Curvature measures and soap bubbles beyond convexity

Daniel Hug Mario Santilli

Abstract

Extending the celebrated results of Alexandrov (1958) and Korevaar–Ros (1988) for smooth sets, as well as the results of Schneider (1979) and the first author (1999) for arbitrary convex bodies, we obtain for the first time the characterization of the isoperimetric sets of a uniformly convex smooth finite-dimensional normed space (i.e. Wulff shapes) in the non-smooth and non-convex setting, based on a natural geometric condition involving the curvature measures. More specifically we show, under a weak mean convexity assumption, that finite unions of disjoint Wulff shapes are the only sets of positive reach $A \subseteq \mathbf{R}^{n+1}$ with finite and positive volume such that, for some $k \in \{0, \dots, n-1\}$, the k-th generalized curvature measure $\Theta^{\phi}_k(A, \cdot)$, which is defined on the unit normal bundle of A with respect to the relative geometry induced by ϕ, is proportional to the top order generalized curvature measure $\Theta^{\phi}_n(A, \cdot)$. If $k = n-1$ the conclusion holds for all sets of positive reach with finite and positive volume. We also prove a related sharp result about the removability of the singularities. This result is based on the extension of the notion of a normal boundary point, originally introduced by Busemann and Feller (1936) for arbitrary convex bodies, to sets of positive reach.

These findings are new even in the Euclidean space.

Several auxiliary and related results are proved, which are of independent interest. They include the extension of the classical Steiner–Weyl tube formula to arbitrary closed sets in a finite dimensional uniformly convex normed vector space, a general formula for the derivative of the localized volume function, which extends and complements recent results of Chambolle–Lussardi–Villa (2021), and general versions of the Heintze–Karcher inequality.

MSC-classes 2020. 49Q10, 52A30, 53C45, 53C60, 53C65, 28A75, 26B25.

Keywords. Tube formula, curvature measures, sets of positive reach, Wulff shapes, removable singularities

Contents

1 Introduction

The following result is fundamental in the geometry of submanifolds: if a smooth hypersurface in a Euclidean space encloses a bounded domain and one of its mean curvature functions is constant, then it is a Euclidean sphere. We refer to this statement as the *soap bubble theorem*. We remark that we are considering *only* hypersurfaces that *enclose* a domain (i.e. they are embedded); otherwise (i.e. for immersed hypersurfaces) it is well known that such a uniqueness result is in general true only in very special situations; see [Hop83] and [Wen86]. The aforementioned fundamental result was proved by Alexandrov for the mean curvature function in [Ale58] and by Korevaar and Ros for the higher-order mean curvature functions in [Ros88] and [Ros87]. Several other proofs were found earlier under various additional hypotheses (e.g. convexity, star-shapedness, mean convexity type assumptions) starting from the pioneering work of Jellett in [Jel53] in the nineteenth century; see [Hsi54] (and the references therein). An analogous result is true for hypersurfaces embedded in finite dimensional uniformly convex normed spaces, provided that the Euclidean mean curvature functions and the Euclidean sphere are replaced by their anisotropic counterparts (in particular the role of the sphere is played by the Wulff shape); see [HLMG09].

A key feature of all the results mentioned so far is that they hold for *smooth* hypersurfaces. In fact, since these results are about hypersurfaces with constant mean curvature functions, the smoothness hypothesis may appear to be natural and somehow unavoidable. However, considering different but equivalent points of view, it turns out that the soap bubble theorem is only part of a more general and more natural problem that does not require any a-priori smoothness assumption. There are at least two standard ways to adjust the soap bubble theorem: via the variational approach based on the notion of a critical point of the isoperimetric functional, and via the integral-geometric approach based on the notion of curvature measures.

Let us first briefly describe the variational approach. A standard computation shows that if a smooth hypersurface with constant mean curvature encloses a bounded domain Ω, then Ω is a critical point of the Euclidean isoperimetric functional among all sets of finite perimeter. Therefore, the Alexandrov theorem can be equivalently stated saying that a critical point of the isoperimetric functional is a sphere, provided it has a smooth boundary. The same is true for the anisotropic counterpart studied in [HLMG09], if a suitable anisotropic isoperimetric functional is considered. From this point of view the assumption of smoothness appears to be a possibly convenient condition rather than a necessary restriction. In fact, one can ask whether it is true that all critical points of the Euclidean isoperimetric functional are Euclidean balls. It should be remarked that the regularity theory in geometric measure theory does not imply that a critical point is automatically smooth. Hence, in the non-smooth framework the problem genuinely involves hypersurfaces which a priori may have severe singularities. A positive resolution of this type of problem is given in [DM19] for the Euclidean case and in [DRKS20] in an anisotropic setting, under some additional hypotheses.

We now focus on the integral-geometric approach, which is the one adopted in the present work. The Weyl tube formula asserts that for all sufficiently small radii $\rho > 0$ the volume of a tubular neighbourhood C_ρ around a domain $C \subseteq \mathbf{R}^{n+1}$ with $\mathcal{C}^2$-boundary ∂C is a polynomial function in ρ; in other words,

$$\mathcal{L}^{n+1}(C_\rho \setminus C) = \sum_{j=0}^{n} \frac{\rho^{j+1}}{j+1} c_{n-j} \qquad \text{for all sufficiently small } \rho > 0.$$

The coefficients c_k can be obtained integrating over ∂C (or, equivalently, over the unit normal bundle of C) the $(n-k)$-th mean curvature function of ∂C (with respect to the exterior normal map) for $k \in \{0, \dots, n\}$. In the special case of smooth convex domains or of convex polytopes in $\mathbf{R}^d$ with $d \in \{2, 3\}$, this formula has already been found in the nineteenth century by Steiner. Now if $k \in \{1, \dots, n\}$, then the k-th curvature measure of C is defined as the Borel measure obtained by integrating over a given Borel subset of ∂C the $(n-k)$-th mean curvature function of ∂C. It has been Federer's fundamental discovery in [Fed59] that the existence of the curvature measures and the validity of the polynomial Weyl tube formula are independent of the smoothness hypothesis. In the same seminal paper, Federer laid the foundation of a theory of *sets of positive reach*, a class that includes all convex bodies, all embedded $\mathcal{C}^2$-submanifolds and much more; indeed the boundary of a domain with positive reach need not even be a topological manifold (see the example described at the end of this section). The soap bubble theorem can be equivalently stated saying that if C is a bounded domain with smooth boundary such that one of the curvature measures of C is a multiple of the area measure associated to the boundary of C, then C is a Euclidean ball. At this point the most compelling problem is to establish a corresponding uniqueness result without any smoothness hypothesis. In fact, this is a classical task in convex geometry. For an arbitrary convex body and for the 0-th mean curvature measure, Diskant accomplished this task and even established a sharp stability result in [Dis68]. A decade later, Schneider [Sch79] resolved the problem for all curvature measures associated with a general convex body (see also [Sch14, Theorem 8.5.7]). Different approaches to prove and generalize Schneider's theorem were found by Kohlmann in [Koh98a] and [Koh98b]. The extension to the anisotropic setting of the results of Schneider and Kohlmann can be found in [Hu99]. On the other hand, as far as we are aware of, up to now no results are available for arbitrary sets of positive reach, and thus the problem has remained unexplored in the non-convex and non-smooth setting. The main goal of this paper is to address this problem in full generality. Our main theorems, Theorem 6.15, Theorem 6.16 and Corollary 6.18 extend Alexandrov's theorem to all sets of positive reach and extend its higher-order version, considered by Korevaar and Ros in the smooth setting, to sets of positive reach under a natural mean convexity hypothesis. Actually we treat this problem directly in the more general setting of uniformly convex finite-dimensional normed spaces, hence generalizing also the main result in [HLMG09] to arbitrary sets of positive reach.

A central notion of this paper are the generalized curvature measures of a set of positive reach in the relative (Minkowski) geometry induced by a uniformly convex $\mathcal{C}^2$-norm ϕ in $\mathbf{R}^{n+1}$. If ϕ is the Euclidean norm, then Federer's tube formula for sets of positive reach allows to introduce the Euclidean curvature measures (see [Fed59] and [Za86]). On the other hand, for non-Euclidean norms a general tube formula for non-smooth and non-convex sets was missing so far. In the recent paper [CdL16], an attempt to obtain an anisotropic tube formula for arbitrary sets of positive reach has been impeded by difficulties to obtain Lipschitz estimates for the nearest point projection (which in the Euclidean setting were established by Federer in [Fed59]); see the comments after Theorem 1.1 in [CdL16, p. 472]. Given this premise, the first task in this paper is to lay the foundation of the theory of curvature measures in the anisotropic setting for sets of positive reach and, more generally, for arbitrary closed sets. For this purpose, let $\boldsymbol{\delta}^\phi_A$ and $\boldsymbol{\nu}^\phi_A$ be the distance function and the Cahn–Hoffman map of A with respect to the metric induced by the conjugate norm ϕ^* of ϕ (see equations (10) and (11) below). Then we define the ϕ-unit normal bundle $N^\phi(A)$ of A by

$$N^\phi(A) = \{(a, \eta) : a \in A,\, \eta \in \partial \mathcal{W}^\phi,\, \boldsymbol{\delta}^\phi_A(a + r\eta) = r \text{ for some } r > 0\},$$

where $\mathcal{W}^\phi = \{\eta \in \mathbf{R}^{n+1} : \phi^*(\eta) \le 1\}$ is called the Wulff shape of ϕ. In general, $\boldsymbol{\nu}^\phi_A$ is a multivalued map and $N^\phi(A)$ is a countably $(\mathcal{H}^n, n)$ rectifiable subset of $A \times \partial \mathcal{W}^\phi$. Employing recent results on the Lipschitz and differentiability properties of $\boldsymbol{\nu}^\phi_A$ provided in [KS21], we introduce the ϕ-principal curvatures

$$-\infty < \kappa^\phi_{A,1}(a, \eta) \le \dots \le \kappa^\phi_{A,n}(a, \eta) \le +\infty$$

of A at $\mathcal{H}^n$ a.e. $(a, \eta) \in N^\phi(A)$, similarly as in the smooth or convex case, as follows: if $\chi^\phi_{A,1}(a + r\eta) \le \dots \le \chi^\phi_{A,n}(a + r\eta)$ denote the eigenvalues of $\mathrm{D}\,\boldsymbol{\nu}^\phi_A(a + r\eta)$, then we define

$$\kappa^\phi_{A,i}(a, \eta) = \frac{\chi^\phi_{A,i}(a + r\eta)}{1 - r\chi^\phi_{A,i}(a + r\eta)} \in (-\infty, +\infty],$$

where the right-hand side is independent of r, if $r > 0$ is chosen sufficiently small, depending on (a, η). We denote by $\widetilde{N}^\phi(A)$ the set of points $(a, \eta) \in N^\phi(A)$ where the principal curvatures exist (hence $\mathcal{H}^n(N^\phi(A) \setminus \widetilde{N}^\phi(A)) = 0$) and define $\widetilde{N}^\phi_d(A)$ to be the set of all $(a,\eta) \in \widetilde{N}^\phi(A)$ such that $\kappa^\phi_{A,d}(a,\eta) < \infty$ and $\kappa^\phi_{A,d+1}(a,\eta) = +\infty$. In particular, $\widetilde{N}^\phi_n(A)$ is the set of all $(a,\eta) \in N^\phi(A)$ such that the ϕ-principal curvatures of A at (a,η) are finite. The i-th ϕ-mean curvature function $\boldsymbol{H}^\phi_{A,i}(a,\eta)$ of A at $\mathcal{H}^n$ a.e. $(a,\eta) \in N^\phi(A)$ is defined by taking certain combinations of the elementary symmetric functions of the ϕ-principal curvatures (if A is a smooth submanifold, then we recover classical definitions); see Definition 3.11. As a consequence of Theorem 3.16, the volume of the tubular neighbourhood $B^\phi(A,\rho) \setminus A = \{x \in \mathbf{R}^{n+1} : 0 < \boldsymbol{\delta}^\phi_A(x) \le \rho\}$ of an arbitrary compact set A can be expressed by the formula

$$\mathcal{L}^{n+1}(B^\phi(A,\rho) \setminus A) = \sum_{j=0}^{n} \frac{1}{j+1} \int_{N^\phi(A)} \phi(\boldsymbol{n}^\phi(\eta))\, J^\phi_A(a,\eta)\, \inf\{\rho, \boldsymbol{r}^\phi_A(a,\eta)\}^{j+1} \boldsymbol{H}^\phi_{A,j}(a,\eta)\, d\mathcal{H}^n(a,\eta).$$

Here $\boldsymbol{r}^\phi_A$ is the reach function of A (see (14)), J^ϕ_A is a Jacobian-type function encoding the tangential properties of the normal bundle (see Definition 3.17) and $\boldsymbol{n}^\phi$ is the Euclidean exterior unit-normal of $\mathcal{W}^\phi$ (see (7)). For an arbitrary compact set A, the right side of the tube formula is in general not a polynomial function in ρ (the volume growth is sub-polynomial) and the mean curvature functions $\boldsymbol{H}^\phi_{A,i}$ are in general not integrable on $N^\phi(A)$ (in spite of the fact that the integral is well defined due to the power of the reach function under the integral). On the other hand, if the ϕ-reach of A (see Definition 5.1) is greater than or equal to some positive threshold $\rho_0 > 0$, then $\boldsymbol{r}^\phi_A(a,\eta) \ge \rho_0$ for every $(a,\eta) \in N^\phi(A)$ and we obtain a polynomial-type formula

$$\mathcal{L}^{n+1}(B^\phi(A,\rho) \setminus A) = \sum_{j=0}^{n} \frac{\rho^{j+1}}{j+1} \int_{N^\phi(A)} \phi(\boldsymbol{n}^\phi(\eta))\, J^\phi_A(a,\eta)\, \boldsymbol{H}^\phi_{A,j}(a,\eta)\, d\mathcal{H}^n(a,\eta)$$

for $\rho \in (0,\rho_0)$. Noting that the functions $J^\phi_A \cdot \boldsymbol{H}^\phi_{A,i}$ are integrable on $N^\phi(A)$ if A has positive reach, we define the m-th generalized ϕ-curvature measure of A as the signed Radon measure supported on $N^\phi(A)$ given by

$$\Theta^\phi_m(A,B) = \frac{1}{n-m+1} \int_{N^\phi(A)\cap B} \phi(\boldsymbol{n}^\phi(\eta))\, J^\phi_A(a,\eta)\, \boldsymbol{H}^\phi_{A,n-m}(a,\eta)\, d\mathcal{H}^n(a,\eta)$$

for every bounded Borel set $B \subseteq \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$ and $m \in \{0,\dots,n\}$. If ϕ is the Euclidean norm, these measures coincide with the classical generalized curvature measures for sets of positive reach (up to the normalization); see [Za86]. Moreover, if ϕ is the Euclidean norm, then the curvature functions and the tube formula from Section 3 agree with those in [HLW04]. However, our approach here is substantially different from the one in [HLW04]. In fact, in [HLW04] the Euclidean principal curvatures of an arbitrary closed set are introduced by means of an approximation with sets of positive reach (see Stachó's approximation Lemma in [HLW04, Lemma 2.3] and [Sta79]). Observe that such an approximation argument is not available in the anisotropic case, since there is no fully fledged theory of sets with positive reach ready to be used in the current more general framework (see the discussion above on [CdL16]).

An important consequence of the tube formula for arbitrary closed sets (see also Corollary 3.18) is the sharp integral-geometric inequality in Theorem 3.20, for which equality is attained only by disjoint unions of finitely many rescaled and translated Wulff shapes (assuming an a priori bound for the mean curvature). Theorem 3.20 generalizes the geometric inequality known as the Heintze–Karcher inequality for sufficiently smooth sets (see [Ros87] and [MR91]) to arbitrary closed sets in Euclidean space, under a natural (weak) mean convexity assumption. For sufficiently smooth sets C, the Heintze–Karcher inequality provides an upper bound for the volume of C by the integral average over the boundary of C of the reciprocal of the mean curvature function of C. For convex bodies (compact convex sets with non-empty interiors) this inequality was proved in [Koh98b] (in the Euclidean case) and in [Hu99, Lemma 2.45] (in the anisotropic framework). The general anisotropic version for arbitrary closed sets given in Theorem 3.20 will be specialized to sets of positive reach in Theorem 5.15, and this result is one of the pillars for the subsequent soap bubble theorems.

In Section 3, we also provide a detailed analysis of the curvature functions of an arbitrary closed set A in relation to the dimension of the fibers of the ϕ-unit normal bundle $N^\phi(A)$ of A. This analysis allows us to obtain the general disintegration formula stated in Theorem 3.27, which in the present generality is new even in the special Euclidean case (see [CH00, Theorem 5.5] for sets with positive reach and [Hu99, Theorems 1.56 and 1.57] for convex bodies).

As another consequence of the tube formula, we obtain differentiability properties of the parallel volume function in Section 4. In Theorem 4.3 we determine the left and right derivatives of the localized volume function of the tubular neighbourhood around a closed set A with respect to ϕ and provide a novel necessary and sufficient condition for the existence of the two-sided derivative. Although this result is not needed in the proof of the soap bubble theorems, we have decided to include it here since it is of interest in itself. Indeed the derivative of the volume function has been the subject of several investigations; see [Sta76], [HLW04], [HLW06], [RW10] and [CLV21]. The most recent contribution [CLV21, Theorem 5.2] treats *arbitrary and possibly asymmetric* norms and establishes formulae for the left and the right derivative of the volume function in terms of *area-integrals* on the boundary of the tubular neighbourhood of a compact set. Theorem 5.2 in [CLV21] can be compared with the third equality in (55) and in (56) (notice, however, that the results in [CLV21] are not localized). On the other hand, the main novelty of our results are the formulae for the left and the right derivatives of the localized volume function in terms of *curvature integrals* on the ϕ-unit normal bundle of A. As a consequence, our result establishes the relation between the (localized) area integrals on the boundary of the tubular neighbourhood of A with the corresponding (localized) curvature integrals on the ϕ-unit normal bundle of A.

In Section 5, we generalize the classical notion of a normal boundary point of a convex body (see [Sch14] and [Sch15] and the references therein) to arbitrary sets of positive reach. It is well known that the boundary of a convex body C is locally the epigraph of a convex function around each of its boundary points. Employing the classical theorem of Alexandrov on the twice differentiability of convex functions, one can see that the normal boundary points are precisely those boundary points where the locally representing function is twice differentiable. Consequently, the notion of a pointwise second fundamental form and pointwise defined mean curvature functions can be introduced at each such boundary point. Some authors refer to the normal boundary points as *Alexandrov points*, and for this reason we denote the set of these points by $\mathcal{A}(C)$. If $\boldsymbol{p} : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$, $\boldsymbol{p}(a, \eta) = a$ for $(a, \eta) \in \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$, is the projection onto the first coordinate, it is known that for an arbitrary convex body $C \subset \mathbf{R}^{n+1}$ it holds that

$$\mathcal{A}(C) = \boldsymbol{p}(\widetilde{N}^{\phi}_n(C)).$$

Moreover, the pointwise defined mean curvature functions of C, associated with the pointwise second fundamental form, coincide with the mean curvature functions defined on $\widetilde{N}^{\phi}_n(C)$; see [Hu98, Lemma 3.1] and [Hu99]. We extend these results to arbitrary sets of positive reach in an arbitrary uniformly convex normed space. A key difference to the case of convex bodies is that the boundary of a set C of positive reach is in general not graphical around each of its points. Therefore we define $\partial^v C$ as the set of points $a \in \partial C$ where the fibre $N(C, a)$ of the Euclidean unit normal bundle of C contains only one vector (the same set is obtained if $\partial^v C$ is defined with respect to the fibres of $N^{\phi}(C)$ and a general norm ϕ). In Theorem 5.7 we show that a set of positive reach C is locally the epigraph of a semiconvex function around each point $a \in \partial^v C$ and we prove that this function is twice differentiable at a if and only if $\kappa^{\phi}_{C,i}(a, \eta) < \infty$ for $i = 1, \ldots, n$, where $N^{\phi}(C, a) = \{\eta\}$. This result opens the way to introduce the notion of an Alexandrov point for a set of positive reach: these are all points of $\partial^v A$ where the semiconvex function locally representing A is twice differentiable. We denote the set of all Alexandrov points of C by $\mathcal{A}(C)$ and, as in the convex case, each Alexandrov point entails pointwise curvature information that we express by the mean curvature functions $\boldsymbol{h}^{\phi}_{C,k}$, for $k \in \{0, \ldots, n\}$; see Definition 5.8. We prove that

$$\mathcal{A}(C) = \boldsymbol{p}(\widetilde{N}^{\phi}_n(C)) \cap \partial^v C$$

and

$$\boldsymbol{h}^{\phi}_{C,k}(a) = \boldsymbol{H}^{\phi}_{C,k}(a, \eta) \quad \text{for every } a \in \mathcal{A}(C) \text{ and } N^{\phi}(C, a) = \{\eta\};$$

see Corollary 5.10. In the remaining part of Section 5, employing the geometric inequality for closed sets from Theorem 3.20, we derive a version of the Heintze–Karcher inequality for sets of positive reach in Theorem 5.15. As a consequence, we obtain Corollary 5.16 which states that the only sets

C of positive reach with finite and positive volume such that

$$\boldsymbol{h}^{\phi}_{C,1}(a) \geq \frac{n\mathcal{P}^{\phi}(C)}{(n+1)\mathcal{L}^{n+1}(C)} \qquad \text{for } \mathcal{H}^n \text{ a.e. } a \in \mathcal{A}(C), \tag{1}$$

are finite unions of rescaled and translated Wulff shapes of radius $\frac{(n+1)\mathcal{L}^{n+1}(C)}{\mathcal{P}^{\phi}(C)}$. Here $\mathcal{P}^{\phi}(C)$ is the ϕ-perimeter of C, namely

$$\mathcal{P}^{\phi}(C) = \int_{\partial^* C} \phi(\boldsymbol{n}(C,a))\, d\mathcal{H}^n(a),$$

where $\partial^* C$ is the reduced boundary of C and $\boldsymbol{n}(C,\cdot)$ is the measure-theoretic Euclidean unit normal of C (notice that a set of positive reach has always locally finite perimeter, see Lemma 2.10). One can easily see that the lower bound in (1) is sharp by considering convex bodies obtained as unions of congruent spherical caps; see Remark 5.17. A key feature of these results is that they provide information on the global geometry of a set C of positive reach requiring only assumptions on points in $\partial^v C$. This is quite surprising in view of the fact that there exist sets C of positive reach with finite volume and non-empty interior such that $\mathcal{H}^n(\partial C \setminus \partial^v C) > 0$ (an explicit example is obtained by taking the function f in the example described at the end of this introduction such that $\{f = 0\}$ has positive $\mathcal{L}^1$ measure). Corollary 5.16 plays a key role for the soap bubble theorems in Section 6, but is also of independent interest.

A special case of our first soap bubble theorem (Theorem 6.15) can be stated as follows. In view of condition (2), we point out that while the Radon measures $\Theta^{\phi}_j(C,\cdot)$, for $j = 0,\dots,n-1$ and a set $C \subset \mathbf{R}^{n+1}$ of positive reach, are signed in general, the measure $\Theta^{\phi}_n(C,\cdot)$ is always non-negative. The hypothesis in (2), as well as the hypothesis in (3) and (4), respectively, of the subsequent theorems, is the natural generalization of the hypothesis of "k-convexity" for smooth domains (see [CW13] and references therein) to the singular setting of the present paper. If C is a convex body in $\mathbf{R}^{n+1}$, then all generalized curvature measures $\Theta^{\phi}_j(C,\cdot)$ are non-negative. In fact, a set $C \subset \mathbf{R}^{n+1}$ of positive reach is convex if and only if $\Theta^{\phi}_j(C,\cdot) \geq 0$ for all $j = 0,\dots,n-1$.

Theorem A (cf. Theorem 6.15)**.** *Let $k \in \{1,\dots,n\}$, and let $C \subset \mathbf{R}^{n+1}$ be a set of positive reach with positive and finite volume. Assume that*

$$\Theta^{\phi}_{n-i}(C,\cdot) \quad \text{is a non-negative measure for } i = 1,\dots,k-1 \tag{2}$$

and

$$\Theta^{\phi}_{n-k}(C,\cdot) = \lambda\, \Theta^{\phi}_n(C,\cdot) \quad \text{for some } \lambda \in \mathbf{R}\setminus\{0\}.$$

Then C is a finite disjoint union of rescaled and translated Wulff shapes of radius $\frac{(n+1)\mathcal{L}^{n+1}(C)}{\mathcal{P}^{\phi}(C)}$ and $\lambda > 0$.

If $k = 1$, then the conclusion holds for every set of positive reach with finite and positive volume and for every $\lambda \in \mathbf{R}$.

In Theorem 6.15 we point out how the common radius of the translated and rescaled Wulff shapes can be expressed in terms of λ, n and k. Conversely, whenever C is a finite disjoint union of rescaled and translated Wulff shapes, each of the generalized curvature measures $\Theta^{\phi}_{n-k}(C,\cdot)$ is proportional to the top order curvature measure $\Theta^{\phi}_n(C,\cdot)$. For the proof of Theorem 6.15, we first establish an anisotropic extension of the Minkowski–Hsiung formulae for arbitrary sets of positive reach, which is provided in Theorem 6.8 and adds to several previous versions available in the literature (see [Koh94, Theorem 3.4], [Fu98, Corollary 3.4], [Hu99, Theorem 2.42]). It is interesting to notice that the hypothesis in (2) is preserved for limits of sequences of smooth sets satisfying a positive lower bound on the reach; see Lemma 6.12 and Corollary 6.13. Henceforth Theorem A might be helpful to study global geometric properties of limits of smooth almost k-th mean convex sets. We remark that if C is a smooth set, then the hypothesis in (2) is redundant, because the existence of an elliptic point in combination with the continuity of the principal curvatures guarantees the non-negativity hypothesis in (2). This is a classical argument outlined in [Ros87]. In the general situation of Theorem A, we do not have any continuity for the curvature functions and it is unclear whether the hypothesis in (2) can be relaxed further (beyond the mean convexity assumption of Theorem 6.15).

If the assumption (2) is strengthened to include also the condition that $\Theta^{\phi}_{n-k}(C,\cdot)$ is a non-negative measure, then we obtain the following version of Theorem A, which is a special case of Theorem 6.16. In the statement of the result, we use the ϕ-curvature measures $\mathcal{C}^{\phi}_j(C,\cdot)$, $j \in \{0,\ldots,n\}$, of C, which are the Radon measures on $\mathbf{R}^{n+1}$ that are obtained as the image measures of the generalized curvature measures $\Theta^{\phi}_j(C,\cdot)$ under the projection map $\boldsymbol{p}$, that is, $\mathcal{C}^{\phi}_j(C,\cdot) = \Theta^{\phi}_j(C,\cdot \times \partial\mathcal{W}^{\phi})$.

Theorem B (cf. Theorem 6.16)**.** *Let $k \in \{1,\ldots,n\}$, and let $C \subset \mathbf{R}^{n+1}$ be a set of positive reach with positive and finite volume. Assume that*

$$\Theta^{\phi}_{n-i}(C,\cdot) \quad \textit{is a non-negative measure for } i = 1,\ldots,k \tag{3}$$

and

$$\mathcal{C}^{\phi}_{n-k}(C,\cdot) = \lambda\, \mathcal{C}^{\phi}_n(C,\cdot) \quad \textit{for some } \lambda > 0.$$

Then C is a finite union of rescaled and translated Wulff shapes of radius $\frac{(n+1)\mathcal{L}^{n+1}(C)}{\mathcal{P}^{\phi}(C)}$.

In the particular case where C is a convex body (and condition (3) is automatically satisfied as pointed out above), Theorem B has already been established in [Hu99, Theorem 2.43]. In the framework of convex bodies the single measure on the left side of the hypothesis $\mathcal{C}^{\phi}_{n-k}(C,\cdot) = \lambda\, \mathcal{C}^{\phi}_n(C,\cdot)$ can even be replaced by a non-negative linear combination of curvature measures $\mathcal{C}^{\phi}_{n-k}(C,\cdot)$ with $k \in \{1,\ldots,n\}$. Related stability results have been proved in [Hu99, Theorems 2.47 and 2.48]. In fact, the results in [Hu99, Section 2.7] completely establish (in generalized form) Conjecture 8.2 stated in [AW21].

Theorems A and B can be seen as measure-theoretic versions of the soap bubble theorem. Employing the notion of pointwise curvature in Alexandrov points, we obtain the following differential-geometric version.

Theorem C (cf. Corollary 6.18)**.** *Suppose that $k \in \{1,\ldots,n\}$, $\lambda \in \mathbf{R} \setminus \{0\}$ and $\varnothing \neq C \subset \mathbf{R}^{n+1}$ is a set of positive reach with positive and finite volume such that $\mathcal{H}^n(\partial C \setminus \partial^v C) = 0$. If $k = 1$ we allow $\lambda \in \mathbf{R}$. Assume that*

$$\boldsymbol{h}^{\phi}_{C,i}(a) \geq 0 \quad \textit{for } i = 1,\ldots,k-1 \textit{ and for } \mathcal{H}^n \textit{ a.e. } a \in \mathcal{A}(C) \textit{ and} \tag{4}$$

$$\boldsymbol{h}^{\phi}_{C,k}(a) = \lambda \quad \textit{for } \mathcal{H}^n \textit{ a.e. } a \in \mathcal{A}(C).$$

Then C is a finite union of rescaled and translated Wulff shapes of radius $\frac{(n+1)\mathcal{L}^{n+1}(C)}{\mathcal{P}^{\phi}(C)}$, provided that $\mathcal{H}^{n-k}\big[\boldsymbol{p}(\widetilde{N}^{\phi}(C) \setminus \widetilde{N}^{\phi}_n(C))\big] = 0$.

The hypothesis $\mathcal{H}^n(\partial C \setminus \partial^v C) = 0$ is equivalent to require that $\mathcal{P}(C) = \mathcal{H}^n(\partial C)$, where $\mathcal{P}(C) = \mathcal{H}^n(\partial^* C)$ is the Euclidean perimeter of C; see Corollary 5.10. Hence the condition $\mathcal{P}(C) = \mathcal{H}^n(\partial C)$ is an alternative way to say in a geometric-measure theoretic sense that ∂C encloses C. As mentioned in the first paragraph of this introduction, this is a fundamental prerequisite to obtain soap bubbles in arbitrary dimension and without any topological assumptions, from the hypothesis that one of the mean curvature functions is constant. The hypothesis $\mathcal{H}^n(\partial C \setminus \partial^v C) = 0$ implies that $\mathcal{H}^n$ a.e. $a \in \boldsymbol{p}(\widetilde{N}^{\phi}_n(C))$ is an Alexandrov point (see again Corollary 5.10). Based on the disjoint decomposition

$$\boldsymbol{p}(\widetilde{N}^{\phi}(C)) = \boldsymbol{p}(\widetilde{N}^{\phi}_n(C)) \cup \boldsymbol{p}(\widetilde{N}^{\phi}(C) \setminus \widetilde{N}^{\phi}_n(C)) \quad \text{and} \quad \boldsymbol{p}(\widetilde{N}^{\phi}_n(C)) \cap \boldsymbol{p}(\widetilde{N}^{\phi}(C) \setminus \widetilde{N}^{\phi}_n(C)) = \varnothing,$$

provided in Theorem 5.6, the set $\boldsymbol{p}(\widetilde{N}^{\phi}(C) \setminus \widetilde{N}^{\phi}_n(C))$ can be seen as the set of singular points of ∂C. Consequently, the hypothesis $\mathcal{H}^{n-k}\big[\boldsymbol{p}(\widetilde{N}^{\phi}(C) \setminus \widetilde{N}^{\phi}_n(C))\big] = 0$ is (in general) a sharp assumption on the smallness of the singular set. For $k = 1$ the example of the union of two congruent spherical caps shows that the condition on the Hausdorff measure cannot be relaxed. On the other hand, for $k = n$ the examples of peaked spheres (see [GHM13] and [FLW19, Section 2 and Theorem 15]), including as a special case a set resembling an American football [FLW19, Section 4], demonstrate that the condition on the Hausdorff measure is sharp. (The construction of corresponding examples for $1 < k < n$ seems to be an interesting problem.) In this sense, Theorem C can also be seen as a result on removable singularities.

We remark that the class of sets of positive reach (with finite and positive volume) satisfying the assumption $\mathcal{H}^n(\partial C \setminus \partial^v C) = 0$ not only includes all sets that can be locally represented as epigraphs of semiconvex functions (in particular all convex bodies), see Lemma 6.20, but it includes many other examples of sets whose boundaries are not topological manifolds. One such example can be constructed by reflecting around the x-axis in $\mathbf{R}^2$ the set $\{(x, f(x)) : x \in \mathbf{R}\}$, where $f : [a, b] \to \mathbf{R}$ is a non-negative smooth function such that $f(a) = f(b) = 0$, $\lim_{x\to a+} f'(x) = +\infty$, $\lim_{x\to b-} f'(x) = -\infty$ and $\{f = 0\}$ is a Cantor set of $\mathcal{L}^1$-measure zero (in this case $\partial C \setminus \partial^v C = \{f = 0\}$).

We conclude this introduction with a few comments on some lines of research, which are naturally related with the results of the present paper.

It is well known that the soap bubble theorem also holds for smooth hypersurfaces in Riemannian manifolds of constant sectional curvature, where the bubbles are realized by geodesic spheres. This was proved long ago by Alexandrov [Ale62] for the mean curvature case by means of his celebrated method of moving planes, and extended by Montiel–Ros in [MR91] to the case of higher-order mean curvature functions employing the integral-geometric approach. Looking at more general Riemannian spaces, soap bubble theorems in warped product spaces have been the subject of intensive research in the last two decades. In this direction a fundamental contribution is made by the work of Brendle in [Bre13], where it is proved that a compact and embedded hypersurface with constant mean curvature in a suitable class of warped product spaces is a slice; in a subsequent work [BE13] Brendle and Eichmair treat the case of constant higher order mean curvature hypersurfaces under special convexity hypothesis. In the smooth setting stability results for the aforementioned soap bubble theorems are currently subject of intensive research; see [CRV21] and [SX22] and the references therein.

While in these investigations the smoothness assumption is crucial, it is natural to aim at extensions of the results of the present paper to non-Euclidean ambient spaces in a non-smooth framework as well. In this respect, several important foundational investigations should be mentioned. Walter (see his survey of related work up to 1981 in [Wal81]), Kleinjohann [Kl80, Kl81] and Bangert [Ba78a, Ba78b, Ba79, Ba82] studied intensively notions of convexity, sets with the unique footpoint property and regularity properties of the associated normal bundles in general Riemannian spaces. In his PhD-thesis (1988) Kohlmann (see also [Koh98a]) considered Alexandrov's soap bubble problem for general convex sets in constant curvature spaces via curvature measures, but in the non-Euclidean case his methods did not allow to resolve the important mean curvature case. Sets with positive reach and curvature measures have been intensively studied in Euclidean space (see [RZ19] and the works cited there). In the Riemannian setting, the theory of curvature measures and its connection to valuation theory is currently developed; see, e.g., the recent contribution by Fu and Wannerer [FW19]. It remains to be explored whether some of these curvature measures can be used for obtaining uniqueness results as considered in the present work. Important structural information about sets with positive reach in Riemannian spaces, such as upper curvature bounds and characterization results, has been derived by Lytchak [Ly04, Ly04], but a complete structural description of general sets with positive reach is not even available in Euclidean spaces so far. However, useful foundational results on distance functions, cut sets and curvatures in Riemannian spaces (or, more specifically, in Cartan–Hadamard manifolds) can be found in the recent works by Kapovitch and Lytchak [KL21] and by Ghomi and Spruck [GS22].

The present work is mainly motivated by classical problems in differential geometry and the calculus of variations. However, local Steiner formulas and differentials of basic geometric functionals of convex bodies, as considered here for more general classes of sets, also play a crucial role in the Brunn–Minkowski theory [Sch14] and its applications. These formulas naturally lead to the curvature measures, which are a major topic of the current investigation, but also to surface area measures, quermassintegrals, and to L_p, Orlicz and dual versions of these fundamental functionals and measures. We refer to the seminal work by Huang, Lutwak, Yang and Zhang [HLYZ16], where several new measures are introduced and connections to classical geometric measures are explored. In [HLYZ16] then all these measures are combined in the investigation of associated Minkowski problems, which have received much attention in recent years (see, e.g., [BLYZ13, BLYZ13, BHP18, HLYZ18, LYZ18, BLYZ19, GHWXY19, GHXY20]).

2 Preliminaries

2.1 Notation and basic facts

In general, but with few exceptions explained below, we follow the notation and terminology of [Fed69] (see [Fed69, pp. 669-676]). In particular we adopt the terminology from [Fed69, 3.2.14] when dealing with rectifiable sets.

If X is a topological space and $S \subseteq X$, then we denote by $\mathrm{int}(S)$ the interior part of S, by ∂S the topological boundary of S and by $\mathrm{clos}(S)$ the closure of S; moreover, the characteristic function of S is $\mathbf{1}_S$. If $Q \subseteq X \times Y$ and $S \subseteq X$, we set $Q|S = \{(x, y) \in Q : x \in S\}$. We denote by $\bullet$ a fixed scalar product on $\mathbf{R}^{n+1}$ and by $|\cdot|$ its associated norm. Hence $\mathbf{S}^n = \{x \in \mathbf{R}^{n+1} : |x| = 1\}$ is the Euclidean unit sphere. The maps $\mathbf{p} : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ and $\boldsymbol{q} : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ are the projection onto the first and the second component respectively, i.e. $\mathbf{p}(x, \eta) = x$ and $\boldsymbol{q}(x, \eta) = \eta$.

If $S \subseteq \mathbf{R}^p$ and $a \in \mathbf{R}^p$, then we denote by $\mathrm{Tan}(S, a)$ and $\mathrm{Nor}(S, a)$ the *tangent and normal cone* of S at a (see [Fed69, 3.1.21]). Always following [Fed69] we use the symbol $\mathrm{Tan}^m(\mathcal{H}^m \llcorner S, a)$ for the cone of all $(\mathcal{H}^m \llcorner S, m)$ *approximate tangent vectors* at a (see [Fed69, 3.2.16]). For an $(\mathcal{H}^m, m)$ rectifiable and $\mathcal{H}^m$ measurable set $S \subseteq \mathbf{R}^p$, the cone $\mathrm{Tan}^m(\mathcal{H}^m \llcorner S, a)$ is an m-dimensional linear subspace for $\mathcal{H}^m$ a.e. $a \in S$. Each Lipschitz function $f : \mathbf{R}^p \to \mathbf{R}^q$ has at $\mathcal{H}^m$ almost all points of $a \in S$ an $(\mathcal{H}^m \llcorner S, m)$ *approximate differential* $\mathrm{ap}\,\mathrm{D}\,f(a) : \mathrm{Tan}^m(\mathcal{H}^m \llcorner S, a) \to \mathbf{R}^q$ (see [Fed69, 3.2.16, 3.2.19]). If this approximate differential exists, for $k \in \{1, \ldots, m\}$ we define the $(\mathcal{H}^m \llcorner S, m)$ *approximate k-th Jacobian* of f aat a as

$$\mathrm{ap}\, J_k^S f(a) = \big\| \textstyle\bigwedge_k \mathrm{ap}\,\mathrm{D}\,f(a) \big\| = \sup\{|[\textstyle\bigwedge_k \mathrm{ap}\,\mathrm{D}\,f(a)](\xi)| : \xi \in \textstyle\bigwedge_k \mathrm{Tan}^m(\mathcal{H}^m \llcorner S, a),\ |\xi| = 1\}, \tag{5}$$

where $\bigwedge_k \mathrm{ap}\,\mathrm{D}\,f(a) : \bigwedge_k \mathrm{Tan}^m(\mathcal{H}^m \llcorner S, a) \to \bigwedge_k \mathbf{R}^q$ is the linear map induced by $\mathrm{ap}\,\mathrm{D}\,f(a)$ (see [Fed69, 1.3.1]). The norms $|\cdot|$ on the right-side of (5) denote the norm induced on $\bigwedge_k \mathrm{Tan}^m(\mathcal{H}^m \llcorner S, a)$ and $\bigwedge_k \mathbf{R}^q$ by the inner products of $\mathrm{Tan}^m(\mathcal{H}^m \llcorner S, a)$ and $\mathbf{R}^q$, respectively (see [Fed69, 1.7.5, 1.7.6]). The approximate Jacobian of a Lipschitz map will be repeatedly used in this paper in applying the following version of Federer's coarea formula for Lipschitz maps on rectifiable sets, for which we refer to [Fed78, pp. 300–301].

Lemma 2.1 (Federer)**.** *If W is an $(\mathcal{H}^m, m)$ rectifiable $\mathcal{H}^m$ measurable subset of $\mathbf{R}^p$, $f : W \to \mathbf{R}^\mathbf{q}$ is a Lipschitzian map, $k \in \{0, \ldots, m\}$ and $S \subseteq \mathbf{R}^q$ is a countable union of Borel subsets of $\mathbf{R}^q$ with finite $\mathcal{H}^k$ measure, then*

$$\int_{W \cap f^{-1}(S)} \phi(x)\ \mathrm{ap}\, J_k^W f(x)\, d\mathcal{H}^m(x) = \int_S \int_{W \cap f^{-1}(\{y\})} \phi(x)\, d\mathcal{H}^{m-k}(x)\, d\mathcal{H}^k(y)$$

for every $\mathcal{H}^m$ measurable function $\phi : W \to [0, \infty]$.

Following Federer [Fed69, page 15] we denote by $\Lambda(n, m)$ the set of all increasing maps from $\{1, \ldots, m\}$ into $\{1, \ldots, n\}$. We now introduce the k-th elementary symmetric functions. For $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ and $k \in \{1, \ldots, n\}$, we define

$$S_k(x) := \sum_{\lambda \in \Lambda(n,k)} x_{\lambda(1)} \cdots x_{\lambda(k)}. \tag{6}$$

Then

$$\Gamma_k^\circ := \{x \in \mathbf{R}^n : S_1(x) > 0, \ldots, S_k(x) > 0\}$$

is an open convex cone whose closure is the (pointed) closed convex cone

$$\Gamma_k := \{x \in \mathbf{R}^n : S_1(x) \geq 0, \ldots, S_k(x) \geq 0\}$$

with apex 0 (see [TW99, Section 2, page 582] or [Sal99, Proposition 1.3.2]).

For a vector x with positive components the next lemma is well known. In the present more general form, it is in fact harder to find an explicit reference.

Lemma 2.2. *Let $k \in \{1, \dots, n\}$. If $x \in \Gamma_k$, then*

$$\left(\frac{S_i(x)}{\binom{n}{i}}\right)^{\frac{1}{i}} \geq \left(\frac{S_j(x)}{\binom{n}{j}}\right)^{\frac{1}{j}} \quad \textit{for } 1 \leq i \leq j \leq k.$$

Proof. See [Sal99, Proposition 1.3.3 (4)]. We indicate an alternative argument here. First, Newton's inequality holds for any $x \in \mathbf{R}^n$, as shown in [Ro89]. Let $S_0(x) := 1$ and $E_r(x) := S_r(x)/\binom{n}{r}$ for $r = 0, \dots, k$. If $x \in \Gamma_k^\circ$, then the asserted inequalities can be obtained (as usually) by repeated application of Newton's inequality $E_\ell(x)E_{\ell+2}(x) \leq E_\ell(x)^2$ via

$$\prod_{\ell=0}^{r-1} \left(E_\ell(x)E_{\ell+2}(x)\right)^{\ell+1} \leq \prod_{\ell=1}^{r} E_\ell(x)^{2\ell} \quad \text{for } r \in \{1, \dots, k-1\}.$$

Since Γ_k is the closure of the open convex cone Γ_k°, the assertion for $x \in \Gamma_k$ follows by an obvious approximation argument. ☐

2.2 Multivalued maps

A map T defined on a set X is called Y*-multivalued*, if $T(x)$ is a subset of Y for every $x \in X$. If $T(x)$ is a singleton, with a little abuse of notation we denote by $T(x)$ the unique element of the set $T(x) \subseteq Y$. Suppose that $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ are finite-dimensional normed vectorspaces and T is a Y-multivalued map such that $T(x) \neq \varnothing$ for every $x \in X$.

(1) We say that T is *weakly continuous at $x \in X$* if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $y \in X$ and $\|y - x\| < \delta$, then

$$T(y) \subseteq T(x) + \{v \in Y : \|v\| < \epsilon\};$$

if, additionally, $T(x)$ is a singleton, then we say that T is continuous at x.

(2) We say that T *is strongly differentiable at $x \in X$* if and only if $T(x)$ is a singleton and there exists a linear map $L : X \to Y$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $y \in X$, $\|y - x\| < \delta$ and $w \in T(y)$, then

$$\|w - T(x) - L(y - x)\| \leq \varepsilon \|y - x\|;$$

cf. [KS21, Definition 2.28]. The linear map L is unique (cf. [KS21, Remark 2.29]) and we denote it by $\mathrm{D}\,T(x)$. Moreover we denote by $\operatorname{dmn} \mathrm{D}\,T$ the set of points $x \in X$ at which T is strongly differentiable. In the following, we simply write "differentiable" when we actually mean "strongly differentiable".

The following general fact on the Borel measurability of the differential of a multivalued map will be useful.

Lemma 2.3. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be finite-dimensional normed vectorspaces, and let T be a Y-multivalued weakly continuous map such that $T(x) \neq \varnothing$ for $x \in X$.*

Then $\{x \in X : T(x)$ is a singleton$\}$ and $\operatorname{dmn} \mathrm{D}\,T$ are Borel subsets of X and $\mathrm{D}\,T : \operatorname{dmn} \mathrm{D}\,T \to \operatorname{Hom}(X, Y)$ is Borel measurable.

Proof. We define $U = \{x \in X : T(x) \text{ is a singleton}\}$ and the function $\operatorname{diam} : \mathbf{2}^Y \setminus \{\varnothing\} \to [0, \infty]$ by $\operatorname{diam} S = \sup\{\|y_1 - y_2\| : y_1, y_2 \in S\}$ for every $S \in \mathbf{2}^Y \setminus \{\varnothing\}$. Noting that $\operatorname{diam} \circ T : X \to [0, +\infty]$ is upper semicontinuous, we conclude that $U = \{x \in X : \operatorname{diam}(T(x)) = 0\}$ is a Borel subset of X.

For positive integers $i, j \in \mathbb{N}$ we define

$$C_{ij} = \left\{(x, L) \in U \times \operatorname{Hom}(X, Y) : \|w - T(x) - L(h)\| \leq \frac{1}{i}\|h\| \quad \text{for } \|h\| < \frac{1}{j} \text{ and } w \in T(x + h)\right\}.$$

We prove that C_{ij} is relatively closed in $U \times \mathrm{Hom}(X, Y)$. By contradiction assume that C_{ij} is not closed. Then there exists $(x_0, L_0) \in (U \times \mathrm{Hom}(X, Y)) \setminus C_{ij}$ and a sequence $(x_k, L_k) \in C_{ij}$ converging to (x_0, L_0). Hence that there exist $h_0 \in X$ with $\|h_0\| < \frac{1}{j}$ and $w_0 \in T(x_0 + h_0)$ such that

$$\|w_0 - T(x_0) - L_0(h_0)\| > \frac{1}{i}\|h_0\|.$$

We define $h_k = x_0 + h_0 - x_k$ for $k \geq 1$ and select $k_0 \geq 1$ so that $\|h_k\| < \frac{1}{j}$ for $k \geq k_0$. Since $(x_k, L_k) \in C_{ij}$, $x_0 + h_0 = x_k + h_k$ and $w_0 \in T(x_k + h_k)$ for $k \geq 1$, we infer that

$$\|w_0 - T(x_k) - L_k(h_k)\| \leq \frac{1}{i}\|h_k\| \qquad \text{for } k \geq k_0.$$

Noting that $T(x_k) \to T(x_0)$ and $h_k \to h_0$ as $k \to \infty$, we deduce that

$$\|w_0 - T(x_0) - L_0(h_0)\| \leq \frac{1}{i}\|h_0\|,$$

and we obtain a contradiction.

Let $G := \{(x, \mathrm{D}\, T(x)) : x \in \mathrm{dmn}\, \mathrm{D}\, T\}$ and $\pi_X : X \times \mathrm{Hom}(X, Y) \to X$, $\pi_X(x, T) = x$ for every $(x, T) \in X \times \mathrm{Hom}(X, Y)$. Noting that

$$G = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} C_{ij},$$

we infer that G is a Borel subset of $U \times \mathrm{Hom}(X, Y)$. Since $\pi_X | G$ is injective, we obtain from [Fed69, 2.2.10, bottom of page 67] that $\{x \in \mathrm{dmn}\, \mathrm{D}\, T : \mathrm{D}\, T(x) \in B\} = \pi_X(G \cap (X \times B))$ is a Borel set in X if $B \subseteq \mathrm{Hom}(X, Y)$ is a Borel set, which implies the remaining assertions. □

Remark 2.4. The case of single-valued continuous functions is treated in [Fed69, page 211] with a similar proof.

The following elementary lemma will be useful in Section 6.

Lemma 2.5. *Let $U \subseteq \mathbf{R}^k$ be open, let $F : U \to \mathbf{R}^k$ be differentiable at $a \in U$, and assume that $\mathrm{D}\, F(a) : \mathbf{R}^k \to \mathbf{R}^k$ is invertible. Let $V \subseteq \mathbf{R}^k$ be open with $x = F(a) \in V$ and assume that $G : V \to \mathbf{R}^k$ is Lipschitz. Further, assume that $F(U) \subseteq V$ and $G \circ F = \mathrm{Id}_U$. Then G is differentiable at x and $\mathrm{D}\, G(x) = \mathrm{D}\, F(a)^{-1}$.*

Proof. Let $v \in \mathbf{R}^k$. Let $\varepsilon > 0$. Then there is some $\delta > 0$ such that if $t \in \mathbf{R}$ with $0 < |t| < \delta$, then $a + tv \in U$ and

$$\left| \frac{F(a + tv) - F(a)}{t} - \mathrm{D}\, F(a)(v) \right| \leq \frac{\varepsilon}{\mathrm{Lip(G)}}.$$

Then we obtain

$$\begin{aligned}
&\left| \frac{G(x + t\,\mathrm{D}\, F(a)(v)) - G(x)}{t} - v \right| \\
&\quad = \left| \frac{G(F(a) + t\,\mathrm{D}\, F(a)(v)) - G(F(a))}{t} - \frac{G(F(a + tv)) - G(F(a))}{t} \right| \\
&\quad = \left| \frac{G(F(a + tv)) - G(F(a) + t\,\mathrm{D}\, F(a)(v))}{t} \right| \\
&\quad \leq \mathrm{Lip}(G) \cdot \left| \frac{F(a + tv) - F(a)}{t} - \mathrm{D}\, F(a)(v) \right| \leq \varepsilon.
\end{aligned}$$

Since $\mathrm{D}\, F(a) : \mathbf{R}^k \to \mathbf{R}^k$ is invertible, this shows that if $w \in \mathbf{R}^k$, then

$$\lim_{t \to 0} \left| \frac{G(x + tw) - G(x)}{t} - \mathrm{D}\, F(a)^{-1}(w) \right| = 0,$$

and hence G is differentiable at x with $\mathrm{D}\, G(x) = \mathrm{D}\, F(a)^{-1}$. □

2.3 Norms and Wulff shapes.

Let ϕ be a norm on $\mathbf{R}^{n+1}$. We say that ϕ is a $\mathcal{C}^k$*-norm* if and only if $\phi \in \mathcal{C}^k(\mathbf{R}^{n+1} \setminus \{0\})$. We say that ϕ is *uniformly convex* if and only if there exists a constant $\gamma > 0$ (ellipticity constant) such that the function $\mathbf{R}^{n+1} \ni u \mapsto \phi(u) - \gamma|u|$ is convex. If ϕ is a uniformly convex $\mathcal{C}^2$-norm then

$$\mathrm{D}^2\,\phi(u)(v,v) \geq \gamma|v|^2$$

for all $u \in \mathbf{R}^{n+1}$ with $|u| = 1$ and for all $v \in \mathbf{R}^{n+1}$ perpendicular to u. In the following, a compact convex set with non-empty interior will be called a *convex body*. The symmetric (with respect to the origin o) convex body $B = \{x \in \mathbf{R}^{n+1} : \phi(x) \leq 1\}$ is the unit ball or *gauge body* associated with ϕ. Conversely, the *gauge function* (norm) of B is just ϕ, i.e.

$$g(B,x) := g_B(x) := \|x\|_B := \min\{\lambda \geq 0 : x \in \lambda B\} = \phi(x).$$

For a compact convex set $\varnothing \neq K \subset \mathbf{R}^{n+1}$, we write $h_K = h(K,\cdot)$ for its *support function*, which is defined by $h(K,x) := h_K(x) := \max\{x \bullet z : z \in K\}$, and we denote by $K^\circ := \{x \in \mathbf{R}^{n+1} : x \bullet y \leq 1 \text{ for } y \in K\}$ the *polar body* of K. Note that K° is again a convex body if the origin o is an interior point of K. Hence we have

$$\phi = g_B = h_{B^\circ}$$

(see [Sch14, Section 1.7.2, page 53]).

For any norm ϕ we denote by ϕ^* *the conjugate norm* of ϕ which is defined by $\phi^*(u) = \sup\{v \bullet u : \phi(v) = 1\}$ for $u \in \mathbf{R}^{n+1}$. Then we also have

$$\phi^* = h_B = g_{B^\circ}.$$

It is well known that if ϕ is a uniformly convex $\mathcal{C}^2$-norm then ϕ^* is a uniformly convex $\mathcal{C}^2$-norm. In geometric terms, this is equivalent to the property that B and B° both have a boundary of class C^2 and positive Gauss curvature everywhere (we then say that these bodies are of class C^2_+). In particular, the *spherical image map (Gauss map)* $\boldsymbol{u}_B : \partial B \to \mathbf{S}^n$ is a diffeomorphism of class C^1 whose inverse is given by the restriction of ∇h_B to $\mathbf{S}^n$. We refer to [DRKS20, Lemma 2.32] for this and other basic facts on ϕ and ϕ^* and to [Sch14, Section 2.5] for the relations between smoothness properties of B and B°. These facts will be tacitly used throughout the paper.

We define the *Wulff shape* (or *Wulff crystal*) of ϕ as

$$\mathcal{W}^\phi = \{x \in \mathbf{R}^{n+1} : \phi^*(x) \leq 1\} = B^\circ.$$

Hence, if ϕ is a uniformly convex $\mathcal{C}^2$-norm, then the Wulff shape of ϕ is a uniformly convex set with $\mathcal{C}^2_+$ boundary. In this case the exterior unit normal (spherical image) map of $\mathcal{W}^\phi$ is the map

$$\boldsymbol{n}^\phi : \partial\mathcal{W}^\phi \to \mathbf{S}^n; \tag{7}$$

we remark (see [DRKS20, 2.32] or [Sch14, Section 2.5]) that $\boldsymbol{n}^\phi = \boldsymbol{u}_{B^\circ}$ is a $\mathcal{C}^1$-diffeomorphism onto $\mathbf{S}^n$ and

$$\nabla\phi(\boldsymbol{n}^\phi(x)) = x \quad \text{for } x \in \partial\mathcal{W}^\phi, \qquad \boldsymbol{n}^\phi(\nabla\phi(u)) = u \quad \text{for } u \in \mathbf{S}^n. \tag{8}$$

Since $\mathrm{D}(\nabla\phi)(u) \bullet u = 0$ for $u \in \mathbf{S}^n$, we notice that

$$\mathrm{Tan}(\mathbf{S}^n, u) = \mathrm{D}(\nabla\phi)(u)[\mathrm{Tan}(\mathbf{S}^n, u)] = \mathrm{Tan}(\partial\mathcal{W}^\phi, \nabla\phi(u)).$$

Moreover, we have $\phi(\boldsymbol{n}^\phi(x)) = \boldsymbol{n}^\phi(x) \bullet x$ for $x \in \partial\mathcal{W}^\phi$. We also point out (cf. [DRKS20, Lemma 2.32(f)]) that the compositions of the gradient maps $\nabla\phi : \mathbf{R}^{n+1} \setminus \{0\} \to \partial B^\circ$ and $\nabla\phi^* : \mathbf{R}^{n+1} \setminus \{0\} \to \partial B$ satisfy the relations

$$\nabla\phi^* \circ \nabla\phi|_{\partial B} = \mathrm{Id}_{\partial B} \qquad \text{and} \qquad \nabla\phi \circ \nabla\phi^*|_{\partial B^\circ} = \mathrm{Id}_{\partial B^\circ}\,. \tag{9}$$

2.4 Distance function and normal bundle

Warning. In this paper we occasionally refer to [KS21]. However, notice that in this paper we use the same symbols with a different meaning (the roles of ϕ and ϕ^* are changed). Hence the definitions below have to be compared carefully with those given in [KS21, Sections 1 and 2].

Convention. If ϕ is the Euclidean norm, then the dependence on ϕ is omitted in all the symbols introduced below.

Let $\varnothing \neq A \subseteq \mathbf{R}^{n+1}$ be a closed set, and let ϕ be a uniformly convex $\mathcal{C}^2$-norm on $\mathbf{R}^{n+1}$. The *ϕ-distance function* $\boldsymbol{\delta}^\phi_A : \mathbf{R}^{n+1} \to \mathbf{R}$ is defined by

$$\boldsymbol{\delta}^\phi_A(x) = \min\{\phi^*(x-c) : c \in A\} = \min\{\lambda \geq 0 : x \in A + \lambda B^\circ\} \qquad \text{for } x \in \mathbf{R}^{n+1}. \tag{10}$$

Next we define the level and sublevel sets at distance $r > 0$ with respect to $\boldsymbol{\delta}^\phi_A$ by

$$S^\phi(A, r) = \{x \in \mathbf{R}^{n+1} : \boldsymbol{\delta}^\phi_A(x) = r\} \qquad \text{and} \qquad B^\phi(A, r) = \{x \in \mathbf{R}^{n+1} : \boldsymbol{\delta}^\phi_A(x) \leq r\}.$$

For $a \in \mathbf{R}^{n+1}$ we set $B^\phi(a, r) = B^\phi(\{a\}, r) = a + rB^\circ$. Moreover, an open neighborhood of A is defined by $U^\phi(A, r) = \{x \in \mathbf{R}^{n+1} : \delta^\phi_A(x) < r\} = \operatorname{int}(B^\phi(A, r))$, and again we set $U^\phi(a, r) = U(\{a\}, r)$. Clearly, $\boldsymbol{\delta}^\phi_A$ is a Lipschitz map; moreover it is a classical fact that $-\boldsymbol{\delta}^\phi_A$ is semiconvex on $\mathbf{R}^{n+1} \setminus B^\phi(A, r)$ for $r > 0$; (cf. [KS21, Lemma 2.41(b)] and the references therein).

The *nearest ϕ-projection* $\boldsymbol{\xi}^\phi_A : \mathbf{R}^{n+1} \to \mathbf{2}^A$ is the A-multivalued map defined by

$$\boldsymbol{\xi}^\phi_A(x) = \{c \in A : \boldsymbol{\delta}^\phi_A(x) = \phi^*(x - c)\} \qquad \text{for } x \in \mathbf{R}^{n+1}.$$

This is a weakly continuous map by [KS21, Lemma 2.41(f)]. By $\operatorname{Unp}^\phi(A)$ we denote the set of all $x \in \mathbf{R}^{n+1} \setminus A$ such that there exists a unique point $c \in A$ with $\phi^*(x - c) = \boldsymbol{\delta}^\phi_A(x)$, i.e., $\operatorname{Unp}^\phi(A) = \{x \in \mathbf{R}^{n+1} \setminus A : \mathcal{H}^0(\boldsymbol{\xi}^\phi_A(x)) = 1\}$. For $x \in \operatorname{Unp}^\phi(A)$ we simply write $\xi^\phi_A(x) = c$ if $\xi^\phi_A(x) = \{c\}$. Notice that $\operatorname{Unp}^\phi(A)$ is a Borel subset of $\mathbf{R}^{n+1}$ by Lemma 2.3 (see [HL00, Lemma 3.12] for a more general fact). It is well known that $\mathbf{R}^{n+1} \setminus (A \cup \operatorname{Unp}^\phi(A))$ equals the set of points in $\mathbf{R}^{n+1} \setminus A$ where $\boldsymbol{\delta}^\phi_A$ is not differentiable (cf. [KS21, Lemma 2.41(c)]. Moreover, $\mathbf{R}^{n+1} \setminus (A \cup \operatorname{Unp}^\phi(A))$ can be covered outside a set of $\mathcal{H}^n$ measure zero by a countable union of n-dimensional graphs of $\mathcal{C}^2$-functions; for the proof of this result one can proceed as in the Euclidean case which is treated in [Haj22].

The ϕ-Cahn–Hoffman map of A is the $\partial\mathcal{W}^\phi$-multivalued function $\boldsymbol{\nu}^\phi_A : \mathbf{R}^{n+1} \setminus A \to \mathbf{2}^{\partial\mathcal{W}^\phi}$ defined by

$$\boldsymbol{\nu}^\phi_A(x) = \boldsymbol{\delta}^\phi_A(x)^{-1}(x - \boldsymbol{\xi}^\phi_A(x)) \qquad \text{for } x \in \mathbf{R}^{n+1} \setminus A. \tag{11}$$

Next we introduce the map $\boldsymbol{\psi}^\phi_A : \mathbf{R}^{n+1} \setminus A \to \mathbf{2}^A \times \mathbf{2}^{\partial\mathcal{W}^\phi}$ by

$$\boldsymbol{\psi}^\phi_A(x) = (\boldsymbol{\xi}^\phi_A(x), \boldsymbol{\nu}^\phi_A(x)) \qquad \text{for } x \in \mathbf{R}^{n+1} \setminus A.$$

Recall the relations

$$\nabla\boldsymbol{\delta}^\phi_A(x) = \nabla\phi^*(x - \boldsymbol{\xi}^\phi_A(x)) \in \partial\mathcal{W}^{\phi^*} \qquad \text{and} \qquad \nabla\phi(\nabla\boldsymbol{\delta}^\phi_A(x)) = \boldsymbol{\nu}^\phi_A(x) \in \partial\mathcal{W}^\phi \tag{12}$$

for $x \in \operatorname{Unp}^\phi(A)$, cf. [KS21, Lemma 2.41(c)] (but recall that the notation and in particular the roles of ϕ and ϕ^* are changed in comparison with [KS21]). The equivalence of these two relations can be seen from (9). It follows from [DRKS20, Lemma 2.32] or from (12) and basic properties of strictly convex bodies that $\boldsymbol{\nu}^\phi_A(x) \bullet \nabla\boldsymbol{\delta}^\phi_A(x) = \phi(\nabla\boldsymbol{\delta}^\phi_A(x)) = 1$ for $x \in \operatorname{Unp}^\phi(A)$.

The *ϕ-unit normal bundle of A* is defined by

$$N^\phi(A) = \{(x, \eta) \in A \times \partial\mathcal{W}^\phi : \boldsymbol{\delta}^\phi_A(x + r\eta) = r \text{ for some } r > 0\} = \{\boldsymbol{\psi}^\phi_A(z) : z \in \operatorname{Unp}^\phi(A)\},$$

and we set

$$N^\phi(A, x) = \{\eta \in \partial\mathcal{W}^\phi : (x, \eta) \in N^\phi(A)\} \qquad \text{for } x \in A\ .$$

We recall (cf. [DRKS20, Lemma 5.2]) that $N^\phi(A)$ is a Borel subset of $\mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$ and it can be covered up to a set of $\mathcal{H}^n$ measure zero by a countable union of n-dimensional graphs of $\mathcal{C}^1$-functions; moreover we have

$$N^\phi(A) = \{(a, \nabla\phi(u)) : (a, u) \in N(A)\}. \tag{13}$$

The *ϕ-reach function* of A is the upper semicontinuous (see [KS21, Lemma 2.35]) function $\boldsymbol{r}^\phi_A : N^\phi(A) \to (0, +\infty]$ given by

$$\boldsymbol{r}^\phi_A(a, \eta) = \sup\bigl\{s > 0 : \boldsymbol{\delta}^\phi_A(a + s\eta) = s\bigr\} \qquad \text{for } (a, \eta) \in N^\phi(A), \tag{14}$$

and *the ϕ-cut locus of A* is given by

$$\mathrm{Cut}^\phi(A) = \big\{a + \boldsymbol{r}^\phi_A(a,\eta)\eta : (a,\eta) \in N^\phi(A)\big\}.$$

The strict convexity of ϕ implies that $\mathbf{R}^{n+1} \setminus (A \cup \mathrm{Unp}^\phi(A)) \subseteq \mathrm{Cut}^\phi(A)$; see Lemma 2.41 (c) and Remark 4.1 in [KS21] as well as further references given there. Moreover $\mathrm{Cut}^\phi(A)$ is always contained in the closure of $\mathbf{R}^{n+1} \setminus (A \cup \mathrm{Unp}^\phi(A))$; cf. [Fr97, Theorem 3B]. We recall that (cf. [DRKS20, Remark 5.10])

$$\mathcal{L}^{n+1}(\mathrm{Cut}^\phi(A)) = 0. \tag{15}$$

On the other hand, the closure of $\mathbf{R}^{n+1} \setminus (A \cup \mathrm{Unp}^\phi(A))$ might have non-empty interior even if A is the closure of the complement of a convex body with $\mathcal{C}^{1,1}$-boundary (see [San21] for this and other critical examples). If A is convex, then $\mathrm{Cut}^\phi(A) = \varnothing$. A related function which will be useful in the sequel is defined by

$$\boldsymbol{\rho}^\phi_A(x) = \sup\{s \geq 0 : \boldsymbol{\delta}^\phi_A(a + s(x-a)) = s\boldsymbol{\delta}^\phi_A(x)\} \qquad \text{for } x \in \mathbf{R}^{n+1} \setminus A \text{ and } a \in \boldsymbol{\xi}^\phi_A(x).$$

This definition does not depend on the choice of $a \in \boldsymbol{\xi}^\phi_A(x)$ and the function $\boldsymbol{\rho}^\phi_A : \mathbf{R}^{n+1} \setminus A \to [1, +\infty]$ is upper semicontinuous; cf. [KS21, Lemma 2.33]. Notice that $\{x : \boldsymbol{\rho}^\phi_A(x) > 1\} \subseteq \mathrm{Unp}^\phi(A)$ (see Lemma 2.33 and Remark 4.1 in [KS21] and

$$\boldsymbol{r}^\phi_A(a,\eta) = r\boldsymbol{\rho}^\phi_A(a + r\eta) \qquad \text{for } (a,\eta) \in N^\phi(A) \text{ and } 0 < r < \boldsymbol{r}^\phi_A(a,\eta), \tag{16}$$

as shown in [KS21, Lemma 2.35].

The following two results from [KS21], which we recall here for the ease of the reader, plays an important role in the next section. The norm ϕ is always assumed to be uniformly convex and $\mathcal{C}^2$ in $\mathbf{R}^{n+1} \setminus \{0\}$.

Lemma 2.6 (cf. [KS21, Corollary 3.10])**.** *Let $\varnothing \neq A \subseteq \mathbf{R}^{n+1}$ be closed, $1 < \lambda < \infty$, $0 < s < t < \infty$, and*

$$A_{\lambda,s,t} = \big\{x \in \mathbf{R}^{n+1} \setminus A : \boldsymbol{\rho}^\phi_A(x) \geq \lambda,\ s \leq \boldsymbol{\delta}^\phi_A(x) \leq t\big\}.$$

Then $\boldsymbol{\xi}^\phi_A | A_{\lambda,s,t}$ is Lipschitz continuous.

Before we can state the next result we need to recall from [KS21] the reach-type function $\underline{\boldsymbol{r}^\phi_A} : N^\phi(A) \to [0, +\infty]$ defined by

$$\underline{\boldsymbol{r}^\phi_A}(a,\eta) = \sup\left\{\sigma r : \sigma > 1,\ 0 < r < \boldsymbol{r}^\phi_A(a,\eta),\ \lim_{\rho \to 0+} \frac{\mathcal{L}^{n+1}(A_\sigma \cap B(a + r\eta, \rho))}{\mathcal{L}^{n+1}(B(a + r\eta, \rho))} = 1\right\} \cup \{0\},$$

where $A_\sigma = \{\boldsymbol{\rho}^\phi_A \geq \sigma\}$. Notice that $\underline{\boldsymbol{r}^\phi_A}(a,\eta) \leq \boldsymbol{r}^\phi_A(a,\eta)$ for every $(a,\eta) \in N^\phi(A)$; see [KS21, Remark 4.10]. This function plays a central role in [KS21] in the study of the structure of the set $\mathrm{dmn}\,\mathrm{D}\,\boldsymbol{\nu}^\phi_A$ which coincides with the set of twice differentiability points of $\boldsymbol{\delta}^\phi_A$ by [KS21, Lemma 2.41(e)]).

Lemma 2.7 (cf. [KS21, Theorem 1.5])**.** *If $\varnothing \neq A \subseteq \mathbf{R}^{n+1}$ is a closed set, then*

$$\mathcal{L}^{n+1}\big(\mathbf{R}^{n+1} \setminus (A \cup \mathrm{dmn}\,\mathrm{D}\,\boldsymbol{\nu}^\phi_A)\big) = 0$$

and

$$\{a + r\eta : 0 < r < \boldsymbol{r}^\phi_A(a,\eta)\} \subseteq \mathrm{dmn}(\mathrm{D}\,\boldsymbol{\nu}^\phi_A) \qquad \text{for } \mathcal{H}^n \text{ almost all } (a,\eta) \in N^\phi(A).$$

The function $\underline{\boldsymbol{r}^\phi_A}$ is Borel measurable. Moreover, if $a + s\eta \in \mathrm{dmn}\,\mathrm{D}\,\boldsymbol{\nu}^\phi_A$ for some $s \in (0, \underline{\boldsymbol{r}^\phi_A}(a,\eta))$, then $a + r\eta \in \mathrm{dmn}\,\mathrm{D}\,\boldsymbol{\nu}^\phi_A$ for all $r \in (0, \underline{\boldsymbol{r}^\phi_A}(a,\eta))$. Finally,

$$\underline{\boldsymbol{r}^\phi_A}(a,\eta) = \boldsymbol{r}^\phi_A(a,\eta) \qquad \text{for } \mathcal{H}^n \text{ a.e. } (a,\eta) \in N^\phi(A).$$

Lemma 2.8. *Suppose $\varnothing \neq A \subseteq \mathbf{R}^{n+1}$ is a closed set, $\xi : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is an arbitrary function such that $\xi(y) \in \boldsymbol{\xi}^{\phi}_A(y)$ for $y \in \mathbf{R}^{n+1}$ and*

$$u(y) = \frac{\nabla\phi^*(y - \xi(y))}{|\nabla\phi^*(y - \xi(y))|} \qquad \text{for } y \in \mathbf{R}^{n+1} \setminus A.$$

For $x \in \operatorname{dmn} \mathrm{D}\,\boldsymbol{\nu}^{\phi}_A$, $r = \boldsymbol{\delta}^{\phi}_A(x)$ and $T = \operatorname{Tan}(S^{\phi}(A, r), x)$ the following statements hold.

(a) *$\boldsymbol{\rho}^{\phi}_A(x) > 1$, $\operatorname{im} \mathrm{D}\,\boldsymbol{\nu}^{\phi}_A(x) \subseteq T$ and $\mathrm{D}\,\boldsymbol{\nu}^{\phi}_A(x)(\boldsymbol{\nu}^{\phi}_A(x)) = 0$.*

(b) *The maps $\mathrm{D}(\nabla\phi)(u(x))|T$ and $\mathrm{D}\,u(x)|T$ are self-adjoint (with respect to the underlying scalar product $\bullet$) automorphisms of T,*

$$\mathrm{D}\,\boldsymbol{\nu}^{\phi}_A(x) = \mathrm{D}(\nabla\phi)(u(x)) \circ \mathrm{D}\,u(x) \quad \text{and} \quad \mathrm{D}\,u(x)(\tau_1) \bullet \tau_2 = \frac{\mathrm{D}^2\,\boldsymbol{\delta}^{\phi}_A(x)(\tau_1, \tau_2)}{|\nabla\boldsymbol{\delta}^{\phi}_A(x)(x)|} \quad \text{for } \tau_1, \tau_2 \in T.$$

(c) *There is a basis $\tau_1, \ldots, \tau_n$ of T of eigenvectors of $\mathrm{D}\,\boldsymbol{\nu}^{\phi}_A(x)|T$ and the corresponding eigenvalues $\chi_1 \leq \ldots \leq \chi_n$ of $\mathrm{D}\,\boldsymbol{\nu}^{\phi}_A(x)$ are real numbers such that*

$$\frac{1}{(1 - \boldsymbol{\rho}^{\phi}_A(x))r} \leq \chi_i \leq \frac{1}{r}\,. \tag{17}$$

2.5 Boundaries and perimeter

Let $A \subseteq \mathbf{R}^{n+1}$ and $a \in A$. For $x \in \mathbf{R}^{n+1}$, $u \in \mathbf{S}^n$ and $r > 0$, we define the open halfspace through x with (inner and outer, respectively) normal u by $H^+(x, u) := \{z \in \mathbf{R}^{n+1} : (z - x) \bullet u > 0\}$ and $H^-(x, u) := \{z \in \mathbf{R}^{n+1} : (z - x) \bullet u < 0\}$. Following [Fed69, Section 4.5.5], we say that a vector $u \in \mathbf{S}^n$ is an *exterior normal of A at a* if

$$\lim_{r \to 0+} \frac{\mathcal{L}^{n+1}\big(H^+(a, u) \cap U(a, r) \cap A\big)}{r^{n+1}} = 0$$

and

$$\lim_{r \to 0+} \frac{\mathcal{L}^{n+1}\big(H^-(a, u) \cap U(a, r) \setminus A\big)}{r^{n+1}} = 0.$$

Clearly, in this definition $U(a, r)$ can be replaced by $B(a, r)$ and the open halfspaces can be replaced by the corresponding closed halfspaces. Recall also from [Fed69, Section 4.5.5] that if u and v are exterior unit normals of A at a, then $u = v$. The set of points where the exterior normal of A exists is denoted by $\partial^m A$; we define $\boldsymbol{n}(A, \cdot) : \partial^m A \to \mathbf{S}^n$ to be the Euclidean exterior normal map of A. We extend this definition by $\boldsymbol{n}(A, x) = 0$ for $x \notin \partial^m A$. Notice the equality $\boldsymbol{n}(\mathcal{W}^{\phi}, \cdot) = \boldsymbol{n}^{\phi}$ on ∂W^{ϕ}.

For an $\mathcal{L}^{n+1}$ measurable set $A \subseteq \mathbf{R}^{n+1}$ one can also consider the *essential boundary* $\partial^* A$; see [AFP00, Definition 3.60]. Recalling the notions of *approximate discontinuity set* S_u and *approximate jump set* J_u of a function $u \in L^1_{\mathrm{loc}}(\mathbf{R}^{n+1})$, see [AFP00, Definitions 3.63 and 3.67], we notice that if $A \subseteq \mathbf{R}^{n+1}$ is an $\mathcal{L}^{n+1}$ measurable set, then $\partial^* A = S_{\mathbf{1}_A}$ and $\partial^m A = J_{\mathbf{1}_A}$, and it follows from [AFP00, Proposition 3.64] and [AFP00, Proposition 3.69] that $\partial^m A$ and $\partial^* A$ are Borel subsets of $\mathbf{R}^{n+1}$, $\boldsymbol{n}(A, \cdot)$ is a Borel function and

$$\partial^m A \subseteq \partial^* A.$$

Employing an argument similar to [San19, Lemma 5.1], one can still prove that if $A \subseteq \mathbf{R}^{n+1}$ is an arbitrary set, then $\partial^m A$ is a Borel subset of $\mathbf{R}^{n+1}$ and $\boldsymbol{n}(A, \cdot)$ is a Borel function.

We recall that an $\mathcal{L}^{n+1}$ measurable subset A of $\mathbf{R}^{n+1}$ is a set of locally finite perimeter in $\mathbf{R}^{n+1}$ if the characteristic function $\mathbf{1}_A$ is a function of locally bounded first variation (see [AFP00, Chapter 3]). If $A \subseteq \mathbf{R}^{n+1}$ is a set of finite perimeter, we denote by $\mathcal{F}A$ the *reduced boundary* of A (see [AFP00, 3.54]). An important result of De Giorgi, see [AFP00, Theorem 3.59], implies that

$$\mathcal{F}A \subseteq \partial^m A.$$

A result of Federer (see [AFP00, Theorem 3.61]) yields that if A is a set of locally finite perimeter, then

$$\mathcal{H}^n(\partial^* A \setminus \mathcal{F}A) = 0. \tag{18}$$

Another result of Federer (see [Fed69, Theorem 4.5.11]) implies that if $A \subseteq \mathbf{R}^{n+1}$ and $\mathcal{H}^n(K \cap \partial A) < \infty$ for every compact set $K \subset \mathbf{R}^{n+1}$, then A is a set of locally finite perimeter.

Definition 2.9. Let $A \subset \mathbf{R}^{n+1}$ be a Borel set with locally finite perimeter, and let ϕ be a uniformly convex $\mathcal{C}^2$-norm on $\mathbf{R}^{n+1}$. The *ϕ-perimeter* of A is the Radon measure $\mathcal{P}^\phi(A, \cdot)$ supported in ∂A such that

$$\mathcal{P}^\phi(A, S) = \int_{S \cap \partial^m A} \phi(\boldsymbol{n}(A, x))\, d\mathcal{H}^n(x) \qquad \text{for Borel sets } S \subseteq \mathbf{R}^{n+1}.$$

The total measure is denoted by $\mathcal{P}^\phi(A, \mathbf{R}^{n+1}) = \mathcal{P}^\phi(A, \partial^m A) = \mathcal{P}^\phi(A) \in [0, \infty]$.

Clearly, we have $\mathcal{P}^\phi(A) > 0$ if and only if $\mathcal{H}^n(\partial^m A) > 0$.

The following lemma will be needed in Section 6. We refer to this section and the references provided there, for the definition and the basic facts concerning sets of positive reach.

Lemma 2.10. (a) *If $A \subset \mathbf{R}^{n+1}$ is a Borel set of locally finite perimeter such that $0 < \mathcal{L}^{n+1}(A) < \infty$, then $\mathcal{H}^n(\mathcal{F}A) > 0$.*

(b) *If $A \subseteq \mathbf{R}^{n+1}$ is a set of positive reach, then $\mathcal{H}^n(K \cap \partial A) < \infty$ for every compact set $K \subseteq \mathbf{R}^{n+1}$, and consequently A is a set of locally finite perimeter.*

Proof. Noting [AFP00, Theorem 3.59] and [AFP00, Theorem 3.36], the statement in (a) directly follows from the isoperimetric inequality in [AFP00, Theorem 3.46].

We now prove (b). We fix $0 < r < \operatorname{reach}(A)$ and note that $S(A, r)$ is a closed $\mathcal{C}^1$-hypersurface and $\xi := \boldsymbol{\xi}_A | S(A, r)$ is a Lipschitz map with $\xi(S(A, r)) = \partial A$. Since $\partial A \cap B(0, s) \subseteq \xi(\xi^{-1}(\partial A \cap B(0, s)))$ and $\xi^{-1}(\partial A \cap B(0, s)) \subseteq S(A, r) \cap B(0, r + s)$ for $s > 0$, we infer that

$$\mathcal{H}^n(B(0, s) \cap \partial A) \le \operatorname{Lip}(\xi)^n \mathcal{H}^n(S(A, r) \cap B(0, s + r)) \quad \text{for } s > 0.$$

The right-hand side is evidently finite, since $S(A, r) \cap B(0, s + r)$ is a compact subset of the closed $\mathcal{C}^1$-hypersurface $S(A, r)$. ☐

It will be sometimes useful to consider another notion of boundary: if $A \subseteq \mathbf{R}^{n+1}$ is a closed set, then we define the *viscosity boundary* of A by

$$\partial^v A = \{a \in \partial A : \mathcal{H}^0(N(A, a) = 1\}.$$

This is precisely the set of boundary points $a \in \partial A$ for which there is a unique (outer unit normal) vector $u \in \mathbf{S}^n$ with $(a, u) \in N(A)$. For $s > 0$ we also define $\partial^v_s A$ to be the set of points $a \in \partial^v A$ such that there exists a closed Euclidean ball B of radius s such that $B \subseteq A$ and $a \in \partial B$. We set

$$\partial^v_+ A = \bigcup_{s > 0} \partial^v_s A.$$

Remark 2.11. We notice that $\partial^v_+ A \subseteq \partial^m A \cap \boldsymbol{p}(N(A)) \subseteq \partial^v A$ and

$$N(A, a) = \{\boldsymbol{n}(A, a)\} \qquad \text{for } a \in \partial^m A \cap \boldsymbol{p}(N(A)).$$

Moreover, $N(\mathbf{R}^{n+1} \setminus \operatorname{int}(A), a) = \{-\boldsymbol{n}(A, a)\}$ for every $a \in \partial^v_+ A$. Finally, if $s > 0$ then $\partial^v_s A$ is a closed subset of ∂A and $B(a - s\boldsymbol{n}(A, a), s) \subseteq A$ for every $a \in \partial^v_s A$.

The following lemma (or rather the consequence of it discussed in Remark 2.14) will be relevant in the special case of sets with positive reach in Section 5.3 (see Remark 5.13).

Lemma 2.12. *Let $A \subseteq \mathbf{R}^{n+1}$ be a closed set and $x, y \in \partial A$. Let $0 < r < s/2$. Suppose that $u, v \in \mathbf{S}^n$ are such that*

$$B(x - ru, r) \subseteq A, \quad U(x + su, s) \cap A = \emptyset, \quad B(y - rv, r) \subseteq A, \quad U(y + sv, s) \cap A = \emptyset.$$

Then

$$|u - v| \le \max\left\{\frac{2(s-2r)}{r(s-r)}, \sqrt{\frac{2}{r(s-r)}}\right\}|x - y|.$$

Proof. Since $y \in A$, we have $y \notin U(x + su, s)$, hence $|y - x - su|^2 \ge s^2$, which yields

$$|y - x|^2 + s^2 - 2s(y - x) \bullet u \ge s^2$$

or

$$(y - x) \bullet u \le \frac{|y - x|^2}{2s}. \tag{19}$$

By symmetry, we also have

$$(x - y) \bullet v \le \frac{|x - y|^2}{2s}. \tag{20}$$

Noting that $x - ru + rv \in B(x - ru, r) \subseteq A$, we conclude from (20) that

$$\begin{aligned}(x - ru + rv - y) \bullet v &\le \frac{|x - y + r(v - u)|^2}{2s} \\ &\le \frac{1}{2s}|x - y|^2 + \frac{r^2}{2s}|u - v|^2 + \frac{r}{s}(x - y) \bullet (v - u).\end{aligned}$$

Exchanging x and y (and using (19)), we also get

$$(y - rv + ru - x) \bullet u \le \frac{1}{2s}|x - y|^2 + \frac{r^2}{2s}|u - v|^2 + \frac{r}{s}(y - x) \bullet (u - v).$$

Now we sum the last two inequalities to obtain

$$(x - y) \bullet (v - u) + r|v - u|^2 \le \frac{1}{s}|x - y|^2 + \frac{r^2}{s}|u - v|^2 + \frac{2r}{s}(x - y) \bullet (v - u),$$

and we infer

$$r\Big(1 - \frac{r}{s}\Big)|u - v|^2 \le \frac{1}{s}|x - y|^2 + \Big(1 - \frac{2r}{s}\Big)|x - y||u - v|. \tag{21}$$

If $\frac{1}{s}|x - y|^2 \le \big(1 - \frac{2r}{s}\big)|x - y||u - v|$, then

$$|u - v|^2 \le \frac{2(s - 2r)}{r(s - r)}|x - y||u - v|.$$

If $\frac{1}{s}|x - y|^2 \ge \big(1 - \frac{2r}{s}\big)|x - y||u - v|$, then

$$|u - v|^2 \le \frac{2}{r(s - r)}|x - y|^2,$$

which yields the asserted upper bound. ∎

Remark 2.13. For convex bodies, Lemma 2.12 is provided in [Hu99, Lemma 1.28] (see also [Hu96, Lemma 2.1] for a less explicit statement and the literature cited there). In this special case, it can be seen from (21) that the Lipschitz constant is bounded from above by $1/r$ (with $s = \infty$).

Remark 2.14. For a closed set $A \subset \mathbf{R}^{n+1}$ and $r, s > 0$, let $X_{r,s}(A)$ denote the set of all $a \in \partial A$ such that $B(a - ru, r) \subseteq A$ and $U(a + su, s) \cap A = \emptyset$ for some $u \in \mathbf{S}^n$. Then $X_{r,s}(A) \subset \partial^m A$, for any $a \in X_{r,s}(A)$ the unit vector u is equal to $\boldsymbol{n}(A, a)$ (and uniquely determined) and $\{\boldsymbol{n}(A, a)\} = N(A, a) \cap \mathbf{S}^n$. If $0 < r \le s/4$, then Lemma 2.12 yields that $\boldsymbol{n}(A, \cdot)|X_{r,s}(A)$ is Lipschitz continuous with Lipschitz constant bounded from above by $3/r$, since

$$\frac{2(s - 2r)}{r(s - r)} \le \frac{2s}{r(s - s/4)} = \frac{2s}{r\frac{3}{4}s} = \frac{8}{3}\frac{1}{r} < \frac{3}{r}$$

and

$$\sqrt{\frac{2}{r(s - r)}} \le \sqrt{\frac{2}{r\frac{3}{4}s}} = \sqrt{\frac{8}{3}\frac{1}{r2r}} = \frac{2}{\sqrt{3}}\frac{1}{r} < \frac{2}{r}.$$

3 A Steiner-type formula for arbitrary closed sets

Throughout this section, we assume that ϕ is a uniformly convex $\mathcal{C}^2$-norm. Recalling Lemma 2.7 we start by introducing the following definition.

3.1 Normal bundle and curvatures

We start introducing the principal curvature of the level sets $S^\phi(A,r)$ of the distance function $\boldsymbol{\delta}^\phi_A$ taking the eigenvalues of the normal vector field $\boldsymbol{\nu}^\phi_A$ defined in equation (11).

Definition 3.1. Suppose $\varnothing \neq A \subseteq \mathbf{R}^{n+1}$ is closed, $x \in \operatorname{dmn}(\mathrm{D}\,\boldsymbol{\nu}^\phi_A)$ and $r = \boldsymbol{\delta}^\phi_A(x)$. Then the eigenvalues (counted with their algebraic multiplicities) of $\mathrm{D}\,\boldsymbol{\nu}^\phi_A(x)|\operatorname{Tan}(S^\phi(A,r),x)$ are denoted by

$$\chi^\phi_{A,1}(x) \leq \ldots \leq \chi^\phi_{A,n}(x).$$

Lemma 3.2. *The set* $\operatorname{dmn}(\mathrm{D}\,\boldsymbol{\nu}^\phi_A) \subseteq \operatorname{Unp}^\phi(A)$ *is a Borel subset of* $\mathbf{R}^{n+1}$ *and the functions* $\chi^\phi_{A,i} : \operatorname{dmn}(\mathrm{D}\,\boldsymbol{\nu}^\phi_A) \to \mathbf{R}$ *are Borel functions for* $i \in \{1,\ldots,n\}$.

Proof. Let $\mathcal{X}$ be the set of all $\varphi \in \operatorname{Hom}(\mathbf{R}^{n+1},\mathbf{R}^{n+1})$ with real eigenvalues. For each $\varphi \in \mathcal{X}$ we define $\lambda_0(\varphi) \leq \ldots \leq \lambda_n(\varphi)$ to be the eigenvalues of φ counted with their algebraic multiplicity, and then we define the map $\lambda : \mathcal{X} \to \mathbf{R}^{n+1}$ by

$$\lambda(\varphi) = (\lambda_0(\varphi),\ldots,\lambda_n(\varphi)) \qquad \text{for } \varphi \in \mathcal{X}.$$

We observe that $\mathcal{X}$ is a Borel set and λ is a continuous map by [HM87, Theorem A]. Moreover we notice that $\operatorname{dmn}(\mathrm{D}\,\boldsymbol{\nu}^\phi_A) = \operatorname{dmn}(\mathrm{D}\,\boldsymbol{\xi}^\phi_A)$ and that this is a Borel subset of $\mathbf{R}^{n+1}$ by Lemma 2.3. For each $x \in \operatorname{dmn}(\mathrm{D}\,\boldsymbol{\xi}^\phi_A)$, we have $\mathrm{D}\,\boldsymbol{\xi}^\phi_A(x)(\boldsymbol{\nu}^\phi_A(x)) = 0$ and $\boldsymbol{\delta}^\phi_A(x)\cdot\boldsymbol{\nu}^\phi_A(x) = x - \boldsymbol{\xi}^\phi_A(x)$, hence

$$\lambda_0(\mathrm{D}\,\boldsymbol{\xi}^\phi_A(x)) = 0 \qquad \text{and} \qquad \lambda_i(\mathrm{D}\,\boldsymbol{\xi}^\phi_A(x)) = 1 - \boldsymbol{\delta}^\phi_A(x)\chi^\phi_{A,n+1-i}(x) \geq 0 \quad \text{for } i = 1,\ldots,n,$$

where also (17) was used. Since the map $\mathrm{D}\,\boldsymbol{\xi}^\phi_A : \operatorname{dmn}(\mathrm{D}\,\boldsymbol{\xi}^\phi_A) \to \mathcal{X}$ is a Borel function, we obtain the assertion. $\square$

Remark 3.3 (cf. [KS21, Lemmas 2.41 and 2.44]). Suppose $\varnothing \neq A \subseteq \mathbf{R}^{n+1}$ is closed, $x \in \operatorname{Unp}^\phi(A)$, $r = \boldsymbol{\delta}^\phi_A(x)$, $0 < t < 1$ and $y = \boldsymbol{\xi}^\phi_A(x) + tr\boldsymbol{\nu}^\phi_A(x) = \boldsymbol{\xi}^\phi_A(x) + t(x - \boldsymbol{\xi}^\phi_A(x))$. Then $y \in \operatorname{Unp}^\phi(A)$,

$$\operatorname{Tan}(S^\phi(A,r),x) = \{v \in \mathbf{R}^{n+1} : v \bullet \nabla\boldsymbol{\delta}^\phi_A(x) = 0\},$$

$$\nabla\boldsymbol{\delta}^\phi_A(x) = \nabla\boldsymbol{\delta}^\phi_A(y) \qquad \text{and} \qquad \operatorname{Tan}(S^\phi(A,r),x) = \operatorname{Tan}(S^\phi(A,tr),y).$$

Remark 3.4. For $(a,\eta) \in N^\phi(A)$ and $0 < r < \boldsymbol{r}^\phi_A(a,\eta)$ we have

$$\operatorname{Tan}(S^\phi(A,r),a+r\eta) = \operatorname{Tan}(\partial\mathcal{W}^\phi,\eta).$$

Setting $u = \frac{\nabla\boldsymbol{\delta}^\phi_A(a+r\eta)}{|\nabla\boldsymbol{\delta}^\phi_A(a+r\eta)|}$, this assertion follows from Remark 3.3, noting that (see (8) and (12))

$$\nabla\phi(u) = \nabla\phi(\nabla\boldsymbol{\delta}^\phi_A(a+r\eta)) = \eta, \qquad \boldsymbol{n}^\phi(\eta) = u.$$

Lemma 3.5. *Suppose* $\varnothing \neq A \subseteq \mathbf{R}^{n+1}$ *is a closed set,* $(a,\eta) \in N^\phi(A)$, $0 < r < s < \boldsymbol{r}^\phi_A(a,\eta)$ *so that* $a + r\eta, a + s\eta \in \operatorname{dmn}\mathrm{D}\,\boldsymbol{\nu}^\phi_K$ *and* $\tau_1,\ldots,\tau_n \in \operatorname{Tan}(\partial\mathcal{W}^\phi,\eta)$.

Then $\mathrm{D}\,\boldsymbol{\nu}^\phi_A(a+r\eta)\tau_i = \chi^\phi_{A,i}(a+r\eta)\tau_i$ *for* $i = 1,\ldots,n$ *if and only if* $\mathrm{D}\,\boldsymbol{\nu}^\phi_A(a+s\eta)\tau_i = \chi^\phi_{A,i}(a+s\eta)\tau_i$ *for* $i = 1,\ldots,n$, *in which case it holds that*

$$\frac{\chi^\phi_{A,i}(a+r\eta)}{1 - r\chi^\phi_{A,i}(a+r\eta)} = \frac{\chi^\phi_{A,i}(a+s\eta)}{1 - s\chi^\phi_{A,i}(a+s\eta)} \quad \text{for } i = 1,\ldots,n.$$

Proof. We define $x = a + r\eta$, $y = a + s\eta$ and $t = \frac{r}{s} \in (0, 1)$. We notice that $\boldsymbol{\xi}_A^\phi$ is differentiable at y and

$$\mathrm{D}\,\boldsymbol{\xi}_A^\phi(y)|\operatorname{Tan}(\partial\mathcal{W}^\phi, \eta) = \mathrm{Id}_{\operatorname{Tan}(\partial\mathcal{W}^\phi,\eta)} - s\,\mathrm{D}\,\boldsymbol{\nu}_A^\phi(y)|\operatorname{Tan}(\partial\mathcal{W}^\phi, \eta).$$

Let $\xi : \mathbf{R}^{n+1} \setminus A \to A$ be such that $\xi(z) \in \boldsymbol{\xi}_A^\phi(z)$ for $z \in \mathbf{R}^{n+1} \setminus A$. Then define $\nu : \mathbf{R}^{n+1} \setminus A \to \partial\mathcal{W}^\phi$ by $\nu(z) = \boldsymbol{\delta}_A^\phi(z)^{-1}(z - \xi(z))$ for $z \in \mathbf{R}^{n+1} \setminus A$. It follows from the strict convexity of ϕ (see [KS21, Remark 2.17]) that

$$\nu(\xi(z) + t(z - \xi(z))) = \nu(z) \qquad \text{for } z \in \mathbf{R}^{n+1} \setminus A.$$

Differentiating this equality in y, we obtain

$$\mathrm{D}\,\nu(x) \circ [\mathrm{D}\,\xi(y) + t(\mathrm{Id}_{\mathbf{R}^{n+1}} - \mathrm{D}\,\xi(y))] = \mathrm{D}\,\nu(y).$$

Assume now that $\mathrm{D}\,\nu(y)\tau_i = \chi_{A,i}^\phi(y)\tau_i$ for $i = 1, \ldots, n$. Then

$$\mathrm{D}\,\boldsymbol{\xi}_A^\phi(y)\tau_i = \tau_i - s \cdot \mathrm{D}\,\boldsymbol{\nu}_A^\phi(y)\tau_i = (1 - s\chi_{A,i}^\phi(y))\tau_i,$$

and hence we get

$$\chi_{A,i}^\phi(y)\tau_i = [1 - (s - r)\chi_{A,i}^\phi(y)]\,\mathrm{D}\,\nu(x)\tau_i \qquad \text{for } i = 1, \ldots, n.$$

Note that by (16) we have $\boldsymbol{\rho}_A^\phi(y)s - s > r - s$, and hence by the lower bound in (17) we get

$$1 + (r - s)\chi_{A,i}^\phi(y) \geq 1 + (r - s)\frac{1}{(1 - \boldsymbol{\rho}_A^\phi(y))s} = 1 - \frac{r - s}{\boldsymbol{\rho}_A^\phi(y)s - s} > 0,$$

that is, $1 - (s - r)\chi_{A,i}^\phi(y) > 0$ for $i = 1, \ldots, n$. We conclude that

$$\mathrm{D}\,\nu(x)\tau_i = \chi_{A,i}^\phi(x)\tau_i, \qquad \chi_{A,i}^\phi(x) = \frac{\chi_{A,i}^\phi(y)}{1 - (s - r)\chi_{A,i}^\phi(y)}$$

and

$$\frac{\chi_{A,i}^\phi(x)}{1 - r\chi_{A,i}^\phi(x)} = \frac{\chi_{A,i}^\phi(y)}{1 - s\chi_{A,i}^\phi(y)}$$

for $i = 1, \ldots, n$ (where this common ratio may be infinite).

The last paragraph shows in particular that $\mathrm{D}\,\nu(x)|\operatorname{Tan}(\partial\mathcal{W}^\phi, \eta)$ and $\mathrm{D}\,\nu(y)|\operatorname{Tan}(\partial\mathcal{W}^\phi, \eta)$ have the same number k of distinct eigenvalues. Denoting by $N_1(x), \ldots, N_k(x)$ and $N_1(y), \ldots, N_k(y)$ the eigenspaces of $\mathrm{D}\,\nu(x)|\operatorname{Tan}(\partial\mathcal{W}^\phi, \eta)$ and $\mathrm{D}\,\nu(y)|\operatorname{Tan}(\partial\mathcal{W}^\phi, \eta)$ respectively, we can also derive from the last paragraph the inclusions $N_i(y) \subseteq N_i(x)$ for $i = 1, \ldots, k$. Since

$$N_1(y) \oplus \cdots \oplus N_k(y) = \operatorname{Tan}(\partial\mathcal{W}^\phi, \eta) = N_1(x) \oplus \cdots \oplus N_k(x),$$

we conclude that $N_i(y) = N_i(x)$ for $i = 1, \ldots, n$ and the proof is completed. ∎

Definition 3.6. Suppose $\varnothing \neq A \subseteq \mathbf{R}^{n+1}$ is closed. We define

$$\widetilde{N}^\phi(A) = \{(a, \eta) \in N^\phi(A) : a + r\eta \in \operatorname{dmn}(\mathrm{D}\,\boldsymbol{\nu}_A^\phi) \text{ for some } r \in (0, \underline{\boldsymbol{r}_A^\phi}(a, \eta))\}$$

and

$$\kappa_{A,i}^\phi(a, \eta) = \frac{\chi_{A,i}^\phi(a + r\eta)}{1 - r\chi_{A,i}^\phi(a + r\eta)} \in (-\infty, \infty]$$

for $(a, \eta) \in \widetilde{N}^\phi(A)$, $0 < r < \underline{\boldsymbol{r}_A^\phi}(a, \eta)$ with $a + r\eta \in \operatorname{dmn}\mathrm{D}\,\boldsymbol{\nu}_A^\phi$ and $i = 1, \ldots, n$. The numbers $\kappa_{A,i}^\phi(a, \eta)$, $i \in \{1, \ldots, n\}$, are called *anisotropic (with respect to ϕ) generalized curvatures of A at (a, η)* or *generalized ϕ-curvatures of A at (a, η)*.

Remark 3.7. Lemma 3.5 demonstrates that the definition of $\kappa^{\phi}_{A,i}(a,\eta)$ does not depend on the choice of r and Lemma 2.7 implies that $\mathcal{H}^n(N^{\phi}(A)\setminus \widetilde{N}^{\phi}(A))=0$. Moreover, Lemma 2.7 ensures that all points of the open segment $\{a+r\eta : 0<r<\underline{\boldsymbol{r}}^{\phi}_A(a,\eta)\}$ are points of differentiability of $\boldsymbol{\nu}^{\phi}_A$ for every $(a,\eta)\in \widetilde{N}^{\phi}(A)$; in other words,

$$\widetilde{N}^{\phi}(A)=\{(a,\eta)\in N^{\phi}(A) : \underline{\boldsymbol{r}}^{\phi}_A(a,\eta)>0,\ a+r\eta\in \operatorname{dmn} \mathrm{D}\,\boldsymbol{\nu}^{\phi}_A \text{ for every } r\in(0,\underline{\boldsymbol{r}}^{\phi}_A(a,\eta))\}.$$

Combining Lemma 3.2 and Lemma 2.7 it follows that $\widetilde{N}^{\phi}(A)$ is a Borel subset of $N^{\phi}(A)$. Moreover by Lemma 3.2 we deduce that the functions $\kappa^{\phi}_{A,i}$ are Borel measurable. Since the eigenvalues $\chi^{\phi}_{A,i}(a+r\eta)$, $i=1,\ldots,n$, are arranged in increasing order, we also have $-\infty<\kappa^{\phi}_{A,1}(a,\eta)\leq\ldots\leq\kappa^{\phi}_{A,n}(a,\eta)\leq\infty$.

Remark 3.8. Using (16), Lemma 2.7, Lemma 2.8 (a), (17) and Definition 3.6, we obtain that

$$-\frac{1}{\boldsymbol{r}^{\phi}_A(a,\eta)}\leq \kappa^{\phi}_{A,i}(a,\eta)\leq+\infty$$

for $(a,\eta)\in\widetilde{N}^{\phi}(A)$ and $i=1,\ldots,n$.

Lemma 3.9. *Let $\varnothing\neq A\subseteq \mathbf{R}^{n+1}$ be closed. Suppose $\tau_i:\widetilde{N}^{\phi}(A)\to\mathbf{R}^{n+1}$, for $i=1,\ldots,n$, are defined so that $\tau_1(a,\eta),\ldots,\tau_n(a,\eta)$ form a basis of $\operatorname{Tan}(\partial\mathcal{W}^{\phi},\eta)$ with*

$$\mathrm{D}\,\boldsymbol{\nu}^{\phi}_A(a+r\eta)(\tau_i(a,\eta))=\chi^{\phi}_{A,i}(a+r\eta)\tau_i(a,\eta)\qquad \text{for } i=1,\ldots,n \text{ and } 0<r<\underline{\boldsymbol{r}}^{\phi}_A(a,\eta). \tag{22}$$

Let $\zeta_i:\widetilde{N}^{\phi}(A)\to\mathbf{R}^{n+1}\times\mathbf{R}^{n+1}$, for $i=1,\ldots,n$, be defined so that

$$\zeta_i(a,\eta)=\begin{cases}(\tau_i(a,\eta),\kappa^{\phi}_{A,i}(a,\eta)\tau_i(a,\eta)), & \text{if } \kappa^{\phi}_{A,i}(a,\eta)<\infty,\\ (0,\tau_i(a,\eta)), & \text{if } \kappa^{\phi}_{A,i}(a,\eta)=+\infty.\end{cases} \tag{23}$$

Let $W\subseteq N^{\phi}(A)$ be an $\mathcal{H}^n$ measurable set with $\mathcal{H}^n(W)<\infty$. Then, for $\mathcal{H}^n$ almost all $(a,\eta)\in W$, the set $\operatorname{Tan}^n(\mathcal{H}^n\llcorner W,(a,\eta))$ is an n-dimensional linear subspace and $\zeta_1(a,\eta),\ldots,\zeta_n(a,\eta)$ form a basis of $\operatorname{Tan}^n(\mathcal{H}^n\llcorner W,(a,\eta))$. Moreover,

$$\operatorname{ap} J^W_n\boldsymbol{p}(a,\eta)=\frac{|\tau_1(a,\eta)\wedge\ldots\wedge\tau_n(a,\eta)|}{|\zeta_1(a,\eta)\wedge\ldots\wedge\zeta_n(a,\eta)|}\,\mathbf{1}_{\widetilde{N}^{\phi}_n(A)}(a,\eta)$$

for $\mathcal{H}^n$ almost all $(a,\eta)\in W$.

Proof. Assume that $W\subseteq N^{\phi}(A)$ is $\mathcal{H}^n$ measurable with $\mathcal{H}^n(W)<\infty$ and $\lambda>1$. For $r>0$ we define

$$W_r=\{(a,\eta)\in W:\boldsymbol{r}^{\phi}_A(a,\eta)=\underline{\boldsymbol{r}}^{\phi}_A(a,\eta)\geq\lambda r\},$$

which is an $\mathcal{H}^n$ measurable subset of W. Furthermore, we denote by W_r^* the set of all $(a,\eta)\in W_r$ such that $\operatorname{Tan}^n(\mathcal{H}^n\llcorner W,(a,\eta))$ is an n-dimensional linear subspace and $\operatorname{Tan}(\mathcal{H}^n\llcorner W,(a,\eta))=\operatorname{Tan}^n(\mathcal{H}^n\llcorner W_r,(a,\eta))$. It follows from [Fed69, 3.2.19] that $\mathcal{H}^n(W_r\setminus W_r^*)=0$. Moreover, by the coarea formula there exists $J\subseteq(0,\infty)$ with $\mathcal{H}^1(J)=0$ such that $\mathcal{H}^n(S^{\phi}(A,r)\setminus\operatorname{Unp}^{\phi}(A))=0$ for $r\notin J$.

We fix $r>0$, $r\notin J$, and define

$$M_r=\{a+r\eta:(a,\eta)\in W_r\}.$$

It follows from (16) that $M_r\subseteq\{x\in S^{\phi}(A,r):\boldsymbol{\rho}^{\phi}_A(x)\geq\lambda\}$ (see also (50)); moreover, M_r is $\mathcal{H}^n$ measurable. By Lemma 2.6 the function $\boldsymbol{\psi}^{\phi}_A|M_r$ is Lipschitz; moreover, we notice that $\boldsymbol{\psi}^{\phi}_A(M_r)=W_r$ and $(\boldsymbol{\psi}^{\phi}_A|M_r)^{-1}(a,\eta)=a+r\eta$ for $(a,\eta)\in W_r$. We denote by M_r^* the set of all $x\in M_r$ such that $\operatorname{Tan}(S^{\phi}(A,r),x)$ is an n-dimensional linear subspace and $\operatorname{Tan}^n(\mathcal{H}^n\llcorner M_r,x)=\operatorname{Tan}(S^{\phi}(A,r),x)$. It follows from Remark 3.3 and [Fed69, 3.2.19] that $\mathcal{H}^n(M_r\setminus M_r^*)=0$. We conclude that

$$\mathcal{H}^n(W_r\setminus(W_r^*\cap\boldsymbol{\psi}^{\phi}_A(M_r^*)))=0.$$

Further, if $(a,\eta) \in (W_r^* \cap \boldsymbol{\psi}_A^\phi(M_r^*)) \cap \widetilde{N}^\phi(A)$ it follows from [San20, Lemma B.2] and Remark 3.4 that $\{\tau_1(a,\eta),\dots,\tau_n(a,\eta)\}$ is a basis of $\operatorname{Tan}(S^\phi(A,r),a+r\eta)$,

$$\mathrm{D}\,\boldsymbol{\psi}_A^\phi(a+r\eta)[\operatorname{Tan}(S^\phi(A,r),a+r\eta)] = \operatorname{Tan}^n(\mathcal{H}^n \llcorner W,(a,\eta))$$

and

$$\mathrm{D}\,\boldsymbol{\psi}_A^\phi(a+r\eta)(\tau_i(a,\eta)) = \begin{cases} \frac{1}{1+r\kappa^\phi_{A,i}(a,\eta)}\zeta_i(a,\eta), & \text{if } \kappa^\phi_{A,i}(a,\eta)<\infty,\\ \frac{1}{r}\zeta_i(a,\eta), & \text{if } \kappa^\phi_{A,i}(a,\eta)=+\infty.\end{cases}$$

This proves that $\{\zeta_1(a,\eta),\dots,\zeta_n(a,\eta)\}$ is a basis of $\operatorname{Tan}^n(\mathcal{H}^n \llcorner W,(a,\eta))$ for $\mathcal{H}^n$ a.e. $(a,\eta)\in W_r$ and for every $r \notin J$.

Since $W \setminus \bigcup_{r>0} W_r$ has $\mathcal{H}^n$ measure zero and $W_r \subseteq W_s$ for $0<s<r$, there exists a sequence $r_i \searrow 0$, $r_i \notin J$, so that $W \setminus \bigcup_{i=1}^\infty W_{r_i}$ has $\mathcal{H}^n$ measure zero, which completes the proof. $\square$

Remark 3.10. The existence of a basis $\tau_1(a,\eta),\dots,\tau_n(a,\eta)$ of $\operatorname{Tan}(\partial\mathcal{W}^\phi,\eta)$ such that (22) is satisfied is based on Lemma 2.8 (c).

We now provide an alternative but equivalent description which leads to some additional information that will be useful in the proof of Theorem 3.27. Let $(a,\eta)\in\widetilde{N}^\phi(A)$, $r\in(0,\underline{\boldsymbol{r}}_A^\phi(a,\eta))$ and $x=a+r\eta$. Let $u:\mathbf{R}^{n+1}\setminus A\to\mathbf{S}^n$ be the map defined in Lemma 2.8 and notice that $u(x)=\boldsymbol{n}^\phi(\eta)$. Hence we have $\operatorname{Tan}(\partial\mathcal{W}^\phi,\eta)=u(x)^\perp=\{z\in\mathbf{R}^{n+1}: z\bullet u(x)=0\}$. We define a symmetric bilinear form (i.e. an inner product)

$$B_\eta : \operatorname{Tan}(\partial\mathcal{W}^\phi,\eta)\times\operatorname{Tan}(\partial\mathcal{W}^\phi,\eta)\to\mathbf{R}$$

setting

$$B_\eta(\tau,\sigma)=\mathrm{D}\,\boldsymbol{n}^\phi(\eta)(\tau)\bullet\sigma=[\mathrm{D}(\nabla\phi)(u(x))|u(x)^\perp]^{-1}(\tau)\bullet\sigma$$

for $\tau,\sigma\in\operatorname{Tan}(\partial\mathcal{W}^\phi,\eta)=u(x)^\perp$. Using Lemma 2.8(b), we see that

$$B_\eta(\mathrm{D}\,\boldsymbol{\nu}_A^\phi(x)(\tau),\sigma)=\mathrm{D}\,u(x)(\tau)\bullet\sigma=\tau\bullet\mathrm{D}\,u(x)(\sigma)=B_\eta(\tau,\mathrm{D}\,\boldsymbol{\nu}_A^\phi(x)(\sigma))\quad\text{for }\tau,\sigma\in u(x)^\perp.$$

This shows that $\mathrm{D}\,\boldsymbol{\nu}_A^\phi(x)|u(x)^\perp$ is self-adjoint with respect to B_η. Hence there is an orthonormal basis $\tau_1(a,\eta),\dots,\tau_n(a,\eta)$ of $u(x)^\perp$ with respect to B_η consisting of eigenvectors of $\mathrm{D}\,\boldsymbol{\nu}_A^\phi(x)$. Henceforth we can assume that $\tau_1(a,\eta),\dots,\tau_n(a,\eta)$ are chosen in this way.

We now consider the natural extension of the inner product B_η to the product space $\operatorname{Tan}(\partial\mathcal{W}^\phi,\eta)\times\operatorname{Tan}(\partial\mathcal{W}^\phi,\eta)=\boldsymbol{n}^\phi(\eta)^\perp\times\boldsymbol{n}^\phi(\eta)^\perp$ given by

$$\overline{B}_\eta((\tau_1,\sigma_1),(\tau_2,\sigma_2))=B_\eta(\tau_1,\sigma_1)+B_\eta(\tau_2,\sigma_2)$$

for $(\tau_1,\sigma_1),(\tau_2,\sigma_2)\in\operatorname{Tan}(\partial\mathcal{W}^\phi,\eta)\times\operatorname{Tan}(\partial\mathcal{W}^\phi,\eta)$ and for every $\eta\in\partial\mathcal{W}^\phi$. With respect to this inner product, the linearly independent vectors $\zeta_1(a,\eta),\dots,\zeta_n(a,\eta)$ from (23) are pairwise orthogonal. Let $|\cdot|_\eta$ denote the norm induced by $\overline{B}_\eta$ on $\bigwedge_m(\boldsymbol{n}^\phi(\eta)^\perp\times\boldsymbol{n}^\phi(\eta)^\perp)$, for $m\in\{1,\dots,n\}$. Then using continuity and compactness one can show that

$$\begin{aligned}0<c:=\inf\big\{|A|_\eta : A\in{\textstyle\bigwedge_m}(\boldsymbol{n}^\phi(\eta)^\perp\times\boldsymbol{n}^\phi(\eta)^\perp),|A|=1,\eta\in\partial\mathcal{W}^\phi, m\in\{1,\dots,n\}\big\}\\ \le\sup\big\{|A|_\eta : A\in{\textstyle\bigwedge_m}(\boldsymbol{n}^\phi(\eta)^\perp\times\boldsymbol{n}^\phi(\eta)^\perp),|A|=1,\eta\in\partial\mathcal{W}^\phi, m\in\{1,\dots,n\}\big\}=:C<\infty.\end{aligned}$$

Thus we obtain the inequality

$$\begin{aligned}|\zeta_1(a,\eta)\wedge\cdots\wedge\zeta_n(a,\eta)| &\ge \frac{1}{C}\,|\zeta_1(a,\eta)\wedge\cdots\wedge\zeta_n(a,\eta)|_\eta\\ &=\frac{1}{C}\,|\zeta_1(a,\eta)\wedge\cdots\wedge\zeta_m(a,\eta)|_\eta\cdot|\zeta_{m+1}(a,\eta)\wedge\cdots\wedge\zeta_n(a,\eta)|_\eta\\ &\ge\frac{c^2}{C}\,|\zeta_1(a,\eta)\wedge\cdots\wedge\zeta_m(a,\eta)|\cdot|\zeta_{m+1}(a,\eta)\wedge\cdots\wedge\zeta_n(a,\eta)|,\end{aligned}\tag{24}$$

which will be used in the proof of Theorem 3.27.

In the following, it will be useful to distinguish how many of the generalized curvatures are finite for a given $(a,\eta)\in\widetilde{N}^{\phi}(A)$.

Definition 3.11. Let $\varnothing\neq A\subseteq\mathbf{R}^{n+1}$ be closed. We define

$$\widetilde{N}^{\phi}_{d}(A)=\{(a,\eta)\in\widetilde{N}^{\phi}(A):\kappa^{\phi}_{A,d}(a,\eta)<\infty,\kappa^{\phi}_{A,d+1}(a,\eta)=\infty\}\quad\text{for } d\in\{1,\ldots,n-1\},$$

$$\widetilde{N}^{\phi}_{0}(A)=\{(a,\eta)\in\widetilde{N}^{\phi}(A):\kappa^{\phi}_{A,1}(a,\eta)=\infty\}$$

and

$$\widetilde{N}^{\phi}_{n}(A)=\{(a,\eta)\in\widetilde{N}^{\phi}(A):\kappa^{\phi}_{A,n}(a,\eta)<\infty\}.$$

For $a\in A$ we set $\widetilde{N}^{\phi}_{d}(A,a)=\{\eta:(a,\eta)\in\widetilde{N}^{\phi}_{d}(A)\}$. Moreover, for $j\in\{0,\ldots,n\}$ we define the map $E^{\phi}_{A,j}:\widetilde{N}^{\phi}(A)\to\mathbf{R}$ by

$$E^{\phi}_{A,j}(a,\eta)=\begin{cases}\displaystyle\sum_{\lambda\in\Lambda(d,j)}\kappa^{\phi}_{A,\lambda(1)}(a,\eta)\cdots\kappa^{\phi}_{A,\lambda(j)}(a,\eta), & \text{if } (a,\eta)\in\widetilde{N}^{\phi}_{d}(A)\text{ and } d\geq j,\\ 0, & \text{if } (a,\eta)\in\widetilde{N}^{\phi}_{d}(A)\text{ and } d<j,\end{cases}$$

and for $j=0$ this means that $E^{\phi}_{A,0}\equiv 1$. Finally, for $r\in\{0,\ldots,n\}$ we define the *r-th ϕ-mean curvature of A* as

$$\boldsymbol{H}^{\phi}_{A,r}=\sum_{j=0}^{r}E^{\phi}_{A,j}\,\mathbf{1}_{\widetilde{N}^{\phi}_{j+n-r}(A)}=\sum_{i=0}^{r}E^{\phi}_{A,r-i}\,\mathbf{1}_{\widetilde{N}^{\phi}_{n-i}(A)}.$$

Remark 3.12. The sets $\widetilde{N}^{\phi}_{d}(A)$ and functions $\boldsymbol{H}^{\phi}_{A,r}$ introduced in Definition 3.11 are Borel measurable (see Remark 3.7). In particular, by definition we have $\boldsymbol{H}^{\phi}_{A,0}=\mathbf{1}_{\widetilde{N}^{\phi}_{n}(A)}$.

The next result will be used repeatedly in Section 6, in the special case of sets with positive reach. We prepare it by recalling a fact from linear algebra, which can be easily deduced from the standard spectral theorem (cf. [DRKS20, Remark 2.25]).

Remark 3.13. Suppose X is an n-dimensional Hilbert space, $M_1,M_2\in\operatorname{Hom}(X,X)$ are self-adjoint and M_1 is positive definite. Then there exist $C\in\operatorname{Hom}(X,X)$ self-adjoint and positive definite, n real numbers $\lambda_1\leq\ldots\leq\lambda_n$ and an orthonormal basis $v_1,\ldots,v_n$ of X such that $C\circ C=M_1$ and

$$(M_1\circ M_2)(C(v_i))=\lambda_i C(v_i)\qquad\text{for } i=1,\ldots,n.$$

Lemma 3.14. *For every closed set $\varnothing\neq A\subseteq\mathbf{R}^{n+1}$ the following statements hold.*

(a) *Suppose $s,r>0$, $x\in\operatorname{dmn}(\mathrm{D}\,\boldsymbol{\nu}^{\phi}_{A})\cap S^{\phi}(A,r)$ and V is an open neighbourhood of x in $\mathbf{R}^{n+1}$ such that $U^{\phi}(x-s\boldsymbol{\nu}^{\phi}_{A}(x),s)\cap S^{\phi}(A,r)\cap V=\varnothing$. Then*

$$\chi^{\phi}_{A,i}(x)\leq\frac{1}{s}\qquad\textit{for } i=1,\ldots,n.$$

(b) *If $(a,\eta)\in\widetilde{N}^{\phi}(A)$, $a\in\partial^{v}_{+}A$ and $r>0$ such that $U^{\phi}(a-r\eta,r)\subseteq\operatorname{int}A$, then $(a,\eta)\in\widetilde{N}^{\phi}_{n}(A)$ and*

$$\kappa^{\phi}_{A,n}(a,\eta)\leq\frac{1}{r}.$$

(c) *Suppose $a\in A^{(n)}\setminus\partial^{v}A$, $(a,\eta)\in\widetilde{N}^{\phi}(A)$, $W\subseteq\mathbf{R}^{n+1}$ is an open set with $a\in W$ and $f:W\cap(a+\eta^{\perp})\to\mathbf{R}$ is a function such that $f(a)=0$, f is continuous at a and* $\operatorname{graph}(f)\subseteq A$. *Then $(a,\eta)\in\widetilde{N}^{\phi}_{n}(A)$.*

Proof. (a) Choose a function $\xi:\mathbf{R}^{n+1}\to\mathbf{R}^{n+1}$ with $\xi(y)\in\boldsymbol{\xi}^{\phi}_{A}(y)$ for every $y\in\mathbf{R}^{n+1}$ and define

$$u(y)=\frac{\nabla\phi^{*}(y-\xi(y))}{|\nabla\phi^{*}(y-\xi(y))|}\qquad\text{for } y\in\mathbf{R}^{n+1}\setminus A.$$

Note that (12) and Remark 3.3 yield $u(x) = \frac{\nabla \boldsymbol{\delta}^\phi_A(x)}{|\nabla \boldsymbol{\delta}^\phi_A(x)|}$. Define the linear subspace $T = \operatorname{Tan}(S^\phi(A,r),x)$ and the open convex set $U = U^\phi(x - s\boldsymbol{\nu}^\phi_A(x), s)$. Since $x \in \operatorname{dmn}(\mathrm{D}\,\boldsymbol{\nu}^\phi_A)$, recalling [KS21, Lemma 2.44], we deduce that $u(x) \perp T = \operatorname{Tan}(\partial U, x)$ and there exist a relatively open subset W of $x+T$ with $x \in W$ and a continuous function $f : W \to \mathbf{R}$ that is pointwise twice differentiable at x such that $f(x) = 0$, $\mathrm{D}\, f(x) = 0$,

$$\{b - f(b)u(x) : b \in W\} \subseteq S^\phi(A,r) \cap V \quad \text{and} \quad \mathrm{D}^2 f(x) = \frac{1}{|\nabla \boldsymbol{\delta}^\phi_A(x)|}\,\mathrm{D}^2\, \boldsymbol{\delta}^\phi_A(x)|(T \times T).$$

Let $g : W \to \mathbf{R}$ be the smooth function such that $g(x) = 0$, $\mathrm{D}\, g(x) = 0$ and $\{b - g(b)u(x) : b \in W\} \subseteq \partial U$. Since $u(x) \bullet \boldsymbol{\nu}^\phi_A(x) = |\nabla \boldsymbol{\delta}^\phi_A(x)|^{-1} > 0$ and $U \cap S^\phi(A,r) \cap V = \varnothing$, we infer that $f(b) \le g(b)$ for $b \in W$. Let $v : \partial U \to \mathbf{S}^n$ be the exterior unit normal of ∂U and notice that $v(x) = u(x)$. From [KS21, Lemma 2.45] and Lemma 2.8(2b) we infer that

$$\mathrm{D}\, u(x)(\tau) \bullet \tau = \mathrm{D}^2 f(x)(\tau,\tau) \le \mathrm{D}^2 g(x)(\tau,\tau) = \mathrm{D}\, v(x)(\tau) \bullet \tau$$

for $\tau \in T$. Recalling again Lemma 2.8(2b), we can apply Remark 3.13 with the automorphisms $M_1 = \mathrm{D}(\nabla\phi)(u(x))|T$ and $M_2 = \mathrm{D}\, u(x)|T$ on T (and the induced scalar product of $\mathbf{R}^{n+1}$) to infer the existence of a selfadjoint linear map $C : T \to T$ and an orthonormal basis $\tau_1, \ldots, \tau_n$ of T such that $\mathrm{D}(\nabla\phi)(u(x))|T = C \circ C$ and $C(\tau_1), \ldots, C(\tau_n)$ is a basis of eigenvectors of $\mathrm{D}\,\boldsymbol{\nu}^\phi_A(x)|T$, i.e. $\mathrm{D}\,\boldsymbol{\nu}^\phi_A(x)(C(\tau_i)) = \chi^\phi_{A,i}(x)\, C(\tau_i)$ for $i \in \{1, \ldots, n\}$. Employing [DRKS20, Lemma 2.33], we notice that

$$\begin{aligned}
s^{-1}|C(\tau)|^2 &= (C^{-1} \circ \mathrm{D}(\nabla\phi \circ v)(x) \circ C)(\tau) \bullet \tau = (C^{-1} \circ \mathrm{D}(\nabla\phi)(v(x)) \circ \mathrm{D}\, v(x) \circ C)(\tau) \bullet \tau \\
&= (C \circ \mathrm{D}\, v(x) \circ C)(\tau) \bullet \tau = \mathrm{D}\, v(x)(C(\tau)) \bullet C(\tau) \\
&\ge \mathrm{D}\, u(x)(C(\tau)) \bullet C(\tau) = C^{-1} \circ \mathrm{D}(\nabla\phi \circ u)(x) \circ C)(\tau) \bullet \tau \\
&= (C^{-1} \circ \mathrm{D}\,\boldsymbol{\nu}^\phi_A(x) \circ C)(\tau) \bullet \tau
\end{aligned}$$

for $\tau \in T$. Evaluating this inequality at $\tau = \tau_i$ for $i = 1, \ldots, n$ we obtain the conclusion.

(b) Choose $0 < s < \underline{\boldsymbol{r}}^\phi_A(a,\eta)$. We observe that $S^\phi(A,s) \cap U^\phi(a - r\eta, r+s) = \varnothing$ and $a + s\eta \in S^\phi(A,s) \cap \partial \mathbf{B}^\phi(a - r\eta, r+s)$. Since $a + s\eta \in \operatorname{dmn} \mathrm{D}\,\boldsymbol{\nu}^\phi_A$ and $\boldsymbol{\nu}^\phi_A(a+s\eta) = \eta$, we can apply (a) with x and s replaced by $a + s\eta$ and $r+s$ respectively, to conclude that

$$\chi^\phi_{A,n}(a+s\eta) \le \frac{1}{r+s} \quad \text{and} \quad \kappa^\phi_{A,n}(a,\eta) = \frac{\chi^\phi_{A,n}(a+s\eta)}{1 - s\chi^\phi_{A,n}(a+s\eta)} \le \frac{\frac{1}{r+s}}{1 - \frac{s}{r+s}} = \frac{1}{r} < +\infty.$$

(c) We may assume that $a = 0$, and we denote by π the orthogonal projection onto $\eta^\perp$. We notice that there is some $s > 0$ such that $U^\phi(-s\eta, s) \cap A = \varnothing$. Then we choose $0 < r < \inf\{s, \underline{\boldsymbol{r}}^\phi_A(a,\eta)\}$ and we notice that $U^\phi(r\eta, r) \cap A = \varnothing$. Set $U_0 = \pi(U^\phi(-s\eta,s))$ and let $g : U_0 \to \mathbf{R}$ be the continuous function such that $g(0) = 0$,

$$\{b + g(b)\eta : b \in U_0\} \subseteq \partial U^\phi(-s\eta, s) \quad \text{and} \quad U^\phi(-s\eta,s) \subseteq \{b + t\eta : t < g(b),\ b \in U_0\}.$$

Since $U^\phi(-s\eta,s) \cap A = \varnothing$ it follows from the continuity of f at 0 that there is $\epsilon > 0$ such that $g(b) \le f(b) \le \frac{r}{2}$ for every $b \in \eta^\perp$ with $|b| < \epsilon$. We can assume that $W = \{b \in \eta^\perp : |b| < \epsilon\} \subseteq U_0$. Since $r\eta \in \operatorname{dmn} \mathrm{D}\,\boldsymbol{\nu}^\phi_A$ it follows from [KS21, Lemma 2.44] that there exists an open neighbourhood V of $r\eta$ in $\mathbf{R}^{n+1}$ such that $S^\phi(A,r) \cap V$ is equal to the graph of a continuous function. Therefore we can choose V small enough so that $\pi(V) \subseteq W$ and

$$S^\phi(A,r) \cap V \subseteq \Big\{y : |(y - r\eta) \bullet \eta| < \frac{r}{4} \cdot |\eta|^2\Big\}.$$

We claim that $[(S^\phi(A,r) \cap V) - r\eta] \cap U^\phi(-s\eta, s) = \varnothing$. In fact, assume by contradiction that there is some $z \in [(S^\phi(A,r) \cap V) - r\eta] \cap U^\phi(-s\eta,s)$. Then we obtain that

$$|z \bullet \eta| < \frac{r}{4} \cdot |\eta|^2 \quad \text{and} \quad \frac{z \bullet \eta}{|\eta|^2} < g(\pi(z)) \le f(\pi(z)) \le \frac{r}{2} < \frac{(z + r\eta) \bullet \eta}{|\eta|^2}.$$

It follows that $\pi(z)+f(\pi(z))\eta$ lies in the open segment joining z with $z+r\eta$. Since $\pi(z)+f(\pi(z))\eta \in A$ and $z+r\eta \in S^\phi(A,r)$, it follows that $r = \boldsymbol{\delta}^\phi_A(z+r\eta) < r$, which is a contradiction. Now we can use (a) to conclude that

$$\chi^\phi_{A,n}(r\eta) \le \frac{1}{s} \quad \text{and} \quad \kappa^\phi_{A,n}(a,\eta) \le \frac{1}{s-r} < \infty.$$

□

Remark 3.15. It follows from Lemma 3.14 that $(N^\phi(A)|\partial^v_+ A)\backslash \widetilde{N}^\phi_n(A) \subseteq N^\phi(A)\backslash \widetilde{N}^\phi(A)$. In particular,

$$\mathcal{H}^n\big((N^\phi(A)|\partial^v_+ A)\setminus \widetilde{N}^\phi_n(A)\big) = 0.$$

3.2 Steiner-type formula and disintegration of Lebesgue measure

The following result extends the Steiner-type formula from [HL00] (see also the literature cited there such as [Sta79]) to the anisotropic setting.

Theorem 3.16 (Steiner-type formula for closed sets)**.** *Let $\varnothing \ne A \subseteq \mathbf{R}^{n+1}$ be a closed set, let $\tau_1,\ldots,\tau_n$ and $\zeta_1,\ldots,\zeta_n$ be functions satisfying the hypotheses in Lemma 3.9 and let J be the function defined on $\mathcal{H}^n$ almost all of $N^\phi(A)$ by*

$$J(a,\eta) = \frac{|\tau_1(a,\eta)\wedge\ldots\wedge\tau_n(a,\eta)|}{|\zeta_1(a,\eta)\wedge\ldots\wedge\zeta_n(a,\eta)|} \in (0,\infty) \qquad \text{for } \mathcal{H}^n \text{ a.e. } (a,\eta)\in N^\phi(A).$$

Then the following statements hold.

(a) *For $0<s<\infty$ and $0<t<s$ let*

$$S^s_t = \{x \in S^\phi(A,t) : \rho^\phi_A(x) \ge s/t\} \qquad \text{and} \qquad N_s = \{(a,\eta)\in N^\phi(A) : \boldsymbol{r}^\phi_A(a,\eta) \ge s\}.$$

Then $\mathcal{H}^n(N_s \cap (B\times\partial\mathcal{W}^\phi)) < \infty$ for every compact set $B\subset\mathbf{R}^{n+1}$ and $\boldsymbol{\psi}^\phi_A|S^s_t$ is a bi-lipschitzian homeomorphism with $\boldsymbol{\psi}^\phi_A[S^s_t] = N_s$ and $(\boldsymbol{\psi}^\phi_A|S^s_t)^{-1}(a,\eta) = a+t\eta$ for each $(a,\eta)\in N_s$.

(b) *If $\tau'_1,\ldots,\tau'_n,\zeta'_1,\ldots,\zeta'_n$ is another set of functions satisfying the hypotheses of Lemma 3.9, then*

$$J(a,\eta) = \frac{|\tau'_1(a,\eta)\wedge\ldots\wedge\tau'_n(a,\eta)|}{|\zeta'_1(a,\eta)\wedge\ldots\wedge\zeta'_n(a,\eta)|}$$

for $\mathcal{H}^n$ a.e. $(a,\eta)\in N^\phi(A)$. Moreover J is $\mathcal{H}^n \llcorner N^\phi(A)$ measurable.

(c) *If $\rho>0$ and $B\subset\mathbf{R}^{n+1}$ is compact, then*

$$\int_{N^\phi(A)\cap(B\times\partial\mathcal{W}^\phi)} \inf\{\rho,\boldsymbol{r}^\phi_A\}^{j+1}\cdot J\cdot|\boldsymbol{H}^\phi_{A,j}|\,d\mathcal{H}^n < \infty \tag{25}$$

for $j=0,\ldots,n$ and

$$\int_{B^\phi(A,\rho)\backslash A} (\varphi\circ\boldsymbol{\psi}^\phi_A)\,d\mathcal{L}^{n+1} \\ = \sum_{j=0}^n \frac{1}{j+1}\int_{N^\phi(A)} \varphi(a,\eta)\,\phi(\boldsymbol{n}^\phi(\eta))\,J(a,\eta)\,\inf\{\rho,\boldsymbol{r}^\phi_A(a,\eta)\}^{j+1}\boldsymbol{H}^\phi_{A,j}(a,\eta)\,d\mathcal{H}^n(a,\eta) \tag{26}$$

for every bounded Borel function $\varphi:\mathbf{R}^{n+1}\times\mathbf{R}^{n+1}\to\mathbf{R}$ with compact support.

Proof. Fix $s>0$ and a compact set $B\subset\mathbf{R}^{n+1}$. We define $B_t = \{x\in\mathbf{R}^{n+1} : \boldsymbol{\delta}^\phi_B(x)\le t\}$ for $t>0$. One can easily check that $\boldsymbol{\psi}^\phi_A[S^s_t] = N_s$, $\boldsymbol{\psi}^\phi_A|S^s_t$ is injective and $(\boldsymbol{\psi}^\phi_A|S^s_t)^{-1}(a,\eta) = a+t\eta$ for $(a,\eta)\in N_s$ and $0<t<s$. Consequently $\boldsymbol{\psi}^\phi_A|S^s_t$ is a bi-lipschitzian homeomorphism by Lemma 2.6 for $0<t<s$. Moreover, we notice that

$$N_s\cap(B\times\partial\mathcal{W}^\phi) \subseteq \boldsymbol{\psi}^\phi_A(S^s_t\cap B_t) \subseteq \boldsymbol{\psi}^\phi_A(S^s_t\cap B_s) \qquad \text{for } 0<t<s$$

and infer that

$$\mathcal{H}^n(N_s \cap (B \times \partial \mathcal{W}^\phi)) \le \operatorname{Lip}(\boldsymbol{\psi}^\phi_A | S^s_t)^n \, \mathcal{H}^n(S^\phi(A,t) \cap B_s) \qquad \text{for } 0 < t < s. \tag{27}$$

Using the coarea formula, we get

$$\int_0^\infty \mathcal{H}^n(S^\phi(A,r) \cap B_s)\, dr = \int_{B_s} J_1 \boldsymbol{\delta}^\phi_A(x)\, d\mathcal{L}^{n+1}(x) < \infty,$$

whence we infer that $\mathcal{H}^n(S^\phi(A,r) \cap B_s) < \infty$ for $\mathcal{L}^1$ a.e. $r > 0$. Consequently, there exists some $t \in (0,s)$ so that $\mathcal{H}^n(S^\phi(A,t) \cap B_s) < \infty$, and hence (27) implies that

$$\mathcal{H}^n(N_s \cap (B \times \partial \mathcal{W}^\phi)) < \infty.$$

This proves (a).

For each $0 < s \le s'$ we define $f_s : N_{s'} \to \mathbf{R}^{n+1}$ by $f_s(a,\eta) = a + s\eta$ for $(a,\eta) \in N_{s'}$. We apply Lemma 3.9 to compute

$$\operatorname{ap} J^{N_{s'}}_n f_s(a,\eta) \cdot \mathbf{1}_{\widetilde{N}^\phi_d(A)}(a,\eta) = J(a,\eta) s^{n-d} \left(\prod_{j=1}^d (1 + s\kappa^\phi_{A,j}(a,\eta)) \right) \mathbf{1}_{\widetilde{N}^\phi_d(A)}(a,\eta) \tag{28}$$

for $\mathcal{H}^n$ a.e. $(a,\eta) \in N_s$. Since $1 + s\kappa^\phi_{A,i}(a,\eta) > 0$ for $i = 1, \ldots, n$ and for $\mathcal{H}^n$ a.e. $(a,\eta) \in N_{2s}$ by Remark 3.8, we conclude that

$$J(a,\eta) = \operatorname{ap} J^{N_{2s}}_n f_s(a,\eta) \sum_{d=0}^n s^{d-n} \left(\prod_{j=1}^d (1 + s\kappa^\phi_{A,j}(a,\eta)) \right)^{-1} \mathbf{1}_{\widetilde{N}^\phi_d(A)}(a,\eta)$$

for $\mathcal{H}^n$ a.e. $(a,\eta) \in N_{2s}$ and for every $s > 0$. Noting that the right-hand side of the last equation does not depend on the choice of $\tau_1, \ldots, \tau_n, \zeta_1, \ldots, \zeta_n$ and defines an $\mathcal{H}^n \llcorner N^\phi(A)$-measurable function, we obtain (b).

We fix $\rho > 0$. Then we define

$$\delta(a,\eta) = \inf\{\rho, \boldsymbol{r}^\phi_A(a,\eta)\} \qquad \text{for } (a,\eta) \in N^\phi(A),$$

$$\Omega = \{(a,\eta,t) : (a,\eta) \in N^\phi(A),\ 0 < t < \boldsymbol{r}^\phi_A(a,\eta)\}$$

and the bijective (locally) Lipschitz map

$$f : \Omega \to \mathbf{R}^{n+1} \setminus (A \cup \operatorname{Cut}^\phi(A)), \qquad f(a,\eta,t) = a + t\eta \quad \text{for } (a,\eta,t) \in \Omega.$$

We choose an arbitrary sequence $s_i \to 0+$ and define

$$\Omega_i = \{(a,\eta,t) : (a,\eta) \in N_{s_i},\ 0 < t < \boldsymbol{r}^\phi_A(a,\eta)\} \qquad \text{for } i \ge 1.$$

Notice that $\Omega = \bigcup_{i=1}^\infty \Omega_i$,

$$\operatorname{Tan}^{n+1}(\mathcal{H}^{n+1} \llcorner \Omega_i, (a,\eta,t)) = \operatorname{Tan}^n(\mathcal{H}^n \llcorner N_{s_i}, (a,\eta)) \times \mathbf{R} \qquad \text{for } \mathcal{H}^{n+1} \text{ a.e. } (a,\eta,t) \in \Omega_i$$

and $\operatorname{Tan}^n(\mathcal{H}^n \llcorner N_{s_i}, (a,\eta))$ is an n-dimensional linear subspace for $\mathcal{H}^n$ a.e. $(a,\eta) \in N_{s_i}$ by [Fed69, 3.2.19]. We apply again Lemma 3.9, for $\mathcal{H}^{n+1}$ a.e. $(a,\eta,t) \in \Omega_i$ and for every $i \ge 1$, to compute

$$\begin{aligned}
&\mathbf{1}_{\widetilde{N}^\phi_d(A)}(a,\eta) \cdot \operatorname{ap} J^{\Omega_i}_{n+1} f(a,\eta,t) \\
&= \mathbf{1}_{\widetilde{N}^\phi_d(A)}(a,\eta) \cdot \frac{|\tau_1(a,\eta) \wedge \ldots \wedge \tau_n(a,\eta) \wedge \eta|}{|\zeta_1(a,\eta) \wedge \ldots \wedge \zeta_n(a,\eta)|} t^{n-d} \left(\prod_{j=1}^d (1 + t\kappa^\phi_{A,j}(a,\eta)) \right) \\
&= \mathbf{1}_{\widetilde{N}^\phi_d(A)}(a,\eta) \cdot (\boldsymbol{n}^\phi(\eta) \bullet \eta) \frac{|\tau_1(a,\eta) \wedge \ldots \wedge \tau_n(a,\eta) \wedge \boldsymbol{n}^\phi(\eta)|}{|\zeta_1(a,\eta) \wedge \ldots \wedge \zeta_n(a,\eta)|} t^{n-d} \left(\prod_{j=1}^d (1 + t\kappa^\phi_{A,j}(a,\eta)) \right)
\end{aligned}$$

$$= \mathbf{1}_{\widetilde{N}_d^\phi(A)}(a,\eta)\cdot \phi(\boldsymbol{n}^\phi(\eta)) J(a,\eta) t^{n-d}\prod_{j=1}^{d}(1+t\kappa_{A,j}^\phi(a,\eta)). \tag{29}$$

Note that

$$B^\phi(A,\rho)\setminus\big(A\cup \mathrm{Cut}^\phi(A)\big) = f\big(\{(a,\eta,t)\in\Omega : 0<t\le\rho\}\big)$$

and recall that $\mathcal{L}^{n+1}(\mathrm{Cut}^\phi(A))=0$. Let $h:\mathbf{R}^{n+1}\to[0,\infty)$ be a Borel function. Then we employ the monotone convergence theorem, the coarea formula, Fubini's theorem and (29) to get

$$\begin{aligned}
&\int_{B^\phi(A,\rho)\setminus A} h(x)\,d\mathcal{L}^{n+1}(x)\\
&\quad=\lim_{i\to\infty}\sum_{d=0}^{n}\int_{N_{s_i}\cap\widetilde{N}_d^\phi(A)}\int_0^{\delta(a,\eta)} h(a+t\eta)\ \mathrm{ap}\,J_{n+1}^{\Omega_i}f(a,\eta,t)\,dt\,d\mathcal{H}^n(a,\eta)\\
&\quad=\sum_{d=0}^{n}\int_{\widetilde{N}_d^\phi(A)}\phi(\boldsymbol{n}^\phi(\eta))\,J(a,\eta)\bigg(\int_0^{\delta(a,\eta)} h(a+t\eta)\,t^{n-d}\prod_{j=1}^{d}(1+t\kappa_{A,j}^\phi(a,\eta))\,dt\bigg)\,d\mathcal{H}^n(a,\eta).
\end{aligned} \tag{30}$$

We notice the equality

$$t^{n-d}\prod_{j=1}^{d}(1+t\kappa_{A,j}^\phi(a,\eta))=\sum_{j=0}^{d}t^{n-d+j}E_{A,j}^\phi(a,\eta) \tag{31}$$

for $(a,\eta)\in\widetilde{N}_d^\phi(A)$ and $t>0$. If $h(x)=(\varphi\circ\boldsymbol{\psi}_A^\phi)(x)$ for every $x\in\mathrm{Unp}^\phi(A)$ for some Borel function $\varphi:\mathbf{R}^{n+1}\times\mathbf{R}^{n+1}\to[0,+\infty)$, then $h(a+t\eta)=\varphi(a,\eta)$ for every $(a,\eta,t)\in\Omega$ and we obtain from (30) and (31) that

$$\begin{aligned}
&\int_{B^\phi(A,\rho)\setminus A}(\varphi\circ\boldsymbol{\psi}_A^\phi)\,d\mathcal{L}^{n+1}\\
&\quad=\sum_{d=0}^{n}\int_{\widetilde{N}_d^\phi(A)}\varphi(a,\eta)\phi(\boldsymbol{n}^\phi(\eta))\,J(a,\eta)\,\sum_{j=0}^{d}\frac{\delta(a,\eta)^{n-d+j+1}}{n-d+j+1}\ E_{A,j}^\phi(a,\eta)\ d\mathcal{H}^n(a,\eta).
\end{aligned} \tag{32}$$

If $B\subset\mathbf{R}^{n+1}$ is a compact set, we can choose $\varphi=\mathbf{1}_{B\times\partial\mathcal{W}^\phi}$ in (32) to infer that

$$\int_{\widetilde{N}_d^\phi(A)\cap(B\times\partial\mathcal{W}^\phi)}\phi(\boldsymbol{n}^\phi(\eta))\,J(a,\eta)\,\sum_{j=0}^{d}\frac{\delta(a,\eta)^{n-d+j+1}}{n-d+j+1}\ E_{A,j}^\phi(a,\eta)\ d\mathcal{H}^n(a,\eta)<\infty \tag{33}$$

for $d=0,\dots,n$.

We now proceed as in [HLW04] to prove that for $d=0,\dots,n$ and $l=0,\dots,d$ there exists a constant $\bar{c}(l,d,n)>0$ so that

$$\sum_{j=0}^{d}\frac{\delta(a,\eta)^{n-d+j+1}}{n-d+j+1}\ E_{A,j}^\phi(a,\eta)\ge\bar{c}(l,d,n)\,\delta(a,u)^{n-d+l+1}\,|E_{A,l}^\phi(a,\eta)|\ge 0 \tag{34}$$

for $(a,\eta)\in\widetilde{N}_d^\phi(A)$. For $i=1,\dots,d$ and $(a,\eta)\in\widetilde{N}_d^\phi(A)$ we define $\kappa_i^+(a,\eta)=\sup\{\kappa_{A,i}^\phi(a,\eta),0\}$ and $\kappa_i^-(a,\eta)=\inf\{\kappa_{A,i}^\phi(a,\eta),0\}$. We set

$$E_j^+(a,\eta)=\sum_{\lambda\in\Lambda(d,j)}\prod_{h=1}^{j}\kappa_{\lambda(h)}^+(a,\eta),\qquad E_j^-(a,\eta)=\sum_{\lambda\in\Lambda(d,j)}\prod_{h=1}^{j}\kappa_{\lambda(h)}^-(a,\eta)$$

for $(a,\eta)\in\widetilde{N}_d^\phi(A)$ and $j=0,\dots,d$ (notice $E_0^+\equiv 1$ and $E_-^0\equiv 1$). We set $k=n-d+1$. For $(a,\eta)\in\widetilde{N}_d^\phi(A)$ we observe that

$$E_{A,j}^\phi(a,\eta)=\sum_{l=0}^{j}E_l^-(a,\eta)E_{j-l}^+(a,\eta)\qquad\text{for } j=0,\dots,d$$

and, noting that $E_s^+(a,\eta)E_l^-(a,\eta)=0$ for $s>d-l$ and $-1\le\delta(a,u)\kappa_i^-(a,\eta)\le 0$ for $i=1,\dots,d$ by Remark 3.8, we employ [HLW04, Lemma 2.4] to conclude

$$\begin{aligned}
\sum_{j=0}^{d}\frac{\delta(a,\eta)^{k+j}}{k+j}\,E^{\phi}_{A,j}(a,\eta) &= \sum_{j=0}^{d}\sum_{l=0}^{j}\frac{\delta(a,\eta)^{k+j}}{k+j}\,E_l^-(a,\eta)E_{j-l}^+(a,\eta)\\
&= \sum_{l=0}^{d}\sum_{s=0}^{d-l}\frac{\delta(a,\eta)^{k+l+s}}{k+l+s}\,E_l^-(a,\eta)E_s^+(a,\eta)\\
&= \sum_{s=0}^{d}E_s^+(a,\eta)\delta(a,\eta)^{k+s}\left(\sum_{l=0}^{d}\frac{\delta(a,\eta)^{l}}{k+l+s}\,E_l^-(a,\eta)\right)\\
&\ge \sum_{s=0}^{d}E_s^+(a,\eta)\delta(a,\eta)^{k+s}c(d,n,s),
\end{aligned}$$

where $c(d,n,s)$ is a positive constant depending only on s, d and n. Since

$$\sup_{(a,\eta)\in\widetilde{N}^{\phi}_d(A)}\delta(a,\eta)^l|E_l^-(a,\eta)|\le\binom{d}{l}\qquad\text{for } l=0,\dots,d,$$

we conclude that

$$\begin{aligned}
\sum_{j=0}^{d}\frac{\delta(a,\eta)^{k+j}}{k+j}\,E^{\phi}_{A,j}(a,\eta) &\ge \sum_{s=0}^{d}E_s^+(a,\eta)\delta(a,\eta)^{k+s}c(d,n,s)\binom{d}{l-s}^{-1}|E^-_{l-s}(a,\eta)|\delta(a,\eta)^{l-s}\\
&\ge \delta(a,\eta)^{k+l}\bar{c}(l,d,n)\sum_{s=0}^{d}E_s^+(a,\eta)|E^-_{l-s}(a,\eta)|\\
&\ge \delta(a,\eta)^{k+l}\bar{c}(l,d,n)|E^{\phi}_{A,l}(a,\eta)|
\end{aligned}$$

for $(a,\eta)\in\widetilde{N}^{\phi}_d(A)$ and $l=0,\dots,d$, where $\bar{c}(l,d,n)=\min\left\{c(d,n,s)\binom{d}{l-s}^{-1}:s=0,\dots,d\right\}>0$.

Let $B\subseteq\mathbf{R}^{n+1}$ be a compact set. It follows from (33) and (34) that

$$\int_{\widetilde{N}^{\phi}_d(A)\cap(B\times\partial\mathcal{W}^{\phi})}\phi(\boldsymbol{n}^{\phi}(\eta))\,J(a,\eta)\,\delta(a,\eta)^{n-d+l+1}\,|E^{\phi}_{A,l}(a,\eta)|\,d\mathcal{H}^n(a,\eta)<\infty \tag{35}$$

for $l=0,\dots,d$ and $d=0,\dots,n$. Since

$$\sum_{i=0}^{n}\frac{\delta(a,\eta)^{i+1}}{i+1}|\boldsymbol{H}^{\phi}_{A,i}(a,\eta)|\le\sum_{d=0}^{n}\sum_{j=0}^{d}\frac{\delta(a,\eta)^{n-d+j+1}}{n-d+j+1}\,|E^{\phi}_{A,j}(a,\eta)|\,\mathbf{1}_{\widetilde{N}^{\phi}_d(A)}(a,\eta),$$

where equality holds if the absolute values are omitted on both sides of this equation, we obtain (25) from (35). In addition, we conclude from (32) that

$$\begin{aligned}
&\int_{\{x\in\mathbf{R}^{n+1}:0<\boldsymbol{\delta}^{\phi}_A(x)\le\rho\}}(\varphi\circ\boldsymbol{\psi}^{\phi}_A)\,d\mathcal{L}^{n+1}\\
&\quad=\int_{N^{\phi}(A)}\varphi(a,\eta)\,\phi(\boldsymbol{n}^{\phi}(\eta))\,J(a,\eta)\,\sum_{r=0}^{n}\frac{\delta(a,\eta)^{r+1}}{r+1}\boldsymbol{H}^{\phi}_{A,r}(a,\eta)\,d\mathcal{H}^n(a,\eta)\\
&\quad=\sum_{r=0}^{n}\frac{1}{r+1}\int_{N^{\phi}(A)}\varphi(a,\eta)\,\phi(\boldsymbol{n}^{\phi}(\eta))\,J(a,\eta)\,\delta(a,\eta)^{r+1}\,\boldsymbol{H}^{\phi}_{A,r}(a,\eta)\,d\mathcal{H}^n(a,\eta)
\end{aligned}\tag{36}$$

for every bounded Borel function $\varphi:\mathbf{R}^{n+1}\times\mathbf{R}^{n+1}\to\mathbf{R}$ with compact support, where we use the integrability property in (25) to obtain the equality in (36). $\square$

It is convenient to introduce the following function.

Definition 3.17. For every non-empty closed set $A \subseteq \mathbf{R}^{n+1}$, we denote by J^ϕ_A the $\mathcal{H}^n \llcorner N^\phi(A)$ measurable function J introduced in Theorem 3.16.

The arguments in the proof of Theorem 3.16 readily provide the following change-of-variable-type formula. This formula plays a central role in the proof of Theorem 3.20.

Corollary 3.18 (Disintegration of Lebesgue measure)**.** *Let A be a closed set, and let $h : \mathbf{R}^{n+1} \to \mathbf{R}$ be a non-negative Borel function. Then*

$$\begin{aligned}\int_{\mathbf{R}^{n+1}\setminus A} & h(x)\, d\mathcal{L}^{n+1}(x) \\ &= \sum_{d=0}^{n} \int_{\widetilde{N}^\phi_d(A)} \phi(\boldsymbol{n}^\phi(\eta))\, J^\phi_A(a,\eta) \int_0^{\boldsymbol{r}^\phi_A(a,\eta)} h(a+t\eta)\, t^{n-d} \prod_{j=1}^{d}(1+t\kappa^\phi_{A,j}(a,\eta))\, dt\, d\mathcal{H}^n(a,\eta).\end{aligned}$$

Proof. During the proof of Theorem 3.16 we have proved (see eq. (30)) that

$$\begin{aligned}\int_{B^\phi(A,\rho)\setminus A} & h(x)\, d\mathcal{L}^{n+1}(x) \\ &= \sum_{d=0}^{n} \int_{\widetilde{N}^\phi_d(A)} \phi(\boldsymbol{n}^\phi(\eta))\, J^\phi_A(a,\eta) \int_0^{\inf\{\rho, \boldsymbol{r}^\phi_A(a,\eta)\}} h(a+t\eta)\, t^{n-d} \prod_{j=1}^{d}(1+t\kappa^\phi_{A,j}(a,\eta))\, dt\, d\mathcal{H}^n(a,\eta)\end{aligned}$$

for every $\rho > 0$. The conclusion now follows letting $\rho \to +\infty$. ☐

3.3 A Heintze–Karcher inequality for closed sets

The following theorem provides a very general version of a Heintze–Karcher inequality under minimal assumptions on a closed set and its complement. Several consequences will be derived for sets with positive reach in Section 6.

We start with a lemma.

Lemma 3.19. *Suppose $A \subseteq \mathbf{R}^{n+1}$ is a closed set, $s_0 > 0$ and $\boldsymbol{r}^\phi_A(a,u) \geq s_0$ for $\mathcal{H}^n$ a.e. $(a,u) \in N^\phi(A)$. Then $\{x \in \mathbf{R}^{n+1} : \boldsymbol{\delta}^\phi_A(x) < s_0\} \subseteq \operatorname{Unp}^\phi(A)$.*

Proof. Let $0 < s < s_0$. Define $\Omega^*_s = \{(a,\eta,t) : (a,\eta) \in N^\phi(A),\ 0 < t < \inf\{s, \boldsymbol{r}^\phi_A(a,\eta)\}\}$ and the bijective map

$$f : \Omega^*_s \to \{x : 0 < \boldsymbol{\delta}^\phi_A(x) < s\} \setminus \operatorname{Cut}^\phi(A), \qquad f(a,\eta,t) = a + t\eta.$$

Moreover we define $\Omega_s = \Omega^*_s \cap \{(a,\eta,t) : \boldsymbol{r}^\phi_A(a,\eta) \geq s_0\}$ and we notice that the hypothesis implies

$$\mathcal{H}^{n+1}(\Omega^*_s \setminus \Omega_s) = 0.$$

Consequently $\mathcal{L}^{n+1}(f(\Omega^*_s) \setminus f(\Omega_s)) = 0$ and, recalling that $\mathcal{L}^{n+1}(\operatorname{Cut}^\phi(A)) = 0$, we conclude that $f(\Omega_s)$ is dense in $\{x : 0 < \boldsymbol{\delta}^\phi_A(x) < s\}$. We choose now $x \in \mathbf{R}^{n+1}$ so that $0 < \boldsymbol{\delta}^\phi_A(x) < s$ and a sequence $(a_i, \eta_i, t_i) \in \Omega_s$ so that $a_i + t_i\eta_i \to x$. Up to subsequences we can assume that there exist $a \in A$, $\eta \in \partial\mathcal{W}^\phi$ and $0 \leq t \leq s$ so that $a_i \to a$, $\eta_i \to \eta$ and $t_i \to t$. Therefore $x = a + t\eta$ and

$$0 < \boldsymbol{\delta}^\phi_A(x) = \lim_{i\to\infty} \boldsymbol{\delta}^\phi_A(a_i + t_i\eta_i) = \lim_{i\to\infty} t_i = t.$$

It follows that $(a,\eta) \in N^\phi(A)$ and the upper semicontinuity of $\boldsymbol{r}^\phi_A$ implies that $\boldsymbol{r}^\phi_A(a,\eta) \geq s_0 > s \geq t$ and $x \in \operatorname{Unp}^\phi(A)$. In conclusion, we have proved that $\{x : \boldsymbol{\delta}^\phi_A(x) < s\} \subseteq \operatorname{Unp}^\phi(A)$ for every $0 < s < s_0$. ☐

Theorem 3.20. *Let $C \subset \mathbf{R}^{n+1}$ be a closed set with $0 < \mathcal{L}^{n+1}(\operatorname{int}(C)) < \infty$. Let $K = \mathbf{R}^{n+1} \setminus \operatorname{int}(C)$ and assume that*

$$\sum_{i=1}^{n} \kappa^\phi_{K,i}(a,\eta) \leq 0 \qquad \text{for } \mathcal{H}^n \text{ a.e. } (a,\eta) \in N^\phi(K). \tag{37}$$

Then

$$(n+1)\mathcal{L}^{n+1}(\operatorname{int}(C)) \leq n \int_{\widetilde{N}^{\phi}_n(K)} J^{\phi}_K(a,\eta) \frac{\phi(\boldsymbol{n}^{\phi}(\eta))}{|\boldsymbol{H}^{\phi}_{K,1}(a,\eta)|}\, d\mathcal{H}^n(a,\eta). \tag{38}$$

If equality holds in (38) *and there exists* $q < \infty$ *so that* $|\boldsymbol{H}^{\phi}_{K,1}(a,\eta)| \leq q$ *for* $\mathcal{H}^n$ *a.e.* $(a,\eta) \in \widetilde{N}^{\phi}_n(K)$, *then there are* $N \in \mathbb{N}$, $c_1,\ldots,c_N \in \mathbf{R}^{n+1}$ *and* $\rho_1,\ldots,\rho_N \geq \frac{n}{q}$ *such that*

$$\operatorname{int}(C) = \bigcup_{i=1}^{N} \operatorname{int}(c_i + \rho_i \mathcal{W}^{\phi}), \qquad \operatorname{int}\big(c_i + \rho_i\mathcal{W}^{\phi}\big) \cap \operatorname{int}\big(c_j + \rho_j \mathcal{W}^{\phi}\big) = \varnothing \quad \textit{for } i \neq j.$$

Proof. Note that (37) implies that $\mathcal{H}^n(N^{\phi}(K) \setminus \widetilde{N}^{\phi}_n(K)) = 0$.

For the proof we may assume that $|\boldsymbol{H}^{\phi}_{K,1}(a,\eta)| > 0$ for $\mathcal{H}^n$ a.e. $(a,\eta) \in \widetilde{N}^{\phi}_n(K)$, since otherwise the inequality (38) is obviously true (with strict inequality). By Remark 3.8 we infer that

$$\boldsymbol{r}^{\phi}_K(a,\eta) \leq -\frac{n}{\boldsymbol{H}^{\phi}_{K,1}(a,\eta)} \qquad \text{for } \mathcal{H}^n \text{ a.e. } (a,\eta) \in \widetilde{N}^{\phi}_n(K)$$

and $1 + t\kappa^{\phi}_{K,i}(a,\eta) > 0$ for $\mathcal{H}^n$ a.e. $(a,\eta) \in \widetilde{N}^{\phi}_n(K)$, $0 < t < \boldsymbol{r}^{\phi}_K(a,\eta)$ and $i = 1,\ldots,n$. Employing the change of variable formula in Corollary 3.18 (with $h \equiv 1$) and the classical arithmetic-geometric mean inequality, we can estimate

$$\begin{aligned}
\mathcal{L}^{n+1}(\operatorname{int}(C)) &= \int_{\widetilde{N}^{\phi}_n(K)} \phi(\boldsymbol{n}^{\phi}(\eta)) J^{\phi}_K(a,\eta) \int_0^{\boldsymbol{r}^{\phi}_K(a,\eta)} \prod_{j=1}^{n} (1 + t\kappa^{\phi}_{K,j}(a,\eta))\, dt\, d\mathcal{H}^n(a,\eta) \\
&\leq \int_{\widetilde{N}^{\phi}_n(K)} \phi(\boldsymbol{n}^{\phi}(\eta)) J^{\phi}_K(a,\eta) \int_0^{\boldsymbol{r}^{\phi}_K(a,\eta)} \Big(1 + \frac{t}{n}\boldsymbol{H}^{\phi}_{K,1}(a,\eta)\Big)^n dt\, d\mathcal{H}^n(a,\eta) \\
&\leq \int_{\widetilde{N}^{\phi}_n(K)} \phi(\boldsymbol{n}^{\phi}(\eta)) J^{\phi}_K(a,\eta) \int_0^{-\frac{n}{\boldsymbol{H}^{\phi}_{K,1}(a,\eta)}} \Big(1 + \frac{t}{n}\boldsymbol{H}^{\phi}_{K,1}(a,\eta)\Big)^n dt\, d\mathcal{H}^n(a,\eta) \\
&= \frac{n}{n+1} \int_{\widetilde{N}^{\phi}_n(K)} J^{\phi}_K(a,\eta)\, \frac{\phi(\boldsymbol{n}^{\phi}(\eta))}{|\boldsymbol{H}^{\phi}_{K,1}(a,\eta)|}\, d\mathcal{H}^n(a,\eta).
\end{aligned} \tag{39}$$

We discuss now the equality case. We assume that $|\boldsymbol{H}^{\phi}_{K,1}(a,\eta)| \leq q$ for $\mathcal{H}^n$ a.e. $(a,\eta) \in \widetilde{N}^{\phi}_n(K)$. If (38) holds with equality, then the inequalities in the derivation of (39) become equalities. In particular, we deduce that

$$\boldsymbol{r}^{\phi}_K(a,\eta) = -\frac{n}{\boldsymbol{H}^{\phi}_{K,1}(a,\eta)} \geq \frac{n}{q} \qquad \text{for } \mathcal{H}^n \text{ a.e. } (a,\eta) \in \widetilde{N}^{\phi}_n(K) \tag{40}$$

and the condition

$$\kappa^{\phi}_{K,1}(a,\eta) = \ldots = \kappa^{\phi}_{K,n}(a,\eta) \qquad \text{for } \mathcal{H}^n \text{ a.e. } (a,\eta) \in \widetilde{N}^{\phi}_n(K). \tag{41}$$

Consequently, we infer from Lemma 3.19 that $\{x \in \mathbf{R}^{n+1} : \boldsymbol{\delta}^{\phi}_K(x) < \frac{n}{q}\} \subseteq \operatorname{Unp}^{\phi}(K)$, which means that $\operatorname{reach}^{\phi}(K) \geq \frac{n}{q}$ (see Definition 5.1). We define $E_r = \{x \in \mathbf{R}^{n+1} : \boldsymbol{\delta}^{\phi}_K(x) \geq r\}$ for $r > 0$ and notice that $\partial E_r = S^{\phi}(K,r)$ for $r > 0$. We fix now $0 < r < \frac{n}{q}$. It follows that ∂E_r is a closed $\mathcal{C}^{1,1}$-hypersurface by [DRKS20, Corollary 5.8] and $\chi^{\phi}_{K,1},\ldots,\chi^{\phi}_{K,n}$ are the anisotropic principal curvatures of ∂E_r with respect to the anisotropic normal $\boldsymbol{\nu}^{\phi}_K|\partial E_r$ (which points towards C). It follows from (40) and (41) that

$$\chi^{\phi}_{K,1}(x) = \ldots = \chi^{\phi}_{K,n}(x) = \frac{1}{r - \boldsymbol{r}^{\phi}_K(\boldsymbol{\xi}^{\phi}_K(x), \boldsymbol{\nu}^{\phi}_K(x))} \geq \frac{1}{r - \frac{n}{q}} = \Big(\frac{qr-n}{q}\Big)^{-1}$$

for $\mathcal{H}^n$ a.e. $x \in \partial E_r$; in particular, $\chi^{\phi}_{K,i}(x) < 0$ for $i = 1,\ldots,n$ and $\mathcal{H}^n$ a.e. $x \in \partial E_r$. Hence, an application of [DRKS20, Lemma 3.2][1], to each of the at most countably many connected components of ∂E_r shows that there exist at most countably many points $c_1, c_2, \ldots \in \mathbf{R}^{n+1}$ and numbers

[1]Notice that the last line of [DRKS20, Lemma 3.2] contains a typo: one should replace the equality $M = \partial\mathbf{B}^F(a,|\lambda|^{-1})$ with $M = \partial\mathbf{B}^{F^*}(a,|\lambda|^{-1})$, which is what the proof shows.

$\lambda_1, \lambda_2 \ldots \geq \frac{n-qr}{q}$ so that

$$E_r = \bigcup_{i=1}^{N} (c_i + \lambda_i \mathcal{W}^\phi) \qquad \text{and} \qquad (c_i + \lambda_i \mathcal{W}^\phi) \cap (c_j + \lambda_j \mathcal{W}^\phi) = \varnothing \quad \text{for } i \neq j,$$

where $N \in \mathbf{N} \cup \{\infty\}$. Since $\mathcal{L}^{n+1}(\mathrm{int}(C)) < \infty$, it follows that $N < \infty$. If $i \in \{1, \ldots, N\}$ and $z \in \partial\mathcal{W}^\phi$, we have

$$\frac{\nabla \boldsymbol{\delta}^\phi_K(c_i + \lambda_i z)}{|\nabla \boldsymbol{\delta}^\phi_K(c_i + \lambda_i z)|} = -\boldsymbol{n}^\phi(z)$$

and, noting (8) and (12), we conclude

$$z = \nabla\phi(\boldsymbol{n}^\phi(z)) = -\nabla\phi(\nabla\boldsymbol{\delta}^\phi_K(c_i + \lambda_i z)) = -\boldsymbol{\nu}^\phi_K(c_i + \lambda_i z).$$

For $0 \leq s < r$ we define the bilipschitz homeomorphism $f_s : \partial E_r \to \partial E_{r-s}$ by $f_s(x) = x - s\boldsymbol{\nu}^\phi_K(x)$ for $x \in \partial E_r$. Then we get

$$f_s(c_i + \lambda_i \partial\mathcal{W}^\phi) = c_i + (\lambda_i + s)\partial\mathcal{W}^\phi \quad \text{for } i \geq 1, \qquad \partial E_{r-s} = \bigcup_{i=1}^{N} \big(c_i + (\lambda_i + s)\partial\mathcal{W}^\phi\big)$$

and

$$\big(c_i + (\lambda_i + s)\partial\mathcal{W}^\phi\big) \cap \big(c_j + (\lambda_j + s)\partial\mathcal{W}^\phi\big) = \varnothing \quad \text{for } i \neq j.$$

Consequently, for $0 \leq s < r$,

$$E_{r-s} = \bigcup_{i=1}^{N} \big(c_i + (\lambda_i + s)\mathcal{W}^\phi\big), \qquad \big(c_i + (\lambda_i + s)\mathcal{W}^\phi\big) \cap \big(c_j + (\lambda_j + s)\mathcal{W}^\phi\big) = \varnothing \quad \text{for every } i \neq j.$$

and the proof is complete. □

3.4 A general disintegration formula

For the next definition we need to recall that if A is a closed set in $\mathbf{R}^{n+1}$ and $a \in A$, then the set

$$\mathrm{Dis}(A, a) := \{u \in \mathbf{R}^{n+1} : \boldsymbol{\delta}_A(a + u) = |u|\}$$

is a closed convex set (see [Fed59, Theorem 4.8 (2)] or [MS19]). If X is a convex set, then $\dim X$ denotes the dimension of the affine hull of X.

Definition 3.21. Let $\varnothing \neq A \subseteq \mathbf{R}^{n+1}$ be a closed set and $i \in \{0, \ldots, n+1\}$. We define the i-th stratum of A as

$$A^{(i)} = \{a \in A : \dim \mathrm{Dis}(A, a) = n + 1 - i\}.$$

Remark 3.22. Evidently, we have $A = \bigcup_{i=0}^{n+1} A^{(i)}$ and $A^{(n+1)} = A \setminus \boldsymbol{p}(N(A))$. Noting that $N(A, a) = \{u/|u| \in \mathbf{S}^n : u \in \mathrm{Dis}(A, a) \setminus \{0\}\}$ for every $a \in \boldsymbol{p}(N(A))$, we can easily deduce from (13) that

$$A^{(i)} = \{a \in A : 0 < \mathcal{H}^{n-i}(N^\phi(A, a)) < \infty\} \qquad \text{for } i = 0, \ldots, n.$$

We recall that $A^{(i)}$ is a countably i-rectifiable Borel set which can be covered outside a set of $\mathcal{H}^i$-measure zero by a countable union of $\mathcal{C}^2$-submanifolds of dimension i, for $i = 0, \ldots, n$; see [MS19].

Lemma 3.23. *If $\varnothing \neq A \subseteq \mathbf{R}^{n+1}$ is a closed set, $m \in \{0, \ldots, n\}$ and $S \subseteq \mathbf{R}^{n+1}$ is a countable union of Borel subsets with finite $\mathcal{H}^m$ measure, then*

$$\mathcal{H}^n\big((N^\phi(A)|A^{(j)}) \setminus \textstyle\bigcup_{l=0}^{j} \widetilde{N}^\phi_l(A)\big) = 0 \qquad \textit{for } j = 0, \ldots, n \tag{42}$$

and

$$\mathcal{H}^n\big((N^\phi(A)|A^{(j)} \cap S) \setminus \textstyle\bigcup_{l=0}^{m-1} \widetilde{N}^\phi_l(A)\big) = 0 \qquad \textit{for } j > m. \tag{43}$$

Proof. Recall that $N_s = \{(a,\eta) \in N^\phi(A) : \boldsymbol{r}^\phi_A(a,\eta) \geq s\}$, for every $s > 0$, has finite $\mathcal{H}^n$ measure on each bounded set (see Theorem 3.16) and therefore is $(\mathcal{H}^n, n)$ rectifiable.

Fix $j \in \{0, \dots, n-1\}$ and notice that $\mathcal{H}^{j+1}(A^{(j)}) = 0$. Therefore we can apply Lemma 2.1 to conclude that

$$\int_{N_s|A^{(j)}} \operatorname{ap} J^{N_s}_{j+1}\boldsymbol{p}(a,\eta)\, d\mathcal{H}^n(a,\eta) = \int_{A^{(j)}} \mathcal{H}^{n-1-j}(N_s|\{a\})\, d\mathcal{H}^{j+1}(a) = 0$$

for every $s > 0$. It follows that $\operatorname{ap} J^{N_s}_{j+1}\boldsymbol{p}(a,\eta) = 0$ for $\mathcal{H}^n$ a.e. $(a,\eta) \in N_s|A^{(j)}$ and $s > 0$. We infer from Lemma 3.9 that

$$\kappa^\phi_{A,j+1}(a,\eta) = +\infty \qquad \text{for } \mathcal{H}^n \text{ a.e. } (a,\eta) \in N^\phi(A)|A^{(j)},$$

which is precisely the assertion in (42).

If $m = 0$, then S is a countable set and $\mathcal{H}^n(N^\phi(A,a)) = 0$ for every $a \in A^{(j)}$ and for every $j \geq 1$. This implies (43) for $m = 0$. Fix now $j > m \geq 1$. Noting that $\mathcal{H}^{n-m}(N^\phi(A,a)) = 0$ for $a \in A^{(j)}$, we apply again the coarea formula in Lemma 2.1 to obtain

$$\int_{N_s|S\cap A^{(j)}} \operatorname{ap} J^{N_s}_{m}\boldsymbol{p}(a,\eta)\, d\mathcal{H}^n(a,\eta) = \int_{S\cap A^{(j)}} \mathcal{H}^{n-m}(N_s|\{a\})\, d\mathcal{H}^{m}(a) = 0$$

for $s > 0$. As above this implies that $\kappa^\phi_{A,m}(a,\eta) = +\infty$ for $\mathcal{H}^n$ a.e. $(a,\eta) \in N^\phi(A)|A^{(j)} \cap S$, which is equivalent to the assertion in (43). $\square$

Remark 3.24. If A, m and S are as in Lemma 3.23 then

$$\boldsymbol{H}^\phi_{n-m}(a,\eta) = \mathbf{1}_{\widetilde{N}^\phi_m(A)}(a,\eta) \qquad \text{for } \mathcal{H}^n \text{ a.e. } (a,\eta) \in N^\phi(A)|A^{(m)},$$

$$\boldsymbol{H}^\phi_{n-m}(a,\eta) = 0 \qquad \text{for } \mathcal{H}^n \text{ a.e. } (a,\eta) \in N^\phi(A)|A^{(j)} \text{ and } j < m$$

and

$$\boldsymbol{H}^\phi_{n-m}(a,\eta) = 0 \qquad \text{for } \mathcal{H}^n \text{ a.e. } (a,\eta) \in N^\phi(A)|(A^{(j)} \cap S) \text{ and } j > m.$$

Lemma 3.25. *For every closed set $A \subseteq \mathbf{R}^{n+1}$ the following statements hold.*

(a) $\mathcal{H}^0(N^\phi(A,a)) \in \{1,2\}$ *for* $a \in \boldsymbol{p}(\widetilde{N}^\phi_n(A))$.

(b) $\mathcal{H}^m\Big(\big\{a \in A^{(m)} : \mathcal{H}^{n-m}\big(N^\phi(A,a) \setminus \widetilde{N}^\phi_m(A,a)\big) > 0\big\}\Big) = 0$ *for* $m \in \{0, \dots, n\}$.

(c) $\mathcal{H}^n\big(\boldsymbol{p}\big[N^\phi(A) \setminus \widetilde{N}^\phi_n(A)\big]\big) = 0$*; in particular,* $\mathcal{H}^0(N^\phi(A,a)) \in \{1,2\}$ *for* $\mathcal{H}^n$ *a.e.* $a \in \boldsymbol{p}(N^\phi(A))$.

(d) $N^\phi(A,a) = -N^\phi(A,a)$ *for* $a \in A$ *with* $\mathcal{H}^0(N^\phi(A,a)) = 2$.

(e) *If* $X = \boldsymbol{p}(N^\phi(A) \setminus \widetilde{N}^\phi_n(A)) \cap \boldsymbol{p}(\widetilde{N}^\phi_n(A))$*, then* $\mathcal{H}^n(N^\phi(A)|X) = 0$.

Proof. (a) Let $(a,\eta) \in \widetilde{N}^\phi_n(A)$ and $0 < r < \underline{\boldsymbol{r}}^\phi_A(a,\eta)$. Then $1 - r\chi^\phi_{A,i}(a + r\eta) > 0$ for $i = 1, \dots, n$ and, since these numbers are the eigenvalues of $\mathrm{D}\,\boldsymbol{\xi}^\phi_A(a + r\eta)|\operatorname{Tan}(S^\phi(A,r), a + r\eta)$, we conclude (noting Remark 3.4 and [Fed69, 3.1.21])

$$\operatorname{Tan}(\partial\mathcal{W}^\phi, \eta) = \mathrm{D}\,\boldsymbol{\xi}^\phi_A(a + r\eta)[\operatorname{Tan}(S^\phi(A,r), a + r\eta)] \subseteq \operatorname{Tan}(A,a).$$

Since $N(A,a) \subseteq \operatorname{Nor}(A,a) \subseteq \operatorname{Nor}(\partial\mathcal{W}^\phi, \eta)$ and $\dim \operatorname{Nor}(\partial\mathcal{W}^\phi, \eta) = 1$, it follows that $\mathcal{H}^0(N^\phi(A,a)) \in \{1,2\}$ and (a) is proved.

(b) First, let $m \in \{1, \dots, n\}$, define $P = \bigcup_{i=m}^n \widetilde{N}^\phi_i(A)$ and recall that $A^{(m)}$ is a countably m-rectifiable set (see above). For $s > 0$ define N_s as in Theorem 3.16 and set $W_s = [N_s \setminus P]|A^{(m)}$, which is an $(\mathcal{H}^n, n)$ rectifiable set. Noting from Lemma 3.9 that $\operatorname{ap} J^{W_s}_m\boldsymbol{p}(a,\eta) = 0$ for $\mathcal{H}^n$ a.e. $(a,\eta) \in W_s$, we conclude from Lemma 2.1 that

$$\int_{A^{(m)}} \mathcal{H}^{n-m}\big((N_s|\{x\}) \setminus (P|\{x\})\big)\, d\mathcal{H}^m(x) = 0,$$

and thus we infer that

$$\mathcal{H}^m\left(\left\{x \in A^{(m)} : \mathcal{H}^{n-m}((N_s|\{x\}) \setminus P|\{x\}) > 0\right\}\right) = 0.$$

Furthermore, we have

$$\{x \in A^{(m)} : \mathcal{H}^{n-m}(N^\phi(A,x) \setminus (P|\{x\})) > 0\} = \bigcup_{i=1}^{\infty}\{x \in A^{(m)} : \mathcal{H}^{n-m}((N_{s_i}|\{x\}) \setminus (P|\{x\})) > 0\}$$

for any positive sequence $s_i \searrow 0$. Since also $\mathcal{H}^n(N^\phi(A,a) \setminus \widetilde{N}^\phi(A,a)) = 0$ for $a \in A^{(0)}$, we obtain that

$$\mathcal{H}^m\left(\left\{a \in A^{(m)} : \mathcal{H}^{n-m}\left(N^\phi(A,a) \setminus \bigcup_{i=m}^{n} \widetilde{N}_i^\phi(A,a)\right) > 0\right\}\right) = 0 \tag{44}$$

for $m \in \{0, \dots, n\}$. Now let $m \in \{0, \dots, n-1\}$. Since by Lemma 3.23 it holds that

$$\mathcal{H}^n\left((N^\phi(A)|A) \cap \bigcup_{i=m+1}^{n} \widetilde{N}_i^\phi(A)\right) = 0,$$

an application of the coarea formula in Lemma 2.1 yields that

$$\int_{A^{(m)}} \mathcal{H}^{n-m}\left(N_s|\{x\} \cap \bigcup_{i=m+1}^{n} \widetilde{N}_i^\phi(A,x)\right) d\mathcal{H}^m(x) = 0 \qquad \text{for every } s > 0,$$

whence, as above, we infer

$$\mathcal{H}^m\left(\left\{a \in A^{(m)} : \mathcal{H}^{n-m}\left(N^\phi(A,a) \cap \bigcup_{i=m+1}^{n} \widetilde{N}_i^\phi(A,a)\right) > 0\right\}\right) = 0. \tag{45}$$

Now the assertion follows from (44) and (45).

(c) Since $\boldsymbol{p}(N^\phi(A)) = \bigcup_{i=0}^n A^{(i)}$, $\mathcal{H}^n(A^{(j)}) = 0$ for $j \in \{0, \dots, n-1\}$ and $\boldsymbol{p}\big[N^\phi(A) \setminus \widetilde{N}_n^\phi(A)\big] = \boldsymbol{p}(N^\phi(A)) \cap \{a : \mathcal{H}^0(N^\phi(A,a) \setminus \widetilde{N}_n^\phi(A,a)) > 0\}$, we conclude that (c) directly follows from (b) with $m = n$.

(d) The cone $\{tu : t > 0,\, u \in N(A,a)\}$ is convex for $a \in \boldsymbol{p}(N^\phi(A))$. Consequently, we have $N(A,a) = -N(A,a)$ for $a \in A$ with $\mathcal{H}^0(N(A,a)) = 2$. Since $\nabla\phi(N(A,a)) = N^\phi(A,a)$ for $a \in \boldsymbol{p}(N^\phi(A))$, we conclude that $N^\phi(A,a) = -N^\phi(A,a)$ for $a \in A$ with $\mathcal{H}^0(N^\phi(A,a)) = 2$.

(e) Finally, let X be the set defined in (e). Employing again the sets N_s as defined in Theorem 3.16, we notice that $\operatorname{ap} J_n^{N_s}\boldsymbol{p}(a,u) > 0$ for $\mathcal{H}^n$ a.e. $(a,u) \in \widetilde{N}_n^\phi(A)$ and, by the coarea formula and (c), we get

$$\int_{(N_s \cap \widetilde{N}_n^\phi(A))|X} \operatorname{ap} J_n^{N_s}\boldsymbol{p}(a,\eta)\, d\mathcal{H}^n(a,\eta) = 0$$

for every $s > 0$. It follows that $\mathcal{H}^n(\widetilde{N}_n^\phi(A)|X) = 0$. Since by (a) and (d) we have that

$$N^\phi(A)|X = (\widetilde{N}_n^\phi(A)|X) \cup \{(a,-\eta) : (a,\eta) \in \widetilde{N}_n^\phi(A)|X\},$$

we obtain (e). $\square$

Remark 3.26. If $A \subseteq \mathbf{R}^{n+1}$ is a convex body (with non-empty interior), then $\mathcal{H}^0(N^\phi(A,a)) = 1$ for every $a \in \boldsymbol{p}(\widetilde{N}_n^\phi(A))$.

We now prove a very general disintegration formula. This result, which is of independent interest, plays a key role in the proof of Corollary 6.18 through Lemma 6.6 and the subsequent Lemma 6.7.

Theorem 3.27. *Let $\varnothing \neq A \subseteq \mathbf{R}^{n+1}$ be a closed set and $m \in \{0, \dots, n\}$. Then there exists a positive real-valued $\mathcal{H}^n \llcorner N^\phi(A)$ measurable function $\rho_{A,m}^\phi$ on $N^\phi(A)$ such that the following statements hold.*

(a) $0 < \rho^{\phi}_{A,m}(a,\eta) \leq c$ *for* $\mathcal{H}^n$ *a.e.* $(a,\eta) \in N^{\phi}(A)$*, where* c *depends only on* ϕ *and* n*.*

(b) *If* $m \in \{0, n\}$ *or if* ϕ *is the Euclidean norm, then* $\rho^{\phi}_{A,m}(a,\eta) = 1$ *for* $\mathcal{H}^n$ *a.e.* $(a,\eta) \in N(A)$*.*

(c) *For every Borel set* $B \subseteq \mathbf{R}^{n+1}$ *with* σ*-finite* $\mathcal{H}^m$ *measure it holds that*

$$\boldsymbol{H}^{\phi}_{A,n-m}(a,\eta)\, \mathbf{1}_{N^{\phi}(A)|B}(a,\eta) = \mathbf{1}_{\widetilde{N}^{\phi}_m(A)|(A^{(m)}\cap B)}(a,\eta) \tag{46}$$

for $\mathcal{H}^n$ *a.e.* $(a,\eta) \in N^{\phi}(A)|B$ *and*

$$\begin{aligned}\int_{N^{\phi}(A)|B} \mathbf{1}_D(a,\eta)\phi(\boldsymbol{n}^{\phi}(\eta))\, J^{\phi}_A(a,\eta)\, \boldsymbol{H}^{\phi}_{A,n-m}(a,\eta)\, d\mathcal{H}^n(a,\eta)\\ = \int_{B\cap A^{(m)}} \int_{N^{\phi}(A,a)} \mathbf{1}_D(a,\eta)\phi(\boldsymbol{n}^{\phi}(\eta))\, \rho^{\phi}_{A,m}(a,\eta)\, d\mathcal{H}^{n-m}(\eta)\, d\mathcal{H}^m(a)\end{aligned}$$

for every Borel set $D \subseteq \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$*.*

Proof. Since $N^{\phi}(A)|B = \bigcup_{j=0}^{n}(N^{\phi}(A)|(B \cap A^{(j)}))$, the equality in (46) follows from Remark 3.24.

Since the case $m = 0$ is easy to check directly, we assume that $m \geq 1$ in the following.

Let $\tau_i : \widetilde{N}^{\phi}(A) \to \mathbf{R}^{n+1}$ and $\zeta_i : \widetilde{N}^{\phi}(A) \to \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$ be the maps defined in Lemma 3.9 for $i = 1, \ldots, n$. For $(a,\eta) \in \widetilde{N}^{\phi}(A)$ we define $T(a,\eta)$ to be the linear space generated by $\zeta_1(a,\eta), \ldots, \zeta_n(a,\eta)$ and we notice that

$$\begin{aligned}1 &\geq \|\boldsymbol{p}|T(a,\eta)\|^m \geq \left\|\textstyle\bigwedge_m\big(\boldsymbol{p}|T(a,\eta)\big)\right\|\\ &\geq \left|\textstyle\bigwedge_m \boldsymbol{p}\left(\dfrac{\zeta_1(a,\eta)\wedge\ldots\wedge\zeta_m(a,\eta)}{|\zeta_1(a,\eta)\wedge\ldots\wedge\zeta_m(a,\eta)|}\right)\right| = \dfrac{|\tau_1(a,\eta)\wedge\ldots\wedge\tau_m(a,\eta)|}{|\zeta_1(a,\eta)\wedge\ldots\wedge\zeta_m(a,\eta)|} > 0\end{aligned} \tag{47}$$

for $(a,\eta) \in \widetilde{N}^{\phi}_m(A)$. Therefore we define

$$\varrho^{\phi}_{A,m}(a,\eta) = \begin{cases} \dfrac{J^{\phi}_A(a,\eta)}{\left\|\bigwedge_m(\boldsymbol{p}|T(a,\eta))\right\|} & \text{for } (a,\eta) \in \widetilde{N}^{\phi}_m(A),\\ 1 & \text{for } (a,\eta) \in N^{\phi}(A) \setminus \widetilde{N}^{\phi}_m(A).\end{cases}$$

Notice that $\rho^{\phi}_{A,n}(a,\eta) = 1$ for $\mathcal{H}^n$ a.e. $(a,\eta) \in N^{\phi}(A)$ by Lemma 3.9.

If $m \in \{1, \ldots, n-1\}$, we combine

$$|\tau_1(a,\eta)\wedge\ldots\wedge\tau_n(a,\eta)| \leq |\tau_1(a,\eta)\wedge\ldots\wedge\tau_m(a,\eta)| \cdot |\tau_{m+1}(a,\eta)\wedge\ldots\wedge\tau_n(a,\eta)|$$

(see [Fed69, 1.7.5]) with the estimate in (24) from Remark 3.10 to get

$$\begin{aligned}\rho^{\phi}_{A,m}(a,\eta) &\leq \frac{|\tau_1(a,\eta)\wedge\ldots\wedge\tau_n(a,\eta)|}{|\zeta_1(a,\eta)\wedge\ldots\wedge\zeta_n(a,\eta)|}\cdot\frac{|\zeta_1(a,\eta)\wedge\ldots\wedge\zeta_m(a,\eta)|}{|\tau_1(a,\eta)\wedge\ldots\wedge\tau_m(a,\eta)|}\\ &\leq \frac{C}{c^2}\cdot\frac{|\tau_{m+1}(a,\eta)\wedge\ldots\wedge\tau_n(a,\eta)|}{|\zeta_{m+1}(a,\eta)\wedge\ldots\wedge\zeta_n(a,\eta)|} = \frac{C}{c^2}\end{aligned}$$

for $(a,\eta) \in \widetilde{N}^{\phi}_m(A)$, which provides the required finite upper bound with constants $0 < c \leq C < \infty$ depending only on n, ϕ.

Let N_s be the set defined in Theorem 3.16 for $s > 0$. By Lemma 3.9, we have $\operatorname{ap} J^{N_s}_m \boldsymbol{p}(a,\eta) = \left\|\bigwedge_m\big(\boldsymbol{p}|T(a,\eta)\big)\right\|$ for $\mathcal{H}^n$ a.e. $(a,\eta) \in N_s$, for $s > 0$. An application of the coarea formula shows that

$$\begin{aligned}\int_{N_s|(B\cap A^{(m)})} \phi(\boldsymbol{n}^{\phi}(\eta))\, J^{\phi}_A(a,\eta)\, \mathbf{1}_D(a,\eta)\, \mathbf{1}_{\widetilde{N}^{\phi}_m(A)}(a,\eta)\, d\mathcal{H}^n(a,\eta)\\ = \int_{B\cap A^{(m)}} \int_{N_s\cap\boldsymbol{p}^{-1}(\{a\})} \phi(\boldsymbol{n}^{\phi}(\eta))\varrho^{\phi}_{A,m}(a,\eta)\mathbf{1}_{\widetilde{N}^{\phi}_m(A)}(a,\eta)\, \mathbf{1}_D(a,\eta)\, d\mathcal{H}^{n-m}(a,\eta)\, d\mathcal{H}^m(a)\end{aligned}$$

for $s > 0$. Applying the monotone convergence theorem in combination with Lemma 3.25 (b) and (46), we obtain

$$\begin{aligned}
&\int_{N^\phi(A)|B} \mathbf{1}_D(a,\eta)\phi(\boldsymbol{n}^\phi(\eta))\, J_A^\phi(a,\eta)\, \boldsymbol{H}^\phi_{A,n-m}(a,\eta)\, d\mathcal{H}^n(a,\eta) \\
&\quad = \int_{\widetilde{N}^\phi_m(A)|(B\cap A^{(m)})} \phi(\boldsymbol{n}^\phi(\eta))\, J_A^\phi(a,\eta)\, \mathbf{1}_D(a,\eta)\, d\mathcal{H}^n(a,\eta) \\
&\quad = \int_{B\cap A^{(m)}} \int_{\widetilde{N}^\phi_m(A,a)} \phi(\boldsymbol{n}^\phi(\eta))\varrho^\phi_{A,m}(a,\eta)\, \mathbf{1}_D(a,\eta)\, d\mathcal{H}^{n-m}(\eta)\, d\mathcal{H}^m(a) \\
&\quad = \int_{B\cap A^{(m)}} \int_{N^\phi(A,a)} \phi(\boldsymbol{n}^\phi(\eta))\varrho^\phi_{A,m}(a,\eta)\, \mathbf{1}_D(a,\eta)\, d\mathcal{H}^{n-m}(\eta)\, d\mathcal{H}^m(a).
\end{aligned}$$

Finally, suppose that ϕ is the Euclidean norm. Then $\{\zeta_{\lambda(1)}(a,\eta)\wedge\ldots\wedge\zeta_{\lambda(m)}(a,\eta) : \lambda\in\Lambda(n,m)\}$ is an orthogonal basis of $\bigwedge_m T(a,\eta)$ for every $(a,\eta)\in\widetilde{N}(A)$. Fix now $(a,\eta)\in\widetilde{N}^\phi_m(A)$ and $\xi\in\bigwedge_m T(a,\eta)$ such that

$$|\xi| = 1 \qquad \text{and} \qquad \xi = \sum_{\lambda\in\Lambda(n,m)} c_\lambda \cdot \frac{\zeta_{\lambda(1)}(a,\eta)\wedge\ldots\wedge\zeta_{\lambda(m)}(a,\eta)}{|\zeta_{\lambda(1)}(a,\eta)\wedge\ldots\wedge\zeta_{\lambda(m)}(a,\eta)|}.$$

It follows from the orthogonality that $|c_\lambda|\le 1$ for every $\lambda\in\Lambda(n,m)$. Therefore, denoting with $\lambda_0\in\Lambda(n,m)$ the map such that $\lambda_0(i)=i$ for every $i\in\{1,\ldots,m\}$, we notice that

$$\textstyle\bigwedge_m \boldsymbol{p}(\xi) = c_{\lambda_0}\dfrac{\tau_1(a,\eta)\wedge\ldots\wedge\tau_m(a,\eta)}{|\zeta_1(a,\eta)\wedge\ldots\wedge\zeta_m(a,\eta)|}$$

and we infer in combination with (47) that

$$\big\|\textstyle\bigwedge_m(\boldsymbol{p}|T(a,\eta))\big\| = \dfrac{|\tau_1(a,\eta)\wedge\ldots\wedge\tau_m(a,\eta)|}{|\zeta_1(a,\eta)\wedge\ldots\wedge\zeta_m(a,\eta)|}.$$

We can now easily conclude that $\varrho^\phi_{A,m}(a,\eta)=1$ for every $(a,\eta)\in\widetilde{N}^\phi_m(A)$ is a suitable choice. □

3.5 Relation between Euclidean and anisotropic curvatures

We consider the map

$$T : \mathbf{R}^{n+1}\times\partial\mathcal{W}^\phi \to \mathbf{R}^{n+1}\times\mathbf{S}^n, \qquad (a,\eta)\mapsto T(a,\eta) = (a,\boldsymbol{n}^\phi(\eta)).$$

By equation (8) in the introduction, T is a $\mathcal{C}^1$-diffeomorphism whose inverse is $T^{-1}(a,u) = (a,\nabla\phi(u))$ for $(a,u)\in\mathbf{R}^{n+1}\times\mathbf{S}^n$. In particular, $\mathrm{D}\,T(a,\eta) : \mathbf{R}^{n+1}\times\mathrm{Tan}(\partial\mathcal{W}^\phi,\eta)\to\mathbf{R}^{n+1}\times\mathrm{Tan}(\mathbf{S}^n,\boldsymbol{n}^\phi(\eta))$ is an isomorphism and

$$\mathrm{D}\,T(a,\eta)(\tau,\upsilon) = (\tau, \mathrm{D}\,\boldsymbol{n}^\phi(\eta)(\upsilon)) \quad \text{for } (\tau,\upsilon)\in\mathbf{R}^{n+1}\times\mathrm{Tan}(\partial\mathcal{W}^\phi,\eta).$$

As already recalled in (13), we have $T(N^\phi(A)) = N(A)$.

Remark 3.28. Suppose X is a normed vector space, μ is a measure over X and $f,g : X\to Y$ are functions differentiable at a. If the m dimensional density of $\mu\llcorner\{x : f(x)\neq g(x)\}$ is zero at a, then

$$\mathrm{D}\,f(a)(v) = \mathrm{D}\,g(a)(v) \qquad \text{for every } v\in\mathrm{Tan}^m(\mu,a),$$

where $\mathrm{Tan}^m(\mu,a)$ is the cone of the (μ,m) approximate tangent vectors of μ at a (see [Fed69, 3.2.16]). One can check this remark from the definitions.

The following result expresses the mean curvature functions $\boldsymbol{H}^\phi_{A,j}\cdot\mathbf{1}_{\widetilde{N}^\phi_n} = E^\phi_{A,j}\cdot\mathbf{1}_{\widetilde{N}^\phi_n(A)}$ in terms of the Euclidean generalized curvatures $\kappa_{A,1},\ldots,\kappa_{A,n}$ of A.

Theorem 3.29. *Let $\varnothing \neq A \subseteq \mathbf{R}^{n+1}$ be a closed set, and let ϕ be a uniformly convex $\mathcal{C}^2$ norm.*

Then $\mathcal{H}^n(T(\widetilde{N}^\phi_d(A))\triangle\widetilde{N}_d(A)) = 0$ for $d = 0, \ldots, n$. Moreover, if $e_1, \ldots, e_n : \widetilde{N}(A) \to \mathbf{R}^{n+1}$ are maps such that

$$\mathrm{D}\,\boldsymbol{\nu}_A(a+ru)(e_i(a,u)) = \chi_{A,i}(a+ru)e_i(a,u) \quad \text{for } i = 1, \ldots, n,\ 0 < r < \underline{\boldsymbol{r}_A}(a,u) \text{ and } (a,u) \in \widetilde{N}(A),$$

then

$$\begin{aligned} E^\phi_{A,j}(a,\eta) = \sum_{\lambda \in \Lambda(n,j)} &\Big(\prod_{i=1}^{j} \kappa_{A,\lambda(i)}(T(a,\eta)) \Big) \\ &\times \Big(\bigwedge_{i=1}^{j} \mathrm{D}(\nabla\phi)(\boldsymbol{n}^\phi(\eta))(e_{\lambda(i)}(T(a,\eta))) \Big) \bullet \Big(\bigwedge_{i=1}^{j} e_{\lambda(i)}(T(a,\eta)) \Big) \end{aligned}$$

for $\mathcal{H}^n$ a.e. $(a,\eta) \in \widetilde{N}^\phi_n(A)$ and $j \in \{1, \ldots, n\}$.

Proof. Given the maps $e_1, \ldots, e_n$ as in the statement of the Theorem, we define the maps $z_i : \widetilde{N}(A) \to \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$, for $i = 1, \ldots, n$, so that

$$z_i(a,u) = \begin{cases} (e_i(a,u), \kappa_{A,i}(a,u)e_i(a,u)), & \text{if } \kappa_{A,i}(a,u) < \infty, \\ (0, e_i(a,u)), & \text{if } \kappa_{A,i}(a,u) = +\infty. \end{cases}$$

Then we choose the maps $\tau_1, \ldots, \tau_n$ and $\zeta_1, \ldots, \zeta_n$ as in Lemma 3.9. Notice that $e_1(a,u). \ldots, e_n(a,u)$ form an orthonormal basis of $\operatorname{Tan}(\mathbf{S}^n, u) = \operatorname{Tan}(\partial\mathcal{W}^\phi, \nabla\phi(u))$.

Suppose $W \subseteq N^\phi(A)$ is $\mathcal{H}^n$ measurable and $\mathcal{H}^n(W) < \infty$ and define W^* as the set of $(a,\eta) \in W$ such that

$$\operatorname{Tan}^n(\mathcal{H}^n \llcorner W, (a,\eta)) = \operatorname{span}\{\zeta_1(a,\eta), \ldots, \zeta_n(a,\eta)\}$$

and

$$\operatorname{Tan}^n(\mathcal{H}^n \llcorner T(W), T(a,\eta)) = \operatorname{span}\{z_1(T(a,\eta)), \ldots, z_n(T(a,\eta))\}.$$

Since T is a bi-lipschitz map it follows from Lemma 3.9 that $\mathcal{H}^n(W \setminus W^*) = 0$. Fix now $(a,\eta) \in W^*$ and define $V = \ker[\boldsymbol{p}| \operatorname{Tan}^n(\mathcal{H}^n \llcorner W, (a,\eta))]$, $d = n - \dim V$, $V' = \ker[\boldsymbol{p}| \operatorname{Tan}^n(\mathcal{H}^n \llcorner T(W), T(a,\eta))]$ and $d' = n - \dim V'$. By [San20][Lemma B.2] we infer that

$$\mathrm{D}\,T(a,\eta)\big(\operatorname{Tan}^n(\mathcal{H}^n \llcorner W, (a,\eta))\big) = \operatorname{Tan}^n(\mathcal{H}^n \llcorner T(W), T(a,\eta)),$$

whence we can easily deduce that $\mathrm{D}\,T(a,\eta)(V) = V'$. This implies in particular that $d = d'$. Consequently it holds that

$$T(W^* \cap \widetilde{N}^\phi_d(A)) \subseteq \widetilde{N}_d(A) \quad \text{and} \quad \widetilde{N}_d(A) \cap T(W^*) \subseteq T(\widetilde{N}^\phi_d(A))$$

for every $d = 0, \ldots, n$. Since $N^\phi(A)$ is a countable union of $\mathcal{H}^n$ measurable sets W with finite $\mathcal{H}^n$ measure, we conclude

$$\mathcal{H}^n\big(T(\widetilde{N}^\phi_d(A))\triangle\widetilde{N}_d(A)\big) = 0.$$

For $r > 0$ we define

$$Q_r = \{(a,\eta) \in N^\phi(A) : \boldsymbol{r}^\phi_A(a,\eta) \geq r,\ \boldsymbol{r}_A(T(a,\eta)) \geq r\}.$$

From the upper semicontinuity of $\boldsymbol{r}^\phi_A$ and $\boldsymbol{r}_A$ (see [KS21, Lemma 2.35]) it follows that Q_r is relatively closed in $N^\phi(A)$ (in particular it is a Borel set) and $\mathcal{H}^n \llcorner Q_r$ is finite on compact sets by Theorem 3.16(a). Notice that $Q_r \subseteq Q_s$ is $s \leq r$ and $\bigcup_{r>0} Q_r = N^\phi(A)$. In addition, for $r > 0$ we define Q^*_r as the set of all $(a,\eta) \in Q_r$ such that $\underline{\boldsymbol{r}_A}(a,\eta) = \boldsymbol{r}_A(a,\eta)$, the n-dimensional density of $\mathcal{H}^n \llcorner (\mathbf{R}^{n+1} \times \mathbf{R}^{n+1}) \setminus Q_r$ is zero at (a,η) and $\operatorname{Tan}^n(\mathcal{H}^n \llcorner Q_r, (a,\eta)) = \operatorname{lin}\{\zeta_1(a,\eta), \ldots, \zeta_n(a,\eta)\}$. By Lemma 3.9, [Fed69, 2.10.19] and Lemma 2.7 it follows that $\mathcal{H}^n(Q_r \setminus Q^*_r) = 0$ for every $r > 0$; moreover, one can check directly from the definitions (see [Fed69, 3.2.16]) that $Q^*_r \subseteq Q^*_s$ if $s \leq r$.

Fix now $(\hat{a},\hat{\eta}) \in \widetilde{N}^{\phi}_d(A) \cap T^{-1}(\widetilde{N}_d(A)) \cap Q^*_r$ for some $r > 0$. For $s > 0$ define $L_s : \mathbf{R}^{n+1} \times \partial\mathcal{W}^{\phi} \to \mathbf{R}^{n+1}$ by $L_s(a,\eta) = a + s\boldsymbol{n}^{\phi}(\eta)$. Choose a function $u : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ such that $u(x) \in \boldsymbol{\nu}_A(x)$ for every $x \in \mathbf{R}^{n+1} \setminus A$. For $0 < s \leq r$,

$$(\nabla\phi \circ u \circ L_s)(a,\eta) = \boldsymbol{q}(a,\eta) = \eta \quad \text{for } (a,\eta) \in Q_s$$

and $\nabla\phi \circ u \circ L_s$ is differentiable at $(\hat{a},\hat{\eta})$ by Remark 3.7. Remark 3.28 yields $\mathrm{D}(\nabla\phi \circ u \circ L_s)(\hat{a},\hat{\eta})(\xi) = \boldsymbol{q}(\xi)$ for $\xi \in \mathrm{Tan}^n(\mathcal{H}^n \llcorner Q_s, (\hat{a},\hat{\eta}))$. Hence, for $1 \leq i \leq d$ and $s < r$ we compute

$$\big[\mathrm{D}(\nabla\phi)(\boldsymbol{n}^{\phi}(\hat{\eta})) \circ \mathrm{D}\,u(\hat{a} + s\boldsymbol{n}^{\phi}(\hat{\eta}))\big]\big(\tau_i(\hat{a},\hat{\eta}) + s\kappa^{\phi}_{A,i}(\hat{a},\hat{\eta})\,\mathrm{D}\,\boldsymbol{n}^{\phi}(\hat{\eta})(\tau_i(\hat{a},\hat{\eta}))\big) = \kappa^{\phi}_{A,i}(\hat{a},\hat{\eta})\tau_i(\hat{a},\hat{\eta}). \quad (48)$$

Noting that

$$\begin{aligned} & s\,\mathrm{D}\,u(\hat{a} + s\boldsymbol{n}^{\phi}(\hat{\eta}))\Big(\sum_{j=1}^{d}\big[\mathrm{D}\,\boldsymbol{n}^{\phi}(\hat{\eta})(\tau_i(\hat{a},\hat{\eta})) \bullet e_j(T(\hat{a},\hat{\eta}))\big]e_j(T(\hat{a},\hat{\eta}))\Big) \\ & \qquad = s\sum_{j=1}^{d}\big[\mathrm{D}\,\boldsymbol{n}^{\phi}(\hat{\eta})(\tau_i(\hat{a},\hat{\eta})) \bullet e_j(T(\hat{a},\hat{\eta}))\big]\frac{\kappa_{A,j}(T(\hat{a},\hat{\eta}))}{1 + s\kappa_{A,j}(T(\hat{a},\hat{\eta}))}e_j(T(\hat{a},\hat{\eta})) \to 0 \quad \text{as } s \to 0 \end{aligned}$$

and

$$\begin{aligned} & s\,\mathrm{D}\,u(\hat{a} + s\boldsymbol{n}^{\phi}(\hat{\eta}))\Big(\sum_{j=d+1}^{n}\big[\mathrm{D}\,\boldsymbol{n}^{\phi}(\hat{\eta})(\tau_i(\hat{a},\hat{\eta})) \bullet e_j(T(\hat{a},\hat{\eta}))\big]e_j(T(\hat{a},\hat{\eta}))\Big) \\ & \qquad = \sum_{j=d+1}^{n}\big[\mathrm{D}\,\boldsymbol{n}^{\phi}(\hat{\eta})(\tau_i(\hat{a},\hat{\eta})) \bullet e_j(T(\hat{a},\hat{\eta}))\big]e_j(T(\hat{a},\hat{\eta})) \quad \text{for } s > 0, \end{aligned}$$

we conclude from (48) that

$$\begin{aligned} & \lim_{s\to 0}\big[\mathrm{D}(\nabla\phi)(\boldsymbol{n}^{\phi}(\hat{\eta})) \circ \mathrm{D}\,u(\hat{a} + s\boldsymbol{n}^{\phi}(\hat{\eta}))\big]\big(\tau_i(\hat{a},\hat{\eta})\big) \qquad (49) \\ & \quad = \kappa^{\phi}_{A,i}(\hat{a},\hat{\eta})\tau_i(\hat{a},\hat{\eta}) \\ & \qquad - \kappa^{\phi}_{A,i}(\hat{a},\hat{\eta})\sum_{j=d+1}^{n}\big[\mathrm{D}\,\boldsymbol{n}^{\phi}(\hat{\eta})(\tau_i(\hat{a},\hat{\eta})) \bullet e_j(T(\hat{a},\hat{\eta}))\big]\,\mathrm{D}(\nabla\phi)(\boldsymbol{n}^{\phi}(\hat{\eta}))(e_j(T(\hat{a},\hat{\eta}))) \end{aligned}$$

for $i \leq d$.

Now choose $d = n$ in the previous paragraph and define the linear maps $T_s : \mathrm{Tan}(\partial\mathcal{W}^{\phi},\hat{\eta}) \to \mathrm{Tan}(\partial\mathcal{W}^{\phi},\hat{\eta})$ by

$$T_s = \mathrm{D}(\nabla\phi)(\boldsymbol{n}^{\phi}(\hat{\eta})) \circ \mathrm{D}\,u(\hat{a} + s\boldsymbol{n}^{\phi}(\hat{\eta})$$

and $T_0 : \mathrm{Tan}(\partial\mathcal{W}^{\phi},\hat{\eta}) \to \mathrm{Tan}(\partial\mathcal{W}^{\phi},\hat{\eta})$ by $T_0(\tau_i(\hat{a},\hat{\eta})) = \kappa^{\phi}_{A,i}(\hat{a},\hat{\eta})\tau_i(\hat{a},\hat{\eta})$ for $i = 1,\ldots,n$. Denoting by $\|\cdot\|$ the operator norm, we notice that $\sup_{s>0}\|T_s\| < \infty$ and $T_s(v) \to T_0(v)$ for each $v \in \mathrm{Tan}(\partial\mathcal{W}^{\phi}.\hat{\eta})$ by (49). Therefore, $\lim_{s\to\infty}\|T_s - T_0\| = 0$ and by continuity

$$\begin{aligned} H^{\phi}_{A,j}(\hat{a},\hat{\eta}) &= \mathrm{trace}\big(\textstyle\bigwedge_j T_0\big) \\ &= \mathrm{trace}\big(\textstyle\bigwedge_j \lim_{s\to 0} T_s\big) = \lim_{s\to 0}\mathrm{trace}\big(\textstyle\bigwedge_j T_s\big). \end{aligned}$$

Computation of the $\mathrm{trace}\big(\bigwedge_j T_s\big)$ by means of the orthonormal basis $\{\bigwedge_{i=1}^{j} e_{\lambda(i)}(T(\hat{a},\hat{\eta})) : \lambda \in \Lambda(n,j)\}$ of $\bigwedge_j \mathrm{Tan}(\partial\mathcal{W}^{\phi},\hat{\eta})$, we get the conclusion. $\square$

4 Differentiability of the volume function

In this section, employing the Steiner-type formula from the previous section, we study the differentiability properties of the (localized) parallel volume function V of an anisotropic tubular neighbourhood around an arbitrary compact (closed) set. In particular, we obtain an expression for the left and right derivative of V in terms of the anisotropic curvatures of the compact set and we deduce a geometric characterization of the differentiability points of V. The results of this section extend [HLW04, eq. (4.5) and (4.6), Corollary 4.5] and complement some of the results in [CLV21], see also Remark 4.4 below and the preceding work [Sta76, RW10, Wi19].

Remark 4.1. Notice that if $\rho > 0$, $y \in S^\phi(A,\rho)$ and $a \in \boldsymbol{\xi}^\phi_A(x)$, that is, $a \in A$ and $\phi^*(y-a) = \boldsymbol{\delta}^\phi_A(y)$, then

$$U^\phi(a,\rho) \cap S^\phi(A,\rho) = \varnothing \quad \text{and} \quad B^\phi(a,\rho) \subseteq B^\phi(A,\rho).$$

Hence we deduce that $\boldsymbol{p}\big(N(B^\phi(A,\rho))\big) = \partial^v B^\phi(A,\rho) = \partial^v_+ B^\phi(A,\rho)$ and the function u defined in Theorem 4.3 satisfies $u(x) = \boldsymbol{n}(B^\phi(A,\rho),x)$ for every $x \in \partial^v_+ B^\phi(A,\rho)$.

The following auxiliary result is used in the proof of Theorem 4.3. At the same time it provides an interesting insight into the nature of the cut points in $\mathrm{Cut}^\phi(A) \cap \mathrm{Unp}^\phi(A)$. In fact, in this regard we recall that there are closed (compact) sets $A \subseteq \mathbf{R}^{n+1}$ such that $\mathcal{H}^n[S^\phi(A,\rho) \setminus (\partial B^\phi(A,\rho) \cup \mathrm{Unp}^\phi(A))] > 0$ for some $\rho > 0$; for instance, let $A \subseteq \mathbf{R}^2$ be the union of two parallel lines (segments) at distance 2 and $\rho = 1$. In view of these simple examples, the second equation in (53) (which holds for every $\rho > 0$!) is quite surprising since $\mathrm{Cut}^\phi(A) \cap \mathrm{Unp}^\phi(A)$ can be a much larger set than $\mathbf{R}^{n+1} \setminus (A \cup \mathrm{Unp}^\phi(A))$. In fact, since $\mathbf{R}^{n+1} \setminus (A \cup \mathrm{Unp}^\phi(A))$ is always an n-dimensional set (see section 2.4), it follows from the example in [BH08] that the set $\mathrm{Cut}^\phi(A) \cap \mathrm{Unp}^\phi(A)$ can be an $(n+1)$-dimensional set!

Lemma 4.2. *Let $A \subseteq \mathbf{R}^{n+1}$ be a closed set and $\rho > 0$. Let $f_\rho : N^\phi(A) \cap \{\boldsymbol{r}^\phi_A \geq \rho\} \to S^\phi(A,\rho)$ be defined by $f_\rho(a,\eta) = a + \rho\eta$. Then*

$$f_\rho(N^\phi(A) \cap \{\boldsymbol{r}^\phi_A \geq \rho\}) = S^\phi(A,\rho), \tag{50}$$

$$f_\rho(N^\phi(A) \cap \{\boldsymbol{r}^\phi_A > \rho\}) = \partial^v_+ B^\phi(A,\rho) \subseteq S^\phi(A,\rho) \cap \mathrm{Unp}^\phi(A), \tag{51}$$

$$S^\phi(A,\rho) \cap \mathrm{Unp}^\phi(A) \subseteq \partial B^\phi(A,\rho), \quad \mathcal{H}^n(S^\phi(A,\rho) \cap \mathrm{Unp}^\phi(A) \setminus \partial^v_+ B^\phi(A,\rho)) = 0, \tag{52}$$

$$\partial^v_+ B^\phi(A,\rho) \cap \mathrm{Cut}^\phi(A) = \varnothing \quad \textit{and} \quad \mathcal{H}^n(S^\phi(A,\rho) \cap \mathrm{Unp}^\phi(A) \cap \mathrm{Cut}^\phi(A)) = 0. \tag{53}$$

Proof. We start with (50). "$\subseteq$": Let $(x,\eta) \in N^\phi(A) \cap \{\boldsymbol{r}^\phi_A \geq \rho\}$. Then $\boldsymbol{\delta}^\phi_A(x+t\eta) = t$ for $0 \leq t < \rho$, hence also for $t = \rho$ since δ^ϕ_A is continuous. Thus, $f_\rho(x,\eta) \in S^\phi(A,\rho)$.

"$\supseteq$": Let $z \in S^\phi(A,\rho)$, that is, $\delta^\phi_A(z) = \rho$ and there is some $x \in A$ with $\phi^*(z-x) = \rho$. Set $\eta = \rho^{-1}(z-x)$. Then $\boldsymbol{\xi}^\phi_A(x+t\eta) = x$ for $0 \leq t < \rho$ by [DRKS20, Lemma 2.38 (g)]. It follows that $(x,\eta) \in N^\phi(A)$, $\boldsymbol{r}^\phi_A(x,\eta) \geq \rho$ and $f_\rho(x,\eta) = z$.

Next we deal with (51). Let $(x,\eta) \in N^\phi(A) \cap \{\boldsymbol{r}^\phi_A > \rho\}$. Then $f_\rho(x,\eta) \in S^\phi(A,\rho)$ by (50). Since $\boldsymbol{r}^\phi_A(x,\eta) > \rho$ it follows from (16) that $\boldsymbol{\rho}^\phi_A(x+\rho\eta) > 1$, and therefore also $f_\rho(x,\eta) = x+\rho\eta \in \mathrm{Unp}^\phi(A)$. This yields $f_\rho(N^\phi(A) \cap \{\boldsymbol{r}^\phi_A > \rho\}) \subseteq S^\phi(A,\rho) \cap \mathrm{Unp}^\phi(A)$.

Now we show that $f_\rho(N^\phi(A) \cap \{\boldsymbol{r}^\phi_A > \rho\}) = \partial^v_+ B^\phi(A,\rho)$ by proving two inclusions. "$\subseteq$": Let $(x,\eta) \in N^\phi(A) \cap \{\boldsymbol{r}^\phi_A > \rho\}$. In view of Remark 4.1 it suffices to show that $x + \rho\eta \in \partial B^\phi(A,\rho)$. For this, choose $\bar\rho \in (\rho, \boldsymbol{r}^\phi_A(x,\eta))$. Then we have already shown that $x + \bar\rho\eta \in S^\phi(A,\bar\rho) \cap \mathrm{Unp}^\phi(A)$ and clearly $\boldsymbol{\xi}^\phi_A(x+\bar\rho\eta) = x$. We now claim that $\boldsymbol{\xi}^\phi_{B^\phi(A,\rho)}(x+\bar\rho\eta) = x+\rho\eta$, which will imply the required inclusion. Since

$$x + \rho\eta \in (x + \bar\rho\eta + (\bar\rho - \rho)B^\circ) \cap B^\phi(A,\rho) \qquad \text{and} \qquad (x + \bar\rho\eta + (\bar\rho-\rho)\,\mathrm{int}(B^\circ)) \cap B^\phi(A,\rho) = \varnothing,$$

we get $x + \rho\eta \in \boldsymbol{\xi}^\phi_{B^\phi(A,\rho)}(x+\bar\rho\eta)$ and $\delta^\phi_{B^\phi(A,\rho)}(x+\bar\rho\eta) = \bar\rho - \rho > 0$. Let $z \in \xi^\phi_{B^\phi(A,\rho)}(x+\bar\rho\eta)$ be arbitrarily chosen. Then

$$z \in (x + \bar\rho\eta + (\bar\rho-\rho)\partial B^\circ) \cap (a + \rho\partial B^\circ) \qquad \text{for some } a \in A$$

and

$$(x + \bar\rho\eta + (\bar\rho-\rho)\,\mathrm{int}(B^\circ)) \cap (a + \rho\,\mathrm{int}(B^\circ)) = \varnothing.$$

Hence $z = a + \rho\eta_1 = x + \bar\rho\eta + (\bar\rho-\rho)\eta_2$ for some $\eta_1, \eta_2 \in \partial B^\circ$. Since B° is smooth and strictly convex, it follows that $\eta_1 = -\eta_2$. This implies that $a = x + \bar\rho\eta + \bar\rho\eta_2 \in \boldsymbol{\xi}^\phi_A(x+\bar\rho\eta)$, thus $a = x$ and $\eta = -\eta_2 = \eta_1$, which yields $z = x + \rho\eta$.

"$\supseteq$": Let $z \in \partial^v_+ B(A,\rho)$. Then there is some $x \in A$ with

$$z \in x + \rho B^\circ \subseteq B^\phi(A,\rho) \qquad \text{and} \qquad z \in S^\phi(A,\rho).$$

Furthermore there are $y \notin B^\phi(A,\rho)$ and $\epsilon > 0$ such that

$$z \in y + \epsilon B^\circ \qquad \text{and} \qquad (y + \epsilon \operatorname{int}(B^\circ)) \cap B^\phi(A,\rho) = \varnothing.$$

This implies that $\boldsymbol{\delta}_A^\phi(y) = \rho + \epsilon$. Moreover,

$$\rho + \epsilon \le \phi^*(y-x) \le \phi^*(y-z) + \phi^*(z-x) \le \epsilon + \rho,$$

and hence $\phi^*(y-x) = \rho+\epsilon$, $\phi^*(y-z) = \epsilon$ and $\phi^*(z-x) = \rho$. Since ϕ^* is strictly convex, we also get $y - z = s(z-x)$ with some $s > 0$ and therefore $z = \frac{1}{1+s}y + \frac{s}{1+s}x$. We set $\eta = (\rho+\epsilon)^{-1}(y-x) \in \partial B^\circ$ and thus get $\boldsymbol{\delta}_A^\phi(y) = \boldsymbol{\delta}_A^\phi(x + (\rho+\epsilon)\eta) = \rho + \epsilon$. From [DRKS20, Lemma 2.38 (g)] we now conclude that $(x,\eta) \in N^\phi(A)$ and $\boldsymbol{r}_A^\phi(x,\eta) \ge \rho + \epsilon > \rho$, which yields $z = f_\rho(x,\eta) \in f_\rho(N^\phi(A) \cap \{\boldsymbol{r}_A^\phi > \rho\})$.

Now we turn to (52). We fix an arbitrary $\rho > 0$ and $x \in S^\phi(A,\rho) \cap \operatorname{Unp}^\phi(A)$. Recall that $\boldsymbol{\delta}_A^\phi$ is semiconcave on $\mathbf{R}^{n+1} \setminus B^\phi(A,s)$ for every $s > 0$ (see section 2.4). Since $x \in S^\phi(A,\rho) \cap \operatorname{Unp}^\phi(A)$, the distance function $\boldsymbol{\delta}_A^\phi$ is differentiable at x with $0 \notin \{\nabla \boldsymbol{\delta}_A^\phi(x)\} = \partial \boldsymbol{\delta}_A^\phi(x)$, where we use that the generalized subgradient coincides with the subgradient from convex analysis for semiconcave functions (see [Fu85, Remark 1.4] and the references to [Cl90] given there). Hence we infer from [Fu85, Theorem 3.3 and Proposition 1.7] that there exist $\epsilon, \delta > 0$, $u \in \mathbf{S}^n$ and a lipschitzian semiconvex function $f : x + u^\perp \to \mathbf{R}$ such that

$$\operatorname{epi}(f) \cap U_{\epsilon,\delta}(x,u) = B^\phi(A,\rho) \cap U_{\epsilon,\delta}(x,u) \quad \text{and} \quad \operatorname{graph}(f) \cap U_{\epsilon,\delta}(x,u) = S^\phi(A,\rho) \cap U_{\epsilon,\delta}(x,u).$$

(One can easily see that $u = \nabla \boldsymbol{\delta}_A^\phi(x)/|\nabla \boldsymbol{\delta}_A^\phi(x)|$, but this is not relevant here). Since semiconvex functions are pointwise twice-differentiable almost everywhere by a classical theorem of Alexandrov, we conclude that for $\mathcal{H}^n$ a.e. $y \in S^\phi(A,\rho) \cap U_{\epsilon,\delta}(x,u)$ there exists an open ball $B \subseteq U_{\epsilon,\delta}(x,u)$ such that $B \cap B^\phi(A,\rho) = \varnothing$ and $y \in \operatorname{clos}(B) \cap B^\phi(A,\rho)$. Therefore it follows from Remark 4.1 that

$$\mathcal{H}^n(S^\phi(A,\rho) \cap U_{\epsilon,\delta}(x,u) \setminus \partial_+^v B^\phi(A,\rho)) = 0.$$

Since x is arbitrarily chosen in $S^\phi(A,\rho) \cap \operatorname{Unp}^\phi(A)$, we obtain the assertions in (52).

The assertions in (53) follow immediately from what we have already shown. □

For a closed set $A \subseteq \mathbf{R}^{n+1}$, a bounded Borel set $D \subset \mathbf{R}^{n+1} \times B^\circ$ and $\rho > 0$, we define

$$P_\rho^\phi(A,D) = \left\{ x \in \operatorname{Unp}(A) \setminus A : \boldsymbol{\psi}_A^\phi(x) \in D, \boldsymbol{\delta}_A^\phi(x) \le \rho \right\}$$

and

$$V(A,D;\rho) := \mathcal{L}^{n+1}(P_\rho^\phi(A,D)).$$

Recall that $\mathcal{W}^\phi = B^\circ$. If $\xi : \mathbf{R}^{n+1} \setminus A \to \partial A$ and $\nu : \mathbf{R}^{n+1} \setminus A \to \partial \mathcal{W}^\phi$ are two Borel functions such that

$$\xi(x) \in \boldsymbol{\xi}_A^\phi(x) \qquad \text{and} \qquad \nu(x) \in \boldsymbol{\nu}_A^\phi(x), \tag{54}$$

then we also define

$$\overline{P}_\rho^\phi(A,D) = \left\{ x \in \mathbf{R}^{n+1} \setminus A : (\xi(x), \nu(x)) \in D, \boldsymbol{\delta}_A^\phi(x) \le \rho \right\}$$

and

$$\overline{V}(A,D;\rho) := \mathcal{L}^{n+1}(\overline{P}_\rho^\phi(A,D)).$$

Clearly, we have $\overline{V}(A,D;\rho) = V(A,D;\rho)$, and if $D = A \times B^\circ$, then $\overline{V}(A,D;\rho) = V(A,D;\rho) = \mathcal{L}^{n+1}(B^\phi(A,\rho) \setminus A)$. Furthermore, note that if $x \in S^\phi(A,\rho) \cap \operatorname{Unp}^\phi(A) \setminus A$, then $\xi(x) = x - \rho \cdot \nu(x)$.

Theorem 4.3. *Let $A \subseteq \mathbf{R}^{n+1}$ be a closed set, and let ϕ be a uniformly convex $\mathcal{C}^2$-norm on $\mathbf{R}^{n+1}$. Let $D \subset \mathbf{R}^{n+1} \times \mathcal{W}^\phi$ be a bounded Borel set, and let $\xi : \mathbf{R}^{n+1} \setminus A \to \partial A$ and $\nu : \mathbf{R}^{n+1} \setminus A \to \partial \mathcal{W}^\phi$ be Borel functions as in (54). Define a Borel function $u : \mathbf{R}^{n+1} \setminus A \to \mathbf{S}^n$ by $u(x) = \boldsymbol{n}^\phi(\nu(x))$*

for $x \in \mathbf{R}^{n+1} \setminus A$. Then, for every $\rho > 0$ the right derivative $V'_+(A, D; \rho)$ and the left derivative $V'_-(A, D; \rho)$ of V at ρ exist and are given by

$$\begin{aligned}
V'_+(A,D;\rho) &= \sum_{d=0}^{n} \int_{\widetilde{N}^\phi_d(A)\cap D\cap\{\boldsymbol{r}^\phi_A>\rho\}} \phi(\boldsymbol{n}^\phi(\eta)) J^\phi_A(a,\eta)\, \rho^{n-d} \prod_{i=1}^{d}(1+\rho\kappa^\phi_{A,i}(a,\eta))\, d\mathcal{H}^n(a,\eta) \\
&= \sum_{i=0}^{n} \rho^i \int_{N^\phi(A)\cap D\cap\{\boldsymbol{r}^\phi_A>\rho\}} \phi(\boldsymbol{n}^\phi(\eta)) J^\phi_A(a,\eta) \boldsymbol{H}^\phi_{A,i}(a,\eta)\, d\mathcal{H}^n(a,\eta) \\
&= \int_{\partial^v_+ B^\phi(A,\rho)} 1_D(\xi(x),\nu(x))\phi(u(x))\, d\mathcal{H}^n(x) \qquad (55)
\end{aligned}$$

and

$$\begin{aligned}
V'_-(A,D;\rho) &= \sum_{d=0}^{n} \int_{\widetilde{N}^\phi_d(A)\cap D\cap\{\boldsymbol{r}^\phi_A\geq\rho\}} \phi(\boldsymbol{n}^\phi(\eta)) J^\phi_A(a,\eta)\, \rho^{n-d} \prod_{i=1}^{d}(1+\rho\kappa^\phi_{A,i}(a,\eta))\, d\mathcal{H}^n(a,\eta) \\
&= \sum_{i=0}^{n} \rho^i \int_{N^\phi(A)\cap D\cap\{\boldsymbol{r}^\phi_A\geq\rho\}} \phi(\boldsymbol{n}^\phi(\eta)) J^\phi_A(a,\eta) \boldsymbol{H}^\phi_{A,i}(a,\eta)\, d\mathcal{H}^n(a,\eta) \\
&= V'_+(A,D;\rho) + \int_{S^\phi(A,\rho)\setminus \mathrm{Unp}^\phi(A)} \big(\mathbf{1}_D(x-\rho\nu(x),\nu(x)) \qquad (56) \\
&\qquad + \mathbf{1}_D(x+\rho\nu(x),-\nu(x))\big)\phi(u(x))\, d\mathcal{H}^n(x).
\end{aligned}$$

Consequently, $V(A, D; \cdot)$ is differentiable at $\rho > 0$ if

$$\int_{N^\phi(A)\cap D\cap\{\boldsymbol{r}^\phi_A=\rho\}} \phi(\boldsymbol{n}^\phi(\eta)) J^\phi_A(a,\eta) \boldsymbol{H}^\phi_{A,i}(a,\eta)\, d\mathcal{H}^n(a,\eta) = 0 \qquad \text{for } i = 0, \dots, n,$$

and this happens for all but countably many $\rho \in (0, \infty)$.

Furthermore, $V(A, D; \cdot)$ is differentiable at $\rho > 0$ if and only if

$$\int_{S^\phi(A,\rho)\setminus \mathrm{Unp}^\phi(A)} \big(\mathbf{1}_D(x-\rho\nu(x),\nu(x)) + \mathbf{1}_D(x+\rho\nu(x),-\nu(x))\big)\phi(u(x))\, d\mathcal{H}^n(x).$$

In particular, $V(A, (\mathbf{R}^{n+1})^2; \cdot)$ is differentiable at $\rho > 0$ if and only if $\mathcal{H}^n(S^\phi(A, \rho) \setminus \mathrm{Unp}^\phi(A)) = 0$.

Proof. Note that if $(a, \eta) \in N^\phi(A)$ and $0 < t < \boldsymbol{r}^\phi_A(a, \eta)$, then $a + t\eta \in \mathrm{Unp}^\phi(A)$ and therefore $\xi(a+t\eta) = \boldsymbol{\xi}^\phi_A(a+t\eta) = a$ and $\nu(a+t\eta) = \boldsymbol{\nu}^\phi_A(a+t\eta) = \eta$. Therefore, choosing $h(x) = \mathbf{1}_D(\xi(x), \nu(x))$ in Corollary 3.18, we get that

$$\begin{aligned}
&\overline{V}(A,D;\rho) \\
&\quad= \sum_{d=0}^{n} \int_{\widetilde{N}^\phi_d(A)\cap D} \phi(\boldsymbol{n}^\phi(\eta)) J^\phi_A(a,\eta) \bigg(\int_0^\rho \mathbf{1}\{t < \boldsymbol{r}^\phi_A(a,\eta)\}\, t^{n-d} \prod_{i=1}^{d}(1+t\kappa^\phi_{A,i}(a,\eta))\, dt \bigg) d\mathcal{H}^n(a,\eta)
\end{aligned}$$

for $\rho > 0$. Since the integrand is non-negative, we can exchange the order of integration by Fubini theorem [Fed69, Theorem2.6.2] and obtain

$$\begin{aligned}
&\overline{V}(A,D;\rho) \\
&\quad= \int_0^\rho \Bigg(\sum_{d=0}^{n} \int_{\widetilde{N}^\phi_d(A)\cap D\cap\{\boldsymbol{r}^\phi_A>t\}} \phi(\boldsymbol{n}^\phi(\eta)) J^\phi_A(a,\eta)\, t^{n-d} \prod_{i=1}^{d}(1+t\kappa^\phi_{A,i}(a,\eta))\, d\mathcal{H}^n(a,\eta) \Bigg)\, dt
\end{aligned}$$

for $\rho > 0$. For $\rho > 0$, we define

$$g(\rho) = \sum_{d=0}^{n} \int_{\widetilde{N}^\phi_d(A)\cap D\cap\{\boldsymbol{r}^\phi_A>\rho\}} \phi(\boldsymbol{n}^\phi(\eta)) J^\phi_A(a,\eta)\, \rho^{n-d} \prod_{i=1}^{d}(1+\rho\kappa^\phi_{A,i}(a,\eta))\, d\mathcal{H}^n(a,\eta)$$

and $f_\rho : N_\rho \to S^\phi(A,\rho)$ by $f_\rho(a,\eta) = a + \rho\eta$ (where N_ρ is defined as in Theorem 3.16). Since

$$\mathbf{1}_{\{\boldsymbol{r}_A^\phi > s\}}(a,\eta) \nearrow \mathbf{1}_{\{\boldsymbol{r}_A^\phi > t\}}(a,\eta) \qquad \text{as } s \searrow t \text{ for } (a,\eta) \in N^\phi(A),$$

an application of the dominated convergence theorem gives that $\lim_{s\downarrow t} g(s) = g(t)$ for $t > 0$. Using the formula (28) in the proof of Theorem 3.16, we get

$$g(t) = \int_{N^\phi(A)\cap D\cap\{\boldsymbol{r}_A^\phi > t\}} \phi(\boldsymbol{n}^\phi(\eta)) \operatorname{ap} J_n^{N_t} f_t(a,\eta)\, d\mathcal{H}^n(a,\eta) < \infty \tag{57}$$

for $t > 0$. Now, noting that $\overline{V}(A,D;\rho) = \int_0^\rho g(s)\,ds$, we readily obtain that

$$\overline{V}'_+(A,D;\rho) = g(\rho) \qquad \text{for } \rho > 0. \tag{58}$$

By Lemma 4.2, we have $f_\rho(N^\phi(A) \cap \{\boldsymbol{r}_A^\phi > \rho\}) = \partial_+^v B^\phi(A,\rho) \subseteq \operatorname{Unp}^\phi(A) \cap S^\phi(A,\rho)$; moreover,

$$\boldsymbol{q}(f_\rho^{-1}(\{x\})) = \{\nu(x)\} = \{\boldsymbol{\nu}_A^\phi(x)\} \qquad \text{for } x \in \partial_+^v B^\phi(A,\rho).$$

Noting (57), we can apply the coarea formula to conclude that

$$g(\rho) = \int_{\partial_+^v B^\phi(A,\rho)} \mathbf{1}_D(\xi(x),\nu(x))\phi(u(x))\, d\mathcal{H}^n(x) \qquad \text{for } \rho > 0.$$

Now we deal with the left derivative. Since

$$\mathbf{1}_{\{\boldsymbol{r}_A^\phi > s\}} \searrow \mathbf{1}_{\{\boldsymbol{r}_A^\phi \geq t\}} \quad \text{as } s \nearrow t,$$

it follows again from the dominated convergence theorem that

$$\lim_{s\uparrow t} g(s) = \sum_{d=0}^{n} \int_{\widetilde{N}_d^\phi(A)\cap D\cap\{\boldsymbol{r}_A^\phi \geq t\}} \phi(\boldsymbol{n}^\phi(\eta)) J_A^\phi(a,\eta)\, t^{n-d} \prod_{i=1}^{d} (1 + t\kappa_{A,i}^\phi(a,\eta))\, d\mathcal{H}^n(a,\eta)$$

for $t > 0$. Hence we deduce again using the formula (28) in the proof of Theorem 3.16 that

$$\begin{aligned}
\overline{V}'_-(A,D;\rho) &= \sum_{d=0}^{n} \int_{\widetilde{N}_d^\phi(A)\cap D\cap\{\boldsymbol{r}_A^\phi \geq \rho\}} \phi(\boldsymbol{n}^\phi(\eta)) J_A^\phi(a,\eta)\, \rho^{n-d} \prod_{i=1}^{d} (1 + \rho\kappa_{A,i}^\phi(a,\eta))\, d\mathcal{H}^n(a,\eta) \\
&= \int_{N^\phi(A)\cap D\cap\{\boldsymbol{r}_A^\phi \geq \rho\}} \phi(\boldsymbol{n}^\phi(\eta)) \operatorname{ap} J_n^{N_\rho} f_\rho(a,u)\, d\mathcal{H}^n(a,\eta) \\
&= \int_{S^\phi(A,\rho)} \int_{f_\rho^{-1}(\{x\})} \mathbf{1}_D(a,\eta)\phi(\boldsymbol{n}^\phi(\eta))\, d\mathcal{H}^0(a,\eta)\, d\mathcal{H}^n(x),
\end{aligned}$$

where (50) and the coarea formula have been used in the last step.

If $x \in S^\phi(A,\rho) \cap \operatorname{Unp}^\phi(A)$, then there is a unique $(a,\eta) \in N^\phi(A) \cap \{\boldsymbol{r}_A^\phi \geq \rho\}$ with $f_\rho(a,\eta) = x$, and we have $\xi(x) = \boldsymbol{\xi}_A^\phi(x) = a$, $\nu(x) = \boldsymbol{\nu}_A^\phi(x) = \eta$ and $u(x) = \boldsymbol{n}^\phi(\eta)$. On the other hand, if $x \in S^\phi(A,\rho) \setminus \operatorname{Unp}^\phi(A)$, then $\mathcal{H}^0\big(\boldsymbol{q}(f_\rho^{-1}(\{x\}))\big) > 1$, and in addition we have

$$\boldsymbol{q}(f_\rho^{-1}(\{x\})) \subseteq N^\phi(S^\phi(A,\rho),x) \qquad \text{for } x \in S^\phi(A,\rho). \tag{59}$$

Consequently, we deduce from Lemma 3.25 that

$$\boldsymbol{q}(f_\rho^{-1}(\{x\})) = \{\nu(x), -\nu(x)\} \qquad \text{for } \mathcal{H}^n \text{ a.e. } x \in S^\phi(A,\rho) \setminus \operatorname{Unp}^\phi(A). \tag{60}$$

Thus we conclude that

$$\overline{V}'_-(A,D;\rho) = \int_{S^\phi(A,\rho)\cap\operatorname{Unp}^\phi(A)} \mathbf{1}_D(\xi(x),\nu(x))\phi(u(x))\, d\mathcal{H}^n(x)$$

$$+\int_{S^\phi(A,\rho)\setminus \mathrm{Unp}^\phi(A)} \big(\mathbf{1}_D(x-\rho\nu(x),\nu(x))\phi(\boldsymbol{n}^\phi(\nu(x))) \\ + \mathbf{1}_D(x+\rho\nu(x),-\nu(x))\phi(\boldsymbol{n}^\phi(-\nu(x)))\big)\,d\mathcal{H}^n(x).$$

It follows from (51) and (52) that the first summand on the right side of the preceding equation equals $\overline{V}'_+(A,D;\rho)$. We now obtain the asserted representation of $\overline{V}'_-(A,D;\rho)$ by observing that $\boldsymbol{n}^\phi(\pm\nu(x)) = \pm u(x)$.

To check the second equality both in (55) and in (56) we notice that

$$\sum_{d=0}^{n}\rho^{n-d}\prod_{i=1}^{d}(1+\rho\kappa^\phi_{A,i}(a,\eta))\,\mathbf{1}_{\widetilde{N}^\phi_d(A)}(a,\eta) = \sum_{i=0}^{n}\rho^i \boldsymbol{H}^\phi_{A,i}(a,\eta) \qquad \text{for } (a,\eta)\in\widetilde{N}^\phi(A)$$

and we use the integrability condition in (25) proved in Theorem 3.16 to interchange summation and integral.

Finally, for $\epsilon>0$ and $s>0$ the set

$$I_{\epsilon,s} = \left\{t\in[s,\infty): \int_{N^\phi(A)\cap\{\boldsymbol{r}^\phi_A=t\}} \phi(\boldsymbol{n}^\phi(\eta))J^\phi_A(a,\eta)|\boldsymbol{H}^\phi_{A,i}(a,\eta)|\,d\mathcal{H}^n(a,u)\geq\epsilon\right\}$$

is finite by the integrability property in (25). This readily implies that

$$\int_{N^\phi(A)\cap\{\boldsymbol{r}^\phi_A=\rho\}} \phi(\boldsymbol{n}^\phi(\eta))J^\phi_A(a,\eta)\boldsymbol{H}^\phi_{A,i}(a,\eta)\,d\mathcal{H}^n(a,\eta)\neq 0$$

for at most countably many different values of ρ. □

Remark 4.4. The derivative of the volume function has been the subject of several investigations; see [Sta76], [HLW04], [HLW06], [RW10] and [CLV21].

In order to compare our characterization of the differentiability of the parallel volume function of a compact set (in the non-localized setting) with the one in [CLV21], we introduce the following notation. If $F\subseteq\mathbf{R}^{n+1}$ is a measurable set, then we write F^1 for the set of all $x\in\mathbf{R}^{n+1}$ for which the $(n+1)$-dimensional density of F at x equals 1. Let $A\subset\mathbf{R}^{n+1}$ be compact and $\rho>0$. Then it follows from (59) and (60) that

$$\mathcal{H}^n\left([S^\phi(A,\rho)\setminus\mathrm{Unp}^\phi(A)]\setminus B^\phi(A,\rho)^1\right)=0.$$

On the other hand, we also have

$$\mathcal{H}^n\left(S^\phi(A,\rho)\cap B^\phi(A,\rho)^1\cap\mathrm{Unp}^\phi(A)\right) = \mathcal{H}^n\left(S^\phi(A,\rho)\cap\mathrm{Unp}^\phi(A)\setminus\partial^v_+B^\phi(A,\rho)\right)=0,$$

where we use (52) and the fact that $B^\phi(A,\rho)^1\cap\partial^v_+B^\phi(A,\rho)=\varnothing$. Thus we see that if Δ denotes the symmetric difference operator for subsets of $\mathbf{R}^{n+1}$, then

$$\mathcal{H}^n\left([S^\phi(A,\rho)\cap B^\phi(A,\rho)^1]\Delta[S^\phi(A,\rho)\setminus\mathrm{Unp}^\phi(A)]\right)=0.$$

More generally, our method allows us to study the differentiation of *local* parallel volumes in an anisotropic setting. Such a localization was suggested in an isotropic framework by results in [Wi19] (note however that [Wi19, Lemma 2.9] is not correct, which affects the proof of [Wi19, Proposition 2.10] for instance).

Remark 4.5. By the arguments in the proof of Theorem 4.3 it also follows that $V'_+(A,D;\cdot)$ is continuous from the right and $V'_-(A,D;\cdot)$ is continuous from the left.

5 Alexandrov points of sets of positive reach

Throughout this section, ϕ is a uniformly convex $\mathcal{C}^2$-norm on $\mathbf{R}^{n+1}$ with corresponding gauge body B. For $\rho>0$ we set $B^\phi(\rho)=\rho B^\circ=\rho\mathcal{W}^\phi$ and $B^\phi=B^\circ=\mathcal{W}^\phi$. Recall that if ϕ is the Euclidean norm, then the upper index ϕ is omitted.

Notation. For $a \in \mathbf{R}^{n+1}$, $u \in \mathbf{S}^n$ and $\varepsilon, \delta > 0$ we define

$$u^\perp = \{x \in \mathbf{R}^{n+1} : x \bullet u = 0\},$$

$$U_\varepsilon(a,u) = \{x \in a + u^\perp : |x-a| < \varepsilon\}, \quad U_{\varepsilon,\delta}(a,u) = \{x + \lambda u : x \in U_\epsilon(a,u),\ \lambda \in (-\delta,\delta)\}.$$

If $f : a + u^\perp \to \mathbf{R}$ is a function (and u is a given orienting normal vector), we write

$$\operatorname{graph}(f) = \{x - f(x)u : x \in a + u^\perp\} \qquad \text{and} \qquad \operatorname{epi}(f) = \{x - su : s \geq f(x),\ x \in a + u^\perp\} \tag{61}$$

for the graph of f and the epigraph of f (with respect to u), respectively.

5.1 Positive ϕ-reach

Definition 5.1. Let $A \subseteq \mathbf{R}^{n+1}$ be a closed set. The *ϕ-reach* of A is defined as the non-negative number

$$\operatorname{reach}^\phi(A) = \sup\{\rho \geq 0 : B^\phi(A,\rho) \subseteq \operatorname{Unp}^\phi(A)\}.$$

The set A is said to have *positive ϕ-reach* if $\operatorname{reach}^\phi(A) > 0$.

In the following, we say that a convex body $L \subset \mathbf{R}^{n+1}$ slides freely inside a convex body $K \subset \mathbf{R}^{n+1}$ if for each $x \in \partial K$ there is some $t \in \mathbf{R}^{n+1}$ such that $x \in L + t \subseteq K$ (see [Sch14, Section 3.2]). In particular, L slides freely inside K if and only if L is a summand of K. It follows from [Sch14, Theorem 3.2.12] that if $L, K \subset \mathbf{R}^{n+1}$ are convex bodies of class C^2_+, then there is some $\rho > 0$ (depending on L, K) such that ρL slides freely inside K.

Lemma 5.2. *Let $A \subset \mathbf{R}^{n+1}$ be a closed set. Let $\phi, \bar\phi$ be uniformly convex C^2 norms on $\mathbf{R}^{n+1}$ with corresponding gauge bodies $B, \bar B$. Let $\rho > 0$ be such that $B^\phi(\rho) = \rho B^\circ$ slides freely inside $B^{\bar\phi} = \bar B^\circ$. If $\operatorname{reach}^{\bar\phi}(A) > r$, then $\operatorname{reach}^\phi(A) > \rho r$.*

Proof. Assume that $\operatorname{reach}^{\bar\phi}(A) > r$ and $B^\phi(\rho) = \rho B^\circ$ slides freely inside $B^{\bar\phi} = \bar B^\circ$. Then sB° slides freely inside $r\bar B^\circ$ if $0 < s \leq \rho r$. Aiming at a contradiction, we assume that there is some $z \in \mathbf{R}^{n+1} \setminus A$ and there are $x_1, x_2 \in \partial A$ with $x_1 \neq x_2$ and such that $\{x_1, x_2\} \subseteq (z + s_0 B^\circ) \cap A$ and $\operatorname{int}(z + s_0 B^\circ) \cap A = \emptyset$ for some $s_0 \in (0, \rho r]$. Then $x_i - z \in \partial(s_0 B^\circ)$ and $N(s_0 B^\circ, x_i - z) = \{v_i\}$ for some unit vector v_i, for $i = 1, 2$ (since B° is smooth). In particular, we have $x_i - z = s_0 \nabla h_{B^\circ}(v_i)$ for $i = 1, 2$. Since B° is of class C^2_+ (and hence a Euclidean ball slides freely inside B°), it follows that $-v_i \in N(A, x_i)$ for $i = 1, 2$. Since $\operatorname{reach}^{\bar\phi}(A) > r$, we conclude that

$$(x_i - r\nabla h_{\bar B^\circ}(v_i) + r\bar B^\circ) \cap A = \{x_i\}. \tag{62}$$

Using first that $x_i - s_0 \nabla h_{B^\circ}(v_i) = z$ for $i = 1, 2$ and then that $s_0 B^\circ$ slides freely inside $r\bar B^\circ$, we get

$$\{x_1, x_2\} \subset z + s_0 B^\circ \subseteq x_i - s_0 \nabla h_{B^\circ}(v_i) + s_0 B^\circ \subseteq x_i - r\nabla h_{\bar B^\circ}(v_i) + r\bar B^\circ.$$

But then (62) yields

$$x_2 \in (x_1 - r\nabla h_{\bar B^\circ}(v_1) + r\bar B^\circ) \cap A = \{x_1\},$$

a contradiction. $\square$

Remark 5.3. Assume that ϕ, ψ are any two uniformly convex $\mathcal{C}^2$-norms. Then $\operatorname{reach}^\phi(A) > 0$ if and only if $\operatorname{reach}^\psi(A) > 0$. We say that a closed set $A \subseteq \mathbf{R}^{n+1}$ is a *set of positive reach* if $\operatorname{reach}(A) > 0$, that is, if A has positive reach with respect to the Euclidean norm. Hence, a set has positive reach if and only if it has positive ϕ-reach for some (and then for any) uniformly convex $\mathcal{C}^2$-norm ϕ on $\mathbf{R}^{n+1}$.

Remark 5.4. The class of sets of positive reach as defined here is precisely the class of sets of positive reach introduced in [Fed59, Definition 4.1]. We recall from [Fed59, Theorem 4.8 (12)] that if $A \subseteq \mathbf{R}^{n+1}$ is a set with positive reach, then

$$N(A) = \{(a,u) : a \in A, u \in \operatorname{Nor}(A,a), |u| = 1\}.$$

Moreover it follows from Lemma 2.6 that $\boldsymbol{\xi}^\phi_A | \{x \in \mathbf{R}^{n+1} : 0 < \boldsymbol{\delta}^\phi_A(x) < \operatorname{reach}^\phi(A)\}$ is a locally Lipschitz map and $\boldsymbol{\psi} | S^\phi(A,r)$ is a locally bilipschitz homeomorphism onto $N^\phi(A)$ for $r \in (0, \operatorname{reach}^\phi(A))$. In particular, $S^\phi(A,r)$ is a $\mathcal{C}^{1,1}$ closed hypersurface for $r \in (0, \operatorname{reach}^\phi(A))$ and $N^\phi(A)$ is a closed Lipschitz n-dimensional submanifold of $\mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$.

In the next section we exploit the fundamental connection between sets of positive reach and semiconvex functions. We will refer to the fundamental work [Fu85] (see also the cited references therein). Here we recall the notion of a semiconvex function.

Definition 5.5. Suppose $U \subseteq \mathbf{R}^k$ is open and convex. Then a function $f : U \to \mathbf{R}$ is called *semiconvex* if there exists $0 \le \kappa < \infty$ such that the function $U \ni y \mapsto \kappa|y|^2 + f(y)$ is convex. We denote the *generalized Clarke gradient* of f at $a \in U$ by $\partial f(a)$ (see [Fu85] or [Cl90]).

5.2 Alexandrov points and pointwise curvatures

The first main result of this section (see Theorem 5.6) states that for a set A of positive reach the set $\boldsymbol{p}(\widetilde{N}^\phi(A))$ (which is the set of curvature points of A) can be partitioned as follows

$$\boldsymbol{p}(\widetilde{N}^\phi(A)) = \boldsymbol{p}(\widetilde{N}^\phi(A) \setminus \widetilde{N}^\phi_n(A)) \cup \boldsymbol{p}(\widetilde{N}^\phi_n(A)) \tag{63}$$

For a convex body this partition is well known, as $\boldsymbol{p}(\widetilde{N}^\phi_n(A))$ is the set of normal boundary points (also known as Alexandrov points); see [Sch14, Notes for Sections 1.5 and 2.6], [Hu98, Lemma 3.1] and the literature cited there. The second goal of this section is to extend the notion of an Alexandrov point to sets of positive reach. This notion will play a central role in most of the subsequent rigidity statements; see for instance Theorem 5.15 and Corollary 5.16 in the next section.

Theorem 5.6. *Let $A \subseteq \mathbf{R}^{n+1}$ be a closed set with* $\operatorname{reach}(A) > 0$.

If $(a, \eta) \in \widetilde{N}^\phi_n(A)$, $u \in N(A, a)$ with $\nabla\phi(u) = \eta$ and $\nu \in \mathbf{S}^n \setminus u^\perp$, then the following statements hold.

(a) *There exist $\epsilon(\nu) > 0$ and a function $f_\nu : U_{\epsilon(\nu)}(a, \nu) \to \mathbf{R}$ which is differentiable at a and such that* $\operatorname{graph}(f_\nu) \subseteq \partial A$.

(b) *There exists a map $\widetilde{\eta} : U_{\epsilon(u)}(a, u) \to \partial\mathcal{W}^\phi$ such that $\widetilde{\eta}(b) \in N^\phi(A, b + f_u(b)u)$ for every $b \in U_{\epsilon(u)}(a, u)$, $\widetilde{\eta}(a) = \eta$, $\widetilde{\eta}$ is differentiable at a and the eigenvalues $\lambda_1 \le \ldots \le \lambda_n$ of $\mathrm{D}\,\widetilde{\eta}(a)$ satisfy*

$$\lambda_i = \kappa^\phi_{A,i}(a, \eta) \qquad \textit{for } i = 1, \ldots, n.$$

Moreover, $\boldsymbol{p}(\widetilde{N}^\phi_n(A)) \cap \boldsymbol{p}(\widetilde{N}^\phi(A) \setminus \widetilde{N}^\phi_n(A)) = \varnothing$.

Proof. (a) Suppose $a = 0$ and $0 < r < \operatorname{reach}^\phi(A)$. Fix $\nu \in \mathbf{S}^n \setminus u^\perp$ and let π_ν be the orthogonal projection onto $\nu^\perp$. We consider the lipschitz function $\psi_\nu : S^\phi(A, r) \to \nu^\perp$, $\pi_\nu \circ \boldsymbol{\xi}^\phi_A$. If H is the halfspace orthogonal to u which does not contain u, we notice the inclusions

$$\mathrm{D}\,\boldsymbol{\xi}^\phi_A(r\eta)[\operatorname{Tan}(S^\phi(A, r), r\eta)] \subseteq \operatorname{Tan}(\boldsymbol{\xi}^\phi_A(S^\phi(A, r)), 0) \subseteq \operatorname{Tan}(A, 0) \subseteq H.$$

Since $\mathrm{D}\,\boldsymbol{\xi}^\phi_A(r\eta)|\operatorname{Tan}(S^\phi(A, r), r\eta)$ is injective, we infer that

$$\operatorname{Tan}(S^\phi(A, r), r\eta) = \mathrm{D}\,\boldsymbol{\xi}^\phi_A(r\eta)[\operatorname{Tan}(S^\phi(A, r), r\eta)] = u^\perp$$

and $\mathrm{D}\,\psi_\nu(r\eta)$ is an isomorphism onto $\nu^\perp$. Consequently, we can apply the (right-) inverse function theorem in [Sl00] to infer the existence of a constant $\epsilon(\nu) > 0$ and a map $\varphi_\nu : U_{\epsilon(\nu)}(0, \nu) \to S^\phi(A, r)$ such that $\varphi_\nu(0) = r\eta$, φ_ν is differentiable at 0 with $\mathrm{D}\,\varphi_\nu(0) = \mathrm{D}\,\psi_\nu(r\eta)^{-1} = \mathrm{D}\,\boldsymbol{\xi}^\phi_A(r\eta)^{-1} \circ (\pi_\nu|u^\perp)^{-1}$ and $\psi_\nu(\varphi_\nu(b)) = b$ for every $b \in U_{\epsilon(\nu)}(a, \nu)$. We define $f_\nu : U_{\epsilon(\nu)}(a, \nu) \to \mathbf{R}$ by

$$f_\nu(b) = \boldsymbol{\xi}^\phi_A(\varphi_\nu(b)) \bullet \nu \qquad \text{for } b \in U_{\epsilon(\nu)}(a, \nu)$$

and we readily check that $f_\nu(0) = 0$, $\mathrm{D}\,f_\nu(0) = (\pi_\nu|u^\perp)^{-1} \bullet \nu$ and $b + f_\nu(b)\nu = \boldsymbol{\xi}^\phi_A(\varphi_\nu(b)) \in \partial A$ for every $b \in U_{\epsilon(\nu)}(a, \nu)$.

(b) We define $\widetilde{\eta} : U_{\epsilon(u)}(a, u) \to \partial\mathcal{W}^\phi$ by

$$\widetilde{\eta}(b) = \frac{1}{r}(\varphi_u(b) - \boldsymbol{\xi}^\phi_A(\varphi_u(b))) \qquad \text{for } b \in U_{\epsilon(u)}(a, u)$$

and notice that $\widetilde{\eta}(0) = \eta$ and $\widetilde{\eta}(b) \in N^{\phi}(A, b + f_u(b)u)$ for every $b \in U_{\epsilon(u)}(a, u)$ and

$$\mathrm{D}\,\widetilde{\eta}(0) = \frac{1}{r}(\mathrm{D}\,\boldsymbol{\xi}^{\phi}_{A}(r\eta)^{-1}|u^{\perp} - \mathbf{1}_{u^{\perp}}).$$

Noting that $\{(1 - r\chi^{\phi}_{A,i}(r\eta))^{-1} : i = 1, \ldots, n\}$ are the eigenvalues of $\mathrm{D}\,\boldsymbol{\xi}^{\phi}_{A}(r\eta)^{-1}|u^{\perp}$, we conclude that the eigenvalues of $\mathrm{D}\,\widetilde{\eta}(0)$ satisfy the equation $\lambda_i = \kappa^{\phi}_{A,i}(a, \eta)$ for $i \in \{1, \ldots, n\}$.

Finally, since $\nabla\phi(u) \bullet u = \phi(u) \neq 0$ for $u \in \mathbf{S}^n$ (see 8), the remaining assertion follows from part (a) and Lemma 3.14. □

A more refined description of a set of positive reach around the points of the viscosity boundary is given by the following result.

Theorem 5.7. *Let $A \subset \mathbf{R}^{n+1}$ be a closed set with $\operatorname{reach}(A) > 0$ and $a \in \partial^v A$. Assume that $N(A, a) = \{u\}$ for some $u \in \mathbf{S}^n$. Then the following statements hold.*

(a) *There are $\varepsilon, \delta > 0$ and a semiconvex lipschitz function $f : a + u^{\perp} \to \mathbf{R}$ such that $f(a) = 0$, f is differentiable at a with $\mathrm{D}\,f(a) = 0$ and (with respect to the orienting normal vector u)*

$$\operatorname{graph}(f) \cap U_{\varepsilon,\delta}(a, u) = \partial A \cap U_{\varepsilon,\delta}(a, u), \quad \operatorname{epi}(f) \cap U_{\varepsilon,\delta}(a, u) = A \cap U_{\varepsilon,\delta}(a, u). \tag{64}$$

(b) *$(a, \nabla\phi(u)) \in \widetilde{N}^{\phi}_{n}(A)$ if and only if f is pointwise twice differentiable at a. In this case, every map $\nu : a + u^{\perp} \to \mathbf{S}^n$ such that $\nu(b) \in N(\operatorname{epi}(f), b - f(b)u)$ for every $b \in a + u^{\perp}$ is differentiable at a and satisfies*

$$\mathrm{D}^2 f(a)(\tau_1, \mathrm{D}(\nabla\phi)(u)(\tau_2)) = \mathrm{D}(\nabla\phi \circ \nu)(a)(\tau_1) \bullet \tau_2 \qquad \textit{for } \tau_1, \tau_2 \in u^{\perp}.$$

Proof. Let $0 < r < \operatorname{reach}(A)$. We start with (a). By [Fed69, Theorem 4.8 (12)] we have $\operatorname{Tan}(A, a) = \{v \in \mathbf{R}^{n+1} : v \bullet u \leq 0\}$. Then $\mathcal{C} = \{v \in \mathbf{R}^{n+1} : v \bullet u \leq -\frac{1}{2}|v|\}$ defines a closed convex cone such that $\mathcal{C} \subseteq \operatorname{int}(\operatorname{Tan}(A, a)) \cup \{0\}$. Lemma 3.5 in [RZ17] shows that there is some $s \in (0, r/2)$ such that

$$(a + \mathcal{C}) \cap B(a, s) \subseteq A. \tag{65}$$

We consider the set

$$M_u = \{z \in \partial A : |u - n_z| < 1/4 \text{ for some } n_z \in \operatorname{Nor}(A, z) \cap \mathbf{S}^n\}.$$

It follows from [RZ17, Proposition 3.1 (iii)] that there is some $s' \in (0, s)$ such that if $z \in \partial A \cap B(a, s')$, then $|u - n_z| < 1/4$ whenever $n_z \in \operatorname{Nor}(A, z) \cap \mathbf{S}^n$. The proof of [RZ17, Theorem 5.9] then shows that $\partial A \cap B(a, s') \subset M_u \cap B(a, s')$ is contained in the graph of a lipschitz semiconvex function $f : a + u^{\perp} \to \mathbf{R}$ with $f(a) = 0$, and hence there are $\varepsilon, \delta > 0$ such that (with respect to the orienting normal vector u)

$$\partial A \cap U_{\varepsilon,\delta}(a, u) \subseteq \operatorname{graph}(f) \cap U_{\varepsilon,\delta}(a, u). \tag{66}$$

In order to verify the remaining assertions, we use [RZ17, Proposition 3.3] (see [Fed59, Theorem 4.18]). Thus we get

$$\begin{aligned} A \cap B(a, r) &\subseteq \left\{a + x + tu : t \leq \tfrac{1}{2r}(|x|^2 + t^2), x \in u^{\perp}, t \in \mathbf{R}, |x + tu| \leq r\right\} \\ &\subseteq \left\{a + x + tu : -r \leq t \leq \tfrac{1}{r}|x|^2, x \in u^{\perp}, t \in \mathbf{R}, |x| \leq r\right\}. \end{aligned} \tag{67}$$

At this point the equalities in (64) follows from elementary topology by combining (65)–(67). Moreover, since $\operatorname{Nor}(\operatorname{epi}(f), a) = \operatorname{Nor}(A, a) = \{tu : t \geq 0\}$, it follows from [Fu85, Remarks 1.4, Lemma 2.9] that the generalized Clarke gradient of f at a contains only 0 and f is differentiable at a with $\mathrm{D}\,f(a) = 0$.

(b) Let $\mathcal{N}$ be the family of all maps $\nu : a + u^{\perp} \to \mathbf{S}^n$ such that $\nu(b) \in N(\operatorname{epi}(f), b - f(b)u)$ for every $b \in a + u^{\perp}$, where f is chosen according to (64) with respect to the orienting normal vector u. For every $\nu \in \mathcal{N}$ we define the function $g_\nu : U_\epsilon(a, u) \to u^{\perp}$ by

$$g_\nu(b) = \frac{\nu(b) - (\nu(b) \bullet u)u}{\nu(b) \bullet u} \quad \text{for } b \in U_\epsilon(a, u).$$

(Notice $\nu(b) \bullet u > 0$ for every $b \in U_\epsilon(a,u)$ since f is lipschitz). Employing [Fu85, Lemma 2.9] we conclude that $g_\nu(b) \in \partial f(b)$ for every $b \in U_\epsilon(a,u)$. In particular, if we assume that $(a, \nabla\phi(u)) \in \widetilde{N}^\phi_n(A)$, then it follows from Theorem 5.6 that there exists at least one map $\nu_0 \in \mathcal{N}$ that is differentiable at a. Therefore g_{ν_0} is differentiable at a and the classical theory of subgradients for (semi)convex functions (see [Ba79] or [KS21, Lemma 2.39]) implies that f is twice differentiable at a.

Suppose f is twice differentiable at a, $a = 0$ and $\nu \in \mathcal{N}$. Then the aforementioned theory of subgradients for (semi)convex functions implies that ν is differentiable at 0 with $\mathrm{D}\,\nu(0) \bullet \nu(0) = 0$ and

$$\mathrm{D}\,\nu(0)(\tau_1) \bullet \tau_2 = \mathrm{D}\,g_\nu(0)(\tau_1) \bullet \tau_2 = \mathrm{D}^2 f(0)(\tau_1, \tau_2) \tag{68}$$

for $\tau_1, \tau_2 \in u^\perp$ and for every $\nu \in \mathcal{N}$. It follows that

$$\mathrm{D}(\nabla\phi \circ \nu)(0)(\tau_1) \bullet \tau_2 = \mathrm{D}\,\nu(0)(\tau_1) \bullet \mathrm{D}(\nabla\phi)(u)(\tau_2) = \mathrm{D}^2 f(0)(\tau_1, \mathrm{D}(\nabla\phi)(u)(\tau_2))$$

for $\tau_1, \tau_2 \in u^\perp$. Now we choose $r, \epsilon' > 0$ so that $0 < \epsilon' < r < r + \epsilon' < \operatorname{reach}(A)$ and we define $J = (r - \epsilon', r + \epsilon')$, $\eta = \nabla\phi \circ \nu$ and the function $F : U_\epsilon(0,u) \times J \to \mathbf{R}^{n+1}$ by

$$F(b,t) = b - f(b)u + t\eta(b) \qquad \text{for } b \in U_\epsilon(0,u) \times J.$$

Then F is differentiable at $(0,r)$ and, noting that $U(\operatorname{reach}^\phi(A)\eta(0), \operatorname{reach}^\phi(A)) \cap \operatorname{epi}(f) \cap U_{\epsilon,\delta}(0,u) = \varnothing$, we conclude from (68), employing the same comparison-of-curvatures argument as the one used in the proof of Lemma 3.14, that all eigenvalues of $\mathrm{D}\,\eta(0)$ are smaller or equal than $\operatorname{reach}^\phi(A)^{-1}$. Consequently $\mathrm{D}\,F(a,r)$ is invertible. Let $\pi : \mathbf{R}^{n+1} \to u^\perp$ be the orthogonal projection onto $u^\perp$. Since the function $G : (\boldsymbol{\delta}^\phi_A)^{-1}(J) \to u^\perp \times \mathbf{R}$, defined by $G(x) = (\pi(\boldsymbol{\xi}^\phi_A(x)), \boldsymbol{\delta}^\phi_A(x))$ for $x \in (\boldsymbol{\delta}^\phi_A)^{-1}(J)$, is Lipschitzian and satisfies

$$F(U_\epsilon(0,u) \times J) \subseteq (\boldsymbol{\delta}^\phi_A)^{-1}(J) \quad \text{and} \quad G \circ F = \mathbf{1}_{U_\epsilon(0,u)\times J},$$

it follows from Lemma 2.5 that G and consequently $\boldsymbol{\xi}_A$ is differentiable at ru. Since $\underline{\boldsymbol{r}}^\phi_A(0, \nabla\phi(u)) \geq \operatorname{reach}^\phi(A) > r$ by [KS21, Lemma 4.16], we infer that $\boldsymbol{\xi}^\phi_A$ is differentiable at $s\nabla\phi(u)$ for every $0 < s < \underline{\boldsymbol{r}}^\phi_A(0, \nabla\phi(u))$. Therefore $(0, \nabla\phi(u)) \in \widetilde{N}^\phi(A)$. Since f is twice differentiable at 0, it follows that $0 \in \partial^v_+ A$ and $(0, \nabla\phi(u)) \in \widetilde{N}^\phi_n(A)$ by Lemma 3.14(b). ☐

Subsequently, we prefer to write C for a set of positive reach. Theorem 5.7 motivates the following definition.

Definition 5.8. Suppose C is a set of positive reach, $a \in \partial^v C$, $N(C,a) = \{u\}$ and $f : a + u^\perp \to \mathbf{R}$ is a semiconvex function locally representing C as in Theorem 5.7. Then a is said to be an *Alexandrov point of* C if f is twice differentiable at a. Moreover, if ϕ is a uniformly convex $\mathcal{C}^2$-norm and $k \in \{1, \ldots, n\}$, then the *pointwise k-th ϕ-mean curvature* of C at a is defined by

$$\boldsymbol{h}^\phi_{C,k}(a) = S_k(\mathrm{D}(\nabla\phi \circ \nu)(a)),$$

where $\nu : a + u^\perp \to \mathbf{S}^n$ is a map differentiable at a such that $\nu(b) \in N(\operatorname{epi}(f), b - f(b)u)$ for $b \in a + u^\perp$ and $S_k(\mathrm{D}(\nabla\phi \circ \nu)(a))$ is the k-th elementary symmetric function of the eigenvalues (counted with multiplicities) of the endomorphism $\mathrm{D}(\nabla\phi \circ \nu)(a)$ on $u^\perp$.

We denote the set of Alexandrov points of C by $\mathcal{A}(C)$.

Remark 5.9. If C is a convex body, then this notion of an Alexandrov point coincides with the classical notion of a normal boundary point of C; see [Sch14, Notes for Sections 1.5 and 2.6] for further background information.

Corollary 5.10. *Suppose $C \subseteq \mathbf{R}^{n+1}$ is a set of positive reach and ϕ is a uniformly convex $\mathcal{C}^2$-norm. Then the following statements hold.*

(a) $\mathcal{A}(C) = \boldsymbol{p}(\widetilde{N}^\phi_n(C)) \cap \partial^v C = \boldsymbol{p}(\widetilde{N}^\phi(C)) \cap \partial^v_+ C$ *and*

$$\boldsymbol{H}^\phi_{C,k}(a,\eta) = \boldsymbol{h}^\phi_{C,k}(a) \qquad \text{for } a \in \mathcal{A}(C) \text{ and } N^\phi(C,a) = \{\eta\}.$$

(b) $\mathcal{H}^n(\partial^v C \setminus \mathcal{A}(C)) = 0$ *and*

$$\mathcal{P}^\phi(C) = \int_{\partial^v_+ C} \phi(\boldsymbol{n}(C,a))\, d\mathcal{H}^n(a) = \int_{N^\phi(C)|\partial^v_+ C} J^\phi_C(a,\eta)\phi(\boldsymbol{n}^\phi(\eta))\, d\mathcal{H}^n(a,\eta). \tag{69}$$

(c) *If* $\mathcal{L}^{n+1}(C) < \infty$ *then*

$$\mathrm{int}(C) \neq \varnothing \quad \iff \quad \mathcal{L}^{n+1}(C) > 0 \quad \iff \mathcal{H}^n(\partial^v_+ C) > 0.$$

(d) $\mathcal{H}^n\big[\boldsymbol{p}(\widetilde{N}^\phi_n(C)) \setminus \mathcal{A}(C)\big] = 0$ *if and only if* $\mathcal{H}^n(\partial C \setminus \partial^v C) = 0$.

Proof. The assertions in (a) follow from Theorem 5.6, Theorem 5.7 and Lemma 3.14. Using (a) we infer

$$\partial^v C \setminus \mathcal{A}(C) = \partial^v C \setminus \boldsymbol{p}(\widetilde{N}^\phi_n(C)) \subseteq \boldsymbol{p}(N^\phi(C)) \setminus \boldsymbol{p}(\widetilde{N}^\phi_n(C)) \subseteq \boldsymbol{p}\left(N^\phi(C) \setminus \widetilde{N}^\phi_n(C)\right),$$

whence we obtain $\mathcal{H}^n(\partial^v C \setminus \mathcal{A}(C)) = 0$ from Lemma 3.25 (c). Since $\partial C = \boldsymbol{p}(N^\phi(C))$ we obtain the first equality in (69) from Remark 2.11 and the second equality by combining Remark 3.15, Lemma 3.9 and the coarea formula.

We prove (c). Clearly, $\mathrm{int}(C) \neq \varnothing$ implies $\mathcal{L}^{n+1}(C) > 0$. Let us assume $\mathcal{L}^{n+1}(C) > 0$. Then it follows from Lemma 2.10 that $\mathcal{H}^n(\partial^m C) > 0$. Since $\partial^v_+ C \subseteq \partial^m C \subseteq \partial^v C$ by Remark 2.11 and $\mathcal{H}^n(\partial^v C \setminus \partial^v_+ C) = 0$ by (b), it follows that $\mathcal{H}^n(\partial^v_+ C) > 0$. It is again clear that $\mathcal{H}^n(\partial^v_+ C) > 0$ implies $\mathrm{int}(C) \neq \varnothing$.

Finally, it follows by (a) that

$$\partial C \setminus \partial^v C \subseteq \big[\partial C \setminus \boldsymbol{p}(\widetilde{N}^\phi_n(C))\big] \cup \big[\boldsymbol{p}(\widetilde{N}^\phi_n(C)) \setminus \partial^v C\big] = \big[\partial C \setminus \boldsymbol{p}(\widetilde{N}^\phi_n(C))\big] \cup \big[\boldsymbol{p}(\widetilde{N}^\phi_n(C)) \setminus \mathcal{A}(C)\big]$$

and, since $\mathcal{H}^n\big[\partial C \setminus \boldsymbol{p}(\widetilde{N}^\phi_n(C))\big] = 0$ by Lemma 3.25 (c), we obtain that $\mathcal{H}^n\big[\boldsymbol{p}(\widetilde{N}^\phi_n(C)) \setminus \mathcal{A}(C)\big] = 0$ implies that $\mathcal{H}^n(\partial C \setminus \partial^v C) = 0$. On the other hand, if $\mathcal{H}^n(\partial C \setminus \partial^v C) = 0$, then Lemma 3.25 (c) implies again that $\mathcal{H}^n\big[\boldsymbol{p}(\widetilde{N}^\phi_n(C)) \setminus \partial^v C\big] = 0$, and hence $\mathcal{H}^n\big[\boldsymbol{p}(\widetilde{N}^\phi_n(C)) \setminus \mathcal{A}(C)\big] = 0$ follows from part (a). ☐

Remark 5.11. Suppose C is a closed convex set with $\mathrm{int}(C) \neq \varnothing$. Then $C^{(n)} \setminus \partial^v C = \varnothing$ and, recalling that $\boldsymbol{p}(\widetilde{N}^\phi_n(C)) \subseteq C^{(n)}$ by Lemma 3.25(a), we infer from Corollary 5.10 that

$$\mathcal{A}(C) = \boldsymbol{p}(\widetilde{N}^\phi_n(C)).$$

5.3 Lower-bounded pointwise mean curvature and bubbling

After some preparations, we will deduce the Heintze-Karcher inequality for sets of positive reach from the more general version for arbitrary closed sets.

Lemma 5.12. *Suppose* $C \subset \mathbf{R}^{n+1}$ *is a set of positive reach with* $\mathrm{int}(C) \neq \varnothing$, $K = \mathbf{R}^{n+1} \setminus \mathrm{int}(C)$ *and*

$$\iota : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$$

is the linear map defined by $\iota(a,\eta) = (a,-\eta)$ *for* $(a,\eta) \in \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$.

Then the following statements hold.

(a) $\boldsymbol{p}(N^\phi(K)) = \partial^v_+ C = \partial^v_+ K$ *and* $N^\phi(K,a) = \{-\nabla\phi(\boldsymbol{n}(C,a))\} = -N^\phi(C,a)$ *for* $a \in \boldsymbol{p}(N^\phi(K))$.

(b) $\widetilde{N}^\phi_n(K) = \widetilde{N}^\phi(K)$ *and* $\mathcal{H}^n(N^\phi(K)|S) = 0$ *for every* $S \subseteq \mathbf{R}^{n+1}$ *with* $\mathcal{H}^n(S) = 0$.

(c) $\kappa^\phi_{K,i}(a,\eta) = -\kappa^\phi_{C,n+1-i}(a,-\eta)$ *for* $\mathcal{H}^n$ *a.e.* $(a,\eta) \in N^\phi(K)$ *and* $i = 1,\dots,n$.

(d) $\boldsymbol{H}^\phi_{K,1}(a,\eta) = -\boldsymbol{H}^\phi_{C,1}(a,-\eta)$ *for* $\mathcal{H}^n$ *a.e.* $(a,\eta) \in N^\phi(K)$.

(e) $J^\phi_K(a,\eta) = \mathrm{ap}\, J^{N^\phi(K)}_n \iota(a,\eta)\, J^\phi_C(a,-\eta)$ *for* $\mathcal{H}^n$ *a.e.* $(a,\eta) \in N^\phi(K)$.

Proof. The statement in (a) readily follows from Remark 2.11.

For the statement in (b) we combine (a) and Lemma 3.14(b) to infer that $\widetilde{N}^\phi_n(K) = \widetilde{N}^\phi(K)$. Moreover if $\mathcal{H}^n(S) = 0$, then we can combine Lemma 2.1 (first applied on N_s for some $s > 0$) with Lemma 3.9 to see that

$$\int_{\widetilde{N}^\phi_n(K)|S} \operatorname{ap} J_n^{N^\phi(K)} \boldsymbol{p}(a,u)\, d\mathcal{H}^n(a,u) = 0$$

and $\operatorname{ap} J_n^{N^\phi(K)} \boldsymbol{p}(a,u) > 0$ for $\mathcal{H}^n$ a.e. $(a,u) \in \widetilde{N}^\phi_n(K)$. We conclude that $\mathcal{H}^n(\widetilde{N}^\phi(K)|S) = 0$ provided that $\mathcal{H}^n(S) = 0$. Since $\mathcal{H}^n(N^\phi(K) \setminus \widetilde{N}^\phi(K)) = 0$, we obtain (b).

Next we prove (c). We define the open set $U = \{y \in \mathbf{R}^{n+1} : 0 < \boldsymbol{\delta}^\phi_K(y) < \operatorname{reach}^\phi(C)\}$. For $0 < \lambda \le 1$, $y \in U$ and $a \in \boldsymbol{\xi}^\phi_K(y)$ we notice that

$$a + \lambda \boldsymbol{\delta}^\phi_K(y) \frac{a - y}{\boldsymbol{\delta}^\phi_K(y)} = (1+\lambda)a - \lambda y,$$

$$N^\phi(K,a) = \left\{ \frac{y-a}{\boldsymbol{\delta}^\phi_K(y)} \right\}, \qquad N^\phi(C,a) = \left\{ \frac{a-y}{\boldsymbol{\delta}^\phi_K(y)} \right\},$$

$$\boldsymbol{\nu}^\phi_C((1+\lambda)a - \lambda y) = \frac{a-y}{\boldsymbol{\delta}^\phi_K(y)}, \qquad -\boldsymbol{\nu}^\phi_C((1+\lambda)a - \lambda y) \in \boldsymbol{\nu}^\phi_K(y).$$

We infer that

$$\boldsymbol{\nu}^\phi_K(y) = \{-\boldsymbol{\nu}^\phi_C((1+\lambda)a - \lambda y) : a \in \boldsymbol{\xi}^\phi_K(y)\} \qquad \text{for } y \in U \text{ and } 0 < \lambda \le 1. \tag{70}$$

Define $S = \boldsymbol{p}\big(\iota(N^\phi(K)) \setminus \widetilde{N}^\phi(C)\big)$ and notice that $\mathcal{H}^n(S) = 0$ by Remark 3.7. It follows from (b) that $\mathcal{H}^n(N^\phi(K)|S) = 0$. Fix now $(a,\eta) \in \widetilde{N}^\phi_n(K)$ with $a \notin S$, $0 < r < \inf\{\boldsymbol{r}^\phi_K(a,\eta), \operatorname{reach}^\phi(C)\}$ and, noting that $1 - r\chi^\phi_{K,i}(a + r\eta) > 0$ for $i = 1, \ldots, n$, we select $0 < \lambda \le 1$ so that

$$\chi^\phi_{K,i}(a + r\eta) < \frac{1}{(1+\lambda)r} \qquad \text{for } i = 1, \ldots, n.$$

Since $a \notin S$, then $(a, -\eta) \in \widetilde{N}^\phi(C)$ and $\boldsymbol{\nu}^\phi_C$ is differentiable at $a - t\eta$ for every $0 < t < \operatorname{reach}^\phi(C)$. Since $a + r\eta \in U$ and $\boldsymbol{\nu}^\phi_C$ is differentiable at $(1+\lambda)a - \lambda(a + r\eta) = a - \lambda r\eta$, we differentiate at $a + r\eta$ the equality in (70), and thus we get

$$\mathrm{D}\,\boldsymbol{\nu}^\phi_K(a + r\eta) = -\mathrm{D}\,\boldsymbol{\nu}^\phi_C(a - \lambda r\eta) \circ ((1+\lambda)\,\mathrm{D}\,\boldsymbol{\xi}^\phi_K(a + r\eta) - \mathrm{Id}_{\mathbf{R}^{n+1}}).$$

If $\tau_1, \ldots, \tau_n$ form a basis of $\operatorname{Tan}(\partial\mathcal{W}^\phi, \eta)$ such that $\mathrm{D}\,\boldsymbol{\nu}^\phi_K(a + r\eta)(\tau_i) = \chi^\phi_{K,i}(a + r\eta)\tau_i$ for $i = 1, \ldots, n$, then we infer

$$\mathrm{D}\,\boldsymbol{\nu}^\phi_C(a - \lambda r\eta)(\tau_i) = \frac{\chi^\phi_{K,i}(a + r\eta)}{(1+\lambda)r\chi^\phi_{K,i}(a + r\eta) - 1}\tau_i \qquad \text{for } i = 1, \ldots, n.$$

Note that $\operatorname{Tan}(\partial\mathcal{W}^\phi_1, -\eta) = \operatorname{Tan}(\partial\mathcal{W}^\phi_1, \eta)$ and recall that $(1+\lambda)r\chi^\phi_{K,i}(a + r\eta) - 1 < 0$ for $i = 1, \ldots n$. Hence, we conclude that

$$\chi^\phi_{C,n+1-i}(a - \lambda r\eta) = \frac{\chi^\phi_{K,i}(a + r\eta)}{(1+\lambda)r\chi^\phi_{K,i}(a + r\eta) - 1}$$

for $i = 1, \ldots, n$. Therefore,

$$\kappa^\phi_{C,n+1-i}(a, -\eta) = \frac{\chi^\phi_{C,n+1-i}(a - \lambda r\eta)}{1 - \lambda r\chi^\phi_{C,n+1-i}(a - \lambda r\eta)} = \frac{\chi^\phi_{K,i}(a + r\eta)}{r\chi^\phi_{K,i}(a + r\eta) - 1} = -\kappa^\phi_{K,i}(a, \eta)$$

for $i = 1, \ldots, n$.

To prove (d) we use (b) and (c), which yields that

$$\boldsymbol{H}_{K,1}^{\phi}(a,\eta) = \sum_{i=1}^{n} \kappa_{K,i}^{\phi}(a,\eta) = -\sum_{i=1}^{n} \kappa_{C,i}^{\phi}(a,-\eta) = -\boldsymbol{H}_{C,1}^{\phi}(a,-\eta)$$

for $\mathcal{H}^n$ a.e. $(a,\eta) \in \widetilde{N}_n^{\phi}(K)$.

Finally, we prove (e). Let $\tau_1, \dots, \tau_n, \zeta_1, \dots, \zeta_n$ be $\mathcal{H}^n \llcorner N^{\phi}(K)$-measurable functions satisfying the hypothesis of Lemma 3.9. Noting (b) and Lemma 3.5, we observe that the proof of (c) shows that

$$\mathrm{D}\,\boldsymbol{\nu}_C^{\phi}(a - t\eta)(\tau_i(a,\eta)) = \chi_{C,n+1-i}^{\phi}(a - t\eta)\tau_i(a,\eta)$$

for $\mathcal{H}^n$ a.e. $(a,u) \in N^{\phi}(K)$ and $0 < t < \mathrm{reach}^{\phi}(C)$. Since $\kappa_{C,i}^{\phi}(a,-\eta) < \infty$ for $\mathcal{H}^n$ a.e. $(a,\eta) \in N^{\phi}(K)$ by (b) and (c), we infer that

$$J_C^{\phi}(a,-\eta) = \frac{|\tau_1(a,\eta) \wedge \dots \wedge \tau_n(a,\eta)|}{|\iota(\zeta_1(a,\eta)) \wedge \dots \wedge \iota(\zeta_n(a,\eta))|}$$

for $\mathcal{H}^n$ a.e. $(a,\eta) \in N^{\phi}(K)$. Since

$$\mathrm{ap}\, J_n^{N^{\phi}(K)}\iota(a,\eta) = \frac{|\iota(\zeta_1(a,\eta)) \wedge \dots \wedge \iota(\zeta_n(a,\eta))|}{|\zeta_1(a,\eta) \wedge \dots \wedge \zeta_n(a,\eta)|},$$

the equation in (e) follows. □

Remark 5.13. The second statement in Lemma 5.12 (b) can also be obtained as follows. Let $0 < s < \mathrm{reach}(C)$. From (a) we get

$$N^{\phi}(K)|S = \bigcup_{\ell \in \mathbb{N}} \left\{(x, -\nabla\phi(\boldsymbol{n}(C,x))) : x \in S \cap X_{\frac{s}{4\ell},s}(C)\right\}.$$

The assertion now follows from Remark 2.14.

Remark 5.14. We also outline an alternative argument for Lemma 5.12 (c). First, we obtain that for $\mathcal{H}^n$ a.e. $(a,\eta) \in \widetilde{N}_n^{\phi}(K)$ also $(a,-\eta) \in \widetilde{N}^{\phi}(C)$ and

$$T = \mathrm{Tan}^n(\mathcal{H}^n \llcorner \widetilde{N}_n^{\phi}(K), (a,\eta)) = \mathrm{lin}\{\zeta_1^K(a,\eta), \dots, \zeta_n^K(a,\eta)\} = \mathrm{lin}\{\zeta_1^C(a,\eta), \dots, \zeta_n^C(a,\eta)\},$$

where

$$\zeta_i^K(a,\eta) = \left(\tau_i^K(a,\eta), \kappa_{K,i}^{\phi}(a,\eta)\tau_i^K(a,\eta)\right) \qquad \text{for } i = 1, \dots, n,$$

the linearly independent vectors $\tau_1^K(a,\eta), \dots, \tau_n^K(a,\eta)$ span an n-dimensional linear subspace V of $\mathbf{R}^{n+1}$ and $\kappa_{K,i}^{\phi}(a,\eta) \in \mathbf{R}$, and where

$$\zeta_i^C(a,\eta) = \begin{cases} \left(\tau_i^C(a,-\eta), -\kappa_{C,i}^{\phi}(a,-\eta)\tau_i^C(a,-\eta)\right), & \text{if } \kappa_{C,i}^{\phi}(a,-\eta) < \infty, \\ \left(0, -\kappa_{C,i}^{\phi}(a,-\eta)\tau_i^C(a,-\eta)\right), & \text{if } \kappa_{C,i}^{\phi}(a,-\eta) = \infty, \end{cases}$$

with linearly independent vectors $\tau_1^C(a,-\eta), \dots, \tau_n^C(a,-\eta)$ which span an n-dimensional linear subspace V' of $\mathbf{R}^{n+1}$.

Since the number of curvatures which are infinite equals the dimension of the kernel of the image of the linear map $\boldsymbol{p}|T$, a comparison of the two representations of T shows that $\kappa_{C,i}^{\phi}(a,-\eta) < \infty$ for $i = 1, \dots, n$. Hence $V = V'$ and $\boldsymbol{p}|T$ is an injective linear map. Therefore the linear map $L : V \to V$ with $L = \boldsymbol{q} \circ \boldsymbol{p}^{-1}$ is well defined and its eigenvalues are $\kappa_{K,i}^{\phi}(a,\eta)$ with corresponding eigenvectors $\tau_i^K(a,\eta)$, but also $-\kappa_{C,i}^{\phi}(a,-\eta)$ with corresponding eigenvectors $\tau_i^C(a,-\eta)$ for $i = 1, \dots, n$. This implies the assertion.

We can now state the Heintze–Karcher inequality for sets of positive reach in the following form. Recall that $\mathcal{H}^n(\partial^v C \setminus \mathcal{A}(C)) = 0$ by Corollary 5.10.

Theorem 5.15. *Suppose $\varnothing \neq C \subseteq \mathbf{R}^{n+1}$ is a set of positive reach with finite volume and assume that*

$$\boldsymbol{h}^{\phi}_{C,1}(a) \geq 0 \quad \text{for } \mathcal{H}^n \text{ a.e. } a \in \mathcal{A}(C).$$

Then

$$(n+1)\mathcal{L}^{n+1}(C) \leq n \int_{\partial^v C} \frac{\phi(\boldsymbol{n}(C,a))}{\boldsymbol{h}^{\phi}_{C,1}(a)}\, d\mathcal{H}^n(a). \tag{71}$$

If $\operatorname{int}(C) \neq \varnothing$, *equality holds in* (71) *and there exists* $q < \infty$ *so that* $\boldsymbol{h}^{\phi}_{C,1}(a) \leq q$ *for* $\mathcal{H}^n$ *a.e.* $a \in \mathcal{A}(C)$, *then there are* $N \in \mathbb{N}$, $c_1, \ldots, c_N \in \mathbf{R}^{n+1}$ *and* $\rho_1, \ldots, \rho_N \geq \frac{n}{q}$ *such that*

$$\operatorname{int}(C) = \bigcup_{i=1}^{N} \operatorname{int}(c_i + \rho_i \mathcal{W}^{\phi}), \qquad \operatorname{dist}^{\phi}\left(c_i + \rho_i \mathcal{W}^{\phi}, c_j + \rho_j \mathcal{W}^{\phi}\right) \geq 2 \operatorname{reach}^{\phi}(C) \quad \text{for } i \neq j.$$

Proof. We assume $\operatorname{int}(C) \neq \varnothing$ (otherwise there is nothing to prove) and we define $K = \mathbf{R}^{n+1} \setminus \operatorname{int}(C)$. Note that $\mathcal{L}^{n+1}(\partial C) = 0$ and $\iota(N^{\phi}(K)) = N^{\phi}(C)|\partial^v_+ C$. By Lemma 5.12 and the assumption, we infer that

$$\boldsymbol{H}^{\phi}_{K,1}(a,\eta) = -\boldsymbol{H}^{\phi}_{C,1}(a,-\eta) \leq 0 \quad \text{for } \mathcal{H}^n \text{ a.e. } (a,\eta) \in N^{\phi}(K).$$

Therefore, applying Theorem 3.20, Lemma 5.12 and the coarea formula, we obtain

$$\begin{aligned}(n+1)\mathcal{L}^{n+1}(C) &\leq n \int_{N^{\phi}(K)} J^{\phi}_K(a,\eta)\, \frac{\phi(\boldsymbol{n}^{\phi}(\eta))}{|\boldsymbol{H}^{\phi}_{K,1}(a,\eta)|}\, d\mathcal{H}^n(a,\eta) \\ &= n \int_{N^{\phi}(K)} \operatorname{ap} J^{N^{\phi}(K)}_n \iota(a,\eta)\, J^{\phi}_C(a,-\eta)\, \frac{\phi(\boldsymbol{n}^{\phi}(\eta))}{\boldsymbol{H}^{\phi}_{C,1}(a,-\eta)}\, d\mathcal{H}^n(a,\eta) \\ &= n \int_{N^{\phi}(C)|\partial^v_+ C} J^{\phi}_C(a,\eta)\, \frac{\phi(\boldsymbol{n}^{\phi}(\eta))}{\boldsymbol{H}^{\phi}_{C,1}(a,\eta)}\, d\mathcal{H}^n(a,\eta).\end{aligned}$$

Since $\phi(\boldsymbol{n}^{\phi}(\eta)) = \phi(\boldsymbol{n}^{\phi}(\nabla\phi(\boldsymbol{n}(C,a)))) = \phi(\boldsymbol{n}(C,a))$ for $(a,\eta) \in N^{\phi}(C)|\partial^v_+ C$, recalling Remark 3.15 and Corollary 5.10, we apply coarea formula in combination with Lemma 3.9 to obtain

$$\int_{N^{\phi}(C)|\partial^v_+ C} J^{\phi}_C(a,\eta)\, \frac{\phi(\boldsymbol{n}^{\phi}(\eta))}{\boldsymbol{H}^{\phi}_{C,1}(a,\eta)}\, d\mathcal{H}^n(a,\eta) = \int_{\partial^v C} \frac{\phi(\boldsymbol{n}(C,a))}{\boldsymbol{h}^{\phi}_{C,1}(a)}\, d\mathcal{H}^n(a),$$

which yields the first part of the assertion of the theorem.

Assume now that $\boldsymbol{h}^{\phi}_{C,1}(a) \leq q$ for $\mathcal{H}^n$ a.e. $a \in \mathcal{A}(C)$ and $\operatorname{int}(C) \neq \varnothing$. Combining Corollary 5.10 with Lemma 5.12 we get that

$$-\boldsymbol{H}^{\phi}_{K,1}(a,\eta) = \boldsymbol{H}^{\phi}_{C,1}(a,-\eta) = \boldsymbol{h}^{\phi}_{C,1}(a) \leq q \qquad \text{for } \mathcal{H}^n \text{ a.e. } (a,\eta) \in N^{\phi}(K).$$

Therefore if the equality holds in (71) then the conclusion follows from the characterization provided by Theorem 3.20. ∎

From Theorem 5.15 we obtain a geometric rigidity result for a set C of positive reach with positive and finite volume under the assumption of a sharp lower bound on the pointwise ϕ mean-curvature at almost all points in $\partial^v C$. For the set C in the next theorem we notice that $\mathcal{P}^{\phi}(C) > 0$ and $\operatorname{int}(C) \neq \varnothing$ by Corollary 5.10.

Corollary 5.16. *Suppose $C \subset \mathbf{R}^{n+1}$ is a set of positive reach with finite and positive volume and define* $\rho = \frac{(n+1)\mathcal{L}^{n+1}(C)}{\mathcal{P}^{\phi}(C)}$. *Assume that*

$$\boldsymbol{h}^{\phi}_{C,1}(a) \geq \frac{n}{\rho} \qquad \text{for } \mathcal{H}^n \text{ a.e. } a \in \mathcal{A}(C). \tag{72}$$

Then there exist $N \in \mathbb{N}$ and $c_1, \ldots, c_N \in \mathbf{R}^{n+1}$ such that

$$\operatorname{int}(C) = \bigcup_{i=1}^{N} \operatorname{int}(c_i + \rho \mathcal{W}^{\phi}), \quad \operatorname{dist}^{\phi}\left(c_i + \rho \mathcal{W}^{\phi}, c_j + \rho \mathcal{W}^{\phi}\right) \geq 2 \operatorname{reach}^{\phi}(C) \quad \text{for } i \neq j.$$

Proof. Notice that there is at least one point $a \in \partial^v_+ C$ with $\boldsymbol{h}^\phi_{C,1}(a) < \infty$, hence we obtain $\rho > 0$. Therefore $0 < \mathcal{P}^\phi(C) < \infty$.

For $\epsilon > 0$ we set

$$Z_\epsilon = \left\{ a \in \mathcal{A}(C) : \boldsymbol{h}^\phi_{C,1}(a) \geq (1+\epsilon)\frac{n}{\rho} \right\}.$$

We claim that $\mathcal{H}^n(Z_\epsilon) = 0$ for $\epsilon > 0$. Suppose that $\mathcal{H}^n(Z_\epsilon) > 0$ for some $\epsilon > 0$. Then we deduce

$$\begin{aligned}
n \int_{\partial^v C} \frac{\phi(\boldsymbol{n}(C,a))}{\boldsymbol{h}^\phi_{C,1}(a)}\, d\mathcal{H}^n(a) &= n \int_{\partial^v C \setminus Z_\epsilon} \frac{\phi(\boldsymbol{n}(C,a))}{\boldsymbol{h}^\phi_{C,1}(a)}\, d\mathcal{H}^n(a) + n \int_{Z_\epsilon} \frac{\phi(\boldsymbol{n}(C,a))}{\boldsymbol{h}^\phi_{C,1}(a)}\, d\mathcal{H}^n(a) \\
&\leq \rho \int_{\partial^v C \setminus Z_\epsilon} \phi(\boldsymbol{n}(C,a))\, d\mathcal{H}^n(a) + (1+\epsilon)^{-1}\rho \int_{Z_\epsilon} \phi(\boldsymbol{n}(C,a))\, d\mathcal{H}^n(a) \\
&< \rho \mathcal{P}^\phi(C) = (n+1)\mathcal{L}^{n+1}(C),
\end{aligned}$$

where we used (72) on $\partial^v C \setminus Z_\epsilon$ and the lower bound for $\boldsymbol{h}^\phi_{C,1}(a)$ on Z_ϵ. This contradicts the inequality in Theorem 5.15 and thus proves the claim.

Since $\mathcal{H}^n(Z_\epsilon) = 0$ for $\epsilon > 0$, we infer that

$$\boldsymbol{h}^\phi_{C,1}(a) = \frac{n\mathcal{P}^\phi(C)}{(n+1)\mathcal{L}^{n+1}(C)} \qquad \text{for } \mathcal{H}^n \text{ a.e. } a \in \partial^v C,$$

whence we infer that

$$n \int_{\partial^v C} \frac{\phi(\boldsymbol{n}(C,a))}{\boldsymbol{h}^\phi_{C,1}(a)}\, d\mathcal{H}^n(a) = (n+1)\mathcal{L}^{n+1}(C),$$

thus (71) holds with equality. We obtain now the conclusion of the theorem by employing the second part of Theorem 5.15. ☐

Remark 5.17. Corollary 5.16 is sharp already in the special isotropic case and for convex bodies. In fact, if we consider the union of two congruent proper antipodal spherical caps of the unit sphere, we obtain a convex body K whose k-th mean curvature on the smooth part of its boundary is constant and smaller than $\frac{\mathcal{H}^n(\partial K)}{(n+1)\mathcal{L}^{n+1}(K)}\binom{n}{k}$. We provide the details for completeness. Let $P \in \mathbf{G}(n+1, n)$ and $\eta \in P^\perp$ with $|\eta| = 1$. For every $0 < \epsilon < 1$ we define

$$\Sigma^+_\epsilon = \mathbf{S}^n \cap \{x : x \bullet \eta \geq \epsilon\}, \quad \Sigma^-_\epsilon = \mathbf{S}^n \cap \{x : x \bullet \eta \leq -\epsilon\},$$

$$P^+_\epsilon = \mathbf{B}(0,1) \cap \{x : x \bullet \eta = \epsilon\}, \quad P^-_\epsilon = \mathbf{B}(0,1) \cap \{x : x \bullet \eta = -\epsilon\},$$

and we denote by K^+_ϵ and K^-_ϵ the convex bodies enclosed by $\Sigma^+_\epsilon \cup P^+_\epsilon$ and $\Sigma^-_\epsilon \cup P^-_\epsilon$ respectively. Then we define

$$K_\epsilon = \{x - \epsilon\eta : x \in K^+_\epsilon\} \cup \{x + \epsilon\eta : x \in K^-_\epsilon\}.$$

Let $X(x) = x$ for every $x \in \mathbf{R}^{n+1}$. Since K^+_ϵ is a set of finite perimeter, we denote by η_ϵ the exterior unit normal and we compute by means of the divergence theorem [Fed69, Gauss–Green Theorem 4.5.6] (alternatively by noting that $K_\epsilon = \operatorname{conv}(\{o\} \cup \Sigma^+_\epsilon) \setminus \operatorname{conv}(\{o\} \cup P_\epsilon)$, where conv denotes the convex hull operator)

$$\begin{aligned}
(n+1)\mathcal{L}^{n+1}(K^+_\epsilon) &= \int_{K^+_\epsilon} \operatorname{div} X \, d\mathcal{L}^{n+1} \\
&= \int_{\Sigma^+_\epsilon \cup P^+_\epsilon} \eta_\epsilon(x) \bullet X(x)\, d\mathcal{H}^n(x) = \mathcal{H}^n(\Sigma^+_\epsilon) - \epsilon\mathcal{H}^n(P^+_\epsilon).
\end{aligned}$$

We conclude that

$$\frac{\mathcal{H}^n(\partial K_\epsilon)}{(n+1)\mathcal{L}^{n+1}(K_\epsilon)} = \frac{\mathcal{H}^n(\Sigma^+_\epsilon)}{(n+1)\mathcal{L}^{n+1}(K^+_\epsilon)} = 1 + \epsilon\frac{\mathcal{H}^n(P^+_\epsilon)}{(n+1)\mathcal{L}^{n+1}(K^+_\epsilon)} > 1$$

for $0 < \epsilon < 1$. Finally, we notice that the k-th mean curvature of K_ϵ equals $\binom{n}{k}$ on the smooth part of ∂K_ϵ.

6 Curvature measures and soap bubbles

6.1 Curvature measures and Minkowski formulae

In the following, we write C for a non-empty set with positive reach in $\mathbf{R}^{n+1}$. For sets with positive reach, the Steiner formula simplifies in the following way (also in the anisotropic setting).

Corollary 6.1 (Anisotropic Steiner formula for sets of positive reach)**.** *Let $\varnothing \neq C \subset \mathbf{R}^{n+1}$ be a set of positive reach. Let $\varphi : N^\phi(C) \to \mathbf{R}$ be a bounded Borel function with compact support. Then*

$$\int_{\{x\in\mathbf{R}^{n+1}:0<\boldsymbol{\delta}^\phi_C(x)\leq\rho\}} (\varphi\circ\boldsymbol{\psi}^\phi_C)\, d\mathcal{L}^{n+1} = \sum_{i=0}^{n} \frac{\rho^{i+1}}{i+1}\int_{N^\phi(C)} \phi(\boldsymbol{n}^\phi(\eta))\, J^\phi_C(a,\eta)\, \boldsymbol{H}^\phi_{C,i}(a,\eta)\, \varphi(a,u)\, d\mathcal{H}^n(a,\eta)$$

for $0<\rho<\mathrm{reach}^\phi(C)$.

Proof. The assertion is a straightforward consequence of Theorem 3.16. □

We can now introduce the generalized curvature measures of a set of positive reach with respect to ϕ. These are real-valued Radon measures (see [Za86, RZ19, Hu99, HL00] and the references cited there).

Definition 6.2. Let $\varnothing \neq C \subset \mathbf{R}^{n+1}$ be a set of positive reach and $m\in\{0,\dots,n\}$. The *m-th generalized curvature measure of C with respect to ϕ* is the real-valued Radon measure $\Theta^\phi_m(C,\cdot)$ on $\mathbf{R}^{n+1}\times\mathbf{R}^{n+1}$ such that

$$\Theta^\phi_m(C,B) = \frac{1}{n-m+1}\int_{N^\phi(C)\cap B} \phi(\boldsymbol{n}^\phi(\eta))\, J^\phi_C(a,\eta)\, \boldsymbol{H}^\phi_{C,n-m}(a,\eta)\, d\mathcal{H}^n(a,\eta)$$

for any bounded Borel subset $B\subset\mathbf{R}^{n+1}\times\mathbf{R}^{n+1}$. Moreover, we set

$$\mathcal{V}^\phi_m(C) = \Theta^\phi_m(C, N^\phi(C)) \qquad \text{for } m\in\{0,\dots,n\}.$$

Remark 6.3. Let $C\subseteq\mathbf{R}^{n+1}$ be a set of positive reach and $m\in\{0,\dots,n\}$. The *m-th curvature measure of C with respect to ϕ* is the real-valued Radon measure $\mathcal{C}^\phi_m(C,\cdot)$ on $\mathbf{R}^{n+1}$ such that

$$\mathcal{C}^\phi_m(C,B) = \Theta^\phi_m(C, B\times\mathbf{R}^{n+1})$$

for any bounded Borel subset $B\subset\mathbf{R}^{n+1}$.

Lemma 6.4. *Suppose C is a set of positive reach, $m\in\{0,\dots,n\}$ and let $\nu : C^{(n)}\to\mathbf{S}^n$ be a Borel map such that $\nu(a)\in N(C,a)$ for every $a\in C^{(n)}$. Let $\eta(a)=\nabla\phi(\nu(a))$. Then*

$$\begin{aligned}
&(n-m+1)\Theta^\phi_m(C, B\cap\widetilde{N}^\phi_n(C))\\
&\quad= \int_{\mathcal{A}(C)} \mathbf{1}_B(a,\eta(a))\phi(\nu(a))\,\boldsymbol{h}^\phi_{C,n-m}(a)\, d\mathcal{H}^n(a)\\
&\qquad+ \int_{C^{(n)}\setminus\partial^v C} \phi(\nu(a))\big[\mathbf{1}_B(a,\eta(a))\boldsymbol{H}^\phi_{C,n-m}(a,\eta(a)) + \mathbf{1}_B(a,-\eta(a))\boldsymbol{H}^\phi_{C,n-m}(a,-\eta(a))\big]\, d\mathcal{H}^n(a)
\end{aligned}$$

for every Borel set $B\subseteq N^\phi(C)$.

Proof. Let $B\subseteq N^\phi(C)$ be a Borel set. Combining Lemma 3.9 and the coarea formula, we get

$$(n-m+1)\Theta^\phi_m(C, B\cap\widetilde{N}^\phi_n(C)) = \int_{\boldsymbol{p}(\widetilde{N}^\phi_n(C))}\int_{N^\phi(C,a)} \mathbf{1}_B(a,\eta)\phi(\boldsymbol{n}^\phi(\eta))\,\boldsymbol{H}^\phi_{C,n-m}(a,\eta)\, d\mathcal{H}^0(\eta)\, d\mathcal{H}^n(a).$$

Since $\boldsymbol{n}^\phi(\nabla\phi(\nu(a)))=\nu(a)$ for $a\in C^{(n)}$, the argument is completed by applying Lemma 3.25 (c) and Corollary 5.10, since $N^\phi(C,a)=\{\pm\nabla\phi(\nu(a))\}$ for $a\in C^{(n)}\setminus\partial^v C$. □

Remark 6.5. The Lebesgue decomposition of the curvature measure $\mathcal{C}^{\phi}_{m}(C,\cdot)$ with respect to $\mathcal{H}^n \llcorner \partial C$ is given by $\Theta^{\phi}_{m}(C,(\cdot\times\mathbf{R}^{n+1})\cap\widetilde{N}^{\phi}_{n}(C))$ (the absolutely continuous part) and $\Theta^{\phi}_{m}(C,(\cdot\times\mathbf{R}^{n+1})\setminus\widetilde{N}^{\phi}_{n}(C))$ (the singular part). It follows from Lemma 3.25 (e) that these parts are indeed singular with respect to each other. For the absolutely continuous part, Lemma 6.4 yields an explicit description. In the case of convex bodies where $C^{(n)} \setminus \partial^v C = \emptyset$, a corresponding analysis can be found in [Hu98] in the isotropic framework.

The following lemma extends [Fa96, Lemma 2.1] from Euclidean curvature measures of convex bodies to generalized curvature measures with respect to a C^2-norm ϕ and sets with positive reach (compare also [CH00, Section 3]). The non-negative Radon measure $|\Theta^{\phi}_{m}(C,\cdot)\llcorner(A\times\partial\mathcal{W}^{\phi})|$ in the next lemma is the total variation of the real-valued Radon measure $\Theta^{\phi}_{m}(C,\cdot)\llcorner(A\times\partial\mathcal{W}^{\phi})$; see [AFP00, Definition 1.4].

Lemma 6.6. *Let $\varnothing \neq C \subset \mathbf{R}^{n+1}$ be a set of positive reach. Let $A \subset \mathbf{R}^{n+1}$ be a $\mathcal{H}^m$ measurable set and $m \in \{0,\dots,n\}$. Then there is a non-negative constant c, depending only on n,ϕ, such that*

$$|\Theta^{\phi}_{m}(C,\cdot)\llcorner(A\times\partial\mathcal{W}^{\phi})| \le c\cdot\mathcal{H}^m(A).$$

Proof. For the proof, one can assume that $\mathcal{H}^m(A) < \infty$. An application of Theorem 3.27 to the positive and the negative part of $\Theta^{\phi}_{m}(C,\cdot)\llcorner(A\times\partial\mathcal{W}^{\phi})$ then yields the assertion. ☐

The following lemma is now an immediate consequence of Lemma 6.6. We do not include the case $m = n$ in the statement of the lemma, since in this case the hypothesis is always satisfied by Lemma 3.25 (c) and the conclusion holds essentially by definition; see Remark 3.12.

Lemma 6.7. *Let $\varnothing \neq C \subset \mathbf{R}^{n+1}$ be a set of positive reach. Let $m \in \{0,\dots,n-1\}$ and assume that*

$$\mathcal{H}^m[\boldsymbol{p}(\widetilde{N}^{\phi}(C)\setminus\widetilde{N}^{\phi}_{n}(C))] = 0.$$

Then

$$(n-m+1)\cdot\Theta^{\phi}_{m}(C,B) = \int_{\widetilde{N}^{\phi}_{n}(C)\cap B}\phi(\boldsymbol{n}^{\phi}(\eta))J^{\phi}_{C}(a,\eta)\boldsymbol{H}^{\phi}_{C,n-m}(a,\eta)\,d\mathcal{H}^n(a,\eta)$$

for every Borel set $B \subseteq N^{\phi}(C)$.

Proof. An application of Lemma 6.6 with $A = \boldsymbol{p}(\widetilde{N}^{\phi}(C)\setminus\widetilde{N}^{\phi}_{n}(C))$ yields that

$$\Theta^{\phi}_{m}(C,\cdot)\llcorner(\boldsymbol{p}(\widetilde{N}^{\phi}(C)\setminus\widetilde{N}^{\phi}_{n}(C))\times\partial\mathcal{W}^{\phi}) = 0,$$

and hence $\Theta^{\phi}_{m}(C, B\cap\widetilde{N}^{\phi}(C)\setminus\widetilde{N}^{\phi}_{n}(C)) = 0$, which is the desired conclusion. ☐

We now prove the anisotropic Minkowski formulae for sets of positive reach. The case of convex bodies has been treated in a different way in [Hu99].

Theorem 6.8. *If $\varnothing \neq C \subset \mathbf{R}^{n+1}$ is a set of positive reach with finite volume and $r \in \{1,\dots,n\}$, then*

$$\begin{aligned}(n-r+1)&\int_{N^{\phi}(C)}\phi(\boldsymbol{n}^{\phi}(\eta))\,J^{\phi}_{C}(a,\eta)\,\boldsymbol{H}^{\phi}_{C,r-1}(a,\eta)\,d\mathcal{H}^n(a,\eta)\\ &= r\int_{N^{\phi}(C)}[a\bullet\boldsymbol{n}^{\phi}(\eta)]J^{\phi}_{C}(a,\eta)\,\boldsymbol{H}^{\phi}_{C,r}(a,\eta)\,d\mathcal{H}^n(a,\eta)\end{aligned}$$

and

$$\int_{N^{\phi}(C)} a\bullet\boldsymbol{n}^{\phi}(\eta)\,J^{\phi}_{C}(a,\eta)\,\boldsymbol{H}^{\phi}_{C,0}(a,\eta)\,d\mathcal{H}^n(a,\eta) = (n+1)\mathcal{L}^{n+1}(C).$$

Proof. We set $u(x) = \frac{\nabla\boldsymbol{\delta}^{\phi}_{C}(x)}{|\nabla\boldsymbol{\delta}^{\phi}_{C}(x)|}$ for $x \in \operatorname{Unp}^{\phi}(C)$ and notice that for $0 < \rho < \operatorname{reach}^{\phi}(C)$ the set $B^{\phi}(C,\rho)$ is a domain with $\mathcal{C}^{1,1}$-boundary $\partial B^{\phi}(C,\rho) = S^{\phi}(C,\rho)$ whose exterior unit normal is given by $u|\partial B^{\phi}(C,\rho)$. Moreover, for $0 < \rho < \operatorname{reach}^{\phi}(C)$ the map $f_\rho : N^{\phi}(C) \to S^{\phi}(C,\rho)$ defined by

$$f_\rho(a,\eta) = a + \rho\eta \qquad \text{for } (a,\eta)\in N^{\phi}(C)$$

is a bi-lipschitz homeomorphism by Remark 5.4. We observe (see proof of Theorem 3.16) that

$$J_n^{N^\phi(C)} f_\rho(a,\eta) = J_C^\phi(a,\eta) \sum_{m=0}^{n} \rho^{n-m}\, \boldsymbol{H}_{C,n-m}^\phi(a,\eta)$$

for $\mathcal{H}^n$ a.e. $(a,\eta) \in N^\phi(C)$. We set

$$I_m(C) = \int_{N^\phi(C)} a \bullet \boldsymbol{n}^\phi(\eta)\, J_C^\phi(a,\eta)\, \boldsymbol{H}_{C,n-m}^\phi(a,\eta)\, d\mathcal{H}^n(a,\eta) \qquad \text{for } m = 0,\dots,n.$$

The divergence theorem and Remark 3.4 yield

$$\begin{aligned}
(n+1)\mathcal{L}^{n+1}(B^\phi(C,\rho)) &= \int_{S^\phi(C,\rho)} x \bullet u(x)\, d\mathcal{H}^n(x) \\
&= \int_{N^\phi(C)} [(a+\rho\eta) \bullet u(a+\rho\eta)]\, J_n^{N^\phi(C)} f_\rho(a,\eta)\, d\mathcal{H}^n(a,\eta) \\
&= \sum_{m=0}^{n} \rho^{n-m} I_m(C) + \sum_{m=0}^{n} (n-m+1)\rho^{n-m+1} \mathcal{V}_m^\phi(C)
\end{aligned}$$

for $0 < \rho < \operatorname{reach}^\phi(C)$. Employing the Steiner formula 6.1, we get

$$(n+1)\mathcal{L}^{n+1}(B^\phi(C,\rho)) = (n+1)\mathcal{L}^{n+1}(C) + (n+1) \sum_{m=0}^{n} \rho^{n-m+1} \mathcal{V}_m^\phi(C)$$

for $0 < \rho < \operatorname{reach}^\phi(C)$. Hence, we infer

$$\sum_{m=0}^{n-1} [I_m(C) - (m+1)\mathcal{V}_{m+1}^\phi(C)]\rho^{n-m} + I_n(C) - (n+1)\mathcal{L}^{n+1}(C) = 0$$

for $0 < \rho < \operatorname{reach}^\phi(C)$. It follows that $I_m(C) = (m+1)\mathcal{V}_{m+1}^\phi(C)$ for $m = 0,\dots,n-1$ and in addition we have $I_n(C) = (n+1)\mathcal{L}^{n+1}(C)$. □

6.2 The soap bubble theorem for sets of positive reach

The following notion of k-convexity generalizes the classical analogous notion used in the Euclidean setting to study isoperimetric-type inequalities for Querrmassintegrals (see [Tru94] or the more recent [CW13]). Analogous concepts also arise in the context of elliptic differential operators (see [TW99] and [Sal99] and the references given there to earlier work for instance by Caffarelli, Nirenberg, Spruck ('85), Garding ('59), Ivochkina ('83, '85), Li ('90)).

Definition 6.9. Let $\varnothing \neq C \subset \mathbf{R}^{n+1}$ be a set of positive reach with $\mathcal{P}^\phi(C) > 0$, and let $r \in \{0,\dots,n\}$. We say that C is *(r,ϕ)-mean convex* if

$$\boldsymbol{h}_{C,i}^\phi(a) \geq 0 \qquad \text{for } \mathcal{H}^n \text{ a.e. } a \in \mathcal{A}(C) \text{ and } i = 1,\dots,r-1 \tag{73}$$

and

$$\boldsymbol{H}_{C,r}^\phi(a,u) \geq 0 \qquad \text{for } \mathcal{H}^n \text{ a.e. } (a,u) \in N^\phi(C). \tag{74}$$

Remark 6.10. Suppose that $\varnothing \neq C \subset \mathbf{R}^{n+1}$ is a set of positive reach with finite volume. Then $\mathcal{P}^\phi(C) > 0$ if and only if $\mathcal{H}^n(\partial_+^v C) > 0$ by Corollary 5.10. This in turn is equivalent to $\operatorname{int}(C) \neq \varnothing$.

Remark 6.11. The $(0,\phi)$-mean convex sets are all sets of positive reach with non-empty interior, since by definition we have $\boldsymbol{H}_{C,0}^\phi = \mathbf{1}_{\widetilde{N}_n^\phi(C)} \geq 0$. Moreover, if C is a set of positive reach and positive perimeter such that $\Theta_{n-k}^\phi(C\cdot)$ are non-negative measures for all $k \in \{1,\dots,r\}$, then C is (r,ϕ)-mean convex.

Before we can add another remark, we need some preparations. Let d_H^ϕ denote the Hausdorff distance on the space of closed subsets of the metric space $(\mathbf{R}^{n+1}, \phi^*)$. We say that a sequence of closed sets $C_i \subseteq \mathbf{R}^{n+1}$, $i \in \mathbb{N}$, converges to a closed set $C \subseteq \mathbf{R}^{n+1}$ as $i \to \infty$ if $d_H^\phi(C_i, C) \to 0$ as $i \to \infty$, which is equivalent to the uniform convergence of $\delta_{C_i}^\phi$ to δ_C^ϕ as $i \to \infty$ on $\mathbf{R}^{n+1}$ (see [Be85]). The following lemma is well known in the Euclidean setting (see Theorem 4.13, Remark 4.14 and Theorem 5.9 in [Fed59] and [RZ01, Section 3.1, pp. 7–9]).

Lemma 6.12. *Let $\varnothing \neq C_i \subset \mathbf{R}^{n+1}$ for $i \in \mathbb{N}$ be a sequence of closed sets converging to a closed set $C \subset \mathbf{R}^{n+1}$. Suppose there is a constant $\rho > 0$ such that $\mathrm{reach}^\phi(C_i) \geq \rho$ for all $i \in \mathbb{N}$. Then $\mathrm{reach}^\phi(C) \geq \rho$ and for each $k \in \{0, \dots, n\}$ the Radon measures $\Theta_k^\phi(C_i, \cdot)$ converge vaguely to the Radon measure $\Theta_k^\phi(C, \cdot)$ as $i \to \infty$.*

Proof. Let $\rho_1 \in (0, \rho)$. Under the assumptions of the lemma, we show that $\mathrm{reach}(C) \geq \rho_1$. Let $x \in \mathbf{R}^{n+1} \setminus C$ with $\delta_C^\phi(x) \leq \rho_1$. There is some $i_1 \in \mathbb{N}$ such that if $i \geq i_1$, then $0 < \delta_{C_i}^\phi(x) \leq \rho_2 < \rho$, where $\rho_2 := \frac{1}{2}(\rho + \rho_1)$. Define $x_i := \xi_{C_i}^\phi(x) + \rho_2 \cdot \nu_{C_i}^\phi(x)$, hence $\delta_{C_i}^\phi(x_i) = \rho_2$, $\delta_{C_i}^\phi(x_i) = \delta_{C_i}^\phi(x)$ and $U^\phi(x_i, \rho_2) \cap C_i = \varnothing$, since $\boldsymbol{r}_{C_i}^\phi(\xi_{C_i}^\phi(x), \nu_{C_i}^\phi(x)) \geq \mathrm{reach}(C_i)$. By compactness, we can find an infinite subset $I \subseteq \mathbb{N}$ such $\xi_{C_i}^\phi(x_i) \to \xi \in \partial C$, $x_i \to z$, $\delta_C^\phi(z) = \rho_2$, $U^\phi(z, \rho_2) \cap C = \varnothing$ and $x \in (\xi, z)$, where $I \ni i \to \infty$. But then clearly $U^\phi(x, \rho_2) \cap C = \varnothing$, $x \in \mathrm{Unp}^\phi(C)$ and $\xi = \xi_C^\phi(x)$. This proves the first assertion.

In view of Corollary 6.1, it is sufficient to show that if $\varphi : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}$ is a continuous function with compact support and $t \in (0, \rho)$, then

$$\int \mathbf{1}\{0 < \delta_{C_i}^\phi(x) \leq t\} \varphi(\xi_{C_i}^\phi(x), \nu_{C_i}^\phi(x)) \, d\mathcal{L}^{n+1}(x) \to \int \mathbf{1}\{0 < \delta_C^\phi(x) \leq t\} \varphi(\xi_C^\phi(x), \nu_C^\phi(x)) \, d\mathcal{L}^{n+1}(x)$$

as $i \to \infty$.

We consider an arbitrary point $x \in \mathbf{R}^{n+1}$. If $0 < \delta_C^\phi(x) < t$, then also $0 < \delta_{C_i}^\phi(x) < t$ if $i \in \mathbb{N}$ is large enough. Moreover, $\xi_{C_i}^\phi(x) \to \xi_C^\phi(x)$ and $\nu_{C_i}^\phi(x) \to \nu_C^\phi(x)$ as $i \to \infty$, which can be obtained by minor adjustments of the proof for [HL00, Lemma 2.2]. Therefore,

$$\mathbf{1}\{0 < \delta_{C_i}^\phi(x) \leq t\} \varphi(\xi_{C_i}^\phi(x), \nu_{C_i}^\phi(x)) \to \mathbf{1}\{0 < \delta_C^\phi(x) \leq t\} \varphi(\xi_C^\phi(x), \nu_C^\phi(x)) \tag{75}$$

as $i \to \infty$.

If $\delta_C^\phi(x) > t$, then also $\delta_{C_i}^\phi(x) > t$ if $i \in \mathbb{N}$ is large enough, hence (75) holds trivially. The same is true if $x \in \mathrm{int}(C)$ which implies that $x \in C_i$ for all sufficiently large $i \in \mathbb{N}$.

Since $\mathcal{L}^{n+1}(\{x \in \mathbf{R}^{n+1} : \delta_C^\phi(x) = t\} \cup \partial C) = 0$, the assertion follows from the dominated convergence theorem. $\square$

The preceding lemma implies that a non-negativity condition closely related to Definition 6.9 is stable with respect to converging sequences of sets of positive reach, as described in the following corollary.

Corollary 6.13. *Let $\rho > 0$ be a fixed constant. Let $\varnothing \neq C_i \subset \mathbf{R}^{n+1}$ for $i \in \mathbb{N}$ be a sequence of closed sets with $\mathrm{reach}(C_i) \geq \rho > 0$ converging to a closed set $C \subset \mathbf{R}^{n+1}$. Then the following statements hold.*

(a) *If $\Theta_k^\phi(C_j, \cdot) \geq 0$ for all $j \in \mathbb{N}$, then also $\Theta_k^\phi(C, \cdot) \geq 0$.*

(b) *If C_j is (r, ϕ)-mean convex and smooth for all $j \in \mathbb{N}$ (so that $\Theta_{n-k}^\phi(C_j, \cdot) \geq 0$ holds for $k = 1, \dots, r$), then $\Theta_{n-k}^\phi(C, \cdot) \geq 0$ for $k = 1, \dots, r$; hence, C is (r, ϕ)-mean convex.*

A major ingredient for the proof of the Alexandrov theorem for sets with positive reach is the next lemma.

Lemma 6.14. *Let $\varnothing \neq C \subset \mathbf{R}^{n+1}$ be a set of positive reach with finite and positive volume. Let $\lambda \in \mathbf{R}$ and $r \in \{1, \dots, n\}$ be such that*

$$\Theta_{n-r}^\phi(C, \cdot) = \lambda \, \Theta_n^\phi(C, \cdot). \tag{76}$$

Then $\boldsymbol{H}^{\phi}_{C,r}(a,\eta) = 0$ *for* $\mathcal{H}^n$ *a.e.* $(a,\eta) \in N^{\phi}(C) \setminus \widetilde{N}^{\phi}_n(C)$ *and*

$$\boldsymbol{H}^{\phi}_{C,r}(a,\eta) = (r+1)\lambda = \frac{n-r+1}{(n+1)\mathcal{L}^{n+1}(C)}\mathcal{V}^{\phi}_{n-r+1}(C) \qquad \text{for } \mathcal{H}^n \text{ a.e. } (a,\eta) \in \widetilde{N}^{\phi}_n(C).$$

Furthermore, the following two statements hold.

(a) *If* $\lambda \neq 0$*,* $r \geq 2$ *and* C *is* $(r-1,\phi)$*-mean convex, then*

$$\infty > \frac{\boldsymbol{h}^{\phi}_{C,1}(a)}{\binom{n}{1}} \geq \left(\frac{\boldsymbol{h}^{\phi}_{C,2}(a)}{\binom{n}{2}}\right)^{\frac{1}{2}} \geq \ldots \geq \left(\frac{\boldsymbol{h}^{\phi}_{C,r}(a)}{\binom{n}{r}}\right)^{\frac{1}{r}} \geq \frac{\mathcal{P}^{\phi}(C)}{(n+1)\mathcal{L}^{n+1}(C)}$$

for $\mathcal{H}^n$ *a.e.* $a \in \partial^v C$*.*

(b) *If* $r = 1$*, then* $\boldsymbol{H}^{\phi}_{C,1}(a,\eta) \geq \frac{n\mathcal{P}^{\phi}(C)}{(n+1)\mathcal{L}^{n+1}(C)}$ *for* $\mathcal{H}^n$ *a.e.* $(a,\eta) \in \widetilde{N}^{\phi}_n(C)$*.*

Proof. From the equality $\Theta^{\phi}_{n-r}(C,\cdot) = \lambda\,\Theta^{\phi}_n(C,\cdot)$ we get

$$\begin{aligned}\int_{B\cap \widetilde{N}^{\phi}_n(C)} &\phi(\boldsymbol{n}^{\phi}(\eta))\, J^{\phi}_C(a,\eta) \left(\frac{1}{r+1}\boldsymbol{H}^{\phi}_{C,r}(a,\eta) - \lambda\right) d\mathcal{H}^n(a,\eta) \\ &+ \frac{1}{r+1}\int_{(N^{\phi}(C)\setminus \widetilde{N}^{\phi}_n(C))\cap B} \phi(\boldsymbol{n}^{\phi}(\eta))\, J^{\phi}_C(a,\eta)\, \boldsymbol{H}^{\phi}_{C,r}(a,\eta)\, d\mathcal{H}^n(a,\eta) = 0\end{aligned}$$

for every bounded Borel set $B \subset N^{\phi}(C)$. Since $J^{\phi}_C(a,\eta)\phi(\boldsymbol{n}^{\phi}(\eta)) > 0$ for $\mathcal{H}^n$ a.e. $(a,\eta) \in N^{\phi}(C)$, we conclude that $\boldsymbol{H}^{\phi}_{C,r}(a,\eta) = 0$ for $\mathcal{H}^n$ a.e. $(a,\eta) \in N^{\phi}(C) \setminus \widetilde{N}^{\phi}_n(C)$ and $\boldsymbol{H}^{\phi}_{C,r}(a,\eta) = (r+1)\lambda$ for $\mathcal{H}^n$ a.e. $(a,\eta) \in \widetilde{N}^{\phi}_n(C)$. Noting that $\boldsymbol{H}^{\phi}_{C,0} = \boldsymbol{1}_{\widetilde{N}^{\phi}_n(C)}$ and employing Theorem 6.8, we derive

$$\begin{aligned}(n-r+1)&\int_{N^{\phi}(C)} \phi(\boldsymbol{n}^{\phi}(\eta))\, J^{\phi}_C(a,\eta)\, \boldsymbol{H}^{\phi}_{C,r-1}(a,\eta)\, d\mathcal{H}^n(a,\eta) \\ &= r\int_{N^{\phi}(C)} [a \bullet \boldsymbol{n}^{\phi}(\eta)] J^{\phi}_C(a,\eta)\, \boldsymbol{H}^{\phi}_{C,r}(a,\eta)\, d\mathcal{H}^n(a,\eta) \\ &= r(r+1)\lambda \int_{\widetilde{N}^{\phi}_n(C)} [a \bullet \boldsymbol{n}^{\phi}(\eta)] J^{\phi}_C(a,\eta)\, d\mathcal{H}^n(a,\eta) \\ &= r(r+1)\lambda(n+1)\mathcal{L}^{n+1}(C), \end{aligned} \tag{77}$$

from which we infer that

$$\boldsymbol{H}^{\phi}_{C,r}(a,\eta) = \frac{n-r+1}{(n+1)\mathcal{L}^{n+1}(C)}\mathcal{V}^{\phi}_{n-r+1}(C) \qquad \text{for } \mathcal{H}^n \text{ a.e. } (a,\eta) \in \widetilde{N}^{\phi}_n(C), \tag{78}$$

since $\mathcal{L}^{n+1}(C) \in (0,\infty)$. In particular, using Corollary 5.10, we obtain

$$\boldsymbol{h}^{\phi}_{C,r}(a) = \frac{n-r+1}{(n+1)\mathcal{L}^{n+1}(C)}\mathcal{V}^{\phi}_{n-r+1}(C) \qquad \text{for } \mathcal{H}^n \text{ a.e. } a \in \partial^v C.$$

Notice that the asserted conclusion for $r = 1$ already follows from (78), since $\mathcal{V}^{\phi}_n(C) \geq \mathcal{P}^{\phi}(C)$.

We assume now $r \geq 2$, $\lambda \neq 0$ and that C is $(r-1,\phi)$-mean convex. The non-negativity property of $\boldsymbol{H}^{\phi}_{C,r-1}$ in combination with (77) implies that $\lambda \geq 0$, that means $\lambda > 0$. Therefore $\boldsymbol{h}^{\phi}_{C,r}(a) > 0$ is satisfied for $\mathcal{H}^n$ a.e. $a \in \partial^v C$. Hence we can apply Lemma 2.2 to conclude that

$$\frac{\boldsymbol{h}^{\phi}_{C,1}(a)}{\binom{n}{1}} \geq \ldots \geq \left(\frac{\boldsymbol{h}^{\phi}_{C,r-1}(a)}{\binom{n}{r-1}}\right)^{\frac{1}{r-1}} \geq \left(\frac{\boldsymbol{h}^{\phi}_{C,r}(a)}{\binom{n}{r}}\right)^{\frac{1}{r}} = \left(\frac{(r+1)\lambda}{\binom{n}{r}}\right)^{\frac{1}{r}} \tag{79}$$

for $\mathcal{H}^n$ a.e. $a \in \partial^v C$. Using again that $\boldsymbol{H}^{\phi}_{C,r-1}(a,\eta) \geq 0$ for $\mathcal{H}^n$ a.e. $(a,\eta) \in N^{\phi}(C)$ and the lower bound for $\boldsymbol{H}^{\phi}_{C,r-1}(a,\eta)$ from (79), we get

$$(n-r+1)\int_{N^{\phi}(C)} \phi(\boldsymbol{n}^{\phi}(\eta))\, J^{\phi}_C(a,\eta)\, \boldsymbol{H}^{\phi}_{C,r-1}(a,\eta)\, d\mathcal{H}^n(a,\eta)$$

$$
\begin{aligned}
&\geq (n-r+1)\int_{N^\phi(C)|\partial_+^v C} \phi(\boldsymbol{n}^\phi(\eta))\, J_C^\phi(a,\eta)\, \boldsymbol{H}^\phi_{C,r-1}(a,\eta)\, d\mathcal{H}^n(a,\eta)\\
&\geq (n-r+1)[(r+1)\lambda]^{\frac{r-1}{r}}\binom{n}{r}^{\frac{1-r}{r}}\binom{n}{r-1}\int_{N^\phi(C)|\partial_+^v C}\phi(\boldsymbol{n}^\phi(\eta))\, J_C^\phi(a,\eta)\, d\mathcal{H}^n(a,\eta)\\
&= (n-r+1)[(r+1)\lambda]^{\frac{r-1}{r}}\binom{n}{r}^{\frac{1-r}{r}}\binom{n}{r-1}\mathcal{P}^\phi(C). \qquad (80)
\end{aligned}
$$

Combining (77) and (80), noting that $(n-r+1)\binom{n}{r}^{-1}\binom{n}{r-1}\frac{1}{r}=1$ and recalling that $\lambda>0$, we conclude

$$
\infty > [\lambda(r+1)]^{\frac{1}{r}} \geq \frac{\mathcal{P}^\phi(C)}{(n+1)\mathcal{L}^{n+1}(C)}\binom{n}{r}^{\frac{1}{r}}.
$$

Now the remaining assertion follows from (79). ☐

The Alexandrov theorem for sets of positive reach is now a corollary of our previous results.

Theorem 6.15. *Let $r\in\{1,\dots,n\}$, and let $C\subset\mathbf{R}^{n+1}$ be an $(r-1,\phi)$-mean convex set with positive and finite volume such that*

$$
\Theta^\phi_{n-r}(C,\cdot)=\lambda\,\Theta^\phi_n(C,\cdot)\quad \text{for some } \lambda\in\mathbf{R}\setminus\{0\}. \qquad (81)
$$

Then $\lambda>0$ and there exist a finite natural number $N\geq 1$ and $c_1,\dots,c_N\in\mathbf{R}^{n+1}$ such that

$$
\operatorname{int}(C)=\bigcup_{i=1}^N \operatorname{int}(c_i+\rho\mathcal{W}^\phi),\quad \operatorname{dist}^\phi\left(c_i+\rho\mathcal{W}^\phi, c_j+\rho\mathcal{W}^\phi\right)\geq 2\operatorname{reach}^\phi(C)\quad \text{for } i\neq j,
$$

where ρ satisfies the relations $\binom{n}{r}^{\frac{1}{r}}(\lambda(r+1))^{-\frac{1}{r}}=\rho=\frac{(n+1)\mathcal{L}^{n+1}(C)}{\mathcal{P}^\phi(C)}$ and $\lambda>0$.

If $r=1$, then the same conclusion is obtained for any $\lambda\in\mathbf{R}$ and any set C of positive reach with positive and finite volume.

Proof. The assertion is implied by a combination of Lemma 6.14 and Corollary 5.16, in particular it follows that $\lambda>0$.

To check the equation $\rho=\binom{n}{r}^{\frac{1}{r}}(\lambda(r+1))^{-\frac{1}{r}}$, we first observes that $\kappa^\phi_{\rho\mathcal{W}^\phi}(a,\eta)=\frac{1}{\rho}$ for $(a,\eta)\in N^\phi(\rho\mathcal{W}^\phi)$ (see for instance [DRKS20, Corollary 2.33]); since C is the disjoint union of N translated copies of $\rho\mathcal{W}^\phi$, one infers that $\boldsymbol{H}^\phi_{C,r}(a,\eta)=\binom{n}{r}\rho^{-r}$ for $(a,\eta)\in N^\phi(C)$. As $\boldsymbol{H}^\phi_{C,r}(a,\eta)=\lambda(r+1)$ for $(a,\eta)\in N^\phi(C)$ by Lemma 6.14, we obtain the aforementioned equation. ☐

In Theorem 6.15 we deal with sets which are non-convex, hence it is natural to state the proportionality assumption (81) on the r-th ϕ-mean curvatures in terms of generalized curvature measures. However, under a slightly more restrictive mean convexity assumption we also get the following variant of Theorem 6.15 in which a corresponding assumption on the proportionality of a curvature measure is imposed in (82). The proof is based on a modified version of Lemma 6.14, whose proof only requires minor adjustments (and uses (63) in first part of the argument).

Theorem 6.16. *Let $r\in\{1,\dots,n\}$, and let $C\subset\mathbf{R}^{n+1}$ be an $(r-1,\phi)$-mean convex set with positive and finite volume and such that $\boldsymbol{H}^\phi_{C,r}(a,\eta)\geq 0$ for $\mathcal{H}^n$ a.e. $(a,\eta)\in N^\phi(C)$. Assume that*

$$
\mathcal{C}^\phi_{n-r}(C,\cdot)=\lambda\,\mathcal{C}^\phi_n(C,\cdot)\quad \text{for some } \lambda>0. \qquad (82)
$$

Then there exist a finite natural number $N\geq 1$ and $c_1,\dots,c_N\in\mathbf{R}^{n+1}$ such that

$$
\operatorname{int}(C)=\bigcup_{i=1}^N \operatorname{int}(c_i+\rho\mathcal{W}^\phi),\quad \operatorname{dist}^\phi\left(c_i+\rho\mathcal{W}^\phi, c_j+\rho\mathcal{W}^\phi\right)\geq 2\operatorname{reach}^\phi(C)\quad \text{for } i\neq j,
$$

where ρ is given as in Theorem 6.15.

We conclude this section discussing the validity of the hypothesis of Theorem 6.15 in terms of Alexandrov points and pointwise curvatures for a large subclass of sets of positive reach, namely those sets C for which $\mathcal{H}^n(\partial C \setminus \partial^v C) = 0$. This class includes all convex bodies, and more generally all closed sets that can be locally represented as the epigraph of a semiconvex function; see Lemma 6.20. But it includes much more; indeed, it is easy to construct sets of positive reach C for which $\mathcal{H}^n(\partial C \setminus \partial^v C) = 0$, but the boundary is not a topological manifold (see [RZ17, Example 7.12] or [ACV08, Example 1]).

The hypotheses in the next statement should be seen in connection with the disjoint union displayed in (63).

Lemma 6.17. *If $k \in \{1, \ldots, n\}$, $\lambda \in \mathbf{R}$ and $\varnothing \neq C \subset \mathbf{R}^{n+1}$ is a set of positive reach such that $\mathcal{H}^n(\partial C \setminus \partial^v C) = 0$ and $\mathcal{P}^\phi(C) > 0$, then the following two statements hold.*

(a) *If $\boldsymbol{h}^\phi_{C,k}(a) = \lambda$ for $\mathcal{H}^n$ a.e. $a \in \mathcal{A}(C)$ and $\mathcal{H}^{n-k}\big[\boldsymbol{p}(\widetilde{N}^\phi(C) \setminus \widetilde{N}^\phi_n(C))\big] = 0$, then*

$$\Theta^\phi_{n-k}(C, \cdot) = \lambda\, \Theta^\phi_n(C, \cdot).$$

(b) *If $\boldsymbol{h}^\phi_{C,1}(a) \geq 0, \ldots, \boldsymbol{h}^\phi_{C,k-1}(a) \geq 0$ for $\mathcal{H}^n$ a.e. $a \in \mathcal{A}(C)$ and $\mathcal{H}^{n-k+1}\big[\boldsymbol{p}(\widetilde{N}^\phi(C) \setminus \widetilde{N}^\phi_n(C))\big] = 0$, then C is $(k-1, \phi)$-mean convex.*

Proof. (a) Let $B \subseteq N^\phi(C)$ be a Borel set. Noting that $\Theta^\phi_n(C, B) = \Theta^\phi_n\big(C, B \cap \widetilde{N}^\phi_n(C)\big)$ by definition of $\boldsymbol{H}^\phi_{C,0}$, we can use Lemma 6.7, Lemma 6.4 and Corollary 5.10 (a), (b) together with $\eta(a) = \nabla\phi(\boldsymbol{n}(C, a))$ for $a \in \partial^v C$ to compute

$$\begin{aligned}
\Theta^\phi_{n-k}(C, B) &= \Theta^\phi_{n-k}\big(C, B \cap \widetilde{N}^\phi_n(C)\big) \\
&= \int_{\partial^v C} \mathbf{1}_B(a, \eta(a))\phi(\boldsymbol{n}(C, a))\boldsymbol{h}^\phi_{C,k}(a)\, d\mathcal{H}^n(a) \\
&= \lambda \int_{\partial^v C} \mathbf{1}_B(a, \eta(a))\phi(\boldsymbol{n}(C, a))\, d\mathcal{H}^n(a) \\
&= \lambda\, \Theta^\phi_n(C, B).
\end{aligned}$$

(b) Arguing as in (a) we can compute

$$\begin{aligned}
\Theta^\phi_{n-k+1}(C, B) &= \Theta^\phi_{n-k+1}\big(C, B \cap \widetilde{N}^\phi_n(C)\big) \\
&= \int_{\partial^v C} \mathbf{1}_B(a, \eta(a))\phi(\boldsymbol{n}(C, a))\boldsymbol{h}^\phi_{C,k-1}(a)\, d\mathcal{H}^n(a) \geq 0
\end{aligned}$$

for every Borel set $B \subseteq N^\phi(C)$. This means that $\boldsymbol{H}^\phi_{C,k-1}(a, \eta) \geq 0$ for $\mathcal{H}^n$ a.e. $(a, \eta) \in N^\phi(C)$ and consequently C is $(\phi, k-1)$-mean convex. ☐

Now with the help of Lemma 6.17, the following result can be easily deduced as a special case of Theorem 6.15.

Corollary 6.18. *Suppose $k \in \{1, \ldots, n\}$, $\lambda \in \mathbf{R} \setminus \{0\}$ and $\varnothing \neq C \subset \mathbf{R}^{n+1}$ is a set of positive reach with finite and positive volume such that*

(1) *$\mathcal{H}^n(\partial C \setminus \partial^v C) = 0$ and $\mathcal{H}^{n-k}\big[\boldsymbol{p}(\widetilde{N}^\phi(C) \setminus \widetilde{N}^\phi_n(C))\big] = 0$,*

(2) *$\boldsymbol{h}^\phi_{C,k}(a) = \lambda$ for $\mathcal{H}^n$ a.e. $a \in \mathcal{A}(C)$,*

(3) *$\boldsymbol{h}^\phi_{C,1}(a) \geq 0, \ldots, \boldsymbol{h}^\phi_{C,k-1}(a) \geq 0$ for $\mathcal{H}^n$ a.e. $a \in \mathcal{A}(C)$.*

Then the conclusion of Theorem 6.15 holds. If $k = 1$, then the same conclusion is true for every $\lambda \in \mathbf{R}$.

Remark 6.19. Corollary 6.18 includes as very special cases the soap bubble theorems of Alexandrov ([Ale58]), Korevaar–Ros ([Ros87] and [Ros88]) and He–Li–Ma–Ge ([HLMG09]). In fact for connected and compact domains with $\mathcal{C}^2$-boundary the hypothesis (c) of Corollary 6.18 can be easily deduced from the existence of an elliptic point, the continuity of the principal curvatures and the Garding theory on hyperbolic polynomials (see [Ros88, page 450] for further details). The continuity of the principal curvatures and the assumption of connectedness play a key role in this argument.

Recall the definition of epigraph from (61).

Lemma 6.20. *Suppose $C \subseteq \mathbf{R}^{n+1}$ is a compact set such that for every $a \in \partial C$ there exists $u \in \mathbf{S}^n$, $\epsilon, \delta > 0$ and a semiconvex function $f : a + u^\perp \to \mathbf{R}$ such that*

$$\mathrm{epi}(f) \cap U_{\varepsilon,\delta}(a,u) = C \cap U_{\varepsilon,\delta}(a,u). \tag{83}$$

Then $\mathrm{reach}(C) > 0$ *and* $\mathcal{H}^n(\partial C \setminus \partial^v C) = 0$.

Proof. For $a \in \partial C$, let $\varepsilon(a), \delta(a) > 0$ and the local representation in terms of a semiconvex function f_a be as in (83). By [Fu85, Theorem 2.3] we know that $\mathrm{reach}(\mathrm{epi}(f_a)) \geq r(a) > 0$. Define $\rho(a) = \frac{1}{4}\min\{\varepsilon(a), \delta(a), r(a)\}$. Then we have $U(a, \rho(a)) \subseteq \mathrm{Unp}(C)$ for every $a \in \partial C$. If $a \in \mathrm{int}(C)$, then there is also a positive number $\rho(a)$ such that $U(a, \rho(a)) \subset \mathrm{int}(C) \subseteq \mathrm{Unp}(C)$. Since the sets $U(a, \rho(a))$, for $a \in C$, are an open cover of the compact set C, we get a finite number of points $a_1, \ldots, a_N \in C$ such that $C \subseteq \bigcup_{i=1}^N U(a_i, \frac{\rho(a_i)}{2})$. Then for $0 < \tau < \inf\{\frac{\rho(a_1)}{2}, \ldots, \frac{\rho(a_N)}{2}\}$, it holds that for every $c \in C$ there is some a_i such that $U(c, \tau) \subseteq U(a_i, \rho(a_i)) \subseteq \mathrm{Unp}(C)$. This shows that $\mathrm{reach}(C) \geq \tau > 0$.

Note that $\partial C = \boldsymbol{p}(N(C))$ (see e.g. [RZ19, Corollary 4.12(a)]). Since $\mathcal{H}^0(N(C, a)) \neq 2$ for $a \in \partial C$ due to (83), it follows from Lemma 3.25 (c) that $\mathcal{H}^n(\boldsymbol{p}(N(C)) \setminus \partial^v C) = 0$, which gives the remaining assertion. ☐

Acknowledgements. D. Hug was supported by DFG research grant HU 1874/5-1 (SPP 2265) and gratefully acknowledges support by ICERM (Brown University).

References

[Ale58] Aleksandr D. Aleksandrov. Uniqueness theorems for surfaces in the large, V (in Russian). *Vestnik Leningrad. Univ.*, 13(19):5–8, 1958. English translation of parts I-V: *Amer. Math. Soc. Transl.*, Ser. 2, 21:341–416, 1962.

[Ale62] Aleksandr D. Alexandrov. A characteristic property of spheres. *Ann. Mat. Pura Appl. (4)*, 58:303–315, 1962.

[ACV08] Luigi Ambrosio, Andrea Colesanti and Elena Villa. Outer Minkowski content for some classes of closed sets. *Math. Ann.*, 342(4):727–748, 2008.

[AFP00] Luigi Ambrosio, Nicola Fusco and Diego Pallara. Functions of bounded variation and free discontinuity problems. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[AW21] Ben Andrews and Yong Wei. Volume preserving flow by powers of the k-th mean curvature. *J. Differential Geom.*, 117(2):193–222, 2021.

[Ba78a] Victor Bangert. Über die Approximation von lokal konvexen Mengen. (German) *Manuscripta Math.* 25(4):397–420, 1978.

[Ba78b] Victor Bangert. Konvexe Mengen in Riemannschen Mannigfaltigkeiten. (German) *Math. Z.* 162(3):263–286, 1978.

[Ba79] Victor Bangert. Analytische Eigenschaften konvexer Funktionen auf Riemannschen Mannigfaltigkeiten. (German) *J. Reine Angew. Math.*, 307/308:309–324, 1979.

[Ba82] Victor Bangert. Sets with positive reach. *Arch. Math. (Basel)* 38(1):54–57, 1982.

[Be85] Gerald Beer. On convergence of closed sets in a metric space and distance functions. *Bull. Austral. Math. Soc.*, 31(3):421–432, 1985.

[BE13] Simon Brendle and Michael Eichmair. Isoperimetric and Weingarten surfaces in the Schwarzschild manifold. *J. Differential Geom.*, 94(3):387–407, 2013.

[BH08] Christopher J. Bishop and Hrant Hakobyan. A central set of dimension 2. *Proc. Amer. Math. Soc.* 136(7):2453–2461, (2008).

[BHP18] Károly J. Böröczky, Martin Henk and Hannes Pollehn. Subspace concentration of dual curvature measures of symmetric convex bodies. *J. Differential Geom.*, 109(3):411–429, 2018.

[BLYZ13] Károly J. Böröczky, Erwin Lutwak, Deane Yang and Gaoyong Zhang. The logarithmic Minkowski problem. *J. Amer. Math. Soc.*, 26(3):831–852, 2013.

[BLYZ19] Károly J. Böröczky, Erwin Lutwak, Deane Yang, Gaoyong Zhang and Yiming Zhao. The dual Minkowski problem for symmetric convex bodies. *Adv. Math.*, 356:106805, 30 pp., 2019,

[Bre13] Brendle, Simon. Constant mean curvature surfaces in warped product manifolds. *Publ. Math. Inst. Hautes Études Sci.*, 117, 2013, 247–269.

[CdL16] Daniel Cibotaru and Jorge de Lira. A note on the area and coarea formulas for general volume densities and some applications. *Manuscripta Math.*, 149(3-4):471–506, 2016.

[CW13] Sun-Yung Alice Chang and Yi Wang. Inequalities for quermassintegrals on k-convex domains. *Adv. Math.*, 248:335–377, 2013.

[CLV21] Antonin Chambolle, Luca Lussardi and Elena Villa. Anisotropic tubular neighborhoods of sets. *Math. Z.*, 299(3-4):1257–1274, 2021.

[CRV21] Giulio Ciraolo, Alberto Roncoroni and Luigi Vezzoni. Quantitative stability for hypersurfaces with almost constant curvature in space forms. *Ann. Mat. Pura Appl. (4)*, 200(5):2043–2083, 2021.

[Cl90] Frank H. Clarke. *Optimization and nonsmooth analysis.* Second edition. Classics in Applied Mathematics, 5. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1990.

[CH00] Andrea Colesanti and Daniel Hug. Hessian measures of semi-convex functions and applications to support measures of convex bodies. *Manuscripta Math.*, 101(2):209–238, 2000.

[FLW19] Hao Fang, Mijia Lai and Weifeng Wo. A note on closed convex hypersurfaces with singularities. *Proc. Amer. Math. Soc.*, 147(6):2661–2671, 2019.

[DM19] Matias Gonzalo Delgadino and Francesco Maggi. Alexandrov's theorem revisited. *Anal. PDE*, 12(6):1613–1642, 2019.

[DRKS20] Antonio De Rosa, Sławomir Kolasiński, and Mario Santilli. Uniqueness of critical points of the anisotropic isoperimetric problem for finite perimeter sets. *Arch. Ration. Mech. Anal.*, 238(3):1157–1198, 2020.

[Dis68] V. I. Diskant. Stability of a sphere in the class of convex surfaces of bounded specific curvature (in Russian). *Sibirskii. Mat. Zh.*, 9:816–824, 1968. English translation: *Siberian Math. J.* 9:610–615, 1968.

[Fa96] Hermann Fallert. Quermaßdichten für Punktprozesse konvexer Körper und Boolesche Modelle. (German) [Densities for quermassintegrals for point processes of convex bodies and Boolean models]. *Math. Nachr.* 181:165–184, 1996.

[Fed59] Herbert Federer. Curvature measures. *Trans. Amer. Math. Soc.*, 93:418–491, 1959.

[Fed69] Herbert Federer. *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[Fed78] Herbert Federer. Colloquium lectures on geometric measure theory. *Bull. Amer. Math. Soc.*, 84(3):291–338, 1978.

[Fr97] David H. Fremlin. Skeletons and central sets. *Proc. London Math. Soc. (3)*, 74(3):701–720, 1997.

[Fu85] Joseph H. G. Fu. Tubular neighborhoods in Euclidean spaces. *Duke Math. J.*, 52(4): 1025–1046, 1985.

[Fu98] Joseph H. G Fu. Some remarks on Legendrian rectifiable currents. *Manuscripta Math.*, 97(2):175–187, 1998.

[FW19] Joseph H. G Fu and Thomas Wannerer. Riemannian curvature measures. *Geom. Funct. Anal.* 29(2):343–81, 2019.

[GHM13] José A. Gálvez, Laurent Hauswirth and Pablo Mira. Surfaces of constant curvature in $\mathbb{R}^3$ with isolated singularities. *Adv. Math.*, 241:103–126, 2013.

[GHWXY19] Richard J. Gardner, Daniel Hug, Wolfgang Weil, Sudan Xing and Deping Ye. General volumes in the Orlicz-Brunn-Minkowski theory and a related Minkowski problem I. *Calc. Var. Partial Differential Equations*, 58(1): Paper No. 12, 35 pp., 2019.

[GHXY20] Richard J. Gardner, Daniel Hug, Sudan Xing and Deping Ye. General volumes in the Orlicz-Brunn-Minkowski theory and a related Minkowski problem II. *Calc. Var. Partial Differential Equations*, 59(1): Paper No. 15, 33 pp., 2020.

[GS22] Mohammad Ghomi and Joel Spruck. Total curvature and the isoperimetric inequality in Cartan–Hadamard manifolds. *J. Geom. Anal.*, 32(2):Paper No. 50, 54 pp., 2022.

[Haj22] Piotr Hajłasz. On an old theorem of Erdős about ambiguous locus. *Colloq. Math.*, 168:249–256, 2022.

[HLMG09] Yijun He, Haizhong Li, Hui Ma and Jianquan Ge. Compact embedded hypersurfaces with constant higher order anisotropic mean curvatures. *Indiana Univ. Math. J.*, 58(2):853–868, 2009.

[Hop83] Heinz Hopf. Differential geometry in the large. Volume 1000 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983. Notes taken by Peter Lax and John Gray. With a preface by S. S Chern.

[Hsi54] Chuan-Chih Hsiung. Some integral formulas for closed hypersurfaces. *Math. Scand.*, 2:286–294, 1954.

[HLYZ16] Yong Huang, Erwin Lutwak, Deane Yang and Gaoyong Zhang. Geometric measures in the dual Brunn-Minkowski theory and their associated Minkowski problems. *Acta Math.*, 216(2):325–388, 2016.

[HLYZ18] Yong Huang, Erwin Lutwak, Deane Yang and Gaoyong Zhang. The Lp-Aleksandrov problem for Lp-integral curvature. *J. Differential Geom.*, 110(1):1–29, 2018.

[Hu96] Daniel Hug. Contributions to affine surface area. *Manuscripta Math.*, 91(3):283–301, 1996.

[Hu98] Daniel Hug. Absolute continuity for curvature measures of convex sets. I. *Math. Nachr.* 195:139–158, 1998.

[Hu99] Daniel Hug. Measures, curvatures and currents in convex geometry. Habilitationsschrift, Albert-Ludwigs-Universität, Freiburg im Breisgau, December 1999. https://doi.org/10.6094/UNIFR/167349

[HL00] Daniel Hug and Günter Last. On support measures in Minkowski spaces and contact distributions in stochastic geometry. *Ann. Probab.*, 28:796–850, 2000.

[HLW04] Daniel Hug, Günter Last and Wolfgang Weil. A local Steiner-type formula for general closed sets and applications. *Math. Z.*, 246(1-2):237–272, 2004.

[HLW06] Daniel Hug, Günter Last and Wolfgang Weil. Polynomial parallel volume, convexity and contact distributions of random sets. *Probab. Theory Related Fields*, 135(2):169–200, 2006.

[HM87] Gary Harris and Clyde Martin. The roots of a polynomial vary continuously as a function of the coefficients. *Proc. Amer. Math. Soc.*, 100(2):390–392, 1987.

[Jel53] M. J.-H. Jellett. Sur la surface dont la courbure moyenne est constante. *J. Math. Pures Appl.*, 18:163–167, 1853.

[KL21] Vitali Kapovitch and Alexander Lytchak. Remarks on manifolds with two-sided curvature bounds. *Anal. Geom. Metr. Spaces*, 9(1):53–64, 2021.

[Kl80] Norbert Kleinjohann. Convexity and the unique footpoint property in Riemannian geometry. *Arch. Math. (Basel)*, 33(6):574–582, 1981.

[Kl81] Norbert Kleinjohann. Nächste Punkte in der Riemannschen Geometrie. (German)*Math. Z.*, 176(3):327–344, 1981.

[Koh94] Peter Kohlmann. Minkowski integral formulas for compact convex bodies in standard space forms. *Math. Nachr.*, 166:217–228, 1994.

[Koh98a] Peter Kohlmann. Characterization of metric balls in standard space forms via curvature. *Math. Z.*, 227(4):729–746, 1998.

[Koh98b] Peter Kohlmann. Characterizations via linear combinations of curvature measures. *Arch. Math. (Basel)*, 70(3):250–256, 1998.

[KS21] Sławomir Kolasiński and Mario Santilli. Regularity of distance functions from arbitrary closed sets. *Math. Ann.*, 2022. https://doi.org/10.1007/s00208-022-02407-7

[LYZ18] Erwin Lutwak, Deane Yang and Gaoyong Zhang. L_p dual curvature measures. *Adv. Math.*, 329:85–132, 2018.

[Ly04] Alexander Lytchak. On the geometry of subsets of positive reach. *Manuscripta Math.* 115(2):199–205, 2004.

[Ly04] Alexander Lytchak. Almost convex subsets. *Geom. Dedicata*, 115:201–218, 2005.

[MR91] Sebastián Montiel and Antonio Ros. Compact hypersurfaces: the Alexandrov theorem for higher order mean curvatures. In *Differential geometry*, volume 52 of *Pitman Monogr. Surveys Pure Appl. Math.*, pages 279–296. Longman Sci. Tech., Harlow, 1991.

[MS19] Ulrich Menne and Mario Santilli. A geometric second-order-rectifiable stratification for closed subsets of Euclidean space. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 19(3):1185–1198, 2019.

[RW10] Jan Rataj and Steffen Winter. On volume and surface area of parallel sets. *Indiana Univ. Math. J.*, 59(5), 1661–1685, 2010.

[RZ01] Jan Rataj and Martina Zähle. Curvatures and currents for unions of sets with positive reach. II. *Ann. Global Anal. Geom.*, 20(1):1–21, 2001.

[RZ17] Jan Rataj and Ludek Zajíček. On the structure of sets with positive reach. *Math. Nachr.*, 290(11-12), 1806–1829, 2017.

[RZ19] Jan Rataj and Martina Zähle. Curvature measures of singular sets. Springer Monographs in Mathematics. Springer, Cham, 2019.

[Ros87] Antonio Ros. Compact hypersurfaces with constant higher order mean curvatures. *Rev. Mat. Iberoamericana*, 3(3-4):447–453, 1987.

[Ros88] Antonio Ros. Compact hypersurfaces with constant scalar curvature and a congruence theorem. *J. Differential Geom.*, 27(2):215–223, 1988. With an appendix by Nicholas J. Korevaar.

[Ro89] Shmuel Rosset. Normalized symmetric functions, Newton's inequalities and a new set of stronger inequalities. *Amer. Math. Monthly*, 96 (9):815–819, 1989.

[Sal99] Paolo Salani. Equazioni Hessiane e k-convessità. Tesi di Dottorato in Matematica. IX ciclo: 1993–1997. Università di Firenze.

[San19] Mario Santilli. Rectifiability and approximate differentiability of higher order for sets. *Indiana Univ. Math. J.*, 68:1013–1046, 2019.

[San20] Mario Santilli. Fine properties of the curvature of arbitrary closed sets. *Ann. Mat. Pura Appl. (4)*, 199(4):1431–1456, 2020.

[San21] Mario Santilli. Distance functions with dense singular sets. *Comm. Partial Differential Equations*, 46(7):1319–1325, 2021.

[SX22] Julian Scheuer and Chao Xia. Stability for Serrin's problem and Alexandroff's theorem in warped product manifolds. Preprint, arXiv: 2207.02491, 2022.

[Sch79] Rolf Schneider. Bestimmung konvexer Körper durch Krümmungsmaße. *Comment. Math. Helv.*, 54(1):42–60, 1979.

[Sch14] Rolf Schneider. Convex bodies: the Brunn–Minkowski theory. Second expanded edition. Encyclopedia of Mathematics and its Applications, 151. Cambridge University Press, Cambridge, 2014.

[Sch15] Rolf Schneider. Curvatures of typical convex bodies—the complete picture. *Proc. Amer. Math. Soc.*, 143(1):387–393, 2015.

[Sl00] B. Slezák. A right inverse function theorem without assuming differentiability. *Studia Sci. Math. Hungar.* 36(1-2):153–164, 2000.

[Sta76] László L. Stachó. On the volume function of parallel sets. *Acta Sci. Math. (Szeged)*, 38(3-4):365–374, 1976.

[Sta79] László L. Stachó. On curvature measures. *Acta Sci. Math. (Szeged)*, 41(1-2):191–207, 1979.

[Tru94] Neil S. Trudinger. Isoperimetric inequalities for quermassintegrals. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 11(4):411–425, 1994.

[TW99] Neil S. Trudinger and Xu-Jia Wang. Hessian measures. II. *Ann. of Math.* (2), 150(2):579–604, 1999.

[Wal81] Rolf Walter. Konvexität in riemannschen Mannigfaltigkeiten. (German) *Jahresber. Deutsch. Math.-Verein.* 83(1):1–31, 1981.

[Wen86] Henry C. Wente. Counterexample to a conjecture of H. Hopf. *Pacific J. Math.*, 121(1):193–243, 1986.

[Wi19] Steffen Winter. Localization results for Minkowski contents. *J. Lond. Math. Soc.* (2), 99(2):553–582, 2019.

[Za86] Martina Zähle. Integral and current representation of Federer's curvature measures. *Arch. Math.*, 46(6):557–567, 1986.

Daniel Hug, Karlsruhe Institute of Technology (KIT), Institute of Stochastics, D-76128 Karlsruhe, Germany, daniel.hug@kit.edu

Mario Santilli, Department of Information Engineering, Computer Science and Mathematics, Università degli Studi dell'Aquila, 67100 L'Aquila, Italy, mario.santilli@univaq.it

Notation Index