Received: 18 October 2022 | Revised: 23 January 2023 | Accepted: 31 January 2023

DOI: 10.1002/nag.3506

RESEARCH ARTICLE

WILEY

The u-p approximation versus the exact dynamic equations for anisotropic fluid-saturated solids. I. Hyperbolicity

Vladimir A. Osinov

Institute of Soil Mechanics and Rock Mechanics, Karlsruhe Institute of Technology, Karlsruhe, Germany

Correspondence
Vladimir A. Osinov, Institute of Soil Mechanics and Rock Mechanics, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany.
Email: vladimir.osinov@kit.edu

Abstract

The numerical solution of dynamic problems for porous fluid-saturated solids is often performed with the use of simplified equations known as the *u*-*p* approximation. The simplification of the equations consists in neglecting some acceleration terms, which is justified for a certain class of problems related, in particular, to geomechanics and earthquake engineering. There exist two *u*-*p* approximations depending on how many acceleration terms are neglected. All comparative studies of the exact and *u*-*p* formulations are focused on the question of how well the *u*-*p* solutions approximate those obtained with the exact equations. In this paper, the equations are compared from a different point of view, addressing the question of well-posedness of boundary value problems. The exact equations must be hyperbolic and satisfy the corresponding hyperbolicity conditions for the boundary value problems to be well posed. The *u*-*p* equations are not of the form to which the conventional definition of hyperbolicity applies. A slight extension of the approach makes it possible to derive hyperbolicity conditions as necessary conditions for well-posedness for the *u*-*p* approximations. The hyperbolicity conditions derived in this paper for the *u*-*p* approximations are formulated in terms of the acoustic tensor of the skeleton. They differ essentially from the hyperbolicity conditions for the exact equations.

KEYWORDS
acoustic tensor, fluid-saturated solid, hyperbolicity, *u*-*p* approximation

1 | INTRODUCTION

The *u*-*p* approximation of the dynamic equations for porous fluid-saturated solids was proposed in the early 1980s in geomechanics with the aim of reducing the number of variables and computational costs for the numerical solution of earthquake-related problems.[1,2] The approximation consists in neglecting some acceleration terms in the governing equations. This makes it possible to eliminate the fluid velocity from the equations and devise a finite-element scheme for the displacement of the solid skeleton, $\mathbf{u}$, and the pore pressure p (hence the name u-p approximation). Because of the rapid increase in computer capacities during the last decades, the advantage of having less variables is nowadays not as important as before. Nevertheless, the numerical implementation of the u-p approximation remains popular[3–12] and even extends beyond conventional finite-element techniques to modern mesh-free methods.[13–16] Since the u-p formulation is an approximation of the exact equations, an important question is how far the solution obtained with the u-p equations for a particular problem of interest deviates from the solution obtained with the exact equations. The difference between the two solutions depends primarily on the permeability of the porous medium and the frequency content of the motion. Although the range of applicability of the u-p approximation was identified from the very beginning,[1] the question of validity of this formulation is still topical.[17,18]

Along with the accuracy of the u-p approximation with respect to the exact equations, another aspect of applicability of the u-p equations – as well as any other differential equations – is well-posedness (correctness) of the boundary value problems. A boundary value problem is said to be well posed if a solution exists, is unique and depends continuously on the initial and boundary data. Otherwise, the problem is said to be ill-posed. Whether a boundary value problem is well-posed depends on the equations themselves and on the form of initial and boundary conditions. Owing to the complexity of boundary value problems for solids with sophisticated plasticity models, there are no proofs of well-posedness for such problems. However, necessary conditions for well-posedness known for similar linear problems can be employed in a more general context for nonlinear problems and serve as criteria for ill-posedness when they are violated. Specifically, well-posedness of a dynamic boundary value problem for a solid with rate-independent constitutive behaviour requires the system of equations to be hyperbolic. Analyses of the governing equations with plasticity models (in particular, for geomaterials) revealed that the dynamic boundary value problems may become ill-posed due to loss of hyperbolicity. This concerns both one-phase solids[19–22] and porous fluid-saturated solids.[23,24] Some general aspects of hyperbolicity for anisotropic fluid-saturated solids were addressed recently in Refs [25, 26].

The above-mentioned papers[23–26] on fluid-saturated solids deal with the full (exact) system of equations involving the fluid velocity components as unknown variables. Well-posedness of the u-p approximation has not been studied so far and is the subject of the present paper. A difficulty in studying the u-p equations is that they are not of the form to which the definition of hyperbolicity applies. The system of equations is of a mixed hyperbolic-parabolic type and requires a special approach. There exist two u-p formulations depending on how many acceleration terms in the equations are neglected. For each u-p formulation, hyperbolicity conditions similar to the conventional definition of hyperbolicity are introduced in the present paper as necessary conditions for well-posedness. The constitutive relations for the solid skeleton are written in the general incrementally linear anisotropic form with an arbitrary stiffness tensor without considering any specific constitutive models. The hyperbolicity conditions for the two u-p formulations are expressed in terms of the acoustic tensor of the skeleton. This allows the two conditions to be easily compared with the known hyperbolicity conditions for dry and saturated solids expressed also in terms of the acoustic tensor.

2 | GOVERNING EQUATIONS

2.1 | Exact formulation

The equations in this paper are written in Cartesian coordinates x_1, x_2, x_3 for small strains with the partial time derivatives in place of the material ones neglecting the convective terms. The constitutive relations for a dry porous solid or a saturated solid under fully drained conditions (no changes in the pore pressure) are assumed to be in the rate form

$$\frac{\partial \sigma_{ji}}{\partial t} = C_{jikl} \frac{\partial v_{sk}}{\partial x_l}, \tag{1}$$

where $\sigma_{ji}, C_{jikl}, v_{sk}$ are, respectively, the components of the stress tensor, the stiffness tensor and the velocity vector, and t is time. The first subscript 's' in v_{sk} stands for 'solid', whereas the second one indicates the component. The summation convention for repeated indices is used throughout the paper. The coefficients C_{jikl} represent either the constant anisotropic stiffness of a linearly elastic solid or the current incremental stiffness of a plastic solid. The stiffness tensor defined by Equation (1) possesses the minor symmetries (the right minor symmetry follows from the fact that the stress rate is independent of the skew-symmetric part of the velocity gradient). The major symmetry is not assumed.

The dynamic deformation of the dry porous solid is described by the constitutive equations (1) and the equations of motion

$$\frac{\partial \sigma_{ji}}{\partial x_j} = (1-n)\varrho_s \frac{\partial v_{si}}{\partial t}, \tag{2}$$

where n is the porosity and ϱ_s is the density of the solid phase.

For a fluid-saturated porous solid, Equation (1) describes changes in the *effective stresses* defined as stresses which depend on the macroscopic deformation of the skeleton and are independent of changes in the pore pressure. For a linearly elastic skeleton, the effective stress components are[27–29]

$$\sigma_{ji} = \sigma_{ji}^{total} + \left(\delta_{ji} - \frac{C_{jikk}}{3K_s}\right) p_f, \tag{3}$$

where σ_{ji}^{total} are the total stress components, p_f is the pore fluid pressure (positive for compression), K_s is the bulk modulus of the solid phase and δ_{ji} is the Kronecker delta. Definition (3) is not applicable to plastic solids with a finite K_s since the coefficients C_{jikk} are not constant in that case. If the solid phase may be considered incompressible compared with the macroscopic stiffness of the skeleton, that is, if $K_s \gg |C_{jikl}|$, the effective stresses (3) with $K_s \to \infty$ reduce to

$$\sigma_{ji} = \sigma_{ji}^{total} + \delta_{ji} p_f. \tag{4}$$

As distinct from Equation (3), definition (4) is valid for both elastic and plastic skeletons. (For the definition of effective stresses in elastoplastic porous media with compressible solid phase, see Ref [30].)

For linearly elastic fluid-saturated solids, the evolution equation for the pore pressure is[28,29]

$$\frac{\partial p_f}{\partial t} = -Q^* \left(\delta_{ji} - n\delta_{ji} - \frac{C_{kkji}}{3K_s}\right) \frac{\partial v_{sj}}{\partial x_i} - Q^* n \frac{\partial v_{fl}}{\partial x_l}, \tag{5}$$

where

$$\frac{1}{Q^*} = \frac{n}{K_f} + \frac{1}{K_s}\left(1 - n - \frac{C_{iijj}}{9K_s}\right), \tag{6}$$

v_{fi} are the pore fluid velocity components (the first subscript 'f' stands for 'fluid', and the second one indicates the component), K_f is the pore fluid bulk modulus and n is the porosity. Equation (5) is written without convective terms, and the porosity gradient is also neglected.

If the stiffness tensor of the skeleton is such that

$$C_{jikk} = C_{kkji} = 3K\delta_{ji} \tag{7}$$

with a scalar K, then the effective stresses (3) can be written as

$$\sigma_{ji} = \sigma_{ji}^{total} + \alpha p_f \delta_{ji}, \tag{8}$$

where

$$\alpha = 1 - \frac{K}{K_s}. \tag{9}$$

Accordingly, Equation (5) for the pore pressure becomes

$$\frac{\partial p_f}{\partial t} = -Q(\alpha - n)\frac{\partial v_{sk}}{\partial x_k} - Qn\frac{\partial v_{fk}}{\partial x_k}, \tag{10}$$

where

$$\frac{1}{Q} = \frac{n}{K_f} + \frac{\alpha - n}{K_s}. \tag{11}$$

Condition (7) is satisfied for an isotropic skeleton with the bulk modulus K, but is weaker than the condition of isotropy.

The presence of the stiffness components C_{jikl} in Equations (3) and (5) not only complicates these equations, as compared to Equations (8) and (10) with the scalar α, but also gives rise to additional tensor components in the equations of motion for the solid phase. In applications, it may be reasonable to use Equations (8) and (10) with a scalar α estimated from C_{jikl} and K_s even though condition (7) is not satisfied exactly. Such an approximation of the equations would simplify them and yet allow one to investigate the influence of the compressibility of the solid phase.

In the present study, we use Equation (8) for the effective stresses and Equation (10) for the pore pressure with a scalar α. We also assume that $\alpha > n$, which is needed for the proof of certain propositions (for justification of this inequality for elastic porous solids, see, e.g.,[31–33] and references therein). This approach is applicable to

- plastic and elastic solids with incompressible solid phase ($K_s \gg |C_{jikl}|, \alpha = 1$),
- linearly elastic solids with compressible solid phase ($\alpha < 1$) as a rigorous or approximate approach depending, respectively, on whether Equation (7) is satisfied or not.

The equations of motion for the solid and fluid phases are[2]

$$\frac{\partial \sigma_{ji}}{\partial x_j} - (\alpha - n)\frac{\partial p_f}{\partial x_i} + \frac{n^2}{k}(v_{fi} - v_{si}) = (1 - n)\varrho_s \frac{\partial v_{si}}{\partial t}, \tag{12}$$

$$-n\frac{\partial p_f}{\partial x_i} - \frac{n^2}{k}(v_{fi} - v_{si}) = n\varrho_f \frac{\partial v_{fi}}{\partial t}, \tag{13}$$

where ϱ_s, ϱ_f are the densities of the solid and fluid phases. The permeability coefficient $k > 0$ in Equations (12) and (13) has the dimension [length^3×time/mass] and is connected with the permeability k' [length/time] conventionally used in geomechanics (hydraulic conductivity) by the relation $k = k'/(\varrho_f g)$, where g is the acceleration due to gravity.

The system of dynamic equations for a fluid-saturated solid studied here involves 13 unknown functions $v_{si}, v_{fi}, \sigma_{ji}, p_f$ and 13 scalar equations: the constitutive equations (1) for the effective stresses, the evolution equation (10) for the pore pressure, and the equations of motion (12), (13) for the solid and fluid phases. This system will be called the exact formulation.

2.2 | The u-p approximations

As mentioned in Introduction, there exist two u-p approximations. They have no special names in the literature and will be referred to in this paper as UP1 and UP2.

In certain cases – for instance, during the earthquake-induced dynamic deformation of soil – the accelerations of the solid and fluid phases calculated with the exact equations are close to each other. This observation led to the idea of replacing the fluid acceleration $\partial v_{fi}/\partial t$ in Equation (13) with the solid acceleration $\partial v_{si}/\partial t$.[1] After this substitution, adding Equations (12) and (13) gives the equations of motion for the whole continuum,

$$\frac{\partial \sigma_{ji}}{\partial x_j} - \alpha\frac{\partial p_f}{\partial x_i} = \varrho \frac{\partial v_{si}}{\partial t}, \tag{14}$$

where

$$\varrho = (1 - n)\varrho_s + n\varrho_f. \tag{15}$$

Equation (14) does not contain the fluid velocity and is the same as for a saturated solid under locally undrained conditions (zero permeability). Equation (13) with $\partial v_{si}/\partial t$ in place of $\partial v_{fi}/\partial t$ becomes

$$n(v_{fi} - v_{si}) = -k\left(\varrho_f \frac{\partial v_{si}}{\partial t} + \frac{\partial p_f}{\partial x_i}\right). \tag{16}$$

Table 1 Unknown functions and governing equations. $\mathbf{v}_s, \mathbf{v}_f, \boldsymbol{\sigma}, p_f$ are the solid and fluid velocity vectors, the effective stress tensor and the pore pressure.

	Unknown functions	Governing equations
Exact formulation	$\mathbf{v}_s, \mathbf{v}_f, \boldsymbol{\sigma}, p_f$	(1), (10), (12), (13)
u-*p* approximation UP1	$\mathbf{v}_s, \boldsymbol{\sigma}, p_f$	(1), (14), (17)
u-*p* approximation UP2	$\mathbf{v}_s, \boldsymbol{\sigma}, p_f$	(1), (14), (18)

The fluid velocity components can be eliminated from the set of unknown variables by substituting them from Equation (16) into Equation (10):

$$\frac{\partial p_f}{\partial t} = -Q\alpha \frac{\partial v_{si}}{\partial x_i} + Qk \frac{\partial}{\partial x_i}\left(\varrho_f \frac{\partial v_{si}}{\partial t} + \frac{\partial p_f}{\partial x_i} \right). \tag{17}$$

In the derivation of Equation (17), the gradients of the permeability and porosity are neglected.

The UP1 approximation involves 10 unknown functions v_{si}, σ_{ji}, p_f and 10 scalar equations: the constitutive equations (1) for the effective stresses, the equations of motion (14) for the whole continuum and the evolution equation (17) for the pore pressure.

The UP1 equations correspond to what is called *u*-*p* approximation in tab. 2.1 of Ref[2]. However, close examination of Ref. [2] and other sources reveals that a further simplification is often made to the equations at the stage of numerical implementation, see, for example, Ref. [2], Equations (3.17) and (3.28) as compared to Equations (2.21) and (2.33). Namely, after neglecting the relative fluid–solid acceleration and obtaining the equations of motion (14) for the whole continuum, the acceleration terms in the Equation (17) for the pore pressure are omitted, which turns this equation into

$$\frac{\partial p_f}{\partial t} = -Q\alpha \frac{\partial v_{si}}{\partial x_i} + Qk \frac{\partial^2 p_f}{\partial x_i \partial x_i}. \tag{18}$$

This leads to another version of the *u*-*p* approximation called here UP2. Omitting the acceleration terms in Equation (17) is equivalent to using the equations of motion (14) for the whole continuum and the quasi-static Darcy law rather than its dynamic version (13). The UP2 approximation, as compared to UP1, includes the same Equations (1) and (14) but a different equation for the pore pressure, Equation (18) instead of Equation (17). The UP2 approximation, despite being less accurate because of the additional simplification, is more popular than UP1 due to numerical convenience.

The unknown functions and the equations for the exact and *u*-*p* approximations are summarized in Table 1.

3 | HYPERBOLICITY OF THE EXACT EQUATIONS

This section presents some results on hyperbolicity for dry and fluid-saturated solids for comparison with hyperbolicity conditions that will be derived below for the *u*-*p* approximations.

Equations (1) and (2) for a dry solid and the exact Equations (1), (10), (12) and (13) for a saturated solid are of the form

$$\frac{\partial U}{\partial t} + \sum_{k=1}^{3} M^{(k)} \frac{\partial U}{\partial x_k} = F, \tag{19}$$

where U is the column vector of dependent variables and $M^{(k)}$ are real matrices. The dependent variables for dry and saturated solids are, respectively, v_{si}, σ_{ji} and $v_{si}, v_{fi}, \sigma_{ji}, p_f$. In the latter case, there is also a non-zero right-hand side F as a function of the dependent variables v_{si}, v_{fi}.

System (19) is called *hyperbolic* if for any real n_1, n_2, n_3 the eigenvalues of the matrix $M = \sum_{k=1}^{3} n_k M^{(k)}$ are real and there is a complete set of linearly independent eigenvectors (Ref. [34], Section 7.3.1). Hyperbolicity defined in this way is sometimes called *strong*, as distinct from *weak* hyperbolicity which does not require the existence of a complete set of eigenvectors (Ref. [35], Section 3.3). The verification of hyperbolicity amounts to the analysis of the eigenvalue problem

$$MU^0 = c\,U^0 \tag{20}$$

with the matrix M defined above, where c is the eigenvalue and U^0 is the eigenvector. Without loss of generality, we take n_1, n_2, n_3 in the definition of the matrix M to be the components of a unit vector $\mathbf{n}$. In this case, the eigenvalues c are the characteristic speeds.

It is known that hyperbolicity of the equations for a one-phase solid is determined by the properties of the acoustic tensor $\mathbf{A}$ with the components $A_{ik} = C_{jikl} n_j n_l$, where n_j are the components of a unit vector $\mathbf{n}$. We will make use of one proposition that gives necessary and sufficient conditions for strong hyperbolicity of the dynamic equations for one-phase solids.

Proposition 1. *System (1), (2) for a one-phase solid is hyperbolic if and only if for each* **n**, *the eigenvalues of the acoustic tensor* **A** *are real and positive with a complete set of eigenvectors.*[26]

For saturated solids, the eigenvalue problem (20) for the exact formulation is

$$\frac{1}{(1-n)\varrho_s}\left[-n_j\sigma^0_{ji} + (\alpha - n)n_i p^0_f\right] = c v^0_{si}, \tag{21}$$

$$\frac{1}{\varrho_f} n_i p^0_f = c v^0_{fi}, \tag{22}$$

$$-C_{jikl} n_l v^0_{sk} = c\sigma^0_{ji}, \tag{23}$$

$$Q(\alpha - n)n_k v^0_{sk} + Qn n_k v^0_{fk} = c p^0_f, \tag{24}$$

where $v^0_{si}, v^0_{fi}, \sigma^0_{ji}, p^0_f$ are the components of the eigenvector U^0. It is seen by inspection that there are five linearly independent eigenvectors associated with $c = 0$:

$$v^0_{si} = 0, \quad v^0_{fi} = 0, \quad \sigma^0_{ji} = s_j s_i, \quad p^0_f = 0, \tag{25}$$

$$v^0_{si} = 0, \quad v^0_{fi} = 0, \quad \sigma^0_{ji} = q_j q_i, \quad p^0_f = 0, \tag{26}$$

$$v^0_{si} = 0, \quad v^0_{fi} = 0, \quad \sigma^0_{ji} = s_j q_i + s_i q_j, \quad p^0_f = 0, \tag{27}$$

$$v^0_{si} = 0, \quad v^0_{fi} = s_i, \quad \sigma^0_{ji} = 0, \quad p^0_f = 0, \tag{28}$$

$$v^0_{si} = 0, \quad v^0_{fi} = q_i, \quad \sigma^0_{ji} = 0, \quad p^0_f = 0, \tag{29}$$

where s_i, q_i are the components of two non-zero vectors $\mathbf{s}$,$\mathbf{q}$ orthogonal to each other and to the vector $\mathbf{n}$. Since the eigenvectors (25)–(29) are linearly independent, the characteristic polynomial of the matrix of the eigenvalue problem (21)–(24) can be written as $c^5 f(c)$, where $f(c)$ is an 8th-degree polynomial in c. Another eight eigenvalues can be found by assuming $c \neq 0$ and substituting σ^0_{ji} and p^0_f from Equations (23) and (24) into Equations (21) and (22). This leads to an eigenvalue problem for the components v^0_{si}, v^0_{fi}:

$$\frac{1}{(1-n)\varrho_s}\left[A_{ik} v^0_{sk} + Q(\alpha - n)^2 n_i n_k v^0_{sk} + Q(\alpha - n)n n_i n_k v^0_{fk}\right] = c^2 v^0_{si}, \tag{30}$$

$$\frac{1}{\varrho_f} Q n_i n_k \left[(\alpha - n)v^0_{sk} + n v^0_{fk}\right] = c^2 v^0_{fi}, \tag{31}$$

where $A_{ik} = C_{jikl} n_j n_l$ are the components of the acoustic tensor. Multiplying Equation (31) by n_i with summation, the eigenvalue problem (30), (31) can be further reduced to a 4×4 system for $v^0_{s1}, v^0_{s2}, v^0_{s3}, v^0_f$, where $v^0_f = n_i v^0_{fi}$:

$$\frac{1}{(1-n)\varrho_s}\left[A_{ik} v^0_{sk} + Q(\alpha - n)^2 n_i n_k v^0_{sk} + Q(\alpha - n)n n_i v^0_f\right] = c^2 v^0_{si}, \tag{32}$$

$$\frac{1}{\varrho_f} Q\left[(\alpha - n)n_k v^0_{sk} + n v^0_f\right] = c^2 v^0_f, \tag{33}$$

or, in matrix form,

$$SU^0 = c^2U^0, \tag{34}$$

where $U^0 = (v^0_{s1}, v^0_{s2}, v^0_{s3}, v^0_f)^T$ is the eigenvector. The matrix S is

$$S = \begin{pmatrix} \frac{1}{(1-n)\varrho_s}A + \frac{(\alpha-n)^2Q}{(1-n)\varrho_s}NN^T & \frac{(\alpha-n)nQ}{(1-n)\varrho_s}N \\ \frac{(\alpha-n)Q}{\varrho_f}N^T & \frac{nQ}{\varrho_f} \end{pmatrix}, \tag{35}$$

where A is the matrix of the acoustic tensor, and $N = (n_1, n_2, n_3)^T$ denotes the column vector of the components of the vector $\mathbf{n}$. The 4×4 matrix S of the eigenvalue problem (32), (33) will be referred to as *the acoustic matrix* of the exact formulation. The eigenvalues of the acoustic matrix are the squared characteristic speeds. This fact establishes a correspondence between the eigenvalues of the matrix M of system (1), (10), (12), (13) and the eigenvalues of the acoustic matrix S. Finding the eigenvalues of S suffices to verify hyperbolicity in the weak sense. For the characteristic speeds in the particular case of an isotropic elastic skeleton, see, for example, Ref. [25], Section 6 and Ref. [36], Section 4.2.

In Propositions 2 and 3 below, adapted from Refs. [25, 26], hyperbolicity is understood in the strong sense.

Proposition 2. *System (1), (10), (12), (13) of the exact formulation is hyperbolic if and only if for each* $\mathbf{n}$*, the eigenvalues of the acoustic matrix* S *are real and positive with a complete set of eigenvectors.*

Proposition 2 provides necessary and sufficient conditions for hyperbolicity in terms of the acoustic matrix S. The proposition is proved in Ref. [26] for incompressible solid phase with $\alpha = 1$, but the proof applies to the equations with $\alpha > n$ as well. In general, there is no correspondence between hyperbolicity for a dry solid determined by the eigenvalues of the acoustic tensor $\mathbf{A}$, and hyperbolicity for the same but saturated solid determined by the eigenvalues of the acoustic matrix S. The dynamic equations for the dry solid may be hyperbolic, while the equations for the saturated solid may not.[24] The following proposition in combination with Proposition 1 shows that there are special cases in which hyperbolicity is guaranteed for both the dry and saturated solids.

Proposition 3. *System (1), (10), (12), (13) of the exact formulation is hyperbolic if for each* $\mathbf{n}$*, the acoustic tensor* $\mathbf{A}$ *of the skeleton is symmetric and positive definite.*[*]

Proposition 3 proved in Ref. [25] provides sufficient conditions for hyperbolicity for the saturated solid in terms of the acoustic tensor of the skeleton. As follows from Proposition 1, these conditions – the symmetry and positive definiteness of the acoustic tensor – also guarantee hyperbolicity for the dry solid, so both systems of equations are hyperbolic in this case.

4 | HYPERBOLICITY CONDITIONS FOR THE U-P APPROXIMATIONS

Replacing $\partial v_{fi}/\partial t$ with $\partial v_{si}/\partial t$ in Equation (13) gives immediately the UP1 approximation as a first-order system for the functions $v_{si}, v_{fi}, \sigma_{ji}, p_f$. This system is, however, not of the form (19), as it does not contain the time derivatives of the fluid velocity, so that the definition of hyperbolicity given in Section 3 for system (19) is not applicable. Reducing the set of unknown functions to v_{si}, σ_{ji}, p_f leads to system (1), (14), (17). The definition of hyperbolicity is not applicable to this system either, because Equation (17) is second order. It is nevertheless possible to derive a hyperbolicity condition for the UP1 equations based on the notion of characteristics. This can be done for either of the two systems (first or second order) in essentially the same way leading eventually to the same result. We proceed with the second-order system (1), (14), (17). For our purposes, we will first reduce the system to a more tractable first-order system in one space variable.

Hyperbolicity of the system

$$\frac{\partial U}{\partial t} + M\frac{\partial U}{\partial x} = F \tag{36}$$

[*] For the symmetry of the acoustic tensor, see Appendix.

in one space variable x is defined by imposing the same conditions on the eigenvalues and eigenvectors of the matrix M as for system (19). Let $n_i, i = 1, 2, 3$, be the components of a unit vector $\mathbf{n}$. For plane wave solutions of system (19) of the form $U(x, t)$, where $x = n_i x_i$, system (19) reduces to system (36) with the matrix $M = \sum_{k=1}^{3} n_k M^{(k)}$. This shows the well-known connection between hyperbolicity of the spatially three-dimensional system (19) and hyperbolicity of the one-dimensional system (36) that describes plane wave solutions: the hyperbolicity condition for system (19) requires system (36) for plane wave solutions to be hyperbolic for all directions $\mathbf{n}$.

Substituting $\partial v_{si}/\partial t$ from Equation (14) into Equation (17) and writing the UP1 equations (14), (1), (17) for plane wave solutions, we obtain, respectively,

$$\varrho \frac{\partial v_{si}}{\partial t} - n_j \frac{\partial \sigma_{ji}}{\partial x} + \alpha n_i \frac{\partial p_f}{\partial x} = 0, \tag{37}$$

$$\frac{\partial \sigma_{ji}}{\partial t} - C_{jikl} n_l \frac{\partial v_{sk}}{\partial x} = 0, \tag{38}$$

$$\frac{\partial p_f}{\partial t} + Q\alpha n_i \frac{\partial v_{si}}{\partial x} - \frac{Qk}{\varrho} \left[\varrho_f n_i n_j \frac{\partial^2 \sigma_{ji}}{\partial x^2} + \left(\varrho - \alpha \varrho_f\right) \frac{\partial^2 p_f}{\partial x^2} \right] = 0. \tag{39}$$

System (37)–(39), being first order in time and second order in space, resembles a parabolic system, but it is not parabolic (cf. Ref. [35], Section 3.4). The system is of a mixed type. It consists of the hyperbolic-like equations (37), (38) without second-order derivatives, and the parabolic-like equation (39). The second-order equation (39) can be written as two first-order equations with a new function h:

$$\frac{\partial p_f}{\partial t} + Q\alpha n_i \frac{\partial v_{si}}{\partial x} - \frac{Qk}{\varrho} \frac{\partial h}{\partial x} = 0, \tag{40}$$

$$\varrho_f n_i n_j \frac{\partial \sigma_{ji}}{\partial x} + \left(\varrho - \alpha \varrho_f\right) \frac{\partial p_f}{\partial x} = h. \tag{41}$$

System (37), (38), (40), (41) for $v_{si}, \sigma_{ji}, p_f, h$ is of the form

$$J \frac{\partial U}{\partial t} + M \frac{\partial U}{\partial x} = F, \tag{42}$$

where the matrix J is singular. The singularity of J does not allow the system to be multiplied on the left by J^{-1} and written like system (36). As will be seen below, the matrix M is singular as well, so we have to deal with system (42) containing two matrices. In such a case, the characteristic speeds $c = \Delta x/\Delta t$ are found from the generalized eigenvalue problem

$$\Delta x J U^0 = \Delta t M U^0, \tag{43}$$

where U^0 is the eigenvector. The generalized eigenvalue problem encompasses the characteristics with $\Delta t = 0$ (infinite characteristic speed) when the matrix J is singular.

For finite characteristic speeds with $\Delta t \neq 0$, the eigenvalue problem (43) for system (37), (38), (40), (41) can be written in component form as

$$c\varrho v_{si}^0 + n_j \sigma_{ji}^0 - \alpha n_i p_f^0 = 0, \tag{44}$$

$$c\sigma_{ji}^0 + C_{jikl} n_l v_{sk}^0 = 0, \tag{45}$$

$$c p_f^0 - Q\alpha n_i v_{si}^0 + \frac{Qk}{\varrho} h^0 = 0, \tag{46}$$

$$\varrho_f n_i n_j \sigma_{ji}^0 + \left(\varrho - \alpha \varrho_f\right) p_f^0 = 0, \tag{47}$$

where $c = \Delta x/\Delta t$ is the characteristic speed, and $v_{si}^0, \sigma_{ji}^0, p_f^0, h^0$ are the components of the eigenvector U^0. The determinant of system (44)–(47) can be written as $c^3 f(c)$, where $f(c)$ is a 6th-degree polynomial in c, so there must be nine finite characteristic speeds, counting multiplicities. It is seen by inspection that there are three linearly independent eigenvectors associated with $c = 0$:

$$v_{si}^0 = 0, \quad \sigma_{ji}^0 = s_j s_i, \quad p_f^0 = 0, \quad h^0 = 0, \tag{48}$$

$$v_{si}^0 = 0, \quad \sigma_{ji}^0 = q_j q_i, \quad p_f^0 = 0, \quad h^0 = 0, \tag{49}$$

$$v_{si}^0 = 0, \quad \sigma_{ji}^0 = s_j q_i + s_i q_j, \quad p_f^0 = 0, \quad h^0 = 0, \tag{50}$$

where s_i, q_i are the components of two non-zero vectors $\mathbf{s}$,$\mathbf{q}$ orthogonal to each other and to the vector $\mathbf{n}$. The eigenvalue $c = 0$ means that the matrix M in Equation (42) is singular, as mentioned earlier.

In order to find non-zero speeds, we assume $c \neq 0$, substitute σ_{ji}^0 from Equation (45) into Equations (44) and (47) and then, assuming $\varrho \neq \alpha\varrho_f$, substitute p_f^0 from Equation (47) into Equation (44). In this way, the components σ_{ji}^0, p_f^0 are eliminated from the equations, and we arrive at a system for v_{si}^0:

$$B_{ik} v_{sk}^0 = \varrho c^2 v_{si}^0, \tag{51}$$

where

$$B_{ik} = A_{ik} + \frac{\alpha\varrho_f}{\varrho - \alpha\varrho_f} n_i n_j A_{jk}, \tag{52}$$

and $A_{ik} = C_{jikl} n_j n_l$ are the components of the acoustic tensor. System (51) is the eigenvalue problem for a matrix B with the components B_{ik}, and ϱc^2 is the eigenvalue. In matrix form,

$$B = A + \frac{\alpha\varrho_f}{\varrho - \alpha\varrho_f} N N^T A = \left(I + \frac{\alpha\varrho_f}{\varrho - \alpha\varrho_f} N N^T \right) A, \tag{53}$$

where A is the matrix of the acoustic tensor, and N is the column vector of the components of the vector $\mathbf{n}$. Since A_{ik} are the components of a tensor (the acoustic tensor $\mathbf{A}$), B_{ik} are also the components of a tensor $\mathbf{B}$ written in tensorial notations as

$$\mathbf{B} = \mathbf{A} + \frac{\alpha\varrho_f}{\varrho - \alpha\varrho_f} \mathbf{n} \otimes \mathbf{n} \cdot \mathbf{A}. \tag{54}$$

Thus, we have seen that the UP1 equations (37), (38), (40), (41) for plane wave solutions have infinite and zero characteristic speeds and, in addition, six characteristic speeds $c = \pm\sqrt{\zeta_i/\varrho}$, $i = 1, 2, 3$, where ζ_i are the eigenvalues of the tensor $\mathbf{B}$.

In the particular case of an isotropic elastic skeleton with the Lamé constants λ and μ, the components of the acoustic tensor and the tensor $\mathbf{B}$ are

$$A_{ik} = (\lambda + \mu) n_i n_k + \mu\delta_{ik}, \tag{55}$$

$$B_{ik} = (\lambda + \mu) n_i n_k + \mu\delta_{ik} + \frac{\alpha\varrho_f}{\varrho - \alpha\varrho_f} (\lambda + 2\mu) n_i n_k. \tag{56}$$

The eigenvalues of the tensor $\mathbf{B}$ are

$$\zeta_1 = \frac{\varrho}{\varrho - \alpha\varrho_f} (\lambda + 2\mu), \quad \zeta_2 = \zeta_3 = \mu. \tag{57}$$

If $\mathbf{n}$ is parallel to one of the coordinate axis, the system of equations splits into two independent systems: one for the longitudinal motion, and the other one for the transverse motion, with the eigenvalues ζ_1 and $\zeta_2 = \zeta_3$, respectively. If $\varrho > \alpha\varrho_f$, the eigenvalue ζ_1 is positive and the characteristic speeds $\pm\sqrt{\zeta_1/\varrho}$ for longitudinal waves are real. The inequality $\varrho > \alpha\varrho_f$, or equivalently

$$(1 - n)\varrho_s > (\alpha - n)\varrho_f, \tag{58}$$

is satisfied in applications. For incompressible solid phase, $\alpha = 1$ and the condition $\varrho > \alpha\varrho_f$ reduces to $\varrho_s > \varrho_f$.

We assume that the boundary value problem of the UP1 approximation for an isotropic elastic fluid-saturated solid with $\varrho > \alpha\varrho_f$ and properly specified initial and boundary conditions is well-posed. Based on this premise and taking into account that the characteristic speeds are determined by the eigenvalues of the tensor $\mathbf{B}$, we postulate the following necessary condition for well-posedness in the general anisotropic case. We say that the UP1 system (1), (14), (17) satisfies *the hyperbolicity condition* if $\varrho \neq \alpha\varrho_f$ and the eigenvalues of the tensor $\mathbf{B}$ defined by Equation (54) are real and positive for all directions $\mathbf{n}$. If the hyperbolicity condition is not satisfied, the boundary value problem is considered to be ill-posed.

Similar to the exact formulation, there are cases in which the UP1 hyperbolicity condition can be shown to be satisfied.

Proposition 4. *If $\varrho > \alpha\varrho_f$ and the acoustic tensor of the skeleton is symmetric and positive definite for all directions $\mathbf{n}$, then the UP1 system* (1), (14), (17) *satisfies the hyperbolicity condition.*

Proof. If $\varrho > \alpha\varrho_f$, then the matrix in brackets in Equation (53) is symmetric and positive definite. The eigenvalues of the product of two real symmetric positive definite matrices are real and positive (Ref. [37], Corollary 7.6.2). □

The UP2 system is similar to the UP1 system. It contains the same two first-order equations (1), (14) and the second-order equation (18), but the latter is different from Equation (17) – in particular, it does not contain the stress components. We proceed along the same lines as before to reduce the UP2 system to a first-order system in one space variable. For plane wave solutions $U(x,t)$, where $x = n_i x_i$, Equation (18) becomes

$$\frac{\partial p_f}{\partial t} + Q\alpha n_i \frac{\partial v_{si}}{\partial x} - Qk\frac{\partial^2 p_f}{\partial x^2} = 0. \tag{59}$$

Introducing a new function g, we write the second-order equation (59) as two first-order equations

$$\frac{\partial p_f}{\partial t} + Q\alpha n_i \frac{\partial v_{si}}{\partial x} - Qk\frac{\partial g}{\partial x} = 0, \tag{60}$$

$$\frac{\partial p_f}{\partial x} = g. \tag{61}$$

System (37), (38), (60), (61) is of the form (42), where both matrices J and M are singular. The singularity of J yields infinite characteristic speeds, while the singularity of M yields zero speeds. The matrix M has the same three linearly independent eigenvectors (48)–(50) associated with $c = 0$, with g^0 in place of h^0. For finite characteristic speeds, the eigenvalue problem (43) consists of Equations (44), (45) and two equations

$$cp_f^0 - Q\alpha n_i v_{si}^0 + Qkg^0 = 0, \tag{62}$$

$$p_f^0 = 0. \tag{63}$$

The determinant of system (44), (45), (62), (63) can be written as $c^3 f(c)$, where $f(c)$ is a 6th-degree polynomial in c. This shows that there are nine finite characteristic speeds, counting multiplicities. Since Equations (62) and (63) provide no information about the characteristic speeds, these are found solely from Equations (44) and (45) with $p_f^0 = 0$. Equations (44) and (45) with $p_f^0 = 0$ are the same as for a one-phase solid with the density ϱ. For $c \neq 0$, substituting σ_{ji}^0 from Equation (45) into Equation (44) leads to the eigenvalue problem

$$A_{ik} v_{sk}^0 = \varrho c^2 v_{si}^0, \tag{64}$$

which gives six characteristic speeds $c = \pm\sqrt{\eta_i/\varrho}$, $i = 1, 2, 3$, where η_i are the eigenvalues of the acoustic tensor. In the isotropic case, the eigenvalues of the acoustic tensor are

$$\eta_1 = \lambda + 2\mu, \quad \eta_2 = \eta_3 = \mu. \tag{65}$$

They are positive and give six real characteristic speeds, counting multiplicities. Assuming that the boundary value problem of the UP2 formulation with properly specified initial and boundary conditions is well-posed in the isotropic case, we postulate the following necessary condition for well-posedness in the general anisotropic case. We say that the UP2 system (1), (14), (18) satisfies *the hyperbolicity condition* if the eigenvalues of the acoustic tensor of the skeleton are real and positive for all directions $\mathbf{n}$.

5 | CONCLUSION

Hyperbolicity of the dynamic equations for fluid-saturated solids with rate-independent constitutive behaviour of the skeleton is a necessary condition for well-posedness of the boundary value problems. The equations of the two u-p approximations, written either as first-order systems for $v_{si}, v_{fi}, \sigma_{ji}, p_f$ or second-order systems for v_{si}, σ_{ji}, p_f, do not belong to the class of equations to which the conventional definition of hyperbolicity can be applied. The hyperbolicity conditions proposed in this paper as necessary conditions for well-posedness for the u-p equations are derived from the characteristic speed analysis of the equations for plane waves, with infinite characteristic speeds being allowed for. It is assumed that

(i) for isotropic elastic solids, the u-p approximations lead to well-posed problems (provided $\varrho > \alpha\varrho_f$ for the UP1 approximation),
(ii) in the general anisotropic case, for the boundary value problems of the u-p approximations to be well-posed, the number of real non-zero finite characteristic speeds must be the same as in the isotropic case.

The requirement (ii) is fulfilled if the hyperbolicity condition for the u-p approximation is satisfied. According to the assumptions (i) and (ii), the boundary value problem of the u-p approximation is considered to be ill-posed if the corresponding hyperbolicity condition is violated.

Although the exact formulation and the two u-p approximations can be used for the modelling of the same physical process with the same constitutive relations, the criteria for well-posedness defined in terms of hyperbolicity are different for the three systems of equations. The hyperbolicity condition for the UP1 equations involves the eigenvalues of the 3×3 matrix B defined by Equation (53), while hyperbolicity of the exact equations is determined by the eigenvalues of the 4×4 acoustic matrix S defined by Equation (35). A marked difference is that the hyperbolicity condition for the UP1 equations does not depend on the compressibility of the pore fluid. Another distinctive feature is the role of the inequality $\varrho > \alpha\varrho_f$ for the UP1 equations. In particular, if $\varrho < \alpha\varrho_f$, then the UP1 equations for an isotropic solid yield imaginary characteristic speeds for longitudinal waves, see Equation (57). The similarity between the exact and the UP1 formulations is that in both cases, hyperbolicity for the dry solid determined by the acoustic tensor of the skeleton does not in general guarantee hyperbolicity for the saturated solid, except when the acoustic tensor is symmetric.

The hyperbolicity condition for the UP2 approximation is basically the same as weak hyperbolicity for the dry solid, as both are determined by the eigenvalues of the acoustic tensor of the skeleton. If the equations for the dry solid are hyperbolic, then the UP2 hyperbolicity condition is fulfilled. This fact simplifies the verification of hyperbolicity. Both the UP1 and UP2 hyperbolicity conditions do not involve the compressibility of the pore fluid.

Hyperbolicity is ensured for all three formulations if the acoustic tensor of the skeleton is symmetric and positive definite for all directions (provided $\varrho > \alpha\varrho_f$ for the UP1 approximation). This is the case, in particular, for a linearly hyperelastic skeleton with a positive strain energy function.

ACKNOWLEDGEMENTS

Open Access funding enabled and organized by Projekt DEAL.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analysed in this study.

ORCID

Vladimir A. Osinov https://orcid.org/0000-0002-6177-9770

REFERENCES

1. Zienkiewicz OC, Chang CT, Bettess P. Drained, undrained, consolidating and dynamic behaviour assumptions in soils. *Géotechnique*. 1980;30(4):385-395.
2. Zienkiewicz OC, Chan AHC, Pastor M, Schrefler BA, Shiomi T. *Computational Geomechanics with Special Reference to Earthquake Engineering*. Wiley; 1999.
3. Zhang HW, Sanavia L, Schrefler BA. An internal length scale in dynamic strain localization of multiphase porous media. *Mech Cohes-Frict Mater*. 1999;4(5):443-460.
4. Elgamal A, Yang Z, Parra E. Computational modeling of cyclic mobility and post-liquefaction site response. *Soil Dyn Earthq Eng*. 2002;22:259-271.

5. Li C, Borja RI, Regueiro RA. Dynamics of porous media at finite strain. *Comput Methods Appl Mech Eng*. 2004;193:3837-3870.
6. Taiebat M, Shahir H, Pak A. Study of pore pressure variation during liquefaction using two constitutive models for sand. *Soil Dyn Earthq Eng*. 2007;27:60-72.
7. Huang Y, Yashima A, Sawada K, Zhang F. Numerical assessment of the seismic response of an earth embankment on liquefiable soils. *Bull Eng Geol Environ*. 2008;67:31-39.
8. Dewoolkar MM, Chan AHC, Ko HY, Pak RYS. Finite element simulations of seismic effects on retaining walls with liquefiable backfills. *Int J Numer Anal Meth Geomech*. 2009;33:791-816.
9. Xia ZF, Ye GL, Wang JH, Ye B, Zhang F. Fully coupled numerical analysis of repeated shake-consolidation process of earth embankment on liquefiable foundation. *Soil Dyn Earthq Eng*. 2010;30:1309-1318.
10. Na SH, Sun WC. Wave propagation and strain localization in a fully saturated softening porous medium under the non-isothermal conditions. *Int J Numer Anal Meth Geomech*. 2016;40(10):1485-1510.
11. Chrisopoulos S, Osinov VA, Triantafyllidis T. Dynamic problem for the deformation of saturated soil in the vicinity of a vibrating pile toe. In: Triantafyllidis T, ed. *Holistic Simulation of Geotechnical Installation Processes, Benchmarks and Simulations*. Springer International Publishing; 2016:53-67.
12. Little M, Rathje E. Key trends regarding the effects of site geometry on lateral spreading displacements. *J Geotech Geoenviron Eng*. 2021;147(12):04021142.
13. Zhang HW, Wang KP, Chen Z. Material point method for dynamic analysis of saturated porous media under external contact/impact of solid bodies. *Comput Methods Appl Mech Eng*. 2009;198:1456-1472.
14. Zheng Y, Gao F, Zhang H, Lu M. Improved convected particle domain interpolation method for coupled dynamic analysis of fully saturated porous media involving large deformation. *Comput Methods Appl Mech Eng*. 2013;257:150-163.
15. Zhang H, Li H, Ye H, Zheng Y. A coupling peridynamic approach for the consolidation and dynamic analysis of saturated porous media. *Comput Mech*. 2019;64:1097-1113.
16. Wei H, Chen JS, Beckwith F, Baek J. A naturally stabilized semi-Lagrangian meshfree formulation for multiphase porous media with application to landslide modeling. *J Eng Mech*. 2020;146(4):04020012.
17. Staubach P, Machacek J. Influence of relative acceleration in saturated sand: analytical approach and simulation of vibratory pile driving tests. *Comput Geotech*. 2019;112:173-184.
18. Argani LP, Gajo A. A novel insight into vertical ground motion modelling in earthquake engineering. *Int J Numer Anal Meth Geomech*. 2022;46:164-186.
19. Loret B, Prévost JH, Harireche O. Loss of hyperbolicity in elastic-plastic solids with deviatoric associativity. *Eur J Mech A/Solids*. 1990;9(3):225-231.
20. Bigoni D, Zaccaria D. On the eigenvalues of the acoustic tensor in elastoplasticity. *Eur J Mech A/Solids*. 1994;13(5):621-638.
21. Osinov VA. Theoretical investigation of large-amplitude waves in granular soils. *Soil Dyn Earthq Eng*. 1998;17(1):13-28.
22. Osinov VA, Gudehus G. Dynamics of hypoplastic materials: theory and numerical implementation. In: Hutter K, Kirchner N, eds. *Dynamic Response of Granular and Porous Materials under Large and Catastrophic Deformations*. Springer; 2003:265-284.
23. Loret B, Harireche O. Acceleration waves, flutter instabilities and stationary discontinuities in inelastic porous media. *J Mech Phys Solids*. 1991;39(5):569-606.
24. Osinov VA. On well-posedness of the dynamic problem for an anisotropic fluid-saturated solid. *Arch Appl Mech*. 2009;79:69-80.
25. Osinov VA. Sufficient conditions for hyperbolicity and consistency of the dynamic equations for fluid-saturated solids. *Arch Appl Mech*. 2021;91:2569-2579.
26. Osinov VA. On hyperbolicity of the dynamic equations for plastic fluid-saturated solids. *Arch Appl Mech*. 2021;91:3327-3339.
27. Carroll MM. An effective stress law for anisotropic elastic deformation. *J Geophys Res*. 1979;84(B13):7510-7512.
28. Cheng AHD. Material coefficients of anisotropic poroelasticity. *Int J Rock Mech Min Sci*. 1997;34(2):199-205.
29. Zhang Q, Borja RI. Poroelastic coefficients for anisotropic single and double porosity media. *Acta Geotechnica*. 2021;16:3013-3025.
30. Zhao Y, Borja RI. A continuum framework for coupled solid deformation–fluid flow through anisotropic elastoplastic porous media. *Comput Methods Appl Mech Eng*. 2020;369:113225.
31. Biot MA, Willis DG. The elastic coefficients of the theory of consolidation. *J Appl Mech*. 1957;24:594-601.
32. Zimmerman RW, Somerton WH, King MS. Compressibility of porous rocks. *J Geophys Res*. 1986;91(B12):12765-12777.
33. Berryman JG. Effective stress for transport properties of inhomogeneous porous rock. *J Geophys Res*. 1992;97(B12):17409-17424.
34. Evans LC. *Partial Differential Equations*. 2nd ed. American Mathematical Society; 2010.
35. Gustafsson B, Kreiss H-O, Oliger J. *Time-dependent Problems and Difference Methods*. 2nd ed. Wiley; 2013.
36. Osinov VA. Non-reflecting boundary conditions for plane waves in anisotropic elasticity and poroelasticity. *Acta Mech*. 2012;223:593-607.
37. Horn RA, Johnson CR. *Matrix Analysis*. 2nd ed. Cambridge University Press; 2013.

How to cite this article: Osinov VA. The u-p approximation versus the exact dynamic equations for anisotropic fluid-saturated solids. I. Hyperbolicity. *Int J Numer Anal Methods*. 2023;1-13. https://doi.org/10.1002/nag.3506

APPENDIX

SYMMETRY OF THE ACOUSTIC TENSOR

As shown in Ref. [25], the necessary and sufficient condition for the symmetry of the acoustic tensor for all directions **n** is

$$C_{ijkl} - C_{klij} = C_{kjil} - C_{ilkj}, \tag{A.1}$$

where C_{jikl} are the components of the stiffness tensor defined by Equation (1). The stiffness tensor defined by Equation (1) possesses both minor symmetries. Condition (A.1) seems at first glance to be weaker than the major symmetry $C_{jikl} = C_{klji}$, so it is erroneously inferred in Ref. [25] that the major symmetry is not necessary for the acoustic tensor to be symmetric for all directions. A closer inspection reveals that condition (A.1) leads to the major symmetry. Indeed, writing Equation (A.1) as

$$C_{jkli} - C_{lijk} = C_{lkji} - C_{jilk} \tag{A.2}$$

and taking into account the minor symmetries, we see that the right-hand side of Equation (A.1) is the same as the left-hand side of Equation (A.2), and hence the left-hand side of Equation (A.1) is equal to the right-hand side of Equation (A.2). The latter equality gives the major symmetry. Thus, the major symmetry follows from Equation (A.1) and is, therefore, necessary for the acoustic tensor to be symmetric for all directions.