



Travelling waves for Maxwell's equations in nonlinear and nonsymmetric media

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Abstract. We look for travelling wave fields

$$E(x, y, z, t) = U(x, y) \cos(kz + \omega t) + \tilde{U}(x, y) \sin(kz + \omega t), \\ (x, y, z) \in \mathbb{R}^3, t \in \mathbb{R}$$

satisfying Maxwell's equations in a nonlinear medium which is not necessarily cylindrically symmetric. The nonlinearity of the medium enters Maxwell's equations by postulating a nonlinear material law $D = \varepsilon E + \chi(x, y, \langle |E|^2 \rangle)E$ between the electric field E , its time averaged intensity $\langle |E|^2 \rangle$ and the electric displacement field D . We derive a new semilinear elliptic problem for the profiles $U, \tilde{U} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$

$$Lu - V(x, y)u = f(x, y, u) \quad \text{with } u = \begin{pmatrix} U \\ \tilde{U} \end{pmatrix}, \text{ for } (x, y) \in \mathbb{R}^2,$$

where $f(x, y, u) = \omega^2 \chi(x, y, |u|^2)u$. Solving this equation we can obtain exact travelling wave solutions of the underlying nonlinear Maxwell equations. We are able to deal with super quadratic and subcritical focusing effects, e.g. in the Kerr-like materials with the nonlinear susceptibility of the form $\chi(x, y, \langle |E|^2 \rangle)E = \chi^{(3)}(x, y) \langle |E|^2 \rangle E$. A variational approach is presented for the semilinear problem. The energy functional associated with the equation is strongly indefinite, since L contains an infinite dimensional kernel. The methods developed in this paper may be applicable to other strongly indefinite elliptic problems and other nonlinear phenomena.

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1. Introduction

We are looking for travelling wave fields

$$E(x, y, z, t) = U(x, y) \cos(kz + \omega t) + \tilde{U}(x, y) \sin(kz + \omega t) \quad (1.1)$$

solving in the absence of charges and currents the Maxwell system $\nabla \times E + \partial_t B = 0$ (Faraday's law), $\nabla \times H = \partial_t D$ (Ampère's law) together with $\operatorname{div} D = 0$ and $\operatorname{div} B = 0$. We require the linear magnetic material law $B = \mu(x, y)H$ and the nonlinear electric material law $D = \varepsilon(x, y)E + \chi(x, y, \langle |E|^2 \rangle)E$ where $\langle |E(x, y, z)|^2 \rangle = \frac{1}{T} \int_0^T |E(x, y, z, t)|^2 dt$ is the average intensity of a time-harmonic electric field over one period $T = 2\pi/\omega$. Taking the curl of Faraday's law and inserting the material laws for B and D together with Ampère's law we find that E has to satisfy the *nonlinear electromagnetic wave equation*

$$\begin{aligned} \nabla \times \left(\frac{1}{\mu(x, y)} \nabla \times E \right) + \partial_{tt} (\varepsilon(x, y)E + \chi(x, y, \langle |E|^2 \rangle)E) = 0 \\ \text{for } (x, y, z, t) \in \mathbb{R}^3 \times \mathbb{R}. \end{aligned} \quad (1.2)$$

Here $U, \tilde{U} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ are the profiles of the travelling waves, $\omega > 0$ is the temporal frequency and $k \in \mathbb{R} \setminus \{0\}$ the spatial wave number in the direction of propagation, $\varepsilon = \varepsilon(x, y)$ is the permittivity of the medium, $\mu = \mu(x, y)$ is the magnetic permeability, and χ is the scalar nonlinear susceptibility which depends on (x, y) and on the time averaged intensity of E only.

Note that having solved the nonlinear electromagnetic wave equation, one obtains the electric displacement field D directly from the constitutive relation

$$D = \varepsilon(x, y)E + \chi(x, y, \langle |E|^2 \rangle)E. \quad (1.3)$$

Moreover the magnetic induction B may be obtained by time integrating Faraday's law with divergence free initial conditions, which are then preserved in time. Finally the magnetic field H is given by $H = \mu^{-1}B$. From (1.2) we get $\operatorname{div}(\partial_{tt}D) = 0$ and the Gauss law $\operatorname{div}(D) = 0$ is satisfied due to the travelling wave ansatz of the form (1.1). Altogether, we find *exact propagation* of the electromagnetic field in the nonlinear medium according to the Maxwell equations with the time-averaged material law (1.3), see also [23, 25, 26].

In physics and mathematical literature there are several simplifications relying on approximations of the nonlinear electromagnetic wave equation. The most prominent one is the scalar or vector nonlinear Schrödinger equation. For instance, one assumes that the term $\nabla(\operatorname{div} E)$ in $\nabla \times (\nabla \times E) = \nabla(\operatorname{div} E) - \Delta E$ is negligible and can be dropped, or one can use the so-called slowly varying envelope approximation. However, this approach may produce non-physical solutions; see e.g. [1, 11] and references therein. Therefore, in this paper, we are interested in exact travelling wave solutions of the Maxwell equations.

We would like to mention that exact propagation of travelling waves of the nonlinear electromagnetic wave problem (1.2) have been studied analytically so far only in cylindrically symmetric media. Namely, if E is an axisymmetric

TE -mode of the form

$$E(x, y, z, t) = U(x, y) \cos(kz + \omega t)$$

with

$$U(x, y) := u(r)(-y/r, x/r, 0), \text{ where } r = \sqrt{x^2 + y^2} \quad (1.4)$$

and the scalar function u only depends on r , then solutions of (1.2) have been considered in a series of papers by Stuart and Zhou [25, 27–32] for asymptotically constant susceptibilities and by McLeod, Stuart and Troy [14] for a cubic nonlinear polarization. Clearly, E has the form of (1.1) with $\tilde{U} = 0$. The search for these solutions reduces to a one-dimensional variational problem or an ODE for $u(r)$, which simplifies the problem considerably. However, if the medium is not necessarily cylindrically symmetric, then it is not clear how to find travelling waves (1.1) with $\tilde{U} = 0$ analytically and whether any variational approach can be provided with this constraint.

The aim of this work is to provide an analysis of the travelling waves of the general form (1.1) propagating in media, which are not necessarily cylindrically symmetric. To the best of our knowledge it is the first analytical study of travelling waves of (1.2), i.e., the Maxwell system with material law (1.3), where the medium is not supposed to be cylindrically symmetric. We present a variational approach which allows to treat (1.2) and to find ground state solutions of the problem with the least possible energy as well as infinitely many geometrically distinct bound states.

Let us briefly comment on the problem of finding time-harmonic standing wave solutions of (1.2) of the form

$$E(x, y, z, t) = U(x, y, z) \cos(\omega t).$$

This leads to the so-called nonlinear *curl-curl problem* and has been recently studied e.g. in [4, 5] on a bounded domain and in [7, 15, 18] on \mathbb{R}^3 , see also the survey [6] and references therein. In the curl-curl problem, however, U is required to be localized in all space directions, i.e., it is supposed to lie in some Lebesgue space over \mathbb{R}^3 . Since travelling waves of the form (1.1) are not localized, they have not been taken into account in these works. We would like to also mention that the study of time-harmonic standing waves in \mathbb{R}^3 in the nonsymmetric case has been presented only in [15, 18]. The methods used there required assumptions about the vanishing properties of the permittivity ε , or even $\varepsilon = 0$, and the double power behaviour of the nonlinear effect, e.g. $\chi(x, y, z, E) = \Gamma(x) \min\{|\langle E \rangle|^{p-2}, |\langle E \rangle|^{q-2}\}$ with $2 < p < 6 < q$. Note that $p = 4$ corresponds to the Kerr-type effect, but only for sufficiently strong fields $|\langle E \rangle| > 1$. In this work, however, we are able to treat the probably most common type of nonlinearity in the physics and engineering literature, the *Kerr nonlinearity*

$$\chi(x, y, \langle |E|^2 \rangle) E = \chi^{(3)}(x, y) \langle |E|^2 \rangle E$$

and we no longer require that the permittivity vanishes at infinity.

The search for travelling waves of the form (1.1) leads to a new nonlinear elliptic problem. Namely, in order to solve the nonlinear electromagnetic wave

equation, we observe that the profiles U, \tilde{U} of the travelling wave satisfy the elliptic problem

$$L \begin{pmatrix} U \\ \tilde{U} \end{pmatrix} - \omega^2 \epsilon(x, y) \begin{pmatrix} U \\ \tilde{U} \end{pmatrix} = \omega^2 \chi \left(x, y, \frac{1}{2}(|U|^2 + |\tilde{U}|^2) \right) \begin{pmatrix} U \\ \tilde{U} \end{pmatrix}, \quad (1.5)$$

where

$$L = \begin{pmatrix} -\partial_{yy} + k^2 & \partial_{xy} & 0 & 0 & 0 & k\partial_x \\ \partial_{xy} & -\partial_{xx} + k^2 & 0 & 0 & 0 & k\partial_y \\ 0 & 0 & -\partial_{xx} - \partial_{yy} & k\partial_x & k\partial_y & 0 \\ 0 & 0 & -k\partial_x & -\partial_{yy} + k^2 & \partial_{xy} & 0 \\ 0 & 0 & -k\partial_y & \partial_{xy} & -\partial_{xx} + k^2 & 0 \\ -k\partial_x & -k\partial_y & 0 & 0 & 0 & -\partial_{xx} - \partial_{yy} \end{pmatrix}.$$

For simplicity we have assumed that the magnetic permeability is a constant given by $\mu = 1$. Let us define

$$\text{dom}(L) := \left\{ \begin{pmatrix} U \\ \tilde{U} \end{pmatrix} \in L^2(\mathbb{R}^2)^6 : L \begin{pmatrix} U \\ \tilde{U} \end{pmatrix} \in L^2(\mathbb{R}^2)^6 \right\}$$

where $L((U, \tilde{U})^T)$ is defined in the sense of distributions. One verifies that the second-order differential operator

$$L : \text{dom}(L) \subset L^2(\mathbb{R}^2)^6 \rightarrow L^2(\mathbb{R}^2)^6 \quad (1.6)$$

is elliptic and self-adjoint, see Sect. 2 for details.

Our aim is to find solutions $u : \mathbb{R}^2 \rightarrow \mathbb{R}^6$ to the following slight generalization of (1.5) given by

$$Lu - V(x)u = f(x, u) \quad \text{for } x \in \mathbb{R}^2 \quad (1.7)$$

involving the operator L , where we assume $f(x, u) = \partial_u F(x, u)$. From now on the space variable in \mathbb{R}^2 will be denoted by x instead of (x, y) .

Observe that, if

$$V(x) = \omega^2 \epsilon(x), \quad F(x, u) = \omega^2 \chi \left(x, \frac{1}{2}|u|^2 \right), \quad x \in \mathbb{R}^2, u \in \mathbb{R}^6 \quad (1.8)$$

then (1.7) leads to (1.5) and we obtain the *exact* propagation of the travelling electromagnetic waves (E, B) , where E is given by (1.1) and B is provided by Faraday's law and a subsequent time integration. On the other hand, (1.7) is more general, since the nonlinear potential F may depend on the direction of u and not necessarily on $|u| = \sqrt{U^2 + \tilde{U}^2}$ as in anisotropic media.

Before we describe our results in more detail, let us comment on the main difference of the current work and previous works [25–30] in cylindrically symmetric media. In these previous works the reduction of the curl-curl-operator to the Laplacian when acting on $E(x, y, z, t) = U(x, y) \cos(kz + \omega t)$ was possible by building-in the constraint $\text{div}(U) = 0$ into the ansatz as in (1.4). In our work we admit two profiles U, \tilde{U} in the ansatz (1.1) and thus reduce the curl-curl-operator to the operator L above. If we would obtain a one-profile solution $(U, 0)$ of (1.5) with $k \neq 0$ then automatically $\text{div}(U) = 0$ follows. However, finding such a one-profile solution might be restricted to cylindrically symmetric media. Two-profile solutions like in our ansatz (1.1) allow us

to treat more general non-cylindrically symmetric media and a large class of nonlinearities.

The model nonlinearity which we have in mind is

$$F(x, u) = \frac{1}{p} \Gamma(x) |u|^p \tag{1.9}$$

with $\Gamma \in L^\infty(\mathbb{R}^2)$ positive, bounded away from 0 and $p > 2$. In particular if $p = 4$ we deal with the Kerr nonlinearity.

We want to find weak solutions of (1.7) which correspond to critical points of the following functional

$$J(u) := \frac{1}{2} b_L(u, u) - \frac{1}{2} \int_{\mathbb{R}^2} V(x) |u|^2 dx - \int_{\mathbb{R}^2} F(x, u) dx \tag{1.10}$$

defined on a Banach space $X \subset L^2(\mathbb{R}^2)^6 \cap L^p(\mathbb{R}^2)^6$ given later. Here $b_L(\cdot, \cdot)$ is the bilinear form associated with L such that $b_L(u, \varphi) = \int_{\mathbb{R}^2} \langle Lu, \varphi \rangle dx$ for all $u \in \text{dom}(L)$ and all $\varphi \in C_0^\infty(\mathbb{R}^2)^6$.

Now let us enlist several difficulties underlying the problem. For $\alpha, \tilde{\alpha} \in C_0^\infty(\mathbb{R}^2)$ and for $\beta, \tilde{\beta} \in C_0^\infty(\mathbb{R}^2)^3$ let us denote

$$\mathring{\nabla} \begin{pmatrix} \alpha \\ \tilde{\alpha} \end{pmatrix} := \begin{pmatrix} \partial_x \alpha \\ \partial_y \alpha \\ k \tilde{\alpha} \\ \partial_x \tilde{\alpha} \\ \partial_y \tilde{\alpha} \\ -k \alpha \end{pmatrix} : \mathbb{R}^2 \rightarrow \mathbb{R}^6, \quad \mathring{\nabla} \times \begin{pmatrix} \beta \\ \tilde{\beta} \end{pmatrix} := \begin{pmatrix} \partial_y \beta_3 - k \tilde{\beta}_2 \\ k \beta_1 - \partial_x \beta_3 \\ \partial_x \beta_2 - \partial_y \beta_1 \\ \partial_y \tilde{\beta}_3 + k \tilde{\beta}_2 \\ -k \beta_1 - \partial_x \tilde{\beta}_3 \\ \partial_x \tilde{\beta}_2 - \partial_y \tilde{\beta}_1 \end{pmatrix} : \mathbb{R}^2 \rightarrow \mathbb{R}^6$$

and observe that $L = \mathring{\nabla} \times \mathring{\nabla} \times, \mathring{\nabla} \times \mathring{\nabla} = 0$ and thus $w := \mathring{\nabla} \begin{pmatrix} \alpha \\ \tilde{\alpha} \end{pmatrix} \in \ker(L)$.

Moreover J may be unbounded from above and from below and its critical points may have infinite Morse index. This is due to the infinite-dimensional kernel of L . In addition to these problems related to the strongly indefinite geometry of J' , we also have to deal with the lack of compactness issues. Namely, the functional J is not (sequentially) weak-to-weak* continuous, i.e. the weak convergence $u_n \rightharpoonup u$ in X does not imply that $J'(u_n) \rightharpoonup J'(u)$ in X^* . In particular, if $J'(u_n) \rightarrow 0$, we do not know whether u is a critical point of J and solves (1.7).

In Theorem 2.3 we show that the spectrum $\sigma(L) = \{0\} \cup [k^2, \infty)$, where 0 is an eigenvalue of infinite multiplicity and $[k^2, \infty)$ consists of absolutely continuous spectrum. This allows us to consider the following general assumptions.

- (V) $V \in L^\infty(\mathbb{R}^2)$ and $0 < \text{ess inf } V \leq \text{ess sup } V < k^2$. Moreover there is $V_0 \in \mathbb{R}$ such that $V - V_0 \in L^{\frac{p}{p-2}}(\mathbb{R}^2)$ for some $p > 2$.
- (F1) $F : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow [0, \infty)$ is differentiable with respect to the second variable $u \in \mathbb{R}^2$, and $f = \partial_u F : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is measurable in $x \in \mathbb{R}^2$, continuous in $u \in \mathbb{R}^2$ for a.e. $x \in \mathbb{R}^2$. Moreover f is \mathbb{Z}^2 -periodic in x i.e. $f(x, u) = f(x + y, u)$ for $x, u \in \mathbb{R}^2, y \in \mathbb{Z}^2$.
- (F2) $|f(x, u)| = o(1)$ as $u \rightarrow 0$ uniformly in $x \in \mathbb{R}^2$.

(F3) There are $p > 2$ and $c_1 > 0$ such that

$$|f(x, u)| \leq c_1(1 + |u|^{p-1}) \quad \text{for all } u \in \mathbb{R}^2 \text{ and a.e. } x \in \mathbb{R}^2.$$

(F4) There is $c_2 > 0$ such that

$$\liminf_{|u| \rightarrow \infty} F(x, u)/|u|^p \geq c_2 \quad \text{for a.e. } x \in \mathbb{R}^2.$$

Assumptions (V), (F1)–(F3) are sufficient to show that J is of class \mathcal{C}^1 .

(F4) provides a lower bound estimate for large $|u|$, however we will need also its stronger variant:

(F4') There is $c_2 > 0$ such that

$$F(x, u) \geq c_2|u|^p \quad \text{for all } u \in \mathbb{R}^2 \text{ and a.e. } x \in \mathbb{R}^2.$$

In order to deal with ground states one has to assume the following assumption:

(F5) If $\langle f(x, u), v \rangle = \langle f(x, v), u \rangle > 0$, then $F(x, u) - F(x, v) \leq \frac{\langle f(x, u), u \rangle^2 - \langle f(x, u), v \rangle^2}{2\langle f(x, u), u \rangle}$. Moreover $\langle f(x, u), u \rangle \geq 2F(x, u)$ for every $x \in \mathbb{R}^2$ and $u \in \mathbb{R}^2$.

Condition (F5) was introduced in [4, 16]. Conditions (F4) and (F5) imply in particular the convexity of F in u , cf. Remark 6.2. Observe that if F is isotropic in u , i.e. $F(x, u) = \chi(x, \frac{1}{2}|u|^2)$, $\chi(x, s) \geq 0$, $\partial_s \chi(x, s)$ is nondecreasing and $\chi(x, 0) = 0$, then (F5) is satisfied. In general, (F5) does not imply the Ambrosetti-Rabinowitz condition (F6) given below, cf. [2]. For instance, if we take $0 < a < b$ then we may find a function χ of class \mathcal{C}^1 such that $\partial_s \chi(s) = s^{\frac{p-2}{2}}$ for $0 \leq s < a$, $\partial_s \chi(s)$ is constant for $s \in [a, b]$, and $\partial_s \chi(s) = cs^{\frac{p-2}{2}}$ for $s > b$ and a suitable $c > 0$. Then (F1)–(F5) are satisfied, but the Ambrosetti-Rabinowitz condition (F6) does not hold. Note also that if f and \tilde{f} satisfy (F1)–(F4), then also $\alpha f + \beta \tilde{f}$ satisfies (F1)–(F4) for $\alpha, \beta > 0$. It is not clear if (F5) considered alone has this positive additivity property, however similarly as in [16, Remark 3.3 (b)] we check that if f and \tilde{f} satisfy (F1)–(F5) simultaneously, then $\alpha f + \beta \tilde{f}$ satisfies the same assumptions.

The main result reads as follows and note that all our assumptions are satisfied for the model nonlinearity given by (1.9).

Theorem 1.1. *Suppose that (V) and (F1)–(F5) hold. If $V = V_0$, or $V(x) > V_0$ for a.e. $x \in \mathbb{R}^3$ and (F4') holds, then there is a nontrivial solution to (1.7) of the form $u_0 = v + w$ with $v \in H^1(\mathbb{R}^2)^6 \setminus \{0\}$ and $w \in L^2(\mathbb{R}^2)^6 \cap L^p(\mathbb{R}^2)^6$ such that $Lw = 0$. Moreover u_0 is a ground state solution, i.e., $J(u_0) = \inf_{\mathcal{N}} J > 0$, where*

$$\mathcal{N} := \left\{ u \in X \setminus \{0\} : J'(u)(u) = 0 \text{ and } J'(u) \left(\overset{\circ}{\nabla} \begin{pmatrix} \alpha \\ \tilde{\alpha} \end{pmatrix} \right) = 0 \text{ for any } \alpha, \tilde{\alpha} \in C_0^\infty(\mathbb{R}^2) \right\}.$$

*If, in addition, f is odd in u and $V = V_0$, then there is an infinite sequence (u_n) of \mathbb{Z}^2 -distinct solutions, i.e., $(\mathbb{Z}^2 * u_n) \cap (\mathbb{Z}^2 * u_m) = \emptyset$ for $n \neq m$, where $\mathbb{Z}^2 * u := \{u(\cdot + z) : z \in \mathbb{Z}^2\}$ for $u \in X$.*

Observe that for our model (1.8), condition (V) provides the relation between the wave number k and the frequency ω .

Note that \mathcal{N} contains all nontrivial critical points and \mathcal{N} is contained in the classical Nehari manifold $\{u \in X \setminus \{0\} : J'(u)(u) = 0\}$, see [20] and Sect. 3 for further properties. Condition (F5) is important to obtain a bounded Palais-Smale sequence for J at level $\inf_{\mathcal{N}} J$. However it is not clear how to obtain a bounded Palais-Smale sequence for J under (F1)–(F4) and the classical Ambrosetti-Rabinowitz condition

(F6) There is $\gamma > 2$ such that for every $u \in \mathbb{R}^2$ and a.e. $x \in \mathbb{R}^2$

$$\langle f(x, u), u \rangle \geq \gamma F(x, u).$$

Instead, we need to consider (F4') together with (F6) in order to prove the boundedness of Palais-Smale sequences, see Lemmas 6.3 and 6.4.

Theorem 1.2. *Suppose that F is convex in $u \in \mathbb{R}^2$, (F1)–(F3), (F4'), (F6) and that (V) holds for a constant function V , i.e., $V(x) = V_0$ with $0 < V_0 < k^2$. Then there is a nontrivial solution to (1.7) of the form $u_0 = v + w$ with $v \in H^1(\mathbb{R}^2)^6 \setminus \{0\}$ and $w \in L^2(\mathbb{R}^2)^6 \cap L^p(\mathbb{R}^2)^6$ such that $Lw = 0$. Moreover u_0 is a least energy solution, i.e., $J(u_0) = c_0$, where*

$$c_0 := \inf \{J(u) : J'(u) = 0 \text{ and } u \in X \setminus \{0\}\} > 0.$$

If, in addition, f is odd in u , then there is an infinite sequence (u_n) of \mathbb{Z}^2 -distinct solutions.

We show that if $u \in X$ solves (1.7), then the total electromagnetic energy per unit interval in x_3 -direction given by

$$\mathcal{L}(t) := \frac{1}{2} \int_{\mathbb{R}^2} \int_a^{a+1} \langle E, D \rangle + \langle B, H \rangle dx_3 d(x_1, x_2) \quad (1.11)$$

is finite; see Corollary 2.4. We do not know, however, whether the fields E , D , B and H are localized, i.e. decay to zero as $|(x_1, x_2)| \rightarrow \infty$, however X lies in $L^2(\mathbb{R}^2)^6 \cap L^p(\mathbb{R}^2)^6$. Therefore E has a weaker decay property and cannot travel in the (x_1, x_2) -plane. The finiteness of the total electromagnetic energy and the localization problem attract a strong attention in the study of self-guided beams of light in nonlinear media; see e.g. [25, 26]. The question of the regularity of the fields remains open.

The first crucial step in our approach is to build the functional and variational setting for the new operator L and the problem (1.7), which will be demonstrated in the next Sect. 2. Next we recognize the strongly indefinite nature of (1.10) and show that it is of the form of the critical point theory presented in [4, 5, 18] and built for curl-curl problems. However, we work in a different functional setting and under a different set of assumptions, so that we have to slightly refine the results, in particular we do not always assume condition (I8) given in Sect. 3. In Sect. 4 we deal with the lack of compactness issue, in particular with the lack of weak-to-weak* continuity of J' . By means of a profile decomposition result (Theorem 5.1), we are able to prove this regularity in some weakly closed topological constraint $\mathcal{M} \subset X$, so that a weak limit point of a bounded Palais-Smale sequence of \mathcal{M} is a critical point of J .

In the last Sect. 6 we show that a variant of Cerami’s condition holds [10]. In particular we show that any Cerami sequence is bounded, and we emphasize that the proof the boundedness is considerably nonstandard and not straightforward, even if (F6) holds. Some technical inequalities have to be worked out, see details in Lemmas 6.3, 6.4. Finally we complete the proof of Theorems 1.1 and 1.2.

2. Variational setting

We introduce the following notation. If $u = \begin{pmatrix} U \\ \tilde{U} \end{pmatrix} \in \mathbb{R}^6$, then $|u| := (|U|^2 + |\tilde{U}|^2)^{1/2}$ and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in \mathbb{R}^N , $N \geq 1$. In the sequel $\langle \cdot, \cdot \rangle_2$ denotes the inner product in $L^2(\mathbb{R}^2)^6$ and $|\cdot|_q$ denotes the usual L^q -norm for $q \in [1, +\infty]$. Furthermore we denote by C a generic positive constant which may vary from one inequality to the next. We always assume that $k \neq 0$.

Let us introduce the following space

$$\begin{aligned} \mathcal{V} &:= \left\{ u = \begin{pmatrix} U \\ \tilde{U} \end{pmatrix} \in H^1(\mathbb{R}^2)^6 : \left\langle u, \overset{\circ}{\nabla} \begin{pmatrix} \alpha \\ \tilde{\alpha} \end{pmatrix} \right\rangle_2 = 0 \text{ for any } \alpha, \tilde{\alpha} \in C_0^\infty(\mathbb{R}^2) \right\} \\ &= \left\{ u = \begin{pmatrix} U \\ \tilde{U} \end{pmatrix} \in H^1(\mathbb{R}^2)^6 : \partial_{x_1} u_1 + \partial_{x_2} u_2 + k\tilde{u}_3 = 0, \right. \\ &\quad \left. \partial_{x_1} \tilde{u}_1 + \partial_{x_2} \tilde{u}_2 - k u_3 = 0 \text{ a.e. in } \mathbb{R}^2 \right\} \end{aligned}$$

and note that it is a closed subspace of $H^1(\mathbb{R}^2)^6$. Let us consider the following norm in \mathcal{V}

$$\|u\| := \left(\sum_{i=1}^3 |\nabla u_i|_2^2 + k^2 |u_i|_2^2 + |\nabla \tilde{u}_i|_2^2 + k^2 |\tilde{u}_i|_2^2 \right)^{1/2}$$

which is equivalent to the standard H^1 -norm. Let \mathcal{W} be the completion of vector fields $w = \overset{\circ}{\nabla} \begin{pmatrix} \Phi \\ \tilde{\Phi} \end{pmatrix}$, $\Phi, \tilde{\Phi} \in C_0^\infty(\mathbb{R}^2)^3$ with respect to the following norm

$$\|w\| := (|w|_2^2 + |w|_p^2)^{1/2}$$

so that $\mathcal{W} \subset L^2(\mathbb{R}^2)^6 \cap L^p(\mathbb{R}^2)^6$. Note that $\mathcal{V} \cap \mathcal{W} = \{0\}$ and we may define a norm on $\mathcal{V} \oplus \mathcal{W}$ as follows

$$\|v + w\| = \|v\| + \|w\|, \quad v \in \mathcal{V}, w \in \mathcal{W}.$$

Theorem 2.1. *The spaces \mathcal{V} and \mathcal{W} are closed subspaces of $L^2(\mathbb{R}^2)^6$ and orthogonal with respect to $\langle \cdot, \cdot \rangle_2$ and $X := \mathcal{V} \oplus \mathcal{W}$ is the completion of $C_0^\infty(\mathbb{R}^2)^6$ with respect to the norm $\|\cdot\|$.*

Proof. It is clear that \mathcal{V} and \mathcal{W} are closed subspaces of $L^2(\mathbb{R}^2)^6$ and orthogonal with respect to $\langle \cdot, \cdot \rangle_2$. Let $\varphi = \begin{pmatrix} \Phi \\ \tilde{\Phi} \end{pmatrix} \in C_0^\infty(\mathbb{R}^2)^6$ and let $\alpha, \tilde{\alpha} \in H^1(\mathbb{R}^2) \cap$

$C^\infty(\mathbb{R}^2)$ be the unique solutions to

$$-\Delta\alpha + k^2\alpha = -(\partial_{x_1}\Phi_1 + \partial_{x_2}\Phi_2 + k\tilde{\Phi}_3)$$

and

$$-\Delta\tilde{\alpha} + k^2\tilde{\alpha} = -(\partial_{x_1}\tilde{\Phi}_1 + \partial_{x_2}\tilde{\Phi}_2 - k\Phi_3)$$

respectively. Since α is the Bessel potential of the C_0^∞ -function $-(\partial_{x_1}\Phi_1 + \partial_{x_2}\Phi_2 + k\tilde{\Phi}_3)$ we find that $\alpha \in W^{l,s}(\mathbb{R}^2)$ for any $l \in \mathbb{N}$ and $1 < s < \infty$, cf. [24, Chapter V Sect. 3]. Arguing similarly we see that the same also holds for $\tilde{\alpha}$. Taking standard mollifiers and applying a cutoff argument, we find $\alpha_n, \tilde{\alpha}_n \in C_0^\infty(\mathbb{R}^2)$ such that $\alpha_n \rightarrow \alpha$ and $\tilde{\alpha}_n \rightarrow \tilde{\alpha}$ in $W^{l,s}(\mathbb{R}^2)$ for any $l \in \mathbb{N}$ and $1 < s < \infty$. In particular this implies

$$\varphi_{\mathcal{W}} := \overset{\circ}{\nabla} \begin{pmatrix} \alpha \\ \tilde{\alpha} \end{pmatrix} = \lim_{n \rightarrow \infty} \overset{\circ}{\nabla} \begin{pmatrix} \alpha_n \\ \tilde{\alpha}_n \end{pmatrix} \in \mathcal{W}$$

and $\varphi_{\mathcal{W}} \in W^{l,s}(\mathbb{R}^2)$ for all $l \in \mathbb{N}$ and $1 < s < \infty$. Moreover, \mathcal{W} is clearly contained in the completion of $C_0^\infty(\mathbb{R}^2)^6$ with respect to the norm $\|\cdot\|$. Since in particular $\varphi_{\mathcal{W}} \in H^1(\mathbb{R}^2)^6$ we obtain by integration by parts

$$\begin{aligned} \left\langle \varphi, \overset{\circ}{\nabla} \begin{pmatrix} \beta \\ \tilde{\beta} \end{pmatrix} \right\rangle &= - \int_{\mathbb{R}^2} (\partial_{x_1}\Phi_1 + \partial_{x_2}\Phi_2 + k\tilde{\Phi}_3)\beta + (\partial_{x_1}\tilde{\Phi}_1 + \partial_{x_2}\tilde{\Phi}_2 - k\Phi_3)\tilde{\beta} \, dx \\ &= \int_{\mathbb{R}^2} (-\Delta\alpha + k^2\alpha)\beta + (-\Delta\tilde{\alpha} + k^2\tilde{\alpha})\tilde{\beta} \, dx \\ &= \left\langle \varphi_{\mathcal{W}}, \overset{\circ}{\nabla} \begin{pmatrix} \beta \\ \tilde{\beta} \end{pmatrix} \right\rangle \end{aligned}$$

for every $\beta, \tilde{\beta} \in C_0^\infty(\mathbb{R}^2)$. Therefore $\varphi_{\mathcal{V}} := \varphi - \varphi_{\mathcal{W}} \in \mathcal{V}$ and we obtain the following Helmholtz-type decomposition

$$\varphi = \varphi_{\mathcal{V}} + \varphi_{\mathcal{W}} \quad \text{with } \varphi_{\mathcal{V}} \in \mathcal{V} \text{ and } \varphi_{\mathcal{W}} \in \mathcal{W}$$

and moreover we have shown that $\varphi \in X$. It remains to show that also \mathcal{V} is contained in the completion of $C_0^\infty(\mathbb{R}^2)^6$ with respect to $\|\cdot\|$. To see this, let $v \in \mathcal{V}$ and take $(\varphi_n) \subset C_0^\infty(\mathbb{R}^2)^6$ such that $\varphi_n \rightarrow v$ in $H^1(\mathbb{R}^2)^6$. Let us decompose $\varphi_n = \varphi_{\mathcal{V}}^n + \varphi_{\mathcal{W}}^n \in \mathcal{V} \oplus \mathcal{W}$ and let $\text{cl}_{L^2 \cap L^p} \mathcal{V}$ denote the closure of \mathcal{V} in $L^2(\mathbb{R}^2)^6 \cap L^p(\mathbb{R}^2)^6$. Observe that $(\text{cl}_{L^2 \cap L^p} \mathcal{V}) \cap \mathcal{W} = \{0\}$, so that there is a continuous $L^2(\mathbb{R}^2)^6 \cap L^p(\mathbb{R}^2)^6$ -projection of $(\text{cl}_{L^2 \cap L^p} \mathcal{V}) \oplus \mathcal{W}$ onto \mathcal{W} . Since $\varphi_n \rightarrow v$ in $L^2(\mathbb{R}^2)^6 \cap L^p(\mathbb{R}^2)^6$, we infer that $\varphi_{\mathcal{W}}^n \rightarrow 0$ in \mathcal{W} . Similarly, arguing with the closure of \mathcal{W} in $H^1(\mathbb{R}^2)^6$ we get $\varphi_{\mathcal{V}}^n \rightarrow v$ in \mathcal{V} . Therefore

$$\|v - \varphi_n\| = \|v - \varphi_{\mathcal{V}}^n\| + \|\varphi_{\mathcal{W}}^n\| \rightarrow 0$$

as $n \rightarrow \infty$ and we conclude that $\mathcal{V} \oplus \mathcal{W}$ is the completion of $C_0^\infty(\mathbb{R}^2)^6$ with respect to the norm $\|\cdot\|$. □

Now we investigate the differential operator L and its spectrum in the following two results.

Proposition 2.2. *The second-order differential operator (1.6) is elliptic and self-adjoint on the domain $\text{dom}(L)$. Its associated bilinear form $b_L : \text{dom}(b_L) \times \text{dom}(b_L) \rightarrow \mathbb{R}$ is given by*

$$b_L(u, v) = \int_{\mathbb{R}^2} \langle \overset{\circ}{\nabla} \times u, \overset{\circ}{\nabla} \times v \rangle dx$$

with $\text{dom}(b_L) = \{u \in L^2(\mathbb{R}^2)^6 \text{ s.t. } \overset{\circ}{\nabla} \times u \in L^2(\mathbb{R}^2)^6\}$.

Proof. Let us show that $\text{dom}(b_L)$ is closed in $L^2(\mathbb{R}^2)^6$. First note the symmetry property $\langle \overset{\circ}{\nabla} \times u, v \rangle = \langle u, \overset{\circ}{\nabla} \times v \rangle$ for all $u, v \in \text{dom}(b_L)$. If $(u_n)_{n \in \mathbb{N}} \in \text{dom}(b_L)$ is a sequence such $u_n \rightarrow u$ in $L^2(\mathbb{R}^2)^6$ and $\overset{\circ}{\nabla} \times u_n \rightarrow \psi$ in $L^2(\mathbb{R}^2)^6$ for some $\psi \in L^2(\mathbb{R}^2)^6$ then the symmetry property implies $\langle \overset{\circ}{\nabla} \times u_n, \phi \rangle = \langle u_n, \overset{\circ}{\nabla} \times \phi \rangle \rightarrow \langle u, \overset{\circ}{\nabla} \times \phi \rangle$ for all $\phi \in \text{dom}(b_L)$. Thus $\psi = \overset{\circ}{\nabla} \times u$ and therefore b_L is a closed symmetric bilinear form. It is therefore the associated bilinear form of a unique selfadjoint operator, cf. [22, Theorem VIII.15]. For all $u, v \in C_0^\infty(\mathbb{R}^2)^6$ we see that

$$\langle Lu, \varphi \rangle = \int_{\mathbb{R}^2} (\overset{\circ}{\nabla} \times \overset{\circ}{\nabla} \times u) \cdot \varphi dx = \int_{\mathbb{R}^2} (\overset{\circ}{\nabla} \times u) \cdot (\overset{\circ}{\nabla} \times \varphi) dx = b_L(u, \varphi).$$

Hence b_L is the bilinear form of the operator $L : \text{dom}(L) \rightarrow L^2(\mathbb{R}^2)^6$ with $\text{dom}(L) = \{u \in L^2(\mathbb{R}^2)^6 \text{ s.t. } Lu \in L^2(\mathbb{R}^2)^6\}$. \square

Theorem 2.3. *The operator $L : \text{dom}(L) \subset L^2(\mathbb{R}^2)^6 \rightarrow L^2(\mathbb{R}^3)^6$ has spectrum $\sigma(L) = \{0\} \cup [k^2, \infty)$, where 0 is an eigenvalue of infinite multiplicity and $[k^2, \infty)$ consists of absolutely continuous spectrum.*

Proof. Let us consider the symbol $\hat{L}(\xi)$ which is a complex hermitian 6×6 matrix for $\xi \in \mathbb{R}^2$. Let us denote by $\sigma(\hat{L}(\xi))$ the matrix eigenvalues of $\hat{L}(\xi)$. We will verify $\sigma(L) = \bigcup_{\xi \in \mathbb{R}^2} \sigma(\hat{L}(\xi))$ by two steps:

- (i) λ is a matrix eigenvalue of $\hat{L}(\xi)$ for some $\xi \in \mathbb{R}^2 \Rightarrow \lambda \in \sigma(L)$
- (ii) $\text{dist}(\lambda, \bigcup_{\xi \in \mathbb{R}^2} \sigma(\hat{L}(\xi))) > 0 \Rightarrow \lambda$ lies in the resolvent set of L

Since the spectrum of L is closed, (i) and (ii) imply the claimed representation of $\sigma(L)$.

Proof of (i): Let $a \in \mathbb{R}^6$ be a unit vector with $\hat{L}(\xi_0)a = \lambda a$. If $\rho \in C_0^\infty(\mathbb{R}^2)$ is a function with $\|\rho\|_{L^2(\mathbb{R}^2)} = 1$ then let $\hat{u}_k(\xi) = k\rho(k(\xi - \xi_0))a$. One finds that $\|\hat{u}_k\|_{L^2(\mathbb{R}^2)^6} = 1$ and

$$\begin{aligned} \|(\hat{L} - \lambda \text{Id})\hat{u}_k\|_{L^2(\mathbb{R}^2)^6}^2 &= \int_{\mathbb{R}^2} |(\hat{L}(\xi) - \hat{L}(\xi_0))a|^2 k^2 \rho^2(k(\xi - \xi_0)) d\xi \\ &= \int_{\mathbb{R}^2} |(\hat{L}(\xi_0 + k^{-1}\eta) - \hat{L}(\xi_0))a|^2 \rho^2(\eta) d\eta \\ &\rightarrow 0 \text{ as } k \rightarrow \infty \end{aligned}$$

by continuity of the symbol and dominated convergence. By Plancherel's theorem we have that $(L - \lambda)u_k \rightarrow 0$ as $k \rightarrow \infty$ in $L^2(\mathbb{R}^2)^6$ with $\|u_k\|_{L^2(\mathbb{R}^2)^6} = 1$ so that $\lambda \in \sigma(L)$ by Weyl's criterion.

Proof of (ii): Let $|\lambda - \lambda_j(\xi)| \geq \delta > 0$ for all $\xi \in \mathbb{R}^2$, all eigenvalues $\lambda_1(\xi), \dots, \lambda_6(\xi)$ of the matrix $\hat{L}(\xi)$ and let us denote the corresponding orthonormal basis of eigenvectors by $v_1(\xi), \dots, v_6(\xi) \in \mathbb{C}^6$. Then, for given $f \in L^2(\mathbb{R}^2)^6$, solving the problem $(L - \lambda \text{Id})u = f$ is equivalent to computing the inverse Fourier transform of

$$\hat{u}(\xi) = \sum_{j=1}^6 \frac{\langle \hat{f}(\xi), v_j(\xi) \rangle_{\mathbb{C}^6}}{\lambda_j(\xi) - \lambda} v_{\lambda}(\xi).$$

By Plancherel's theorem this amounts to

$$\|u\|_{L^2(\mathbb{R}^2)^6}^2 = \|\hat{u}\|_{L^2(\mathbb{R}^2)^6}^2 = \int_{\mathbb{R}^2} \sum_{j=1}^6 \frac{|\langle \hat{f}(\xi), v_j(\xi) \rangle_{\mathbb{C}^6}|^2}{|\lambda_j(\xi) - \lambda|^2} \leq \frac{6}{\delta^2} \|f\|_{L^2(\mathbb{R}^2)^6}^2.$$

This shows that λ belongs to the resolvent set of L .

The remaining part of the proof is dedicated to the computation of the matrix eigenvalue $\lambda_1(\xi), \dots, \lambda_6(\xi)$. It turns out that the characteristic polynomial of $\hat{L}(\xi)$ is given by $\lambda^2(|\xi|^2 + k^2 - \lambda)^4$ and hence $\sigma(L) = \{0\} \cup [k^2, \infty)$. Indeed, taking the Fourier-transform we find the symbol of L , whose characteristic polynomial is given by

$\det(\hat{L}(\xi) - \lambda \text{Id})$

$$= \det \begin{pmatrix} \xi_2^2 + k^2 - \lambda & -\xi_1 \xi_2 & 0 & 0 & 0 & ik\xi_1 \\ -\xi_1 \xi_2 & \xi_1^2 + k^2 - \lambda & 0 & 0 & 0 & ik\xi_2 \\ 0 & 0 & |\xi|^2 - \lambda & ik\xi_1 & ik\xi_2 & 0 \\ 0 & 0 & -ik\xi_1 & \xi_2^2 + k^2 - \lambda & -\xi_1 \xi_2 & 0 \\ 0 & 0 & -ik\xi_2 & -\xi_1 \xi_2 & \xi_1^2 + k^2 - \lambda & 0 \\ -ik\xi_1 & -ik\xi_2 & 0 & 0 & 0 & |\xi|^2 - \lambda \end{pmatrix}.$$

Interchanging column 3 and 6 as well as line 3 and 6 we get a block structure which leads to

$$\begin{aligned} \det(\hat{L}(\xi) - \lambda \text{Id}) &= \det \begin{pmatrix} \xi_2^2 + k^2 - \lambda & -\xi_1 \xi_2 & ik\xi_1 \\ -\xi_1 \xi_2 & \xi_1^2 + k^2 - \lambda & ik\xi_2 \\ -ik\xi_1 & -ik\xi_2 & |\xi|^2 - \lambda \end{pmatrix} \\ &\quad \times \det \begin{pmatrix} |\xi|^2 - \lambda & ik\xi_1 & ik\xi_2 \\ -ik\xi_1 & \xi_2^2 + k^2 - \lambda & -\xi_1 \xi_2 \\ -ik\xi_2 & -\xi_1 \xi_2 & \xi_1^2 + k^2 - \lambda \end{pmatrix} \\ &= \lambda^2(|\xi|^2 + k^2 - \lambda)^4 \end{aligned}$$

since both matrices have the same determinant $-\lambda(|\xi|^2 + k^2 - \lambda)^2$. Thus the matrix eigenvalues are given by $\sigma(\hat{L}(\xi)) = \{0, |\xi|^2 + k^2\}$. The zero eigenvalue of infinite multiplicity is generated by all vector fields $\overset{\circ}{\nabla} \begin{pmatrix} \alpha \\ \tilde{\alpha} \end{pmatrix}$ with $\alpha, \tilde{\alpha} \in C_c^\infty(\mathbb{R}^2)$. □

Let $p_{\mathcal{V}} : X \rightarrow \mathcal{V}$, $p_{\mathcal{W}} : X \rightarrow \mathcal{W}$ denote the projections of X onto \mathcal{V} , \mathcal{W} , respectively. Usually we just write $u = v + w \in \mathcal{V} \oplus \mathcal{W}$, where $v = p_{\mathcal{V}}(u) \in \mathcal{V}$ and $w = u - v = p_{\mathcal{W}}(u) \in \mathcal{W}$. Observe that \mathcal{V} and \mathcal{W} are both subspaces of $\text{dom}(b_L)$ and that for $v + w \in X = \mathcal{V} \oplus \mathcal{W}$ we have

$$b_L(v + w, v + w) = b_L(v, v) = \|v\|^2.$$

For $u \in \text{dom}(b_L)$ using the duality pairing between $\text{dom}(b_L)'$ and $\text{dom}(b_L)$ we can define Lu by setting $\langle Lu, \phi \rangle_{\text{dom}(b_L)' \times \text{dom}(b_L)} := b_L(u, \phi)$ for all $\phi \in \text{dom}(b_L)$. In this way $L : X \rightarrow \text{dom}(b_L)'$ is well-defined and has the kernel \mathcal{W} . Note that L restricted to \mathcal{V} coincides with the vector Schrödinger operator $-\Delta + k^2$ acting diagonally on elements of \mathcal{V} .

We say that $u \in \begin{pmatrix} U \\ \tilde{U} \end{pmatrix} \in X$ is a *weak solution* to (1.7) provided that

$$b_L(u, \varphi) - \int_{\mathbb{R}^2} V(x) \langle u, \varphi \rangle dx - \int_{\mathbb{R}^2} \langle f(x, u), \varphi \rangle dx = 0 \tag{2.1}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^2)^6$. From now on we assume that (F1)–(F4) and (V) are satisfied. Observe that for $u = v + w \in \mathcal{V} \oplus \mathcal{W}$ we get

$$J(u) := \frac{1}{2} \|v\|^2 - \frac{1}{2} \int_{\mathbb{R}^2} V(x) |u|^2 dx - \int_{\mathbb{R}^2} F(x, u) dx.$$

Corollary 2.4. *$J : X \rightarrow \mathbb{R}$ is of class \mathcal{C}^1 and $u \in X$ is a critical point of J if and only if u is a weak solution to (1.7). Moreover, if (1.8) holds and E of the form (1.1) is a travelling wave field with the profiles U and \tilde{U} , where $u = \begin{pmatrix} U \\ \tilde{U} \end{pmatrix} \in X$ is a critical point of J , then the total electromagnetic energy per unit interval on the x_3 -axis is finite, i.e.*

$$\mathcal{L}(t) = \frac{1}{2} \int_{\mathbb{R}^2} \int_a^{a+1} \langle E, D \rangle + \langle B, H \rangle dx_3 d(x_1, x_2) < \infty.$$

Proof. The first statement is a consequence of Theorem 2.1. According to the material laws

$$\begin{aligned} \langle E, D \rangle &= -\varepsilon(x_1, x_2) \omega^2 |E|^2 + \chi(x_1, x_2, \langle |E|^2 \rangle) |E|^2 \\ &= \left(-V(x_1, x_2) + \chi(x_1, x_2, \frac{1}{2} |u|^2) \right) (|U|^2 \cos^2(kx_3 + \omega t) \\ &\quad + |\tilde{U}|^2 \sin^2(kx_3 + \omega t)) \end{aligned}$$

and by the Faraday’s law

$$\begin{aligned} \langle B, H \rangle &= |B|^2 \\ &= \frac{1}{\omega^2} |\nabla \times (U(x_1, x_2) \sin(kx_3 + \omega t) - \tilde{U}(x_1, x_2) \cos(kx_3 + \omega t))|^2 \\ &\leq \frac{1}{\omega^2} |\overset{\circ}{\nabla} \times u|^2. \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{L}(t) &\leq \frac{1}{2\omega^2} b_L(u, u) + \frac{1}{2} \int_{\mathbb{R}^2} \left(-V(x_1, x_2) + \chi \left(x_1, x_2, \frac{1}{2} |u|^2 \right) \right) \\ &\quad \cdot \int_a^{a+1} (|U|^2 \cos^2(kx_3 + \omega t) + |\tilde{U}|^2 \sin^2(kx_3 + \omega t)) dx_3 d(x_1, x_2) \\ &\leq \frac{1}{2} \left(1 + \frac{1}{\omega^2} \right) b_L(u, u) < \infty. \end{aligned}$$

□

3. Abstract variational approach

Our approach is based on the critical point theory from [4, 5, 18], however sometimes we do not assume the monotonicity assumption (I8) given below and we present results with emphasis on where this condition is crucial.

Let $J : X \rightarrow \mathbb{R}$ be a functional of the form

$$J(u) := \frac{1}{2} \|u^+\|^2 - I(u) \quad \text{for } u = u^+ + u^- \in X^+ \oplus X^-, \quad (3.1)$$

such that X is a reflexive Banach space with the norm $\|\cdot\|$ and a topological direct sum decomposition $X = X^+ \oplus X^-$, where X^+ is a Hilbert space with a scalar product $\langle \cdot, \cdot \rangle$. For $u \in X$ we denote by $u^+ \in X^+$ and $u^- \in X^-$ the corresponding summands so that $u = u^+ + u^-$. We may assume $\langle u, u \rangle = \|u\|^2$ for any $u \in X^+$ and $\|u\|^2 = \|u^+\|^2 + \|u^-\|^2$. We define the topology \mathcal{T} on X as the product of the norm topology in X^+ and the weak topology in X^- . Hence $u_n \xrightarrow{\mathcal{T}} u$ if and only if $u_n^+ \rightarrow u^+$ and $u_n^- \rightharpoonup u^-$. Let us define the set

$$\mathcal{M} := \{u \in X : J'(u)|_{X^-} = 0\} = \{u \in X : I'(u)|_{X^-} = 0\}. \quad (3.2)$$

Clearly \mathcal{M} contains all critical points of J and we assume the following conditions introduced in [4, 5]:

- (I1) $I \in C^1(X, \mathbb{R})$ and $I(u) \geq I(0) = 0$ for any $u \in X$.
- (I2) I is \mathcal{T} -sequentially lower semicontinuous: if $u_n \xrightarrow{\mathcal{T}} u$ then $\liminf I(u_n) \geq I(u)$.
- (I3) If $u_n \xrightarrow{\mathcal{T}} u$ and $I(u_n) \rightarrow I(u)$ then $u_n \rightarrow u$.
- (I4) $\|u^+\| + I(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$.
- (I5) If $u \in \mathcal{M}$ then $I(u) < I(u + v)$ for every $v \in X^- \setminus \{0\}$.
- (I6) There exists $r > 0$ such that $a := \inf_{u \in X^+, \|u\|=r} J(u) > 0$.
- (I7) $I(t_n u_n)/t_n^2 \rightarrow \infty$ if $t_n \rightarrow \infty$ and $u_n^+ \rightarrow u^+ \neq 0$ as $n \rightarrow \infty$.

Observe that if I is strictly convex, continuous and satisfies (I4), then (I2) and (I5) are clearly satisfied. Moreover, for any $u \in X^+$ we find $m(u) \in \mathcal{M}$ which is the unique global maximizer of $J|_{u+X^-}$. From now on we assume (I1)–(I7). Note that m needs not be C^1 , and \mathcal{M} needs not be a differentiable manifold since I' is only continuous, however from [5, Proof of Theorem 4.4] we observe $m : X^+ \rightarrow \mathcal{M}$ is a homeomorphism and $\tilde{J} := J \circ m : X^+ \rightarrow \mathbb{R}$ is of class C^1 .

We introduce the following min-max level

$$c := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J(\gamma(t)), \quad (3.3)$$

where

$$\Gamma := \{\gamma \in C([0, 1], \mathcal{M}) : \gamma(0) = 0, \|\gamma(1)^+\| > r, \text{ and } J(\gamma(1)^+) < 0\}. \quad (3.4)$$

If we look for ground state solutions with the least possible energy, we consider a Nehari-type constraint \mathcal{N} and the following condition

- (I8) $\frac{t^2-1}{2} I'(u)[u] + I(u) - I(tu+v) = \frac{t^2-1}{2} I'(u)[u] + tI'(u)[v] + I(u) - I(tu+v) \leq 0$ for every $u \in \mathcal{N}$, $t \geq 0$, $v \in X^-$, where

$$\mathcal{N} := \{u \in X \setminus X^- : J'(u)|_{\mathbb{R}u \oplus X^-} = 0\} = \{u \in \mathcal{M} \setminus X^- : J'(u)[u] = 0\} \subset \mathcal{M}.$$

The constraint \mathcal{N} is a generalized Nehari-type manifold inspired by a constrained considered in [21] for the Schrödinger problem. We state the following result obtained in [18, Theorem 3.3] under assumptions (I1)–(I8).

Theorem 3.1. *Suppose J of the form (3.1) satisfies (I1)–(I7). Then J has a sequence $(u_n) \subset \mathcal{M}$ such that $J(u_n) \rightarrow c \geq a > 0$ and $J'(u_n)(1 + \|u_n^+\|) \rightarrow 0$ as $n \rightarrow \infty$. If, in addition, (I8) holds, then $c = \inf_{\mathcal{N}} J$.*

Proof. For the reader’s convenience, we shortly sketch the proof. Recall that by (I5)–(I7), $\tilde{J}(u) \geq J(u) \geq a$ for $u \in X^+$ and $\|u\| = r$, and $\tilde{J}(tu)/t^2 \rightarrow -\infty$ as $t \rightarrow \infty$. Thus \tilde{J} has the mountain pass geometry and similarly as in [5, Theorem 4.4] we may define the mountain pass level

$$c_{\mathcal{M}} := \inf_{\sigma \in \tilde{\Gamma}} \sup_{t \in [0,1]} \tilde{J}(\sigma(t)) = \inf_{\gamma \in \tilde{\Gamma}} \sup_{t \in [0,1]} J(\gamma(t)) \geq a, \tag{3.5}$$

where

$$\tilde{\Gamma} := \{ \sigma \in \mathcal{C}([0, 1], X^+) : \sigma(0) = 0, \|\sigma(1)\| > r, \text{ and } \tilde{J}(\sigma(1)) < 0 \}.$$

By the mountain pass theorem there exists a Cerami sequence (v_n) for \tilde{J} at the level $c_{\mathcal{M}}$, see [3, 10]. Since $\tilde{J}'(v_n) = J'(m(v_n))$ (see [5, 18]), then setting $u_n := m(v_n)$ we obtain

$$J'(u_n)(1 + \|u_n^+\|) = \tilde{J}'(v_n)(1 + \|v_n\|) \rightarrow 0.$$

If (I8) holds, then by [18, Theorem 3.3 (b)], $c = \inf_{\mathcal{M}} J$. □

Now we present a multiplicity result. Recall that for a topological group acting on X , the orbit of $u \in X$ is often denoted by $G * u$, i.e.,

$$G * u := \{gu : g \in G\}.$$

A set $A \subset X$ is called G -invariant if $gA \subset A$ for all $g \in G$. A functional $J : X \rightarrow \mathbb{R}$ is called G -invariant and a map $T : X \rightarrow X^*$ is called G -equivariant if $J(gu) = J(u)$ and $T(gu) = gT(u)$ for all $g \in G, u \in X$. In our application we use the action given by \mathbb{Z}^2 -translations.

Assume that G is a topological group such that

- (G) G acts on X by isometries and discretely in the sense that for each $u \neq 0$, $(G * u) \setminus \{u\}$ is bounded away from u . Moreover, J is G -invariant and X^+, X^- are G -invariant.

Observe that \mathcal{M} is G -invariant and $m : X^+ \rightarrow \mathcal{M}$ is G -equivariant.

We shall use the notation

$$\begin{aligned} \tilde{J}^\beta &:= \{u \in X^+ : \tilde{J}(u) \leq \beta\}, & \tilde{J}_\alpha &:= \{u \in X^+ : \tilde{J}(u) \geq \alpha\}, \\ \tilde{J}_\alpha^\beta &:= \tilde{J}_\alpha \cap \tilde{J}^\beta, & \mathcal{K} &:= \{u \in X^+ : \tilde{J}'(u) = 0\} \end{aligned}$$

and call $G * u$ a *critical orbit* whenever $u \in \mathcal{K}$. Moreover the following variant of the Cerami condition between the levels $\alpha, \beta \in \mathbb{R}$ has been introduced in [18]:

- (M) $^\beta_\alpha$ (a) Let $\alpha \leq \beta$. There exists M_α^β such that $\limsup_{n \rightarrow \infty} \|u_n\| \leq M_\alpha^\beta$ for every $(u_n) \subset X^+$ satisfying $\alpha \leq \liminf_{n \rightarrow \infty} \tilde{J}(u_n) \leq \limsup_{n \rightarrow \infty} \tilde{J}(u_n) \leq \beta$ and $(1 + \|u_n\|)\tilde{J}'(u_n) \rightarrow 0$.

- (b) Suppose in addition that the number of critical orbits in \tilde{J}_α^β is finite. Then there exists $m_\alpha^\beta > 0$ such that if $(u_n), (v_n)$ are two sequences as above and $\|u_n - v_n\| < m_\alpha^\beta$ for all n large, then $\liminf_{n \rightarrow \infty} \|u_n - v_n\| = 0$.

Note that if J is even, then m is odd and \mathcal{M} is symmetric, i.e. $\mathcal{M} = -\mathcal{M}$. Note also that $(M)_\alpha^\beta$ is a condition on \tilde{J} . We slightly generalize [18, Theorem 3.5 (b)].

Theorem 3.2. *Suppose J of the form (3.1) satisfies (I1)–(I7), J is even and $\dim(X^+) = \infty$. If $(M)_0^\beta$ holds for every $\beta > 0$, then J has infinitely many distinct critical orbits.*

In fact, in [18, Theorem 3.5 (b)] condition (I8) has been assumed as well, which is redundant for this part of the result. Indeed, by the inspection of the proof of [18, Theorem 3.5 (b)], the following min-max values

$$\beta_k := \inf_{A \in \Sigma, i^*(A) \geq k} \sup_{u \in A} \tilde{J}(u), \quad k = 1, 2, \dots,$$

are well-defined and finite, where $\Sigma := \{A \subset X^+ : A = -A \text{ and } A \text{ is compact}\}$, and for $A \in \Sigma$, we define a variant of Benci's pseudoindex [3, 18].

$$i^*(A) := \min_{h \in \mathcal{H}} \gamma(h(A) \cap S(0, r))$$

where r is as in (I6), $S(0, r) := \{u \in X^+ : \|u\| = r\}$, γ is Krasnoselskii's genus and

$$\begin{aligned} \mathcal{H} &:= \{h : X^+ \rightarrow X^+ \text{ is a homeomorphism, } h(-u) = -h(u) \text{ and} \\ &\tilde{J}(h(u)) \leq \tilde{J}(u) \text{ for all } u\}. \end{aligned}$$

We observe also that $\beta_k \geq a$. Clearly (I6) implies that

$$\sup_{u \in A} \tilde{J}(u) \geq \sup_{u \in A \cap S(0, r)} \tilde{J}(u) \geq \inf_{u \in S(0, r)} \tilde{J}(u) \geq a$$

for any $A \in \Sigma$ such that $i^*(A) \geq k$. Hence, if we assume for contradiction that there is a finite number of distinct orbits $\{G * u : u \in \mathcal{K}\}$, then arguing as in the proof of [18, Theorem 3.5 (b)], β_k are critical values and $a \leq \beta_1 < \beta_2 < \dots$, which is impossible by the assumption. Therefore Theorem 3.2 holds true.

4. Application of the abstract variational approach

Recall that we assume (F1)–(F4) and (V). We assume, in addition, that F is convex in $u \in \mathbb{R}^2$. Let $X^+ := \mathcal{V}$ and $X^- := \mathcal{W}$. Let us define $I : X \rightarrow \mathbb{R}$ such that

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^2} V(x)|u|^2 dx + \int_{\mathbb{R}^2} F(x, u) dx,$$

hence $J(v + w) = \frac{1}{2}\|v\|^2 - I(v + w)$ for $v \in \mathcal{V}$ and $w \in \mathcal{W}$ so that J is of the form (3.1). In this section we check that (I1)–(I7) are satisfied.

Let us define the following map

$$H^1(\mathbb{R}^2)^6 \ni u \mapsto I_F(u) := \int_{\mathbb{R}^2} F(x, u) \, dx,$$

which is continuous and convex.

Lemma 4.1. *Conditions (I1)–(I4) are satisfied.*

Proof. In view of (F1), (F2), (F3) and (V) we easily get (I1) and by the convexity assumption on F also (I2). To see that (I3) holds let us suppose that $u_n \xrightarrow{\mathcal{I}} u$ and $I(u_n) \rightarrow I(u)$. Then by the fact that $F \geq 0$ and weak lower semicontinuity

$$\int_{\mathbb{R}^2} V(x)|u_n|^2 \, dx \rightarrow \int_{\mathbb{R}^2} V(x)|u|^2 \, dx, \quad I_F(u_n) \rightarrow I_F(u) \tag{4.1}$$

and passing to a subsequence $(V(x)^{1/2}u_n) \rightharpoonup (V(x)^{1/2}u)$ in $L^2(\mathbb{R}^2)^6$. Thus $u_n \rightarrow u$ in $L^2(\mathbb{R}^2)^6$ and $u_n(x) \rightarrow u(x)$ for a.e. $x \in \mathbb{R}^2$. Then, by Vitali’s convergence theorem

$$\begin{aligned} I_F(u_n) - I_F(u_n - u) &= \int_{\mathbb{R}^N} \int_0^1 -\frac{d}{ds} F(x, u_n - su) \, ds \, dx \\ &= \int_{\mathbb{R}^N} \int_0^1 f(x, u_n - su)u \, ds \, dx \\ &\rightarrow \int_0^1 \int_{\mathbb{R}^N} f(x, u - su)u \, dx \, ds \\ &= \int_{\mathbb{R}^N} \int_0^1 -\frac{d}{ds} F(x, u - su) \, ds \, dx \\ &= \int_{\mathbb{R}^N} F(x, u) \, dx = I_F(u) \end{aligned}$$

as $n \rightarrow \infty$. From (4.1) we infer that $I_F(u_n - u) \rightarrow 0$. Observe that, by (F4) and $F \geq 0$, for any $\varepsilon > 0$ we find a constant $\tilde{c}_\varepsilon > 0$ such that

$$F(x, u) + \varepsilon|u|^2 \geq \tilde{c}_\varepsilon|u|^p \quad \text{for } x, u \in \mathbb{R}^2. \tag{4.2}$$

Therefore

$$\int_{\mathbb{R}^2} F(x, u_n - u) \, dx + |u_n - u|_2^2 \geq \tilde{c}_1|u_n - u|_p^p$$

and we get $u_n \rightarrow u$ in $L^p(\mathbb{R}^2)^6$, which completes the proof of (I3). Now note that if $I(v_n + w_n)$ is bounded with $(v_n) \subset \mathcal{V}$ and $w_n \subset \mathcal{W}$ such that v_n is bounded, then w_n is bounded in $L^2(\mathbb{R}^2)^6$ and by (4.2), w_n is bounded in $L^p(\mathbb{R}^2)^6$. Thus (I4) holds. \square

Let $v \in \mathcal{V}$. Since $\mathcal{W} \ni w \mapsto I(v + w) \in \mathbb{R}$ is strictly convex and coercive, $I(u) \geq I(0) = 0$, then $I(v + \cdot)$ attains a unique global minimum at some $w(v) \in \mathcal{W}$. Hence the set

$$\begin{aligned} \mathcal{M} &:= \{u \in X : J'(u)|_{\mathcal{W}} = 0\} = \{u \in X : I'(u)|_{\mathcal{W}} = 0\} \\ &= \{u \in X : u = p_{\mathcal{V}}(u) + w(p_{\mathcal{V}}(u))\}, \end{aligned}$$

obviously contains all critical points of J and there holds:

(I5) If $u \in \mathcal{M}$ then $I(u) < I(u + w)$ for every $w \in \mathcal{W} \setminus \{0\}$.

Since $I(v + w(v)) \leq I(v)$ for $v \in \mathcal{V}$, we see that w maps bounded sets into bounded sets and in view of (I2) and (I3) one can easily show the continuity of w .

Now we show the following properties which imply the linking geometry in the spirit of Benci and Rabinowitz [8].

Lemma 4.2. *The following conditions are satisfied.*

(I6) *There exists $r > 0$ such that $a := \inf_{v \in \mathcal{V}, \|v\|=r} J(v) > 0$.*

(I7) *$I(t_n u_n)/t_n^2 \rightarrow \infty$ if $t_n \rightarrow \infty$ and $p_{\mathcal{V}}(u_n) \rightarrow v \neq 0$ as $n \rightarrow \infty$.*

Proof. Observe that (F2) and (F3) imply that for any $\varepsilon > 0$ there is $c_\varepsilon > 0$ such that

$$|f(x, u)| \leq \varepsilon|u| + c_\varepsilon|u|^{p-1} \text{ and } F(x, u) \leq \varepsilon|u|^2 + c_\varepsilon|u|^p \text{ for } x, u \in \mathbb{R}^2. \quad (4.3)$$

Then, taking $0 < \varepsilon < k^2 - \text{ess sup } V$, by (V) we get

$$J(v) \geq \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 + (k^2 - \text{ess sup } V - \varepsilon)|v|^2 dx - C_\varepsilon|v|_p^p$$

for $v \in \mathcal{V}$, and we obtain (I6) by the Sobolev embedding of $H^1(\mathbb{R}^2)^6$ into $L^p(\mathbb{R}^2)^6$ and by choosing $r > 0$ sufficiently small. Now suppose that $I(t_n u_n)/t_n^2$ is bounded, $t_n \rightarrow \infty$ and $p_{\mathcal{V}}(u_n) \rightarrow v \neq 0$ as $n \rightarrow \infty$. Note that if $u_n = v_n + w_n$ with $v_n = p_{\mathcal{V}}(u_n) \in \mathcal{V}$ and $w_n \in \mathcal{W}$, then by (4.2)

$$I(t_n u_n)/t_n^2 \geq \frac{1}{2} \int_{\mathbb{R}^2} (\text{ess inf } V - \varepsilon)|v_n + w_n|^2 dx + t_n^{p-2} c_\varepsilon |v_n + w_n|_p^p.$$

Since $I(t_n u_n)/t_n^2$ is bounded and taking $0 < \varepsilon < \text{ess\,inf } V$ we obtain that $v_n + w_n \rightarrow 0$ in $L^p(\mathbb{R}^2)^6$. Since $v_n + w_n$ is also bounded in $L^2(\mathbb{R}^2)^6$ it has a weakly convergent subsequence and the weak L^2 -limit of this subsequence has to coincide with the the strong L^p -limit. Thus, dropping subsequence indices, we get in total that $w_n \rightharpoonup -v \neq 0$ in \mathcal{W} and obtain a contradiction. \square

In view of Theorem 3.1, there is a sequence $u_n = v_n + w_n \in \mathcal{M}$ such that $J'(u_n) \rightarrow c \geq a > 0$ and $J'(u_n)(1 + \|v_n\|) \rightarrow 0$ as $n \rightarrow \infty$. We will show that (u_n) is bounded if, in addition, (F5) or (F6) holds, see Lemma 6.4 in Sect. 6. The next section however is devoted to profile decompositions of bounded sequences in \mathcal{M} , which will be important to show that there is a nontrivial weak limit point of the sequence (u_n) up to translations, which is a critical point.

5. Profile decompositions in \mathcal{M}

In addition to (F1)–(F4) and (V) we assume also that $V = V_0$ or $V(x) > V_0$ for a.e. $x \in \mathbb{R}^3$. Let us define $I_0 : X \rightarrow \mathbb{R}$ such that

$$I_0(u) = \frac{1}{2} \int_{\mathbb{R}^2} V_0 |u|^2 dx + \int_{\mathbb{R}^2} F(x, u) dx.$$

Clearly I_0 satisfies analogous conditions (I1)–(I4) and for $v \in \mathcal{V}$, $I_0(v + \cdot)$ attains a unique global minimum at some $w_0(v) \in \mathcal{W}$. Thus, similarly as in (3.2), we define

$$\mathcal{M}_0 := \{u \in X : I'_0(u)|_{\mathcal{W}} = 0\}.$$

Recall that profile decompositions of bounded sequences in $H^1(\mathbb{R}^N)$ have been obtained for instance by Nawa [19], Hmidi and Keraani [13], and by Gérard [12] in $\dot{H}^s(\mathbb{R}^N)$. A similar result cannot be obtained in $X = \mathcal{V} \oplus \mathcal{W}$, since \mathcal{W} is not locally compactly embedded into $L^p(\mathbb{R}^2)$ for $p \geq 1$. However we prove the following decomposition result in the topological constraints $\mathcal{M}, \mathcal{M}_0 \subset X$.

Theorem 5.1. *If (u_n) is bounded in \mathcal{M} , then, passing to a subsequence, there is $K \in \mathbb{N} \cup \{\infty\}$ and there are $\tilde{u}_i \in \mathcal{M}$ and sequences $(\tilde{u}_i)_{i \geq 1} \subset \mathcal{M}_0, (y_n^i)_{n \geq i} \subset \mathbb{Z}^2$ such that $y_n^0 = 0, |y_n^i - y_n^j| \rightarrow \infty$ as $n \rightarrow \infty$ for $i \neq j$, and the following conditions hold:*

- (a) *If $K < \infty$, then $\tilde{u}_i \neq 0$ for $1 \leq i \leq K$ and $\tilde{u}_i = 0$ for $i > K$. If $K = \infty$, then $\tilde{u}_i \neq 0$ for all $i \geq 1$.*
- (b) *$u_n(\cdot + y_n^i) \rightharpoonup \tilde{u}_i$ in X for any $0 \leq i < K + 1$ ⁽¹⁾.*
- (c) *$u_n(\cdot + y_n^i) \rightarrow \tilde{u}_i$ in $L^p_{loc}(\mathbb{R}^2)^6$ and a.e. in \mathbb{R}^2 for any $0 \leq i < K + 1$.*

¹If $K = \infty$, then $K + 1 = \infty$ as well.

(d) *There holds*

$$\lim_{n \rightarrow \infty} \|p_{\mathcal{V}}(u_n)\|^2 \geq \sum_{i=0}^{\infty} \|p_{\mathcal{V}}(\tilde{u}_i)\|^2, \tag{5.1}$$

$$\lim_{n \rightarrow \infty} I(u_n) \geq I(\tilde{u}_0) + \sum_{i=1}^{\infty} I_0(\tilde{u}_i), \tag{5.2}$$

$$\lim_{n \rightarrow \infty} I_F(u_n) = \sum_{i=0}^{\infty} I_F(\tilde{u}_i), \tag{5.3}$$

$$\lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} I_F \left(u_n - \sum_{i=0}^k \tilde{u}_i(\cdot - y_n^i) \right) = 0. \tag{5.4}$$

Proof. For a measurable set $A \subset \mathbb{R}^2$ we use the notation χ_A to denote the characteristic function of A . Let $u_n = v_n + w(v_n) \in \mathcal{M}$, where $v_n = p_{\mathcal{V}}(u_n) \in \mathcal{V}$. Passing to a subsequence $\lim_{n \rightarrow \infty} \|v_n\|$, $\lim_{n \rightarrow \infty} I(u_n)$ and $\lim_{n \rightarrow \infty} I_F(u_n)$ exist and are finite.

Part 1. Profile decomposition for (v_n) . Take $r > \sqrt{2}$. Since $(v_n) \subset H^1(\mathbb{R}^2)$ is bounded, we claim that, passing to a subsequence, there is $K \in \mathbb{N} \cup \{\infty\}$ and there is a sequence $(\tilde{v}_i)_{i=0}^K \subset H^1(\mathbb{R}^2)$, for $0 \leq i < K + 1$ there are a sequence $(y_n^i) \subset \mathbb{Z}^2$ and positive numbers $(c_i)_{i=1}^K$ such that $y_n^0 = 0$ and for any $0 \leq i < K + 1$ one has

$$\begin{aligned} v_n(\cdot + y_n^i) &\rightharpoonup \tilde{v}_i \text{ in } H^1(\mathbb{R}^2) \text{ and} \\ v_n(\cdot + y_n^i)\chi_{B(0,n)} &\rightarrow \tilde{v}_i \text{ in } L^2(\mathbb{R}^2)^6 \cap L^p(\mathbb{R}^2)^6, \end{aligned} \tag{5.5}$$

$$\tilde{v}_i \neq 0 \text{ if } i \geq 1, \tag{5.6}$$

$$\begin{aligned} |y_n^i - y_n^j| &\geq n - 2r \text{ for } j \neq i, 0 \leq i, j < K + 1 \\ &\text{and sufficiently large } n, \end{aligned} \tag{5.7}$$

$$\begin{aligned} \int_{B(y_n^i, r)} |\phi_n^{i-1}|^2 dx &\geq c_i \geq \frac{1}{2} \sup_{y \in \mathbb{R}^N} \int_{B(y, r)} |\phi_n^{i-1}|^2 dx > 0 \text{ for } i \geq 1 \\ &\text{and sufficiently large } n, \end{aligned} \tag{5.8}$$

where

$$\phi_n^i := v_n - \sum_{j=0}^i \tilde{v}_j(\cdot - y_n^j) \quad \text{for } i \geq 0, n \geq 1.$$

Indeed, passing to a subsequence we may assume that

$$\begin{aligned} v_n &\rightharpoonup \tilde{v}_0 \quad \text{in } H^1(\mathbb{R}^2)^6 \\ v_n \chi_{B(0,n)} &\rightarrow \tilde{v}_0 \quad \text{in } L^2(\mathbb{R}^2)^6 \cap L^p(\mathbb{R}^2)^6. \end{aligned}$$

The latter convergence follows from the fact that for any n , $H^1(B(0, n))$ is compactly embedded into $L^p(B(0, n))$ and we find sufficiently large k_n such that

$$|(v_{k_n} - \tilde{v}_0)\chi_{B(0,n)}|_2 + |(v_{k_n} - \tilde{v}_0)\chi_{B(0,n)}|_p < \frac{1}{n}.$$

The subsequence (u_{k_n}) is then relabelled by (u_n) .

Take $\phi_n^0 := v_n - \tilde{v}_0$ and if

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B(y,r)} |\phi_n^0|^2 dx = 0,$$

then we finish the proof of our claim with $K = 0$. Otherwise, passing to a subsequence, we find $(y_n^1) \subset \mathbb{Z}^2$ and a constant $c_1 > 0$ such that

$$\int_{B(y_n^1,r)} |\phi_n^0|^2 dx \geq c_1 \geq \frac{1}{2} \sup_{y \in \mathbb{R}^N} \int_{B(y,r)} |\phi_n^0|^2 dx > 0. \tag{5.9}$$

Note that (y_n^1) is unbounded and we may assume that $|y_n^1| \geq n - r$. Since $(v_n(\cdot + y_n^1))$ is bounded in $H^1(\mathbb{R}^2)$, we find $\tilde{v}_1 \in H^1(\mathbb{R}^2)$ such that up to a subsequence $v_n(\cdot + y_n^1) \rightharpoonup \tilde{v}_1$. In view of (5.9), we get $\tilde{v}_1 \neq 0$, and again we may assume that $v_n(\cdot + y_n^1)\chi_{B(0,n)} \rightarrow \tilde{v}_1$ in $L^2(\mathbb{R}^2)^6 \cap L^p(\mathbb{R}^2)^6$. We set $\phi_n^1 := \phi_n^0 - \tilde{v}_1(\cdot - y_n^1) = v_n - \tilde{v}_0 - \tilde{v}_1(\cdot - y_n^1)$ and observe that if

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^2} \int_{B(y,r)} |\phi_n^1|^2 dx = 0,$$

then we finish the proof of our claim with $K = 1$. Otherwise, passing to a subsequence, we find $(y_n^2) \subset \mathbb{Z}^2$ and a constant $c_2 > 0$ such that

$$\int_{B(y_n^2,r)} |\phi_n^1|^2 dx \geq c_2 \geq \frac{1}{2} \sup_{y \in \mathbb{R}^2} \int_{B(y,r)} |\phi_n^1|^2 dx > 0 \tag{5.10}$$

and $|y_n^2| \geq n - r$. Moreover $|y_n^2 - y_n^1| \geq n - 2r$. Otherwise $B(y_n^2, r) \subset B(y_n^1, n)$ and the convergence $\tilde{v}_0\chi_{B(y_n^2,r)} \rightarrow 0$ and $v_n(\cdot + y_n^1)\chi_{B(0,n)} \rightarrow \tilde{v}_1$ in $L^2(\mathbb{R}^2)^6$ contradict (5.10). Then we find $\tilde{v}_2 \neq 0$ such that passing to a subsequence

$$\begin{aligned} \phi_n^1(\cdot + y_n^2), v_n(\cdot + y_n^2) &\rightharpoonup \tilde{v}_2 \text{ in } H^1(\mathbb{R}^2) \text{ and} \\ v_n(\cdot + y_n^2)\chi_{B(0,n)} &\rightarrow \tilde{v}_2 \text{ in } L^2(\mathbb{R}^2)^6 \cap L^p(\mathbb{R}^2)^6. \end{aligned}$$

Again, if

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^2} \int_{B(y,r)} |\phi_n^2|^2 dx = 0,$$

where $v_n^2 := \phi_n^1 - \tilde{v}_2(\cdot - y_n^2)$, then we finish proof with $K = 2$. Continuing the above procedure we finally find $K \in \mathbb{N} \cup \{\infty\}$ such that for $0 \leq i < K + 1$, (5.5)–(5.8) hold. In view of the Brezis-Lieb Lemma [9],

$$\begin{aligned} \limsup_{n \rightarrow \infty} |v_n|_p &= |\tilde{v}_0|_p + \limsup_{n \rightarrow \infty} |\phi_n^0|_p = |\tilde{v}_0|_p + |\tilde{v}_1|_p + \limsup_{n \rightarrow \infty} |\phi_n^1|_p \\ &= |\tilde{v}_0|_p + \dots + |\tilde{v}_i|_p + \limsup_{n \rightarrow \infty} |\phi_n^i|_p \end{aligned}$$

for $i \geq 0$. If there is $i \geq 0$ such that

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^2} \int_{B(y,r)} |\phi_n^i|^2 dx = 0,$$

then $K = i$ and setting $\tilde{v}_k = 0$ for $k > i$ we get by Lions' Lemma [33, Lemma 1.21]

$$\lim_{n \rightarrow \infty} |\phi_n^i|_p = 0.$$

Since $\phi_n^k = \phi_n^i$ for $k \geq i$, we find in particular

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} |\phi_n^k|_p = 0. \quad (5.11)$$

In the case $K = \infty$ we want to show that (5.11) is still satisfied. We argue as in [17, Proof of Theorem 1.4]. Suppose, for a contradiction, that $\limsup_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} |\phi_n^k|_p > 0$. Then we find $\delta > 0$ and increasing sequences $(i_k), (n_k) \subset \mathbb{N}$ such that

$$\int_{\mathbb{R}^2} |\phi_{n_k}^{i_k}|^p dx > \delta$$

and

$$\sup_{y \in \mathbb{R}^2} \int_{B(y, r)} |\phi_{n_k}^{i_k}|^2 dx \leq \limsup_{n \rightarrow \infty} \left(\sup_{y \in \mathbb{R}^2} \int_{B(y, r)} |\phi_n^{i_k}|^2 dx \right) + \frac{1}{i_k}. \quad (5.12)$$

Note that by (5.8) we have

$$\begin{aligned} c_{k+1} &\leq \int_{B(y_n^{k+1}, r)} |\phi_n^k|^2 dx \\ &\leq 2 \int_{B(y_n^{k+1}, r)} |\phi_n^i|^2 dx + 2 \int_{B(y_n^{k+1}, r)} \left| \sum_{j=i+1}^k \tilde{v}_j(\cdot - y_n^j) \right|^2 dx \\ &\leq 4c_{i+1} + 2(k-i) \sum_{j=i+1}^k \int_{B(y_n^{k+1} - y_n^j, r)} |\tilde{v}_j|^2 dx \end{aligned}$$

for any $0 \leq i < k$. Taking into account (5.7) and letting $n \rightarrow \infty$ we get $c_{k+1} \leq 4c_{i+1}$. Take $k \geq 1$ and $n > 4r$. Again by (5.8) and (5.7) we obtain

$$\begin{aligned} \frac{1}{16} \sup_{y \in \mathbb{R}^N} \int_{B(y, r)} |\phi_n^k|^2 dx &\leq \frac{1}{8} c_{k+1} \leq \frac{1}{2k} \sum_{i=0}^{k-1} c_{i+1} \leq \frac{1}{2k} \sum_{i=0}^{k-1} \int_{B(y_n^{i+1}, r)} |\phi_n^i|^2 dx \\ &\leq \frac{1}{k} \sum_{i=0}^{k-1} \int_{B(y_n^{i+1}, r)} |v_n|^2 + \left| \sum_{j=0}^i \tilde{v}_j(\cdot - y_n^j) \right|^2 dx \\ &= \frac{1}{k} \int_{\bigcup_{i=0}^{k-1} B(y_n^{i+1}, r)} |v_n|^2 dx \\ &\quad + \frac{1}{k} \int_{\mathbb{R}^N} \left| \sum_{i=0}^{k-1} \sum_{j=0}^i \tilde{v}_j(\cdot - y_n^j) \chi_{B(y_n^{i+1}, r)} \right|^2 dx \\ &\leq \frac{1}{k} |v_n|_2^2 + \frac{1}{k} \left| \sum_{i=0}^{k-1} \sum_{j=0}^i \tilde{v}_j(\cdot - y_n^j) \chi_{B(y_n^{i+1}, r)} \right|_2^2. \end{aligned}$$

Observe that by (5.7) and since $n > 4r$ we have

$$B(y_n^{i+1} - y_n^j, r) \subset \mathbb{R}^2 \setminus B(0, n - 3r) \text{ for } 0 \leq j \leq i < k$$

and

$$\begin{aligned} & \left| \sum_{i=0}^{k-1} \sum_{j=0}^i \bar{v}_j(\cdot - y_n^j) \chi_{B(y_n^{i+1}, r)} \right|_2 \\ & \leq \sum_{i=0}^{k-1} \sum_{j=0}^i \left| \bar{v}_j \chi_{B(y_n^{i+1} - y_n^j, r)} \right|_2 \leq \sum_{i=0}^{k-1} \sum_{j=0}^i \left| \bar{v}_j \chi_{\mathbb{R}^2 \setminus B(0, n-3r)} \right|_2 \\ & \leq k \sum_{j=0}^{k-1} \left| \bar{v}_j \chi_{\mathbb{R}^2 \setminus B(0, n-3r)} \right|_2 \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Hence

$$\limsup_{n \rightarrow \infty} \left(\sup_{y \in \mathbb{R}^2} \int_{B(y, r)} |\phi_n^k|^2 dx \right) \leq \frac{32}{k} \limsup_{n \rightarrow \infty} |v_n|_2^2. \tag{5.13}$$

Therefore we obtain from (5.12)

$$\lim_{k \rightarrow \infty} \left(\sup_{y \in \mathbb{R}^N} \int_{B(y, r)} |\phi_{n_k}^{i_k}|^2 dx \right) = 0,$$

and in view of [33, Lemma 1.21] we obtain that $\phi_{n_k}^{i_k} \rightarrow 0$ in $L^p(\mathbb{R}^2)^6$ as $k \rightarrow \infty$, which is a contradiction. Therefore passing to a subsequence (5.11) holds.

Part 2. Profile decomposition for (u_n) . Note that $(w(v_n))_{n \in \mathbb{N}}$ and $(w_0(v_n))_{n \in \mathbb{N}}$ are bounded and we may assume

$$w(v_n)(\cdot + y_n^i) \rightharpoonup \tilde{w}_i \text{ in } L^2(\mathbb{R}^2)^6 \cap L^p(\mathbb{R}^2)^6 \text{ for } i \geq 0 \tag{5.14}$$

for some $\tilde{w}_i \in \mathcal{W}$, $i \geq 0$. Let us define $\tilde{u}_0 := \tilde{v}_0 + w(\tilde{v}_0)$ and $\tilde{u}_i := \tilde{v}_i + w_0(\tilde{v}_i)$ if $i \geq 1$.

Part 2. Claim 1. There holds

$$\sum_{i=1}^{\infty} I_F(\tilde{u}_i) \leq \sum_{i=1}^{\infty} I_0(\tilde{u}_i) < +\infty. \tag{5.15}$$

Indeed, observe that for $i \geq 1$

$$(v_n + w(v_n))(\cdot + y_n^i) \chi_{B(0, \frac{n-2r}{2})} \rightharpoonup \tilde{v}_i + \tilde{w}_i \text{ weakly in } L^2(\mathbb{R}^2)^6 \cap L^p(\mathbb{R}^2)^6$$

so that the weak lower semicontinuity of I_0 implies

$$\begin{aligned} \sum_{i=1}^k I_0(\tilde{u}_i) & \leq \sum_{i=1}^k I_0(\tilde{v}_i + \tilde{w}_i) \leq \sum_{i=1}^k \liminf_{n \rightarrow \infty} I_0((v_n + w(v_n))(\cdot + y_n^i) \chi_{B(0, \frac{n-2r}{2})}) \\ & \leq \liminf_{n \rightarrow \infty} \sum_{i=1}^k I_0((v_n + w(v_n)) \chi_{B(y_n^i, \frac{n-2r}{2})}) \\ & \leq \liminf_{n \rightarrow \infty} I_0(v_n + w(v_n)) = \liminf_{n \rightarrow \infty} I_0(u_n) \end{aligned}$$

for any $k \geq 0$. Since (u_n) and hence $(I_0(u_n))$ is bounded, (5.15) holds.

Part 2. Claim 2. Up to a subsequence, there holds

$$\sup_{k \geq 0} \limsup_{n \rightarrow \infty} I_F \left(\sum_{i=0}^k \tilde{u}_i(\cdot - y_n^i) \right) \leq \sum_{i=0}^{\infty} I_F(\tilde{u}_i). \quad (5.16)$$

Observe that for given $k \geq 0$

$$\begin{aligned} I_F \left(\sum_{i=0}^k \tilde{u}_i(\cdot - y_n^i) \right) &= I_F \left(\sum_{i=0}^k \tilde{u}_i(\cdot - y_n^i) \chi_{\cup_{j=0}^k B(y_n^j, \frac{n-2r}{2})} \right) \\ &\quad + I_F \left(\sum_{i=0}^k \tilde{u}_i(\cdot - y_n^i) \chi_{\mathbb{R}^2 \setminus \cup_{j=0}^k B(y_n^j, \frac{n-2r}{2})} \right). \end{aligned} \quad (5.17)$$

Concerning the first sum on the right hand side of (5.17) note that, for given $0 \leq j \leq k$

$$\left| \sum_{0 \leq i \leq k, i \neq j} \tilde{u}_i(\cdot - (y_n^i - y_n^j)) \chi_{B(0, \frac{n-2r}{2})} \right|_p \leq \sum_{0 \leq i \leq k, i \neq j} \left| \tilde{u}_i \chi_{B(y_n^i - y_n^j, \frac{n-2r}{2})} \right|_p \rightarrow 0 \quad (5.18)$$

as $n \rightarrow \infty$, since

$$B\left(y_n^i - y_n^j, \frac{n-2r}{2}\right) \subset \mathbb{R}^2 \setminus B\left(0, \frac{n-2r}{2}\right)$$

for $i \neq j$. The convergence in (5.18) also holds for $p = 2$. Taking into account the uniform continuity of I_F on bounded sets we obtain

$$\begin{aligned} &I_F \left(\sum_{i=0}^k \tilde{u}_i(\cdot - y_n^i) \chi_{\cup_{j=0}^k B(y_n^j, \frac{n-2r}{2})} \right) \\ &= I_F \left(\sum_{i=0}^k \sum_{j=0}^k \tilde{u}_i(\cdot - y_n^i) \chi_{B(y_n^j, \frac{n-2r}{2})} \right) \\ &= I_F \left(\sum_{i=0}^k \tilde{u}_i(\cdot - y_n^i) \chi_{B(y_n^i, \frac{n-2r}{2})} + \sum_{0 \leq i \neq j \leq k} \tilde{u}_i(\cdot - y_n^i) \chi_{B(y_n^j, \frac{n-2r}{2})} \right) \\ &\rightarrow \sum_{i=0}^k I_F(\tilde{u}_i) \end{aligned}$$

as $n \rightarrow \infty$. Finally, concerning the second sum on the right hand side of (5.17) observe that for any $i \geq 0$

$$\begin{aligned} \left| \tilde{u}_i(\cdot - y_n^i) \chi_{\mathbb{R}^2 \setminus \cup_{j=0}^k B(y_n^j, \frac{n-2r}{2})} \right|_p &\leq \left| \tilde{u}_i(\cdot - y_n^i) \chi_{\mathbb{R}^2 \setminus B(y_n^i, \frac{n-2r}{2})} \right|_p \\ &= \left| \tilde{u}_i \chi_{\mathbb{R}^2 \setminus B(0, \frac{n-2r}{2})} \right|_p \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. The same convergence also holds for $p = 2$. Taking the $\limsup_{n \rightarrow \infty}$ and the $\sup_{k \geq 0}$ in (5.17) we conclude the proof of (5.16).

Part 2. Claim 3. For any $k \geq 0$

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^2} |v_n + w(v_n)|^2 dx - \sum_{i=0}^k \int_{\mathbb{R}^2} |\tilde{v}_i + \tilde{w}_i|^2 dx \\ & \geq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^2} |v_n + w(\tilde{v}_0) + \sum_{i=1}^k w_0(\tilde{v}_i)(\cdot - y_n^i)|^2 dx - \sum_{i=0}^k \int_{\mathbb{R}^2} |\tilde{u}_i|^2 dx \end{aligned} \tag{5.19}$$

as $n \rightarrow \infty$. Indeed, observe that

$$\begin{aligned} & \int_{\mathbb{R}^2} |v_n + w(v_n)|^2 dx - \sum_{i=0}^k \int_{\mathbb{R}^2} |\tilde{v}_i + \tilde{w}_i|^2 dx \\ & - \int_{\mathbb{R}^2} |v_n + w(\tilde{v}_0) - \sum_{i=1}^k w_0(\tilde{v}_i)(\cdot - y_n^i)|^2 dx + \sum_{i=0}^k \int_{\mathbb{R}^2} |\tilde{u}_i|^2 dx \\ & = \int_{\mathbb{R}^2} |w(v_n)|^2 dx - \sum_{i=0}^k \int_{\mathbb{R}^2} |\tilde{w}_i|^2 dx - \int_{\mathbb{R}^2} |w(\tilde{v}_0) - \sum_{i=1}^k w_0(\tilde{v}_i)(\cdot - y_n^i)|^2 dx \\ & + \int_{\mathbb{R}^2} |w(\tilde{v}_0)|^2 dx + \sum_{i=1}^k \int_{\mathbb{R}^2} |w_0(\tilde{v}_i)|^2 dx \\ & \geq \sum_{i=0}^k \int_{B(y_n^i, \frac{n-2r}{2})} |w(v_n)|^2 dx - \sum_{i=0}^k \int_{\mathbb{R}^2} |\tilde{w}_i|^2 dx \\ & + 2 \sum_{i=1}^k \int_{\mathbb{R}^2} \langle w(\tilde{v}_0), w_0(\tilde{v}_i)(\cdot - y_n^i) \rangle dx \\ & - 2 \sum_{1 \leq i < j \leq k} \int_{\mathbb{R}^2} \langle w_0(\tilde{v}_i)(\cdot - y_n^i), w_0(\tilde{v}_j)(\cdot - y_n^j) \rangle dx. \end{aligned}$$

Note that for any $i \geq 1$ and $R > 0$

$$\begin{aligned} & \int_{\mathbb{R}^2} |\langle w(\tilde{v}_0), w_0(\tilde{v}_i)(\cdot - y_n^i) \rangle| dx \\ & \leq \int_{B(0,R)} |\langle w(\tilde{v}_0), w_0(\tilde{v}_i)(\cdot - y_n^i) \rangle| dx + |w(\tilde{v}_0)\chi_{\mathbb{R}^2 \setminus B(0,R)}|_2 |w_0(\tilde{v}_i)|_2 \\ & \leq |w(\tilde{v}_0)|_2 |w_0(\tilde{v}_i)\chi_{B(-y_n^i,R)}|_2 + |w(\tilde{v}_0)\chi_{\mathbb{R}^2 \setminus B(0,R)}|_2 |w_0(\tilde{v}_i)|_2 \\ & = |w(\tilde{v}_0)\chi_{\mathbb{R}^2 \setminus B(0,R)}|_2 |w_0(\tilde{v}_i)|_2 + o(1) \end{aligned}$$

as $n \rightarrow \infty$. Letting $R \rightarrow \infty$ we obtain $\int_{\mathbb{R}^2} |\langle w(\tilde{v}_0), w_0(\tilde{v}_i)(\cdot - y_n^i) \rangle| dx = o(1)$ and we conclude the claim, since $w(v_n)(\cdot + y_n^i)\chi_{B(0, \frac{n-2r}{2})} \rightarrow \tilde{w}_i$ in $L^2(\mathbb{R}^2)^6$.

Part 2. Claim 4. For any $k \geq 0$

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} (V(x) - V_0) \left| \sum_{i=0}^k \tilde{u}_i(\cdot - y_n^i) \right|^2 dx = \int_{\mathbb{R}^2} (V(x) - V_0) |\tilde{u}_0|^2 dx. \tag{5.20}$$

Note that if $i \geq j$ and $i \geq 1$, then for any $R > 0$ we have

$$\begin{aligned} & \int_{\mathbb{R}^2} |V(x) - V_0| |\langle \tilde{u}_i(\cdot - y_n^i), \tilde{u}_j(\cdot - y_n^j) \rangle| dx \\ & \leq \int_{B(-y_n^i, R)} |V(x + y_n^i) - V_0| |\langle \tilde{u}_i, \tilde{u}_j(\cdot - (y_n^j - y_n^i)) \rangle| dx \\ & \quad + |(V(x) - V_0)\chi_{\mathbb{R}^2 \setminus B(0, R)}|_{\frac{p}{p-2}} |\tilde{u}_i|_p |\tilde{u}_j|_p \\ & = |(V(x) - V_0)\chi_{\mathbb{R}^2 \setminus B(0, R)}|_{\frac{p}{p-2}} |\tilde{u}_i|_p |\tilde{u}_j|_p + o(1) \end{aligned}$$

as $n \rightarrow \infty$. Letting $R \rightarrow \infty$ we conclude the claim.

Part 2. Claim 5. Up to a subsequence we have

$$w(v_n)\chi_{B(0, \frac{n-2r}{2})} \rightarrow w(\tilde{v}_0) \text{ in } L^2(\mathbb{R}^2)^6 \cap L^p(\mathbb{R}^2)^6, \quad (5.21)$$

$$w(v_n)(\cdot + y_n^i)\chi_{B(0, \frac{n-2r}{2})} \rightarrow w_0(\tilde{v}_i) \text{ in } L^2(\mathbb{R}^2)^6 \cap L^p(\mathbb{R}^2)^6, \quad (5.22)$$

as $n \rightarrow \infty$. Let us denote $\nu_n^k := v_n + w(\tilde{v}_0) + \sum_{i=1}^k w_0(\tilde{v}_i)(\cdot - y_n^i)$. Since the map

$$L^p(\mathbb{R}^2)^6 \ni u \mapsto \frac{1}{2} \int_{\mathbb{R}^2} (V(x) - V_0)|u|^2 dx + I_F(u)$$

is uniformly continuous on bounded sets, then in view of (5.11) for every $\varepsilon > 0$ there is $k_0 \geq 0$ such that for any $k \geq k_0$ there is $n_0 = n_0(k)$ such that for $n \geq n_0$ one obtains

$$\begin{aligned} \varepsilon + \frac{1}{2} \int_{\mathbb{R}^2} (V(x) - V_0) \left| \sum_{i=0}^k \tilde{u}_i(\cdot - y_n^i) \right|^2 dx + I_F \left(\sum_{i=0}^k \tilde{u}_i(\cdot - y_n^i) \right) \\ + \frac{1}{2} \int_{\mathbb{R}^2} V_0 |\nu_n^k|^2 dx \\ \geq \frac{1}{2} \int_{\mathbb{R}^2} (V(x) - V_0) |\nu_n^k|^2 dx + I_F(\nu_n^k) + \frac{1}{2} \int_{\mathbb{R}^2} V_0 |\nu_n^k|^2 dx = I(\nu_n^k) \\ \geq I(u_n) = \frac{1}{2} \int_{\mathbb{R}^2} (V(x) - V_0) |u_n|^2 dx + I_F(u_n) + \frac{1}{2} \int_{\mathbb{R}^2} V_0 |u_n|^2 dx. \end{aligned} \quad (5.23)$$

Moreover

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^2} (V(x) - V_0) |u_n|^2 dx \geq \int_{\mathbb{R}^2} (V(x) - V_0) |\tilde{v}_0 + \tilde{w}_0|^2 dx,$$

and

$$\begin{aligned} \liminf_{n \rightarrow \infty} I_F(u_n) & \geq \liminf_{n \rightarrow \infty} \sum_{i=0}^k I_F((v_n + w(v_n))\chi_{B(y_n^i, \frac{n-2r}{2})}) \\ & \geq \sum_{i=0}^k \liminf_{n \rightarrow \infty} I_F((v_n + w(v_n))\chi_{B(y_n^i, \frac{n-2r}{2})}) \\ & \geq \sum_{i=0}^k I_F(\tilde{v}_i + \tilde{w}_i) \end{aligned} \quad (5.24)$$

for any $k \geq 0$. Then

$$\begin{aligned}
 \liminf_{n \rightarrow \infty} I(u_n) &\geq \frac{1}{2} \int_{\mathbb{R}^2} (V(x) - V_0) |\tilde{v}_0 + \tilde{w}_0|^2 dx + \sum_{i=0}^k I_F(\tilde{v}_i + \tilde{w}_i) \\
 &\quad + \liminf_{n \rightarrow \infty} \frac{1}{2} \int_{\mathbb{R}^2} V_0 |v_n + w(v_n)|^2 dx \\
 &= I(\tilde{v}_0 + \tilde{w}_0) + \sum_{i=1}^k I_0(\tilde{v}_i + \tilde{w}_i) - \sum_{i=0}^k \frac{1}{2} \int_{\mathbb{R}^2} V_0 |\tilde{v}_i + \tilde{w}_i|^2 dx \\
 &\quad + \liminf_{n \rightarrow \infty} \frac{1}{2} \int_{\mathbb{R}^2} V_0 |v_n + w(v_n)|^2 dx \\
 &\geq I(\tilde{u}_0) + \sum_{i=1}^k I_0(\tilde{u}_i) - \sum_{i=0}^k \frac{1}{2} \int_{\mathbb{R}^2} V_0 |\tilde{v}_i + \tilde{w}_i|^2 dx \\
 &\quad + \liminf_{n \rightarrow \infty} \frac{1}{2} \int_{\mathbb{R}^2} V_0 |v_n + w(v_n)|^2 dx \\
 &\geq \frac{1}{2} \int_{\mathbb{R}^2} (V(x) - V_0) |\tilde{u}_0|^2 dx + \sum_{i=0}^k I_F(\tilde{u}_i) \\
 &\quad + \liminf_{n \rightarrow \infty} \frac{1}{2} \int_{\mathbb{R}^2} V_0 |\nu_n^k|^2 dx,
 \end{aligned}$$

where the last inequality follows from (5.19).

On the other hand, taking into account (5.23) and then (5.20), (5.16) we get

$$\begin{aligned}
 \liminf_{n \rightarrow \infty} I(u_n) &\leq \varepsilon + \frac{1}{2} \int_{\mathbb{R}^2} (V(x) - V_0) |\tilde{u}_0|^2 dx + \sum_{i=0}^{\infty} I_F(\tilde{u}_i) \\
 &\quad + \sup_{k \geq 0} \liminf_{n \rightarrow \infty} \frac{1}{2} \int_{\mathbb{R}^2} V_0 |\nu_n^k|^2 dx,
 \end{aligned}$$

hence

$$\begin{aligned}
 \liminf_{n \rightarrow \infty} I(u_n) &= \frac{1}{2} \int_{\mathbb{R}^2} (V(x) - V_0) |\tilde{u}_0|^2 dx + \sum_{i=0}^{\infty} I_F(\tilde{u}_i) \\
 &\quad + \sup_{k \geq 0} \liminf_{n \rightarrow \infty} \frac{1}{2} \int_{\mathbb{R}^2} V_0 |\nu_n^k|^2 dx. \tag{5.25}
 \end{aligned}$$

Therefore we get equalities in the above considerations, in particular $\tilde{u}_i = \tilde{v}_i + \tilde{w}_i$ for $i \geq 0$ and in (5.19) equality holds for $k = \infty$, so that $|w(v_n)(\cdot + y_n^i) \chi_{B(0, \frac{n-2r}{2})}|_2 \rightarrow |\tilde{w}_i|_2$ passing to a subsequence, and then $w(v_n)(\cdot + y_n^i) \chi_{B(0, \frac{n-2r}{2})} \rightarrow \tilde{w}_i$ in $L^2(\mathbb{R}^2)^6$ for $i \geq 0$. Moreover from equality in (5.24) we get

$$\begin{aligned}
 \liminf_{n \rightarrow \infty} I_F(u_n \chi_{B(0, \frac{n-2r}{2})}) &= I_F(\tilde{v}_0 + w(\tilde{u}_0)), \\
 \liminf_{n \rightarrow \infty} I_F(u_n(\cdot + y_n^i) \chi_{B(0, \frac{n-2r}{2})}) &= I_F(\tilde{v}_i + w_0(\tilde{u}_i))
 \end{aligned}$$

for $i \geq 1$. Taking into account (5.5) and arguing as in the proof of (I3) from Lemma 4.1, passing to a subsequence, we obtain

$$\begin{aligned} u_n \chi_{B(0, \frac{n-2r}{2})} &\rightarrow \tilde{v}_0 + w(\tilde{v}_0) \text{ in } L^p(\mathbb{R}^2)^6, \\ u_n(\cdot + y_n^i) \chi_{B(0, \frac{n-2r}{2})} &\rightarrow \tilde{v}_i + w_0(\tilde{v}_i) \text{ in } L^p(\mathbb{R}^2)^6. \end{aligned}$$

Therefore (5.21) and (5.22) are satisfied.

Part 3. Proof of (d). Now we complete the proof. Note that we have already proved (a)–(c). Since $v_n(\cdot + y_n^i) \chi_{B(0, \frac{n-2r}{2})} \rightarrow \tilde{v}_i$ and arguing similarly as in (5.24) we obtain (5.1). Taking into account (5.25) we find (5.2) and using the fact that the equality holds in (5.24) for $k = \infty$ we conclude (5.3).

In order to show (5.4), note that $u_n(\cdot + y_n^i) \rightarrow \tilde{u}_i$ for a.e. in \mathbb{R}^2 and for every $0 \leq i < K + 1$. Starting with $i = 0$ and arguing similarly as in proof of Lemma 4.1 we get

$$\lim_{n \rightarrow \infty} (I_F(u_n) - I_F(u_n - \tilde{u}_0)) = I_F(\tilde{u}_0).$$

Since we know from the beginning of the proof that $\lim_{n \rightarrow \infty} I_F(u_n)$ exists and is finite, we therefore can rearrange and find

$$\lim_{n \rightarrow \infty} I_F(u_n) = I_F(\tilde{u}_0) + \lim_{n \rightarrow \infty} I_F(u_n - \tilde{u}_0). \quad (5.26)$$

Next we define for $0 \leq j < K$

$$r_n^j := u_n - \sum_{i=0}^j \tilde{u}_i(\cdot - y_n^i)$$

and observe that $r_n^j(\cdot + y_n^{j+1}) \rightarrow \tilde{u}_{j+1}$ and $r_n^j(\cdot + y_n^{j+1}) - r_n^{j+1}(\cdot + y_n^{j+1}) = \tilde{u}_{j+1}$. Starting with $j = 0$ we obtain again, similarly as in proof of Lemma 4.1, that

$$\lim_{n \rightarrow \infty} (I_F(r_n^0(\cdot + y_n^1)) - I_F(r_n^1(\cdot + y_n^1))) = I_F(\tilde{u}_1)$$

and since $r_n^0 = u_n - \tilde{u}_0$ and by the shift-invariance of I_F this implies

$$\lim_{n \rightarrow \infty} I_F(u_n - \tilde{u}_0) = \lim_{n \rightarrow \infty} I_F(r_n^0) = I_F(\tilde{u}_1) + \lim_{n \rightarrow \infty} I_F(r_n^1). \quad (5.27)$$

Now, for any $0 \leq j < K$

$$\lim_{n \rightarrow \infty} (I_F(r_n^j(\cdot + y_n^{j+1})) - I_F(r_n^{j+1}(\cdot + y_n^{j+1}))) = I_F(\tilde{u}_{j+1})$$

and hence

$$\lim_{n \rightarrow \infty} I_F(r_n^j) = I_F(\tilde{u}_{j+1}) + \lim_{n \rightarrow \infty} I_F(r_n^{j+1}). \quad (5.28)$$

Collecting (5.26), (5.27) and (5.28) for $0 \leq j < K$ we obtain

$$\lim_{n \rightarrow \infty} I_F(u_n) = \sum_{i=0}^{j+1} I_F(\tilde{u}_i) + \lim_{n \rightarrow \infty} I_F(r_n^{j+1}).$$

Then by (5.3)

$$\lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} I_F(r_n^k) = 0$$

and we finally obtain (5.4). \square

Corollary 5.2. *The map J' is weak-to-weak* continuous on \mathcal{M} , i.e., if $(u_n) \subset \mathcal{M}$ and $u_n \rightharpoonup u$ for some $u \in X$, then $J'(u_n)(\varphi) \rightarrow J'(u)(\varphi)$ for any $\varphi \in X$. Similarly, J'_0 is weak-to-weak* continuous on \mathcal{M}_0 where $J_0(u) = \frac{1}{2} \|p_{\mathcal{V}}(u)\|^2 - I_0(u)$.*

Proof. Since (u_n) is bounded, in view of Theorem 5.1(c) for $i = 0$, up to a subsequence $u_n(x) \rightarrow u(x)$ for a.e. $x \in \mathbb{R}^2$. By Vitaly's convergence theorem we infer that $J'(u_n)(\varphi) \rightarrow J'(u)(\varphi)$ and $J'_0(u_n)(\varphi) \rightarrow J'_0(u)(\varphi)$ for any $\varphi \in X$. □

6. Proof of Theorem 1.1 and Theorem 1.2

Lemma 6.1. *Suppose that (F5) holds. If $u \in X$, $\psi \in \mathcal{W}$ and $t \geq 0$, then*

$$J(u) \geq J(t(u + \psi)) + J'(u) \left(\frac{1 - t^2}{2} u - t^2 \psi \right).$$

In particular (I8) holds.

Proof. We define a map $\varphi : [0, +\infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$\varphi(t, x) := \left\langle f(x, u), \frac{t^2 - 1}{2} u(x) + t^2 \psi(x) \right\rangle + F(x, u(x)) - F(x, t(u(x) + \psi(x))). \tag{6.1}$$

If $u(x) \neq 0$, then (F5) implies that $\varphi(0, x) \leq 0$ and if $u(x) = 0$ then also $\varphi(0, x) = 0$. Next, (F4) implies that $\varphi(t, x) \rightarrow -\infty$ as $t \rightarrow \infty$. Then, for every fixed $x \in \mathbb{R}^2$, there is a global maximum point $t_0 = t_0(x) \geq 0$ of $\varphi(x, \cdot)$. Thus $\varphi'(t_0, x) = 0$ and by (F5) we obtain that $\varphi(t_0, x) \leq 0$; see similar arguments in proof of [15][Proposition 4.1]. Therefore

$$\begin{aligned} & J(t(u + \psi)) + J'(u) \left(\frac{1 - t^2}{2} u - t^2 \psi \right) - J(u) \\ &= -\frac{t^2}{2} \int_{\mathbb{R}^2} V(x) |\psi|^2 dx + \int_{\mathbb{R}^2} \varphi(t, x) dx \leq 0 \end{aligned}$$

and we conclude. □

Remark 6.2. Observe that the inequality $\varphi(1, x) \leq 0$ implies the convexity of F in u . Therefore this assumption is not explicitly stated in Theorem 1.1.

Lemma 6.3. *Suppose that (F6) holds. For any $\varepsilon > 0$ there is a constant $c_\varepsilon > 0$ such that if $u = v + w \in \mathcal{V} \oplus \mathcal{W}$, then*

$$J(u) \geq J(tv) + J'(u) \left(\frac{1 - t^2}{2} u + t^2 w \right) - \varepsilon t^2 |u|_2 |w|_2 - t^2 c_\varepsilon |u|_p^{p-1} |w|_p$$

for any $0 \leq t \leq \sqrt{1 - \frac{2}{\gamma}}$.

Proof. Observe that by (F6)

$$\begin{aligned}\varphi(t, x) &= \left\langle f(x, u), \frac{t^2 - 1}{2} u \right\rangle - t^2 \langle f(x, u), w \rangle + F(x, u) - F(x, tv), \\ &\leq -t^2 \langle f(x, u), w \rangle - F(x, tv) \leq -t^2 \langle f(x, u), w \rangle\end{aligned}$$

Then we obtain

$$\begin{aligned}J(tv) + J'(u) \left(\frac{1 - t^2}{2} u + t^2 w \right) - J(u) &= -\frac{t^2}{2} \int_{\mathbb{R}^2} V(x) |w|^2 dx + \int_{\mathbb{R}^2} \varphi(t, x) dx \\ &\leq -t^2 \int_{\mathbb{R}^2} \langle f(x, u), w \rangle dx \\ &\leq \varepsilon t^2 |u|_2 |w|_2 + t^2 c_\varepsilon |u|_p^{p-1} |w|_p,\end{aligned}$$

where the last inequality follows from (4.3) and we conclude. \square

Lemma 6.4. *If (F5), or (F4') and (F6) hold, then $(M)_0^\beta$ is satisfied for every $\beta > 0$.*

Proof. (a) Take $\beta > 0$ and suppose that $u_n = v_n + w_n \in \mathcal{M}$ is such that

$$0 \leq \liminf_{n \rightarrow \infty} J(u_n) \leq \limsup_{n \rightarrow \infty} J(u_n) \leq \beta \text{ and } J'(u_n)(1 + \|v_n\|) \rightarrow 0$$

as $n \rightarrow \infty$. Observe that by (4.2)

$$\frac{1}{2} \|v_n\|^2 - \left(\frac{1}{2} \operatorname{ess\,inf} V - \varepsilon \right) |v_n|_2^2 \geq J(u_n) + \left(\frac{1}{2} \operatorname{ess\,inf} V - \varepsilon \right) |w_n|_2^2 + \tilde{c}_\varepsilon |u_n|_p^p$$

and if we take $\varepsilon = \frac{1}{4} \operatorname{ess\,inf} V$ then

$$\begin{aligned}\|v_n\|^2 - \frac{1}{2} \operatorname{ess\,inf} V |v_n|_2^2 &\geq J(u_n) + \frac{1}{2} \|v_n\|^2 \\ -\frac{1}{4} \operatorname{ess\,inf} V |v_n|_2^2 + \frac{1}{4} \operatorname{ess\,inf} V |w_n|_2^2 + \tilde{c}_\varepsilon |u_n|_p^p.\end{aligned}\tag{6.2}$$

Let $s_n := \left(\frac{1}{2} \|v_n\|^2 - \frac{1}{4} \operatorname{ess\,inf} V |v_n|_2^2 + \frac{1}{4} \operatorname{ess\,inf} V |w_n|_2^2 + \tilde{c}_\varepsilon |u_n|_p^p \right)^{1/2}$ and suppose by a contradiction that (u_n) is unbounded, that is, passing to a subsequence $s_n \rightarrow \infty$. Let $\tilde{v}_n := v_n/s_n$ and we may assume that $\tilde{v}_n \rightarrow \tilde{v}$ in X for some $\tilde{v} \in \mathcal{V}$, and $\tilde{v}_n(x) \rightarrow \tilde{v}(x)$ a.e. in \mathbb{R}^2 .

Let us show that $\inf_{n \in \mathbb{N}} |\tilde{v}_n|_p > 0$ by a contradiction argument both in the case where (F5) and (F4'), (F6) holds. Therefore assume (passing to a subsequence) that $\tilde{v}_n \rightarrow 0$ in $L^p(\mathbb{R}^2)^6$. By (4.3) we obtain

$$\int_{\mathbb{R}^2} F(x, s\tilde{v}_n) dx \rightarrow 0 \text{ for any } s > 0.\tag{6.3}$$

First assume that (F5) is satisfied. Take $\gamma \in (0, 1)$ such that

$$\operatorname{ess\,sup} V \leq \frac{1}{2}\gamma \operatorname{ess\,inf} V + (1 - \gamma)k^2 \quad (6.4)$$

and $s = 2\sqrt{\beta/\gamma}$. In view of Lemma 6.1, the fact that $J'(u_n)(1 + \|v_n\|) \rightarrow 0$ and (6.3) we obtain the following estimate

$$\begin{aligned} \beta &\geq \liminf_{n \rightarrow \infty} J(u_n) \\ &\geq \liminf_{n \rightarrow \infty} J(s\tilde{v}_n) + \liminf_{n \rightarrow \infty} J'(u_n) \left(\frac{1 - (s/s_n)^2}{2} u_n + (s/s_n)^2 w_n \right) \\ &= \liminf_{n \rightarrow \infty} J(s\tilde{v}_n) \geq \frac{s^2}{2} \liminf_{n \rightarrow \infty} (\|\tilde{v}_n\|^2 - \operatorname{ess\,sup} V|\tilde{v}_n|_2^2). \end{aligned}$$

Using $\|\tilde{v}_n\|^2 - \frac{1}{2} \operatorname{ess\,inf} V|\tilde{v}_n|_2^2 \geq 1$ by (6.2) and (6.4) we obtain the contradiction

$$\beta \geq \frac{s^2}{2} \gamma \liminf_{n \rightarrow \infty} \left(\|\tilde{v}_n\|^2 - \frac{1}{2} \operatorname{ess\,inf} V|\tilde{v}_n|_2^2 \right) \geq \frac{s^2}{2} \gamma = 2\beta.$$

Next, assume that (F4') and (F6) are satisfied. Then

$$J(u_n) - \frac{1}{2} J'(u_n)(u_n) \geq \frac{\gamma - 2}{2} \int_{\mathbb{R}^2} F(x, u_n) \, dx$$

and (u_n) is bounded in $L^p(\mathbb{R}^3)^6$. Hence (w_n) is bounded in $L^p(\mathbb{R}^3)^6$. Note that $s_n \geq \delta|u_n|_2$ for some constant $\delta > 0$. Hence also $s_n \geq \delta|w_n|_2$ and taking $\gamma \in (0, 1)$ as in (6.4), $s = 2\sqrt{\beta/\gamma}$ and $\varepsilon = \frac{1}{8}\gamma\delta^2$, in view of Lemma 6.3, the fact that $J'(u_n)(1 + \|v_n\|) \rightarrow 0$ and (6.3) we obtain the following estimate

$$\begin{aligned} \beta &\geq \liminf_{n \rightarrow \infty} J(u_n) \\ &\geq \liminf_{n \rightarrow \infty} J(s\tilde{v}_n) + \liminf_{n \rightarrow \infty} J'(u_n) \left(\frac{1 - (s/s_n)^2}{2} u_n + (s/s_n)^2 w_n \right) \\ &\quad + \liminf_{n \rightarrow \infty} \left(-\varepsilon(s/s_n)^2|u_n|_2|w_n|_2 - (s/s_n)^2 c_\varepsilon|u_n|_p^{p-1}|w_n|_p \right) \\ &\geq \liminf_{n \rightarrow \infty} J(s\tilde{v}_n) - \varepsilon s^2 \delta^{-2} = \frac{s^2}{2} \liminf_{n \rightarrow \infty} (\|\tilde{v}_n\|^2 - \operatorname{ess\,sup} V|\tilde{v}_n|_2^2) - \varepsilon s^2 \delta^{-2}. \end{aligned}$$

Using as before $\|\tilde{v}_n\| - \frac{1}{2} \operatorname{ess\,inf} V|\tilde{v}_n|_2 \geq 1$, the definition (6.4) of γ and the definition of ε we obtain the contradiction

$$\beta \geq \frac{s^2}{2} \gamma \liminf_{n \rightarrow \infty} \left(\|\tilde{v}_n\|^2 - \frac{1}{2} \operatorname{ess\,inf} V|\tilde{v}_n|_2^2 \right) - \varepsilon s^2 \delta^{-2} \geq s^2 \left(\frac{\gamma}{2} - \varepsilon \delta^{-2} \right) = \frac{3}{2} \beta.$$

In both cases we have led the assumption that $\tilde{v}_n \rightarrow 0$ in $L^p(\mathbb{R}^2)^6$ to a contradiction. Hence, we continue by having $\inf_{n \in \mathbb{N}} |\tilde{v}_n|_p > 0$. Let $\text{cl}_{L^p} \mathcal{V}$ denote the closure of \mathcal{V} in $L^p(\mathbb{R}^2)^6$. Observe that $\text{cl}_{L^p} \mathcal{V} \cap \mathcal{W} = \{0\}$. Indeed let $w \in \text{cl}_{L^p} \mathcal{V} \cap \mathcal{W}$, $(\varphi_n) \subset \mathcal{V}$ and $\varphi_n \rightarrow w$ in $L^p(\mathbb{R}^2)^6$. Then

$$\int_{\mathbb{R}^2} \left\langle w, \overset{\circ}{\nabla} \begin{pmatrix} \alpha \\ \tilde{\alpha} \end{pmatrix} \right\rangle dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} \left\langle w - \varphi_n, \overset{\circ}{\nabla} \begin{pmatrix} \alpha \\ \tilde{\alpha} \end{pmatrix} \right\rangle dx = 0$$

for any $\alpha, \tilde{\alpha} \in C_0^\infty(\mathbb{R}^2)$. By the density argument we infer that $w = 0$. Hence using the continuity of the projection $\text{cl}_{L^p} \mathcal{V} \oplus \mathcal{W} \rightarrow \text{cl}_{L^p} \mathcal{V}$ there is a constant $c' > 0$ such that

$$|u_n|_p^p \geq c' |v_n|_p^p. \tag{6.5}$$

Then by (4.2) we get

$$\int_{\mathbb{R}^2} F(x, u_n) dx \geq -\varepsilon |u_n|_2^2 + \tilde{c}_\varepsilon |u_n|_p^p \geq -\varepsilon |u_n|_2^2 + \tilde{c}_\varepsilon c' |v_n|_p^p.$$

Therefore taking $\varepsilon = \frac{1}{2} \text{ess inf } V$ we see that

$$\begin{aligned} J(u_n)/s_n^2 &\leq \frac{1}{2} \|\tilde{v}_n\|^2 - \frac{1}{2s_n^2} \text{ess inf } V |u_n|_2^2 - \int_{\mathbb{R}^2} \frac{F(x, u_n(x))}{s_n^2} dx \\ &\leq \frac{1}{2} \|\tilde{v}_n\|^2 - \tilde{c}_\varepsilon c' s_n^{p-2} |v_n|_p^p \leq \frac{1}{2} \|\tilde{v}_n\|^2 - \tilde{c}_\varepsilon c' s_n^{p-2} \inf_{n \in \mathbb{N}} |v_n|_p^p \\ &\rightarrow -\infty. \end{aligned}$$

Thus we get a contradiction, and therefore (u_n) must be bounded so that part (a) of $(M)_0^\beta$ holds.

Now let us show part (b) of $(M)_0^\beta$. This part only applies in the case where $V(x) \equiv V_0$. So let us take another sequence $u'_n = v'_n + w'_n \in \mathcal{M}$ is such that

$$0 \leq \liminf_{n \rightarrow \infty} J(u'_n) \leq \limsup_{n \rightarrow \infty} J(u'_n) \leq \beta \text{ and } J'(u'_n)(1 + \|v'_n\|) \rightarrow 0$$

as $n \rightarrow \infty$. Suppose in addition that the number of critical orbits in J_0^β is finite. Let

$$m := \inf \{ \|v - v'\| : J'(m(v)) = J'(m(v')) = 0, v \neq v' \}$$

and since $G = \mathbb{Z}^2$ is discrete (i.e., condition (G) holds), $m > 0$. Now we split the proof into two cases.

Case 1 $|v_n - v'_n|_p \rightarrow 0$. Then

$$\begin{aligned} \|v_n - v'_n\|^2 - \text{ess sup } V |v_n - v'_n|_2^2 &\leq (J'(u_n) - J'(u'_n))(u_n - u'_n) \\ &\quad + \int_{\mathbb{R}^2} \langle f(x, u_n) - f(x, u'_n), v_n - v'_n \rangle dx. \end{aligned}$$

Since $(u_n), (u'_n)$ are bounded Palais-Smale sequences the first term on the right hand side converges to 0 and by (4.3) the last integral on the right hand side tends to 0. Hence, using $\text{ess sup } V < k^2$ we see that $\|v_n - v'_n\|^2 \rightarrow 0$ as $n \rightarrow \infty$ and the proof is finished.

Case 2 $\limsup |v_n - v'_n|_p > 0$. Then by Lion’s Lemma [33, Lemma 1.21], up to a \mathbb{Z}^2 translation, $v_n \rightharpoonup v$ and $v'_n \rightharpoonup v'$ for some $v, v' \in \mathcal{V}$ such that $v \neq v'$. We may assume that $w_n \rightharpoonup w$ and $w'_n \rightharpoonup w'$ for some $w, w' \in \mathcal{W}$. In view of Corollary 5.2, $J'(v + w) = J'(v' + w') = 0$ and thus $\|v - v'\| \geq m$. In other words: if $\|v - v'\| < m$ then we are in the previous Case 1 and the proof is completed. \square

Proof of Theorem 1.1. Suppose that $(u_n) \subset \mathcal{M}$ sequence obtained in Theorem 3.1. By Lemma 6.4 it is a bounded Palais-Smale sequence for the functional J . Passing to a subsequence, we find $\tilde{u}_0 \in \mathcal{M}$ and sequences $(\tilde{u}_i)_{i \geq 0} \subset \mathcal{M}_0$, $(y_n^i)_{n \geq i} \subset \mathbb{Z}^2$ such that the statements of Theorem 5.1 hold. Observe that by Corollary 5.2, \tilde{u}_0 is a critical point of J . Similarly we show that \tilde{u}_i is a critical point of J_0 for $i \geq 1$. Indeed, take any $\varphi \in X$ and since $|y_n^i| \rightarrow \infty$ and $V - V_0 \in L^{\frac{p}{p-2}}(\mathbb{R}^2)$ we obtain $|V(\cdot + y_n^i) - V_0|_{\frac{p}{p-2}} \rightarrow 0$ as $n \rightarrow \infty$ and

$$\begin{aligned} J'_0(u_n(\cdot + y_n^i))(\varphi) &= J'_0(u_n)(\varphi(\cdot - y_n^i)) = J'(u_n)(\varphi(\cdot - y_n^i)) \\ &\quad + \int_{\mathbb{R}^2} (V - V_0)\langle u_n, \varphi(\cdot - y_n^i) \rangle dx \\ &= J'(u_n)(\varphi(\cdot - y_n^i)) \\ &\quad + \int_{\mathbb{R}^2} (V(\cdot + y_n^i) - V_0)\langle u_n(\cdot + y_n^i), \varphi \rangle dx = o(1). \end{aligned}$$

Since $u_n(x + y_n^i) \rightarrow \tilde{u}_i(x)$ for a.e. $x \in \mathbb{R}^2$ by Vitaly’s convergence theorem we infer that $J'_0(\tilde{u}_i)(\varphi) = 0$. We may assume that $\lim_{n \rightarrow \infty} I_F(u_n)$ exists and is positive, since otherwise (for a subsequence) $u_n \rightarrow 0$ in $L^p(\mathbb{R}^2)^6$ and

$$\begin{aligned} J(u_n) &= J(u_n) - \frac{1}{2}J'(u_n)(u_n) + o(1) \\ &= \int_{\mathbb{R}^2} \frac{1}{2}\langle f(x, u_n), u_n \rangle - F(x, u_n) dx + o(1) = o(1) \end{aligned}$$

we get a contradiction. Therefore, having $\lim_{n \rightarrow \infty} I_F(u_n) > 0$, we know that $\tilde{u}_0 \neq 0$ or $\tilde{u}_1 \neq 0$.

We first treat the case where $V = V_0$, hence $J = J_0$. Due to shift-invariance in this case, we may assume w.l.o.g. that $\tilde{u}_0 \neq 0$. Clearly, by Fatou’s lemma

$$\begin{aligned} \inf_{\mathcal{N}_0} J_0 &= \lim_{n \rightarrow \infty} J_0(u_n) = \lim_{n \rightarrow \infty} \left(J_0(u_n) - \frac{1}{2}J'_0(u_n)(u_n) \right) \\ &\geq J_0(\tilde{u}_0) - \frac{1}{2}J'_0(\tilde{u}_0)(\tilde{u}_0) = J_0(\tilde{u}_0), \end{aligned}$$

thus $J_0(\tilde{u}_0) = \inf_{\mathcal{N}_0} J_0$ and $\tilde{u}_0 \in \mathcal{N}_0$ is a ground state solution. If, in addition, f is odd in u , then in view of Lemma 6.4 and Theorem 3.2 there is an infinite sequence of \mathbb{Z}^2 -distinct solutions.

Next we consider the case that $V(x) > V_0$ for a.e. $x \in \mathbb{R}^2$ and (F4') is satisfied. Again we apply Theorem 5.1. Let $v_n := p_V(u_n)$ and $\tilde{v}_i := p_V(\tilde{u}_i)$

for $n \geq 1$ and $0 \leq i < K + 1$. In view of (5.1) from Theorem 5.1, for any $0 \leq k < K + 1$ we get

$$\lim_{n \rightarrow \infty} \|v_n\|^2 \geq \sum_{i=0}^k \|\tilde{v}_i\|^2 \geq 2J(\tilde{u}_0) + \sum_{i=1}^k 2J_0(\tilde{u}_i) \geq 2k \inf_{\mathcal{N}_0} J_0,$$

hence $K < \infty$. Observe that

$$\begin{aligned} & J'(u_n) \left[v_n - \sum_{i=0}^K \tilde{v}_i(\cdot - y_n^i) \right] \\ &= \left\| v_n - \sum_{i=0}^K \tilde{v}_i(\cdot - y_n^i) \right\|^2 + b_L \left(\sum_{i=0}^K \tilde{v}_i(\cdot - y_n^i), v_n - \sum_{i=0}^K \tilde{v}_i(\cdot - y_n^i) \right) \\ &\quad - I'(u_n) \left[v_n - \sum_{i=0}^K \tilde{v}_i(\cdot - y_n^i) \right] \\ &= \left\| v_n - \sum_{i=0}^K \tilde{v}_i(\cdot - y_n^i) \right\|^2 + \sum_{j=0}^K b_L \left(\tilde{v}_j, v_n(\cdot + y_n^j) - \sum_{i=0}^K \tilde{v}_i(\cdot - (y_n^i - y_n^j)) \right) \\ &\quad - I'(u_n) \left[v_n - \sum_{i=0}^K \tilde{v}_i(\cdot - y_n^i) \right] \\ &= \left\| v_n - \sum_{i=0}^K \tilde{v}_i(\cdot - y_n^i) \right\|^2 - I'(u_n) \left[v_n - \sum_{i=0}^K \tilde{v}_i(\cdot - y_n^i) \right] + o(1), \end{aligned}$$

since $v_n(\cdot + y_n^j) \rightarrow \tilde{v}_j$, and for $i \neq j$, $b_L(\tilde{v}_j, \tilde{v}_i(\cdot - (y_n^i - y_n^j))) \rightarrow 0$ as $n \rightarrow \infty$. Since (F4') holds, then by (5.4) we get $u_n - \sum_{i=0}^K \tilde{u}_i(\cdot - y_n^i) \rightarrow 0$ in $L^p(\mathbb{R}^2)^6$, hence $v_n - \sum_{i=0}^K \tilde{v}_i(\cdot - y_n^i) \rightarrow 0$ in $L^p(\mathbb{R}^2)^6$. Then

$$\begin{aligned} & I'(u_n) \left[v_n - \sum_{i=0}^K \tilde{v}_i(\cdot - y_n^i) \right] \\ &= V_0 \left| v_n - \sum_{i=0}^K \tilde{v}_i(\cdot - y_n^i) \right|_2^2 \\ &\quad + \sum_{j=0}^K \int_{\mathbb{R}^2} V_0 \left\langle \tilde{v}_j, v_n - \sum_{i=0}^K \tilde{v}_i(\cdot - (y_n^i - y_n^j)) \right\rangle dx \\ &\quad + \int_{\mathbb{R}^2} (V(x) - V_0) \left\langle v_n, v_n - \sum_{i=0}^K \tilde{v}_i(\cdot - y_n^i) \right\rangle \\ &\quad + I'_F(u_n) \left[v_n - \sum_{i=0}^K \tilde{v}_i(\cdot - y_n^i) \right] \\ &= V_0 \left| v_n - \sum_{i=0}^K \tilde{v}_i(\cdot - y_n^i) \right|_2 + o(1) \end{aligned}$$

and we get

$$\begin{aligned}
 J'(u_n) \left[v_n - \sum_{i=0}^K \tilde{v}_i(\cdot - y_n^i) \right] &= \left\| v_n - \sum_{i=0}^K \tilde{v}_i(\cdot - y_n^i) \right\|^2 \\
 &\quad - V_0 \left| v_n - \sum_{i=0}^K \tilde{v}_i(\cdot - y_n^i) \right|_2^2 + o(1) \\
 &\geq \frac{k^2 - V_0}{k^2} \left\| v_n - \sum_{i=0}^K \tilde{v}_i(\cdot - y_n^i) \right\|^2 + o(1)
 \end{aligned}$$

and since (u_n) is a bounded Palais-Smale sequence we obtain

$$v_n - \sum_{i=0}^K \tilde{v}_i(\cdot - y_n^i) \rightarrow 0 \quad \text{in } H^1(\mathbb{R}^2)^6.$$

Arguing similarly as above we get

$$I'(u_n) \left[u_n - \sum_{i=0}^K \tilde{u}_i(\cdot - y_n^i) \right] = V_0 \left| u_n - \sum_{i=0}^K \tilde{u}_i(\cdot - y_n^i) \right|_2^2 + o(1)$$

and

$$u_n - \sum_{i=0}^K \tilde{u}_i(\cdot - y_n^i) \rightarrow 0 \quad \text{in } L^2(\mathbb{R}^2)^6.$$

Hence, using the divergence property $|y_n^i - y_n^j| \rightarrow \infty$ as $n \rightarrow \infty$ for $i \neq j$, we get

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \|v_n\|^2 &= \sum_{i=0}^K \|\tilde{v}_i\|^2, \\
 \lim_{n \rightarrow \infty} |u_n|_2^2 &= \sum_{i=0}^K |\tilde{u}_i|_2^2,
 \end{aligned}$$

and in view of (5.3) we finally get

$$c = \inf_{\mathcal{N}} J = \lim_{n \rightarrow \infty} J(u_n) = J(\tilde{u}_0) + \sum_{i=1}^K J_0(\tilde{u}_i) \geq J(\tilde{u}_0) + K \inf_{\mathcal{N}_0} J_0.$$

Let us denote by ξ the ground state solutions of J_0 obtained above. Observe that $J(t\xi + w(t\xi)) \rightarrow -\infty$ as $t \rightarrow \infty$, $J(0 + w(0)) = 0$, hence we find $t > 0$ such that

$$J(t\xi + w(t\xi)) \geq c,$$

where c is given by (3.3). Then, in view of Lemma 6.1 and Theorem 3.1

$$\inf_{\mathcal{N}_0} J_0 = J_0(\xi) \geq J_0(t\xi + w(t\xi)) > J(t\xi + w(t\xi)) \geq c = \inf_{\mathcal{N}} J.$$

Therefore $K = 0$ and $\tilde{u}_0 \neq 0$ is a critical point of J . Observe that $\tilde{u}_0 \in \mathcal{N}$ and (using once more Fatou's Lemma)

$$\begin{aligned} c &= \lim_{n \rightarrow \infty} J(u_n) = \lim_{n \rightarrow \infty} \left(J(u_n) - \frac{1}{2} J'(u_n)(u_n) \right) \\ &\geq \lim_{n \rightarrow \infty} J(\tilde{u}_0) - \frac{1}{2} J'(\tilde{u}_0)(\tilde{u}_0) = J(\tilde{u}_0) \end{aligned}$$

so that $J(\tilde{u}_0) = \inf_{\mathcal{N}} J$ and \tilde{u}_0 is a ground state solution. \square

Proof of Theorem 1.2. Here $V = V_0$ and we argue as in proof of Theorem 1.1 and we use the notation introduced therein. Suppose that $(u_n) \subset \mathcal{M}_0$ is the bounded Palais-Smale sequence obtained in Sect. 4. Passing to a subsequence, we find sequences $(\tilde{u}_i)_{i \geq 0} \subset \mathcal{M}_0$, $(y_n^i)_{n \geq i} \subset \mathbb{Z}^2$ such that all the statements of Theorem 5.1 hold. Observe that by Corollary 5.2, \tilde{u}_0 is a critical point of J and \tilde{u}_i is a critical point of J_0 . Since $\lim_{n \rightarrow \infty} I_F(u_n) > 0$, $\tilde{u}_0 \neq 0$ or $\tilde{u}_1 \neq 0$. Note that

$$\mathcal{K}_0 := \{u \in X \setminus \{0\} : J'_0(u) = 0\} \subset \mathcal{M}_0$$

and we already know that $\mathcal{K}_0 \neq \emptyset$ since $\tilde{u}_0 \neq 0$ or $\tilde{u}_1 \neq 0$. It is easy to show that $c_0 := \inf_{\mathcal{K}_0} J_0 > 0$ and, since \mathcal{K} is nonempty, we may take any minimizing sequence $(\xi_n) \subset \mathcal{K}_0$ such that $J_0(\xi_n) \rightarrow c_0$. Similarly as above applying the profile decomposition from Theorem 5.1 to (ξ_n) , passing to a subsequence we find $(z_n) \subset \mathbb{Z}^2$ such that $\xi_n(\cdot + z_n) \rightarrow \xi \in \mathcal{K}_0$ and $\xi_n(\cdot + z_n) \rightarrow \xi$ a.e. on \mathbb{R}^2 . Due to Fatou's Lemma we have

$$\lim_{n \rightarrow \infty} J_0(\xi_n) = \lim_{n \rightarrow \infty} \left(J_0(\xi_n) - \frac{1}{2} J'_0(\xi_n)(\xi_n) \right) \geq J_0(\xi) - \frac{1}{2} J'_0(\xi)(\xi) = J_0(\xi),$$

$J_0(\xi) = c_0$ and we conclude. If, in addition, f is odd in u , then in view of Lemma 6.4 and Theorem 3.2 there is an infinite sequence of \mathbb{Z}^2 -distinct solutions. \square

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Declarations

Conflicts of interest The authors declare that they have no conflict of interests.

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