitian parts. In fact, we conjecture that the Hermitian parts of the master-field matrices (with real eigenvalues) contain information about the emerging spacetime [13]. But what the information in the anti-Hermitian parts corresponds to is not clear for the moment (one suggestion has been put forward in Ref. [18]).

Consider, therefore, the Hermitian parts

$$\widehat{a}^{\,\mu}_{\kappa\text{-num-sol-HERM}} \equiv \frac{1}{2}\left[\widehat{a}^{\,\mu}_{\kappa\text{-num-sol}} + \left(\widehat{a}^{\,\mu}_{\kappa\text{-num-sol}}\right)^{\dagger}\right]. \tag{4.5}$$

Calculating the absolute values of these matrix entries for the $f_{\text{penalty}} = 33.8776$ approximate solution (denoted "κ-num-sol-HERM"), we observe no obvious band-diagonal structure in Fig. 1.

Now, change the basis, in order to diagonalize and order the $\mu = 1$ matrix. This gives the following transformed matrices denoted by a prime:

$$\widehat{a}'^{\,\mu}_{\kappa\text{-num-sol-HERM}}\,, \quad \text{for} \quad \mu = 1, \ldots, 10\,. \tag{4.6}$$

Considering the absolute values of these matrix entries, there is not yet a strong signal for a diagonal/band-diagonal structure (density plots are given in Fig. 2 for the $f_{\text{penalty}} = 33.8776$ approximate solution). But let us take a closer look anyway.

Using the ratio R defined in (4.2), we obtain the values given in Table 2. On the last row of this table with $f_{\text{penalty}} = 33.8776$, we see R_μ values scattered around unity but no clear pattern of band-diagonality. It may, however, be that this numerical solution is simply not yet good enough and that the f_{penalty} value needs to be reduced significantly. At this moment, we are not able to do that for the full $(D,\, N) = (10,\, 4)$ bosonic master-field equation (3.5) with $F = 1$. But the simplified equation with $F = 0$ does allow us to push f_{penalty} further down, as will be discussed in Sec. 4.3 .

4.3 Numerical results from the simplified algebraic equation

Here, we report on approximate numerical solutions of the simplified ($F = 0$) bosonic master-field equation (3.5), where we start with the same procedure and self-made Random-Step (RS) routine as has been used for the full bosonic master-field equation in Sec. 4.2. The random-step calculations for the simplified bosonic master-field equation are much simpler because there are fewer real variables (now 150 instead of 300) and because the difficult Pfaffian term is altogether absent in the simplified equation.

Random-step results are presented in Table 3 for penalty-function values of order 10^3, 10^2, 10^1, and 1. With the computer as described in the caption of Table 1, these results for the simplified equation have taken about 30 hours, which may be compared with the approximately 6 months for the full-algebraic-equation results of Table 1.

But the simplified algebraic equation also allows for the implementation of the `NMinimize` routine of Mathematica 12.1 (cf. Ref. [32]) with the downhill-simplex method of Nelder and Mead [33, 34]. (It is important to use a trick to force the routine `NMinimize` to be purely numerical; see App. A.1 for details.) These NMinimize results are also shown in Table 3 for penalty-function values of order 10^{-1} and down to 10^{-9}. *A priori*, there is the worry that changing the numerical method gives solutions belonging to different valleys. But this does not appear to be the case here, as shown by the two configurations with $f_{\text{penalty}} \sim 0.6$ obtained in the different runs (this will be especially clear in the later Table 4).

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Table 3: Approximate numerical solutions of the simplified $(D,\, N) = (10,\, 4)$ bosonic master-field equation (3.5) with $F = 0$ and pseudorandom constants given by (C.1) and (C.2). Two numerical routines have been used: a self-made Random-Step (RS) routine, which could be partially parallelized, and the `NMinimize` (NM) routine of Mathematica 12.1 [32] with the downhill-simplex method of Nelder and Mead [33, 34], which was not parallelized. The combined numerical calculations took a few days.

f_{penalty}	MaxAbsRes	$\{\widehat{r}_1^{\,1},\, \widehat{s}_1^{\,1},\, \widehat{r}_{10}^{\,15},\, \widehat{s}_{10}^{\,15}\}$	method
996.965	7.15	$\{-0.0678381, 0, -0.350114, 0\}$	RS
100.480	2.38	$\{0.161385, 0, -0.340025, 0\}$	RS
10.1415	0.916	$\{-0.124413, 0, 0.15631, 0\}$	RS
1.21690	0.255	$\{0.0904698, 0, 0.233820, 0\}$	RS
0.590364	0.189	$\{0.0183176, 0, 0.265982, 0\}$	RS
0.599656	0.175	$\{-0.00264037, 0, 0.266334, 0\}$	NM
0.149264	0.0879	$\{-0.075562, 0, 0.247046, 0\}$	NM
1.12955×10^{-2}	3.28×10^{-2}	$\{-0.139296, 0, 0.128158, 0\}$	NM
2.85246×10^{-3}	1.40×10^{-2}	$\{-0.195029, 0, 0.111522, 0\}$	NM
9.51062×10^{-5}	2.77×10^{-3}	$\{-0.216581, 0, 0.122857, 0\}$	NM
4.12196×10^{-5}	1.28×10^{-3}	$\{-0.218765, 0, 0.121856, 0\}$	NM
2.32875×10^{-6}	3.75×10^{-4}	$\{-0.223436, 0, 0.121505, 0\}$	NM
1.33301×10^{-7}	7.66×10^{-5}	$\{-0.223504, 0, 0.121471, 0\}$	NM
1.14189×10^{-8}	2.57×10^{-5}	$\{-0.223699, 0, 0.121473, 0\}$	NM
2.78202×10^{-9}	1.40×10^{-5}	$\{-0.223690, 0, 0.121466, 0\}$	NM

The solution on the last row of Table 3 with $f_{\text{penalty}} = 2.78202 \times 10^{-9}$ will be denoted "κ-num-sol-simpl". We, then, have 150 real numbers defining the following Hermitian matrices:

$$\widehat{a}^{\,\mu}_{\kappa\text{-num-sol-simpl}}\,, \quad \text{for} \quad \mu = 1,\, \ldots\,,\, 10\,. \tag{4.7}$$

A density plot of this solution is shown in Fig. 3 and a density plot of the transformed matrices in Fig. 4.

As to the apparent band-diagonality in the density plots of Fig. 4, we can again quantify the analysis by considering the ratio R defined in (4.2). The obtained values are given in Table 4. Apparently, the R_μ values have stabilized for $f_{\text{penalty}} \lesssim 10^{-4}$. A clear change is observed between the R_μ values for $f_{\text{penalty}} \gtrsim 1$, with an approximately equal number of values above 1 as below 1, and the R_μ values for $f_{\text{penalty}} \lesssim 1/100$, with all or nearly all values above 1. This issue will be discussed further in Sec. 4.4.

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Table 4: Ratios R_μ, as defined by (4.2), calculated with the absolute values of the entries in the transformed matrices $\widehat{a}'^{\,\mu}_{\kappa\text{-num-sol-simpl}}$ from the approximate numerical solutions of Table 3. The ten ratios are presented in two batches of five, so that the ratios on the last row can be easily compared with Fig. 4.

f_{penalty}	$\{R_1,\ R_2,\ R_3,\ R_4,\ R_5\}$	$\{R_6,\ R_7,\ R_8,\ R_9,\ R_{10}\}$
996.965	$\{\infty, 1.22, 0.825, 1.17, 0.540\}$	$\{0.915, 1.31, 1.07, 0.592, 0.618\}$
100.480	$\{\infty, 1.74, 1.36, 1.08, 0.631\}$	$\{0.720, 0.936, 1.42, 0.667, 0.617\}$
10.1415	$\{\infty, 1.10, 1.09, 0.718, 1.11\}$	$\{0.704, 0.932, 1.21, 1.53, 0.541\}$
1.21690	$\{\infty, 0.527, 1.76, 1.22, 0.652\}$	$\{0.641, 1.00, 0.657, 1.21, 0.350\}$
0.590364	$\{\infty, 0.627, 3.13, 1.19, 0.793\}$	$\{1.01, 0.900, 0.927, 1.09, 0.480\}$
0.599656	$\{\infty, 0.767, 2.65, 1.38, 0.900\}$	$\{1.14, 0.876, 0.964, 1.12, 0.497\}$
0.149264	$\{\infty, 1.16, 2.49, 1.36, 1.28\}$	$\{0.949, 1.10, 1.45, 1.44, 0.762\}$
1.12955×10^{-2}	$\{\infty, 1.81, 5.27, 1.63, 1.07\}$	$\{1.97, 0.92, 1.98, 3.29, 1.09\}$
2.85246×10^{-3}	$\{\infty, 4.00, 3.81, 1.95, 1.14\}$	$\{2.59, 1.67, 1.90, 3.49, 1.36\}$
9.51062×10^{-5}	$\{\infty, 6.07, 2.94, 2.47, 1.70\}$	$\{3.12, 2.86, 1.95, 3.87, 1.63\}$
4.12196×10^{-5}	$\{\infty, 6.08, 2.91, 2.47, 1.72\}$	$\{3.10, 2.95, 1.94, 3.89, 1.61\}$
2.32875×10^{-6}	$\{\infty, 6.06, 2.87, 2.49, 1.75\}$	$\{3.11, 3.07, 1.93, 3.91, 1.60\}$
1.33301×10^{-7}	$\{\infty, 6.06, 2.86, 2.48, 1.75\}$	$\{3.10, 3.08, 1.93, 3.90, 1.60\}$
1.14189×10^{-8}	$\{\infty, 6.06, 2.86, 2.49, 1.75\}$	$\{3.10, 3.09, 1.93, 3.90, 1.60\}$
2.78202×10^{-9}	$\{\infty, 6.06, 2.86, 2.49, 1.75\}$	$\{3.10, 3.09, 1.93, 3.90, 1.60\}$

4.4 Discussion of the numerical results

In Table 5, we summarize the presently available numerical results for $N = 3$ and $N = 4$ at different values of D. There are two tentative conclusions from the results collected in Table 5: the strength of the band-diagonality structure appears to be diminished by the increase of the number of dimensions [for the $N = 4$ and $F = 0$ results, from $D = 2$ to $D = 10$] and that strength also appears to be diminished by the inclusion of dynamic fermions [for the $D = 3$ and $N = 3$ results, from $F = 0$ (without dynamic fermions) to $F = 1$ (with dynamic fermions)].

Note that f_{penalty} values of order 10^{-36} for the $(D,\ N) = (2,\ 4)$ simplified algebraic equation relied on the use of the routine `FindMinimum` of Mathematica 12.1 [32], which is partially algebraic. That same routine was also used for the $(D,\ N) = (3,\ 3)$ results mentioned in Table 5. It will be hard to achieve these kind of accuracies with purely numerical methods, but perhaps less radical values of f_{penalty} suffice, as shown in Table 4 for the simplified $(D,\ N) = (10,\ 4)$ bosonic master-field equation.

For the $(D,\ N) = (10,\ 4)$ case, let us now discuss the possible appearance of a band-diagonal structure, after one of the master-field matrices has been diagonalized and ordered. First, we consider the results of the simplified $(D,\ N) = (10,\ 4)$ bosonic master-field equation in Table 4.

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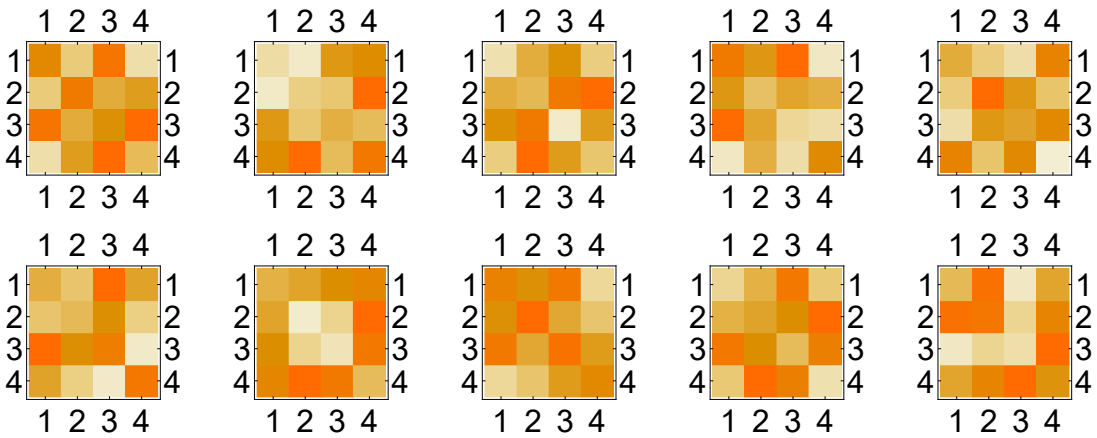

Figure 3: Results from the simplified $(D,\, N) = (10,\, 4)$ bosonic master-field equation (3.5) with $F = 0$ and the pseudorandom constants given by (C.1) and (C.2). Shown are the density plots of $\text{Abs}\big[\widehat{a}^{\,\mu}_{\kappa\text{-}\underline{\text{num-sol-simpl}}}\big]$ from the approximate solution having $f_{\text{penalty}} = 2.78202 \times 10^{-9}$, as given by Table 3. The panels on the top row are for $\mu = 1, \ldots, 5$ and those on the bottom row for $\mu = 6, \ldots, 10$.

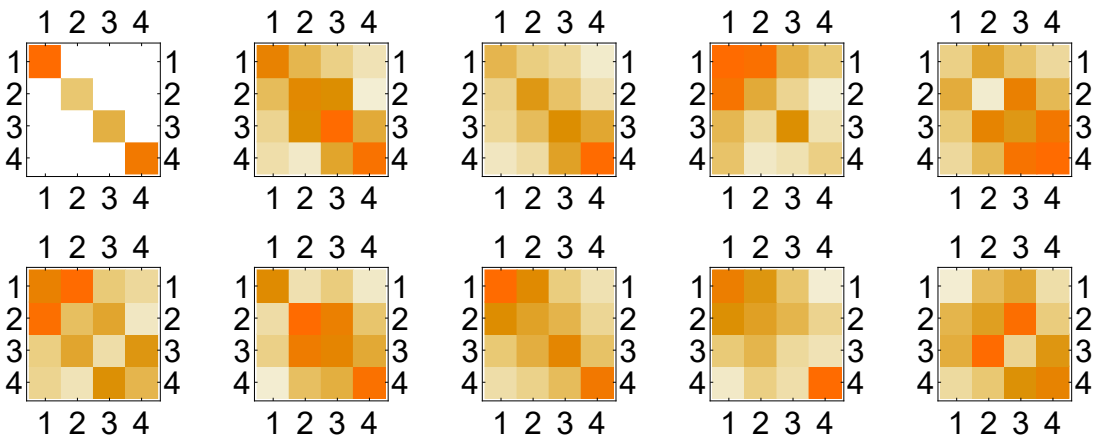

Figure 4: Density plots of $\text{Abs}\big[\widehat{a}'^{\,\mu}_{\kappa\text{-}\underline{\text{num-sol-simpl}}}\big]$ from the matrices of Fig. 3 with a change of basis to diagonalize and order one of them ($\mu = 1$).

Specifically, let us look for a diagonal/band-diagonal pattern in the density plots of Fig. 4 and count how many have a band-diagonal pattern and how many not (having instead a "scattered" pattern, so that these matrices perhaps do not contribute to an emerging classical spacetime):

$$10 \;=\; 1\,\Big[\text{diagonal}\Big] + n_{\text{b-d}}\,\Big[\text{band-diagonal}\Big] + (9 - n_{\text{b-d}})\,\Big[\text{scattered}\Big]\,, \tag{4.8a}$$

$$n_{\text{b-d}} \;\in\; \big\{0,\, 1,\, 2,\, \ldots, 9\big\}\,. \tag{4.8b}$$

Setting the band-diagonality criterium at 1.50, we obtain the following value for the number of band-diagonal dimensions from the best approximate numerical solution of Table 4:

$$n_{\text{b-d}}\,\Big|^{f_{\text{penalty}}=3.02994\times10^{-9},\; R_\mu>1.50}_{\text{simpl-alg-eq}} = 9\,, \quad \text{for } \mu = 2,\, \ldots,\, 9\,. \tag{4.9}$$

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Table 5: Numerical results from the bosonic master-field equation (3.5) for $N = 3$ and $N = 4$, at different values of D and with ($F = 1$) or without ($F = 0$) dynamic fermions. The hash superscript on "max" indicates the restriction to non-diagonalized directions ($\mu \neq 1$ in the present paper). The ratio "R" is defined by (4.2) for $N = 4$ and (E.10) for $N = 3$.

$(D,\, N)$	algebraic eq. (3.5)	f_{penalty}	$\max^{\#}\left(R_\mu\right)$	source
(3, 3)	simplified ($F = 0$)	$\mathrm{O}\left(10^{-69}\right)$	O (10)	Sec. 4.2 of Ref. [17]
(3, 3)	full ($F = 1$)	$\mathrm{O}\left(10^{-64}\right)$	O (5)	Sec. 4.3 of Ref. [17]
(2, 4)	simplified ($F = 0$)	$\mathrm{O}\left(10^{-36}\right)$	O (10)	Sec. 4.1 of Ref. [16]
(10, 4)	simplified ($F = 0$)	$\mathrm{O}\left(10^{-9}\right)$	O (6)	Sec. 4.3 here
(10, 4)	full ($F = 1$)	O (10)	O (1)	Sec. 4.2 here

A first conjecture is that, without dynamical fermions, all nine non-diagonalized matrices obtain some form of band-diagonal structure.

Second, we turn to the results of the full $(D,\, N) = (10,\, 4)$ bosonic master-field equation in Table 2. Taking the results on the last row at face value, we set the criterium for band-diagonality at an *ad hoc* value of 1.10 and get

$$n_{\text{b-d}}\Big|_{\text{full-alg-eq}}^{f_{\text{penalty}}=33.8776,\ R_\mu>1.10} \stackrel{?}{=} 3\,, \quad \text{for } \mu = 2,\, 4,\, 7\,. \tag{4.10}$$

Remark that the same three directions are singled out by the numerical solution with $f_{\text{penalty}} = 108.094$ in Table 2. In order to confirm or disprove the tentative result from (4.10), we need to improve the numerical solution by reducing f_{penalty} significantly (i.e., f_{penalty} values far below 10). Recall that we needed f_{penalty} values of order 10^{-4} or better for the $(D,\, N) = (10,\, 4)$ simplified-algebraic-equation results in Table 4, but perhaps the full-algebraic-equation results stabilize for somewhat larger f_{penalty} values.

A second conjecture is that a 3+6 split of the non-diagonalized matrices (with $n_{\text{b-d}} = 3$) requires nontrivial fermionic dynamics and possibly supersymmetry.

5. Conclusion

In this article, we have explored the hypothesis that a new phase replaces the Friedmann big bang singularity resulting from our current theories, general relativity and the standard model of elementary particle physics. For such a new phase, we need a new theory which extends general relativity and the standard model, and we have used nonperturbative superstring theory in the guise of the IIB matrix model [7, 8]. The model consists of $N \times N$ traceless Hermitian matrices, with 10 bosonic matrices and 16 fermionic matrices.

The first task at hand is to determine how the IIB matrix model gives rise to a classical spacetime. It appears that the required information is encoded in the master-field matrices of the model [13]. The next task is to calculate these master-field matrices and to determine what type of spacetime they give. This is, of course, extremely difficult. Still, we have been able to determine the needed master-field equation and to show the existence of nontrivial solutions for relatively small

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values of the matrix size N. However, further progress at large values of N appears to be hard and perhaps new (analytic) insights may be called for.

If the IIB matrix model indeed gives a new phase replacing the big bang, then we not only need to get an emerging spacetime but also emergent matter. It may be that matter fields appear as appropriate perturbations of the master field. These perturbations must have a very special structure, so that genuine fields appear in the infrared [for example, a scalar field $\phi(x)$ with the proper spacetime dependence]. This special structure is clarified by the toy-model calculation of App. A in Ref. [13].

Let us now return to an issue briefly mentioned in Sec. 3.3, namely that of a length scale. In fact, it may be that the proper IIB matrix model has only dimensionless matrices A^μ and Ψ_α without length scale whatsoever and that the IIB matrix model produces a phase with conformal symmetry. This would fit in nicely with the recent suggestion [28] of a "tamed" big bang as a topological quantum phase transition. Then, the Friedmann big bang singularity would be replaced by a gapless phase which evolves into a gapped state corresponding to our present Universe. The dimensionless IIB matrix model could give rise to such a gapless phase and perhaps also to those "matter" fields (e.g., a 3-form gauge field) that produce the q-type vacuum variable. The initial evolution of the universe would then be driven by the motion of $q(t)$ away from the starting configuration $q = 0$, with normal matter (e.g., the quarks and leptons of the standard model) created from later $q(t)$ oscillations [28]. The subsequent dynamics should also provide a length scale, an energy scale, and a mass scale by breaking the conformal symmetry [29–31].

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A. Large-N factorization

A.1 Setup: Two expectation values

We consider the following two bosonic observables:

$$w_{11} \equiv \frac{1}{N}\,\mathrm{Tr}\left(A^1\,A^1\right), \tag{A.1a}$$

$$w_{11+11} \equiv \left[\frac{1}{N}\,\mathrm{Tr}\left(A^1\,A^1\right)\right]\left[\frac{1}{N}\,\mathrm{Tr}\left(A^1\,A^1\right)\right], \tag{A.1b}$$

and wish to calculate their expectation values (with short-hand notations W_{11} and W_{11+11}),

$$W_{11} \equiv \langle w_{11}\rangle = \frac{1}{Z}\int [DA]\,w_{11}\,, \tag{A.2a}$$

$$W_{11+11} \equiv \langle w_{11+11}\rangle = \frac{1}{Z}\int [DA]\,w_{11+11}\,, \tag{A.2b}$$

$$Z = \int dA\,\left(\mathcal{P}_{D,\,N}\right)^F\,e^{-\,S_{\text{bos}}} \equiv \int [DA]\,, \tag{A.2c}$$

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Table 6: Numerical results from the bosonic ($F = 0$) and supersymmetric ($F = 1$) models for $D = 4$. The relevant quantities have been defined in (A.1) and (A.2). All numerical results are only approximative, the least reliable being those of the fourth column with $F = 1$.

	$\{D, N, F\} = \{4, 4, 0\}$	$\{D, N, F\} = \{4, 6, 0\}$	$\{D, N, F\} = \{4, 4, 1\}$
Z	4.70×10^{12}	6.07×10^{21}	5.47×10^{18}
$\int [DA]\, w_{11}$	2.71×10^{12}	4.43×10^{21}	4.19×10^{18}
$\int [DA]\, w_{11+11}$	1.62×10^{12}	3.28×10^{21}	2.89×10^{18}
W_{11}	0.577	0.730	0.767
W_{11+11}	0.345	0.540	0.529

where the measure dA is defined by (2.1a), the bosonic action S_{bos} by (2.1b), and the Pfaffian $\mathcal{P}_{D,\,N}$ by (2.4). The discrete parameter F takes values in $\{0,\ 1\}$.

There are then three multi-dimensional integrals to perform in (A.2), each having $D\,(N^2 - 1)$ dimensions. For these integrals (assumed to be convergent, see below), we use the `NIntegrate` routine of Mathematica 12.1 (cf. Ref. [32]) with the Adapted-Monte-Carlo method and split the calculation into many calculations by taking successive shells. It is important to use a trick to force the routine `NIntegrate` to avoid any algebraic steps and to stay purely numerical. The trick can be explained by a simple example:

```
f[y_] := NIntegrate[2*z, {z, 0, y}];
result = NIntegrate[If[x == 0 || x != 0, f[x]], {x, 0, 1}];
```

where the first line defines a quadratic function f that only works properly for a numerical variable and where the "If" conditional on the second line forces the integration variable x to be numeric.

With the obtained expectation values W_{11} and W_{11+11}, we determine the following quantity:

$$\Delta W_{11+11} \equiv W_{11+11} - (W_{11})^2\,, \tag{A.3}$$

which tests for large-N factorization (3.1).

Our main results are for the $(D,\,N) = (4,\,4)$ bosonic model with $F = 0$. These results have been extended in two "directions": larger matrices ($N = 6$) and the inclusion of the fermion dynamics ($F = 1$). For the two bosonic cases ($F = 0$ and $N = 4,\,6$), the three integrals of (A.2) have been proven to be convergent [21]. For the supersymmetric case ($F = 1$ and $N = 4$), the two integrals (A.2a) and (A.2b) have not been proven to be convergent [21] but they may still be.

A.2 Bosonic model for (D, N) = (4, 4) and (D, N) = (4, 6)

The simplest model we have studied is the $(D,\,N) = (4,\,4)$ bosonic model, where the $SU(4)$ generators are given in App. B. Preliminary numerical results for the quantities defined in Sec. A.1 appear in the second column of Table 6. From these results, we get for the quantity defined by (A.3) the following numerical value:

$$\Delta W_{11+11}^{(D=4,\,N=4,\,F=0)} \approx 0.0121\,, \tag{A.4}$$

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which shows the approximate equality of $\langle w_{11+11} \rangle$ and $\langle w_{11} \rangle \langle w_{11} \rangle$ at the 4 % level.

We can extend the model by going to a larger matrix size, $N = 6$. The $SU(6)$ generators are then given by the 15 generators from App. B embedded in 6×6 matrices and 20 additional generators. From the results given in the third column of Table 6, we get

$$\Delta W_{11+11}^{(D=4,\, N=6,\, F=0)} \approx 0.0071\,, \tag{A.5}$$

which shows a cancellation at the 1 % level.

A.3 Supersymmetric model for (D, N) = (4, 4)

For $D = 4$, the Pfaffian can be written as the determinant of a $2\,(N^2-1) \times 2\,(N^2-1)$ complex matrix [19]:

$$\mathcal{P}_{4,\,N} = \det \begin{pmatrix} \mathbf{X}_4 + i\,\mathbf{X}_3 & i\,\mathbf{X}_2 + \mathbf{X}_1 \\ i\,\mathbf{X}_2 - \mathbf{X}_1 & \mathbf{X}_4 - i\,\mathbf{X}_3 \end{pmatrix}, \tag{A.6}$$

in terms of matrices $\mathbf{X}_\mu$ in the adjoint representation of $SU(N)$,

$$\left(\mathbf{X}_\mu\right)^{ab} = f^{abc}\, A_\mu^c\,, \tag{A.7a}$$

$$f^{abc} = -2\,i\,\mathrm{Tr}\left(A^a \left[A^b,\, A^c\right]\right), \tag{A.7b}$$

where the f^{abc} are the $SU(N)$ structure constants. The Pfaffian for the $(D,\, N) = (4,\, 4)$ case is then given by the determinant of a 30×30 complex matrix. This determinant cannot be calculated algebraically but can be evaluated numerically (for this reason, we must force the routine `NIntegrate` to stay purely numerical, as discussed in App. A.1).

From the results given in the fourth column of Table 6, we get

$$\Delta W_{11+11}^{(D=4,\, N=4,\, F=1)} \approx -0.059\,, \tag{A.8}$$

which shows a cancellation at the 11 % level. Incidentally, there is no problem with obtaining a negative number, which may anyway still change to a positive number with increasing accuracy.

A.4 Discussion of the factorization results

It is instructive to compare our results (A.4) and (A.5) for the $D = 4$ bosonic model and to write them as follows:

$$\Delta W_{11+11}^{(D=4,\, N=4,\, F=0)} \approx \frac{0.194}{16}\,, \tag{A.9a}$$

$$\Delta W_{11+11}^{(D=4,\, N=6,\, F=0)} \approx \frac{0.256}{36}\,. \tag{A.9b}$$

This shows that we have mild evidence for an $1/N^2$ behavior of the remnant term, with an $1/N^2$ coefficient for $N = 4$ and $N = 6$ that has the same sign (plus) and is of the same order of magnitude (one tenth). For the moment, we do not have similar results for the $D = 4$ supersymmetric model.

As mentioned in Sec. 3.1, Ref. [25] has already obtained extensive numerical results in support of large-N factorization, also for the four-dimensional version of the IIB matrix model. These

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numerical results, which could even reach a matrix size of $N = 48$, considered Wilson-loop-type and Polyakov-line-type observables,

$$o_{\text{loop}}(k) \equiv \frac{1}{N}\,\text{Tr}\left(e^{i\,k\,A^1}\,e^{i\,k\,A^2}\,e^{-i\,k\,A^1}\,e^{-i\,k\,A^2}\right), \tag{A.10a}$$

$$o_{\text{line}}(k) \equiv \frac{1}{N}\,\text{Tr}\left(e^{i\,k\,A^1}\right), \tag{A.10b}$$

for a real dimensionless parameter k. In the above observables, there appear the $SU(N)$ group elements $e^{i\,k\,A^\mu}$, as may be appropriate for the study of a Yang–Mills gauge theory with, for example, the "area-law" behavior of $\langle o_{\text{loop}}\rangle$ as shown in Fig. 4 of Ref. [25]. We have, instead, considered observables (A.1) directly made out products of the Lie-algebra elements A^μ, as may be appropriate for the study of the emergence of spacetime. In any case, the results for both types of observables, even with a smaller number of dimensions ($D = 4$) than needed ($D = 10$), confirm the property of large-N factorization (3.1), which is crucial for the existence of a large-N bosonic master field as discussed in Sec. 3.

B. SU(4) generators

We now give the explicit realization used for the $SU(4)$ generators:

$$t_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad t_2 = \frac{1}{2}\begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad t_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$t_4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad t_5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad t_6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix},$$

$$t_7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad t_8 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad t_9 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$t_{10} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad t_{11} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad t_{12} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix},$$

$$t_{13} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad t_{14} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad t_{15} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{B.1}$$

These generators obey the trace condition (2.1e).

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C. Pseudorandom numbers for (D, N) = (10, 4)

In this appendix, we give the particular realization (the "κ-realization") of the pseudorandom numbers used for the approximate numerical solutions of Sec. 4.

Specifically, we take the following 4 real pseudorandom numbers $\widehat{p}_\mu$ for the master momenta:

$$\widehat{p}_{\kappa\text{-realization}} = \left\{-\frac{111}{250}, \frac{19}{200}, -\frac{63}{200}, \frac{189}{1000}\right\}, \tag{C.1}$$

and the following 150 real pseudorandom numbers $\widehat{\eta}_c^\mu$ entering the Hermitian master-noise matrices:

$$\begin{aligned}\{\widehat{\eta}_1^1, \ldots, \widehat{\eta}_{15}^1\}\big|_{\kappa\text{-realization}} = \Big\{&\frac{1}{1000}, -\frac{9}{20}, \frac{353}{1000}, \frac{12}{25}, -\frac{987}{1000}, -\frac{51}{200}, \frac{1}{4}, -\frac{63}{1000},\\ &-\frac{131}{200}, -\frac{367}{1000}, -\frac{169}{200}, \frac{171}{250}, -\frac{151}{250}, -\frac{369}{500}, \frac{593}{1000}\Big\},\end{aligned} \tag{C.2a}$$

$$\begin{aligned}\{\widehat{\eta}_1^2, \ldots, \widehat{\eta}_{15}^2\}\big|_{\kappa\text{-realization}} = \Big\{&-\frac{153}{250}, -\frac{47}{250}, \frac{897}{1000}, -\frac{61}{500}, \frac{269}{1000}, -\frac{237}{1000}, \frac{1}{125}, -\frac{13}{100},\\ &-\frac{103}{200}, \frac{367}{500}, \frac{1}{10}, -\frac{71}{1000}, \frac{69}{1000}, -\frac{123}{125}, -\frac{17}{25}\Big\},\end{aligned} \tag{C.2b}$$

$$\begin{aligned}\{\widehat{\eta}_1^3, \ldots, \widehat{\eta}_{15}^3\}\big|_{\kappa\text{-realization}} = \Big\{&\frac{7}{100}, \frac{431}{1000}, \frac{17}{20}, -\frac{59}{125}, -\frac{437}{1000}, \frac{69}{500}, -\frac{991}{1000}, -\frac{49}{125},\\ &\frac{83}{500}, -\frac{279}{1000}, \frac{49}{125}, \frac{121}{250}, \frac{313}{500}, \frac{871}{1000}, \frac{7}{125}\Big\},\end{aligned} \tag{C.2c}$$

$$\begin{aligned}\{\widehat{\eta}_1^4, \ldots, \widehat{\eta}_{15}^4\}\big|_{\kappa\text{-realization}} = \Big\{&-\frac{11}{125}, -\frac{469}{1000}, -\frac{439}{500}, -\frac{483}{1000}, -\frac{11}{25}, \frac{41}{200}, \frac{409}{500}, -\frac{343}{1000},\\ &\frac{3}{8}, -\frac{407}{500}, -\frac{141}{250}, -\frac{31}{1000}, \frac{293}{500}, -\frac{12}{125}, \frac{429}{1000}\Big\},\end{aligned} \tag{C.2d}$$

$$\begin{aligned}\{\widehat{\eta}_1^5, \ldots, \widehat{\eta}_{15}^5\}\big|_{\kappa\text{-realization}} = \Big\{&\frac{74}{125}, \frac{249}{500}, -\frac{511}{1000}, -\frac{43}{250}, -\frac{129}{1000}, \frac{19}{100}, \frac{49}{200}, -\frac{463}{500},\\ &\frac{23}{100}, \frac{433}{1000}, \frac{821}{1000}, \frac{199}{1000}, -\frac{96}{125}, -\frac{92}{125}, \frac{183}{250}\Big\},\end{aligned} \tag{C.2e}$$

$$\begin{aligned}\{\widehat{\eta}_1^6, \ldots, \widehat{\eta}_{15}^6\}\big|_{\kappa\text{-realization}} = \Big\{&-\frac{143}{250}, \frac{2}{5}, \frac{921}{1000}, -\frac{313}{500}, -\frac{603}{1000}, -\frac{449}{1000}, \frac{609}{1000}, -\frac{39}{1000},\\ &\frac{443}{1000}, \frac{383}{500}, -\frac{17}{500}, \frac{27}{1000}, \frac{181}{500}, \frac{941}{1000}, -\frac{181}{1000}\Big\},\end{aligned} \tag{C.2f}$$

$$\begin{aligned}\{\widehat{\eta}_1^7, \ldots, \widehat{\eta}_{15}^7\}\big|_{\kappa\text{-realization}} = \Big\{&-\frac{129}{250}, -\frac{147}{1000}, \frac{387}{1000}, -\frac{611}{1000}, \frac{313}{1000}, \frac{191}{200}, -\frac{61}{250}, -\frac{11}{25},\\ &-\frac{969}{1000}, \frac{927}{1000}, \frac{489}{500}, \frac{361}{500}, \frac{57}{1000}, -\frac{4}{125}, \frac{83}{100}\Big\},\end{aligned} \tag{C.2g}$$

$$\begin{aligned}\{\widehat{\eta}_1^8, \ldots, \widehat{\eta}_{15}^8\}\big|_{\kappa\text{-realization}} = \Big\{&-\frac{273}{1000}, -\frac{441}{1000}, \frac{427}{500}, -\frac{317}{1000}, -\frac{407}{1000}, -\frac{57}{200}, \frac{23}{1000}, \frac{93}{200},\\ &\frac{163}{250}, \frac{36}{125}, \frac{333}{1000}, -\frac{18}{25}, \frac{321}{1000}, \frac{307}{500}, -\frac{193}{1000}\Big\},\end{aligned} \tag{C.2h}$$

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$$\left\{\widehat{\eta}_1^9\,,\,\ldots\,,\,\widehat{\eta}_{15}^9\right\}\Big|_{\kappa\text{-realization}} = \left\{\frac{1}{250},\,-\frac{2}{125},\,-\frac{981}{1000},\,\frac{112}{125},\,-\frac{137}{500},\,\frac{547}{1000},\,-\frac{201}{500},\,\frac{101}{500},\right.$$
$$\left.-\frac{63}{250},\,\frac{62}{125},\,\frac{597}{1000},\,-\frac{114}{125},\,-\frac{39}{200},\,\frac{197}{200},\,-\frac{7}{100}\right\}, \tag{C.2i}$$

$$\left\{\widehat{\eta}_1^{10}\,,\,\ldots\,,\,\widehat{\eta}_{15}^{10}\right\}\Big|_{\kappa\text{-realization}} = \left\{\frac{969}{1000},\,\frac{4}{5},\,-\frac{243}{1000},\,\frac{29}{250},\,-\frac{38}{125},\,-\frac{151}{500},\,\frac{17}{25},\,\frac{119}{1000},\right.$$
$$\left.\frac{463}{1000},\,-\frac{33}{200},\,\frac{257}{500},\,\frac{219}{250},\,-\frac{319}{1000},\,\frac{119}{500},\,\frac{577}{1000}\right\}. \tag{C.2j}$$

The corresponding matrices $\widehat{\eta}^{\mu}_{\kappa\text{-realization}}$ have been given in App. B of Ref. [18].

Remark that, following Ref. [16], we have chosen *rational* numbers for the random constants in (C.1) and (C.2). The reason is that we can then easily write down their *exact* values, whereas arbitrary real numbers would require an infinite number of digits (or implicit defining relations, such as for the irrational numbers $\sqrt{3}$ and π).

D. Coefficients from the full algebraic equation for (D, N) = (10, 4)

Denoting the approximate numerical solution from the last row of Table 1 by "κ-num-sol", the 300 real numbers defining the matrices (4.3) are [displaying ten batches of 30 numbers each]:

$$\left\{\widehat{r}_1^1\,,\,\widehat{s}_1^1\,,\,\ldots\,,\,\widehat{r}_1^{15}\,,\,\widehat{s}_1^{15}\,,\,\widehat{r}_2^1\,,\,\widehat{s}_2^1\,,\,\ldots\,,\,\widehat{r}_{10}^{15}\,,\,\widehat{s}_{10}^{15}\right\}\Big|_{\kappa\text{-num-sol}}^{(f_{\text{penalty}}=33.8776)} =$$

$\{-0.0692546, -0.58643, -1.05642, -0.601179, -0.148017, 0.607459,$
$-0.295285, -0.140475, -0.59204, 0.274455, -0.508568, 0.309931,$
$-0.0620355, 0.381277, 0.211426, 0.178033, -0.112937, -0.138966,$
$0.469841, 0.463514, -0.00762073, -0.155881, 0.0143421, 0.466782,$
$0.580538, 0.439774, 0.0995092, -0.622271, -0.68924, -0.437244,$

$0.426018, -0.509828, -0.439085, 0.293402, 0.486298, -0.658821,$
$-0.219118, 0.0364208, 0.824188, 0.429625, 0.276073, 0.179746,$
$-0.068707, 0.0925217, -0.118121, -0.186855, -0.00933079, 0.219118,$
$0.520674, 0.762843, 0.0327783, 0.240407, -0.730098, -0.499508,$
$-0.0487667, 0.269917, 0.328138, 0.133357, -0.226639, -0.21064,$

$0.266147, 0.0693405, -0.0752673, 0.324464, -0.468063, -0.537566,$
$-0.522858, -0.719695, -0.0162757, -0.289673, 0.0600267, -0.473248,$
$0.565031, 0.0352853, 0.609935, 0.341532, 0.251534, -0.141831,$
$-0.145027, -0.146535, 0.020147, 0.0975357, 0.143888, -0.389464,$
$0.00800391, -0.44477, -0.491185, -0.120212, -0.521562, 0.0655693,$

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−0.769702, 0.393627, 0.0849714, −0.57002, −0.00425748, 0.154496,
0.487595, −0.0372879, −0.342562, 0.289481, −0.0750309, 0.611541,
0.163813, −0.17035, −0.0421598, 0.165941, 0.451901, −0.353845,
−0.065826, 0.708222, 0.294372, 0.425108, 0.384654, 0.0226756,
0.570911, −0.324845, −0.261706, 0.035435, −0.483642, 0.251261,

−0.342193, −0.50229, 0.125031, −0.64019, −0.466247, 0.53052,
0.275081, −0.730265, 0.853657, −0.121877, −0.85128, −0.246815,
0.0756539, 0.120902, 0.268202, 0.403389, −0.685643, 0.503552,
0.0864294, −0.224326, 0.352859, −0.248242, −0.0473426, 0.0953277,
0.23623, 0.39878, 0.407025, 0.376143, 0.0485194, 0.0974565,

−0.241239, 0.0523708, −0.0378564, 0.394186, −0.0230116, 0.285019,
0.376322, −0.14455, 0.141922, 0.284971, 0.214979, −0.650774,
0.160804, −0.0462938, 0.325296, 0.337869, −0.0518918, −0.273317,
−0.44725, −0.187702, −0.452864, −0.00378001, −0.388691, 0.358931,
0.382993, −0.866705, 0.255596, 0.631997, 0.013364, −0.311463,

−0.457347, 0.289158, −0.344167, 0.0958883, 0.246078, −0.254109,
−0.68349, 0.111889, 0.391893, −0.360248, −0.0215233, −0.259903,
0.543266, 0.0653004, −0.653161, 0.270118, −0.436127, 0.19117,
−0.588626, 0.188583, 0.684876, −0.0178182, 0.323925, −0.412551,
0.255511, −0.197965, −0.25607, 0.110931, 0.192163, −0.111291,

−0.141328, 0.270626, −0.417135, 0.0142941, −0.524982, −0.269947,
−0.636919, −0.237791, 0.367274, 0.809111, 0.202189, 0.314854,
−0.237622, 0.229124, −0.472898, 0.190654, −0.216502, 0.559077,
0.18242, 0.657434, 0.719802, −0.307403, −0.0636525, −0.511735,
0.43477, 0.231476, 0.832258, 0.131869, 0.436263, 0.666609,

−0.1219, −0.0734223, 0.29402, 0.248577, −0.101651, −0.0870295,
−0.315557, 0.292283, −0.347289, −0.0371514, −0.265792, 0.290594,
0.140305, 0.45099, 0.296089, 0.392803, −0.0969607, −0.0578319,
0.62338, 0.361966, 0.422547, 0.570988, −0.588562, 0.41253,
0.0313179, 0.0259956, −0.361538, −0.730508, −0.43556, 0.133992,

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$$
\begin{aligned}
&-0.494957, -0.389868, -0.953034, -0.177046, -0.358249, 0.186803,\\
&-0.522166, 0.0464763, 0.355931, -0.048642, -0.105042, -0.283861,\\
&-0.216209, -0.0980749, 0.622007, 0.707796, 0.204977, -0.505334,\\
&-0.233755, 0.7281, 0.372945, 0.588971, -0.176769, 0.422182,\\
&-0.102989, 0.366496, -0.743647, 0.526077, -0.350589, -0.045876\big\}\,, \qquad \text{(D.1)}
\end{aligned}
$$

where up to 6 significant digits are shown. The superscript with the f_{penalty} value has been omitted in the main text.

Just in order to avoid any misunderstanding: the above 300 numbers are only given for illustrative purposes, because they are, most likely, different from the correct values (as mentioned in the last paragraph of Sec. 4.2).

E. Nontrivial critical points for the case (D, N) = (3, 3)

E.1 Critical-point setup: General case

This appendix aims at being more or less self-contained. Consider a generalized version of the IIB matrix model [7, 8] with a different number of bosonic matrices ($D = 3,\ 4,\ 6,\ 10$) and different matrix sizes ($N \geq 2$), whereas the genuine model has $D = 10$ and $N \gg 1$. After the fermionic matrices have been integrated out, the partition function reads [7, 8, 19]

$$
\begin{aligned}
Z_{D,\,N} &= \int \prod_{c=1}^{g} \prod_{\mu=1}^{D} dA^c_\mu \; e^{-\,S_{\text{eff},\,D,\,N}[A]}\,, && \text{(E.1a)}\\
S_{\text{eff},\,D,\,N}[A] &= S_{\text{bos},\,D,\,N}[A] - \log \mathcal{P}_{D,\,N}[A]\,, && \text{(E.1b)}\\
S_{\text{bos},\,D,\,N}[A] &= -\frac{1}{2}\,\text{Tr}\Big(\big[A^\mu,\,A^\nu\big]\,\big[A^\mu,\,A^\nu\big]\Big)\,, && \text{(E.1c)}\\
A_\mu &= A^c_\mu\, t_c\,, \quad A^c_\mu \in \mathbb{R}\,, \quad t_c \in \text{su}(N)\,, && \text{(E.1d)}\\
\text{Tr}\left(t_c \cdot t_d\right) &= \frac{1}{2}\,\delta_{cd}\,, && \text{(E.1e)}\\
g &\equiv N^2 - 1\,, && \text{(E.1f)}
\end{aligned}
$$

where repeated Greek indices are summed over (just as having an implicit Euclidean "metric") and the quantity $\mathcal{P}_{D,\,N}$ in (E.1b) will be discussed shortly. The commutators entering the bosonic action term (E.1c) are defined by $[X,\,Y] \equiv X \cdot Y - Y \cdot X$ for square matrices X and Y of equal dimension. The expansion (E.1d), for real coefficients A^c_μ, uses the $N \times N$ traceless Hermitian $SU(N)$ generators t_c with normalization (E.1e).

The Gaussian-type integration of the fermionic matrices, for $D = 3,\ 4,\ 10$, produces the Pfaffian $\mathcal{P}_{D,\,N}[A]$, which is given explicitly by a sum over permutations or by a sum involving the Levi–Civita symbol. Including the $D = 6$ case, the quantity $\mathcal{P}_{D,\,N}[A]$ is a homogenous polynomial in the bosonic coefficients A^c_μ, where the order K, for values $D \in \{3,\ 4,\ 6,\ 10\}$ with supersymmetry, is given by

$$
K = \left(D - 2\right)\left(N^2 - 1\right). \qquad \text{(E.2)}
$$

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Further discussion of the polynomial $\mathcal{P}_{D,\,N}[A]$ appears in Sec. 2.1 and Refs. [19, 20].

The issue of the convergence of the integrals in (E.1a) has been studied by the authors of Ref. [21], with the conclusion that there is absolute convergence for $D = 4,\, 6,\, 10$. In any case, it may be of mathematical interest to look for the critical points (even for $D = 3$) and, more precisely, to establish their existence. Incidentally, critical points of the matrix model have also been discussed in a recent paper [27], which, however, appears to consider only critical points of the bosonic action (E.1c).

Here, we present results for the existence of nontrivial critical points of the effective bosonic action (E.1b). Specifically, we get explicit solutions for a special case with low values of D and N. Remark that this effective action incorporates the fermionic "quantum fluctuations" exactly.

E.2 Critical-point setup: Special case with (D, N) = (3, 3)

Consider the matrix model (E.1) with the following parameters:

$$\{D,\, N\} = \{3,\, 3\}\,. \tag{E.3}$$

The eight generators t_c are proportional to the 3×3 Gell-Mann matrices λ_c from the "eightfold-way" (1961) and we take explicitly

$$\widehat{t}_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \widehat{t}_2 = \frac{1}{2}\begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \widehat{t}_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\widehat{t}_4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \qquad \widehat{t}_5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \widehat{t}_6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix},$$

$$\widehat{t}_7 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \widehat{t}_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \tag{E.4}$$

where the hat distinguishes these generators from those in App. B.

The main reason for considering this special case is that there is now an explicit compact result for the Pfaffian [19]:

$$\begin{aligned} \mathcal{P}_{3,\,3}[A] \;=\; & -\frac{3}{4}\,\mathrm{Tr}\Big(\big[A^\mu,\, A^\nu\big]\,\{A^\rho,\, A^\sigma\}\Big)\,\mathrm{Tr}\Big(\big[A^\mu,\, A^\nu\big]\,\{A^\rho,\, A^\sigma\}\Big) \\ & +\frac{6}{5}\,\mathrm{Tr}\Big(A^\mu\,\big[A^\nu,\, A^\rho\big]\Big)\,\mathrm{Tr}\Big(A^\mu\,\big[\{A^\nu,\, A^\sigma\},\, \{A^\rho,\, A^\sigma\}\big]\Big), \end{aligned} \tag{E.5}$$

which corresponds to a homogenous eighth-order polynomial in the bosonic coefficients A^c_μ. This expression also contains anticommutators, defined by $\{X,\, Y\} \equiv X\cdot Y + Y\cdot X$ for square matrices X and Y of equal dimension.

The critical points are then obtained from the following algebraic equation:

$$\frac{\delta S_{\text{eff},\,3,\,3}\big[\,A\,\big]}{\delta\, A^{\,\mu}_{\,lk}} \;=\; \Big[A^{\,\nu},\, \big[A^{\,\nu},\, A^{\,\mu}\big]\Big]_{kl} - \frac{1}{\mathcal{P}_{3,\,3}\,(A)}\,\frac{\partial\,\mathcal{P}_{3,\,3}\,(A)}{\partial\, A^{\,\mu}_{\,lk}} = 0\,, \tag{E.6}$$

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with expression (E.5) for the Pfaffian. Multiplying (E.6) by $\mathcal{P}_{3,\,3}\,(A)$ gives an 11-th order homogeneous polynomial.

Explicit solutions of (E.6) can be obtained from the procedure of Ref. [17] (see, in particular, the last paragraph of Sec. 4.1 in that reference) by setting the master momenta and the master noise there to zero, $\widehat{p}_k = 0$ and $\widehat{\eta}^{\,\mu}_{kl} = 0$. Very briefly, the procedure is to build a penalty function as the sum of the squares of the 24 component equations (without further overall numerical factors) and to use the numerical minimization routine `FindMinimum` from Mathematica 12.1 (cf. Ref. [32]). Typically, we use a 36-digit working precision (see Sec. E.3.3 for further comments).

E.3 Critical-point results for (D, N) = (3, 3)

E.3.1 First critical-point solution

The coefficients of a first nontrivial solution (denoted by an overbar) of the algebraic equation (E.6) are:

$$\left\{\begin{matrix} \overline{A}^{\,1}_1\,, & \overline{A}^{\,2}_1 \\ \overline{A}^{\,3}_1\,, & \overline{A}^{\,4}_1 \\ \overline{A}^{\,5}_1\,, & \overline{A}^{\,6}_1 \\ \overline{A}^{\,7}_1\,, & \overline{A}^{\,8}_1 \end{matrix}\right\} = \left\{\begin{matrix} 0.570574083490128575476108\,, & 0.658442593948256740170834 \\ 0.591172615092998049901068\,, & 0.459965150810118963329943 \\ 0.027603429537029960433096 0\,, & -0.662039140319935188116796 \\ -0.151421059211537951401624\,, & 1.033311376176621679214 42 \end{matrix}\right\}, \tag{E.7a}$$

$$\left\{\begin{matrix} \overline{A}^{\,1}_2\,, & \overline{A}^{\,2}_2 \\ \overline{A}^{\,3}_2\,, & \overline{A}^{\,4}_2 \\ \overline{A}^{\,5}_2\,, & \overline{A}^{\,6}_2 \\ \overline{A}^{\,7}_2\,, & \overline{A}^{\,8}_2 \end{matrix}\right\} = \left\{\begin{matrix} 0.809211845911449987314691\,, & 0.833691639722612054210554 \\ 0.617000774572453785555974\,, & -0.164275893241917159317804 \\ 0.830645113307734538465843\,, & -0.340131664055950118091891 \\ 0.644115685767825480672465\,, & 1.544396729389621169949 17 \end{matrix}\right\}, \tag{E.7b}$$

$$\left\{\begin{matrix} \overline{A}^{\,1}_3\,, & \overline{A}^{\,2}_3 \\ \overline{A}^{\,3}_3\,, & \overline{A}^{\,4}_3 \\ \overline{A}^{\,5}_3\,, & \overline{A}^{\,6}_3 \\ \overline{A}^{\,7}_3\,, & \overline{A}^{\,8}_3 \end{matrix}\right\} = \left\{\begin{matrix} 0.504330365038665104746314\,, & 1.183381209631055128937 47 \\ 0.087965265502996901688 1395\,, & 0.416121944105811292070174 \\ 0.614213738185278850877443\,, & -0.945948255419088870571396 \\ 0.811508170531654323184674\,, & 0.926928461994400651864667 \end{matrix}\right\}, \tag{E.7c}$$

where only 24 significant digits have been shown. All information of this particular solution is contained in the 24 real numbers $\overline{A}^{\,c}_{\mu}$. Let us now have a closer look at what the nature of this solution is, while showing less digits than above.

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From (E.7), we have the following three matrices:

$$\overline{A}_1 = \begin{pmatrix} 0.223 & 0.285 + 0.329\,i & 0.296 + 0.230\,i \\ 0.285 - 0.329\,i & 0.374 & 0.014 - 0.331\,i \\ 0.296 - 0.230\,i & 0.014 + 0.331\,i & -0.597 \end{pmatrix}, \tag{E.8a}$$

$$\overline{A}_2 = \begin{pmatrix} 0.768 & 0.405 + 0.417\,i & 0.309 - 0.082\,i \\ 0.405 - 0.417\,i & 0.124 & 0.415 - 0.170\,i \\ 0.309 + 0.082\,i & 0.415 + 0.170\,i & -0.892 \end{pmatrix}, \tag{E.8b}$$

$$\overline{A}_3 = \begin{pmatrix} 0.673 & 0.252 + 0.592\,i & 0.044 + 0.208\,i \\ 0.252 - 0.592\,i & -0.138 & 0.307 - 0.473\,i \\ 0.044 - 0.208\,i & 0.307 + 0.473\,i & -0.535 \end{pmatrix}, \tag{E.8c}$$

and the corresponding matrices with absolute values of the entries (in a short-hand notation):

$$\text{Abs}\left[\overline{A}_1\right] = \begin{pmatrix} 0.223 & 0.436 & 0.375 \\ 0.436 & 0.374 & 0.331 \\ 0.375 & 0.331 & 0.597 \end{pmatrix}, \tag{E.9a}$$

$$\text{Abs}\left[\overline{A}_2\right] = \begin{pmatrix} 0.768 & 0.581 & 0.319 \\ 0.581 & 0.124 & 0.449 \\ 0.319 & 0.449 & 0.892 \end{pmatrix}, \tag{E.9b}$$

$$\text{Abs}\left[\overline{A}_3\right] = \begin{pmatrix} 0.673 & 0.643 & 0.213 \\ 0.643 & 0.138 & 0.564 \\ 0.213 & 0.564 & 0.535 \end{pmatrix}. \tag{E.9c}$$

Inspection of the matrices (E.9), in particular, shows that the far-off-diagonal entries at positions $[1,\,3]$ and $[3,\,1]$ are not especially small.

We can quantify this conclusion by calculating the average band-diagonal value from 2+3+2 entries in (E.9) and the average off-band-diagonal value from 1+1 entries. Then, determine the ratio $\widehat{R}_\mu$ of the average band-diagonal value over the average off-band-diagonal value. Summarizing this procedure by an equation, we have the following definition of the ratio $\widehat{R}$ for a symmetric 3×3 matrix $\widehat{M}$ with nonnegative entries $\widehat{m}[i,\,j]$:

$$\widehat{R} \equiv \frac{1}{7}\left(\sum_{i=1}^{3} \widehat{m}[i,\,i] + 2\sum_{j=1}^{2} \widehat{m}[j,\,j+1]\right)\frac{1}{\widehat{m}[1,\,3]}, \tag{E.10}$$

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where we have used the symmetry of M to simplify the expression. In this way, we get, from the three matrices (E.9), the following three ratios:

$$\left\{\widehat{R}_1,\ \widehat{R}_2,\ \widehat{R}_3\right\}_{\text{Abs}\left[\overline{A}_\mu\right]} = \{1.04,\ 1.72,\ 2.53\}\ , \tag{E.11}$$

which are all of order unity.

Following earlier work [16], we diagonalize and order $\overline{A}_1$ (the transformed matrices are denoted by a prime) and get

$$\overline{A}_1{}' = \begin{pmatrix} -0.835 & 0 & 0 \\ 0 & -0.0169 & 0 \\ 0 & 0 & 0.852 \end{pmatrix}, \tag{E.12a}$$

$$\overline{A}_2{}' = \begin{pmatrix} -0.719 & -0.097+0.481\,i & -0.0586+0.0059\,i \\ -0.097-0.481\,i & -0.356 & 0.362+0.256\,i \\ -0.0586-0.0059\,i & 0.362-0.256\,i & 1.08 \end{pmatrix}, \tag{E.12b}$$

$$\overline{A}_3{}' = \begin{pmatrix} -0.658 & -0.466+0.088\,i & -0.0484-0.0255\,i \\ -0.466-0.088\,i & -0.326 & 0.060+0.413\,i \\ -0.0484+0.0255\,i & 0.060-0.413\,i & 0.984 \end{pmatrix}. \tag{E.12c}$$

It is now clear that, for $\overline{A}_2{}'$ and $\overline{A}_3{}'$, the far-off-diagonal entries at positions [1, 3] and [3, 1] have rather small absolute values. The corresponding ratios are:

$$\left\{\widehat{R}_1,\ \widehat{R}_2,\ \widehat{R}_3\right\}_{\text{Abs}\left[\overline{A}'_\mu\right]} = \{\infty,\ 9.76,\ 9.81\}\ , \tag{E.13}$$

where the last two ratios are of order ten.

Similarly, we can diagonalize and order $\overline{A}_2$ (the transformed matrices are denoted by a double prime) and we get the following ratios:

$$\left\{\widehat{R}_1,\ \widehat{R}_2,\ \widehat{R}_3\right\}_{\text{Abs}\left[\overline{A}''_\mu\right]} = \{12.2,\ \infty,\ 13.1\}\ , \tag{E.14}$$

where both nontrivial ratios are again of order ten. The same result is also obtained if we diagonalize and order $\overline{A}_3$ (the transformed matrices are denoted by a triple prime), with the following ratios:

$$\left\{\widehat{R}_1,\ \widehat{R}_2,\ \widehat{R}_3\right\}_{\text{Abs}\left[\overline{A}'''_\mu\right]} = \{11.3,\ 12.1,\ \infty\}\ . \tag{E.15}$$

The conclusion is that the matrix solution of (E.6) with coefficients (E.7) has, upon diagonalization and ordering of one matrix, a clear diagonal/band-diagonal structure, even for the small matrix size considered, $N = 3$.

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E.3.2 Second critical-point solution

The coefficients of a second nontrivial solution (denoted by a tilde) of the algebraic equation (E.6) are:

$$\begin{Bmatrix} \widetilde{A}_1^1\,,\ \widetilde{A}_1^2 \\ \widetilde{A}_1^3\,,\ \widetilde{A}_1^4 \\ \widetilde{A}_1^5\,,\ \widetilde{A}_1^6 \\ \widetilde{A}_1^7\,,\ \widetilde{A}_1^8 \end{Bmatrix} = \begin{Bmatrix} -0.048611843991979067535190 3\,,\ 0.786258758852912152288640 \\ 0.052906715842560850972220 8\,,\ 0.405878380158241357550552 \\ 0.089432519780555329930594 7\,,\ -0.958079473974395296656234 \\ 0.012306827564406056241443 1\,,\ 0.590582765278256159281746 \end{Bmatrix}, \tag{E.16a}$$

$$\begin{Bmatrix} \widetilde{A}_2^1\,,\ \widetilde{A}_2^2 \\ \widetilde{A}_2^3\,,\ \widetilde{A}_2^4 \\ \widetilde{A}_2^5\,,\ \widetilde{A}_2^6 \\ \widetilde{A}_2^7\,,\ \widetilde{A}_2^8 \end{Bmatrix} = \begin{Bmatrix} 0.529762679031679212023180\,,\ 0.769580268233783935440839 \\ -0.233354688796790875044128\,,\ -0.541601511215317101360294 \\ 0.331576461482468810651328\,,\ -0.779354061134799362179733 \\ 0.531179126831174470587476\,,\ 0.652651364568218169075602 \end{Bmatrix}, \tag{E.16b}$$

$$\begin{Bmatrix} \widetilde{A}_3^1\,,\ \widetilde{A}_3^2 \\ \widetilde{A}_3^3\,,\ \widetilde{A}_3^4 \\ \widetilde{A}_3^5\,,\ \widetilde{A}_3^6 \\ \widetilde{A}_3^7\,,\ \widetilde{A}_3^8 \end{Bmatrix} = \begin{Bmatrix} 0.036093184837888882630144 0\,,\ 1.620340883306093525097 32 \\ -1.025981689589663579579 16\,,\ 0.286848079464457062317327 \\ 0.379578994127041469609087\,,\ -1.370505188625878290732 35 \\ 1.427469434159830215997 59\,,\ 0.554733364863768377772749 \end{Bmatrix}, \tag{E.16c}$$

where, again, only 24 significant digits have been shown. This second solution leads to the same conclusions as the first solution. Very briefly, the ratios (E.10) from the original matrices are:

$$\left\{\widehat{R}_1,\ \widehat{R}_2,\ \widehat{R}_3\right\}_{\mathrm{Abs}\left[\,\widetilde{A}_\mu\right]} = \{1.70,\ 1.30,\ 1.28\}\ , \tag{E.17}$$

and those of the transformed matrices:

$$\left\{\widehat{R}_1,\ \widehat{R}_2,\ \widehat{R}_3\right\}_{\mathrm{Abs}\left[\,\widetilde{A}'_\mu\right]} = \{\infty,\ 10.4,\ 9.04\}\ , \tag{E.18a}$$

$$\left\{\widehat{R}_1,\ \widehat{R}_2,\ \widehat{R}_3\right\}_{\mathrm{Abs}\left[\,\widetilde{A}''_\mu\right]} = \{10.2,\ \infty,\ 9.41\}\ , \tag{E.18b}$$

$$\left\{\widehat{R}_1,\ \widehat{R}_2,\ \widehat{R}_3\right\}_{\mathrm{Abs}\left[\,\widetilde{A}'''_\mu\right]} = \{16.8,\ 18.0,\ \infty\}\ . \tag{E.18c}$$

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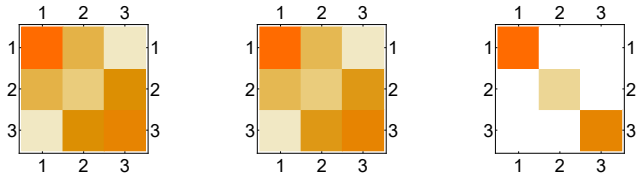

Figure 5: Density plots of $\text{Abs}\big[\widetilde{A}_\mu'''\big]$ from the matrices (E.19). Shown are $\mu = 1,\, 2\,, 3$ from left to right.

Purely for illustrative purposes, we give the matrices with the strongest band-diagonal structure:

$$\widetilde{A}_1''' = \begin{pmatrix} -0.603 & -0.083 - 0.250\,i & -0.0029 + 0.0213\,i \\ -0.083 + 0.250\,i & 0.178 & 0.369 - 0.147\,i \\ -0.0029 - 0.0213\,i & 0.369 + 0.147\,i & 0.425 \end{pmatrix}, \tag{E.19a}$$

$$\widetilde{A}_2''' = \begin{pmatrix} -0.730 & 0.239 - 0.130\,i & -0.0223 + 0.0013\,i \\ 0.239 + 0.130\,i & 0.215 & 0.189 + 0.359\,i \\ -0.0223 - 0.0013\,i & 0.189 - 0.359\,i & 0.515 \end{pmatrix}, \tag{E.19b}$$

$$\widetilde{A}_3''' = \begin{pmatrix} -1.53 & 0 & 0 \\ 0 & 0.228 & 0 \\ 0 & 0 & 1.30 \end{pmatrix}, \tag{E.19c}$$

where the off-band-diagonal entries in $\widetilde{A}_1'''$ and $\widetilde{A}_2'''$ are indeed rather small (cf. Fig. 5).

E.3.3 Technical remarks

The results presented here have been obtained from the numerical minimization routine `FindMinimum` of Mathematica 12.1 (cf. Ref. [32]) with a 36-digit working precision, but, for comparison, we have also used a 24-digit working precision and a 48-digit working precision. The maximum absolute value of the equation residues drops with increasing working precision, as well as the value of the penalty function f_{penalty}; see Table 7 for details. Most importantly, the 24-digit numbers in (E.7) are reproduced by the 48-digit-working-precision calculation.

Following Ref. [17], we will call the solutions (E.7) and (E.16) "quasi-exact," as their number of digits can, in principle, be increased arbitrarily.

E.4 Discussion of critical-point solutions

The main results from the present appendix are two-fold: the existence of nontrivial solutions of the stationarity equation (E.6) for $(D,\, N) = (3,\, 3)$ and a clear diagonal/band-diagonal structure in these solutions. The chosen values of D and N are obviously far below the values (2.7) needed for the genuine IIB matrix model [7, 8]. But it is an important point of principle to have established the existence of these critical-point solutions, even for small values of D and N. We can reverse

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Table 7: First solution of the $(D, N) = (3, 3)$ extremal equation (E.6) as presented in Sec. E.3.1. The residues of the 24 component equations eq-$\widehat{A}_\mu^c$ are computed (they all vanish for a perfect solution). The quantity MaxAbsRes is the maximum of the absolute values of these residues and the function f_{penalty} is the sum of their squared absolute values. The expression eq-A_μ^c follows from the left-hand side of (E.6) by performing a matrix multiplication with $\widehat{t}_c$, taking the trace, and multiplying the result by two [here, the $\widehat{t}_c$ are the $SU(3)$ generators from App. E.2]. The calculations use the `FindMinimum` routine from Mathematica 12.1 [32] with working precision (WP), accuracy goal (AG), and precision goal (PG) as shown.

{WP, AG, PG}	$f_{\text{penalty}} \equiv \sum \left\lvert \text{eq-}A_\mu^c \right\rvert^2$	MaxAbsRes $\equiv \max\left\{\left\lvert \text{eq-}A_\mu^c \right\rvert\right\}$
{24, 12, 12}	$\mathrm{O}\left(10^{-28}\right)$	$\mathrm{O}\left(10^{-14}\right)$
{36, 24, 24}	$\mathrm{O}\left(10^{-55}\right)$	$\mathrm{O}\left(10^{-28}\right)$
{48, 36, 36}	$\mathrm{O}\left(10^{-91}\right)$	$\mathrm{O}\left(10^{-46}\right)$

the argument: assume that there were no such critical-point solutions for $D = 3$ and $N = 3$, then it would be difficult to imagine that there could be critical-point solutions for $D = 10$ and $N \gg 1$, as needed for the IIB matrix model.

A few follow-up remarks about our critical-point calculation are as follows. First, we were not able to obtain a nontrivial critical-point solution of (E.6) if the sign of the Pfaffian term was reversed (thereby removing the supersymmetry of the model).

Second, without a Pfaffian term in (E.6), there is obviously an infinity of diagonal solutions.

Third, comparing the $D = N = 3$ critical solutions obtained in the present appendix with the master-field solutions from Ref. [17], we see that the randomness of the master momenta $\widehat{p}_k$ and the master-noise matrices $\widehat{\eta}_{kl}^{\,\mu}$ appears to have reduced somewhat the strength of the band-diagonality in the master-field solutions [compare the $\mathrm{O}(5)$ ratio value on the second row of Table 5 with the larger nontrivial values in (E.18), for example].

Fourth, it is possible to interpret the bosonic critical-point solution as "classical" (cf. Refs. [7, 27]) and the $\{\widehat{p}_k, \widehat{\eta}_{kl}^{\,\mu}\}$ randomness entering the master-field equation as "bosonic quantum fluctuations" (the "fermionic quantum fluctuations" have already been included in the effective bosonic action). But this is only a qualitative interpretation, as there is no small dimensionless coupling constant in the IIB matrix model (see our previous remarks in Sec. 3.3).

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