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On the Global Topology of Moduli Spaces of Riemannian Metrics with Holonomy Sp(n)

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Abstract

We discuss aspects of the global topology of moduli spaces of hyperkähler metrics. If the second Betti number is larger than 4, we show that each connected component of these moduli spaces is not contractible. Moreover, in certain cases, we show that the components are simply connected and determine the second rational homotopy group. By that, we prove that the rank of the second homotopy group is bounded from below by the number of orbits of MBM-classes in the integral cohomology.

An explicit description of the moduli space of these hyperkähler metrics in terms of Torelli theorems will be given. We also provide such a description for the moduli space of Einstein metrics on the Enriques manifold. For the Enriques manifold, we also give an example of a desingularization process similar to the Kummer construction of Ricci-flat metrics on a Kummer $K3$ surface.

We will use these theorems to provide topological statements for moduli spaces of Ricci-flat and Einstein metrics in any dimension larger than 3. For a compact simply connected manifold N we show that the moduli space of Ricci flat metrics on $N \times T^k$ splits homeomorphically into a product of the moduli space of Ricci flat metrics on N and the moduli of sectional curvature flat metrics on the torus T^k.

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Contents

If I have seen farther than others, it is because I have stood on the shoulders of giants.

Isaac Newton

If I have not seen as far as others, it is because there were giants standing on my shoulders.

Hal Abelson

0 Motivation

The purpose of this chapter is to provide an informal motivation for the work presented in this thesis. The results and an introduction of the thesis are then the content of the next chapter.

Question 1. *What is the 'best' Riemannian metric on a compact manifold?*

The question has been raised by various authors like Yau, Hopf, and Thom. A general reference for this question and some of the following material is the chapter 'Best Metric' in [18], see also [64].

One way to conceptualize Question 1 is to consider the space of Riemannian metrics $\mathcal{R}(M)$ on a compact manifold M. The best metrics are then the critical, or minimal, points of a given functional $\mathcal{F}\colon \mathcal{R}(M) \to \mathbb{R}$. Examples are

- $\mathcal{F}_{\mathrm{Curv}}(g) \coloneqq \int_M |\mathrm{R}_g|^{d/2}\mathrm{dvol}(g)$ and
- $\mathcal{F}_{\mathrm{Scal}}(g) \coloneqq \int_M \mathrm{scal}(g)\mathrm{dvol}(g)$.

Both functionals measure the total curvature of a Riemannian metric g, the first with respect to the norm of the curvature tensor R_g and the second with respect to the scalar curvature. In dimension 2 the classical Gauss-Bonnet theorem states that $\mathcal{F}_{\mathrm{Scal}}(g) = 4\pi\chi(M)$, where $\chi(M)$ is the Euler characteristic of the manifold M. In particular $\mathcal{F}_{\mathrm{Scal}}(g)$ is constant. Thus, from now on we assume the dimension to be larger than 2.

First, let us consider the functional $\mathcal{F}_{\mathrm{Curv}}(g) = \int_M |\mathrm{R}_g|^{d/2}\mathrm{dvol}(g)$ on a 4-dimensional manifold M. If the Euler characteristic $\chi(M)$ is non-negative, the generalized Gauss-Bonnet theorem (see [18, 15.7.8])

$$\frac{1}{8\pi^2}\int_M \left(|\mathrm{R}_g|^2 - |\mathrm{Ric}_g - \frac{\mathrm{scal}(g)}{4}g|^2\right)\mathrm{dvol}g(g) = \chi(M),$$

provides a lower bound $\mathcal{F}_{\mathrm{Curv}}(g) \geq \chi(M)$. Equality holds if and only if

$$\mathrm{Ric}_g - \frac{\mathrm{scal}(g)}{\dim(M)}g = 0. \tag{1}$$

This equation is famous in physics, as for Lorentzian metrics it is the field equation in general relativity of vacuum spacetime, see for instance [19, Chapter 3]. For this reason Riemannian metrics solving (1) will be called *Einstein metrics*. The scalar curvature of such a metric turns out to be constant, and $\lambda := \frac{\mathrm{scal}(g)}{\dim(M)}$ is known as the *Einstein constant*. The global minima of the functional $\mathcal{F}_{\mathrm{Curv}}$ of a 4-dimensional manifold M with $\chi(M) \geq 0$ are thus the *Einstein metrics*.

For higher dimensional manifolds the functional $\mathcal{F}_{\mathrm{Curv}}$ provides more challenges, see for instance [19, 4.H]. The total scalar curvature functional $\mathcal{F}_{\mathrm{Scal}}$ turns out to be practical in every dimension. However, in contrast to $\mathcal{F}_{\mathrm{Curv}}$ the functional $\mathcal{F}_{\mathrm{Scal}}$ never admits a global minimum [19, 4.32 Theorem] and we are thus interested in its critical points.

Let us now explain what being a critical point actually means. The space $\mathcal{R}(M)$ is an open cone in the vector space of smooth and symmetric 2-tensors $\Gamma(M, S^2TM)$ endowed with the topology of smooth convergence, see [19, 4.2]. One may think of $\mathcal{R}(M)$ as a smooth manifold where the tangent space at a point g is naturally identified with $\Gamma(M, S^2TM)$, similar to the case of an open subspace in $\mathbb{R}^n$ but now the dimensions are infinite. A *critical point g* for a functional $\mathcal{F}$ is one where all the directional derivatives are zero, i.e.

$$D_g\mathcal{F} \cdot h = \lim_{t \to 0} \frac{\mathcal{F}(g + th) - \mathcal{F}(g)}{t} = 0$$

for every $h \in \Gamma(M, S^2TM)$.

Also motivated by physics, David Hilbert was the first to compute the derivative of $\mathcal{F}_{\mathrm{Scal}}$ in 1915 [70], see also [19, 4.C]. He showed

$$D_g\mathcal{F}_{\mathrm{Scal}} \cdot h = \int_M \langle \mathrm{Ric}_g - \frac{\mathrm{scal}(g)}{2} g, -h\rangle \mathrm{dvol}(g).$$

One deduces that the critical metrics are exactly those where $\mathrm{Ric}_g = 0$. Such metrics are called *Ricci-flat*. They are Einstein metrics with Einstein constant $\lambda = 0$. In fact, if one considers the functional $\mathcal{F}_{\mathrm{Scal}}$ to be defined on metrics with fixed volume, the critical points are again Einstein metrics, compare [18, Chapter 11].

The discussion suggests that Einstein or Ricci-flat metrics are natural candidates for being 'best metrics'. This is supported by various applications, most notably with respect to the Ricci flow. For this flow Einstein metrics are the stationary points and thus play a fundamental role in Perelman's and Hamilton's solution to Thurston's geometrization conjecture, see [8].

Having a notion of best metrics, a priori seems to provide us with a natural choice of metric on every manifold. However, determining Einstein and especially Ricci-flat metrics in practice can be challenging. Let us say what some of the problems are and what makes them mysterious.

In dimensions 3 and 4 there are obstructions for a manifold to admit Einstein metrics, see for instance [9, Section 4]. In higher dimension it is an open question whether every compact manifold admits an Einstein metric. For Ricci-flat metrics on the other hand there are known obstructions, for instance every such manifold is finitely covered by a product $M \times T^k$, where M is simply connected and T^k a torus, see [50]. Aside of some results on the existence of Ricci-flat metrics, constructing explicit examples which are *not* sectional curvature flat seems out of reach. Only for some cases there are approximate solutions. Furthermore, if they exist on a manifold M, Ricci-flat metrics are usually not unique and thus choosing a 'best metric' on M requires further clarifications.

To solve some of these issues it is natural to study the space of Einstein, respectively Ricci-flat metrics, as a subset of $\mathcal{R}(M)$. However, since passing from a metric to an isomorphic metric provides no new information on the geometry, it is often more natural and interesting to consider the space of isomorphism classes of metrics. This is known as the *moduli space of Riemannian metrics*. It is defined as the quotient space

$$\mathcal{M}(M) \coloneqq \mathcal{R}(M)/\mathrm{Diff}(M),$$

with respect to the pullback action of $\mathrm{Diff}(M)$ on $\mathcal{R}(M)$. The *moduli spaces of Einstein metrics* $\mathcal{E}(M)$ and of *Ricci-flat metrics* $\mathcal{M}^{\mathrm{Ric}=0}(M)$ are the subspaces of $\mathcal{M}(M)$ which consist of isomorphism classes of Einstein, respectively Ricci-flat, metrics. Now that we have established a notion of 'best metrics' our next question is the following.

Question 2. *What do $\mathcal{E}(M)$ and $\mathcal{M}^{\mathrm{Ric}=0}(M)$ look like as topological spaces?*

To provide answers for Question 2 is challenging and the main goal of this thesis. It turns out to be fruitful to go back to Question 1 before going into more depths on Question 2.

Aside from describing best metrics on a compact manifold as minimal or critical points of a functional, one may say that a metric g is 'better' than another metric g' if g is compatible with more structures than g'. Here, by structure we mean any quantity which is given by a globally defined tensor s, like certain forms, endomorphisms etc. We call s compatible with a metric g if $\nabla s = 0$, where ∇ denotes the Levi-Civita connection of g. A tensor s for which $\nabla s = 0$ is often called parallel or constant.

A standard example of a compatible structure can be found on a *Kähler manifold* X. Such a manifold has an atlas of holomorphic charts, called *complex structure*, and is endowed with a Riemannian metric g so that the complex structure is compatible with the metric. The complex structure can be viewed as a globally defined $(1,1)$-tensor. Indeed, since X is a complex manifold, each tangent space T_pX is a complex vector space and multiplication by i induces an $\mathbb{R}$-linear endomorphism I on T_pX. Such an endomorphism is a tensor of type $(1,1)$. If X is Kähler, this is an isometry with respect to g. One can prove that $\nabla I = 0$ in this case. On the other hand, by a corollary [108, Theorem 5.5] of the Newlander-Nirenberg theorem, any parallel endomorphism I of the tangent bundle with $I^2 = -Id$ is induced by a complex structure. Thus, to endow a smooth Riemannian manifold (M, g) with a Kähler structure, one needs to find a certain parallel tensor on M. We often identify the complex structure with the induced endomorphism I and view a Kähler manifold as a triple $X = (M, g, I)$. The additional structures on Kähler manifolds have led to a rich theory that extends the standard theory of Riemannian geometry.

To every Riemannian manifold (M, g) there is an associated group which determines the parallel tensors. The group is known as the *holonomy group* and is denoted $\mathrm{Hol}(M,g)$. It is the group of parallel transports on a tangent space T_pM induced by loops at a point $p \in M$. It acts naturally on T_pM and, with respect to a chosen isomorphism $T_pM \cong \mathbb{R}^n$, the group can be viewed as a subgroup of $\mathrm{O}(n)$ with its natural representation on $\mathbb{R}^n$. For the induced representation of $\mathrm{Hol}(g)$ on tensors $\bigotimes_{k=1}^r \mathbb{R}^n \otimes \bigotimes_{l=1}^s (\mathbb{R}^n)^*$ one finds that invariant elements of this representation are in a 1-to-1 correspondence with globally defined parallel tensors on M. This is known as the *holonomy principle*, see [85, Proposition 2.5.2]. In terms of this principle, the smaller $\mathrm{Hol}(g)$ is, the more parallel tensors exist on (M,g). For Question 1 we may thus say that g is 'better' than g' if $\mathrm{Hol}(g) \subset \mathrm{Hol}(g')$.

A priori, there could be a lot of possible subgroups of $\mathrm{O}(n)$ which can arise as holonomy groups. Surprisingly, this is not the case. For simply connected manifolds which are neither a product nor a symmetric space, the list of possible holonomy groups is the following by a fundamental theorem of Berger [19, 10.92].

Berger's List of Holonomy Groups

Holonomy Group	dim	Associated Structure	Comment
$\mathrm{SO}(n)$	n	Generic Case	Orientable
$\mathrm{U}(n)$	$2n$	Kähler Manifold	Kähler
$\mathrm{SU}(n)$	$2n$	Calabi-Yau	Ricci-Flat-Kähler
$\mathrm{Sp}(n)$	$4n$	Hyperkähler	Ricci-Flat-Hyperkähler
$\mathrm{Sp}(1) \cdot \mathrm{Sp}(n)$	$4n$	Quaternionic	Kähler-Einstein
G_2	7	G_2-manifold	Ricci-flat
$\mathrm{Spin}(7)$	8	$\mathrm{Spin}(7)$-manifold	Ricci-flat

For a compact and simply connected n-dimensional Riemannian manifold (M, g), having holonomy other than $\mathrm{SO}(n)$ and $\mathrm{U}(\frac{n}{2})$ imposes strong curvature constrains on the metric. Note that for $\mathrm{Hol}(g) = \mathrm{U}(\frac{n}{2})$ the dimension needs to be even. For the next cases we assume that $n = 4k$. The quaternionic case $\mathrm{Sp}(1) \cdot \mathrm{Sp}(k)$ is Einstein but never Ricci-flat and the cases G_2 and $\mathrm{Spin}(7)$ imply Ricci-flatness but are restricted to dimension 7 and 8, see [85] for these metrics.

We are mostly interested in metrics with holonomy $\mathrm{Sp}(k)$. In our heuristic they would be considered best metrics, since one has the inclusions

$$\mathrm{Sp}\,(k) \subset \mathrm{SU}\,(2k) \subset \mathrm{U}\,(2k) \subset \mathrm{SO}(4k).$$

A metric with $\mathrm{Hol}(g) = \mathrm{Sp}(k)$ is Ricci-flat. This condition already holds for metrics with $\mathrm{Hol}(g) \subset \mathrm{SU}(2k)$, see [108, Theorem 11.5]. A Riemannian metric with holonomy $\mathrm{Sp}(k)$ is said to be *hyperkähler*. The name comes from the fact that there are three compatible complex structures I, J, K with respect to g so that

$$I^2 = J^2 = K^2 = IJK = -Id.$$

In particular, (M, g) is Kähler in at least three different ways. The complex structures I, J, K are furthermore special in the sense that each admits a *unique holomorphic symplectic form* up to some constant in $\mathbb{C}^*$. For the complex structure I this form is given by

$$g(J\cdot, \cdot) + ig(K\cdot, \cdot).$$

Simply connected Kähler manifolds with such a holomorphic form are known as *irreducible holomorphic symplectic manifolds* (IHSM). They serve as the algebro geometric counterpart of hyperkähler manifolds. Finding examples of hyperkähler manifolds often results in providing examples of irreducible holomorphic symplectic manifolds. The most basic example of a manifold admitting a hyperkähler metric is the $K3$-*manifold* M. The corresponding complex structures modeled on M are known as $K3$-*surfaces*. These are the complex 2-dimensional versions of IHS-manifolds. $K3$-surfaces have been a popular research topic for many years, not just as examples of hyperkähler manifolds but also in many other areas. For instance, they played a central role in the solutions to the Weil conjectures as well as in mirror symmetry, see [39], [83], respectively [14, 13, 12].

Hyperkähler metrics, just like Ricci-flat metrics, often come in families. We are thus interested in the *space of hyperkähler metrics* $\mathcal{R}^{\mathrm{HK}}(M)$ and in particular in the *moduli space of hyperkähler metrics*

$$\mathcal{M}^{\mathrm{HK}}(M) \coloneqq \mathcal{R}^{\mathrm{HK}}(M)/\mathrm{Diff}(M),$$

as the hyperkähler condition is preserved under isomorphisms.

The two notions of 'best metric', i.e. the one given by critical points of the functionals $\mathcal{F}_{\mathrm{Scal}}$ or $\mathcal{F}_{\mathrm{Curv}}$ in $\dim = 4$, and the one provided by the condition $\mathrm{Hol}(g) \subset \mathrm{Hol}(g')$, now neatly come together. Since hyperkähler metrics are Ricci-flat there are natural inclusions

$$\mathcal{M}^{\mathrm{HK}}(M) \subset \mathcal{M}^{\mathrm{Ric}=0}(M) \subset \mathcal{E}(M). \tag{2}$$

In general it is not clear when these inclusions are strict, in fact, it is a famous open question whether there are any Ricci-flat metrics with holonomy $\mathrm{SO}(n)$. But in dimension 4, where the $K3$-manifold is the only manifold that admits hyperkähler metrics, it is known by a result of Hitchin [72] that

$$\mathcal{M}^{\mathrm{HK}}(K3) = \mathcal{M}^{\mathrm{Ric}=0}(K3) = \mathcal{E}(K3).$$

For higher dimensional manifolds the inclusions (2) are simply given by possibly adding connected components, see Lemma 7.1.1.

Like in Question 2 we ask the following.

Question 3. *What does $\mathcal{M}^{\mathrm{HK}}(M)$ look like as a topological space?*

In this thesis we are mainly concerned with partially giving answers to Question 2 and 3. In the next section we put this into a wider context.

Le but de cette thèse est de munir son auteur
du titre de Docteur.

Adrien Douady

1 Introduction

The main goal of this thesis is to compute homotopy groups of the moduli space of hyperkähler metrics on a hyperkählerian manifold. A *hyperkähler manifold* here means a $4n$-dimensional compact Riemannian manifold (M, g) with holonomy $\mathrm{Hol}(g) = \mathrm{Sp}(n)$, while by a *hyperkählerian manifold* we refer to a differentiable manifold which can be endowed with a hyperkähler metric. The *moduli* *space of hyperkähler metrics* is defined as the quotient space

$$\mathcal{M}^{\mathrm{HK}}(M) := \{\ g \in \mathcal{R}(M) \mid g \text{ is a hyperkähler metric } \}/\mathrm{Diff}(M),$$

where $\mathcal{R}(M)$ is the *space of Riemannian metrics* on which the group of diffeomorphisms $\mathrm{Diff}(M)$ acts by pullback. We endow it with the topology of smooth convergence.

1.1 Backgrounds and State of the Art

We begin by introducing backgrounds on the space $\mathcal{M}^{\mathrm{HK}}(M)$ and put it into context with other results on moduli spaces with Ricci curvature constraints.

The space of all Riemannian metrics $\mathcal{R}(M)$ is an infinite dimensional convex cone inside the space of smooth symmetric 2-tensors [122]. The moduli space $\mathcal{M}^{\mathrm{HK}}(M)$ on the other hand is significantly smaller, more precisely, it is known to be an orbifold of dimension $3(\mathrm{b}_2(M) - 3) + 1$, see for instance [19, 12.88 Theorem] respectively [19, 12.98 Proposition].

The space can be encountered in mathematical physics, most notably in the context of Mirror Symmetry, see [80, 131, 136] and for $K3$-surfaces [12, 13], [14, Chapter VIII.22]. In general the moduli space $\mathcal{M}^{\mathrm{HK}}(M)$ provides a good framework for the question: How many hyperkähler metrics does a hyperkählerian manifold M admit? Determining its homotopy groups then provides insights to the relaxed question: How many hyperkähler metrics does M admit up to deformations of these metrics in $\mathcal{M}^{\mathrm{HK}}(M)$?

This point of view naturally lends itself to an active research program, which considers questions on the topology of moduli spaces of Riemannian metrics with various curvature constraints, like positive scalar curvature, non-negative Ricci curvature, zero sectional curvature, Einstein. For an introduction see [130, 128]. The following works are concerned with non-negative and positive Ricci curved metrics [94, 40, 41, 140, 58, 139, 129, 25, 26]. For negative Ricci curvature see [97]. Theorems on the moduli space of sectional curvature flat metrics can be found in [129, 54]. For other spaces of Riemannian metrics we refer to [129] and the references there in.

Since the Ricci curvature of hyperkähler manifolds vanishes, we are naturally interested in the moduli space of Ricci-flat metrics $\mathcal{M}^{\mathrm{Ric}=0}(M)$, respectively the moduli space of Einstein metrics $\mathcal{E}(M)$. In particular, for the moduli space of Einstein metrics $\mathcal{E}(M)$ there has been an intensive research interest for many years, see for instance the survey [9] and [19, Chapter 12]. See also [124, 115, 116] for interesting results on compactifications of the moduli space of hyperkähler metrics on the $K3$-manifold.

However, insights into their homotopy groups remain scarce. For Einstein metrics with positive Einstein constant there are examples of metrics that are isolated points in $\mathcal{E}(M)$, for instance, the standard metric on S^4 [19, Chapter 12.H]. It is unknown if there are also isolated Ricci-flat metrics. In fact, even the basic question whether there exists a compact and simply connected manifold M for which the components $\mathcal{M}^{\mathrm{HK}}(M)$ and $\mathcal{M}^{\mathrm{Ric}=0}(M)$ of $\mathcal{E}(M)$ are contractible or not seems not to be answered in the literature.

The only results in that direction we are aware of are by Giansiracusa on the Nielsen Realization Problem [55, 56] and the related work by Giansiracusa, Kupers and Tshishiku [57] from 2021. They contain results on the (co)-homology of the Teichmüller space of Einstein metrics $\mathcal{T}^{\mathrm{Met}}(K3)$ of the $K3$-manifold as well as a non-vanishing result of the 4th-Betti number for the related homotopy moduli space. Their results are based on the following classical theorem [19, 12.K], [125, 99].

Theorem 1.1.1 (Metric-Torelli-Type-Theorem)**.** *The moduli space of unit volume Einstein metrics on the $K3$-manifold is homeomorphic to an open and dense subspace of*

$$\Gamma \backslash \mathrm{O}(3,19) / (\mathrm{O}(3) \times \mathrm{O}(19)),$$

where Γ is a discrete subgroup of $\mathrm{O}(3,19)$.

In fact, by a result of Hitchin [72], Einstein metrics on the $K3$-manifold are the sames as hyperkähler metrics. In a general setting, understanding hyperkähler metrics seems to be a more feasible task than understanding the general case of Einstein metrics. One major reason for this is that hyperkähler manifolds can be studied using algebraic geometry. The algebro geometric analogues are known as irreducible holomorphic symplectic manifolds (IHSM), which in the case of the $K3$-manifold are just the $K3$-surfaces.

Theorem 1.1.1 is based on Torelli theorems for $K3$-surfaces which have been generalized to IHS-manifolds by Huybrechts, Markman and Verbitsky, see [82, 132, 135, 42]. We refer to theorems which are proven by Torelli theorems, like

Theorem 1.1.1, as metric Torelli theorems to distinguish them from original Torelli theorems which are concerned with questions on complex structures.

Using these Torelli theorems Amerik and Verbitsky [2] gave a partial generalization of Theorem 1.1.1 to higher dimensional hyperkähler manifolds, see also the recent work by Looijenga [100] and Section 1.3. There are also metric Torelli type theorems for non-compact hyperkähler manifolds, more precisely for so called gravitational instantons [95, 34, 35, 37, 36]. However, in this work we focus on the compact case.

The aim of this work is to answer some of the open question stated above. More precisely, to provide insights on the global topology of $\mathcal{M}^{\mathrm{HK}}(M)$, $\mathcal{M}^{\mathrm{Ric}=0}(M)$, and $\mathcal{E}(M)$ in terms of their homotopy groups and metric Torelli theorems.

1.2 Homotopy Groups of Moduli Spaces of Hyperkähler Metrics

We now turn to a discussion of the main results of this thesis.

Concerning contractibility of the moduli space of hyperkähler metrics we prove the following.

Theorem A. *Let M be a hyperkählerian manifold with $\mathrm{b}_2(M) > 4$ and let $\mathcal{M}^{\mathrm{HK}}_o(M)$ be a connected component of the moduli space of hyperkähler metrics $\mathcal{M}^{\mathrm{HK}}(M)$. Then $\mathcal{M}^{\mathrm{HK}}_o(M)$ is not contractible.*

It is an open question if a hyperkählerian manifold N with second Betti number $\mathrm{b}_2(N) = 3$ or $\mathrm{b}_2(N) = 4$ exists. The discussion below shows that when $\mathrm{b}_2(N) = 3$ the space $\mathcal{M}^{\mathrm{HK}}(N)$ would be a finite union of points.

For a general hyperkählerian manifold M we cannot say anything about the number of connected components of $\mathcal{M}^{\mathrm{HK}}(M)$ other than that they are finite, by a result of Huybrechts [78]. In case of the $K3$-manifold however, it is known that this space is connected. Here we obtain a result on the full moduli space of Einstein metrics.

Theorem B. *Let M be the $K3$-manifold. Then the moduli space of Einstein metrics $\mathcal{E}(M)$ is simply connected and the second rational homotopy group is*

$$\pi_2(\mathcal{E}(M)) \otimes \mathbb{Q} \cong \mathrm{H}^2(\Gamma, \mathbb{Q}) \oplus \mathbb{Q},$$

where Γ is an arithmetic subgroup of $\mathrm{O}(3,19)$, given by the automorphism group of the lattice $\mathrm{H}^2(M,\mathbb{Z})$ with its cup-pairing.

Higher dimensional examples of hyperkählerian manifolds are difficult to construct. In each dimension $4n$ there are only two known examples [15], except for two further examples in dimensions 12 and 20, see [117] and [1]. One of the two families is given by constructing so called Hilbert schemes or Douady spaces of length n defined over complex $K3$-surfaces. These are irreducible holomorphic symplectic manifolds and they exist in each dimension $4n$. Their Betti number is $\mathrm{b}_2 = 23$ for $n \geq 2$ and for $n = 1$ the underlying manifold is just the $K3$-manifold which has $\mathrm{b}_2 = 22$.

Theorem C. *Let M be the underlying manifold of a Douady space of length n, denoted $X^{[n]}$, on a $K3$-surface X. Then the connected component of $\mathcal{M}^{\mathrm{HK}}(M)$ which contains a metric which is Kähler with respect to $X^{[n]}$ is simply connected.*

Moreover, the rank of the second homotopy group can be bounded from below. For $X^{[n]}$ the rank is at least 1, for $X^{[2]}$ at least 3 and for $X^{[3]}$ at least 5.

Since the inclusions $\mathcal{M}^{\mathrm{HK}}(M) \subset \mathcal{M}^{\mathrm{Ric}=0}(M) \subset \mathcal{E}(M)$ are given by adding connected components (Lemma 7.1.1), the above results on $\mathcal{M}^{\mathrm{HK}}(M)$ also provide information on the topology of the moduli spaces $\mathcal{M}^{\mathrm{Ric}=0}(M)$ and $\mathcal{E}(M)$.

By considering products with tori T^k we can use these results to provide further statements on moduli spaces of Ricci-flat metrics in every dimension larger than 3. For instance, inspired by the work of Tuschmann and Wiemeler [129] on the moduli space of non-negative Ricci curvature metrics $\mathcal{M}^{\mathrm{Ric}\geq 0}(N \times T^k)$, we will prove the following.

Theorem D. *Let N be a simply connected compact manifold admitting a Ricci flat metric and let T^k be the k-dimensional torus. Then there is a homeomorphism*

$$\mathcal{M}^{\mathrm{Ric}=0}(N \times T^k) \cong \mathcal{M}^{\mathrm{Ric}=0}(N) \times \mathcal{M}^{\sec=0}(T^k),$$

where $\mathcal{M}^{\sec=0}(T^k)$ is the moduli space of sectional curvature flat metrics on T^k.

If N is hyperkählerian we can combine our work with the result [129, Proposition 5.5] on the rational homotopy groups of the space $\mathcal{M}^{\sec=0}(T^k)$. Moreover, for hyperkählerian N we will see that

$$\mathcal{M}^{\mathrm{Ric}\geq 0}(N \times T^k) = \mathcal{M}^{\mathrm{Ric}=0}(N \times T^k).$$

1.3 Metric Torelli Theorems

In this section we turn to the discussion on generalizations of Theorem 1.1.1.

The second cohomology group $\mathrm{H}^2(M, \mathbb{Z})$ of a hyperkählerian manifold M is torsion free. Moreover, $\mathrm{H}^2(M, \mathbb{R})$ is naturally endowed with a non-degenerate bilinear pairing q_M of signature $(3, \mathrm{b}_2(M) - 3)$. When restricted to integral classes $\mathrm{H}^2(M, \mathbb{Z})$ the pairing q_M is integer valued and is known as the Beauville-Bogomolov form.

Consider the Grassmann space of positive definite 3-dimensional linear subspaces in $\mathrm{H}^2(M, \mathbb{R})$ and denote it by $\mathrm{Gr}^+(3, \mathrm{H}^2(M, \mathbb{R}))$. The metric period map $\mathcal{P}^{\mathrm{Met}}$ takes this as its target space by associating to a hyperkähler metric g the 3-dimensional space

$$\mathcal{P}^{\mathrm{Met}}(g) := \mathrm{span}\{[\omega_I], [\omega_J], [\omega_K]\} \in \mathrm{Gr}^+(3, \mathrm{H}^2(M, \mathbb{R}))$$

spanned by the Kähler classes $[\omega_I], [\omega_J], [\omega_K]$ associated to a hyperkähler triple (I, J, K) with respect to g.

From now on we restrict to the case of metrics with unit volume. This does not change the homotopy type of the moduli spaces.

The metric Teichmüller space is defined as

$$\mathcal{T}^{\mathrm{Met}}(M) := \{\ g \in \mathcal{R}^{\mathrm{HK}}(M) \mid g \text{ unit volume }\}/\mathrm{Diff}_0(M),$$

where $\mathrm{Diff}_0(M)$ is the group of diffeomorphisms isotopic to the identity. We will show that $\mathcal{P}^{\mathrm{Met}}$ induces an injection on each connected component of the Teichmüller space $\mathcal{T}_o^{\mathrm{Met}}(M)$. Furthermore, we will explicitly determine the image of this map while giving a detailed proof of the following theorem by Amerik and Verbitsky, which already appeared in 2015 [2, Theorem 4.9], see also [100].

Theorem 1.3.1. *Let $\mathcal{T}_o^{\mathrm{Met}}(M)$ be a connected component of the metric Teichmüller space. Then there is a subset $S_0 \subset \mathrm{H}^2(M,\mathbb{Z})$ so that the metric Period map induces a homeomorphism*

$$\mathcal{T}_o^{\mathrm{Met}}(M) \cong \mathrm{Gr}^+(3, \mathrm{H}^2(M,\mathbb{R})) - \bigcup_{z\in S_o} \mathrm{Gr}^+(3, z^\perp).$$

Here $z^\perp$ denotes the orthogonal complement of z in $\mathrm{H}^2(M,\mathbb{R})$ with respect to q_M. The set S_0 is the set of so called MBM-classes. Roughly speaking these classes are induced by 'minimal' rational curves which determine the Kähler cone of an irreducible holomorphic symplectic manifold (IHSM), more details will be given later.

For a connected component $\mathcal{M}_o^{\mathrm{HK}}(M)$ of the moduli space of unit volume hyperkähler metrics we will then obtain the following corollary.

Corollary A. *For the connected component $\mathcal{M}_o^{\mathrm{HK}}(M)$ there is a discrete subgroup $\Gamma \subset \mathrm{O}(3, \mathrm{b}_2(M) - 3)$ so that*

$$\mathcal{M}_o^{\mathrm{HK}}(M) \cong {}_{\Gamma} \setminus \Big(\mathrm{Gr}^+(3, \mathrm{H}^2(M,\mathbb{R})) - \textstyle\bigcup_{z\in S_o} \mathrm{Gr}^+(3, z^\perp)\Big). \tag{1.1}$$

The group Γ is determined by an IHS*-structure compatible with a metric in*

$\mathcal{M}_o^{\mathrm{HK}}(M)$.

We refer to Definition 4.4.4 respectively Theorem 4.4.2 for the definition of the group Γ and to Theorem 5.4.1 for a more detailed version of Corollary A. The corollary yields that $\mathcal{M}_o^{\mathrm{HK}}(M)$ is homeomorphic to an open and dense subset of the bi-quotient

$$\Gamma \backslash \mathrm{O}(3, \mathrm{b}_2(M) - 3) / (\mathrm{O}(3) \times \mathrm{O}(\mathrm{b}_2(M) - 3)). \tag{1.2}$$

These descriptions of the moduli space are our base for analyzing the global topology of the moduli space of hyperkähler metrics. Corollary A, respectively equation (1.2) generalize Theorem 1.1.1.

If M is the $K3$-manifold one can say even more. Points in (1.2) which are not associated to a smooth Einstein metric are known to correspond naturally to certain Ricci-flat orbifold metrics. Moreover, the space (1.2) is isomorphic to the completion $\overline{\mathcal{E}(K3)}^{L^2}$ of $\mathcal{E}(K3)$ with respect to the L^2-metric, a naturally defined metric on $\mathcal{E}(K3)$, see [90] and [7].

Theorem E. *The moduli space of unit volume Einstein metrics, including orbifold metrics, $\overline{\mathcal{E}(K3)}^{L^2}$ is simply connected and $\mathrm{b}_4\left(\overline{\mathcal{E}(K3)}^{L^2}\right)$ is non-zero.*

Closely related to the $K3$-manifold is the Enriques manifold S. It is a $\mathbb{Z}_2$ quotient of the $K3$-manifold and is known to admit Ricci-flat metrics. While these metrics are never hyperkähler, we will prove that every Einstein metric g on S is locally hyperkähler, and thus that g is Ricci-flat. Furthermore, we show that g is Kähler with respect to a unique complex structure, see Lemma 7.3.4.

We will use this fact and Torelli theorems for $K3$-surfaces to provide an explicit description of the moduli space $\mathcal{E}(S)$ similar to the one described above for hyperkähler metrics. This is closely related to the moduli space of Enriques surfaces $\mathcal{M}_{\mathrm{Enr}}$, see [110, 74, 14]. See [62] for a result on the topology of $\mathcal{M}_{\mathrm{Enr}}$.

Moreover, we will show that the famous Kummer construction, which approximates Ricci-flat metrics on a Kummer $K3$-surface by gluing Eguchi-Hanson spaces with a singular flat $K3$-surface, also works for a related Enriques surface.

1.4 Structure

Chapter 2 is a preliminary chapter where we collect statements we use throughout the text. First, we discuss Riemannian holonomy groups, for which we also recommend [85, 19]. We provide backgrounds on Grassmann spaces as they appear in equation (1.1). Moreover, in Section 2.4 we discuss lattices in the context of an abelian group endowed with an integral bilinear pairing as they naturally appear on $\mathrm{H}^2(M, \mathbb{Z})$ for hyperkählerian manifolds.

What we have not included are backgrounds in complex algebraic and Kähler geometry. Here we generally refer to books like [60, 75, 108, 85, 14]

In Chapter 3 we give an introduction to the theory of hyperkähler manifolds. We start this chapter by introducing the group $\mathrm{Sp}(n)$. We introduce hyperkähler manifolds and the closely irreducible holomorphic symplectic manifolds (IHSM) and state fundamental topological properties. In the remaining sections we are concerned with examples. Moreover, in the case of the $K3$-surface we discuss how Hodge structures can be used to study IHS-structures and state some of the related Torelli theorems for $K3$-surfaces. Standard references for this chapter are [65, 76, 77, 19, 85, 83].

The discussion on Hodge structures will be generalized and described in more detail in Chapter 4 for IHS-manifolds. In this chapter we also introduce and discuss the complex period map which is defined on the Teichmüller space of IHS-structures.

Based on the results on Teichmüller spaces and period maps, Theorem 1.3.1 and Corollary A will be proven in Chapter 5.

In Chapter 6 we will discuss the global topology of moduli spaces of hyperkähler metrics proving Theorems A, B, C while also providing more detailed versions of them.

Chapter 7 is about generalizing the results from hyperkählerian manifolds to other types of manifolds. Here we consider Ricci-flat manifolds and their moduli spaces and prove Theorem D. For the Enriques manifold we state a metric Torelli type theorem and discuss a desingularization process of a Ricci-flat orbifold metric. In this context we will also prove Theorem E.

Oh, he seems like an okay person, except for being a little strange in some ways. All day he sits at his desk and scribbles, scribbles, scribbles. Then, at the end of the day, he takes the sheets of paper he's scribbled on, scrunches them all up, and throws them in the trash can.

John von Neumann's housekeeper

2

Preliminaries

We recall basic facts and discuss some preliminary results which will be used throughout the text. In the first section the notion of holonomy for Riemannian manifolds is introduced. Here we discuss the holonomy principle and Berger's classification Theorem. The second section is about vector spaces endowed with a non-degenerate bilinear pairing (V, q). We will then consider Grassmann spaces in the third section. In the last section we discuss lattice theory, the integral version of (V, q).

2.1 Riemannian Holonomy Groups

Consider a connected Riemannian manifold (M, g) and its Levi-Civita connection ∇. For a smooth path $\gamma\colon I \to M$ on a closed interval I there exists the notion of *parallel transport* which we now recall. Let $a, b \in I$ be the boundary points of I and fix a vector v in the tangent space $T_{\gamma(a)}M$. The Picard-Lindelöf theorem ensures the existence and uniqueness of a vector field V along γ so that the following is true

- $V(a) = v$ and
- $\nabla_{\frac{d}{dt}\gamma(t)} V(t) = 0$ for all $t \in I$.

The second condition says that V is parallel. The parallel transport of $v = V(a)$ along γ is $\mathrm{P}_\gamma^{a,b}(v) \coloneqq V(b) \in T_{\gamma(b)}M$. Every such γ with boundary points a, b thus determines a map

$$\mathrm{P}_\gamma^{a,b}\colon T_{\gamma(a)}M \to T_{\gamma(b)}M$$

which turns out to be a linear isometry with respect to the Riemannian metric on the tangent spaces.

Instead of considering only smooth paths, the above notion readily generalizes to piecewise smooth (p.w.s) paths. Considering only loops, that is paths with the same start and end point, gives rise to the notion of holonomy.

Definition 2.1.1. *For a Riemannian manifold (M, g) the holonomy group at a* *point $p \in M$ is defined as the group of isometries induced by parallel transports along loops, i.e.*

$$\mathrm{Hol}(M,g,p) \coloneqq \left\{ \mathrm{P}^{0,1}_{\gamma} \in \mathrm{O}\left(T_pM\right) \;\middle|\; \gamma\colon [0,1] \to M \text{ p.w.s with } \gamma(0)=\gamma(1)=p \right\}.$$

The reduced holonomy group is the subgroup $\mathrm{Hol}_0(M,g,p)$ *consisting of isometries induced by contractible loops.*

The group $\mathrm{Hol}(M,g,p)$ comes with a natural representation on T_pM. By choosing an isomorphism $T_pM \cong \mathbb{R}^n$ the group $\mathrm{Hol}(M,g,p)$ also acts on euclidean space $\mathbb{R}^n$. Thus, we may identify $\mathrm{Hol}(M,g,p)$ with a subgroup of $\mathrm{O}(n)$. In fact, it is a Lie subgroup, which in addition is closed and connected if M is compact and simply connected [85, Theorem 3.2.8]. When changing the base point p to some other point q the two holonomy groups will turn out to be conjugate to one another in $\mathrm{O}(n)$. Thus, the notion of holonomy without prescribing a base point exists only up to conjugation. In the following we can always choose a base point freely and we often simply write $\mathrm{Hol}(g)$ ignoring the base point.

One of the main features of the holonomy group is that it tells us which tensorial structures on the tangent space T_pM, like certain forms, endomorphisms etc., can be extended to global objects on the manifold. To make this more precise, note that the holonomy representation $\mathrm{Hol}(g)$ on T_pM naturally extends to a representation on $\mathcal{T}^{r,s}(T_pM) \coloneqq \bigotimes_{i=1}^{r} T_pM \otimes \bigotimes_{i=1}^{s} T_pM^*$. On the other hand the Levi-Civita connection induces a connection, also denoted ∇, on the vector bundle $\bigotimes_{i=1}^r TM \otimes \bigotimes_{i=1}^s TM^*$. A globally defined tensor s is then a section of this bundle and is said to be *parallel* if $\nabla s = 0$.

Theorem 2.1.1 (Holonomy Principle)**.** *For a parallel tensor s the holonomy representation $\mathrm{Hol}(M, g, p)$ at a point p leaves the corresponding element $s_p \in \mathcal{T}^{r,s}(T_pM)$ invariant. On the other hand, if $s_p \in \mathcal{T}^{r,s}(T_pM)$ is an invariant element under the holonomy representation, then s_p extends to a globally defined parallel tensor on M.*

See [85, Proposition 2.5.2] for a proof of this theorem. Using this principle one can show that a Riemannian manifold of dimension $2n$ is a Kähler manifold if and only if $\mathrm{Hol}(g) \subset \mathrm{U}(n)$. While a priori the list of possible holonomy groups might be extremely large, it turns out that there are strong constraints.

Theorem 2.1.2 (Berger's Classification Theorem)**.** *Let (M, g) be a simply connected compact n-dimensional Riemannian manifold that is not a symmetric space and whose holonomy representation is irreducible. Then one of the following is the case:*

(i) $\mathrm{Hol}(g) \cong \mathrm{SO}(n)$

(ii) $n = 2m$ *with* $m \geq 2$ *and* $\mathrm{Hol}(g) \cong \mathrm{U}(m)$ *in* $\mathrm{SO}(2m)$

(iii) $n = 2m$ *with* $m \geq 2$ *and* $\mathrm{Hol}(g) \cong \mathrm{SU}(m)$ *in* $\mathrm{SO}(2m)$

(iv) $n = 4m$ *with* $m \geq 2$ *and* $\mathrm{Hol}(g) \cong \mathrm{Sp}(m)$ *in* $\mathrm{SO}(4m)$

(v) $n = 4m$ *with* $m \geq 2$ *and* $\mathrm{Hol}(g) \cong \mathrm{Sp}(m)\mathrm{Sp}(1)$ *in* $\mathrm{SO}(4m)$

(vi) $n = 7$ *with* $\mathrm{Hol}(g) \cong \mathrm{G}_2$ *in* $\mathrm{SO}(7)$

(vii) $n = 8$ *with* $\mathrm{Hol}(g) \cong \mathrm{Spin}(7)$ *in* $\mathrm{SO}(8)$.

For a reference see for instance [19, 10.92] and [85, Theorem 3.4.1]. Let us make some comments on the assumptions of Theorem 2.1.2. First, if M is not simply connected then the theorem still holds if one replaces $\mathrm{Hol}(g)$ with $\mathrm{Hol}_0(g)$. For symmetric Riemannian manifolds there is a separate list classifying the possible groups for $\mathrm{Hol}_0(g)$. Since Ricci-flat symmetric spaces are already sectional curvature flat by [19, 7.61 Theorem], we will not focus on this case and refer to [19, §10.K]. The assumption on irreducible holonomy representation can be dealt with in terms of the following theorem, see [85, Theorem 3.2.7] for instance.

Theorem 2.1.3 (De Rham Decomposition)**.** *Let (M, g) be a complete, simply connected Riemannian manifold. Then there exist complete simply connected Riemannian manifolds $(M_1, g_1), \cdots, (M_k, g_k)$ and an isometry*

$$(M, g) \to (M_1 \times \cdots \times M_k \times \mathbb{R}^l, g_1 + \cdots + g_k + g_F),$$

where $g_1 + \cdots + g_k + g_F$ is the Riemannian product metric and g_F is a flat metric on $\mathbb{R}^l$. Furthermore, $\mathrm{Hol}(g) = \mathrm{Hol}(g_1) \times \cdots \times \mathrm{Hol}(g_k)$ and the representation of each $\mathrm{Hol}(g_i)$ is irreducible.

Proving the existence of compact manifolds admitting metrics with holonomy in the list of Theorem 2.1.2 is difficult except for the cases $(i), (ii), (v)$. The case (i) is the generic case for Riemannian manifolds, while (ii) is the generic case for Kähler manifolds. An example for case (v) is provided by the quaternionic projective space $P\mathbb{H}^n$ endowed with a Riemannian metric similar to the Fubini-

Study metric. Finding an example for $\mathrm{Hol}(g) = \mathrm{SU}(m)$ is already much more challenging. The key to finding such examples is provided by Yau's solution to the Calabi conjecture [141]. We state the theorem in the form in which we will need it, but remark that there are more detailed and general versions.

Theorem 2.1.4 (Calabi-Yau Theorem)**.** *Let X be a compact Kähler manifold with vanishing first real Chern class. Then each Kähler class $[\alpha]$ is represented by a unique Kähler form ω whose corresponding Riemannian metric is Ricci-flat.*

One can now construct an example with $\mathrm{Hol}(g) = \mathrm{SU}(m)$ as follows. Let X be a hypersurface of $\mathbb{C}P^n$ with $n \geq 3$ given as the zero set of a degree $n+1$ homogeneous polynomial. From the adjunction formula [75, Corollary 2.4.9] it follows that the canonical bundle is trivial so that the first Chern class vanishes. The Calabi-Yau theorem shows that X admits a Ricci-flat Kähler metric g. The proof of the following theorem is relatively straightforward, see [108, Theorem 11.5] for instance.

Theorem 2.1.5. *A Kähler manifold (M, g) of real dimension $2m$ is Ricci-flat if and only if $\mathrm{Hol}(g) \subset \mathrm{SU}(m)$.*

For the Ricci-flat metric g on X we now show that $\mathrm{Hol}(g) = \mathrm{SU}(m)$. This will follow from the classification Theorem 2.1.2 if we show that X does not split as a product as in Theorem 2.1.3. The Lefschetz hyperplane theorem [24] shows, for $n \geq 3$, that X is simply connected. In addition, for $n \geq 4$ it shows that the second Betti number is $\mathrm{b}_2(X) = 1$. By Poincaré duality we conclude that the

Betti numbers $b_i(X)$ are 0 in odd degrees. If X is the product of two odd dimensional manifolds they would generate a non-vanishing Betti number in odd degrees, so we can exclude this case. In case that X is the Riemannian product of two even dimensional manifolds (X_1, g_1) and (X_2, g_2) then each $\mathrm{Hol}(g_i)$ is contained in $\mathrm{SU}(\dim_{\mathbb{C}}(X_i))$. The X_i are therefore also Kähler manifolds. If $n \geq 4$ we arrive at a contradiction since $b_2(X) = b_2(X_1) + b_2(X_2) \geq 2$. In the case $n = 3$ the manifold X would be a product of two 2-dimensional manifolds, each of which would be flat and simply connected, a contradiction. Therefore, by Theorem 2.1.2 we have $\mathrm{Hol}(g) = \mathrm{SU}(n-1)$.

In fact, for $n = 3$ we have $\mathrm{Hol}(g) = \mathrm{Sp}(1)$ since $\mathrm{SU}(2) = \mathrm{Sp}(1)$, see Section 3.1. In this case X is a so called $K3$-surface. These and their higher dimensional analogs, which are NOT the examples X with $n \geq 4$, but the ones with $\mathrm{Hol}(g) = \mathrm{Sp}(m)$, will be called hyperkähler manifolds and are at the center of our interest. Higher dimensional examples were found by Beauville and O'Grady for which we refer to Section 3.5.

The remaining two groups G_2 and $\mathrm{Spin}(7)$ also imply Ricci-flatness and the first compact examples were given by Joyce, see [85].

The groups $\mathrm{SU}(m), \mathrm{Sp}(m), \mathrm{G}_2, \mathrm{Spin}(7)$ are called the Ricci-flat holonomy groups.

Let us end this section by mentioning that metrics whose holonomy is one of the above groups are in general not unique on a fixed manifold. It is thus natural to study the space of all metrics having fixed holonomy group $\mathcal{R}^{\mathrm{Hol=G}}(M)$ and related spaces. Sometimes one can construct 'coordinates' on these spaces in the following way. From the holonomy principle it is clear that the dimension of the space of parallel forms $P(g)$ does not change as one varies the metric g in $\mathcal{R}^{\mathrm{Hol=G}}(M)$. But the space $P(g)$ viewed as a subspace in cohomology $\mathrm{H}^*(M,\mathbb{R})$ is not fixed and can be used as a parameter for the metric g. In this way certain Grassmann spaces of $\mathrm{H}^*(M,\mathbb{R})$ are natural parameter spaces for $\mathcal{R}^{\mathrm{Hol=G}}(M)$. For the case $\mathrm{Hol}(g) = \mathrm{Sp}(n)$ we will see that the second cohomology is naturally endowed with an integral valued and indefinite bilinear form q_M. The relative position of $P(g)$ to the integral structure $(\mathrm{H}^*(M,\mathbb{Z}), q_M)$ can provide a lot of information on $\mathcal{R}^{\mathrm{Hol=G}}(M)$ and the metric itself.

2.2 Pseudo Euclidean Geometry

Euclidean geometry is often understood as the study of a finite dimensional $\mathbb{R}$-vector space V endowed with a positive definite scalar product. In this section we consider the case when we drop the assumption of positive definiteness.

Definition 2.2.1. *A pseudo euclidean space is a finite dimensional* $\mathbb{R}$*-vector space* V *endowed with a symmetric and non-degenerate bilinear pairing* $q\colon V \times V \to \mathbb{R}$*, which we will also call a scalar product.*

By an *inner product space* we will more generally mean a vector space defined over a field with a symmetric non-degenerate bilinear pairing. We will often denote the scalar product with $(\cdot,\cdot)$ or $q(\cdot,\cdot)$. For $v \in V$ we set $v^2 := (v,v)$ and call it the *length* of v. Like in the euclidean case one can also take the *orthogo-*

nal complement of a subset $U \subset V$ defined by

$$U^{\perp} \coloneqq \{\ v \in V \mid (v, u) = 0 \text{ for all } u \in U\ \}.$$

If U is a subspace one checks that $(U^{\perp})^{\perp} = U$ and $\dim(U) + \dim(U^{\perp}) = \dim(V)$, since $(\cdot, \cdot)$ is non-degenerate. However, not all statements from Euclidean geometry on orthogonal complements carry over to the indefinite case. Usually one needs to be aware to which regions of V the subset U belongs, in the sense that a pseudo euclidean space is naturally separated into the following cones

- the *positive cone* $\mathrm{Pos}(V) \coloneqq \{v \in V \mid v^2 > 0\}$,
- the *negative cone* $\mathrm{Neg}(V) \coloneqq \{v \in V \mid v^2 < 0\}$,
- the *isotropic cone* $V^0 \coloneqq \{v \in V \mid v^2 = 0\}$.

In case U is a subspace of V with $U \cap V^0 = 0$ we have $U \cap U^{\perp} = 0$. Whenever the latter is the case one obtains an orthogonal decomposition

$$V = U \oplus U^{\perp}.$$

On the other hand, if $U \cap U^{\perp} \neq 0$, then $U + U^{\perp}$ does not need to be equal to V. Also note that if (r, s) denotes the signature of the bilinear pairing $(\cdot, \cdot)$, then r is the dimension of a maximal subspace in $\mathrm{Pos}(V)$ and s the dimension of a maximal subspace in $\mathrm{Neg}(V)$. For a subspace U of maximal dimension in $\mathrm{Pos}(V)$ one then has $U^{\perp} \subset \mathrm{Neg}(V)$.

Orthogonal transformations are defined as in the Euclidean case, i.e. an invertible linear map $g\colon V \to V$ is *orthogonal* if $(gv, gv) = (v, v)$ for all $v \in V$.

Definition 2.2.2. *The orthogonal group* $\mathrm{O}(V, q)$ *of a pseudo Euclidean space* (V, q) *is the group of orthogonal transformations.*

The group $\mathrm{O}(V, q)$ is a Lie group, and non-compact if and only if q is indefinite. In case of the standard space $\mathbb{R}^{r+s}$ with scalar product given by $q_{st} = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 \cdots - x_{r+s}^2$ we also write $\mathrm{O}(r, s)$ for the orthogonal group. Note that each orthogonal group is isomorphic to one of the $\mathrm{O}(r, s)$ according to the sign, for more details see for instance [68].

Among all orthogonal transformations reflections are of particular importance. This is not only true for Euclidean geometry, but also for the indefinite case. A *reflection* r_a along a vector a, with $a^2 \neq 0$, in a pseudo euclidean space (V, q) is a map of the following form. Let $H_a := a^\perp$ be what is called the *reflection hyperplane*, then

$$V = a \cdot \mathbb{R} \oplus H_a.$$

Thus, for each $v \in V$ there exist unique $s \in \mathbb{R}$ and $b \in H_a$ such that $v = sa + b$. Then define $r_a(v) = -sa + b$. Equivalently, a reflection is a map of the form

$$v \mapsto v - 2\frac{(v, a)}{(a, a)}a.$$

In the euclidean case it is not too hard to prove that the orthogonal group is generated by reflections. For the indefinite case this is much more difficult, but well known, see [53] for a constructive proof and some of the history.

Theorem 2.2.1 (Cartan-Dieudonné)**.** *Every orthogonal transformation in a* n*-dimensional inner product space is the composition of at most* n *reflections.*

There are two natural and continuous homomorphisms defined on $\mathrm{O}(r,s)$ with values in $\{\pm 1\}$. One is given by the determinant, which does not need any further introduction. The other is given by the spinor norm which is less known. This notion exists for more general cases than we will define it. Here we are only interested in the real case. We set $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and $(\mathbb{R}^*)^2 := \{ \ x^2 \mid x \in \mathbb{R}^* \ \}$.

Definition 2.2.3. *The spinor norm on the inner product space (V,q) is the homomorphism*

$$\mathrm{spn}\colon \mathrm{O}(V,q) \to \mathbb{R}^*/(\mathbb{R}^*)^2 \cong \{\pm 1, \cdot\}$$

induced by sending a reflection r_a to $-q(a,a)$ in $\mathbb{R}^*/(\mathbb{R}^*)^2$. We denote by $\mathrm{O}^+(V)$ the kernel of spn*.*

Often one finds the spinor norm to be defined by the term $q(a,a)$ instead of $-q(a,a)$ for a reflection r_a. However, in our situation it turns out to be more convenient to work with $-q(a,a)$. Thus, if $g \in \mathrm{O}(V,q)$ is generated by reflections $g = r_{a_1} \cdots r_{a_n}$, then $\mathrm{spn}(g) = 1$ if and only if the number of reflections along a vector of positive length in $(r_{a_1}, \cdots, r_{a_n})$ is even. Aside from this the spinor norm has many more applications. A rather obvious one is that it can be used to distinguish between connected components of $\mathrm{O}(r,s)$. For that, we recall the following well known fact from [68, p.131].

Proposition 2.2.1. *Let $r,s > 0$. Then $\mathrm{O}(r) \times \mathrm{O}(s)$ is a maximal compact subgroup of $\mathrm{O}(r,s)$. In particular $\mathrm{O}(r) \times \mathrm{O}(s)$ is homotopy equivalent to $\mathrm{O}(r,s)$.*

As an immediate consequence we find that $\mathrm{O}(r, s)$ has 4 connected components. Now the determinant and the spinor norm can be used to distinguish between these components, since both are continuous maps into the discrete space $\{1, -1\}$. In the next section on Grassmann manifolds we will see another interpretation of the spinor norm.

2.3 Grassmann Spaces

Grassmann spaces parametrize certain linear subspaces in a fixed vector space. In this section we recall basic facts about these spaces. We are particularly interested in the case when these Grassmann spaces are defined over a pseudo Euclidean space (V, q).

Definition 2.3.1. *Let V be an n-dimensional $\mathbb{R}$-vector space. The Grassmann space of k-dimensional subspaces and the one of oriented k-dimensional subspaces are defined as*

- $\mathrm{Gr}(k, V) \coloneqq \{H \subset V \mid H$ *a k-dimensional linear subspace*$\}$,
- $\mathrm{Gr}^{\mathrm{o}}(k, V) \coloneqq \{H \subset V \mid H$ *a k-dimensional oriented linear subspace*$\}$.

The general linear group $\mathrm{Gl}(V)$ acts transitively on $\mathrm{Gr}(k, V)$. Thus, if S denotes the stabilizer of this action, one has $\mathrm{Gr}(k, V) \cong \mathrm{Gl}(V)/S$. In this way we endow $\mathrm{Gr}(k, V)$ with a topology. Then $\mathrm{Gr}(k, V)$ has the structure of a compact manifold, as we will see below.

If V is endowed with an indefinite scalar product, we define the following Grassmann spaces.

Definition 2.3.2. *Let (V, q) be a pseudo Euclidean space. The Grassmann space of positive k-dimensional subspaces $\mathrm{Gr}^+(k, V)$ is the subspace of $\mathrm{Gr}(k, V)$ consisting of elements on which q is positive definite. The Grassmann space of positive oriented k-dimensional subspaces $\mathrm{Gr}^{+,\mathrm{o}}(k, V)$ is the subspace of $\mathrm{Gr}^{\mathrm{o}}(k, V)$ consisting of oriented positive definite subspaces.*

It is not hard to see that $\mathrm{Gr}^+(k, V)$ as well as $\mathrm{Gr}^{+,\mathrm{o}}(k, V)$ are open subspaces in $\mathrm{Gr}(k, V)$ and $\mathrm{Gr}^{\mathrm{o}}(k, V)$ respectively. Hence, both spaces are non-compact manifolds. Furthermore, they can also be described as homogeneous spaces.

Lemma 2.3.1. *Let (V, q) be an inner product space of dimension n with q having signature (r, s). Then we have the following descriptions*

- $\mathrm{Gr}(k, V) \cong \mathrm{O}(n)/\left(\mathrm{O}(k) \times \mathrm{O}(n-k)\right)$,
- $\mathrm{Gr}^{\mathrm{o}}(k, V) \cong \mathrm{O}(n)/\left(\mathrm{SO}(k) \times \mathrm{O}(n-k)\right)$.

If $k = r$, we have

- $\mathrm{Gr}^+(k, V) \cong \mathrm{O}(r, s)/\left(\mathrm{O}(r) \times \mathrm{O}(s)\right)$,
- $\mathrm{Gr}^{+,\mathrm{o}}(k, V) \cong \mathrm{O}(r, s)/\left(\mathrm{SO}(r) \times \mathrm{O}(s)\right)$.

If $k < r$, we have

- $\mathrm{Gr}^+(k, V) \cong \mathrm{O}(r, s)/\left(\mathrm{O}(k) \times \mathrm{O}(r-k, s)\right)$,
- $\mathrm{Gr}^{+,\mathrm{o}}(k, V) \cong \mathrm{O}(r, s)/\left(\mathrm{SO}(k) \times \mathrm{O}(r-k, s)\right)$.

Proof. It is enough to consider the standard space $(\mathbb{R}^{r+s}, q_{st})$ with the standard basis $\{e_i\}$. The group $\mathrm{O}(r, s)$ acts smoothly and transitively on each of the Grassmann spaces. Depending on which structure should be preserved the stabilizer of $\mathrm{span}\{e_1, \cdots, e_k\}$ is then easily identified. □

As a consequence we get the following.

Lemma 2.3.2. *Let (V, q) be an inner product space of dimension n and signature (r, s) with $r, s > 0$. Then there is a homeomorphism* $\mathrm{Gr}^+(r, V) \cong \mathbb{R}^{r(n-r)}$.

Proof. By the previous lemma $\mathrm{Gr}^+(r, V)$ is the homogeneous space given by a non-compact Lie group modulo its maximal compact subgroup and thus homeomorphic to some $\mathbb{R}^N$ by [71, Theorem 14.3.11]. □

At the end of this section we will prove this fact by constructing a global chart.

The forgetful map $\mathrm{Gr}^{+,\mathrm{o}}(k, V) \to \mathrm{Gr}^+(k, V)$ is a 2-sheeted covering. In case that the positive definite subspaces are of maximal dimension, we find that the covering is trivial by Lemma 2.3.2. In particular $\mathrm{Gr}^{+,\mathrm{o}}(k, V)$ consists of 2 connected components. We can use this to give another interpretation of the spinor norm.

Lemma 2.3.3. *Let (V, q) be a pseudo Euclidean space with* $\mathrm{sgn}(q) = (r, s)$*. If $k = r$ an orthogonal transformation $g \in \mathrm{O}(V, q)$ preserves the connected components of* $\mathrm{Gr}^{+,\mathrm{o}}(k, V)$ *if and only if* $\mathrm{spn}(g) = 1$.

Proof. Since spn is a continuous function it is enough to check the statement for a single element in each connected component of $\mathrm{O}(r, s)$. □

Next, we show how to construct coordinate charts for the Grassmann spaces. Let U_1 be a k-dimensional linear subspace of V and U_2 a subspace such that

$V = U_1 \oplus U_2$. Then consider the map $\mathrm{Hom}(U_1, U_2) \to \mathrm{Gr}(k, V)$ with

$$f \mapsto \mathrm{Graph}(f) := \{w + f(w) \mid w \in U_1\}.$$

By choosing a basis, one may identify $\mathrm{Hom}(U_1, U_2)$ with the space of matrices $\mathrm{Matr}((n-k) \times k)$ which then again can be thought of as $\mathbb{R}^{k(n-k)}$. Choosing U_1 and U_2 appropriately one can construct an atlas on $\mathrm{Gr}(k, V)$ using the above function. One checks that the transition functions are given by quotients of polynomials. Consequently, $\mathrm{Gr}(k, V)$ is not just a manifold, but also an algebraic variety. As a side remark, this can also be seen in terms of the famous Plücker embedding [60, p.209], which realizes $\mathrm{Gr}(k, V)$ as a projective variety.

Using the above charts we can construct a global chart for $\mathrm{Gr}^+(k, V)$ if k is the maximal dimension of a positive definite subspace. Consider the standard space $(\mathbb{R}^{k+s}, q_{std})$ with $W_1 = \mathrm{span}\{e_1, \cdots, e_k\}$ and $W_2 = \mathrm{span}\{e_{k+1}, \cdots, e_{k+s}\}$. Then every positive k-space is the graph of a map $A\colon W_1 \to W_2$. If this were not the case, we would find a space $H \in \mathrm{Gr}^+(k, V)$ whose projection onto W_1 would not be surjective. But then $W_1 + H$ is a positive definite subspace of V with $\dim(W_1 + H) > \dim(H)$. However, this is not possible since H is of maximal dimension. Now note that $x + Ax$ is in the positive cone if and only if

$$\sum_{i=1}^{k} x_i^2 - \sum_{i=k+1}^{k+s} (Ax)_i^2 > 0.$$

For the map A this just means that the operator norm with respect to the standard euclidean norm satisfies $\|A\| < 1$. Thus, we have found a global chart

$$\{A \in \mathrm{Matr}((n-k) \times k) \mid \|A\| < 1\} \xrightarrow{\cong} \mathrm{Gr}^+(k, V).$$

2.4 Lattice Theory

Lattice theory can be viewed as a variation of the theory of pseudo Euclidean spaces, by replacing the $\mathbb{R}$-vector space V by a free $\mathbb{Z}$-module Λ and the scalar product by an integer valued bilinear pairing. Lattices have their roots in number theory, but they have become an important ingredient in various geometric applications. We give a brief overview of lattice theory. Standard sources are [113], [120], [44]. A good survey can be found in [83, Chapter 14].

A lattice will mean the following to us.

Definition 2.4.1. *A lattice Λ is a free abelian group of finite rank, together with a symmetric, non-degenerate bilinear pairing $(\cdot,\cdot)\colon \Lambda \times \Lambda \to \mathbb{Z}$. A lattice is even if the so called length $\lambda^2 := (\lambda, \lambda)$ is even for all $\lambda \in \Lambda$, otherwise it is odd.*

Note, that the above notion of lattice differs from the one given in the theory of Lie groups as a discrete subgroup with finite co-volume.

Just like in the case of inner product spaces one can take *orthogonal complements*. For $\lambda \in \Lambda$ we define

$$\lambda^\perp := \{\ \lambda' \in \Lambda \mid (\lambda, \lambda') = 0\ \}.$$

Clearly, if $\lambda_1 = n\lambda_2$ for $\lambda_1, \lambda_2 \in \Lambda$ and $n \in \mathbb{Z}$ then $\lambda_1^\perp = \lambda_2^\perp$. We say that an element $\lambda \in \Lambda$ is *primitive* if $\lambda = n\lambda'$ implies that $n = \pm 1$. More generally, a sublattice $\Lambda' \subset \Lambda$ is a *primitive sublattice* if Λ/Λ' is torsion free.

Closely related to the notion of a lattice is that of a quadratic module.

Definition 2.4.2. *Let V denote a module over a commutative ring R. A function $q\colon V \to R$ with*

- *$q(av) = a^2 q(v)$ for all $v \in V$ and $a \in R$,*
- *$(x, y) \mapsto q(x + y) - q(x) - q(y)$ a bilinear pairing on V,*

is called a quadratic form, and the tupel (V, q) a quadratic module. Furthermore, if $R = \mathbb{Z}$ and q takes values in $2\mathbb{Z}$ we say that (V, q) is an even quadratic module.

If (V, q) is an even quadratic module we endow V with the bilinear pairing

$$(x, y) \coloneqq \frac{1}{2} \left(q(x + y) - q(x) - q(y) \right).$$

On the other hand, if Λ is a lattice, then setting $q(x) \coloneqq (x, x)$ defines a quadratic form on Λ. In case that $R = \mathbb{F}$ is a field of characteristic not equal to 2, the notion of a quadratic module over $\mathbb{F}$ and that of an inner product space (dropping any assumptions on definiteness) defined over $\mathbb{F}$ are in bijective correspondence [120, p.27].

Every lattice Λ determines an inner product space over $\mathbb{F}$ by extending $(\cdot, \cdot)$ $\mathbb{F}$-linearly to $\Lambda \otimes_{\mathbb{Z}} \mathbb{F}$. Consider the case when $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = \mathbb{R}$, then there is a natural embedding $\Lambda \hookrightarrow \Lambda \otimes \mathbb{F}$ which preserves the bilinear pairing. Thus, we may view Λ as a sublattice of the inner product space $\Lambda \otimes \mathbb{Q}$.

Definition 2.4.3. *Let Λ be a lattice. The dual lattice Λ^* is the sublattice of $\Lambda \otimes \mathbb{Q}$ consisting of those elements $x \in \Lambda \otimes \mathbb{Q}$ for which (x, λ) is an integer for every $\lambda \in \Lambda$.*

Note that the dual lattice is in general not really a lattice, since the bilinear pairing extended to Λ^* takes values in $\mathbb{Q}$ instead of $\mathbb{Z}$.

Definition 2.4.4. *A lattice Λ is unimodular if the dual lattice Λ^* is a lattice and equal to Λ.*

It is not too hard to see, that Λ is a unimodular lattice if and only if

$$\Lambda \to \mathrm{Hom}(\Lambda, \mathbb{Z}) \text{ with } \lambda \mapsto (\cdot, \lambda)$$

is an isomorphism of groups. Then, if A denotes the intersection matrix $((e_i, e_j))_{i,j}$ of Λ for some basis $\{e_i\}$, one checks that unimodularity of Λ is equivalent to $\det(A) = \pm 1$.

Next, let us define some invariants of lattices. We start with those coming from the induced pseudo Euclidean space.

Definition 2.4.5. *Let $(\Lambda, (\cdot, \cdot))$ be a lattice and $\Lambda_{\mathbb{R}} = (\Lambda \otimes \mathbb{R}, (\cdot, \cdot))$ the $\mathbb{R}$-linear extension. Then we define*

- $\mathrm{rank}(\Lambda) \coloneqq \dim(\Lambda_{\mathbb{R}})$ *the rank,*
- $\mathrm{sgn}(\Lambda) \coloneqq (r, s)$ *where (r, s) is the signature of $\Lambda_{\mathbb{R}}$,*
- $\tau(\Lambda) \coloneqq r - s$ *the index of Λ.*

The next invariants capture more of the integral structure.

Definition 2.4.6. *The discriminant group of a lattice* Λ *is the group* $D_\Lambda :=$ Λ^*/Λ*. The discriminant form* q_D *is the quadratic form on* D_Λ *defined by*

- $q_D([x]) := (x,x)$ *mod* $\mathbb{Z}$ *if* Λ *is odd,*
- $q_D([x]) := (x,x)$ *mod* $2\mathbb{Z}$ *if* Λ *is even.*

Furthermore, if A *denotes the intersection matrix of* Λ *in some basis, the discriminant of* Λ *is defined as* $\operatorname{disc}(\Lambda) := \det(A)$.

First note that $\operatorname{disc}(\Lambda)$ is well defined. To see this recall that if $A' = XAX^T$ is the intersection matrix obtained by a base change one has $X \in \mathrm{Gl}(n,\mathbb{Z})$ and hence $\det A' = \det A \cdot (\det X)^2 = \det A$. By computations in a basis of Λ it is also not hard to prove that D_Λ is a finite group. In fact, $|\operatorname{disc}(\Lambda)|$ is equal to the number of elements of D_Λ.

Let us give some examples.

Example 2.4.1. *By* $I_{s,r}$ *we denote the lattice* $\mathbb{Z}^{r+s}$ *with intersection matrix given by the diagonal matrix with* 1*'s in the first* r *entries and* -1*'s in the remaining* s *entries of the diagonal.*

Example 2.4.2. *The hyperbolic lattice, denoted* U*, is the lattice with intersection matrix*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is even and unimodular with $\operatorname{sgn}(U) = (1,1)$ *and* $\operatorname{disc}(U) = -1$.

Example 2.4.3. *The E_8 lattice is given by the intersection matrix*

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

It is even and unimodular with $\mathrm{disc}(E_8) = 1$ *and* $\mathrm{sgn}(E_8) = (8, 0)$.

One can obtain other examples by taking direct sums. Another important construction is given by the *twist* $\Lambda(m)$ of a given lattice Λ, which is obtained when changing the scalar product of Λ by multiplying it with an integer $m \in \mathbb{Z}$, i.e. $(\cdot, \cdot)_{\Lambda(m)} \coloneqq m \cdot (\cdot, \cdot)_\Lambda$. Here is a classification result.

Theorem 2.4.1. [83, Corollary 1.3 Chapter 14] *Let Λ be an indefinite unimodular lattice of signature (r, s) and $\tau \coloneqq r - s$ the index.*

- *If Λ is even, then $\tau \equiv 0$ mod 8 and according to the sign of τ*

$$\Lambda \cong E_8^{\oplus \frac{\tau}{8}} \oplus U^{\oplus s} \textit{ or } \Lambda \cong E_8(-1)^{\oplus \frac{-\tau}{8}} \oplus U^{\oplus r}.$$

- *If Λ is odd, then $\Lambda \cong I_{r,s}$.*

We will also be interested in the automorphisms of lattices.

Definition 2.4.7. *Let* Λ *be a lattice. The orthogonal group, or automorphism group of* Λ *is defined as the group of isomorphisms, i.e.*

$$\mathrm{O}(\Lambda, (\cdot,\cdot)) \coloneqq \left\{ g\colon \Lambda \xrightarrow{\cong} \Lambda \mid (g\lambda, g\lambda') = (\lambda, \lambda') \text{ for all } \lambda, \lambda' \in \Lambda \right\}.$$

We will often write $\mathrm{O}(\Lambda)$ *for this group if it is clear what the bilinear pairing is.*

Note that $\mathrm{O}(\Lambda)$ can be viewed as a discrete subgroup of $\mathrm{O}(\Lambda \otimes \mathbb{R}) \cong \mathrm{O}(r, s)$, where (r, s) is the sign of Λ. If $\mathrm{O}(r, s)$ is semisimple, which is the case if $r + s \geq 3$ [109, Apendix A], then $\mathrm{O}(\Lambda)$ is an arithmetic subgroup by [109, 5.1.11].

Just like for $\mathrm{O}(r, s)$ reflections play an important role for $\mathrm{O}(\Lambda)$. However, $\mathrm{O}(\Lambda)$ is in general not generated by reflections, but the subgroup generated by those is a large subgroup of $\mathrm{O}(\Lambda)$. To see this one notes that it is a normal subgroup and of finite index by Margulis normal subgroup theorem, see [96] and [104, Chapter IV]. We are particularly interested in the subgroup of $\mathrm{O}(\Lambda)$ generated by reflections along (-2)-*classes*, i.e. along those classes $c \in \Lambda$ with $c^2 = -2$. We will need the following notion.

Definition 2.4.8. *The stable orthogonal group* $\tilde{\mathrm{O}}(\Lambda)$ *is the kernel of the natural homomorphism* $\rho\colon \mathrm{O}(\Lambda) \to \mathrm{O}(D_\Lambda, q_D)$.

It often happens that ρ is surjective. This is for example the case whenever Λ is an even indefinite lattice such that the number of generators $l(D_\Lambda)$ of D_Λ satisfies $l(D_\Lambda) + 2 \leq \mathrm{rank}(\Lambda)$, see [113, Thm. 1.14.2]. If r_c is a reflection along a (-2)-class $c \in \Lambda$, then $r_c = id_\Lambda + (\cdot.c)c$. Since $(x, c) \in \mathbb{Z}$ for all $x \in \Lambda^*$ one finds that $\rho(r_c)(x) = x \bmod \Lambda$ for all $x \in \Lambda^*$. Thus, $r_c \in \tilde{\mathrm{O}}(\Lambda)$ and $\mathrm{spn}(r_c) = 1$.

Definition 2.4.9. *Let* $\mathrm{O}^+(\Lambda) \coloneqq \mathrm{O}(\Lambda) \cap \mathrm{O}^+(\Lambda \otimes \mathbb{R})$*, the subgroup of* $\mathrm{O}(\Lambda)$ *of elements with trivial spinor norm. Moreover, let* $\tilde{\mathrm{O}}^+(\Lambda)$ *be the stable orthogonal subgroup with trivial spinor norm, i.e.* $\tilde{\mathrm{O}}^+(\Lambda) \coloneqq \tilde{\mathrm{O}}(\Lambda) \cap \mathrm{O}^+(\Lambda)$*.*

Sometimes it happens that $\tilde{\mathrm{O}}^+(\Lambda)$ determines the group generated by reflections along (-2)-classes. The following result is due to Kneser [87, Satz 4], see also [62, Theorem 1.1].

Theorem 2.4.2. *Let* Λ *be a lattice of signature* (r, s) *with* $r, s \geq 2$*. Assume that there exists a sublattice* Λ' *of rank at least* 5 *with* $\mathrm{disc}(\Lambda')$ *not a multiple of* 3*. Furthermore, assume there is another sublattice* Λ'' *of rank at least* 6 *such that* $\mathrm{disc}(\Lambda'')$ *is not even. Then the group generated by reflections along* (-2)*-classes is* $\tilde{\mathrm{O}}^+(\Lambda)$*.*

Proof. In [87, Satz 4] the above theorem is stated for pairs of reflections along classes of length 2, i.e. so that the lattice generated by $r_c \cdot r_b$ with $c^2 = b^2 = 2$ is equal to $\mathrm{SO}(\Lambda) \cap \tilde{\mathrm{O}}^+(\Lambda)$. However, by changing Λ with $\Lambda(-1)$ we obtain the same statement for (-2)-classes. Furthermore, since $r_c \cdot g$ changes the determinant of every element g in $\mathrm{O}(\Lambda)$, we get the statement stated above. □

Theorem 2.4.2 applies for instance for the lattice $\Lambda \cong E_8(-1)^{\oplus \frac{-\tau}{8}} \oplus U^{\oplus r}$ as in Theorem 2.4.1 with $r, s \geq 3$. Since Λ is unimodular we get that the group generated by reflections along (-2)-classes is exactly the subgroup $\mathrm{O}^+(\Lambda)$ in that case. For $r = 3$ and $s = 19$ this lattice is known as the $K3$-lattice, which we will come across in the next chapter.

Let us end this chapter with a statement about orbits of the $\mathrm{O}(\Lambda)$ action on Λ. To state the theorem we need the following notion for a non-zero element λ in Λ. The *divisibility* of λ, denoted $\mathrm{div}(\lambda)$, is the positive integer generating the subgroup $(\lambda, \Lambda) \subset \mathbb{Z}$. Note that $\frac{1}{\mathrm{div}(\lambda)} \cdot \lambda$ is an element of Λ^*. The following is a result due to Eichler, see [44, Paragraph 10] or [61, Lemma 5.3].

Theorem 2.4.3 (Eichler Criterion)**.** *Let Λ be a lattice containing $U \oplus U$ as a sublattice. Then the $\tilde{\mathrm{O}}(\Lambda)$-orbit of a primitive element $\lambda \in \Lambda$ is determined by the length λ^2 and its image $[\frac{1}{\mathrm{div}(\lambda)}\lambda]$ in D_Λ.*

$i^2 = j^2 = k^2 = ijk = -1$

Sir William Rowan Hamilton

3
Hyperkähler Manifolds

The goal of this chapter is to introduce backgrounds on hyperkähler manifolds. We focus on the case of compact hyperkähler manifolds as they appear in Berger's classification theorem. Closely related to these are so called irreducible holomorphic symplectic manifolds. They can be studied in terms of algebraic geometry. The most basic examples of these are the so called $K3$-surfaces, which we will introduce among related higher dimensional examples. Furthermore, we state Torelli theorems for $K3$-surfaces.

3.1 The Group Sp(n)

We start by introducing the notion of an $\mathbb{H}$-*Hermitian inner product space.* For that recall that the skew field of *quaternions* is defined as

$$\mathbb{H} \coloneqq \{\ x_0 + x_1 i + x_2 j + x_3 k\ :\ x_0, x_1, x_2, x_3 \in \mathbb{R}\ \},$$

where the i, j, k satisfy $i^2 = j^2 = k^2 = ijk = -1$. On $\mathbb{H}$ there is a *conjugate map* given by $\bar{x} \coloneqq \overline{x_0 + x_1 i + x_2 j + x_3 k} \coloneqq x_0 - x_1 i - x_2 j - x_3 k$ and the real part of $x = x_0 + x_1 i + x_2 j + x_3 k$ is defined as $\Re(x) \coloneqq x_0$.

Let V be an $\mathbb{H}$-vector space, that is V is a right $\mathbb{H}$-module. An $\mathbb{H}$-*Hermitian inner product* on V is a positive definite pairing $\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{H}$, which is additive in both entries such that for λ in $\mathbb{H}$ and v, w in V

$$\langle v\lambda, w \rangle = \bar{\lambda} \langle v, w \rangle \text{ with } \langle v, w\lambda \rangle = \langle v, w \rangle \lambda \text{ and } \langle v, w \rangle = \overline{\langle w, v \rangle}.$$

For the standard space $\mathbb{H}^n$ the *standard Hermitian inner product* is $\sum \bar{v}_i \cdot w_i$.

Definition 3.1.1. *The unitary quaternionic group of an* $\mathbb{H}$*-Hermitian vector space* $(V, \langle \cdot, \cdot \rangle)$ *is the group of* $\mathbb{H}$*-linear isomorphisms of* V *which preserve* $\langle \cdot, \cdot \rangle$*,*

$$\mathrm{Sp}\,(V) \coloneqq \{G \in GL\,(V, \mathbb{H})\ \mid \langle Gv, Gw \rangle = \langle v, w \rangle \textit{ for all } v, w \in V\}\,.$$

For the standard Hermitian inner product space we set $\mathrm{Sp}(n) \coloneqq \mathrm{Sp}(\mathbb{H}^n)$.

The group $\mathrm{Sp}(V)$ is often called the *compact symplectic group*. In the following we will see why that is. First note that one can reduce scalars to view the $\mathbb{H}$-module V as an $\mathbb{R}$-vector space as well as a $\mathbb{C}$-vector space. However, the complex numbers $\mathbb{C}$ embed in several ways into $\mathbb{H}$. For that let $c \coloneqq (x_1, x_2, x_3) \in \mathbb{R}^3$ such that $x_1^2+x_2^2+x_3^2 = 1$ and set $i_c \coloneqq x_1 i + x_2 j + x_3 k$. Then the map $\mathbb{C} \to \mathbb{H}$ given by $a + ib \mapsto a + i_c b$ is an injective homomorphism of rings and each such c induces a complex vector space structure on V.

A complex vector space can also be viewed as an $\mathbb{R}$-vector space endowed with an endomorphism I such that $I^2 = -Id$. The endomorphism is then called an *almost complex structure*. The *space of associated almost complex structures* on V is then given by

$$\mathrm{C}(V) \coloneqq \left\{ x_1 I + x_2 J + x_3 K \colon x_1^2 + x_2^2 + x_3^2 = 1 \right\} \subset \mathrm{Gl}(V, \mathbb{R}),$$

where the maps I, J, K are induced by multiplication with i, j, k respectively. For $c \in \mathbb{R}^3$ of unit length, the induced almost complex structure I_c in $\mathrm{C}(V)$ is an isometry for an $\mathbb{H}$-Hermitian inner product. Moreover, I_c induces a $\mathbb{C}$-Hermitian inner product $\langle \cdot, \cdot \rangle_{I_c}$ on V. For instance, if $c = (1, 0, 0)$, one has $\langle v, w \rangle_{I_c} = x_0 + i x_1$ when $\langle v, w \rangle = x_0 + x_1 i + x_2 j + x_3 k$. We also obtain a euclidean metric $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ on V by taking the real part $\langle v, w \rangle_{\mathbb{R}} \coloneqq \Re(\langle v, w \rangle)$. Then $\langle \cdot, \cdot \rangle_{I_c} \coloneqq \langle \cdot, \cdot \rangle_{\mathbb{R}} + i \langle I_c \cdot, \cdot \rangle_{\mathbb{R}}$ is a $\mathbb{C}$-Hermitian inner product with respect to I_c. On the other hand, for any euclidean metric, for which I, J, K are isometries, one gets an induced $\mathbb{H}$-Hermitian metric on V provided by

$$\langle v, w\rangle \coloneqq \langle v, w\rangle_{\mathbb{R}} + i\langle Iv, w\rangle_{\mathbb{R}} + j\langle Jv, w\rangle_{\mathbb{R}} + k\langle Kv, w\rangle_{\mathbb{R}}.$$

This is a straightforward computation, but one needs to take care of the order of operations. For a complex structure $I_c \in \mathrm{C}(V)$ we thus find embeddings $\mathrm{Sp}(V) \subset \mathrm{U}(V) \subset \mathrm{O}(V)$. Furthermore, let $J_c = hJh^{-1}$ and $K_c = hKh^{-1}$ where $h \in \mathrm{Sp}(1)$ is some unitary quaternion such that $I_c = hIh^{-1}$. Then the form

$$\sigma(v, w) \coloneqq \langle J_c v, w\rangle_{\mathbb{R}} + i\langle K_c v, w\rangle_{\mathbb{R}}.$$

defines a complex symplectic form on V with respect to the complex structure I_c. Recall that a *symplectic form* is a non-degenerate alternating bilinear pairing. The form σ is preserved by $\mathrm{Sp}(V)$. In general, the group of complex linear automorphisms V that preserve a complex symplectic form is denoted $\mathrm{Sp}(V, \mathbb{C})$ and called the *complex symplectic group*.

Proposition 3.1.1. *Let V be an $\mathbb{H}$-Hermitian vector space. Then for each associated $\mathbb{C}$-Hermitian structure on V one has*

$$\mathrm{Sp}(V) = \mathrm{Sp}(V, \mathbb{C}) \cap \mathrm{SU}(V).$$

Proof. We have seen that $\mathrm{Sp}(V) \subset \mathrm{Sp}(V, \mathbb{C}) \cap \mathrm{U}(V)$. Since σ is non-degenerate the form $\sigma^{\dim_{\mathbb{C}} V}$ is a volume form. The volume form is preserved under $\mathrm{Sp}(V, \mathbb{C})$, thus every element in $\mathrm{Sp}(V, \mathbb{C})$ has determinant 1 and therefore also every element in $\mathrm{Sp}(V)$. On the other hand, let $G \in \mathrm{Sp}(V, \mathbb{C}) \cap \mathrm{SU}(V)$. Then G preserves the Hermitian metric and the symplectic form

$$\sigma = \langle J_c \cdot, \cdot \rangle_{\mathbb{R}} + i \langle K_c \cdot, \cdot \rangle_{\mathbb{R}}$$

and hence also the forms $\langle J_c \cdot, \cdot \rangle_{\mathbb{R}}$ and $\langle K_c \cdot, \cdot \rangle_{\mathbb{R}}$. As G is an isometry for $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ it will also preserve the complex structures J_c, K_c and $I_c = J_c K_c$. Thus G preserves

$$\langle v, w \rangle = \langle v, w \rangle_{\mathbb{R}} + i \langle I_c v, w \rangle_{\mathbb{R}} + j \langle J_c v, w \rangle_{\mathbb{R}} + k \langle K_c v, w \rangle_{\mathbb{R}}$$

the $\mathbb{H}$-Hermitian form. $\square$

Example 3.1.2. *Consider the case $n = 1$ and recall, for example from [68, p.57], that*

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : \; a, b \in \mathbb{C} \text{ and } |a|^2 + |b|^2 = 1 \right\}.$$

Now the group $\mathrm{Sp}(1)$ is isomorphic to the group of quaternions of unit length. Then the map

$$x_0 + x_1 i + x_2 j + x_3 k \mapsto \begin{pmatrix} x_0 + i x_1 & x_2 + i x_3 \\ -x_2 + i x_3 & x_0 - i x_1 \end{pmatrix}.$$

induces an isomorphism $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$.

From [16] respectively [28, Ch.VIII § 13] we obtain the following statement on the complex representation of $\mathrm{Sp}(n)$ on complex forms.

Theorem 3.1.1. *Let V be a $\mathbb{C}$-Hermitian vector space of complex dimension $2n$ and σ a complex symplectic form. Then the representation of $\mathrm{Sp}(V)$ on $\bigwedge^k V$ decomposes as follows*

$$\bigwedge^k V \cong P_k \oplus P_{k-2} \wedge \sigma \oplus P_{k-4} \wedge \sigma^2 \oplus \cdots ,$$

with $0 \leq l \leq n$ and P_l being irreducible and non-trivial representations for $l > 0$. Furthermore, the $\sigma^{\frac{l}{2}}$ are up to complex constants the unique invariant forms in their respective degree.

3.2 Hyperkähler Manifolds

There are different definitions for a Riemannian manifold (M^{4n}, g) to be hyperkähler. What they usually have in common is the requirement that the holonomy group $\mathrm{Hol}(g)$ is contained in $\mathrm{Sp}(n)$. Here we focus on the strong case.

Definition 3.2.1. *A hyperkähler manifold is a simply connected Riemannian manifold (M, g) of dimension $4n$ such that $\mathrm{Hol}(g) = \mathrm{Sp}(n)$. We say that a manifold M is hyperkählerian if there exists a hyperkähler metric on M.*

Later we will see that a compact manifold with holonomy $\mathrm{Hol}(g) = \mathrm{Sp}(n)$ is automatically simply connected.

We are mostly interested in compact hyperkähler manifolds. Only in the late Sections 7.5 and 7.6 we will come across a non-compact example. Thus, from now on we will assume that M is compact. One of our main interests for studying hyperkähler manifolds comes from the following.

Theorem 3.2.1. *A hyperkähler manifold has vanishing Ricci-curvature.*

Proof. Already the condition $\mathrm{Hol}(g) \subset \mathrm{SU}(2n)$ implies Ricci-flatness, see [75, Proposition 4.A.18] or Theorem 2.1.5. $\square$

The name hyperkähler comes from the following very useful fact.

Proposition 3.2.1. *Let (M, g) be a $4n$-dimensional Riemannian manifold. Then $\mathrm{Hol}(g) \subset \mathrm{Sp}(n)$ if and only if there are parallel complex structures $I, J, K \in \mathrm{End}(TM)$ satisfying the quaternionic multiplication relations*

$$I^2 = J^2 = K^2 = IJK = -Id.$$

In particular, g is a Kähler metric with respect to each I, J, K.

Proof. If such complex structures exist the holonomy must be contained in $\mathrm{Sp}(n)$, as it acts trivially on the $\mathbb{H}$-Hermitian inner product

$$g_p(\cdot,\cdot) + i g_p(I_p\cdot,\cdot) + j g_p(J_p\cdot,\cdot) + k g_p(K_p\cdot,\cdot).$$

On the other hand, assume that $\mathrm{Hol}_p(M, g)$ at a point p in M is isomorphic to a subgroup of $\mathrm{Sp}(n)$. Since $\mathrm{Sp}(1)$ is a subgroup of $\mathrm{Sp}(n)$ the elements $i, j, k \in \mathrm{Sp}(1)$ induce complex structures I_p, J_p, K_p on T_pM. By the holonomy principle we can extend these complex structures to parallel almost complex structures on TM. But any parallel almost complex structure is integrable and thus a complex structure [108, Theorem 5.5]. □

A triple of complex structures (I, J, K) as in the above proposition will often be called a *hyperkähler triple*. Associated to such a triple one obtains a triple of Kähler forms

$$\omega_I \coloneqq g(I\cdot, \cdot),\ \omega_J \coloneqq g(J\cdot, \cdot),\ \omega_K \coloneqq g(K\cdot, \cdot).$$

However, note that if $a^2 + b^2 + c^2 = 1$ for real numbers a, b, c then

$$I^{a,b,c} \coloneqq aI + bJ + cK$$

is also a parallel almost complex structure. We find that a hyperkähler manifold (M, g) admits a whole 2-sphere of complex structures, each of which is Kähler with respect to g. One can enhance this fact as follows:

The Twistor Space: For a hyperkähler manifold (M, g) let $I^{a,b,c}$ be a parallel complex structure as described above. Then we identify the space

$$\mathbb{T}_w \coloneqq \left\{ I^{a,b,c} \in \mathrm{End}(M) \mid a^2 + b^2 + c^2 = 1 \right\}$$

with $\mathbb{C}P^1$ to endow it with a complex structure $I_{\mathbb{C}P^1}$. The space $\mathbb{T}_w$ is called the *twistor line*.

For $\mathbb{X} \coloneqq M \times \mathbb{T}_w$ we define a complex structure as follows. At $\left(p, I^{a,b,c}\right)$ we define the almost complex structure $\mathbb{I}_{\left(p, I^{a,b,c}\right)} \coloneqq I_p^{a,b,c} \oplus I_{\mathbb{C}P^1}$ on the tangent space $T_{\left(p, I^{a,b,c}\right)}\mathbb{X} = T_pM \oplus T_{I^{a,b,c}}\mathbb{T}_w$. Then by the Newlander-Nirenberg theorem [111] one can show that $\mathbb{I}$ is an integrable almost complex structure on $\mathbb{X}$. Therefore, $\mathbb{X}$ is given the structure of a complex manifold so that the projection $\pi\colon \mathbb{X} \to \mathbb{T}_w$ is a holomorphic proper surjection. In other words π is a family of complex manifolds and has fiber $X = (M, I^{a,b,c})$ over the point $I^{a,b,c}$. The space $\mathbb{X}$ is called the *twistor space*. Sometimes we write $\mathbb{X}(g)$ and $\mathbb{T}_w(g)$ to emphasise the dependence on the metric.

3.3 Irreducible Holomorphic Symplectic Manifolds

The algebro geometric counterpart of hyperkähler manifolds are the so called irreducible holomorphic symplectic manifolds which are defined as follows:

Definition 3.3.1. *A simply connected compact complex manifold $X = (M, I)$ is called an irreducible holomorphic symplectic manifold* (IHSM) *if X is Kähler and if, up to some constant in $\mathbb{C}^*$, there is a unique non-degenerate holomorphic* 2*-form.*

A non-degenerate holomorphic 2-form σ is also called a *holomorphic symplectic form*. For $n = \dim_{\mathbb{R}}(X)$ we find that $\sigma^{\frac{n}{2}}$ is a holomorphic volume form. Therefore, it induces a trivialization for the canonical bundle K_X of X and henceforth $c_1(X) = c_1(K_X) = 0$, by [108, Proposition 10.4]. The following result is taken from [16], see also [19, 14.20 Theorem].

Theorem 3.3.1. *Every complex structure I in the twistor line $\mathbb{T}_w$ of a hyperkähler manifold (M, g) admits up to a constant a unique holomorphic symplectic-form, i.e. $X = (M, I)$ is irreducible holomorphic symplectic.*

Proof. From Proposition 3.1.1 we know that $\mathrm{Hol}(g) \cong \mathrm{Sp}(n, \mathbb{C}) \cap \mathrm{SU}(2n)$ preserves a complex symplectic form. By the holonomy principle this gives rise to a globally defined holomorphic symplectic form. From Theorem 3.1.1 we know that this form is unique if the holonomy is exactly $\mathrm{Sp}(n)$. $\square$

Thus, every hyperkähler manifold gives rise to a family of IHS-manifolds parameterized by $\mathbb{T}_w \cong \mathbb{C}P^1$. On the other hand, the next theorem shows that each Kähler class on an IHSM gives rise to a hyperkähler manifold.

Theorem 3.3.2. *Let $X = (M, I)$ be an* IHS*-manifold. Then each Kähler class $[\omega_I]$ is represented by a Kähler form ω_I whose corresponding Riemannian metric is hyperkähler.*

Proof. From the Calabi-Yau theorem 2.1.4 we know that $[\omega_I]$ can be represented by a Kähler form of a Ricci-flat Kähler metric g. By [19, 14.17 Lemma] every holomorphic tensor field is parallel. In particular, this is the case for the holomorphic symplectic form σ. Thus, the holonomy group is contained in $\mathrm{Sp}(n)$. Berger's classification theorem 2.1.2 then implies equality $\mathrm{Hol}(g) = \mathrm{Sp}(n)$. $\square$

The IHS-manifolds can be understood as irreducible components of compact Kähler manifolds with vanishing first real Chern class in terms of a splitting theorem due to Beauville [16, Théorème 1].

Theorem 3.3.3 (Beauville-Bogomolov Decomposition)**.** *Let X be a compact complex Kähler manifold with Ricci-flat Kähler metric. Then the universal cover $\tilde{X}$ is biholomorphic and isometric to the product $\Pi_i X_i \times \Pi_j Y_j \times \mathbb{C}^k$, where the metric on $\mathbb{C}^k$ is flat and X_i, Y_j are simply connected compact manifolds with*

- *X_i are irreducible holomorphic symplectic with* $\mathrm{Hol}(X_i) = \mathrm{Sp}(\frac{1}{2}\dim_{\mathbb{C}}(X_i))$,
- *Y_i are Calabi-Yau, i.e.* $\mathrm{Hol}(Y_j) = \mathrm{SU}(\dim_{\mathbb{C}}(Y_j))$.

Furthermore, there exists a finite cover X' of X such that X' is isometric and biholomorphic to $\Pi_i X_i \times \Pi_j Y_j \times T$ where T is a flat torus.

It appears that a version of the above theorem can already be attributed to Calabi from 1957, see [31]. By considering IHS-manifolds we can now show that the assumption of simply connectedness in our definition of hyperkähler manifolds is superfluous. For the statement below see also [19, 14.21 Lemma] and [16].

Proposition 3.3.1. *Let (M, g) be a compact $4n$-dimensional Riemannian manifold with* $\mathrm{Hol}(g) = \mathrm{Sp}(n)$*. Then M is simply connected. Furthermore, the holomorphic Euler characteristic for every* IHS*-manifold X of real dimension $4n$ is* $\chi(X, \mathcal{O}_X) = n + 1$.

Proof. Endow M with some parallel complex structure I so that $X = (M, I, g)$ is a Kähler manifold. From [19, 14.17 Lemma] we know that every holomorphic k-form on X is parallel. On the other hand from Theorem 3.1.1 we know that up to a constant there is only one such k-form. Henceforth, the dimension $\mathrm{h}^{k,0}$ of the space of holomorphic k-forms $\mathrm{H}^0(X, \Omega^k)$ is 1 if k is even and 0 when k is odd for $0 \leq k \leq 2n$. Since $\Omega^0 = \mathcal{O}_X$ we find by the Hodge symmetries that

$$\chi(X, \mathcal{O}_X) = \sum_k (-1)^k \dim(\mathrm{H}^k(X, \mathcal{O}_X)) = \sum_k (-1)^k \dim(\mathrm{H}^0(X, \Omega^k)) = n + 1.$$

Next, consider the universal cover $(\tilde{X}, \tilde{g})$. From [19, p.281 10.16.] we know that

$\mathrm{Hol}(\tilde{g}) = \mathrm{Hol}_0(g)$. As $\mathrm{Hol}_0(g)$ is the connected component of $\mathrm{Hol}(g) = \mathrm{Sp}(n)$ we have $\mathrm{Hol}(\tilde{g}) = \mathrm{Sp}(n)$. Theorem 3.3.3 now shows that the universal covering is finite. Therefore, $\tilde{X}$ is compact and IHSM. From the above it follows that also $\chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) = n + 1$. If s denotes the number of sheets of the universal covering $\tilde{X} \to X$ the Hirzebruch-Riemann-Roch theorem [75, Theorem 5.1.1] implies that $\chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) = s \cdot \chi(X, \mathcal{O}_X)$. In our case s must therefore be 1. □

3.4 The Beauville-Bogomolov Form

Maybe the most important invariant of a hyperkähler manifold (M, g) is the second cohomology. We first note that $\mathrm{H}^2(M, \mathbb{Z})$ is torsion free, since M is simply connected. The second Betti number $b_2(M)$ of M is at least 3. This follows from the fact that $\mathrm{H}^2(M, \mathbb{R})$ contains the subspace generated by the Kähler classes $[\omega_I], [\omega_J], [\omega_K]$. If we now consider an IHS-manifold $X = (M, I)$ associated to (M, g) then the second cohomology is endowed with a quadratic form.

Definition 3.4.1. *Let X be an* IHS*-manifold of complex dimension $2n$. Denote by σ the holomorphic symplectic form, scaled such that $\int \sigma \wedge \bar{\sigma} = 1$. Then there is a quadratic form $f_X \colon \mathrm{H}^2(M, \mathbb{R}) \to \mathbb{R}$, called Beauville-Bogomolov form (BB-form) which is defined by*

$$f_X(\alpha) := \frac{n}{2} \int_X \alpha \wedge \alpha \wedge (\sigma \wedge \overline{\sigma})^{n-1} + (1-n) \left(\int_X \alpha \wedge \sigma^{n-1} \wedge \overline{\sigma}^n \right) \left(\int_X \alpha \wedge \sigma^n \wedge \overline{\sigma}^{n-1} \right).$$

The associated bilinear pairing is also called BB-pairing, and is defined as

$$q_X(\alpha, \beta) \coloneqq \frac{1}{2}(f_X(\alpha+\beta) - f_X(\alpha) - f_X(\beta)).$$

First we note that the BB-pairing behaves well with respect to the Hodge decomposition. That is, for an IHS-manifold X we are interested in the space

$$(\mathrm{H}^{2,0}(X) \oplus \mathrm{H}^{0,2}(X))_{\mathbb{R}} \coloneqq (\mathrm{H}^{2,0}(X) \oplus \mathrm{H}^{0,2}(X)) \cap \mathrm{H}^2(M, \mathbb{R})$$

and the *space of real* $(1,1)$-*classes* $\mathrm{H}^{1,1}(M, \mathbb{R}) \coloneqq \mathrm{H}^{1,1}(M) \cap \mathrm{H}^2(M, \mathbb{R})$.

Lemma 3.4.1. *Let X be an* IHSM, *then the space* $\mathrm{H}^{1,1}(M, \mathbb{R})$ *is orthogonal to* $(\mathrm{H}^{2,0}(X) \oplus \mathrm{H}^{0,2}(X))_{\mathbb{R}}$ *with respect to the BB-pairing q_X.*

Proof. This is a straightforward calculation. First one can use the fact that any element in $(\mathrm{H}^{2,0}(X) \oplus \mathrm{H}^{0,2}(X))_{\mathbb{R}}$ is of the form $\lambda\sigma + \bar{\lambda}\bar{\sigma}$, since it is the invariant part of $\sigma\mathbb{C} \oplus \bar{\sigma}\mathbb{C}$ with respect to conjugation. For the computation of some of the integrals one uses the fact that an integral of a form which is not of type $(2n, 2n)$ is 0, see [60, p.32]. □

The following theorem now shows that the BB-form endows $\mathrm{H}^2(X, \mathbb{Z})$ with the structure of a lattice. The statement goes back to [16, 47, 52], see also [65, p.184 Proposition 23.14].

Theorem 3.4.2. *For an* IHS-*manifold X there exists a positive constant c_X such that $c_X \cdot q_X$ endows* $\mathrm{H}^2(X, \mathbb{Z})$ *with the structure of a primitive lattice. Furthermore, the sign of this lattice is* $(3, b_2(X) - 3)$.

In the spirit of the above theorem we will rescale the BB-form to obtain a lattice structure on $\mathrm{H}^2(X, \mathbb{Z})$ without changing the notation q_X.

The definition of the BB-form strongly depends on the holomorphic symplectic form and thus on the complex structure. However, from [65, p.212] and [65, Corollary 23.17] we know that there is a positive constant c_n depending only on the dimension such that for $\alpha \in \mathrm{H}^2(X, \mathbb{R})$ one has

$$q_X(\alpha, \alpha) = c_n \int_X \alpha^2 \sqrt{td(X)},$$

where $\sqrt{td(X)}$ denotes the square root of the Todd class of X. From [73] for instance we know that $\sqrt{\hat{A}(X)} = \sqrt{td(X)}$ and furthermore that the $\hat{A}$-genus of X is $\frac{\dim_{\mathbb{R}} X}{4} + 1$. The important thing to note is that $\sqrt{\hat{A}(X)}$ is a polynomial in the Pontryagin classes which only depend on the smooth structure of X. Thus, we have that the BB-form is in fact independent of any complex structure.

Corollary 3.4.1. *For a hyperkählerian manifold M the second cohomology $\mathrm{H}^2(M, \mathbb{Z})$ is naturally endowed with an integral valued bilinear pairing of signature $(3, b_2(M) - 3)$.*

Note that in real dimension 4 this is just the cup pairing.

3.5 Examples of Hyperkähler Manifolds

The first examples of IHS-manifolds are the $K3$-surfaces which are the only examples in dimension 4.

Definition 3.5.1. *A $K3$-surface is a simply connected compact complex surface that admits a nowhere vanishing holomorphic 2-form.*

In a way IHS-manifolds are the higher dimensional analogs of $K3$-surfaces. An example of such a $K3$-surface is given by the *Fermat quartic*

$$F \coloneqq \left\{[z_1 \colon z_2 \colon z_3 \colon z_4] \in \mathbb{C}P^3 \mid z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0\right\}$$

or more generally any hypersurface of degree 4 in $\mathbb{C}P^3$, we have seen this example already in Section 2.1. One can construct other examples by the so called Kummer construction.

Example 3.5.1 (Kummer Surfaces)**.** *Let Λ be a rank 4 lattice in $\mathbb{R}^4$ and let T^4 be the torus $\mathbb{C}^2/\Lambda$. Then on T^4 the group $\mathbb{Z}_2$ is acting by taking (z_1, z_2) to $(-z_1, -z_2)$. The resulting orbit space, $X \coloneqq T^4/\mathbb{Z}_2$, is a complex orbifold with 16 singular points. Each singularity has a neighborhood which is diffeomorphic to $\mathbb{C}^2/\pm 1$. For such a singularity there exists a resolution $\pi\colon T^*\mathbb{C}P^1 \to \mathbb{C}^2/\pm 1$, i.e. a map which is a biholomorphism away from the singular point 0 in $\mathbb{C}^2/\pm 1$ and its fiber $\pi^{-1}(0) \cong \mathbb{C}P^1$. Thus, for each singular point s in X there exists a neighborhood U_s which can be cut out and a copy of $T^*\mathbb{C}P^1$ can be glued in accordingly, thus replacing the singular point s with a $\mathbb{C}P^1$. The resulting space* $\mathrm{Kum}(T^4)$ *is the blow up of X at the 16 singular points and is known as a Kummer surface.*

One can verify that $\mathrm{Kum}(T^4)$ is a $K3$-surface by noting that the 2-form $\sigma = dz_1 \wedge dz_2$ on $\mathbb{C}^2$ descents to a form on $(\mathbb{C}^2 - \{0\}) / \pm 1$. Lifted to $T^*\mathbb{C}P^1$ one can show by direct computations in local charts that the lift extends to $T^*\mathbb{C}P^1$ defining a holomorphic symplectic form. Then σ descents also to X and can be lifted to a holomorphic symplectic form on $\mathrm{Kum}(T^4)$. For simply connectedness we refer to [123].

It is interesting to note that the Fermat quartic is isomorphic to a Kummer surface, see [83, Example 3.18]. However, not every Kummer surface is a hypersurface, they might not even be algebraic and in general there is an abundance of different types of $K3$-surfaces. But when one forgets about complex structures and considers the underlying differentiable manifolds only, one has the following well known theorem [14, (8.6) Corollary].

Theorem 3.5.1. *Any two $K3$-surfaces are diffeomorphic.*

Thus, a manifold diffeomorphic to a $K3$-surface will be called *$K3$-manifold*.

Proposition 3.5.2. *Let M be the $K3$-manifold. Then the Euler number is $\chi(M) = 24$ and the second homology group with its cup pairing is isomorphic as a lattice to the so called $K3$-lattice $\Lambda_{K3} \coloneqq E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U$, i.e. the unimodular and even lattice of sign $(3, 19)$.*

Proof. Let X be a complex $K3$-surface. From Proposition 3.3.1 we have $\chi(X, \mathcal{O}_X) = 2$. The second Chern class $c_2(X)$ can now be computed by Noether's formula [60, p.472]

$$2 = \chi(X, \mathcal{O}_X) = \frac{1}{12}(c_1(X)^2 + c_2(X)) = \frac{c_2(X)}{12}.$$

Since the second Chern class is the Euler class of the tangent bundle we get for the topological Euler characteristic $\chi(X) = 24$ and thus $b_2(X) = 22$. The second

Stiefel Withney class ω_2 is just the image of the first Chern class in $\mathrm{H}^2(X, \mathbb{Z}_2) = 0$. Furthermore, by Wu's formula [107, p.132] we know

$$(x, x) = \langle x \cup x, [X] \rangle = (Sq^2(x), [X]) = (\omega_2, x) \mod 2.$$

Since $\omega_2 = 0$ we have that the cup pairing is even. On the other hand by Poincaré duality we know that the pairing is unimodular. The sign of X can be computed by the Thom-Hirzebruch index theorem [14, I 3.1 Theorem] which yields

$$\tau(\mathrm{H}^2(X, \mathbb{Z})) = \frac{p_1(X)}{3} = \frac{c_1(X)^2 - c_2(X)}{3} = -16.$$

By the classification theorem of even unimodular lattices, Theorem 2.4.1, we know that $\mathrm{H}^2(M, \mathbb{Z})$ is isomorphic to the lattice Λ_{K3}. □

The Fermat quartic F is a complex submanifold of $\mathbb{C}P^3$ and therefore Kähler. Thus F is an example of an IHS-manifold. Since the first Betti number is even, as it vanishes, we know by [14, (3.1) Theorem p. 144] that in fact every $K3$-surface admits a Kähler metric and thus all of them are IHS. $K3$-surfaces were the first manifolds for which one could apply the Calabi-Yau Theorem 2.1.4 producing hyperkähler manifolds and also the first examples of non-flat Ricci-flat metrics. In fact, hyperkähler metrics are the only type of Ricci-flat metrics that can occur on M. For the following see [19, 6.40] and [72].

Theorem 3.5.2. *Let M be the $K3$-manifold and g a Riemannian metric on M. Then the following are equivalent:*

- *g is Einstein*
- *g is Ricci-flat*
- *g has zero scalar curvature*
- *g has non-negative scalar curvature*
- *g is hyperkähler.*

Higher Dimensional Examples

We first give the 2 examples found in dimensions $4k$ by Beauville. For that we need the notion of the Douady space of length n. A complex manifold X can be viewed as a ringed space $(X, \mathcal{O}_X)$ where $\mathcal{O}_X$ is the structure sheaf, i.e. the sheaf of holomorphic functions on X. For a complex subspace $(A, \mathcal{O}_A)$ in the category of ringed spaces, the length is then defined as $\sum_{a \in A} \dim(\mathcal{O}_{A,a})$, where $\mathcal{O}_{A,a}$ denotes the stalk of $\mathcal{O}_A$ at a. The *Douady space of length n*, denoted $X^{[n]}$, is the space which parametrizes all 0-dimensional subspaces of length n in X. It can be described more readily as follows:

Let $X^n := X \times \cdots \times X$ be the n-fold product of X on which the symmetric group S_n acts by permutations. The nth-symmetric product is the quotient space $X^{(n)} := X^n / S_n$ which is a complex orbifold. The singular set of $X^{(n)}$ is given by

$$\Delta = \bigcup_{i \neq j} \{ \ (x_1, \cdots, x_n) \in X^n \mid x_i = x_j \ \} \, .$$

Consider the map $\rho \colon X^{[n]} \to X^{(n)}$ which takes a 0-dimensional subspace $(A, \mathcal{O}_A)$ of length n to $(a_1, \cdots, a_1, \cdots, a_k, \cdots, a_k)$ in $X^{(n)}$ where each $a_j \in A$ appears exactly $\dim(\mathcal{O}_{A,a_j})$ times. Fogarty proves in [51] that $X^{[n]}$ is a smooth complex manifold and ρ induces a biholomorphism $X^{[n]} - \rho^{-1}(\Delta) \cong X^{(n)} - \Delta$. Thus, one may think of ρ as a desingularization of $X^{(n)}$. For $n = 2$ the space $X^{[n]}$ is obtained by the blow up along Δ. If the space X is projective the Douady space agrees with the *Hilbert scheme* $\mathrm{Hilb}^n(X)$ which parametrizes subschemes of length n. The following two examples are due to Beauville, see [16] and [15].

Hyperkähler Manifold of $K3^{[n]}$-type: For a $K3$-surface X the Hilbert Scheme $X^{[n]} := \mathrm{Hilb}^n(X)$ is an IHS-manifold. More precisely, in case X is not projective one should speak of the Douady space described above. The second cohomology lattice of $X^{[n]}$ is isomorphic to

$$\Lambda_{K3^{[n]}} := E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U \oplus I_{1,0}(-2(n-1)),$$

see [65] and [16, Proposition 6]. For the following see also [85, p.166].

Generalized Kummer Variety: Let $Y := T^4$ be a 2-dimensional complex torus which we consider as an abelian Lie group. Let $Y^{(m)}$ be its symmetric product and $\sigma\colon Y^{(m)} \to Y$ the natural map given by summing the m-points. Denote by $K^{m-1}(Y)$ the kernel of the composition $\mathrm{Hilb}^m(Y) \to Y^{(m)} \to Y$. Then $K^{m-1}(Y)$ is a $(2m-2)$-dimensional complex submanifold of $\mathrm{Hilb}^{(m)}(Y)$ and is moreover IHSM by [16, Prop. 8] with $b_2 = 7$.

There are certain methods to construct new examples out of old ones. One is given by deforming an IHS-manifold X (see also next chapter), e.g. one constructs a smooth proper morphism $\pi\colon \mathcal{X} \to \mathcal{S}$ between connected complex spaces $\mathcal{X}$ and $\mathcal{S}$ so that X is the fiber of a designated point $s_0 \in \mathcal{S}$. One can then show that the fiber $Y = \pi^{-1}(t)$, the *deformation* of X, is also an IHS-manifold if t is sufficiently close to s_0, see [65, Proposition 22.7] and in general [65, Section 22.1] for more details. A *smooth deformation* is one where the spaces $\mathcal{X}$ and $\mathcal{S}$ are smooth. For a brief overview on deformation theory see [19, Chapter I Section 10], [85, 4.9.2] and [75, Chapter 6]. In the case of deforming $\mathrm{Hilb}^n(X)$ we introduce the following notion.

Definition 3.5.2. *An* IHS*-manifold Y will be said to be of type $K3^{[n]}$ if there is a $K3$-surface X such that Y is a smooth deformation of* $\mathrm{Hilb}^n(X)$.

If X is an IHS-manifold sometimes a birational IHS-manifolds can be constructed by the so called Mukai flop for which we refer to [65, Example 21.7] and [65, Example 21.8]. Without going into details on birationality let us make the following comment to that notion. The condition of birationality for IHS-manifolds is rather strong compared to the general case. For instance, from [65, Proposition 21.6] we obtain that for IHS-manifolds X, X' a birational map $f\colon X \dashrightarrow X'$ can be described as follows. There exists open subspaces U and U' of X and X' respectively with $\mathrm{codim}_{\mathbb{C}}(X \setminus U) \geq 2$ and $\mathrm{codim}_{\mathbb{C}}(X' \setminus U') \geq 2$ such that f induces an isomorphism $U \to U'$. Let us mention that when X is not projective one should in fact use the term bimeromorphic instead of birational. However, we will also use the term birational and mean bimeromorphic in the non-projective case.

Lemma 3.5.3. *For birational* IHS*-manifolds X and X' there exists a natural isometry* $(\mathrm{H}^2(X', \mathbb{Z}), q_{X'}) \to (\mathrm{H}^2(X, \mathbb{Z}), q_X)$ *which preserves the Hodge decomposition. Furthermore, X' is a smooth deformation of X.*

Proof. The first statement is [76, Lemma 2.6] and the second [65, Proposition 27.8], see also [79]. □

Up until now all known examples of IHS-manifolds are smooth deformations of a manifold of type $K3^{[n]}$ or a generalized Kummer variety, except for two exceptional examples in complex dimension 10 [1] and 6 [117] both found by O'Grady. Now it is important to note that a smooth deformation $f\colon \mathcal{X} \to \mathcal{S}$ is a surjective proper holomorphic map between smooth spaces $\mathcal{X}$ and $\mathcal{S}$. The Ehresmann fibration theorem, see for instance [75, Corollary 6.2.3], states that f is a differentiable fiber bundle. Henceforth, although the fibers are not necessarily isomorphic as complex manifolds, they are all diffeomorphic. Therefore, there are only two known examples of hyperkählerian manifolds in dimensions $4n$, except for the two examples given by O'Grady in real dimensions 12 and 20.

3.6 Torelli Theorems for $K3$-Surfaces

Let us now briefly explain how one can study and distinguish non-isomorphic $K3$-surfaces.

Since a $K3$-surface $X = (M, I)$ is Kähler we may consider the weight 2 Hodge decomposition

$$\mathrm{H}^2(X; \mathbb{C}) = \mathrm{H}^{2,0}(X) \oplus \mathrm{H}^{1,1}(X) \oplus \mathrm{H}^{0,2}(X)$$

and its relative position to the lattice $\mathrm{H}^2(M, \mathbb{Z})$. For instance consider the so called Neron Severi lattice $\mathrm{NS}(X) \coloneqq \mathrm{H}^{1,1}(X) \cap \mathrm{H}^2(M, \mathbb{Z})$ and $Y = \mathrm{Kum}(T^4)$ a Kummer surface. Let e_i for $i = 1, \cdots, 16$ denote the Poincaré duals of the

16 classes E_i provided by the $\mathbb{C}P^1$ we glued in to replace the singular points in the Kummer constructions 3.5.1. The Kummer lattice Λ_K is then the smallest primitive sublattice of $\mathrm{H}^2(Y,\mathbb{Z})$ that contains all e_i. A $K3$-surface X is then a Kummer surface if and only if there exists a primitive embedding $\Lambda_K \to \mathrm{NS}(X)$, see [83, Section 14 Theorem 3.17] and [112].

The Hodge structure therefore provides a lot of information. In general, the *Torelli problems* ask to which extent the Hodge structure determine a $K3$-surface. Also, whether every Hodge structure is induced by one. That is, given a decomposition $\mathrm{H}^2(M,\mathbb{C}) = \mathrm{H}^{2,0} \oplus \mathrm{H}^{1,1} \oplus \mathrm{H}^{0,2}$ with $\dim \mathrm{H}^{2,0} = 1$ and $\overline{\mathrm{H}^{2,0}} = \mathrm{H}^{0,2}$, then is there a $K3$-surface which has this as its Hodge structure. These questions have been answered affirmatively by various authors, see [30], [101], [119], [121] and [81] for a modern survey. For the following theorem see for instance [14, p.332 (11.1) Theorem] and was formulated in this form by Burns and Rapoport [30].

Theorem 3.6.1 (Strong Torelli Theorem for $K3$-Surfaces)**.** *Let X and X' be $K3$-surfaces so that there exists an isometry $\phi\colon \mathrm{H}^2(X,\mathbb{Z}) \to \mathrm{H}^2(X',\mathbb{Z})$ such that*

- *ϕ is an isomorphism of Hodge structures*
- *ϕ takes some Kähler class of X to a Kähler class of X'.*

Then there exists a unique biholomorphism $f\colon X' \to X$ such that $f^* = \phi$.

One can translate the above theorem into a statement about injectivity of a map which is known as a period map. See also [85, Section 7.3.2] for a brief discussion and the chapter on $K3$-surfaces in [14] or [83] for a detailed treatment of the following. We use the language of marked spaces to make this more precise. A *marked $K3$-surface* is a pair (X,ϕ) where X is a $K3$-surface and

$\phi\colon \mathrm{H}^2(X,\mathbb{Z}) \to \Lambda_{K3}$ an isomorphism of lattices. The *moduli space of marked $K3$-surfaces* is then the set

$$\mathcal{M}^m_{K3} \coloneqq \{(X,\phi) \text{ marked K3-surfaces }\}/\sim$$

where isomorphic marked spaces are identified, i.e. $(X,\phi) \sim (X',\phi')$ if there is a biholomorphism $f\colon X \to X'$ such that in cohomology $\phi \circ f^* = \phi'$. Although it is not clear from the definition, $\mathcal{M}^m_{K3}$ has a natural topology which makes it a non-Hausdorff complex manifold. The definition of this topology is not trivial, uses deformation theory and the period map defined below, we refer to [14, Section VIII] for details. Define the *period domain* as the complex manifold

$$\mathbb{P}\mathrm{er} \coloneqq \{\sigma \in \mathbb{P}_{\mathbb{C}}(\Lambda_{K3}) \mid (\sigma,\bar{\sigma}) > 0 \text{ and } (\sigma,\sigma) = 0\}.$$

In the next section we will see that $\mathbb{P}\mathrm{er}$ parametrizes the Hodge structures which arise by $K3$-surfaces. The *period map* $\mathcal{P}\colon \mathcal{M}^m_{K3} \to \mathbb{P}\mathrm{er}$ is then defined by mapping (X,ϕ) to the so called *period point* $\phi_{\mathbb{C}}(\mathrm{H}^{2,0}(X))$. The local Torelli theorem now states that $\mathcal{P}$ is a local isomorphism of complex manifolds and is attributed to Andreotti and Weil, see [92, Thm 17]. The surjectivity of this map gives a positive answer to the question whether every possible Hodge structure is induced by a $K3$-surface and was proven by Todorov [126]. To phrase the second statement of Theorem 3.6.1 in terms of an injectivity result, we need to refine the period map and the period domain. For that we use the following very useful description of the Kähler cone. Recall that the *Kähler cone* of a Kähler manifold X is the set of Kähler classes $\mathrm{K\ddot{a}h}(X)$ in $\mathrm{H}^2_{DR}(X)$.

Theorem 3.6.2. *For a $K3$-surface X let $\Delta := \{c \in \mathrm{H}^2(X,\mathbb{Z}) \cap \mathrm{H}^{1,1}(X) \mid c^2 = (c,c) = -2\}$. Then*

$$\mathrm{K\ddot{a}h}(X) = \left\{ x \in \mathrm{H}^2(X,\mathbb{R}) \cap \mathrm{H}^{1,1}(X) \mid (x,c) > 0 \textit{ for all } c \in \Delta \right\}.$$

In the next section we will see that $x \in \mathrm{H}^2(X,\mathbb{C})$ is of type $(1,1)$ if and only if x is orthogonal to $\mathrm{H}^{2,0}(X)$. Motivated by Theorem 3.6.2 we define $\Delta(\Pi) := \{c \in \Lambda \cap \Pi^\perp \mid c^2 = -2\}$ for a point Π in $\mathbb{P}\mathrm{er}$. A *Kähler chamber* of Π in $\Lambda_{K3} \otimes \mathbb{R}$ is a connected component of the set

$$\left\{x \in (\Lambda_{K3} \otimes \mathbb{R}) \cap \Pi^\perp \mid (x,c) \neq 0 \text{ for all } c \in \Delta(\Pi)\right\}.$$

Let $KC(\Pi)$ denote the *set of Kähler chambers* determined by Π. The *refined period space* is then the space

$$\widetilde{\mathbb{P}\mathrm{er}} := \{(\Pi, C) \mid \Pi \in \mathbb{P}\mathrm{er} \text{ and } C \in KC(\pi)\}.$$

We also obtain a *refined period map* $\widetilde{\mathcal{P}} \colon \mathcal{M}^m_{K3} \to \widetilde{\mathbb{P}\mathrm{er}}$ by associating to a marked space (X,ϕ) the pair $(\phi_{\mathbb{C}}(\mathrm{H}^{2,0}(X)), \phi_{\mathbb{C}}(\mathrm{K\ddot{a}h}(X)))$. The strong Torelli theorem for $K3$-surfaces can then be refined to the statement that the refined period map is bijective, which is a consequence of a result due to Looijenga [98].

The *Weyl group* of a $K3$-surface X is the group

$$\mathrm{W}(X) \coloneqq \{r_c \in \mathrm{O}\left(\mathrm{H}^2(X, \mathbb{R})\right) \mid r_c \text{ a reflection along } c \in \Delta\},$$

i.e. the group of reflections along (-2)-classes of type $(1,1)$. The group acts transitively on the set of Kähler chambers while preserving the Hodge structure, see [83, Proposition 5.5]. From this and the Strong Torelli theorem it follows that two marked $K3$-surfaces (X, ϕ) and (X', ϕ') are biholomorphic if $\mathcal{P}(X, \phi) = \mathcal{P}(X', \phi')$, see also [14, p.333 (11.2) Theorem]. This is known as the Weak Torelli Theorem.

Recall that a hyperkähler metric g defined on the $K3$-manifold M induces a 2-sphere of complex structures on M. A complex structure obtained in this way endows M with the structure of a complex $K3$-surface. Thus, the Torelli theorems provide an extremely useful tool when studying hyperkähler metrics and can be used to prove the following theorem on the moduli space of hyperkähler metrics, which is originally due to Todorov [125] and Looijenga [99].

Theorem 3.6.3. *The moduli space of hyperkähler metrics of unit volume on the $K3$-manifold is homeomorphic to*

$$\mathrm{O}(\Lambda_{K3}) \setminus \left(\mathrm{Gr}^+(3, \Lambda \otimes \mathbb{R}) - \bigcup_{z\ (-2)-classes} \mathrm{Gr}^+(3, z^\perp)) \right).$$

Note, by Theorem 3.5.2 the above is also a description for the moduli space of unit volume Einstein metrics on the $K3$-manifold.

In the following sections we will see analogous results on period maps for general IHS-manifolds. However, instead of working with marked spaces we will introduce and use the notion of Teichmüller spaces. In doing so we will recover a description of the Teichmüller space of hyperkähler metrics due to Amerik and Verbistky [3] and generalize the above theorem to higher dimensional hyperkählerian manifolds.

Before ending this chapter note that points in $\bigcup_z \mathrm{Gr}^+(3, z^\perp)$ do not correspond to any hyperkähler metrics according to Theorem 3.6.3. A natural question is if these points still represent geometrically meaning full objects. This has been answered in the case of the K3-manifold by Kobayashi and Todorov, see [90], and [7] for a different approach. They show that if one allows not only smooth hyperkähler metrics but also metrics having certain singularities, then points in the above space naturally correspond to these singular metrics. For more details to this we refer to [90] and [7], see also Sections 7.5 and 7.6.

Was eine Kurve ist, glaubt jeder Mensch zu wissen, bis er so viel Mathematik gelernt hat, daß ihn die unzähligen möglichen Abnormitäten verwirrt gemacht haben.

Felix Klein

4

Teichmüller Spaces of Irreducible Holomorphic Symplectic Structures

We consider the space of complex structures on a fixed hyperkählerian manifold M and discuss the Teichmüller space $\mathcal{T}^{\mathrm{Cpl}}(M)$ of irreducible holomorphic symplectic structures on M. The Teichmüller space can be understood as a type of moduli space, parametrizing the complex structures up to special isomorphisms. We then explain how IHS-structures can be studied in terms of their induced Hodge structure and introduce the notions of a period domain and the complex period map. The main goal of this chapter is then to state a surjectivity result of Huybrechts [76], and discuss injectivity statements by the work of Verbitsky [132, 135] and Markman [42]. See also [65] for some backgrounds.

4.1 Teichmüller Space and Deformations

We introduce the complex Teichmüller space on a compact smooth manifold M of even dimension. Furthermore, we briefly discuss deformation theory and its relation to the Teichmüller space. For more details we refer to [32], see also [122].

Recall that an *almost complex structure* on M is a section

$$I \colon M \to \mathrm{End}(TM)$$

of the endomorphism bundle $\mathrm{End}(TM)$, such that $I^2 = -Id$. The *space of all almost complex structures* on M, denoted $\mathcal{AC}omp(M)$, is endowed with the topology of smooth convergence. With this choice of topology $\mathcal{AC}omp(M)$ has the structure of an infinite dimensional Fréchet manifold.

Almost complex structures are of particular interest if they are induced by a complex structure. It is well known that I is induced in such a way if and only if I is integrable, see [111]. Recall that *integrablity* means that the Lie bracket of vector fields preserves the splitting $TM_{\mathbb{C}} = TM^{1,0} \oplus TM^{0,1}$ into eigensubbundles of I, i.e. integrability is the condition

$$[V, W] \in TM^{1,0},$$

for all vector fields V, W in $TM^{1,0}$. In this way we may view the *space of all complex structures* on M, denoted $\mathcal{C}omp(M)$, as a closed subspace of $\mathcal{AC}omp(M)$. Let us also state that by a complex structure we often mean its associated integrable almost complex structure and vice versa. A complex manifold X is in this sense a tuple $X = (M, I)$.

It turns out to be convenient for us not to consider the space of all complex structures, but restrict to those which are Kähler and admit, up to a constant, a unique holomorphic symplectic form, i.e. irreducible holomorphic symplectic structures. Thus, we define $\mathcal{C}omp_{IHS}(M)$ to be the subspace of $\mathcal{C}omp(M)$ consisting of complex structures which are IHS.

The space $\mathcal{C}omp_{IHS}(M)$, as all the others above, comes with an action by the *diffeomorphism group* $\mathrm{Diff}(M)$ provided by

$$\phi^* I := d\phi^{-1} \circ I \circ d\phi,$$

for ϕ in $\mathrm{Diff}(M)$. Naturally one would be interested in studying the orbit space, also known as the *moduli space of complex structures*. However, this space is often very ill behaved, see [134] for the case that M is hyperkählerian. A more fruitful notion is provided by the Teichmüller space.

Definition 4.1.1. *The full Teichmüller space of complex structures is*

$$\mathcal{T}^{\mathrm{Cpl}}_{\mathrm{Full}}(M) \coloneqq \mathcal{C}omp(M)/\mathrm{Diff}_0(M),$$

where $\mathrm{Diff}_0(M)$ *denotes the identity component of* $\mathrm{Diff}(M)$*.*

Since $\mathcal{T}^{\mathrm{Cpl}}_{\mathrm{Full}}(M)$ possibly also contains elements which are not induced by hyperkähler metrics we are more interested in the following subspace of $\mathcal{T}^{\mathrm{Cpl}}_{\mathrm{Full}}(M)$.

Definition 4.1.2. *The Teichmüller space of irreducible holomorphic symplectic structures is*

$$\mathcal{T}^{\mathrm{Cpl}}(M) \coloneqq \mathcal{C}omp_{IHS}(M)/\mathrm{Diff}_0(M)$$

and will be called the complex Teichmüller space.

The Teichmüller spaces above can be studied in terms of deformation theory of complex manifolds. A deformation of $X = (M, I)$, in this context, is a holomorphic submersion $f\colon \mathcal{X} \to \mathcal{S}$ between complex spaces $\mathcal{X}, \mathcal{S}$, possibly singular and non-reduced, such that $f^{-1}(0) \cong X$, where 0 is some base point of $\mathcal{S}$. The deformation f is also called a *family over* $\mathcal{S}$ if each fiber gives rise to a smooth complex manifold. However, we will mostly not need this general version, but consider the case when $\mathcal{X}$ and $\mathcal{S}$ are complex manifolds only. Of particular importance is then the notion of deformation equivalence, which we define now.

Given two complex manifolds $X = (M, I)$ and $X' = (M, I')$. Then X and X' are *disk deformation equivalent*, denoted $X \sim_{disk} X'$, if and only if the following holds. There is a proper holomorphic submersion $\pi\colon \mathcal{X} \to \Delta$, with connected fibers over the unit disk $\Delta \subset \mathbb{C}$ satisfying the following property:

There exists $t_1, t_2 \in \Delta$ with

- $\pi^{-1}(t_1)$ is biholomorphic to X,
- $\pi^{-1}(t_2)$ is biholomorphic to X'.

Two complex manifolds $Y = (M, J)$ and $Y' = (M, J')$ are *deformation equivalent*, if there is a sequence $X_1, \cdots, X_n$ of disk deformation equivalent spaces such that

$$Y \sim_{disk} X_1 \sim_{disk} X_2 \sim \cdots \sim X_n \sim_{disk} Y'.$$

From [32, Corollary 6] we get the following theorem.

Theorem 4.1.1. *Two complex manifolds X and X' are in the same connected component of $\mathcal{T}^{\mathrm{Cpl}}_{\mathrm{Full}}(M)$ if and only if they are deformation equivalent.*

For $\mathcal{T}^{\mathrm{Cpl}}(M)$ one can say more. The *Kuranishi space* $\mathfrak{B}(X)$ is the germ, in the sense of complex spaces, which parametrizes all small deformations of the complex manifold $X = (M, I)$. This means that there is a family $\pi \colon \mathcal{F} \to \mathfrak{B}(X)$ such that any deformation $f \colon \mathcal{X} \to \mathcal{S}$ of X is induced by a pullback of π in a small neighborhood of the base point $0 \in \mathcal{S}$.

There exists a surjective map $\mathfrak{B}(X) \to \mathcal{U}_I$, where $\mathcal{U}_I$ is some neighborhood of I in $\mathcal{T}^{\mathrm{Cpl}}_{\mathrm{Full}}(M)$. In general this map is not a homeomorphism. However, for Kähler manifolds with trivial canonical bundle one obtains that this map is indeed an isomorphism [32, Prop.15]. In particular, this is the case for IHS-manifolds. Furthermore, by [19, 14.31 Theorem] respectively [20] we get that $\mathfrak{B}(X)$ is smooth. This implies the following proposition.

Proposition 4.1.1. *The complex Teichmüller space $\mathcal{T}^{\mathrm{Cpl}}(M)$ is locally homeo-*

morphic to some $\mathbb{C}^n$.

Later we will see that $\mathcal{T}^{\mathrm{Cpl}}(M)$ is not Hausdorff.

4.2 The Complex Period Domain

Our goal is to study the Teichmüller space $\mathcal{T}^{\mathrm{Cpl}}(M)$ by using Hodge decompositions. For IHS-manifolds these Hodge decompositions are parametrized by what is called a period domain. In this section we introduce the weight 2 Hodge structures of IHS-manifolds and the corresponding period domain. We will always mean the weight 2 Hodge structures of an IHS-manifold when we speak of its Hodge structure, ignoring the ones in other degrees.

Let $X = (M, I)$ be an IHS-manifold. By definition X is Kähler, and thus we get an induced Hodge structure on $\mathrm{H}^2(M, \mathbb{C})$, i.e. a decomposition of complex vector spaces

$$\mathrm{H}^2(M, \mathbb{C}) = \mathrm{H}^{2,0}(I) \oplus \mathrm{H}^{0,2}(I) \oplus \mathrm{H}^{1,1}(I).$$

Recall that this decomposition only depends on the complex structure I and not on any Kähler metric. We also recall, for instance from [75, Corollary 2.6.21], that $\mathrm{H}^{2,0}(I)$ is isomorphic to $\mathrm{H}^0(X, \Omega^2)$, the space of holomorphic 2-forms. For an IHSM this space is spanned by a unique holomorphic 2-form σ and thus we have

$$\mathrm{H}^{2,0}(I) = \sigma \cdot \mathbb{C} \text{ and } \mathrm{H}^{0,2}(I) = \bar{\sigma} \cdot \mathbb{C}.$$

With respect to the BB-form q recall from Lemma 3.4.1 that the space of $(1,1)$-classes $\mathrm{H}^{1,1}(I)$ is orthogonal to $\mathrm{H}^{2,0}(I) \oplus \mathrm{H}^{0,2}(I)$. Another straightforward computation shows that

$$q(\sigma,\sigma)=0 \text{ and } q(\sigma,\bar{\sigma})>0.$$

We find that the Hodge decomposition of an IHSM is completely determined by the holomorphic 2-form σ. On the other hand, any vector v in $\mathrm{H}^2(M,\mathbb{C})$ satisfying $q(v,v)=0$ and $q(v,\bar{v})>0$ determines a Hodge structure on $\mathrm{H}^2(M,\mathbb{C})$ by setting

$$\mathrm{H}^{2,0}:=v\cdot\mathbb{C},\quad \mathrm{H}^{2,0}:=\bar{v}\cdot\mathbb{C},\quad \mathrm{H}^{1,1}:=(v\cdot\mathbb{C}\oplus\bar{v}\cdot\mathbb{C})^{\perp}.$$

Clearly, if v' is a scalar multiple of v the induced Hodge structures are the same. A Hodge decomposition like above is said to be of IHSM-*type*.

Definition 4.2.1. *The complex period domain is defined as the projectivization*

$$\mathbb{P}\mathrm{er}(M):=\mathbb{P}\left(\{v\in\mathrm{H}^2(M,\mathbb{C})\mid q(v,v)=0 \textit{ and } q(v,\bar{v})>0\}\right).$$

The complex period domain is an open subspace of the quadric $\{q(v,v)=0\}$ in $\mathbb{P}(\mathrm{H}^2(M,\mathbb{C}))$. In particular, $\mathbb{P}\mathrm{er}(M)$ has the structure of a complex manifold.

By the previous discussion we may think of $\mathbb{P}\mathrm{er}(M)$ as the space of Hodge structures of IHSM-type. There is another way to think about $\mathbb{P}\mathrm{er}(M)$ which turns out to be very convenient for us. See also [65, Section 25.4] for the following statements.

Lemma 4.2.1. *The following map is a diffeomorphism,*

$$\begin{aligned} \mathbb{P}\mathrm{er}(M) &\to \mathrm{Gr}^{+,\mathrm{o}}(2, \mathrm{H}^2(M, \mathbb{R})) \\ [v] &\mapsto \mathrm{span}_{\mathbb{R}} \left(\mathrm{Re}(v), \mathrm{Im}(v)\right), \end{aligned}$$

with orientation provided by the basis $(\mathrm{Re}(v), \mathrm{Im}(v))$*.*

Proof. An inverse of the above map can be defined as follows. For a positive oriented 2-space P we choose an oriented orthonormal basis (v_1, v_2). Then we set $[v]$ to be the line generated by $v = v_1 + iv_2$. □

The Grassmann space $\mathrm{Gr}^{+,\mathrm{o}}(2, \mathrm{H}^2(M, \mathbb{R}))$ gives another way to view $\mathbb{P}\mathrm{er}(M)$, namely in terms of a homogeneous space as it follows from Lemma 2.3.1 and Theorem 3.4.2.

Lemma 4.2.2. *The Grassmann space* $\mathrm{Gr}^{+,\mathrm{o}}(2, \mathrm{H}^2(M, \mathbb{R}))$ *is homeomorphic to*

$$\mathrm{O}(3, \mathrm{b}_2 - 3)/\left(\mathrm{SO}(2) \times \mathrm{O}(1, \mathrm{b}_2 - 3)\right),$$

where b_2 *is the second Betti number of* M*.*

Overall we obtain the following topological description of $\mathbb{P}\mathrm{er}(M)$.

Corollary 4.2.3. *The complex period domain* $\mathbb{P}\mathrm{er}(M)$ *is a connected and simply connected complex manifold of complex dimension* $\mathrm{b}_2 - 2$*.*

Proof. The only thing left to be proved is connectedness and simply connectedness. Recall that $\mathrm{O}(3, \mathrm{b}_2 - 3)$ and $\mathrm{O}(1, \mathrm{b}_2 - 3)$ have 4 connected components, which can be distinguished by the determinant and the spinor norm. The inclusion $\mathrm{O}(1, \mathrm{b}_2 - 3) \hookrightarrow \mathrm{O}(3, \mathrm{b}_2 - 3)$ respects these and thus it also respects the connected components. We conclude that

$$\mathrm{O}(3, \mathrm{b}_2 - 3) / \left(\mathrm{SO}(2) \times \mathrm{O}(1, \mathrm{b}_2 - 3)\right)$$

is connected.

Let $\mathrm{SO}^o(3, \mathrm{b}_2 - 3)$ denote the connected component of the identity in $\mathrm{O}(3, \mathrm{b}_2 - 3)$ and $\mathrm{SO}^o(1, \mathrm{b}_2 - 3)$ the corresponding one for $\mathrm{O}(1, \mathrm{b}_2 - 3)$. Then $\mathbb{P}\mathrm{er}(M)$ is homeomorphic to

$$\mathrm{SO}^o(3, \mathrm{b}_2 - 3) / \left(\mathrm{SO}(2) \times \mathrm{SO}^o(1, \mathrm{b}_2 - 3)\right).$$

The quotient map $\mathrm{SO}^o(3, \mathrm{b}_2 - 3) \to \mathrm{SO}^o(3, \mathrm{b}_2 - 3) / \left(\mathrm{SO}(2) \times \mathrm{SO}^o(1, \mathrm{b}_2 - 3)\right)$ is a fiber bundle with fiber $\mathrm{SO}(2) \times \mathrm{SO}^o(1, \mathrm{b}_2 - 3)$. By i denote the inclusion $\mathrm{SO}(2) \times \mathrm{SO}^o(1, \mathrm{b}_2 - 3) \hookrightarrow \mathrm{SO}^o(3, \mathrm{b}_2 - 3)$. From the long exact sequence on homotopy groups we get the exact sequence

$$\cdots \to \pi_1(\mathrm{SO}(2) \times \mathrm{SO}^o(1, \mathrm{b}_2 - 3)) \xrightarrow{i_*} \pi_1(\mathrm{SO}^o(3, \mathrm{b}_2 - 3)) \to \pi_1(\mathbb{P}\mathrm{er}(M))) \to 0.$$

We show that i_* is surjective which then implies that $\pi_1(\mathbb{P}\mathrm{er}(M))) = 0$.

Recall that $\mathrm{SO}(p) \times \mathrm{SO}(q)$ is a maximal subgroup of $\mathrm{SO}^o(p, q)$, in particular they are homotopy equivalent. Now consider the following commutative diagram:

$$\begin{array}{ccc}
\pi_1(\mathrm{SO}(2)) \times \pi_1(\mathrm{SO}(\mathrm{b}_2-3)) & \longrightarrow & \pi_1(\mathrm{SO}(2)) \times \pi_1(\mathrm{SO}^o(1,\mathrm{b}_2-3)) \\
\downarrow & & \downarrow{\scriptstyle i_*} \\
\pi_1(\mathrm{SO}(3)) \times \pi_1(\mathrm{SO}(\mathrm{b}_2-3)) & \longrightarrow & \pi_1(\mathrm{SO}^o(3,\mathrm{b}_2-3))
\end{array}$$

All maps are induced by inclusions, furthermore, the horizontal maps are isomorphisms. The inclusion $\mathrm{SO}(2) \hookrightarrow \mathrm{SO}(3)$ is surjective on the level of fundamental groups. This can be seen by the long exact sequence of homotopy groups associated to the fibration $\mathrm{SO}(3) \to \mathrm{SO}(3)/\mathrm{SO}(2) \cong S^2$. Thus, we conclude that i^* is surjective as well. $\square$

4.3 The Complex Period Map Part I

We define the complex period map and state a surjectivity theorem of Huybrechts in [76]. Furthermore, we introduce the birational Teichmüller space and give first statements about the fibers of the complex period map based on the work of Verbitsky [132, 135], see also [82].

Definition 4.3.1. *The complex period map on a hyperkählerian manifold M is*

$$\mathcal{P}^{\mathrm{Cpl}} \colon \mathcal{T}^{\mathrm{Cpl}}(M) \to \mathbb{P}\mathrm{er}(M),$$

defined by mapping the complex structure I to $[\sigma]$, where σ is the nowhere vanishing holomorphic 2-form of the IHS-*manifold (M, I).*

Recall that $\mathrm{H}^{2,0}(I)$ is spanned by $[\sigma]$ and thus $\mathcal{P}^{\mathrm{Cpl}}(I)$ is the complex line

$\mathrm{H}^{2,0}(I)$, when considered as a point in $\mathbb{P}\mathrm{er}(M)$. Huybrechts proved that the complex period is surjective. More precisely he showed the following theorem, see [76, Section 8],[77].

Theorem 4.3.1. *Let $\mathcal{T}_o^{\mathrm{Cpl}}(M)$ be a connected component of $\mathcal{T}^{\mathrm{Cpl}}(M)$. Then the period map restricted to the connected component*

$$\mathcal{P}^{\mathrm{Cpl}}\colon \mathcal{T}_o^{\mathrm{Cpl}}(M) \to \mathbb{P}\mathrm{er}(M),$$

is surjective.

In general the map $\mathcal{P}^{\mathrm{Cpl}}$ fails to be injective, even when restricted to connected components. However, Verbitsky proves that the complex period map is generically injective on connected components. In [132, 135] (see also [82]), he constructs a Hausdorff version of $\mathcal{T}^{\mathrm{Cpl}}(M)$ over which the period map factorizes. Let us explain what this means.

Definition 4.3.2. *Let X be a topological space which is not Hausdorff. Two points x, y in X are inseparable, sometimes also called non-Hausdorff, if for all open neighborhoods U_x of x and U_y of y one has*

$$U_x \cap U_y \neq \emptyset.$$

The notion of inseparability is in the general case not an equivalence relation, as it fails to be transitive. However, by studying the period map with lots of insights into the Teichmüller space, Verbitsky proved that inseparability turns out to be an equivalence relation for the space $\mathcal{T}^{\mathrm{Cpl}}(M)$.

Definition 4.3.3. *The birational Teichmüller space $\mathcal{T}_b^{\mathrm{Cpl}}(M)$ is defined as*

$$\mathcal{T}_b^{\mathrm{Cpl}}(M) \coloneqq \mathcal{T}^{\mathrm{Cpl}}(M)/\sim,$$

where two points are identified if they are inseparable.

The choice for the name will come apparent by the following two theorems also proven by Verbitsky in [132, 135], see also [82].

Theorem 4.3.2. *The period map $\mathcal{P}^{\mathrm{Cpl}}$ is a local isomorphism which factors through*

$$\mathcal{P}_b^{\mathrm{Cpl}} \colon \mathcal{T}_b^{\mathrm{Cpl}}(M) \to \mathbb{P}\mathrm{er}(M).$$

The map $\mathcal{P}_b^{\mathrm{Cpl}}$ is a trivial covering, i.e. a homeomorphism on connected components.

From the same work [132, 135] we get a description of the fibers of $\mathcal{P}^{\mathrm{Cpl}}$.

Theorem 4.3.3. *Let $\mathcal{T}_o^{\mathrm{Cpl}}(M)$ be a connected component of $\mathcal{T}^{\mathrm{Cpl}}(M)$ and consider the period map $\mathcal{P}^{\mathrm{Cpl}}$ restricted to $\mathcal{T}_o^{\mathrm{Cpl}}(M)$. Then each fiber of $\mathcal{P}^{\mathrm{Cpl}}$ in $\mathcal{T}_o^{\mathrm{Cpl}}(M)$ is finite. Furthermore, the fibers with more than one point correspond exactly to the non-Hausdorff points of $\mathcal{T}_o^{\mathrm{Cpl}}(M)$. If I and I' are contained in the same fiber in $\mathcal{T}_o^{\mathrm{Cpl}}(M)$, then I is birational to I'.*

Note that the theorem does not say that if I and I' are birational that they also belong to the same fiber.

Later we will provide an even more detailed picture of the fibers of $\mathcal{P}^{\mathrm{Cpl}}$. This will be done once we have introduced the notion of MBM-classes and Kähler chambers.

4.4 The Kähler Cone of an IHS-manifold

Recall that a Kähler class on a complex manifold $X := (M, I)$ is a class $[\omega]$ in the de Rham cohomology $\mathrm{H}^2_{DR}(M)$ for which there exists a Kähler form ω, with respect to I, representing $[\omega]$. By de Rham's theorem we will identify de Rham cohomology with singular cohomology $\mathrm{H}^2(M, \mathbb{R})$. In the following we will be interested in the set of Kähler classes as the complex structure I may vary while M is fixed. Recall the definition:

Definition 4.4.1. *The Kähler cone* $\mathrm{K\ddot{a}h}(I)$ *is defined as the set of Kähler classes in* $\mathrm{H}^2(M, \mathbb{R})$.

On a compact Kähler manifold every Kähler class $[\omega]$ is real and of type $(1, 1)$, meaning $[\omega] \in \mathrm{H}^{1,1}(I) \cap \mathrm{H}^2(M, \mathbb{R})$. If X is furthermore assumed to be IHSM, then the second cohomology group is endowed with the BB-form q. For a Kähler form ω, one then finds

$$q([\omega], [\omega]) = c \int \omega^2 \wedge (\sigma \wedge \bar{\sigma})^{n-1} > 0,$$

with c a positive constant. Thus, $\mathrm{K\ddot{a}h}(I)$ is contained in the following cone:

Definition 4.4.2. *The positive cone of* $X = (M, I)$ *is the cone of positive* $(1, 1)$

classes which contains a Kähler class, i.e. Pos*(I) is the connected component of*

$$\{v \in \mathrm{H}^2(M,\mathbb{R}) \mid q(v,v) > 0\} \cap \mathrm{H}^{1,1}(I),$$

which contains $\mathrm{Käh}(I)$*.*

Recall that a *cone* in a $\mathbb{R}$-vector space V is a subset K such that $\mathbb{R}_{>0}\cdot K = K$. A *convex cone* is then a cone K if for every $v, w \in K$ also $t_1 v + t_2 w$ is contained in K whenever $t_1, t_2 > 0$. Clearly $\mathrm{Pos}(I)$ is a convex cone. From [75, Cor. 3.1.8] we get that the same is true for the Kähler cone, more precisely we get:

Lemma 4.4.1. *The Kähler cone* $\mathrm{Käh}(I)$ *on a compact complex Kähler manifold is a connected convex cone, which furthermore is open in* $\mathrm{H}^{1,1}(I) \cap \mathrm{H}^2(M,\mathbb{R})$*.*

Note that connected cones are contractible and so is $\mathrm{Käh}(I)$. Also note that the boundary of a convex cone is in principle enough to recover the cone itself.

Definition 4.4.3. *A wall or face of a cone $K \subset V$ is a subset F of the boundary ∂K, for which there exists a codimension* 1 *subspace H of V with $\partial K \cap H = F$ so that F is has non-empty interior in H.*

It is not true that the boundary $\partial\mathrm{Käh}(I)$ is entirely decomposed into faces. This is most easily seen in the extreme case when $\mathrm{Käh}(I) = \mathrm{Pos}(I)$, in which case $\mathrm{Käh}(I)$ has no faces at all. However, for IHS-manifolds the boundary of the Kähler cone is decomposed into a so called round part, which will turn out to be part of the boundary of the positive cone, and into faces. The faces can be determined in terms of rational curves, which we will explain now. First we recall that by a *rational curve* we mean a generically injective holomorphic map

$$f\colon \mathbb{C}P^1 \to X.$$

The image of f defines a class C in $\mathrm{H}_2(M,\mathbb{Z})$. The bilinear pairing q induces an isomorphism between rational homology and cohomology

$$\mathrm{H}_2(M,\mathbb{Q}) \cong \mathrm{H}^2(M,\mathbb{Q}).$$

With this identification in mind, we may view C as rational cohomology class. The following theorem is due to Huybrechts and Boucksom, see [79] and [27].

Theorem 4.4.1. *Let $X = (M, I)$ be an* IHS*-manifold. Then the Kähler cone* $\mathrm{Käh}(I)$ *is the subcone of* $\mathrm{Pos}(I)$ *which is given by those elements which are strictly positive on all rational curves, i.e.*

$$\mathrm{Käh}(I) = \{\omega \in \mathrm{Pos}(I) \mid q(\omega, C) > 0 \textit{ for all rational curves } C\}.$$

A face of $\mathrm{Käh}(I)$ is therefore of the form $\mathrm{Pos}(I) \cap C^{\perp}$, where $C^{\perp}$ is the orthogonal complement of the rational curve C in $\mathrm{H}^2(M,\mathbb{R})$. In other words, the above theorem states that the Kähler cone is a connected component of

$$\mathrm{Pos}(I) - \bigcup_{C \text{ rational curve}} C^{\perp}.$$

Also note that a connected component K of $\mathrm{Pos}(I) - \bigcup_{C \text{ rational curve}} C^{\perp}$ is determined by the sequence of plus and minus signs $q(K, C)$ as C ranges over the set of all rational curves and $\mathrm{K\ddot{a}h}(I)$ is the one with only positive signs.

An important question now is to which extent one can determine the cone $\mathrm{K\ddot{a}h}(I)$ from data induced by the period $\mathcal{P}^{\mathrm{Cpl}}(I)$. Note, that for the positive cone this is easy since

$$\mathrm{Pos}(I) = \mathcal{P}^{\mathrm{Cpl}}(I)^{\perp} \cap \{v \in \mathrm{H}^2(M, \mathbb{R}) \mid q(v, v) > 0\}.$$

For the Kähler cone this is much more difficult. A problem one faces is that determining rational curves is very hard and in full generality they furthermore behave badly under deformations. However, there is a partial solution to this problem due to Amerik and Verbitsky, see [49] for a survey. We first need the following definition, which will also be of interest at other parts.

Definition 4.4.4. *The monodromy group* $\mathrm{Mon}^2(I)$ *of an* IHS*-manifold* $X = (M, I)$ *is the subgroup of* $\mathrm{O}(\mathrm{H}^2(M, \mathbb{Z}), q)$ *induced by the monodromy action of Gauss-Manin local systems, for all deformations of* X *over a connected complex analytic base.*

For the notion of local systems and Gauss-Manin connections see for instance [46], and [43] for more on the monodromy group. We will mostly not use Definition 4.4.4 but think of the monodromy group like it is presented in the following theorem, which is [132, 135, Thm. 7.2].

Theorem 4.4.2. *Let* $X = (M, I)$ *be an* IHS*-manifold and let* $\mathcal{T}_o^{\mathrm{Cpl}}(M)$ *be the connected component of* $\mathcal{T}^{\mathrm{Cpl}}(M)$ *containing* I*. Furthermore, let* $\mathrm{Diff}_I(M)$ *de-*

note the subgroup of $\mathrm{Diff}(M)$ *which preserves the connected component* $\mathcal{T}_o^{\mathrm{Cpl}}(M)$*. Then* $\mathrm{Mon}^2(I)$ *is the group generated by all isometries* $\phi^*\colon \mathrm{H}^2(M,\mathbb{Z}) \to \mathrm{H}^2(M,\mathbb{Z})$ *with* $\phi \in \mathrm{Diff}_I(M)$*.*

Note that $\mathrm{Mon}^2(I)$ only depends on the connected component $\mathcal{T}_o^{\mathrm{Cpl}}(M)$. We can now introduce the notion of MBM-classes, short for minimal birational monodromy.

Definition 4.4.5. *A class* $z \in \mathrm{H}^2(M,\mathbb{Z})$ *with* $z^2 < 0$ *of type* $(1,1)$ *is an* MBM-*class if and only if there exists some* $\gamma \in \mathrm{Mon}^2(I)$ *and a complex structure* I' *birational to* I *such that* $\gamma(z)^\perp \cap \partial\mathrm{K\ddot{a}h}(I')$ *is a face of the Kähler cone* $\mathrm{K\ddot{a}h}(I')$*.*

The name minimal in MBM comes from the notion of special rational curves on (M, I) which are minimal in the following sense. They cannot be bend and broken, see [38, Chapter 3] for this notion, behave well under deformation and are sufficient to determine the Kähler cone. Amerik and Verbitsky were able to prove the following important theorem, see [2, Thm 6.2] which improves Theorem 4.4.1.

Theorem 4.4.3. *Let* S *denote the subset of* MBM-*classes in* $\mathrm{H}^2(M,\mathbb{Z})$*. Then the Kähler cone is a connected component of*

$$\mathrm{Pos}(I) - \bigcup_{z\in S} z^\perp.$$

Although the definition of MBM-classes still depends on the complex structure, i.e. they are not determined by the period point in any obvious way, Amerik and Verbitsky showed that it only depends on the deformation type, see [2, Cor. 5.13].

Theorem 4.4.4. *Let $X = (M, I)$ and $X' = (M, I')$ be deformation equivalent and z an* MBM*-class with respect to I. If z is also of type* $(1,1)$ *with respect to I', then z is* MBM *with respect to I'.*

As a consequence, we find that MBM-classes only depend on the connected components of the Teichmüller space. To be more precise, if z is MBM for one complex structure I in $\mathcal{T}_o^{\mathrm{Cpl}}(M)$, then z is MBM for all $I' \in \mathcal{T}_o^{\mathrm{Cpl}}(M)$ for which z is of type $(1,1)$. Thus, when a connected component of $\mathcal{T}^{\mathrm{Cpl}}(M)$ is fixed, we define the set of MBM classes as $\mathrm{MBM}(\mathcal{T}_o^{\mathrm{Cpl}}(M))$ to be the set of those $z \in \mathrm{H}^2(M,\mathbb{Z})$ which are MBM for some I' in $\mathcal{T}_o^{\mathrm{Cpl}}(M)$. A class $z \in \mathrm{MBM}(\mathcal{T}_o^{\mathrm{Cpl}}(M))$ is then MBM with respect to I if and only if z is in $\mathcal{P}^{\mathrm{Cpl}}(I)^{\perp}$. Thus, in the following we will often call a class MBM if it is MBM for some IHS-structure on M. For a IHS-manifold $X = (M, I)$ we then also often say that a MBM-class c is of type $(1,1)$ to emphasize that c is MBM with respect to I.

In general it is still difficult to determine which classes are of MBM-type. For the $K3$-surface however, these are precisely the (-2)-classes in $\mathrm{H}^2(M,\mathbb{Z})$, recall Theorem 3.6.2.

4.5 The Complex Period Map Part II

The main goal of this section is to discuss the fibers of the complex period map. For that we consider $\mathcal{P}^{\mathrm{Cpl}}$ restricted to a connected component $\mathcal{T}_o^{\mathrm{Cpl}}(M)$ of the complex Teichmüller space.

From the previous section we know that the Kähler cone of an IHSM $X = (M, I)$ is a connected component of $\mathrm{Pos}(I) - \bigcup_z z^\perp$ with z running through the set of MBM-classes of type $(1,1)$.

Definition 4.5.1. *A connected component of* $\mathrm{Pos}(I) - \bigcup_z z^\perp$ *is called a Kähler chamber.*

Let I' be a complex structure in $\mathcal{T}_o^{\mathrm{Cpl}}(M)$ such that I' is in the fiber of $\mathcal{P}^{\mathrm{Cpl}}(I)$. Since I' is deformation equivalent to I with $\mathrm{H}^{1,1}(I) = \mathrm{H}^{1,1}(I')$, the two complex structures share the same MBM-classes. Thus, the Kähler cone of I' is possibly another connected component of

$$\mathrm{Pos}(I) - \bigcup_z z^\perp.$$

From Theorem 4.4.3 it follows that if $\mathrm{Käh}(I) \cap \mathrm{Käh}(I') \neq \emptyset$, then $\mathrm{Käh}(I) = \mathrm{Käh}(I')$.

Let $\mathrm{Mon}^2_{\mathrm{Hdg}}(I)$ be the subgroup of $\mathrm{Mon}^2(I)$ which consists of those orthogonal transformations of $\mathrm{H}^2(M, \mathbb{Z})$ which preserve the Hodge structure induced by I. This group acts on the set of Kähler chambers, i.e. it takes Kähler chambers to Kähler chambers. Markman makes the following definition, compare [42].

Definition 4.5.2. *A Kähler type chamber is a subcone of* $\mathrm{Pos}(I)$ *of the form*

$$g[f^*\mathrm{Käh}(I')],$$

where f is a bimeromorphisms of I with I' and $g \in \mathrm{Mon}^2_{\mathrm{Hdg}}(I)$.

We want to show that Kähler-type chambers and Kähler chambers are the same. For that we need the following definition.

Definition 4.5.3. *A class $\alpha \in \mathrm{Pos}(I)$ is very general if*

$$\alpha^{\perp} \cap \mathrm{H}^{1,1}(I, \mathbb{Z}) = 0.$$

By [42, Lemma 5.11, Page 278] we have the following lemma.

Lemma 4.5.1. *Every very general class is contained in some Kähler type chamber.*

We can now prove our claim.

Corollary 4.5.2. *Let K be a Kähler chamber. Then K is a Kähler type chamber and vice versa.*

Proof. It is clear that a Kähler type chamber is a Kähler chamber. The set of very general elements is given by the complement of all $c^{\perp}$ in $\mathrm{Pos}(I)$ where c runs through all intergral classes of type $(1,1)$ except 0. Thus, it is an open and dense subset of $\mathrm{Pos}(I)$. The Kähler chamber K is also open, and thus it contains a very general class α. By the previous lemma α is contained in some $g[f^*\mathrm{Käh}(I')]$. Therefore, $g^{-1}\alpha \in f^*\mathrm{Käh}(I')$. Since $\mathrm{Mon}^2_{\mathrm{Hdg}}(I)$ acts on the set of Kähler chambers, $g[f^*\mathrm{Käh}(I')] = K$. $\square$

One might wonder about the terminology chosen here. Markman however defined the notion of Kähler type chambers before the notion of MBM-classes were introduced.

The following theorem on the set of Kähler chambers $KC(I)$ is also due to Markman, see [42, Theorem 5.16, Page 281].

Theorem 4.5.1. *Let $X = (M, I)$ be an* IHSM *and $\mathcal{P}^{\mathrm{Cpl}}$ be the complex period map restricted to the connected component $\mathcal{T}_o^{\mathrm{Cpl}}(M)$ containing I. Then there is a* $\mathrm{Mon}^2_{\mathrm{Hdg}}(I)$*-equivariant bijection*

$$p\colon {\mathcal{P}^{\mathrm{Cpl}}}^{-1}\left(\mathcal{P}^{\mathrm{Cpl}}(I)\right) \to KC(I)$$

given by sending I' to its Kähler cone $\mathrm{K\ddot{a}h}(I')$*.*

Thus, a Kähler chamber is a Kähler cone for some complex structure in the fiber of $\mathcal{P}^{\mathrm{Cpl}}(I)$. Furthermore, we find that the period map is injective on the subspace of $\mathcal{T}_o^{\mathrm{Cpl}}(M)$ consisting of those complex structures for which $\mathrm{Pos}(I) = \mathrm{K\ddot{a}h}(I')$.

We can now conclude this chapter by the following theorem.

Theorem 4.5.2 (Global Torellli Theorem of IHS-manifolds)**.** *Let $\mathcal{T}_o^{\mathrm{Cpl}}(M)$ be a connected component of the Teichmüller space of* IHS*-structures on a hyperkählerian manifold M. The complex period map*

$$\mathcal{P}^{\mathrm{Cpl}}\colon \mathcal{T}_o^{\mathrm{Cpl}}(M) \to \mathbb{P}\mathrm{er}(M)$$

is surjective. Furthermore, the elements of a fiber ${\mathcal{P}^{\mathrm{Cpl}}}^{-1}(I)$ consist of birational IHS*-manifolds which are in a one-to-one correspondence with the set of Kähler chambers $KC(I)$.*

Global Torelli type theorems as they exist for $K3$-surfaces are however known to be wrong for general IHS-manifolds, see [65, section 25.5].

The shortest path between two truths in the real domain passes through the complex domain

Jacques Hadamard

5

Teichmüller and Moduli Spaces of Hyperkähler Metrics

By replacing the complex structures from the previous chapter with hyperkähler metrics we introduce the notion of the metric Teichmüller space $\mathcal{T}^{\mathrm{Met}}(M)$. For that we fix a hyperkählerian manifold M throughout this chapter. We consider the relation between the complex and metric Teichmüller spaces $\mathcal{T}^{\mathrm{Cpl}}(M)$ and $\mathcal{T}^{\mathrm{Met}}(M)$. Furthermore, we define a metric period on $\mathcal{T}^{\mathrm{Met}}(M)$ for which we prove injectivity and determine the image. By that we recover a theorem of Amerik and Verbitsky [3, Thm 4.9]. We then pass to the moduli space of unit volume hyperkähler metrics for which we provide an explicit description.

5.1 The Metric Period Map Part I

Let $\mathcal{R}(M)$ denote the *space of Riemannian metrics.* We view it as a subspace of the space of smooth symmetric 2-tensors on M and endow it with the topology of smooth convergence. In $\mathcal{R}(M)$ we consider the subspace of hyperkähler metrics. It turns out to be convenient to also fix a scaling, for example by considering metrics of unit volume or unit diameter only. Both versions work equally well for our purpose. We make the following definition.

Definition 5.1.1. *Let $\mathcal{R}^{\mathrm{HK}}(M)$ denote the subspace of $\mathcal{R}(M)$ consisting of hyperkähler metrics of unit volume, i.e. $g \in \mathcal{R}^{\mathrm{HK}}(M)$ if and only if*

$$\mathrm{Hol}(g) = \mathrm{Sp}\left(\frac{n}{4}\right) \text{ and } \mathrm{vol}(g) = 1,$$

where n is the dimension of M.

Note that $\mathcal{R}^{\mathrm{HK}}(M) \times \mathbb{R}_{>0}$ can be interpreted as the space of hyperkähler metrics without any restriction on the volume. All that follows can then easily be generalized to this space by adding the factor $\mathbb{R}_{>0}$.

Recall that a differential form α is parallel or g-parallel with respect to a metric g if $\nabla\alpha = 0$, where ∇ is the extension of the Levi-Civita connection to tensors. We will use these forms to distinguish elements in $\mathcal{R}^{\mathrm{HK}}(M)$. This will then ultimately result in a period map similar to the one for complex structures.

Lemma 5.1.1. *Let g be a hyperkähler metric on M. A 2-form ω on M is g-parallel if and only if there exists a complex structure I on M such that*

$$g(I\cdot,\cdot) = \lambda\omega(\cdot,\cdot)$$

for some $\lambda \in \mathbb{R}$.

Proof. From [137] we know that the parallel 2-forms are spanned by the Kähler classes $[\omega_I], [\omega_J], [\omega_K]$ corresponding to a hyperkähler triple (I, J, K). Let $\omega^{\sharp_g}$ be the g parallel endomorphism obtained by raising the index of ω with respect to g. Thus, there exist $a, b, c \in \mathbb{R}$ such that $\omega^{\sharp_g} = aI + bJ + cK$. Set $\lambda \coloneqq 1/\sqrt{a^2+b^2+c^2}$ and define the complex structure

$$\tilde{I} \coloneqq \lambda(aI + bJ + cK).$$

Then $\lambda\omega = g(\tilde{I}\cdot,\cdot)$. □

Therefore, the g-parallel 2-forms are spanned by the Kähler forms associated to the metric g. Next, note that a parallel form α is harmonic. This can be seen by noting that the Hodge-dual $\star\alpha$ is also parallel. Since parallel forms are closed one finds for a parallel k-form α and $l = n(k-1)+1$ that

$$\Delta_{\mathrm{Hdg}}\alpha = d((-1)^l \star d\star)\alpha + ((-1)^l \star d\star)d\alpha = 0.$$

Hodge theory states that a harmonic 2-form can be viewed as an element of $\mathrm{H}^2_{DR}(M)$. Furthermore, with respect to de Rham's isomorphism theorem we identify de Rham cohomology with singular cohomology $\mathrm{H}^2(M,\mathbb{R})$. Thus, for a metric g we may identify parallel forms with their corresponding classes in $\mathrm{H}^2(M,\mathbb{R})$.

Definition 5.1.2. *Given a hyperkähler metric g we denote the space of g-parallel 2-forms by H_g. We view it as a subspace in* $\mathrm{H}^2(M,\mathbb{R})$.

With respect to the BB-form q on M we have.

Lemma 5.1.2. *The Beauville-Bogomolv form is positive definite on H_g.*

Proof. Let $\alpha \in H_g$. Then there exists a complex structure I in the twistor space of g and a non-zero constant c such that $\alpha = c\omega_I$, where ω_I is the Kähler form of g. If σ denotes the nowhere vanishing holomorphic 2-form of $X = (M, I)$ we get

$$q(\alpha,\alpha) = \frac{n}{2}\int c^2\omega_I^2 \wedge (\sigma\wedge\bar{\sigma})^{n-1} > 0.$$

□

The metric g also induces an orientation on H_g. To see this pick a hyperkähler triple (I,J,K) for g and consider the basis $(\omega_I,\omega_J,\omega_K)$ of associated Kähler forms. The orientation induced by this basis is independent of the choice of hyperkähler triple, since for another choice (I', J', K') one has

$$I' = hIh^{-1},\ J' = hJh^{-1}, K' = hKh^{-1},$$

with h a unitary quaternionic transformation in $\mathrm{Sp}(1)$.

We can now define a first version of the metric period map.

Definition 5.1.3. *The metric period map*

$$\mathcal{P}^{\mathrm{Met}} \colon \mathcal{R}^{\mathrm{HK}}(M) \to \mathrm{Gr}^{+,\mathrm{o}}(3, \mathrm{H}^2(M, \mathbb{R})),$$

is defined by sending a metric g to $H_g = \mathrm{span}\{\omega_I, \omega_J, \omega_K\}$.

We now have the following lemma.

Lemma 5.1.3. *The metric period map $\mathcal{P}^{\mathrm{Met}} \colon \mathcal{R}^{\mathrm{HK}}(M) \to \mathrm{Gr}^{+,\mathrm{o}}(3, \mathrm{H}^2(M, \mathbb{R}))$ is continuous.*

Proof. By the holonomy principle we may identify the space H_g with the $\mathrm{Hol}(g)$ invariant subspace of $T_pM^* \wedge T_pM^*$, where the point p is fixed. Let $(e_1(g), \cdots, e_n(g))$ be an orthonormal basis of (T_pM, g_p), which by the Gram-Schmidt process can be chosen in such a way that the corresponding map

$$\begin{aligned} \mathcal{R}(M) &\to T_pM \times \cdots \times T_pM \\ g &\mapsto (e_1(g), \cdots, e_n(g)) \end{aligned}$$

is continuous. We get an induced basis $\{e^i(g) \wedge e^j(g)\}$ of $T_pM^* \wedge T_pM^*$ which depends continuously on g. For the space of $\mathrm{Hol}(g)$-invariant 2-forms we get a basis $(b^1(g), b^2(g), b^3(g))$ by

$$b^i(g) = \sum b^i_{kl} e^k(g) \wedge e^l(g),$$

where (b^i_{kl}) are induced by a basis of the invariant subspace of the standard $\mathrm{Sp}(n/4)$ action on $\bigwedge^2 \mathbb{R}^n$. By extending the $b^i(g)$ to global 2-forms we find

$$\mathrm{span}\{b^1(g), b^2(g), b^3(g)\} = H_g.$$

The $b^i(g)$ define continuous functions on $\mathcal{R}^{\mathrm{HK}}(M)$ which then proves the claim.

$\square$

5.2 Metric Teichmüller Space vs Complex Teichmüller Space

The diffeomorphism group $\mathrm{Diff}(M)$ acts on $\mathcal{R}^{\mathrm{HK}}(M)$ by pullbacks. It also acts on $\mathrm{Gr}^{+,\mathrm{o}}(3, \mathrm{H}^2(M, \mathbb{R}))$ induced by the natural action on $\mathrm{H}^2(M, \mathbb{R})$. For a diffeomorphism ϕ one computes

$$\phi^* H_g = H_{\phi^* g},$$

i.e. that the metric period map $\mathcal{P}^{\mathrm{Met}}$ is $\mathrm{Diff}(M)$-equivariant. We now want to compare the complex period map with the metric one. Recall that $\mathrm{Diff}_0(M)$ denotes the connected component of $\mathrm{Diff}(M)$ which contains the identity.

Definition 5.2.1. *The metric Teichmüller space on M is defined as*

$$\mathcal{T}^{\mathrm{Met}}(M) \coloneqq \mathcal{R}^{\mathrm{HK}}(M)/\mathrm{Diff}_0(M).$$

A diffeomorphism ϕ in $\mathrm{Diff}_0(M)$ induces the identity in cohomology. Thus, $\mathcal{P}^{\mathrm{Met}}$ induces a well defined map

$$\mathcal{P}^{\mathrm{Met}}\colon \mathcal{T}^{\mathrm{Met}}(M) \to \mathrm{Gr}^{+,\mathrm{o}}(3, \mathrm{H}^2(M, \mathbb{R})),$$

which we still call the metric period map. Recall that the complex period map can be understood as a map

$$\mathcal{P}^{\mathrm{Cpl}}\colon \mathcal{T}^{\mathrm{Cpl}}(M) \to \mathrm{Gr}^{+,\mathrm{o}}(2, \mathrm{H}^2(M, \mathbb{R})).$$

The two period maps are related in the following way. Let g be a hyperkähler metric and I a compatible complex structure. Then $\mathcal{P}^{\mathrm{Cpl}}(I) \subset \mathcal{P}^{\mathrm{Met}}(g)$. Moreover, every 2-dimensional subspace of $\mathcal{P}^{\mathrm{Met}}(g)$ is the image of some compatible complex structure. More precisely one has the following lemma.

Lemma 5.2.1. *Let g be a hyperkähler metric and I a compatible complex structure. Furthermore, let J, K be complex structures such that (I, J, K) is a g compatible hyperkähler triple. Then $\mathcal{P}^{\mathrm{Cpl}}(I) = \mathrm{span}\{\omega_J, \omega_K\}$ and*

$$\mathcal{P}^{\mathrm{Met}}(g) = \mathcal{P}^{\mathrm{Cpl}}(I) \oplus \omega_I \cdot \mathbb{R}.$$

The sum is orthogonal with respect to the BB-pairing.

Proof. For the first claim we show that $\sigma := \omega_J + i\omega_K$ is holomorphic and symplectic. Clearly the form is non-degenerate. Next, let X, Y be vector fields of $TM_{\mathbb{C}}$. Then we compute

$$\begin{aligned}\sigma(IX, Y) &= \omega_J(IX, Y) + i\omega_K(IX, Y)\\ &= g(IJX, Y) + ig(KIX, Y)\\ &= g(-KX, Y) + ig(JX, Y)\\ &= i\sigma(X, Y).\end{aligned}$$

This proves that σ is of type $(2,0)$. Since $d\sigma = 0$ we also get $\bar{\partial}\sigma = 0$ and thus σ is holomorphic. By uniqueness of such forms, we get $\mathcal{P}^{\mathrm{Cpl}}(I) = \mathrm{span}\{\mathrm{Re}(\sigma), \mathrm{Im}(\sigma)\}$. Since ω_I is of type $(1,1)$ and σ of type $(2,0)$ the statement of orthogonality also follows. $\square$

The two Teichmüller spaces $\mathcal{T}^{\mathrm{Cpl}}(M)$ and $\mathcal{T}^{\mathrm{Met}}(M)$ are related by a third Teichmüller space as we now explain.

Definition 5.2.2. *The Teichmüller space of Kähler-Einstein metrics is given by*

$$\mathcal{T}^{\mathrm{Met,Cpl}}(M) := \{(I, \omega) \in \mathcal{T}^{\mathrm{Cpl}}(M) \times \mathrm{H}^2(M, \mathbb{R}) \mid \omega \in \mathrm{K\ddot{a}h}(I)\}.$$

The space $\mathcal{T}^{\mathrm{Met,Cpl}}(M)$ comes with two naturally defined maps. One is given by the projection

$$\pi^{Cpl} \colon \mathcal{T}^{\mathrm{Met,Cpl}}(M) \to \mathcal{T}^{\mathrm{Cpl}}(M),$$

whose fiber over $I \in \mathcal{T}^{\mathrm{Cpl}}(M)$ is the Kähler cone $\mathrm{Käh}(I)$. The other map is defined by associating to $(I, [\omega])$ the unique hyperkähler metric inside the Kähler class $[\omega]$ by means of the Calabi-Yau theorem. This yields a locally trivial S^2-fiber bundle

$$\pi^{Met} \colon \mathcal{T}^{\mathrm{Met,Cpl}}(M) \to \mathcal{T}^{\mathrm{Met}}(M).$$

The fiber $F(g)$ of π^{Met} at a point g may then be identified with the twistor line $\mathbb{T}_w(g)$, since π^{Cpl} is injective on $F(g)$ with $\pi^{Cpl}(F(g)) = \mathbb{T}_w(g)$.

We now use $\mathcal{T}^{\mathrm{Met,Cpl}}(M)$ to show that there is a one-to-one relation between the connected components of $\mathcal{T}^{\mathrm{Cpl}}(M)$ and $\mathcal{T}^{\mathrm{Met}}(M)$.

Lemma 5.2.2. *Let g_0, g_1 be two metrics contained in the same connected component $\mathcal{T}_o^{\mathrm{Met}}(M)$ of $\mathcal{T}^{\mathrm{Met}}(M)$. Suppose I_0 is a compatible complex structure with respect to g_0 and I_1 compatible with g_1. Then I_0 and I_1 are contained in the same connected component of $\mathcal{T}_o^{\mathrm{Cpl}}(M)$. On the other hand, if I_0 and I_1 belong to the same connected component, then any two hyperkähler metrics g_0, g_1 which are Kähler with respect to I_0 and I_1 respectively, belong to the same connected component of $\mathcal{T}^{\mathrm{Met}}(M)$.*

Proof. We prove the more general fact that the domain of a surjective open map is connected if the fibers and the codomain are connected. The statement will then follow by applying this claim to π^{Met} and π^{Cpl} when restricted to the respective preimages of connected components in $\mathcal{T}^{\mathrm{Met}}(M)$ and $\mathcal{T}^{\mathrm{Cpl}}(M)$. Let A and B be topological spaces with B being connected. Furthermore, let $\pi \colon A \to B$ be open and surjective with connected fibers. If $U \cup V = A$ for open subsets

U and V then $\pi(U) \cup \pi(V)$ is an open cover of B. Since B is connected, there exists $p \in \pi(U) \cap \pi(V)$. Thus, $\pi^{-1}(p) \cap U \neq \emptyset$ and $\pi^{-1}(p) \cap V \neq \emptyset$. Then $(\pi^{-1}(p) \cap U) \cup (\pi^{-1}(p) \cap V)$ is an open cover of $\pi^{-1}(p)$. By assumption $\pi^{-1}(p)$ is connected and thus $U \cap V \neq \emptyset$. □

Recall that the set of MBM-classes S_o of an IHS-manifold (M, I) only depends on the connected component of the complex Teichmüller space containing I. By Lemma 5.2.2 we obtain that also the corresponding connected component $\mathcal{T}_o^{\mathrm{Met}}(M)$ determines the set of MBM classes S_o.

5.3 The Metric Torelli Theorem

Before computing the image of the metric period map, recall that the Grassmann space $\mathrm{Gr}^{+,o}(3, \mathrm{H}^2(M, \mathbb{R}))$ consists of 2 connected components, both homeomorphic to $\mathrm{Gr}^{+}(3, \mathrm{H}^2(M, \mathbb{R})) \cong \mathbb{R}^{3(n-3)}$. The tautological bundle

$$\{(v, H) \in \mathrm{H}^2(M, \mathbb{R}) \times \mathrm{Gr}^{+,o}(3, \mathrm{H}^2(M, \mathbb{R})) \mid v \in H\} \to \mathrm{Gr}^{+,o}(3, \mathrm{H}^2(M, \mathbb{R}))$$

is therefore trivial. Thus, choosing an orientation for one H in $\mathrm{Gr}^{+}(3, \mathrm{H}^2(M, \mathbb{R}))$ induces a natural orientation for all the other subspaces. Henceforth, each component $\mathcal{T}_o^{\mathrm{Met}}(M)$ gives rise to a preferred choice of orientation on all spaces in $\mathrm{Gr}^{+}(3, \mathrm{H}^2(M, \mathbb{R}))$, induced by the orientation coming from $\mathcal{P}^{\mathrm{Met}}(g)$ for some $g \in \mathcal{T}_o^{\mathrm{Met}}(M)$.

Theorem 5.3.1. *Let M be hyperkählerian. Fix a connected component $\mathcal{T}_o^{\mathrm{Met}}(M)$ of the metric Teichmüller space and let S be the set of* MBM-*classes associated* *to* $\mathcal{T}_o^{\mathrm{Met}}(M)$*. Then* $H \in \mathrm{Gr}^+(3, \mathrm{H}^2(M,\mathbb{R}))$ *is in the image of the metric period map if and only if* H *is not orthogonal to any* MBM*-class in* S*.*

Proof. For a hyperkähler metric g we have

$$\mathcal{P}^{\mathrm{Met}}(g) = H_g = \mathcal{P}^{\mathrm{Cpl}}(I) \oplus \omega_I \cdot \mathbb{R},$$

for some complex structure I. Any class c in $\mathrm{H}^2(M,\mathbb{Z})$ orthogonal to H_g is in particular orthogonal to $\mathcal{P}^{\mathrm{Cpl}}(I)$ and thus of type $(1,1)$. Since MBM-classes of type $(1,1)$ cannot be orthogonal to a Kähler class we obtain that H_g is also not orthogonal to any MBM-class.

On the other hand, let H be a 3-dimensional positive subspace in $\mathrm{H}^2(M,\mathbb{R})$ not orthogonal to any MBM-class. Let P be a 2-dimensional subspace of H and $[w]$ a class in H such that

$$H = P \oplus [w] \cdot \mathbb{R}.$$

We endow P with the orientation, such that the above expression induces the same orientation of H. By surjectivity of the complex period map there is a complex structure I such that $\mathcal{P}^{\mathrm{Cpl}}(I) = P$. Furthermore, we have that

$$[\omega] \in \mathrm{Pos}(I) - \bigcup_{z-\mathrm{MBM}} z^\perp,$$

where z runs through the set of MBM-classes of type $(1,1)$. By Theorem 4.5.1 each connected component of $\mathrm{Pos}(I) - \bigcup_{z-\mathrm{MBM}} z^\perp$ is the Kähler cone for some $I' \in {\mathcal{P}^{\mathrm{Cpl}}}^{-1}(P)$. Therefore, $[\omega]$ is a Kähler class with respect to I'. By the

Calabi-Yau theorem there exists a hyperkähler metric g with Kähler form $\omega_{I'} = [\omega]$ and

$$\mathcal{P}^{\mathrm{Met}}(g) = \mathcal{P}^{\mathrm{Cpl}}(I') \oplus \omega_{I'} \cdot \mathbb{R} = H$$

which proves the claim. $\square$

The above theorem is equivalent to the surjectivity of the metric period map

$$\mathcal{P}^{\mathrm{Met}} \colon \mathcal{T}_o^{\mathrm{Met}}(M) \to \mathrm{Gr}^+(3, \mathrm{H}^2(M, \mathbb{R})) - \bigcup_{z-\mathrm{MBM}} \mathrm{Gr}^+(3, z^\perp),$$

with an adjusted codomain. We can also show injectivity for this map. We need the following lemma.

Lemma 5.3.1. *Let g be a hyperkähler metric. Then in the twistor line $\mathbb{T}_w(g)$ there is a complex structure I such that the Kähler cone is equal to the positive cone, i.e.*

$$\mathrm{Käh}(I) = \mathrm{Pos}(I).$$

To prove this lemma we remind ourselves of the following fact first.

Remark 5.3.2. *Let V be a finite dimensional $\mathbb{R}$-vector space and $\{V_i\}$ a countable collection of subspaces of V. Then $V = \bigcup_i V_i$ if and only if V is contained in one of the V_i.*

Proof. If V is contained in one of the V_i then the statement is trivial. Thus assume that the V_i are proper subspaces and $V \subset \bigcup V_i$. Let λ_n be the $\dim(V) =$

n-dimensional Lebesgue measure on V. Then

$$\infty = \lambda_n(V) \leq \sum_i \lambda_n(V_i) = 0,$$

a contradiction. $\square$

We can now prove the lemma.

Proof of Lemma 5.3.1. We show that there is a complex structure inside the twistor line that does not admit any MBM-class of type $(1,1)$. Since the Kähler cone is cut out of the positive cone by orthogonal complements of MBM-classes of type $(1,1)$, the statement will follow. From surjectivity of the complex period map we find for every 2-dimensional subspace P in $H_g = \mathcal{P}^{HK}(g)$ a complex structure $I \in \mathbb{T}_w(g)$ with $\mathcal{P}^{Cpl}(I) = P$. Furthermore, a class z is of type $(1,1)$ with respect to I if and only if $P \subset z^\perp$. Assume that for every 2-dimensional subspace P in H_g there is an MBM-class z such that $P \subset z^\perp$. We get

$$H_g = \bigcup_{z-\mathrm{MBM}} z^\perp \cap H_g.$$

However, $z^\perp \cap H_g$ must be an honest subspace of H_g by Theorem 5.3.1. But H_g cannot be the union of a countable collection of subspaces of smaller dimension, by Remark 5.3.2. Thus, there is at least one complex structure in the twistor space for which no MBM-class is of type $(1,1)$. $\square$

Theorem 5.3.2. *Let g and g' be metrics contained in the same connected component $\mathcal{T}_o^{\mathrm{Met}}(M)$. Then $\mathcal{P}^{\mathrm{Met}}(g) = \mathcal{P}^{\mathrm{Met}}(g')$ if and only if there is a diffeomor-*

phism $\phi \in \mathrm{Diff}_0(M)$ *such that* $\phi^* g' = g$.

Proof. Assume that $H_g = H_{g'}$. Pick an I in the twistor space of g such that $\mathrm{Käh}(I) = \mathrm{Pos}(I)$ and put $P = \mathcal{P}^{\mathrm{Cpl}}(I)$, a 2-dimensional subspace in $H_{g'}$. Let I' be in the twistor space of g' such that $\mathcal{P}^{\mathrm{Cpl}}(I') = P$. Since $\mathrm{Käh}(I) = \mathrm{Pos}(I)$, the fiber ${\mathcal{P}^{\mathrm{Cpl}}}^{-1}(P)$ is unique by Theorem 4.5.1. Hence, $[I] = [I']$ in $\mathcal{T}^{\mathrm{Cpl}}(M)$, i.e. there is a diffeomorphism ϕ such that $\phi^* I' = I$. Let ω_I be the Kähler form of g with respect to I and correspondingly $\omega'_{I'}$ the one for g' and I'. From $H_g = H_{g'}$ we get

$$\mathcal{P}^{\mathrm{Cpl}}(I) \oplus [\omega_I] \cdot \mathbb{R} = \mathcal{P}^{\mathrm{Cpl}}(I') \oplus [\omega'_{I'}] \cdot \mathbb{R}.$$

Since the orientations of H_g and $H_{g'}$ are the same and since $vol(g) = vol(g') = 1$ we get for the cohomology classes $[\omega_I] = [\omega'_{I'}]$. Now g' is a Ricci-flat metric and thus also $\phi^* g'$. Furthermore, $\phi^* g'$ is Kähler with respect to I and the Kähler class of $\phi^* g'$ is $[\phi^* \omega'_{I'}]$. But

$$[\phi^* g'(I' \cdot, \cdot)] = \phi^* [\omega'_{I'}] = \phi^* [\omega_I] = [\omega_I],$$

since ϕ acts trivial on cohomology. Therefore, the Kähler class of $\phi^* g'$ with respect to I is the same for the Kähler class of ω_I. By the uniqueness of the Calabi-Yau theorem we have $g = \phi^* g'$. $\square$

Putting everything together we proved the following corollary.

Corollary 5.3.3 (Metric Torelli Theorem)**.** *Let* $\mathcal{T}_o^{\mathrm{Met}}(M)$ *be a connected component of the metric Teichmüller space and* S_o *the associated set of* MBM-*classes.*

Then the metric period map induces a homeomorphism

$$\mathcal{P}^{\mathrm{Met}}\colon \mathcal{T}_o^{\mathrm{Met}}(M) \to \mathrm{Gr}^+(3, \mathrm{H}^2(M, \mathbb{R})) - \bigcup_{z \in S_o} \mathrm{Gr}^+(3, z^\perp).$$

The above corollary was first proven by Amerik and Verbistky [3, Thm 4.9] with basically the same methods. See also [100].

5.4 The Moduli Space of Hyperkähler Metrics

While the metric Teichmüller space is interesting in its own right, we are mostly interested in the moduli space of hyperkähler metrics. Which is the space of hyperkähler metrics up to isometry. The formal definition is the following:

Definition 5.4.1. *Let M be hyperkählerian. Then the moduli space of unit volume hyperkähler metrics on M is defined as*

$$\mathcal{M}^{\mathrm{HK}}(M) \coloneqq \mathcal{R}^{\mathrm{HK}}(M)/\mathrm{Diff}(M).$$

The group of isotopy classes, defined below, will tell us how to obtain the moduli space from the Teichmüller space.

Definition 5.4.2. *The mapping class group of M is defined as*

$$\mathrm{MCG}(M) \coloneqq \mathrm{Diff}(M)/\mathrm{Diff}_0(M).$$

It is straight forward to check that $\mathcal{M}^{\mathrm{HK}}(M) \cong \mathcal{T}^{\mathrm{Met}}(M)/\mathrm{MCG}(M)$. Furthermore, if $\mathrm{MCG}_o(M)$ denotes the largest subgroup preserving $\mathcal{T}_o^{\mathrm{Met}}(M)$, then

$$\mathcal{M}_o^{\mathrm{HK}}(M) \coloneqq \mathcal{T}_o^{\mathrm{Met}}(M)/\mathrm{MCG}_o(M)$$

is homeomorphic to a connected component of $\mathcal{M}^{\mathrm{HK}}(M)$.

Our next step is to express the metric Torelli theorem for connected components of $\mathcal{M}^{\mathrm{HK}}(M)$. In order to state the theorem, recall from Definition 4.4.4 and Theorem 4.4.2 that $\mathrm{Mon}^2(I) \subset \mathrm{O}(\mathrm{H}^2(M,\mathbb{Z}))$ denotes the monodromy group of an IHS-manifold (M, I) and that this group only depends on the component $\mathcal{T}_o^{\mathrm{Cpl}}(M)$ containing I.

Theorem 5.4.1. *Let M be hyperkählerian and $\mathcal{M}_o^{\mathrm{HK}}(M)$ a connected component of the moduli space of hyperkähler metrics on M. Then $\mathcal{M}_o^{\mathrm{HK}}(M)$ is homeomorphic to*

$$\mathrm{Mon}^2(I) \setminus \left(\mathrm{Gr}^+(3, \mathrm{H}^2(M,\mathbb{R})) - \bigcup_{z \in S_o} \mathrm{Gr}^+(3, z^\perp) \right),$$

where I is some complex structure compatible with a metric in $\mathcal{M}_o^{\mathrm{HK}}(M)$ and S_0 the set of associated MBM-*classes.*

Proof. The metric period map $\mathcal{P}^{\mathrm{Met}}$ is $\mathrm{Diff}(M)$-equivariant. Since $\mathrm{Diff}_0(M)$ acts trivially on $\mathcal{T}^{\mathrm{Met}}(M)$ as well as on the Grassmann space, we get that

$$\mathcal{T}^{\mathrm{Met}}(M) \to \mathrm{Gr}^+(3, \mathrm{H}^2(M,\mathbb{R}))$$

is $\mathrm{MCG}(M)$-equivariant. Therefore, we get an induced isomorphism by Corollary 5.3.3

$$\mathcal{M}_o^{\mathrm{HK}}(M) \to \Gamma_o^2(M) \setminus \left(\mathrm{Gr}^+(3, \mathrm{H}^2(M, \mathbb{R})) - \bigcup_{z \in S_o} \mathrm{Gr}^+(3, z^\perp) \right),$$

where $\Gamma_o^2(M)$ is the image of the natural action $\mathrm{MCG}_o(M) \to \mathrm{O}(\mathrm{H}^2(M, \mathbb{Z}))$. From Lemma 5.2.2 it follows that the group $\mathrm{MCG}_o(M)$ is also the largest subgroup acting trivially on $\mathcal{T}_o^{\mathrm{Cpl}}(M)$. Then by Theorem 4.4.2 we get

$$\Gamma_o^2(M) = \mathrm{Mon}^2(I),$$

for some complex structure I in $\mathcal{T}_o^{\mathrm{Cpl}}(M)$. □

Note that $\mathrm{Mon}^2(I)$ is a subgroup of the arithmetic group $\mathrm{O}(\mathrm{H}^2(M, \mathbb{Z})) \subset \mathrm{O}(3, \mathrm{b}_2(M) - 3)$. As an immediate consequence we get that each connected component of $\mathcal{M}^{\mathrm{HK}}(M)$ is homeomorphic to an open and dense subspace of

$$\Gamma \diagdown {}^{\textstyle \mathrm{O}(3, \mathrm{b}_2(M) - 3)} \diagup \mathrm{O}(3) \times \mathrm{O}(\mathrm{b}_2(M) - 3), \tag{5.1}$$

with Γ a discrete subgroup of $\mathrm{O}(3, \mathrm{b}_2(M) - 3)$. From the unpublished [133, Theorem 2.6] we get that $\mathrm{Mon}^2(I)$ is always an arithmetic subgroup. Independent of this statement we find:

Corollary 5.4.1. *The moduli space of unit volume hyperkähler metrics* $\mathcal{M}^{\mathrm{HK}}(M)$ *is a non-compact* $3(\mathrm{b}_2(M) - 3)$*-dimensional orbifold.*

Proof. The space $\mathrm{O}(3, \mathrm{b}_2(M) - 3) / \left(\mathrm{O}(3) \times \mathrm{O}(\mathrm{b}_2(M) - 3)\right)$ is the associated symmetric space to $\mathrm{O}(3, \mathrm{b}_2(M) - 3)$. The group Γ is contained in the arithmetic subgroup $\mathrm{O}(\mathrm{H}^2(M, \mathbb{Z}))$, which acts properly and discontinuously on its associated symmetric space, see [22, 2.1 p.139]. The orbit space of such an action on a smooth manifold is an orbifold. $\square$

The orbifold statement can be proven by purely local considerations, see [19, Thm 12.88] and [93]. However, the above statement shows that $\mathcal{M}^{\mathrm{HK}}(M)$ is what some people call a good orbifold, i.e. a global quotient.

Another consequence of the above result is, that $\mathcal{M}^{\mathrm{HK}}(M)$ is Hausdorff. This stands in contrast to the moduli space of complex structures and even to the complex Teichmüller space which are neither Hausdorff. In fact $\mathcal{M}^{\mathrm{HK}}(M)$ can be endowed with a metric naturally, namely the Gromov-Hausdorff metric.

We recall that the *Gromov Hausdorff distance* defines a metric on the space of isomorphism classes of compact metric spaces. Since elements of $\mathcal{M}^{\mathrm{HK}}(M)$ correspond to such classes we may also endow $\mathcal{M}^{\mathrm{HK}}(M)$ with the Gromov Hausdorff metric. By [86, Prop. 5.10] we find that the topology induced by the Gromov Hausdorff metric agrees with the original one.

Groups, as men, will be known by their actions

Guillermo Moreno

6

Topological Aspects of the Moduli Space of Hyperkähler Metrics

We show that the topologies of the Teichmüller space as well as the corresponding moduli space are not trivial in case $\mathrm{b}_2(M) > 4$. For instance, we will prove that the components of the Teichmüller space is simply connected and that the second homotopy group is induced by the set of primitive MBM-classes. Moreover, we show that in certain cases the connected components of the moduli space are simply connected as well. For the simply connected components we prove that the rank of the second rational homotopy group is bounded from below by the number of MBM-classes up to monodromy.

6.1 Topological Aspects of the Metric Teichmüller Space

First, we note that if there exists a hyperkählerian manifold with $\mathrm{b}_2(M) = 3$ the moduli space is just a union of points by Theorem 5.4.1. From now on we will restrict to the case $\mathrm{b}_2(M) \geq 4$.

Recall that the metric Teichmüller space $\mathcal{T}^{\mathrm{Met}}(M)$ on a hyperkählerian manifold M is the space of hyperkähler metrics on M up to diffeomorphisms isotopic to the identity. The metric Torelli Theorem 5.3.3 established that a connected component $\mathcal{T}_o^{\mathrm{Met}}(M)$ is homeomorphic to

$$\mathrm{Gr}^+(3, \mathrm{H}^2(M, \mathbb{R})) - \bigcup_{z \in S_o} \mathrm{Gr}^+(3, z^\perp), \tag{6.1}$$

with $S_o \subset \mathrm{H}^2(M, \mathbb{Z})$ being the set of MBM-classes associated to $\mathcal{T}_o^{\mathrm{Met}}(M)$. For an MBM-class z the class $-z$ is also MBM and we only need one of the two to describe (6.1). Furthermore, it is enough to reduce the case to primitive MBM-classes.

Definition 6.1.1. *Denote by S_+ a set of primitive* MBM*-classes associated to $\mathcal{T}_o^{\mathrm{Met}}(M)$ so that if z is a primitive* MBM*-class either $z \in S_+$ or $-z \in S_+$ but* *not both.*

The set of primitive MBM-classes is then given by $S_+ \cup -S_+$ and the space (6.1) does not change if we replace S_o by S_+.

In Section 2.3 we introduced affine charts for the Grassmann spaces by interpreting a linear subspace $H \in \mathrm{Gr}(k, V)$ as a graph of a $(n-k) \times k$ matrix $A = (a_{i,j})$. In our situation, where $k = 3$, this means that the space H can be considered as the span of the column vectors in the following $n \times 3$ matrix

$$\begin{pmatrix} & I_{3\times 3} & \\ a_{1,1} & a_{1,2} & a_{1,3} \\ \vdots & \vdots & \vdots \\ a_{(n-3),1} & a_{(n-3),2} & a_{(n-3),3} \end{pmatrix}.$$

Here, $I_{3\times 3}$ denotes the 3×3 identity matrix.

Definition 6.1.2. *Let $\mathbb{A}^{3(n-3)}$ denote the real affine space formed by the above matrices. Furthermore, let $\mathbb{B}$ be the image of $\mathrm{Gr}^+(3, \mathrm{H}^2(M, \mathbb{R}))$ of some affine chart $\mathrm{Gr}(3, \mathrm{H}^2(M, \mathbb{R})) \supset U \to \mathbb{A}^{3(n-3)}$ for which $\mathrm{Gr}^+(3, \mathrm{H}^2(M, \mathbb{R})) \subset U$.*

From Section 2.3 recall that $\mathbb{B}$ is isomorphic to an open unit ball in $\mathbb{A}^{3(n-3)}$. Now a subspace of the form $\mathrm{Gr}(3, z^\perp)$ in the above chart is given by the affine linear subspace in $\mathbb{A}^{3(n-3)}$ consisting of those matrices where each column is orthogonal to $z \in \mathbb{R}^n$. Let us denote such an affine linear subspace of $\mathbb{A}^{3(n-3)}$ by $\mathbb{A}_z$. The Metric Torelli theorem can now be read as follows.

Lemma 6.1.1. *The component $\mathcal{T}_o^{\mathrm{Met}}(M)$ is homeomorphic to*

$$\left(\mathbb{A}^{3(n-3)} - \bigcup_{z \in S_+} \mathbb{A}_z\right) \cap \mathbb{B}.$$

Furthermore, if K is a compact subset of $\mathbb{B}$ only finitely many of the $\mathbb{A}_z$ intersect K and the codimension of finite intersections of the $\mathbb{A}_z$ is a multiple of 3.

Before proving the above lemma let us make the following comment. The homeomorphism statement is an immediate consequence of the discussion above. The fact that

$$\operatorname{codim}\left(\mathbb{A}_{z_{i_1}} \cap \cdots \cap \mathbb{A}_{z_{i_k}}\right) = 3k,$$

for distinct $z_{i_1}, \cdots, z_{i_k} \in S_+$ and $0 \leq k \leq (n-3)$ follows by noting that $\mathbb{A}_{z_{i_1}} \cap \cdots \cap \mathbb{A}_{z_{i_k}}$ are given by the set of matrices in $\mathrm{Matr}(n \times 3)$ for which each column is orthogonal to all of the z_i.

The difficult part is the statement about locally finite intersections, as this requires more information on the set of primitive MBM-classes $S_+ \cup -S_+$. What we need is boundedness of this set with respect to the BB-form. Fortunately, this was proven in [4, Theorem 3.17] and we can prove our claim.

Proof of Lemma 6.1.1. Assume that $\mathrm{Gr}^+(3, z_1^\perp)$ and $\mathrm{Gr}^+(3, z_2^\perp)$ intersect with H being an element in the intersection. Then z_1 and z_2 are in the orthogonal complement $H^\perp$. Since H is positive definite and of maximal dimension, the complement $H^\perp$ is a negative definite subspace. Thus, the sphere $S(-r)$ of constant radius $-r$ in $\mathrm{H}^2(M, \mathbb{R})$ intersected with $H^\perp$ is compact and therefore $S(-r) \cap H^\perp \cap \mathrm{H}^2(M, \mathbb{Z})$ finite. One concludes that only finitely many of the

$\mathrm{Gr}^+(3, z^\perp)$, with z constant length $-r$, contain the subspace H. Now define the space

$$\mathcal{X} \coloneqq \{(H, \omega) \mid \omega \in S(-r) \cap H^\perp\} \subset \mathrm{Gr}^+(3, \mathrm{H}^2(M, \mathbb{R})) \times \mathrm{H}^2(M, \mathbb{R}).$$

The projection $\pi_{Gr} \colon \mathcal{X} \to \mathrm{Gr}^+(3, \mathrm{H}^2(M, \mathbb{R}))$ is closed and the preimage of H with respect to π_{Gr} is compact. Since $\mathcal{X}$ is also Hausdorff the map π_{Gr} is proper, i.e. preimages of compact sets are compact. Let π_{H^2} denote the projection $\mathcal{X} \to \mathrm{H}^2(M, \mathbb{R})$ and define $S_K \coloneqq \pi_{\mathrm{H}^2}(\pi_{Gr}^{-1}(K))$ for a compact set K. Then S_K is compact and therefore $S_K \cap \mathrm{H}^2(M, \mathbb{Z})$ finite. Let $z^2 = -r$ and assume $Gr^+(k, z^\perp)$ to intersects K. Then z is in $S_K \cap \mathrm{H}^2(M, \mathbb{Z})$. Thus, there can only be finitely many $\mathrm{Gr}^+(3, z^\perp)$, with $z^2 = -r$, which intersect K. But then there exists also only finitely many $\mathrm{Gr}^+(3, z^\perp)$ intersecting K with $z \in \bigcup_{r=1}^N S(-r) \cap \mathrm{H}^2(M, \mathbb{Z})$. By [4, Theorem 3.7] there exists an $N > 0$ such that $-N < z^2 < 0$ for all primitive MBM-classes $z \in S_+ \cup -S_+$. $\square$

6.2 First and Second Homotopy Group of the Metric Teichmüller Space

We have seen that each component of the metric Teichmüller space can be described as the complement of an arrangement of codimension 3 smooth submanifolds. As we will see in the proof of the following theorem, taking such complements does not have an effect on the fundamental group.

Theorem 6.2.1. *Each connected component of the metric Teichmüller space is* *simply connected.*

Proof. We prove that the inclusion $\left(\mathbb{A}^{3(n-3)} - \bigcup_{z\in S_+} \mathbb{A}_z\right) \cap \mathbb{B} \hookrightarrow \mathbb{B}$ is 2-connected. Since $\mathbb{B}$ is contractible we only need to show injectivity for $\pi_0(\cdot)$ and $\pi_1(\cdot)$. Let $k = 0, 1$ and assume $h\colon S^k \times [0,1] \to \mathbb{B}$ to be a homotopy such that $h_i(S^k)$ does not intersect any of the $\mathbb{A}_z$ for $i = 0, 1$. We can also assume that h is smooth and furthermore by the transversality theorem, see for instance [66, section 2], that h is transversal to $\bigcup \mathbb{A}_z$. One can arrange this without changing h at the boundary $S^k \times \{0,1\}$. We obtain that

$$\dim(\mathrm{Image}(dh_p)) + \dim(T_{h(p)}\mathbb{A}_z) < \dim(T_{h(p)}\mathbb{B})$$

and thus $h^{-1}(\bigcup \mathbb{A}_z)$ is empty. This proves the claim. $\square$

Next, we want to describe the second homotopy group of $\mathcal{T}_o^{\mathrm{Met}}(M)$. Since $\mathcal{T}_o^{\mathrm{Met}}(M)$ is simply connected we do not need to worry about base points and can identify the group of free homotopies $[S^2, \mathcal{T}_o^{\mathrm{Met}}(M)]$ with $\pi_2(\mathcal{T}_o^{\mathrm{Met}}(M))$. Recall that for the definition of π_2 one usually works in the category of pointed spaces.

We now construct a 2-sphere around each $\mathbb{A}_z$. The set of these will turn to provide us with a generating set of $\pi_2(\mathcal{T}_o^{\mathrm{Met}}(M))$.

Consider the space $\mathbb{B} - \mathbb{A}_z$ for a fixed $z \in S_+$. Let $\mathbb{H}_p$ be some 3-dimensional affine subspace in $\mathbb{A}^{3(n-3)}$ which intersects $\mathbb{A}_z$ transversely in p. Endow $\mathbb{H}_p$ with some Euclidean metric and let S^2_z denote the 2-sphere of radius ε centered at p in $\mathbb{H}_p$. We can choose ε small enough so that $S^2_z \subset \mathbb{B} - \mathbb{A}_z$. Clearly, S^2_z is a generator of $\pi_2(\mathbb{B} - \mathbb{A}_z)$ since S_z^2 is homotopy equivalent to $\mathbb{B} - \mathbb{A}_z$. We can do this for every $z \in S_+$. Since the intersection of the $\mathbb{A}_z$ is locally finite by Lemma 6.1.1, a compact disk $D \subset \mathbb{B}$ only intersects finitely many of the $\mathbb{A}_z$. For an $\mathbb{A}_z$ intersecting D we can assume, by possibly changing p and ε, that S_z^2 is a subset of $D \cap (\mathbb{B} - \bigcup \mathbb{A}_z)$. By doing this we can in general assume that each S_z^2 is contained in $\mathbb{B} - \bigcup_{z \in S_+} \mathbb{A}_z$.

Each S_z^2 induces a non-trivial element in $\pi_2(\mathbb{B} - \bigcup \mathbb{A}_z)$, since if there was a homotopy contracting S_z^2 it would also contract S_z^2 in $\mathbb{B} - \mathbb{A}_z$ which is not possible. Moreover, $S_{z_1}^2$ cannot be deformed by a homotopy to $S_{z_2}^2$ for distinct $z_1, z_2 \in S_+$. Therefore, there is a surjective map

$$\pi_2\left(\mathbb{B} - \bigcup_{z \in S_+} \mathbb{A}_z\right) \to \bigoplus_{z \in S_+} \mathbb{Z}. \tag{6.2}$$

We want to prove that this is an isomorphism. Since the space is simply connected it suffices to compute the second homology group. Now the work on stratified Morse theory [59] gives a way to compute the homology of the complement of a finite collection of affine subspaces in $\mathbb{R}^n$. When computing the homology of the metric Teichmüller space in case of the $K3$-surface, Giansiracusa notes in [55, 56, Lemma 5.3] that the computations can be extended to more general situations. With Lemma 6.1.1 we can use the same argument of Giansiracusa to extend his result on $K3$-surfaces to general hyperkählerian manifolds. To state the theorem we introduce the following notions. Let

$$\mathcal{P} := \{\left(\mathbb{A}_{z_{i_1}} \cap \cdots \cap \mathbb{A}_{z_{i_k}}\right) \cap \mathbb{B} \mid i_1 < \cdots < i_k \text{ for } k \in \mathbb{N}\}$$

be partially ordered by inclusion. Furthermore, define recursively on $\mathcal{P} \cup \{\mathbb{B}\}$ the function μ by setting $\mu(\mathbb{B}) = 1$ and $\mu(x) \coloneqq -\sum_{v \supsetneq x} \mu(v)$ for $x \in \mathcal{P}$.

Theorem 6.2.2. *The integral homology of $\mathcal{T}_o^{\mathrm{Met}}(M)$ is torsion free. For the second homology group one has $H_2\left(\mathcal{T}_o^{\mathrm{Met}}(M), \mathbb{Z}\right) \cong \bigoplus_{S_+} \mathbb{Z}$. Furthermore, the Poincaré polynomial, with coefficients in $\mathbb{N}_0 \cup \{\infty\}$, is given by*

$$1 + \sum_{v \in \mathcal{P}} t^{f(v)} |\mu(v)|,$$

where $f(v) = \frac{2}{3}\left(3(n-3) - \dim(v)\right)$.

Proof. We essentially follow the argument of Giansiracus in [55, Lemma 5.3]. In [59, 1.6 Theorem B P.239] we find a way to compute the homology of a complement of a finite affine subspace arrangements of fixed codimension c in $\mathbb{R}^N$, with the additional property that the intersections have codimension a multiple of c. Then the theorem states that the integral homology is torsion free and that the Poincaré polynomial is given by

$$1 + \sum_{v \in \mathcal{P}} t^{f(v)} |\mu(v)|,$$

with $f(v) = \frac{c-1}{c}\left(N - \dim(v)\right)$, $\mathcal{P}$ being the partially ordered set induced by the intersections of the affine subspaces and μ defined just like above. The theorem is proved by considering stratified Morse theory and the Morse function they use is given by the distance function $\mathrm{dist}(\cdot, p)^2$ for a generic point p. We want to apply the same argument to $\mathbb{B} - \bigcup_{z \in S_+} \mathbb{A}_z$.

Recall that we can consider $\mathbb{B}$ as a unit ball in Euclidean space $\mathbb{R}^{k(n-k)}$. After possibly moving some of the $\mathbb{A}_z$ by a small homotopy we may assume that its center p is not contained in $\bigcup_{z\in S_+}\mathbb{A}_z$. Let $B_r(p)$ denote the ball of radius r around p in $\mathbb{B}$. The closure of $B_r(p)$, for $0 < r < 1$, is compact and therefore there are only finitely many of the $\mathbb{A}_z$ intersecting $B_r(p)$ by Lemma 6.1.1. The computations in [59, 1.6 Theorem B P.239] carry over to

$$\left(\mathbb{B} - \bigcup_{z\in S_+}\mathbb{A}_z\right) \cap B_r(p).$$

Taking the colimit for $r \to 1$ then produces the desired result. □

As a corollary we obtain the second homotopy group, and can also determine the induced action of $\mathrm{O}(\mathrm{H}^2(M,\mathbb{Z}))$ on this group.

Corollary 6.2.3. *For a connected component $\mathcal{T}_o^{\mathrm{Met}}(M)$ of the metric Teichmüller space we have*

$$\pi_2\left(\mathcal{T}_o^{\mathrm{Met}}(M)\right) \cong \bigoplus_{z\in S_+}\mathbb{Z}.$$

Furthermore, the action of $\mathrm{O}(\mathrm{H}^2(M,\mathbb{Z}))$ on $\pi_2(\mathcal{T}_o^{\mathrm{Met}}(M))$ is determined by the action on the set of primitive MBM*-classes.*

Proof. From 6.2.2 we know that the map 6.2 is an isomorphism by the Hurewicz isomorphism theorem. The claim on the action is clear when one considers the basis $\{[S_z^2]\}$ of $\pi_2\left(\mathbb{B} - \bigcup_{z\in S_o}\mathbb{A}_z\right)$ described earlier. □

6.3 The Fundamental Group of the Moduli Space of Hyperkähler Metrics of $K3^{[n]}$-type

We discuss the topology of the connected components of the moduli space of hyperkähler metrics $\mathcal{M}^{\mathrm{HK}}(M)$ which by Theorem 5.4.1 are of the form

$$\mathrm{Mon}^2(I) \setminus \left(\mathrm{Gr}^+(3, \mathrm{H}^2(M, \mathbb{R})) - \bigcup_{z \in S_+} \mathrm{Gr}^+(3, z^\perp) \right).$$

Having understood some of the topology of $\mathcal{T}_o^{\mathrm{Met}}(M)$ we thus want to understand the effects of the action of $\mathrm{Mon}^2(I)$ on this space.

Lemma 6.3.1. *Let Γ be a discrete subgroup of $\mathrm{O}(\mathrm{H}^2(M, \mathbb{Z}))$. If Γ is generated by reflections r_a and $-r_b$ along elements $a, b \in \mathrm{H}^2(M, \mathbb{Z})$ of non-zero length, then*

$$\Gamma \setminus \left(\mathrm{Gr}^+(3, \mathrm{H}^2(M, \mathbb{R})) - \bigcup_{z \in S_+} \mathrm{Gr}^+(3, z^\perp) \right)$$

is simply connected.

Proof. By the work of Armstrong on the fundamental group of orbit spaces [10] the fundamental group is isomorphic to Γ modulo the subgroup Γ_F generated by elements which have a fixed point. For a reflection r_a along $a \in \mathrm{H}^2(M, \mathbb{R})$ let H_a denote the reflection hyperplane in $\mathrm{H}^2(M, \mathbb{R})$. Let v be some positive vector in H_a and x a positive vector not orthogonal to any class in S_o. Then $\mathrm{span}\{x, r_a(x), v\}$ corresponds to an element in $\mathcal{T}_o^{\mathrm{Met}}(M)$ and is fixed by r_a. Since $-Id$ acts trivial on the Grassmann space also the elements of the form $-r_a$ have a fixed point. Thus, $\Gamma = \Gamma_F$ which proves the claim. $\square$

For an IHS-manifold $X = (M, I)$ computing the group $\mathrm{Mon}^2(I)$ is in general difficult. However, in case of the $K3$-surface it is well known that $\mathrm{Mon}^2(I) = \mathrm{O}^+(\mathrm{H}^2(X, \mathbb{Z}))$, see [21]. By Theorem 2.4.2 we know that in this case $\mathrm{Mon}^2(I)$ is generated by reflections along (-2)-classes. In [105] Markman generalized this result to IHS-manifolds of $K3^{[n]}$-type, see also [43, Section 9 p. 302]. He proves that if $X = (M, I)$ is of $K3^{[n]}$-type, then $\mathrm{Mon}^2(I)$ is generated by elements of the form r_a and $-r_a$, where r_a is a reflection along a (± 2)-class a, see [105, Theorem 1.2]. Furthermore, he shows that $\mathrm{Mon}^2(I)$ is given by the inverse image of $\{Id, -Id\}$ with respect to the natural map

$$\mathrm{O}^+(\mathrm{H}^2(M, \mathbb{Z})) \to \mathrm{O}(D_{\mathrm{H}^2(M,\mathbb{Z})}),$$

[105, Lemma 4.2]. For $n = 2$ or in case $n - 1$ is a prime power, it is noted that $\mathrm{Mon}^2(I) = \mathrm{O}^+(\mathrm{H}^2(M, \mathbb{Z}))$. In this case we know from Theorem 2.4.2 that $\mathrm{Mon}^2(I)$ is generated by reflections along (-2)-classes. Regardless of this case as an immediate corollary from Markman's work and Lemma 6.3.1 we get the following.

Corollary 6.3.2. *Let M be the underlying manifold of a $K3^{[n]}$-type* IHSM *X. Then the connected component of the moduli space of hyperkähler metrics corresponding to X is simply connected.*

We cannot tell if all components of $\mathcal{M}^{\mathrm{HK}}(M)$ are simply connected. However, this is the case if all IHS-structures on M are deformations of some Hilbert scheme over a $K3$-surface.

6.4 On the Homotopy Groups of the Moduli Space of Hyperkähler Metrics

In this section we prove that all connected components of $\mathcal{M}^{\mathrm{HK}}(M)$ have non-trivial homotopy groups. In case that $\mathcal{M}^{\mathrm{HK}}_o(M)$ is simply connected we can determine certain factors in the second homotopy group generated by MBM-classes. If $\mathcal{M}^{\mathrm{HK}}_o(M)$ is not simply connected we are done.

Fix a connected component $\mathcal{M}^{\mathrm{HK}}_o(M)$ and set

$$\mathcal{T} \coloneqq \mathcal{T}^{\mathrm{Met}}_o(M), \text{ and } \Gamma \coloneqq \mathrm{Mon}^2(I).$$

Thus, $\mathcal{M}^{\mathrm{HK}}_o(M) \cong \Gamma \setminus \mathcal{T}$. Recall that the action of Γ on $\mathcal{T}$ is in general not free. We can overcome this problem by the so called *Borel construction*. Let $\mathrm{B}\,\Gamma$ denote the *classifying space* of Γ, i.e. $\mathrm{B}\,\Gamma$ is the connected CW-complex with fundamental group isomorphic to Γ and vanishing higher homotopy groups. The space $\mathrm{B}\,\Gamma$ is unique only up to homotopy. The universal cover of $\mathrm{B}\,\Gamma$ will be denoted by $\mathrm{E}\,\Gamma$. Then Γ acts freely via deck transformations on $\mathrm{E}\,\Gamma$. Since $\mathrm{B}\,\Gamma$ has vanishing higher homotopy groups, it follows from [29, 11.14. Corollary] that $\mathrm{E}\,\Gamma$ is contractible.

Definition 6.4.1. *The homotopy quotient of the group action of* Γ *on* $\mathcal{T}$ *is defined as* $\mathcal{M}^h \coloneqq \mathcal{T} \times_\Gamma \mathrm{E}\,\Gamma \coloneqq (\mathcal{T} \times \mathrm{E}\,\Gamma)/\Gamma$.

The space $\mathcal{T} \times \mathrm{E}\,\Gamma$ is homotopy equivalent to $\mathcal{T}$ and has a free Γ action, but the homotopy quotient is in general not homotopy equivalent to $\Gamma \setminus \mathcal{T}$. The rational cohomology, however, turns out not to be effected by this construction.

Lemma 6.4.1. *Let X be a smooth manifold and Γ a discrete group acting properly discontinuously with finite stabilizers on X. Then* $\mathrm{H}^i(X \times_\Gamma \mathrm{E}\,\Gamma; \mathbb{Q}) \cong \mathrm{H}^i(X/\Gamma; \mathbb{Q})$.

This lemma seems to be well known. However, for completeness we include a proof here.

Proof. Consider the induced map $f\colon X \times_\Gamma \mathrm{E}\,\Gamma \to X/\Gamma$ and the commuting diagram

$$\begin{tikzcd} X \times \mathrm{E}\,\Gamma \arrow[r, "\tilde{\pi}"] \arrow[d, "p"] & X \times_\Gamma \mathrm{E}\,\Gamma \arrow[d, "f"] \\ X \arrow[r, "\pi"] & X/\Gamma. \end{tikzcd}$$

We compute that the fiber $f^{-1}([x])$ over $[x] \in X/\Gamma$ is isomorphic to $\mathrm{E}\,\Gamma/\Gamma_x$, with Γ_x being the stabilizer of $x \in X$. By assumption Γ_x is finite and since Γ_x acts freely on $\mathrm{E}\,\Gamma$ we further know that $\mathrm{E}\,\Gamma/\Gamma_x$ is a classifying space for the group Γ_x, i.e. $\mathrm{E}\,\Gamma/\Gamma_x \cong \mathrm{B}\,\Gamma_x$. From [48, Corollary 4.3.2] we know that the group cohomology of a finite group is torsion. Thus, for $i > 0$ we get

$$\mathrm{H}^i(f^{-1}([x]), \mathbb{Q}) \cong \mathrm{H}^i(\Gamma_x, \mathbb{Z}) \otimes \mathbb{Q} \cong 0. \tag{6.3}$$

As a consequence, we find that the higher direct image $R^i f(\underline{\mathbb{Q}})$ is trivial for every $i > 0$. For that, recall that $R^i f(\underline{\mathbb{Q}})$ is the sheafification of the presheaf $U \mapsto \mathrm{H}^i(f^{-1}(U), \mathbb{Q})$ whose stalks are given by (6.3). The E_2^{**} page of the Leray spectral sequence of f is now readily computed to be $\mathrm{E}_2^{\mathrm{p},0} = \mathrm{H}^p(X/\Gamma, \mathbb{Q})$ and $\mathrm{E}_2^{\mathrm{p,q}} = 0$ if $q > 0$. By the structure of the 2-page we immediately find $\mathrm{E}_\infty^{\mathrm{p,q}} =$

$\mathrm{E}_2^{\mathrm{p,q}}$. This spectral sequence converges to $\mathrm{H}^i(X \times_\Gamma \mathrm{E}\,\Gamma, \mathbb{Q})$, see [60, p.462] for instance, and we conclude the claim. $\square$

Another way to think about the homotopy quotient $\mathcal{T} \times_\Gamma \mathrm{E}\,\Gamma$ is by noting that it is the associated bundle to the universal covering $\mathrm{E}\,\Gamma \to \mathrm{B}\,\Gamma$. Thus, the map $\mathcal{T} \times_\Gamma \mathrm{E}\,\Gamma \to \mathrm{B}\,\Gamma$ is a fiber bundle with fiber $\mathcal{T}$. From the long exact sequence of homotopy groups we get

- $\pi_1(\mathcal{M}^h) \cong \Gamma$,
- $\pi_2(\mathcal{M}^h) \cong \pi_2(\mathcal{T}) \cong \bigoplus_{S_+} \mathbb{Z}$.

Now, if $\mathcal{M}_o^{\mathrm{HK}}(M)$ is simply connected, then by Lemma 6.4.1 we find

$$\pi_2(\mathcal{M}_o^{\mathrm{HK}}(M)) \otimes \mathbb{Q} \cong \mathrm{H}^2(\mathcal{M}^h, \mathbb{Q}). \tag{6.4}$$

Our goal is to compute the right-hand side. This can be done by using a classical theorem due to Eilenberg and Mac Lane [45]. The theorem states that the second cohomology is determined by the action of the fundamental group on the second homotopy group and an invariant whose definition we now recall.

The k^3-invariant: Let X be a topological space and $\omega_1, \omega_2, \omega_3 \colon [0,1] \to X$ be continuous maps representing loops in $\pi_1(X, x)$, i.e. $\omega_i(0) = \omega_i(1) = x$ for $i = 1, 2, 3$. Let Δ denote the standard 3-simplex with ordered vertices v_0, v_1, v_2, v_3. On the 1-skeleton define a map κ as follows. The edge $[v_0, v_1]$ gets mapped according to ω_1, the edge $[v_1, v_2]$ according to ω_2 and $[v_0, v_2]$ according to the product $\omega_1 * \omega_2$. Similarly, we define the map for the other edges, i.e. we map $[v_1, v_3]$ along $\omega_2 * \omega_3$, $[v_0, v_3]$ along $\omega_1 * \omega_2 * \omega_3$, and $[v_2, v_3]$ along ω_3. Note that

κ can be extended to each face of the simplex Δ, and thus on the full 2 skeleton of Δ, which is a 2-sphere. We obtain a continuous map $k^3(\omega_1, \omega_2, \omega_3)\colon S^2 \to X$. Choosing representatives for each class in $\pi_1(X, x)$ defines a function

$$k^3\colon \pi_1(X) \times \pi_1(X) \times \pi_1(X) \to \pi_2(X).$$

Eilenberg and Mac Lane proved that this defines a cohomology class, denoted k^3, in the group cohomology $\mathrm{H}^3(\pi_1(X); \pi_2(X))$ independent of the chosen representatives.

Lemma 6.4.1. *Let Γ be a discrete group acting on a simply connected space T. Then the k^3-invariant for the homotopy quotient $X \coloneqq T \times_\Gamma \mathrm{E}\,\Gamma$ is trivial.*

Proof. Let Δ denote the standard 3-simplex and let $\omega_1, \omega_2, \omega_3$ be loops representing classes in $\pi_1(X, x)$. Furthermore, let $p\colon X \to \mathrm{B}\,\Gamma$ be the projection and correspondingly $p_*(\omega_1), p_*(\omega_2), p_*(\omega_3)$ the induced loops in $\mathrm{B}\,\Gamma$. Note that the universal covering $\mathrm{E}\,\Gamma \to \mathrm{B}\,\Gamma$ factors as

$$\mathrm{E}\,\Gamma \hookrightarrow T \times \mathrm{E}\,\Gamma \to T \times_\Gamma \mathrm{E}\,\Gamma \to \mathrm{B}\,\Gamma. \tag{6.5}$$

The k^3-invariant induces a map $k \coloneqq k^3\left(p_*(\omega_1), p_*(\omega_2), p_*(\omega_3)\right) : S^2 \to \mathrm{B}\,\Gamma$, by precomposing with p. Then k can be lifted to the universal covering $\tilde{k}\colon S^2 \to \mathrm{E}\,\Gamma$. By composition with respect to (6.5) we get an induced map $S^2 \to T \times_\Gamma \mathrm{E}\,\Gamma$. This map can be chosen to represent $k^3([\omega_1], [\omega_2], [\omega_3]) \in \pi_2(X)$, since p induces an isomorphism on the fundamental group. The space $\mathrm{E}\,\Gamma$ is contractible and thus $k^3([\omega_1], [\omega_2], [\omega_3]) = 0$. □

We can now compute the second rational homotopy group of the moduli space $\mathcal{M}_o^{\mathrm{HK}}(M)$ in case it is simply connected.

Theorem 6.4.2. *Let M be hyperkählerian and $\mathcal{M}_o^{\mathrm{HK}}(M)$ be a connected component of the moduli space of unit volume hyperkähler metrics. Let S_+ denote the associated set of primitive* MBM*-classes up to sign and S_+/Γ the orbit space with $\Gamma = \mathrm{Mon}^2(I)$. If $\mathcal{M}_o^{\mathrm{HK}}(M)$ is simply connected then there is an isomorphism*

$$\pi_2(\mathcal{M}_o^{\mathrm{HK}}(M), \mathbb{Z}) \cong \mathrm{H}^2(\Gamma, \mathbb{Q}) \oplus \bigoplus_{S_+/\Gamma} \mathbb{Q}.$$

If $b_2(M) \geq 5$ then the set S_+/Γ is finite.

The finiteness of the set S_+/Γ was proven by Amerik and Verbitsky and is stated in [49, p.91 section 5] see also [4, Corollary 1.4].

Proof. By (6.4) we need to compute $\mathrm{H}^2(\mathcal{M}^h, \mathbb{Q})$. From Lemma 6.4.1 we know that the k^3-invariant of $\mathcal{M}^h = \mathcal{T} \times_\Gamma \mathrm{E}\,\Gamma$ vanishes. The work of Eilenberg and Mac Lane [45, p.280] now states that

$$\mathrm{H}^2(\mathcal{M}^h, \mathbb{Q}) \cong \mathrm{H}^2(\Gamma, \mathbb{Q}) \oplus \mathrm{Hom}(\pi_2(\mathcal{M}^h)/\pi_2^0, \mathbb{Q}),$$

with $\pi_2^0 \coloneqq \langle \alpha - \omega\alpha \mid \alpha \in \pi_2 \text{ and } \omega \in \pi_1 \rangle$. From Corollary 6.2.3 we know the group action on $\pi_2(\mathcal{T}) \cong \pi_2(\mathcal{M}^h)$ by which we conclude

$$\pi_2(\mathcal{M}^h)/\pi_2^0 \cong \bigoplus_{S_+/\Gamma} \mathbb{Z}.$$

□

The above theorem applies for M being of $K3^{[n]}$-type. When $n = 1$ or 2 we can say more about S_+/Γ. For $n = 1$ this is just a $K3$-surface and the set of MBM-classes are precisely the (-2)-classes. Since $\mathrm{O}^+(\mathrm{H}^2(M,\mathbb{Z}))$ is an index 2 subgroup of $\mathrm{O}(\mathrm{H}^2(M,\mathbb{Z}))$ which does not contain $(-id)$ we know by Eichlers Criterion Theorem 2.4.3 that there is just a single orbit. For $n = 2$ the set of MBM-classes can also be determined up to $\mathrm{Mon}^2(I)$ in which case S_+/Γ consists of 3 elements, see [49, p.91 section 5] and [5]. From [5, Corollary] we know that there are 5 different types of MBM-classes on an IHSM of type $K3^{[3]}$. Thus, we obtain as a corollary

Corollary 6.4.2. *For $K3^{[n]} := (M, I)$ being of type $K3^{[n]}$ we have $\mathcal{M}_o^{\mathrm{HK}}(K3^{[n]})$ is simply connected and the dimension of the second rational homotopy group has at least the number of orbits of* MBM-*classes under the monodromy action. For $n = 1, 2, 3$ we furthermore have*

- $\pi_2(\mathcal{M}_o^{\mathrm{HK}}(K3)) \otimes \mathbb{Q} \cong \mathbb{Q} \oplus \mathrm{H}^2(\mathrm{O}^+(\Lambda_{K3}), \mathbb{Q})$,
- $\pi_2(\mathcal{M}_o^{\mathrm{HK}}(K3^{[2]})) \otimes \mathbb{Q} \cong \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathrm{H}^2(\mathrm{O}^+(\Lambda_{K3^{[2]}}), \mathbb{Q})$,
- $\pi_2(\mathcal{M}_o^{\mathrm{HK}}(K3^{[3]})) \otimes \mathbb{Q} \cong \bigoplus_{k=1}^5 \mathbb{Q} \oplus \mathrm{H}^2(\mathrm{O}^+(\Lambda_{K3^{[2]}}), \mathbb{Q})$.

We cannot say anything about the group $\mathrm{H}^2(\Gamma, \mathbb{Q})$, not even in the $K3^{[n]}$-case. The only thing we can say about the group cohomology of Γ is that the first Betti number vanishes, i.e. $b_1(\Gamma) = 0$ which follows from Kazhdan's Property (T) and $b_4(X) \geq 1$, we refer to Corollary 6.4.4 for the precise statement.

Theorem 6.4.2 shows that a connected component of the moduli space of hyperkähler metrics has a non-trivial homotopy group if it possesses an MBM-class. All known examples of IHS-manifolds admit such classes, but it is an open question if this is true in general, compare [49]. To also cover the situation where no such classes exist, we are let to study the space $\Gamma \setminus \mathrm{Gr}^+(3, \Lambda \otimes \mathbb{R})$ for some non-degenerate lattice Λ of signature $(3, \mathrm{b}_2(M) - 3)$. Recall that this space is isomorphic to the bi-quotient

$$\Gamma \diagdown \mathrm{SO}^o(3, b_2(M) - 3) \diagup (\mathrm{SO}(3) \times \mathrm{SO}(\mathrm{b}_2(M) - 3)).$$

Before computing some of the topology let us mention that there is another reason to study this space even when M admits MBM-classes. The reason for that is that elements of $\mathrm{Gr}^+(3, \mathrm{H}^2(M, \mathbb{R}))$ might still represent geometrical meaningful objects, even when they are orthogonal to an MBM-class. This is classically known in case of the $K3$-surface, where these objects correspond to metrics with certain orbifold type singularities, see Sections 7.5, 7.6 for a further discussion.

The following lemma is largely due to Giansiracusa, Kupers and Tshishiku [57, Proposition 11].

Lemma 6.4.3. *Let Γ be a lattice in $\mathrm{SO}^o(p, q)$, defined over $\mathbb{Q}$ with $p + q > 4$, and*

$$X = \Gamma \diagdown \mathrm{SO}^o(p, q) \diagup \mathrm{SO}(p) \times \mathrm{SO}(q).$$

Then the 4th-Betti number of X is non-trivial.

Proof. Consider G a connected semisimple linear algebraic group defined over $\mathbb{Q}$. Let K be the maximal compact subgroup of $G(\mathbb{R})$ and U the maximal compact subgroup of $G(\mathbb{C})$. Then $Y := G(\mathbb{R})/K$ is a symmetric space and $X_u := G(\mathbb{C})/U$ the compact dual symmetric space. For any lattice $\Gamma \subset G(\mathbb{Q})$ there is a homomorphism

$$\mu\colon \mathrm{H}^*(X_u, \mathbb{C}) \to \mathrm{H}^*(\Gamma \setminus Y, \mathbb{C}),$$

known as the Borel-Matsushima homomorphism see [106] and [23]. In [57, Proposition 11] it is shown that for $G = \mathrm{SO}^o(p,q)$ the map is injective in degrees less than $p + q - 1$. In this case the compact dual symmetric space is $X_u = \mathrm{SO}(p+q)/\mathrm{SO}(p) \times \mathrm{SO}(q)$ for which we now compute some of its cohomology. Consider the fiber bundle $\mathrm{SO}(p+q) \to \mathrm{SO}(p+q)/\mathrm{SO}(p) \times \mathrm{SO}(q)$ and the induced long exact sequence

$$\cdots \to \pi_k(\mathrm{SO}(p+q)) \to \pi_k(X_u) \to \pi_{k-1}(\mathrm{SO}(p)) \times \pi_{k-1}(\mathrm{SO}(q)) \to \cdots .$$

We find that X_u is simply connected. Since tensoring with $\mathbb{Q}$ preserves exactness we find by using Bott periodicity [29, p.467] that $\pi_i(X_u) \otimes \mathbb{Q} = 0$ for $i = 1, 2, 3$ and get the exact sequence $0 \to \pi_4(X_u) \to \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q} \to 0$. Thus, $\pi_4(X_u) \otimes \mathbb{Q} \cong \mathbb{Q}$ and hence $\mathrm{H}^4(X_u, \mathbb{Q}) \cong \mathbb{Q}$. □

This shows that if X in Lemma 6.4.3 is simply connected then a higher rational homotopy group does not vanish. Using this lemma we find the statement on the 4th-Betti number $\mathrm{b}_4(\Gamma)$ as mentioned earlier.

Corollary 6.4.4. *If Γ is a lattice in $\mathrm{SO}^o(p,q)$ with $p + q > 4$, then the 4th-Betti number of Γ is non-trivial. Furthermore, for $q \geq p \geq 3$ the first Betti number of Γ vanishes.*

Proof. The space $\mathrm{SO}^o(p,q)/\mathrm{SO}(p) \times \mathrm{SO}(q)$ is a model for the classifying space of proper Γ-actions, denoted $E_{\mathcal{FIN}}\Gamma$, see [102, Theorem 4.4]. It further follows that there is, up to Γ-homotopy, a unique Γ-map $E_{\mathcal{TR}}\Gamma \to E_{\mathcal{FIN}}\Gamma$, where $E_{\mathcal{TR}}\Gamma = E\Gamma$. By [103, Lemma 4.14] the induced map on the orbit spaces, $\Gamma\backslash E\Gamma \to \Gamma\backslash E_{\mathcal{FIN}}\Gamma$, yields an isomorphism in rational cohomology and thus

$$\mathrm{H}^*(X,\mathbb{Q}) \cong \mathrm{H}^*(\Gamma,\mathbb{Q}).$$

Compare the above to the proof of [129, Proposition 5.5].

For the first Betti number we note that $\mathrm{SO}^o(p,q)$ has Kazhdan's Property (T) in the case $q \geq p \geq 3$. This follows from [17, Theorem 3.5.4] since $\mathrm{SO}^o(p,q)$ is simple and the Lie algebra $\mathfrak{so}(p,q)$ is not isomorphic to $\mathfrak{so}(n,1)$ or $\mathfrak{su}(n,1)$, see [109, §A.2]. The group Γ is a lattice in $\mathrm{SO}^o(p,q)$ and by [17, Theorem 1.7.1] also has Kazhdan's Property (T). From [17, Theorem 1.3.1] we know that Γ is finitely generated and by [17, Theorem 3.2.1] it follows that $\mathrm{H}^1(\Gamma,\mathbb{R}) = 0$. □

Everything put together we proved the following.

Corollary 6.4.5. *Let M be hyperkählerian with $b_2(M) > 4$. Then every connected component of the moduli space of hyperkähler metrics has non-trivial topology. Furthermore, if the connected component is simply connected, then the second Betti number is at least the number of* MBM*-classes up to the monodromy action. If the connected component does not support any* MBM*-classes then the* 4*th-betti number is non-trivial.*

The introduction of numbers as coordinates is an act of violence.

Hermann Weyl

7
Moduli Spaces of Ricci-flat Metrics

In this chapter we discuss applications to other moduli spaces. Moreover, we study Ricci-flat metrics on torus products $N \times T^k$ as well as Einstein metrics on the Enriques manifold. For the Enriques manifold we provide a metric Torelli theorem and a smoothing process of a singular Ricci-flat orbifold metric.

7.1 Hyperkähler Metrics in the Moduli Space of Einstein Metrics

Our first goal is to show that the moduli space of hyperkähler metrics provides information on the *moduli space of Ricci-flat metrics*, which is defined as

$$\mathcal{M}^{\mathrm{Ric}=0}(M) := \{\, g \in \mathcal{R}(M) \mid g \text{ is Ricci-flat and of unit volume} \,\}/\mathrm{Diff}(M).$$

More general than Ricci-flat metrics, one may also consider Einstein metrics. Recall that a Riemannian metric g is *Einstein* if there is a constant $\lambda \in \mathbb{R}$ such that

$$\mathrm{Ric}_g = \lambda g.$$

The *moduli space of Einstein metrics* is defined as the quotient space

$$\mathcal{E}(M) := \{\, g \in \mathcal{R}(M) \mid g \text{ is Einstein and of unit volume} \,\}/\mathrm{Diff}(M).$$

As in Section 5.1 the assumptions on the volume are not important for the homotopy type. The natural inclusions

$$\mathcal{M}^{\mathrm{HK}}(M) \subset \mathcal{M}^{\mathrm{Ric}=0}(M) \subset \mathcal{E}(M). \tag{7.1}$$

turn out to preserve the connected components in the following sense:

Lemma 7.1.1. *The moduli space of hyperkähler metrics $\mathcal{M}^{\mathrm{HK}}(M)$ is a union of connected components in $\mathcal{M}^{\mathrm{Ric}=0}(M)$. The same is true for $\mathcal{M}^{\mathrm{Ric}=0}(M)$ in $\mathcal{E}(M)$.*

Proof. It is known that $\mathcal{E}(M)$ is a union of path connected components on which the scalar curvature function is constant [19, 12.52 Corollary]. Henceforth, the space $\mathcal{M}^{\mathrm{Ric}=0}(M)$ is a union of connected components in $\mathcal{E}(M)$. From [85, Corollary 3.6.3] we know that every hyperkähler metric admits a parallel spinor. By [138, Theorem 3.1] we know that the space of metrics with parallel spinors is a union of connected components in the moduli space of Einstein metrics. According to [6] the holonomy of a metric defines a constant function, up to conjugation, on the connected components of the space of Riemann metrics admitting a parallel spinor. Thus, if g is a hyperkähler metric, then every other metric in the connected component of the moduli space of Einstein metrics is also hyperkähler. Since all hyperkähler metrics are Ricci-flat we proved our claim. $\square$

One can now apply our results from Chapter 6 to the moduli space of Einstein metrics. For instance, from Corollary 6.4.5 we find the following statement.

Corollary 7.1.2. *If M is hyperkählerian, then every connected component of $\mathcal{E}(M)$ which contains a hyperkähler metric has non-trivial topology.*

In dimension 4, where M is the $K3$-manifold, we already know from Theorem 3.5.2 that $\mathcal{M}^{\mathrm{HK}}(M) = \mathcal{M}^{\mathrm{Ric}=0}(M) = \mathcal{E}(M)$.

Corollary 7.1.3. *Let M be the $K3$-manifold. Then $\mathcal{E}(M)$ is connected and simply connected with $\mathrm{b}_2(\mathcal{E}(M)) \geq 1$.*

Proof. This follows from the discussion above and Corollary 6.4.2. $\square$

In general, it is an open question if the inclusions in (7.1) are strict.

7.2 Moduli Spaces of Ricci-Flat Metrics on Products with Tori

For a simply connected compact manifold N we consider Ricci-flat metrics on the product $N \times T^k$ where T^k denotes a k-dimensional torus. The *structure theorem of compact Ricci-flat manifolds* (M, g) states that there is a finite Riemannian cover

$$(N \times T^k, \tilde{g}) \to (M, g),$$

with $\tilde{g}$ a product metric of a Ricci-flat metric g_N on N and a sectional curvature flat metric g_T on T^k, see [50]. This also follows from the Cheeger-Gromoll splitting theorem [33]. Following the proof of the structure theorem in [50] our first goal is to show that a Ricci-flat metric g on $N \times T^k$ is already isometric to a product metric. We need the following lemma on the isometry group $\mathrm{I}(M, g)$ taken from [19, 1.84], see also [50].

Lemma 7.2.1. *Let (M, g) be a Ricci-flat manifold and let $\mathrm{I}(M, g)^\circ$ be the connected component of $\mathrm{I}(M, g)$ which contains the identity. Then $\mathrm{I}(M, g)^\circ$ is a torus group of rank $\mathrm{b}_1(M)$.*

The Lie algebra of Killing vector fields is known to agree with the Lie algebra of $\mathrm{I}(M, g)^\circ$ [91, Theorem 3.4]. By compactness of $\mathrm{I}(M, g)^\circ$ we know that the exponential map is surjective, thus every point in $\mathrm{I}(M, g)^\circ$ is connected with the identity element by a one parameter subgroup. We conclude that every isometry in $\mathrm{I}(M, g)^\circ$ is induced by a Killing vector field. Using this fact and the de Rham Decomposition Theorem 2.1.3 we will prove our first claim.

Lemma 7.2.2. *Let $M = N \times T^k$ be the topological product of a compact and simply connected manifold N and a k-dimensional torus T^k. Then for a Ricci-flat metric g on M there is a Ricci-flat metric g_N on N and a sectional curvature flat metric g_T on T^k such that g is isometric to the product metric $g_N + g_T$.*

Proof. The metric g lifts to a Ricci-flat metric $\tilde{g}$ on the universal cover $\tilde{M} = N \times \mathbb{R}^k$. By de Rham Decomposition Theorem 2.1.3 we find that $\tilde{g}$ is isometric to a product metric $g_N + g_F$, with g_N Ricci-flat on N and g_F a flat metric on $\mathbb{R}^k$.

Let X be a Killing vector field on M and denote by $\tilde{X}$ the Killing vector field obtained by lifting X to the universal cover $\tilde{M}$. Since $\tilde{M}$ is a product we have a natural splitting $T\tilde{M} = TN \oplus T\mathbb{R}^k$. For $\tilde{X}_N$, the factor of $\tilde{X}$ on TN, it is by [19, 1.81 Theorem c)] straightforward to prove that it is a Killing field with respect to g_N. Since $\mathrm{b}_1(N) = 0$ we know from Lemma 7.2.1 that N does not possess any non-trivial Killing fields so that $\tilde{X}_N = 0$. We find that the space of Killing vector fields $\mathrm{KV}(M)$ acts by translations on the $(\mathbb{R}^k, g_F)$ factor of $\tilde{M}$ and an orbit of a point $(n,t) \in \tilde{M}$ is given by $\{n\} \times \mathbb{R}^k$.

Denote by Γ the decktransformation group which acts by isometries on $(N \times T^k, g_N + g_F)$. Note that the action of Γ commutes with the action of $\mathrm{KV}(M)$ on $\tilde{M}$, as this action is induced by lifts, i.e. $\gamma\tilde{X}(p) = \tilde{X}(p\gamma)$ for a point $p \in \tilde{M}$ and $\gamma \in \Gamma$. We may view Γ and $\mathrm{KV}(M)$ as subgroups of the isometry group $I(\tilde{M}, g)$ so that

$$\mathrm{KV}(M)/\mathrm{KV}(M) \cap \Gamma = \mathrm{I}(M,g)^\circ.$$

Since $\mathrm{I}(M,g)^\circ$ is compact we find that $\mathrm{KV}(M) \cap \Gamma$ is a lattice in $\mathrm{KV}(M) \cong \mathbb{R}^k$. Furthermore, as Γ is a free abelian group of rank k we know that Γ is contained in $\mathrm{KV}(M)$. Thus, we have proven that Γ acts by translations on the factor of $\tilde{M}$ leaving invariant the first. The de Rham splitting therefore preserves the action of the decktransformation group and thus the splitting descends to $N \times T^k$. $\square$

The above Lemma proves the existence of an inverse F^{-1} to the map

$$F\colon \mathcal{M}^{\mathrm{Ric}=0}(N) \times \mathcal{M}^{\mathrm{sec}=0}(T^k) \to \mathcal{M}^{\mathrm{Ric}=0}(N \times T^k)$$

which takes a pair (g_N, g_T) to the product metric $g_N + g_T$. While F is easily seen to be continuous, proving that F^{-1} is continuous is more involved. However, we can conclude continuity by relying on the work of Tuschmann and Wiemeler [129]. There they consider a non-negatively Ricci-curved manifold (M, g) with $\pi_1(M) = \mathbb{Z}^n$. By the Cheeger and Gromoll splitting theorem [33] (M, g) is isometric to a bundle with simply connected non-negative Ricci curved fiber (N, h) over a flat n-dimensional torus (T^n, h'). They prove that the maps

- $\mathcal{M}^{\mathrm{Ric}\geq 0}(M) \to \mathcal{M}^{\mathrm{Ric}\geq 0}(N)$, $g \mapsto h$
- $\mathcal{M}^{\mathrm{Ric}\geq 0}(M) \to \mathcal{M}^{\mathrm{sec}=0}(T^k)$, $g \mapsto h'$

are continuous [129, Theorem 5.1]. In fact, they show that these maps are retractions if $\dim M \geq 5$, [129, Corollary 5.3] and [129, Theorem 5.4]. Since $\mathcal{M}^{\mathrm{Ric}=0}(M)$ embeds continuously into $\mathcal{M}^{\mathrm{Ric}\geq 0}(M)$ we find that F is continuous as well. Thus, we have proven the following.

Corollary 7.2.3. *Let N be a compact simply connected manifold and T^k the k-dimensional torus. Then the splitting of Ricci-flat metrics on $N \times T^k$ into*

Ricci-flat and sectional curvature flat factors induces a homeomorphism

$$\mathcal{M}^{\mathrm{Ric}=0}(N \times T^k) \cong \mathcal{M}^{\mathrm{Ric}=0}(N) \times \mathcal{M}^{\mathrm{sec}=0}(T^k).$$

Before drawing consequences on the topology of $\mathcal{M}^{\mathrm{Ric}=0}(N \times T^k)$ let us make the following remark.

Proposition 7.2.4. *Let M be hyperkählerian and T^k the k-dimensional torus. For a Riemannian metric g on $M \times T^k$ the following are equivalent:*

- *g has zero scalar curvature,*
- *g has non-negative scalar curvature,*
- *g has non-negative Ricci curvature,*
- *g is Ricci-flat.*

Proof. For a $4n$-dimensional hyperkählerian manifold M the $\hat{A}$-genus is equal to $n+1$, see [73, p.614]. The projection $\pi\colon M \times T^k \to T^k$ thus has fiber a manifold which has non-trivial $\hat{A}$-genus over every point in T^k. This means that π is a surjective map onto an enlargeable manifold with non-zero $\hat{A}$-degree. By [63, Theorem B] the space $M \times T^k$ does not admit a positive scalar curvature metric and furthermore any metric with non-negative scalar curvature is Ricci-flat. Clearly non-negative Ricci curvature implies non-negative scalar curvature and so does zero scalar curvature. $\square$

As a consequence of the above proposition we have.

Corollary 7.2.5. *Let N be hyperkählerian. Then*

$$\mathcal{M}^{Ric\geq 0}(N \times T^k) = \mathcal{M}^{\mathrm{Ric}=0}(N \times T^k) \cong \mathcal{M}^{\mathrm{Ric}=0}(N) \times \mathcal{M}^{\mathrm{sec}=0}(T^k).$$

Tuschmann and Wiemeler proved in the already mentioned work [129, Proposition 5.5] the following theorem.

Theorem 7.2.6. *If T^k is a k-dimensional torus, then the following holds:*

- *The moduli space of flat metrics $\mathcal{M}^{\mathrm{sec}=0}(T^k)$ on T^k is simply connected.*
- *If $k = 1, 2, 3$, then $\mathcal{M}^{\mathrm{sec}=0}(T^k)$ is contractible.*
- *If $k = 4$, then $\pi_3(\mathcal{M}^{\mathrm{sec}=0}(T^k)) \otimes \mathbb{Q} \cong \mathbb{Q}$.*
- *If $k > 4$, and $k \neq 8, 9, 10$, then $\pi_5(\mathcal{M}^{\mathrm{sec}=0}(T^k)) \otimes \mathbb{Q} \cong \mathbb{Q}$.*

By the splitting in Corollary 7.2.3 one can now easily combine the above results on $\mathcal{M}^{\mathrm{sec}=0}(T^k)$ with our results of Section 6.4. For instance:

Corollary 7.2.7. *Let N be compact, simply connected and T^k a torus with $k \geq 0$. Then every connected component which contains a hyperkähler metric of the moduli space of Ricci-flat metrics $\mathcal{M}^{\mathrm{Ric}=0}(N \times T^k)$ is not contractible.*

The connected components which contain a hyperkähler metric, induced by a Hilbert scheme of length n over a $K3$-surface, are simply connected and the second rational homotopy group of these is not 0. If M is a Ricci-flat manifold, possibly not hyperkählerian, we know that $\mathcal{M}^{\mathrm{Ric}=0}(M \times T^k)$ has non-trivial topology if $k \geq 4$ and $k \neq 8, 9, 10$.

7.3 Einstein Metrics on the Enriques Manifold

We discuss Einstein metrics on the Enriques manifold M and prove a metric Torelli theorem for the moduli space of Einstein metrics on M. We begin by recalling basic facts on the theory of Enriques surfaces.

7.3.1 Enriques Surfaces

There are several definitions of Enriques surfaces, all equivalent. We choose the following.

Definition 7.3.1. *An Enriques surface is a compact complex surface S with $\pi_1(S) = \mathbb{Z}_2$ and trivial real first Chern class.*

These surfaces belong to the family of irreducible compact complex surfaces for which the Calabi-Yau theorem provides non-flat but Ricci-flat Kähler metrics. The universal cover of an Enriques surface is a $K3$-surface. From [72] we know that the only other manifold, aside from the $K3$-manifold, admitting non-flat Ricci-flat Kähler metrics is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ quotient of the $K3$-manifold.

Much of the theory on Enriques surfaces can be traced back to the theory of $K3$-surfaces, such as Torelli theorems, see [110], [74], [14] for example. A consequence of the Torelli theorems is that any two Enriques surfaces are deformation equivalent [74, Theorem 4.3] and hence diffeomorphic. Thus, the underlying smooth manifold of an Enriques surface will be called the *Enriques manifold*. Like in the case of $K3$-surface, every Enriques surface admits a Kähler metric

[14, (3.1)Theorem p.144]. This also follows from the fact that every Enriques surface is projective [14, V Sect. 23], which is not true for $K3$-surfaces.

Since the universal cover of the Enriques manifold is the $K3$-manifold, we also get some of the topological invariants like the topological Euler number $\chi(S) = 12$ and the second Betti number $\mathrm{b}_2(S) = 10$. The universal coefficient theorem then implies

$$\mathrm{H}^2(S, \mathbb{Z}) \cong \mathrm{Hom}(\mathrm{H}_2(M), \mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{Z}_2, \mathbb{Z}) \cong \mathbb{Z}^{10} \oplus \mathbb{Z}_2.$$

The $\mathbb{Z}_2$-factor is generated by the first Chern class. This can be seen as follows.

First we note that $c_1(S)$ is torsion as the real Chern class vanishes by definition. Now the irregularity of S is $\mathrm{h}^{0,1} = 0$ and thus the vanishing of $c_1(S)$ would imply that the canonical bundle is trivial, in which case S admits a holomorphic symplectic form. This would imply that $\chi(S, \mathcal{O}_S) = 2$. Similar to the proof of Proposition 3.3.1 this contradicts the fact that the universal cover $\tilde{S}$ also has holomorphic Euler characteristic $\chi(\tilde{S}, \mathcal{O}_{\tilde{S}}) = 2$.

Next, note that for a torsion element $t \in \mathrm{H}^2(S, \mathbb{Z})$ one has $(c, t) = 0$ for every class $c \in \mathrm{H}^2(S, \mathbb{Z})$. The intersection paring is thus well defined on the free part $\mathrm{H}^2(S, \mathbb{Z})_0 \coloneqq \mathrm{H}^2(S, \mathbb{Z})/\mathbb{Z}_2$.

Lemma 7.3.1. *The cup-pairing on* $\mathrm{H}^2(S, \mathbb{Z})_0$ *is isomorphic to the so called Enriques lattice* $\Lambda_E \coloneqq E_8(-1) \oplus U$.

Proof. Since S admits no nowhere vanishing holomorphic 2-form we know $\mathrm{h}^{2,0} = 0$ and $\mathrm{h}^{0,2} = 0$. Therefore, $\mathrm{h}^{1,1} = 10$, the rank of $\mathrm{H}^2(S, \mathbb{Z})$. Thus every class in $\mathrm{H}^2(S,\mathbb{Z})_0$ is of type $(1,1)$. By the Lefschetz theorem on $(1,1)$-classes every element $d \in \mathrm{H}^2(S,\mathbb{Z})_0$ is represented by a divisor D. From Riemann-Roch for surfaces and divisors [67, V Theorem 1.6] we find

$$d^2 = D \cdot D = 2\chi(S, \mathcal{O}_S(D)) - 2\chi(S, \mathcal{O}_S) + D \cdot K_S.$$

Since $c_1(S)$ is torsion we have $D \cdot K_S = (d, c_1(S)) = 0$ and thus d^2 is even. The Hodge index theorem [75, Corollary 3.3.16] then yields the signature to be $(1,9)$. From the classification Theorem 2.4.1 of unimodular lattices we conclude the claim. $\square$

The following example is closely related to the Kummer construction from Example 3.5.1, see also [118, p.193].

Example 7.3.2. *Let* $\mathbb{T}_1 = \mathbb{C}/\Gamma_1$ *and* $\mathbb{T}_2 = \mathbb{C}/\Gamma_2$ *be complex* 1*-dimensional tori and let* $\mathbb{T}^2 = \mathbb{T}_1 \times \mathbb{T}_2$ *be their product. Define an involution on* $\mathbb{T}^2$ *by*

$$i \colon \mathbb{T}^2 \to \mathbb{T}^2 \ (z_1, z_2) \mapsto (-z_1, z_2).$$

Recall that the action of $\mathbb{Z}_2$ *on* $\mathbb{T}^2$ *given by* $(z_1, z_2) \mapsto (-z_1, -z_2)$ *gives rise to a complex orbifold* $\mathbb{T}^2/\pm 1$ *and the blow up* $\mathrm{Kum}(\mathbb{T}^2)$ *of the* 16*-singular points in* $\mathbb{T}^2/\pm 1$ *is called the Kummer surface. On* $\mathrm{Kum}(\mathbb{T}^2)$ *the map* i *induces a holomorphic involution* $\mathrm{Kum}(\mathbb{T}^2) \to \mathrm{Kum}(\mathbb{T}^2)$ *which however is not fixed point free, since it acts trivial on* $(0, z_2)$*. To obtain a free involution we introduce another involution. Let* $s = (s_1, s_2)$ *be a singular point in* $\mathbb{T}^2/\pm 1$ *with nei-*

ther s_1 nor s_2 being represented by 0*. Then the translation by s, i.e. the map $\tau\colon (z_1, z_2) \mapsto (z_1 + s_1, z_2 + s_2)$ also gives rise to an involution on* $\mathrm{Kum}(\mathbb{T}^2)$*. The composition $\iota := i \circ \tau$ induces then a fixed point free holomorphic involution on* $\mathrm{Kum}(\mathbb{T}^2)$*. The quotient*

$$\mathrm{Kum}(\mathbb{T}^2)/ < \iota >$$

is thus an Enriques surface.

Finding fixed point free involutions is in general an effective tool to construct Enriques surfaces. One can reduce the problem of finding such to a question of lattice theory, which we will now explain, more details of this are given in [110].

For a fixed point free holomorphic involution i on a $K3$-surface X the quotient map $p\colon X \to X/ < i >$ is the universal covering of the Enriques surface $S := X/ < i >$. The image $\Lambda_M := p^*(\mathrm{H}^2(S, \mathbb{Z}))$ is isomorphic to $\Lambda_E(2)$. It is clear that Λ_M is contained in the invariant part of i^*. But in fact equality is true, i.e. $\Lambda_M = \{x \in \mathrm{H}^2(X, \mathbb{Z})|\ i^*(x) = x\ \}$. Furthermore, one can show that the orthogonal complement $\Lambda_N := \Lambda_M^\perp$ is the sublattice on which i^* acts by -1, i.e. $\Lambda_N = \{x \in \mathrm{H}^2(X, \mathbb{Z})|\ i^*(x) = -x\ \}$. Both lattices Λ_M and Λ_N are primitive with $\Lambda_N \cong E_8(-2) \oplus U(2) \oplus U$ and we have an orthogonal splitting

$$\mathrm{H}^2(X, \mathbb{R}) \cong (\Lambda_M \otimes \mathbb{R}) \oplus (\Lambda_N \otimes \mathbb{R}),$$

compare [110]. The above question is now answered as follows.

Lemma 7.3.3. *For a $K3$-surface X there is a 1-to-1 relation between fixed point free involutions $i\colon X \to X$ and involutions* $i^*\colon \mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^2(X, \mathbb{Z})$

preserving the Kähler cone such that

(i) the invariant part $\Lambda_M := \{ \ x \mid i^*(x) = x \ \}$ *is isomorphic to* $\Lambda_E(2)$*,*

(ii) $\Lambda_M \subset \mathrm{H}^{1,1}(X)$ *and* Λ_M *is not orthogonal to any* (-2)*-class of type* $(1,1)$*.*

Proof. Let i be a fixed point free involution on X. Part (i) follows from the previous discussion. The second statement can be seen as follows. Assume that c is a (-2)-class of type $(1,1)$ orthogonal to Λ_M. By the Lefschetz theorem on $(1,1)$-classes c is represented by a divisor C. Furthermore, Riemann-Roch for line bundles on surfaces [60, p.472] implies that either $\mathrm{h}^0(C)$ or $\mathrm{h}^0(-C)$ is positive and hence C or $-C$ effective. We can assume that C is effective. Since i is a biholomorphism $i(C)$ is also effective and $[C] + [i(C)]$ is contained in $\Lambda_M \cap \Lambda_N = \{0\}$, hence $[C] = -[i(C)]$. This contradicts effectiveness of C and $i(C)$.

Now let i^* be an involution of $\mathrm{H}^2(X, \mathbb{Z})$ satisfying (i) and (ii). By the global Torelli theorem 3.6.1 there is a unique holomorphic involution i whose induced map in cohomology is i^*. That i is fixed point free follows from [114, Theorem 4.2.2], see also the proof of [110, Theorem 7.2]. □

7.3.2 Einstein Metrics

An Enriques surface S is Kähler and thus by the Calabi-Yau Theorem 2.1.4 admits a Ricci-flat Kähler metric, as the first real Chern class vanishes. We can show that every Einstein metric is of this form.

Lemma 7.3.4. *Let* M *be the Enriques manifold and* g *a Riemannian metric defined on* M*. Then the following are equivalent:*

- *g is Einstein.*
- *g is Ricci-flat and Kähler with respect to a unique complex structure.*
- $\mathrm{Hol}(g) \cong \mathrm{SU}(2) \rtimes \mathbb{Z}_2$ *and is contained in* $\mathrm{U}(2)$*.*

Proof. The lift $\tilde{g}$ of an Einstein metric g to the universal cover $\tilde{M}$ is also Einstein. Since $\tilde{M}$ is the $K3$-manifold we know that $\tilde{g}$ is Ricci-flat by Theorem 3.5.2 and hence g as well. Furthermore, we also know from the same result that $\mathrm{Hol}(\tilde{g}) \cong \mathrm{SU}(2)$ and thus $\mathrm{Hol}_0(g) \cong \mathrm{SU}(2)$. By [19, 10.112] and [19, 10.114 Proposition] we find that $\mathrm{Hol}(g)$ is a subgroup of $\mathrm{U}(2)$. Furthermore, the canonical homomorphism $h\colon \pi_1(M) \to \mathrm{Hol}(g)/\mathrm{Hol}_0(g)$ is surjective [85, Proposition 2.2.6]. From Proposition 3.3.1 we know that $\mathrm{Hol}_0(g)$ is a proper subgroup of $\mathrm{Hol}(g)$ and thus h is an isomorphism. By Berger's classification theorem [19, 10.92] the identity component of $\mathrm{Hol}(g)$, which is $\mathrm{Hol}_0(g)$, is identified with $\mathrm{SU}(2)$ in $\mathrm{U}(2)$. Thus, $\mathrm{Hol}(g)$ is isomorphic to a $\mathbb{Z}_2$-extension G of $SU(2)$ in $\mathrm{U}(2)$. Since $G/\mathrm{SU}(2) \cong \mathbb{Z}_2$ there is an element $t \in G - \mathrm{SU}(2)$ and an element $h \in \mathrm{SU}(2)$ such that $t^2 h$ is the identity. Thus, $\det t = \pm 1$ and hence

$$\mathrm{Hol}(g) \cong \{\ A \in \mathrm{U}(2) \mid\ \det A = \pm 1\ \}.$$

Therefore, there is at least one complex structure for which g is Kähler. Let I be such a compatible complex structure, then $g(I\cdot, \cdot)$ is invariant under the $\mathrm{Hol}(g)$ action. On the other hand a $\mathrm{Hol}(g)$-invariant 2-form gives rise to a complex structure after possibly rescaling. Thus, we need to find the dimension of $\mathrm{Hol}(g)$-invariant 2-forms. It suffices to consider the standard representation for which

$$\{e^1 \wedge e^2 + e^3 \wedge e^4, e^1 \wedge e^3 + e^4 \wedge e^2, e^1 \wedge e^4 + e^2 \wedge e^3\}$$

is a basis of SU(2) invariant 2-forms. The element $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{U}(2)$ only preserves $e^1 \wedge e^2 + e^3 \wedge e^4$ and thus the space spanned by parallel complex structures has dimension 1. Henceforth, the complex structure is unique. $\square$

Note that an Einstein metric on the Enriques manifold is never hyperkähler, but locally it always is.

7.4 Torelli Theorem For Einstein Metrics on the Enriques Manifold

Inspired by the proof of the global Torelli theorem for Enriques surfaces [110], [74] the goal of this section is to give a description of the moduli space of Einstein-Metrics in terms of a Torelli type theorem similar to the one for hyperkähler metrics. That is, we will define a period map on the moduli space of Einstein metrics, prove injectivity and determine the image of this map. Other than in the case of hyperkähler metrics we will not use any Teichmüller space but use the language of marked spaces instead. Recall:

Definition 7.4.1. *A marked space is a tuple (X, ϕ), where $X = (M, I)$ is a complex manifold and*

$$\phi \colon \mathrm{H}^2(X, \mathbb{Z}) \to \Lambda,$$

an isomorphism called marking to a fixed lattice Λ. A marked Einstein manifold is a triple (M, g, ϕ) where M is a smooth manifold, ϕ a marking and g an Einstein metric of unit volume on M.

For a marked $K3$-surface we always choose the $K3$-lattice $\Lambda_{K3} = E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$ and for Enriques surfaces the Enriques lattice $\Lambda_E = E_8(-1) \oplus U$ in the respective isomorphism class for the markings. Associated to the marked spaces we introduce the following moduli space.

Definition 7.4.2. *Let M be the Enriques manifold. Then the moduli space of marked Einstein metrics on M is the set*

$$\mathcal{E}_m(M) \coloneqq \{(N, g, \phi) \mid (N, g, \phi) \text{ marked Einstein manifold } \}/\sim$$

where N is diffeomorphic to M and (N_1, g_2, ϕ_1) is equivalent to (N_2, g_2, ϕ_2) if there is a diffeomorphism $\psi\colon N_2 \to N_1$ with $\psi^* g_1 = g_2$ and such that

$$\begin{tikzcd} \mathrm{H}^2(N_1, \mathbb{Z}) \arrow[rr, "\psi^*"] \arrow[dr, "\phi_1"'] & & \mathrm{H}^2(N_2, \mathbb{Z}) \arrow[dl, "\phi_1"] \\ & \Lambda_E & \end{tikzcd}$$

commutes.

Note that $\mathcal{E}_m(M)$ is not endowed with any topology. However, there is a bijection $\mathcal{E}(M) \to \mathcal{E}_m(M)/\mathrm{O}(\Lambda_E)$ by taking g to the triple (M, g, ϕ) with ϕ being some marking for M.

The next step is to define a period map on $\mathcal{E}_m(M)$ by reducing it to the case of $K3$-surfaces. For that we want to relate marked Enriques surfaces with marked $K3$-surfaces. We fix once and for all an involution on the $K3$-lattice Λ_{K3} $=$

$E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U$ given by

$$i\colon E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U \to E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U$$
$$(x_1, x_2, u_1, u_2, u_3) \mapsto (x_2, x_1, u_2, u_1, -u_3).$$

By Λ_M denote the eigenlattice to the eigenvalue 1 and by Λ_N respectively the one for (-1), then Λ_M is isomorphic to the twist of the Enriques lattice Λ_E by 2, i.e. $\Lambda_M \cong \Lambda_E(2)$ and furthermore $\Lambda_N \cong \Lambda_E(2) \oplus U$. We also fix a primitive embedding ι of $\Lambda_E(2) = E_8(-2) \oplus U(2)$ into Λ_{K3} given by $(x, u) \mapsto (x, x, u, u, 0)$. The following theorem is proven in [110, Theorem (1.4)].

Theorem 7.4.1. *Let $j_1, j_2\colon \Lambda_E(2) \to \Lambda_{K3}$ be two primitive embeddings. Then any isometry $\phi\colon \Lambda_E \to \Lambda_E$ extends to an isometry $\tilde{\phi}\colon \Lambda_{K3} \to \Lambda_{K3}$ such that $\tilde{\phi} \circ j_1 = j_2 \circ \phi$.*

For a marked Enriques surface (S, ϕ) let $p\colon \tilde{S} \to S$ denote the universal covering. Then by the above theorem there exists a marking $\tilde{\phi}$ on $\tilde{S}$ such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{H}^2(S, \mathbb{Z}) \arrow[r, "\phi"] \arrow[d, "p^*"] & \Lambda_E \arrow[r, "id"] & \Lambda_E(2) \arrow[d, "\iota"] \\
\mathrm{H}^2(\tilde{S}, \mathbb{Z}) \arrow[rr, "\tilde{\phi}"] & & \Lambda_{K3}.
\end{tikzcd}$$

The marking $\tilde{\phi}$ is unique only up to the action of $\mathrm{O}(\Lambda_N)$.

Recall that the metric period map $\mathcal{P}^{\mathrm{Met}}$ for a hyperkähler manifold maps a hyperkähler metric g to the space $H_g = \mathrm{span}\{[\omega_I], [\omega_J], [\omega_K]\}$ given by the span of the associated Kähler classes.

Definition 7.4.3. *The metric period map on the Enriques manifold is*

$$\mathcal{P}_E^{\mathrm{Met}} \colon \mathcal{E}_m(M) \to (\mathrm{O}(\Lambda_M) \times \mathrm{O}(\Lambda_N)) \setminus \left(\mathrm{Gr}^+(3, \Lambda_{K3} \otimes \mathbb{R})\right)$$

defined by taking (N, g, ϕ) *to* $\tilde{\phi}_{\mathbb{R}}(H_{\tilde{g}})$*, where* $\tilde{\phi}_{\mathbb{R}}$ *is the* $\mathbb{R}$*-linear extension of* $\tilde{\phi}$*.*

Now we want to show that the metric period map $\mathcal{P}_E^{\mathrm{Met}}$ is an injection. This will follow once we have proven the following theorem.

Theorem 7.4.2. *Let* (S_1, g_1, ϕ_1) *and* (S_2, g_2, ϕ_2) *be marked Enriques Einstein manifolds. Then* $\mathcal{P}_E^{\mathrm{Met}}(S_1, g_1, \phi_1) = \mathcal{P}_E^{\mathrm{Met}}(S_2, g_2, \phi_2)$ *if and only if* g_1 *is isometric to* g_2*.*

Proof. For marked Enriques Einstein spaces (S_1, g_1, ϕ_1) and (S_2, g_2, ψ_2) assume that $\mathcal{P}_E^{\mathrm{Met}}(S_1, g_1, \phi_1) = \mathcal{P}_E^{\mathrm{Met}}(S_2, g_2, \psi_2)$. Then there exists $(\gamma_1, \gamma_2) \in (\mathrm{O}(\Lambda_M) \times \mathrm{O}(\Lambda_N))$ such that $\tilde{\phi}_{\mathbb{R},1}(H_{\tilde{g}_1}) = (\gamma_1, \gamma_2) \cdot \tilde{\psi}_{\mathbb{R},2}(H_{\tilde{g}_2})$. We put $\tilde{\phi}_2 \coloneqq \gamma_1 \circ \gamma_2 \circ \tilde{\psi}_2$ and get another marking ϕ_2 for (S_2, g_2) with $\mathcal{P}_E^{\mathrm{Met}}(S_1, g_1, \phi_1) = \mathcal{P}_E^{\mathrm{Met}}(S_2, g_2, \phi_2)$. Now let J_1 and J_2 denote (by Lemma 7.3.4) the unique complex structures associated to g_1 and g_2, furthermore let $\tilde{J}_1, \tilde{J}_2$ be their respective lifts to the universal covering. Denote by ω_1, ω_2 the Kähler forms and by $\tilde{\omega}_1, \tilde{\omega}_2$ their lifts. Then

$$H_{\tilde{g}_1} = \mathcal{P}^{\mathrm{Cpl}}(\tilde{J}_1) \oplus [\tilde{\omega}_1] \cdot \mathbb{R} \quad \text{and} \quad H_{\tilde{g}_2} = \mathcal{P}^{\mathrm{Cpl}}(\tilde{J}_2) \oplus [\tilde{\omega}_2] \cdot \mathbb{R}.$$

and since the markings are isometries we get

$$\tilde{\phi}_{\mathbb{R},i}(H_{\tilde{g}_i}) = \tilde{\phi}_{\mathbb{R},i}(\mathcal{P}^{\mathrm{Cpl}}(\tilde{J}_i)) \oplus \tilde{\phi}_{\mathbb{R},i}([\tilde{\omega}_i]) \cdot \mathbb{R},$$

for $i = 1, 2$. Now let f_1 and f_2 be the non-trivial decktransformations of S_1 and S_2 respectively. Then $\tilde{\omega}_i$ is in the invariant part of f_i^* in $\mathrm{H}^2(S_i, \mathbb{R})$ and $\tilde{\phi}_i(\tilde{\omega}_i) \in \Lambda_M \otimes \mathbb{R}$. On the other hand, the 2-plane $\tilde{\phi}_{\mathbb{R},i}(\mathcal{P}^{\mathrm{Cpl}}(J_i))$ is contained in $\Lambda_N \otimes \mathbb{R}$ as f_i^* changes the sign of the nowhere vanishing holomorphic 2-form on S_i. Henceforth $\tilde{\phi}_{\mathbb{R},1}([\tilde{\omega}_1]) \cdot \mathbb{R} = \tilde{\phi}_{\mathbb{R},2}([\tilde{\omega}_2]) \cdot \mathbb{R}$. After possibly changing the markings with an element in $\mathrm{O}(\Lambda_M) \times \mathrm{O}(\Lambda_N)$ we can assume $\phi_{\mathbb{R},1}([\tilde{\omega}_1]) = \phi_{\mathbb{R},2}([\tilde{\omega}_2])$ and $\tilde{\phi}_{\mathbb{R},1}(\mathcal{P}^{\mathrm{Cpl}}(J_1)) = \tilde{\phi}_{\mathbb{R},2}(\mathcal{P}^{\mathrm{Cpl}}(J_2))$ as oriented 2-planes. By the global Torelli theorem for $K3$-surfaces 3.6.1 there exists a biholomorphism $b\colon (\tilde{S}_2, \tilde{J}_2) \to (\tilde{S}_1, \tilde{J}_1)$ such that $\tilde{\phi}_1 = b^* \circ \tilde{\phi}_2$. Furthermore, we know that $b^*(\tilde{\omega}_1) = b^*(\tilde{\omega}_2)$.

We need to show that $b \circ f_2 = f_1 \circ b$. The map $f_2^{*-1} \circ b^* \circ f_1^*$ is a Hodge isometry on $\mathrm{H}^2(M, \mathbb{C})$ which takes $[\tilde{\omega}_1]$ to $[\tilde{\omega}_2]$. Again from the global Torelli theorem 3.6.1 we find that $f_2^{*-1} \circ b^* \circ f_1^*$ is induced by a unique biholomorphic map and thus $b = f_1 \circ b \circ f_2^{-1}$. Therefore, b is an isometry between $\tilde{g}_1$ and $\tilde{g}_2$ which preserves the deck transformations. Hence, g_1 is isomorphic to g_2. $\square$

Our next goal is to determine the image of $\mathcal{P}_E^{\mathrm{Met}}$. For that we recall the definition of the Weyl group of a $K3$-surface.

Definition 7.4.4. *For a $K3$-surface X let $W(X)$ be the subgroup of* $\mathrm{O}(\mathrm{H}^2(X, \mathbb{Z}))$ *generated by reflections along (-2)-classes of type* $(1, 1)$.

If $X' = (M, I')$ is another $K3$-surface with the same period point $\mathcal{P}^{\mathrm{Cpl}}(X')$ as the $K3$-surface X, then $W(X) = W(X')$ as they share the same (-2)-classes. Furthermore, the action of $W(X)$ leaves invariant the period $\mathcal{P}^{\mathrm{Cpl}}(X)$ as it acts by the identity on $\mathrm{H}^{2,0}(I) \oplus \mathrm{H}^{0,2}(I)$. On the other hand, it acts transitively on the set of Kähler chambers, i.e. the closure of a connected component of $\mathrm{Pos}(X) - \bigcup_{z\ (-2)-class\ of\ type-(1,1)} z^{\perp}$. In fact, it is a fundamental domain of the action on the positive cone of $(1,1)$-classes $\mathrm{Pos}(X)$ which contains the Kähler cone, see [83, Proposition 5.5]. We can now determine the image of $\mathcal{P}_E^{\mathrm{Met}}$.

Theorem 7.4.3. *A positive definite* 3*-plane* $H \in \mathrm{Gr}^+(3, \Lambda_{K3} \otimes \mathbb{R})$ *is contained in the image of* $\mathcal{P}_E^{\mathrm{Met}}$ *if and only if the following conditions are satisfied*

(i) $\dim(H \cap (\Lambda_N \otimes \mathbb{R})) = 2$,

(ii) $H \cap (\Lambda_N \otimes \mathbb{R})$ *is not orthogonal to any* (-2)*-class in* Λ_N,

(iii) H *is not orthogonal to any* (-2)*-class in* Λ_{K3}.

Proof. For a marked Enriques Einstein manifold (S, g, ϕ) we know that the lift $\tilde{g}$ is a hyperkähler metric and $H_{\tilde{g}} = \mathcal{P}^{\mathrm{Cpl}}(\tilde{J}) \oplus [\tilde{\omega}] \cdot \mathbb{R}$ where $\tilde{J}$ is the lift of the unique complex structure for which g is Kähler and $[\tilde{\omega}]$ the corresponding Kähler class. Then $\mathcal{P}_E^{\mathrm{Met}}(S, g, \phi) = \tilde{\phi}_{\mathbb{R}}(\mathcal{P}^{\mathrm{Cpl}}(\tilde{J})) \oplus \tilde{\phi}_{\mathbb{R}}([\tilde{\omega}]) \cdot \mathbb{R}$ and $\mathcal{P}^{\mathrm{Cpl}}(\tilde{J}) \subset \Lambda_N \otimes \mathbb{R}$. On the other hand, $\tilde{\omega}$ is contained in the invariant part and hence (i) is satisfied. The second condition follows from Lemma 7.3.3. Since $\tilde{\phi}_{\mathbb{R}}^{-1}(\mathcal{P}_E^{\mathrm{Met}}(S, g, \phi)) = \mathcal{P}^{\mathrm{Met}}(\tilde{g})$ we know that H cannot be orthogonal to any (-2)-class.

Now let H be such that (i), (ii) and (iii) are satisfied. Define $P \coloneqq H \cap (\Lambda_N \otimes \mathbb{R})$. By the surjectivity of the complex period map there exists a marked $K3$-

surface $(\tilde{S}, \tilde{\phi})$ with $\tilde{\phi}_{\mathbb{R}}(\mathcal{P}^{\mathrm{Cpl}}(\tilde{S}))) = P$. By assumption

$$(H \cap (\Lambda_M \otimes \mathbb{R})) \cap \tilde{\phi}\left(\mathrm{Pos}(\tilde{S}) - \bigcup_{(-2)-\text{class orthogonal to } P} z^{\perp}\right) \neq \emptyset.$$

Pick some ω in the above intersection. Then $\tilde{\phi}^{-1}(\omega)$ is contained in a Kähler chamber of $\tilde{S}$. Choose γ in the Weyl group $W(\tilde{S})$ so that $\gamma^{-1} \circ \tilde{\phi}^{-1}(\omega)$ is contained in the Kähler cone $\mathrm{Käh}(\tilde{S})$ and define a new marking $\psi := \phi \circ \gamma$ of $\tilde{S}$. By the Calabi-Yau theorem $\psi^{-1}(\omega)$ is represented by a unique Einstein metric h with $\psi_{\mathbb{R}}(\mathcal{P}^{\mathrm{Met}}(h)) = H$. We need to show that there is a fixed point free involution for which h is an isometry. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^2(\tilde{S}, \mathbb{Z}) & \xrightarrow{j} & \mathrm{H}^2(\tilde{S}, \mathbb{Z}) \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\ \Lambda_{K3} & \xrightarrow{\iota} & \Lambda_{K3}, \end{array}$$

where $j := \psi^{-1} \circ \iota \circ \psi$. Then j is induced by a fixed point free holomorphic involution on $i \colon \tilde{S} \to \tilde{S}$ by Lemma 7.3.3 and as i fixes the Kähler-form of h it is an isometry with respect to h. Thus, h descents to an Einstein metric on the Enriques surface $S := \tilde{S}/ < i >$. □

As a corollary we obtain the following.

Corollary 7.4.4. *The moduli space of unit volume Einstein metrics on the Enriques manifold is homeomorphic to the subspace of* $(\mathrm{O}(\Lambda_M) \times \mathrm{O}(\Lambda_N)) \setminus (\mathrm{Gr}^+(3, \Lambda_{K3} \otimes \mathbb{R}))$ *where the image is given by those* 3*-planes* H *with*

(i) $\dim(H \cap (\Lambda_N \otimes \mathbb{R})) = 2$,

(ii) $H \cap \Lambda_N \otimes \mathbb{R}$ *is not orthogonal to any* (-2)*-class in* Λ_N*,*

(iii) H *is not orthogonal to any* (-2)*-class in* Λ_{K3}*.*

Proof. Let M be the Enriques manifold and $\tilde{M}$ its universal covering. Fix some marking ϕ for M and let $\tilde{\phi}\colon \mathrm{H}^2(\tilde{M}, \mathbb{Z}) \to \Lambda_{K3}$ be a lifted marking. Let $\mathcal{R}^{\mathrm{Ein}}(M)$ denote the space of Einstein-metrics of unit volume on M. Then consider the following commutative diagram

$$\begin{tikzcd}
\mathcal{R}^{\mathrm{Ein}}(M) \arrow[r, "L"] \arrow[d] & \mathcal{R}^{\mathrm{HK}}(\tilde{M}) \arrow[dr, "\tilde{\phi}_{\mathbb{R}} \circ \mathcal{P}^{\mathrm{Met}}"] & \\
\mathcal{E}(M) \arrow[r, "\Phi"] & \mathcal{E}_m(M)/\mathrm{O}(\Lambda_E) \arrow[r, "\mathcal{P}_E^{\mathrm{Met}}"] & \left(\mathrm{O}(\Lambda_M) \times \mathrm{O}(\Lambda_N)\right) \setminus \left(\mathrm{Gr}^+(3, \Lambda_{K3} \otimes \mathbb{R})\right).
\end{tikzcd}$$

Where L is the map which lifts a metric to its universal cover and $\Phi\colon \mathcal{E}(M) \to \mathcal{E}_m(M)/\mathrm{O}(\Lambda_E)$ the bijection induced by the map which takes g to the marked Einstein manifold (M, g, ϕ). The composed map $\tilde{\phi}_{\mathbb{R}} \circ \mathcal{P}^{\mathrm{Met}} \circ L$ is continuous. Thus, if we endow $\mathcal{E}_m(M)/\mathrm{O}(\Lambda_E)$ with the topology induced by Φ we get that also $\mathcal{P}_E^{\mathrm{Met}}$ is continuous. The claim then follows by Theorems 7.4.2 and 7.4.3.

□

7.5 Desingularization of a Flat Orbifold Metric on an Enriques Surface

Recall the Kummer type construction of an Enriques surface in Example 7.3.2. Given a torus $\mathbb{T}^2 = \mathbb{C}/\Gamma_1 \times \mathbb{C}/\Gamma_2$ we constructed a fixed point free involution ι on the singular space $X \coloneqq \mathbb{T}^2/\pm 1$.

The Kummer construction 3.5.1 produces a $K3$-surface $\tilde{S} = \mathrm{Kum}(\mathbb{T}^2)$ by replacing small neighborhoods of each singular point in X with a copy of $T^*\mathbb{C}P^1$. The Example 7.3.2 shows that if one does the same for the 8 singularities in $Y := X/ < \iota >$ one finds an Enriques surface S so that

$$\begin{tikzcd}
\tilde{S} \arrow[r] \arrow[d] & S = \tilde{S}/\langle\iota\rangle \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}$$

commutes with the horizontal maps being universal coverings and the vertical resolution maps.

The standard flat metric on $\mathbb{C}^2$ descends to a flat orbifold metric on X as well as on Y. By gluing Eguchi-Hanson metrics, which are defined on $T^*\mathbb{C}P^1$, with the flat metric on X away from the singular points one obtains an approximated Ricci flat metric on X. The metric can then be perturbed inside its Kähler class to obtain a Ricci-flat metric close to the glued one. Furthermore, in this way one can construct a whole sequence of such metrics which then converges to the flat metric on X in the Gromov-Hausdorff sense. This construction is well known, see [88] and [85, Example 7.3.14] for instance. By following [88] we now show that essentially the same is true for the Enriques surface S and Y.

Eguchi-Hanson Metric

We begin by recalling some facts on the Eguchi-Hanson metric, for more details we refer to [88, p. 293], see also [127].

The *Eguchi-Hanson-Metric* is a non-compact hyperkähler metric on $T^*\mathbb{C}P^1$

which belongs to the family of so called *ALE metrics*, short for asymptotically locally Euclidean. Roughly speaking the ALE-condition means that on the complement $T^*\mathbb{C}P^1 \setminus B_R$ of a large ball B_R the metric locally approximates the flat 4-dimensional Euclidean metric as the radius R increases.

There is a map $p\colon T^*\mathbb{C}P^1 \to \mathbb{C}^2/\pm 1$ which is a biholomorphism away from the singular point and its fiber, i.e. there is an isomorphism

$$\left(\mathbb{C}^2 - 0\right)/\pm 1 \cong \mathbb{T}^*\mathbb{C}P^1 - E,$$

where $E \coloneqq p^{-1}(0)$. The map p is then called a *resolution* of the singularity in $\mathbb{C}^2/\pm 1$. Moreover, the subspace E is called *exceptional divisor*. It is the zero section of $T^*\mathbb{C}P^1$. Therefore, $E \cong \mathbb{C}P^1$ and the self-intersection number is $E^2 = -2$, see [84, Theorem 7.5.1] for more on resolutions of quotient singularities.

On $(\mathbb{C}^2 - 0)/\pm 1$ the Eguchi-Hanson metric can be written in the form $\sqrt{-1}\partial\bar{\partial}\phi$. More precisely, the Kähler-potential ϕ is given by

$$\phi = \phi_{EH,a}(\rho) = \rho^2\sqrt{1 + \frac{a^2}{\rho^4}} + a\log\frac{\rho^2}{\sqrt{\rho^4 + a^2} + a},$$

where $\rho = \sqrt{|z_1|^2 + |z_2|^2}$ for some positive number $a > 0$. Note that for different a one obtains different metrics, which all go under the label Eguchi-Hanson metric. For $a = 0$ we remark that the function $\phi_{EH,a}$ is the Kähler potential for the flat metric on $(\mathbb{C}^2 - \{0\})/\pm 1$. One can show that the Eguchi Hanson spaces converge to the flat orbifold metric on $\mathbb{C}^2/\pm 1$ in the Gromov-Hausdorff topology as a goes to 0. On the exceptional divisor E this has the following effect.

For each $a > 0$ the space E is a totally geodesic submanifold with constant curvature $\frac{1}{a}$ and $Vol(E) = 4\pi a$, i.e. it is a scaled version of the Fubini-Study metric on $\mathbb{C}P^1$. Thus as a approaches 0 the curvature on $T^*\mathbb{C}P^1$ starts to concentrate around E.

Glueing Eguchi-Hanson Space with Flat Metric

Our goal is to replace small neighborhoods around each of the eight singular points in Y with an Eguchi-Hanson space. To make this gluing possible, we need to match the flat metric on Y. This whole process is done as follows.

For fixed $\delta > 0$, let U_i denote the metric ball of radius $1 + \delta$ in Y with center the singular point s_i. By possibly rescaling the metric on Y in the first place, we may assume that the U_i are pairwise disjoint and thus isomorphic to $B_{1+\delta}$ the metric ball of radius $1 + \delta$ in $\mathbb{C}^2/\pm$ centered at 0. Choose a smooth function $f\colon \mathbb{R} \to \mathbb{R}$ with the following properties

$$f(\rho) = 1 \ \text{ for } \ \rho \leq 1 - \delta, \quad f(\rho) = 0 \ \text{ for } \ \rho \geq 1, \quad \text{and} \quad |f'(\rho)| \leq \frac{2}{\delta}.$$

For $a_i > 0$ sufficiently small, we define

$$F_i(\rho) \coloneqq (1 - f(\rho))\rho^2 + f(\rho)\phi_{EH,a_i}(\rho).$$

Then F_i gives rise to a Kähler potential on $B_{1+\delta}$ with $\sqrt{-1}\partial\bar{\partial}F_i$ being the Eguchi-Hanson metric on $B_{1-\delta'} - \{0\}$, where $\delta < \delta' < 1$ and being the flat metric on $B_{1+\delta} - B_1$. For an open neighborhood V in $T^*\mathbb{C}P^1$ containing E we then have isomorphisms $U_i - \{s_i\} \cong B_{1+\delta} - \{0\} \cong V - E$. For each F_i we get a metric defined on $V_i := (V, \omega_i)$ which is the Eguchi-Hanson metric in a small neighborhood of E.

We can thus replace each U_i with V_i to obtain a metric denoted ω_a on the Enriques Surface

$$S = Y \cup \bigcup_{i=1}^{8} V_i,$$

where $a = (a_1, \cdots, a_8)$ and each a_i corresponds to the parameter of the Eguchi-Hanson metric on V_i. Thus producing a flat metric on $Y - \bigcup_{i=1}^{8} V_i$ and a Ricci-flat metric close to the exceptional divisors E. However, the metric is not Ricci-flat in the regions where we interpolate between the flat metrics and the Eguchi-Hanson metrics.

Perturbing the Metric ω_a

The next step is to find a metric close to ω_a being Ricci-flat on the whole of S. For that, let σ denote the nowhere vanishing holomorphic 2-form on $\tilde{S}$. The volume form $\sigma \wedge \bar{\sigma}$ descends to a volume form on S, and we define the function $F := log\frac{\sigma \wedge \bar{\sigma}}{\omega_a^2}$ on S. Due to Yau's solution to the Calabi-conjecture [141] there is a smooth function ϕ_a defined on S such that

$$(\omega_a + \sqrt{-1}\partial\bar{\partial}\phi_a)^2 = e^F \omega_a^2,$$

and furthermore, $\mu_a := \omega_a + \sqrt{-1}\partial\bar{\partial}\phi_a$ defining a Kähler form. Now consider the universal covering $\tilde{S} \to S$. The lift $\tilde{\omega}_a$ of the glued metric ω_a, can also be understood by basically the same gluing procedure namely by replacing $\tilde{U}_i \cong U_i \sqcup U_i$ with $\tilde{V}_i \cong V_i \sqcup V_i$. Furthermore, we can lift the functions F and ϕ_a to $\tilde{S}$ so that

$$\tilde{\omega}^2 = (\tilde{\omega}_a + \sqrt{-1}\partial\bar{\partial}\tilde{\phi}_a)^2 = \sigma \wedge \bar{\sigma}.$$

From [75, Corollary 4.B.23] we know that $\tilde{\omega}$ is Ricci-flat. The apriori estimates for $\tilde{\phi}_a$ are the same for ϕ_a and have been carried out in [88, Theorem 18], see also [142, 69, 11, 89]. In particular, for $\mu_a \coloneqq \omega_a + \sqrt{-1}\partial\bar{\partial}\phi_a$ we find from [88, p.302] that

$$(1 - C|a|^{\frac{1}{2}})\omega_a \leq \mu_a \leq (1 + C|a|^{\frac{1}{2}})\omega_a,$$

for some positive constant C. As in [88, p.303] it follows that μ_a converges to the flat orbifold metric on Y in the Gromov-Hausdorff sense with smooth convergence outside the singular set as $|a| \to 0$.

The Period Point of μ_a

Now consider the case as $|a| \to 0$, where $a = (a_1, \cdots, a_8)$ and a_i the parameter for an Eguchi-Hanson space glued in Y. Denote by $\tilde{\mu}_a$ the lift of the metric μ_a constructed in the previous section to the $K3$-surface $\tilde{S}$.

Let $\pi\colon S \to X$ be the resolution of X, i.e. the map which is a biholomorphism

$$\pi\colon S - \bigcup E_i \to X - \{X_s\},$$

where the $E_i \cong \mathbb{C}P^1$ are the exceptional divisors and X_s the singular points in X. Denote by κ a flat orbifold-Kähler form on X. Then by [90, Theorem 1] the singular form $\pi^*\kappa$ defines a closed current, and thus a cohomology class in $\mathrm{H}^2(\tilde{S},\mathbb{R})$. From [88] we find that the cohomology class of the lift μ_a is given by

$$[\mu_a] = [\pi^*\kappa] - \sum_i^8 a_i \,\mathrm{PD}[E_i] - \sum_i^8 a_i \,\mathrm{PD}[E_i'],$$

where $\mathrm{PD}[E_i]$ and $\mathrm{PD}[E_i']$ are the Poincaré-duals of the exceptional divisors E_i, E_i', where $E_i + E_i'$ is the lift of an exceptional divisor on the Enriques surface. Recall that each $\mathrm{PD}[E_i]$ defines a (-2)-class of type $(1,1)$ and we compute

$$([\mu_a], PD[E_i]) = -2a_i.$$

Thus, the limit $[\mu_0] \coloneqq \lim_{a_i\to 0}([m_a])$ is contained in the boundary of the Kähler cone of $\tilde{S}$.

The period point $\mathcal{P}^{\mathrm{Met}}(\mu_a) = \mathcal{P}^{\mathrm{Cpl}}(\tilde{S}) \oplus [\mu_a]\cdot\mathbb{R}$ converges to a 3-space H in $\bigcap_i \mathrm{Gr}^+(3,\mathrm{PD}[E_i]^\perp)$. For the space H we have $\dim(H\cap(\Lambda_N\otimes\mathbb{R})) = 2$ and $H\cap(\Lambda_N\otimes\mathbb{R})$ is not orthogonal to any (-2)-class in Λ_N, where Λ_N the sublattice of $\mathrm{H}^2(\tilde{S},\mathbb{Z})$ on which the involution i acts by multiplication with -1. But H is now orthogonal to the (-2)-classes provided by the exceptional divisors.

7.6 Holes In the Moduli Space of Hyperkähler Metrics

For a compact hyperkähler manifold, recall that the moduli space of unit volume hyperkähler metrics $\mathcal{M}^{\mathrm{HK}}(M)$ is homeomorphic to a subspace of $\Gamma\backslash\mathrm{Gr}^+(3,\mathrm{H}^2(M,\mathbb{R})$ obtained by 'cutting out' subspaces of the form $\mathrm{Gr}^+(3,z^\perp)$ where z is an MBM-class.

We have seen that sometimes these subspaces, which we refer to as *holes*, induce non-trivial elements in the second rational homotopy group of $\mathcal{M}^{\mathrm{HK}}(M)$, e.g. M being of type $K3^{[n]}$.

In case of the $K3$-manifold, Kobayashi and Todorov [90] prove by generalizing the gluing construction for Kummer surfaces to generalized $K3$-surfaces that elements in the holes can naturally be described in terms of certain Ricci-flat orbifold metrics, see also [88].

Definition 7.6.1. *A generalized $K3$-surface is a compact complex surface X with at worst isolated singularities of the form $\mathbb{C}^2/G$ where G is a finite subgroup of* $\mathrm{SU}(2)$*, so that its minimal resolution $\tilde{X}$ is a $K3$-surface.*

Now a positive definite 3-space H in $\mathrm{H}^2(M,\mathbb{R})$, possibly orthogonal to a (-2)-class, corresponds to a Ricci-flat orbifold metric in the following way. For any 2-plane $P \subset H$ there exists a generalized $K3$-surface X and a Ricci-flat Kähler orbifold metric g on X. The metric only depends on H and so does the underlying smooth orbifold structure of X. Furthermore, on $\tilde{X}$ there exists a sequence of Ricci-flat Kähler metrics converging in the Gromov-Hausdorff topology to the orbifold (X, g). Away from the singular points the convergence is also smooth. The space

$$\mathcal{M}^O \coloneqq \mathrm{O}(\Lambda_{K3}) \diagdown \mathrm{O}(3,19) \diagup \mathrm{O}(3) \times \mathrm{O}(19)$$

can then be understood as the moduli space of Ricci-flat metrics including these singular metrics [90]. By a result of Anderson [7], this space can also be interpreted as the completion of $\mathcal{M}^{\mathrm{HK}}(M)$ when endowed with the L^2-metric. For that we first note that $\mathcal{M}^O$ has a natural metric induced by the symmetric metric on $\mathrm{O}(3,19)/\mathrm{O}(3) \times \mathrm{O}(19)$. The L^2-metric is a $\mathrm{Diff}(M)$-equivariant metric defined on the space of Einstein metrics $\mathcal{R}^{\mathrm{Ein}}(M)$. For the definition we refer to [7]. Anderson proves that the metric period map

$$\mathcal{P}^{\mathrm{Met}} \colon \mathcal{M}^{\mathrm{HK}}(M) \to \mathrm{O}(\Lambda_{K3}) \diagdown \mathrm{O}(3,19) \diagup \mathrm{O}(3) \times \mathrm{O}(19)$$

extends to an isometry on the completion $\overline{\mathcal{M}^{\mathrm{HK}}(M)}^{L^2}$. For the topology of this space we can say the following.

Corollary 7.6.1. *The moduli space of Ricci-flat metrics, including orbifold metrics, on the $K3$-manifold is simply connected and the 4th-Betti number is at least 1.*

Proof. The moduli space is by the above discussion identified with $\mathcal{M}^O$ and Lemma 6.4.3 shows that the 4th-Betti number does not vanish.

The space $\mathcal{M}^O$ is homeomorphic to $\mathrm{O}(\Lambda_{K3}) \setminus \mathrm{Gr}^+(3, \Lambda_{K3} \otimes \mathbb{R})$. From Theorem 2.4.2 we know that $\tilde{\mathrm{O}}^+(\Lambda_{K3})$ is generated by reflections along (-2)-classes. Since Λ_{K3} is unimodular $\tilde{\mathrm{O}}^+(\Lambda_{K3})$ is just the index 2-subgroup of $\mathrm{O}(\Lambda_{K3})$ with spinor norm 1, i.e. $\tilde{\mathrm{O}}^+(\Lambda_{K3}) = \mathrm{O}^+(\Lambda_{K3})$. As $-Id \in \mathrm{O}(\Lambda_{K3}) \setminus \mathrm{O}^+(\Lambda_{K3})$ acts trivially on $\mathrm{Gr}^+(3, \Lambda_{K3} \otimes \mathbb{R})$ there is a homeomorphism

$$\mathcal{M}^O \cong \mathrm{O}^+(\Lambda_{K3}) \setminus \mathrm{Gr}^+(3, \Lambda_{K3} \otimes \mathbb{R}).$$

Lemma 6.3.1 shows that $\mathcal{M}^O$ is simply connected. □

It is natural to ask if points in the 'holes' of other moduli spaces also correspond to geometric objects. For the Enriques manifold we expect that the picture is similar to the $K3$-case. For instance, we make the following conjecture.

Conjecture: Let $H \in \mathrm{Gr}^+(3, \Lambda_{K3} \otimes \mathbb{R})$ and $P := H \cap (\Lambda_N \otimes \mathbb{R})$ such that $\dim P = 2$. Furthermore, assume that P is not orthogonal to any (-2)-class in Λ_N. Let X denote the generalized $K3$-surface endowed with the Ricci-flat orbifold metric g associated to H and P. Then on X there exists a fixed point free biholomorphic involution i which is also an orbifold isometry for the metric g. Furthermore, if $\tilde{S}$ is the minimal resolution of X and S the minimal resolution of $Y := X/\langle i \rangle$ then

$$\begin{tikzcd} \tilde{S} \arrow[r] \arrow[d] & S = \tilde{S}/\langle i \rangle \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

commutes, with the vertical arrows being universal coverings and the horizontal resolutions.

Symbols

$\mathcal{F}_{\mathrm{Curv}}$	Total Curvature Functional
$\mathcal{F}_{\mathrm{Scal}}$	Total Scalar Curvature Functional
R	Riemann Curvature Tensor
Ric	Ricci Curvature Tensor
scal	Scalar Curvature
dvol	Volume Form
Gl	General Linear Group
O	Orthogonal Group
SO	Special Orthogonal Group
U	Unitary Group
SU	Special Unitary Group
Sp	Unitary Quaternionic Group
$\mathcal{T}^{\mathrm{Cpl}}$	Complex Teichmüller Space/ of IHS-structures
$\mathcal{T}^{\mathrm{Cpl}}_{\mathrm{Full}}$	Teichmüller Space of complex structures
$\mathcal{T}^{\mathrm{Cpl}}_{b}$	Birational Complex Teichmüller Space
$\mathcal{T}^{\mathrm{Cpl}}_{o}$	Connected component of Complex Teichmüller Space
$\mathcal{P}^{\mathrm{Cpl}}$	Complex Period Map
$\mathcal{P}^{\mathrm{Cpl}}_{b}$	Birational Period Map
$\mathcal{AC}omp$	Space of Almost Complex Structures
$\mathcal{C}omp$	Space of Complex Structures
$\mathbb{P}\mathrm{er}$	Complex Period Domain
$\mathcal{C}omp_{IHS}$	Space of IHS-Structures
$\mathcal{C}omp_{K}$	Space of Complex Structures which are Kähler
Käh	Kähler Cone
Pos	Positive Cone
$\mathfrak{B}$	Kuranishi Space

Diff	Group of orientation preserving diffeomorphisms
Diff_I	Group of Diffeormorphisms preserving a connected component of complex Teichmüller space
Diff_0	Identity component of $\mathrm{Diff}(M)$
Δ	Unit Disk in $\mathbb{C}$
Ω^2	Sheaf of Holomorphic 2-Forms
Mon^2	Monodromy Group
$\mathrm{Mon}^2_{\mathrm{Hdg}}$	Hodge Monodromy Group
KC	Set of Kähler Chambers
$\mathcal{R}$	Space of Riemannian Metrics
$\mathcal{R}^{\mathrm{HK}}$	Space of Hyperkähler Metrics
$\mathcal{P}^{\mathrm{Met}}$	Metric Period Map
$\mathcal{T}^{\mathrm{Met}}$	Metric Teichmüller Space
$\mathcal{T}^{\mathrm{Met}}_o$	Connected Component of Metric Teichmüller Space
$\mathcal{T}^{\mathrm{Met,Cpl}}$	Teichmüller Space of Hyperkähler Structures
$\mathcal{M}^{\mathrm{HK}}$	Moduli Space of Hyperkähler Metrics
$\mathcal{M}^{\mathrm{HK}}_o$	Component of Moduli Space of Hyperkähler Metrics
MCG	Mapping Class Group
MCG_o	Subgroup of the Mapping Class Group
$\mathbb{T}_w$	Twistor Line
$\mathbb{X}$	Twistor Space
Kum	Kummer Surface
spn	Spinor norm
Gr	Grassmann Space
Gr^{o}	Grassmann Space of Oriented Subspaces
$\mathrm{Gr}^{+,\mathrm{o}}$	Grassmann Space of Positive definite and Oriented Subspaces
Gr^{+}	Grassmann Space of positive definite subspaces
D_Λ	Discriminant group
q_D	Discriminant form
$\mathrm{disc}(\Lambda)$	Discriminant
$\tilde{\mathrm{O}}$	Stable Orthogonal Group
O^+	Orthogonal transformation with spinor norm 1
$\tilde{\mathrm{O}}^+$	Subgroup of stable Orthogonal Group with trivial spinor norm

$\mathcal{P}$	Poset induced by affine subspaces
$\mathrm{B}\,\Gamma$	Classifying Space of the Group Γ
$\mathrm{E}\,\Gamma$	Universal Cover of $\mathrm{B}\Gamma$
$\mathcal{M}^{\mathrm{Ric}=0}$	Moduli Space of Ricci flat metrics
$\mathcal{M}^{\mathrm{sec}=0}$	Moduli Space of Sectional Curvature Flat Metrics
$\mathcal{E}$	Moduli Space of Einstein Metrics
$\mathrm{I}(M,g)^{\circ}$	Connected Component of the Isometry Group
$\mathrm{I}(M,g)$	Isometry Group
$\mathrm{KV}(M)$	Space of Killing Vector Fields
Λ_{K3}	K3-Lattice
Λ_{E}	Enrique's-Lattice
$\mathcal{E}_m$	Moduli-Space of Marked Einstein metrics
$\mathcal{P}_E^{\mathrm{Met}}$	Metric Period Map on the Enrique's manifold
$\mathbb{H}$	Quaternions
$\mathcal{O}_X$	Structure Sheaf
Ω^k	Sheaf of Holomorphic k Forms
Hol	Holonomy Group
Hol_0	Reduced Holonomy Group
$\mathcal{M}_{K3}^m$	Moduli Space of Marked K3 Surfaces
$\widetilde{\mathbb{P}\mathrm{er}}$	Refined Period Domain
b	Betti Number
h	Hodge Number
H	Cohomology Group
End	Endomorphisms
Hilb	Hilbert Scheme
MBM	MBM-classes
IHSM	Irreducible holomorphic symplectic manifold
IHS	Irreducible holomorphic symplectic

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