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Rudolph Clausius (1822–1888) and His Concept of Mathematical Physics

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A contribution on the occasion of Rudolph Clausius' 200th anniversary

Rudolph Clausius is well known as a pioneer of the mechanical theory of heat (1857) and as the creator of the concept of entropy (1865). Oftentimes, he is also called the discoverer of the second law of thermodynamics although some argue that this law was already established by Sadi Carnot in 1824 (while still based on the caloric theory). But beyond any doubt, it was Clausius who gave in 1850 the first mathematically correct formulation of the first law (in its differential form that is still valid today, $dQ = dU + pdV$) and a particularly stringent exposition of both the necessity and independence of the two laws, indeed a logical masterpiece. This paper focuses on his concept of mathematical physics for the development of theoretical physics, contributions that have changed physics well beyond the field of thermodynamics.

1. Introduction

Max Planck (1858–1947), who often stated that Clausius's works were particularly important for his scientific development, wrote in 1921 in his comments to the reprint of Clausius' famous article of 1850 "On the moving force of heat, and the laws which may be derived from them for heat itself" as Ostwald's "Classic of the Exact Sciences" Nr. 99: "May the form of the presentation appear emendable at places, the content of the theorems presented in this work rightfully remains valid today."[1] And already in 1887, Planck had written in his prize-winning essay *The Principle of the Conservation of Energy* for the Göttingen Academy: "One should date the period, in which the mechanical theory of heat gained preponderance, with this epoch-making treatise."[2]

Rudolph Clausius was born on January 2, 1822, in Köslin (today Koszalin, Poland), as son of a schoolteacher and protestant reverend. After his university studies in Berlin (1840–1844), he gained his Doctorate in Physics from the University of Halle in 1848.[3,4] Between 1843 and 1855, he worked first for three and a half years as a house teacher in a wealthy Jewish family (Louis Riess) in Berlin, after 1845 also at the Friedrichs-Werdersche Gymnasium, and after 1850 at the Vereinigte Artillerie- und Ingenieursschule in Berlin,[5] before he became a professor at the newly founded Eidgenössische Polytechnikum in Zürich, where he stayed for 12 years (1855–1867), see also **Figure 1**.[6] After only 2 years in Würzburg (1867–1869),[7] he moved to the University of Bonn,[8] where he stayed until his death in 1888.[9] Clausius, who preferred to publish his scientific work here in the *Annalen* (indeed, his bibliography lists over 70 papers in the *Annalen*),[9] was of a generation who pioneered the new discipline of theoretical physics in the 19th century. In particular, it is important to stress that Clausius, through his entire life, always used the expression "mathematical physics" to describe his concept. While this wording was also used by Franz Neumann (1798–1895), Wilhelm Eduard Weber (1804–1891) and Gustav Robert Kirchhoff (1824–1887), it was Clausius who fully developed his concept after 1850 in the fields of thermodynamics and potential theory, and after 1868 also in electrodynamics.

2. Development of His Concept of Mathematical Physics

Clausius' stated this concept already in his first works on atmospheric light scattering (1847–1848) and on elasticity (1849): the verification of physical hypotheses using mathematically derived predictions, by means of which the validity of the underlying assumptions can be validated. In this approach, mathematics plays the central role, of course: it is the mathematical formalism that enables us to formulate the hypotheses in form of equations, such that (quantitative) predictions can be made, which would then be compared with experimental data (observations). Thus, Clausius already wrote in his very first article of 1847 on the brightness of the sky:[10] "It is therefore necessary that, by mathematical considerations, one obtains a clear preliminary picture

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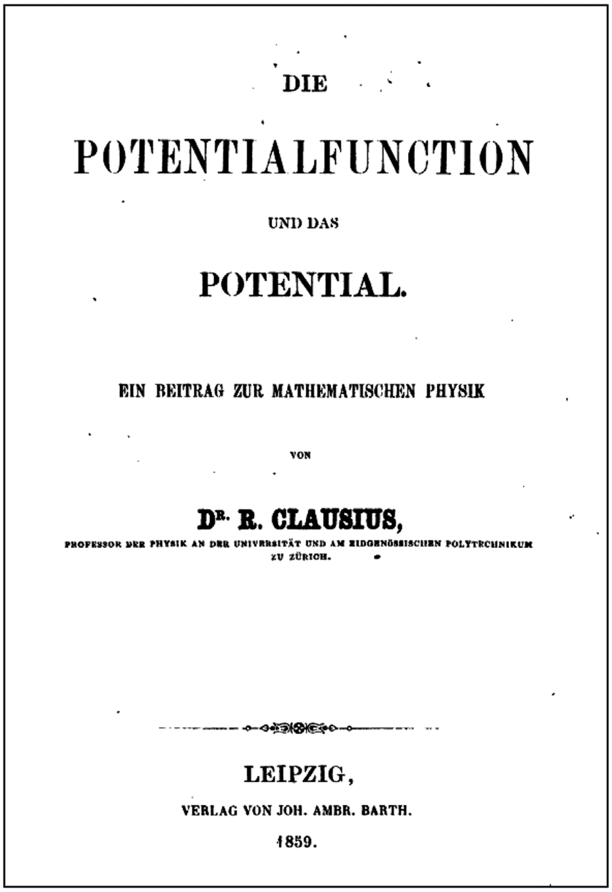
of what amount of light intensity and distribution we can expect" and he continued "... thus a few well-selected and accurately performed experiments may suffice, if they support the hypotheses underlying the calculation, or else demonstrate their inadmissibility, which is just as useful." It is interesting to note that Clausius published his first studies in a mathematical journal (*Crelle's Journal für die reine und angewandte Mathematik*), and he also stated in two letters from 1847 and 1851 that his university teacher Peter Gustav Lejeune Dirichlet (1805–1859) had read his manuscripts before submission.[11,12] In a shorter version of this study, which was also published in 1847 (in what was then *Poggendorff's Annalen der Physik und Chemie*), he wrote:[13] "... it is just the consequent realization of any hypothesis together with all its implications, which may best lead to results that can be compared to reality, and thus can contribute to the confirmation or refutation of the hypothesis itself." A very similar argumentation is found in Clausius' next article, this time on the theory of elasticity (1849), when he discusses observed discrepancies between the well-established equations and very recent measurements:[14] "... first of all, however, it must be investigated whether the assumptions, that have been made to derive the equations, are also necessary for them. And that, I believe, is indeed not the case." The molecular rotations postulated by Clausius in this study on the "*mathematical* [emphasis added] treatment of elasticity"[14] would later be confirmed by his Gustav Wiedemann (1826–1899) through precise experiments. And in another paper on the origin of the blue color of the sky, which he also published in 1849, he writes:[15] "... however I find this theory nowhere developed *mathematically* [emphasis added] ... whereas another kind of explanation ... seems to be accepted." It is therefore no surprise that Clausius argues along the same lines in his famous article of 1850,[16] when he discusses the apparent contradictions between Carnot's and Joule's theories—a logical masterpiece that has been the subject of dedicated studies,[17] where he writes: "I believe, nevertheless, that we ought not to suffer ourselves to be daunted by these difficulties; but that, on the contrary, we must look steadfastly into this theory which calls heat a motion, as in this way alone we can arrive at the means of establishing it or refuting it."[18]

In fact, Clausius had already done detailed mathematical calculations on vibrating and rotating molecules, as early as 1849 (the relevant notes are found in his scientific papers kept at the Deutsche Museum in Munich[3]), and also how the observed deviations from the ideal gas law (combined laws of Gay-Lussac and Boyle-Mariotte), that had become obvious in 1847 by the precise measurements of Henri-Victor Regnault (1810–1878) and Heinrich Gustav Magnus (1802–1870), may reveal information on molecular properties like their size and distance. Repeatedly, Clausius pointed to these early considerations and calculations, for example, in 1857 in his ground-breaking work "On the kind of motion we call heat"[19] (where again, he also underlines that these are "*mathematical* [emphasis added] considerations"), or in the second edition of his book *The Mechanical Theory of Heat*.[20] His concept of mathematical physics went well beyond the field of thermodynamics and became a great success.[21]

Even in the first years, Clausius did not keep his program a secret. Hermann von Helmholtz (1821–1894) recalled in his obituary of Clausius for the Berlin Physical Society in 1889:[22] "I saw him almost daily in the winter of 1848–49, since we used to dine together with G. Wiedemann in the same restaurant. From the very beginning, he was decidedly inclined towards *mathematical physics* [emphasis added]". Helmholtz also wrote in 1854—on the occasion of a public dispute with Clausius,[23] who had pointed out mathematical errors[24] in Helmholtz' work of 1847 *On the Conservation of Force*—to his friend Carl Ludwig (1816–1895): "Since the non-mathematical physicists know that I am not a trained mathematician, they will believe Clausius when he accuses me in such a way of the gravest blunders."[25] Clausius' method was already well accepted, even by the elder generation of physicists in Berlin: not only could he publish his first "mathematical physics" papers also in *Poggendorff's Annalen* already in 1847, but he also apparently succeeded to convince his mentor Gustav Magnus, who always advocated a very strict separation of experimental and mathematical physics.[22] Helmholtz himself turned toward mathematical physics after 1854: he published a first mathematical paper of his own (in *Crelle's Journal*!)

Figure 1. Rudolph Clausius around 1860 in Zurich (where he had married in 1859). The book that he holds in his hand is very probably the first edition of his book *Die Potentialfunction und das Potential* (Barth, Leipzig, 1859). Source: ETH Bibliothek Zürich, Bildarchiv, Public Domain Mark (Creative Commons).


DIE

POTENTIALFUNCTION

UND DAS

POTENTIAL.

EIN BEITRAG ZUR MATHEMATISCHEN PHYSIK

VON

DR. R. CLAUSIUS,

PROFESSOR DER PHYSIK AN DER UNIVERSITÄT UND AM EIDGENÖSSISCHEN POLYTECHNIKUM ZU ZÜRICH.

LEIPZIG,

VERLAG VON JOH. AMBR. BARTH.

1859.

Figure 2. Title page of Clausius' first book *Die Potentialfunction und das Potential* [*A Contribution to Mathematical Physics*] (Barth, Leipzig, 1859).

in 1857 "On the integrals of the hydrodynamical equations which correspond to the motion of vortices,"[26] in 1870 he even wrote to his friend Emil du Bois-Reymond (1818–1896) that he was tired of physiology and now interested only in "mathematical physics,"[21] and in his late lectures on the theory of heat he made extensive use of mathematical physics.[27] However, even Max Planck complained as late as 1921 in his preface to *Helmholtz' Treatises on Thermodynamics* about the many mathematical mistakes "nearly impossible to understand, in all editions made by the author himself" that he had to correct.[28]

Interestingly, even in the minutes of Clausius' habilitation in December 1850 in Berlin, we find a hint about his concept. The examiner Johann Franz Encke (1791–1865), a well-known astronomer, wrote there:[4,5] "The investigations ... touch the field of our knowledge about the inner constitution of bodies so intimately, that a definitive result at this point cannot at all, and even for some time to come, be hoped for. Therefore, it will be important with such meritorious investigations, that need to be started in *mathematical physics* [emphasis added], to see whether the underlying hypotheses can be taken for granted in the same way."

Also Clausius' first book (**Figure 2**) *Die Potentialfunction und das Potential*, published in 1859, is subtitled *Ein Beitrag zur mathematischen Physik* (*A Contribution to Mathematical Physics* [emphasis added]).[29] This textbook on potential theory, a topic widely discussed at the time,[24,30] would become a bestseller, seeing four editions in Clausius' lifetime, and containing nearly everything that even today, young physicists have to learn about potential theory, including the theorems of Green (1828) and Gauss (1839), and various applications. Strange enough, however, this book, which is the first textbook on potential theory in Germany, is never mentioned when Clausius is lauded, even though it had an enormous influence on the theoretical physics education in the second half of the 19th century. Its second edition was also translated into French (*De la fonction potentielle et du potentiel*, Gauthier-Villars, Paris, 1870), and here even subtitled *Introduction à la physique mathématique* (Introduction to *mathematical physics* [emphasis added]) (note the change of accent from "contribution" to "introduction").[31]

It is actually an interesting question, how Clausius had arrived at his concept of mathematical physics. Surely he did not learn it from his mentor Gustav Magnus, who devoted himself as a physico-chemist—in the school of Jöns Jakob Berzelius (1779–1848)—exclusively to experimental work.[32,33] However, the historian of science Stefan Wolff pointed out already in 1995, that Clausius' intense autodidactic studies of the French theoreticians (Cauchy, Fourier, Lacroix, Lamé, Laplace, Navier, Poisson), which he emphasized in his curriculum vitae occasioned by his promotion and his habilitation, as well as in some letters from the years 1847–50, must have played an important role.[4] Indeed, Clausius's meticulous excerpts from these works are all preserved at the Deutsche Museum in Munich and bear witness of his very intense studies.[3]

At least equally important for the development of his approach of mathematical physics must have been the lectures by the mathematicians Peter Gustav Lejeune-Dirichlet, Jakob Steiner (1796–1863), Martin Ohm (1792–1872), and Enno Heeren Dirksen (1788–1850), that Clausius attended in Berlin in the years 1840–1843.[3,5] Steiner even included in his unpublished papers a manuscript that Clausius must have given him already as a student in 1842 (this is actually the first known scientific contribution of Rudolph Clausius—in mathematics!): "In addition to these two solutions, Herr Stud. Clausius has found two more, and has also proven that no more are possible."[34,35] It was indeed Steiner who recommended Clausius as professor in Zürich to the Schweizerische Schulrat in 1854.[6] In the same process, also Johann Christian Poggendorff (1796–1877) recommended Clausius for the position in Zürich, as the best "candidate especially in *mathematical physics* [emphasis added]."[21] But it was certainly his teacher Dirichlet, who worked in the mid-1840s also on potential theory, who had the strongest impact on Clausius' vision of mathematical physics. Note that Dirichlet has been called "the father of mathematical physics in Germany"—and the same authors state that "amongst his direct disciples, one must name Clausius in the first place."[36] Helmholtz wrote in 1859 to Rudolf Lipschitz (1832–1903):[37] "Clausius must be a disciple of Dirichlet."[38]

Clausius scrutinized already as a student many mathematical studies on electrodynamics and magnetism, among others the work of Carl Friedrich Gauß (1777–1855) and of Wilhelm Eduard Weber (1804–1891), and in 1875–1877 he would propose his own "electrodynamical fundamental law,"[39,40] after pointing out in 1868 a fundamental mistake in the law of Bernhard Riemann (1826–1866, published after his death). Also, he lectured already as a young man in 1844 in Magnus's colloquium on (the just published German translation of) Clapeyron's article of 1834 "Sur la puissance motrice du feu," through which he became acquainted with Carnot's work from 1824.[3,4] Furthermore, the famous essay of George Green (1793–1841) on potential theory from 1828—a rare work originally printed in less than 100 copies, probably not accessible for Clausius—was published in *Crelle's Journal* in 1854.

Possibly the teachers of his high-school years in Stettin (present-day Szczecin), especially Justus Günther Grassmann (1779–1852) and Ludwig Giesebrecht (1792–1873) may have contributed to the development of Clausius' concept.[3] In Berlin, Clausius attended lectures in on philosophy (logics) by Friedrich Adolf Trendelenburg (1802–1872), on geography by Carl Ritter (1779–1859) as well as on history by Leopold von Ranke (1795–1886) and by August Böckh (1785–1867)—a very impressive spectrum of topics, given the fact that his main interest was physics.[3,5]

However, Clausius never explicitly elaborated on his concept of mathematical physics in publications or correspondence, except for the many introductory passages in his early works. A public lecture in Zürich in the year 1860 entitled "On the difference between the old and the new physics"[41] appears to be lost, in any case it was not printed—in contrast to his other public lectures. But it was also in Zürich in 1858, that Clausius started his academic lectures on "Mathematical Physics."[42]

3. Clausius and Experimental Physics

It is important to stress that, Clausius himself never published any experimental results, a fact which occasionally was held against him. For example, Robert Bunsen (1811–1899) wrote in an evaluation of Clausius on the occasion of a position to be filled in Heidelberg in 1854—it was actually offered to Gustav Robert Kirchhoff (1824–1887): "His works up to this point are of a purely *mathematical-physical* [emphasis added] nature, supported by the observations of others."[21] Nevertheless, Clausius was very well familiar with most aspects of experimental physics, all through his life. He took care himself of the experimental apparatus in Zürich (1855–1867) and in Bonn (1869–1888), where he wrote many letters to the Berlin Kultusministerium in order to improve the situation of the "Physikalische Kabinett." He was also deeply interested in—and contributed to—the unification of the physical units, including his participation in the Electrical Congress in Paris in 1881. Above all, he always used experimental observation as a foundation and touchstone for his theoretical investigations. This is also particularly true for his famous paper of 1850,[16] where he compared his theoretical calculations with Joule's measurements of the mechanical equivalent of heat—and it is in this very paper that Clausius published for the first time the second law of thermodynamics, that heat can "never pass by itself" from a cooler body to a warmer one: a well-known and apparently self-evident observation, that however in his hands turned into one of the most important principles of physics! Still, it should take him nearly 15 years to derive the correct mathematical form of this statement.[43]

It was probably Clausius's closest friend, John Tyndall, who wrote a very personal obituary in 1889 for the "Institution of Civil Engineers" in London, that ends with the beautiful words:[44] "It is true he never experimented, but his clear vision and practical understanding could not on that account overlook the importance of this branch of physical investigation. ... A theorist Clausius was, but always in touch with practice, and therein lies to a great extent the importance which his work has had in the development of the physical sciences."

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Conflict of Interest

The author declares no conflict of interest.

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